

(in average 1000 times quicker than the ADD heuristics). So CPL dominates the ADD heuristics completely, both with respect to solution time and solution quality.

The Lagrangian heuristic, LH, produces solutions with relative errors between 0.4% and 3.2% (in average 1.5%), with solution times in average 20 times shorter than the ADD heuristics, but of course significantly longer than CPL.

Comparison to other tests is difficult, since other computers and codes are used. The Benders approach in [11] seems to be slower than the Lagrangian approach. However, on modern computers and with modern MIP-codes, its performance may well improve.

Conclusion

The capacitated facility location problem with staircase costs has many important applications. Computational results indicate that it is possible to find near-optimal solutions to such problems of reasonable size in a reasonable time, i. e. that this better model can be used instead of, for example, the ordinary capacitated facility location problem in appropriate situations.

See also

- ▶ [Combinatorial Optimization Algorithms in Resource Allocation Problems](#)
- ▶ [Competitive Facility Location](#)
- ▶ [Facility Location with Externalities](#)
- ▶ [Facility Location Problems with Spatial Interaction](#)
- ▶ [Global Optimization in Weber's Problem with Attraction and Repulsion](#)
- ▶ [MINLP: Application in Facility Location-allocation](#)
- ▶ [Multifacility and Restricted Location Problems](#)
- ▶ [Network Location: Covering Problems](#)
- ▶ [Optimizing Facility Location with Rectilinear Distances](#)
- ▶ [Production-distribution System Design Problem](#)
- ▶ [Resource Allocation for Epidemic Control](#)
- ▶ [Single Facility Location: Circle Covering Problem](#)
- ▶ [Single Facility Location: Multi-objective Euclidean Distance Location](#)
- ▶ [Single Facility Location: Multi-objective Rectilinear Distance Location](#)
- ▶ [Stochastic Transportation and Location Problems](#)
- ▶ [Voronoi Diagrams in Facility Location](#)
- ▶ [Warehouse Location Problem](#)

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Farkas Lemma

FI

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Article Outline

Keywords

See also

References

Keywords

Inequality systems; Certificate; Theorem of the alternative; Skew-symmetric matrix; Orthogonal matrix

Farkas' lemma is the most well-known *theorem of the alternative* or *transposition theorem* (cf. ► **Linear optimization: Theorems of the alternative**). Given an $m \times n$ matrix A and a vector b (of dimension m) it states that either the set

$$S := \{y: y^T A \geq 0, y^T b < 0\}$$

or the set

$$T := \{x: Ax = b, x \geq 0\}$$

is empty but not both sets are empty. This result has a long history and it has had a tremendous impact on the development of the duality theory of linear and nonlinear optimization.

J. Farkas (1847–1930) was professor of Theoretical Physics at the Univ. of Kolozsvár in Hungary. His interest in the subject is explained in the first two sentences of his paper [5]:

The natural and systematic treatment of analytic mechanics has to have as its background the inequality principle of virtual displacements first formulated by Fourier and later by Gauss. The possibility of such a treatment requires, however, some knowledge of homogeneous linear inequalities that may be said to have been entirely missing up to now.

J.B.J. Fourier [7] seems to have been the first who established that a mechanical system has a stable equilibrium state if and only if some homogeneous system of inequalities, like in the definition of the above set S , has no solution. This observation became known as the *mechanical principle of Fourier*. By Farkas' lemma this happens if and only if the set T is nonempty.

It is almost obvious that if the set T is not empty, then the set S will be empty and we have equilibrium.

This follows easily by noting that the sets S and T cannot be both nonempty: if $y \in S$ and $x \in T$ then the contradiction

$$y^T b = y^T (Ax) = (y^T A)x \geq 0$$

follows, because $y^T A \geq 0$ and $x \geq 0$. This shows that the condition 'T is not empty' is certainly a sufficient condition for equilibrium. The hard part is to prove that this is also a necessary condition for equilibrium. The proof has a long history. First, the condition without proof for special cases was given by A. Cournot in 1827 and for the general case by M. Ostrogradsky in 1834. Farkas published his condition first in 1894 and 1895, but the proof contains a gap. A second attempt, in 1896, is also incomplete. The first complete proof was published in Hungarian, in 1898 [3], and in German in 1899 [4]. This proof is included in Farkas' best known paper [5]. For more details and references, see the historical overviews [9] and [10].

Nowadays (1998) many different proofs of Farkas' lemma are known. For quite recent proofs, see, e.g., [1,2,8]. An interesting derivation has been given by A.W. Tucker [11], based on a result that will be referred to as *Tucker's theorem*. (See ► **Tucker homogeneous systems of linear relations**.) The theorem states that for any *skew-symmetric matrix* K (i. e., $K = -K^T$) there exists a vector x such that

$$Kx \geq 0, \quad x \geq 0, \quad x + Kx > 0.$$

By taking

$$K = \begin{pmatrix} 0 & 0 & A & -b \\ 0 & 0 & -A & b \\ -A^T & A^T & 0 & 0 \\ b^T & -b^T & 0 & 0 \end{pmatrix},$$

Tucker's theorem implies the existence of nonnegative vectors z_1 , z_2 and x and a nonnegative scalar t such that

$$Ax - tb \geq 0, \tag{1}$$

$$-Ax + tb \geq 0, \tag{2}$$

$$\begin{aligned} -A^T z_1 + A^T z_2 &\geq 0, \\ b^T z_1 - b^T z_2 &\geq 0, \end{aligned} \tag{3}$$

and

$$\begin{aligned} z_1 + Ax - tb &> 0, \\ z_2 - Ax + tb &> 0, \\ x - A^\top z_1 + A^\top z_2 &> 0, \\ t + b^\top z_1 - b^\top z_2 &> 0. \end{aligned} \quad (4)$$

If $t = 0$, then, putting $y = z_2 - z_1$, (3) and (4) yield a vector in the set S . If $t > 0$, since the above inequalities are all homogeneous, one may take $t = 1$ and then (1) and (2) give a vector in the set T . This shows that at least one of the two sets S and T is nonempty, proving the hard part of Farkas' lemma.

It is worth mentioning a *result* of C.G. Broyden [1] who showed that Tucker's theorem, and hence also Farkas' lemma, follows from a simple property of orthogonal matrices. The result states that for any orthogonal matrix Q (so $QQ^\top = Q^\top Q = I$) there exists a unique sign matrix D and a positive vector x such that $Qx = Dx$; a *sign matrix* is a diagonal matrix whose diagonal elements are equal to either plus one or minus one.

The key observation here is that if K is a skew-symmetric matrix, then

$$Q = (I + K)^{-1}(I - K)$$

is an orthogonal matrix, where I denotes the identity matrix; Q is known as the *Cayley transform* of K [6]. The proof of this fact is straightforward. First, for each vector x one has

$$x^\top(I + K)x = x^\top x,$$

whence $I + K$ is an invertible matrix. Furthermore, using $K^\top = -K$, one may write

$$\begin{aligned} Q^\top Q &= (I + K)(I - K)^{-1}(I + K)^{-1}(I - K) \\ &= (I + K)(I - K^2)^{-1}(I - K). \end{aligned}$$

Multiplying both sides from the left with $(I - K)$ one gets

$$\begin{aligned} (I - K)QQ^\top &= (I - K^2)(I - K^2)^{-1}(I - K) \\ &= (I - K), \end{aligned}$$

and multiplying both sides with $(I - K)^{-1}$ one finds $QQ^\top = I$, showing that Q is orthogonal indeed.

Therefore, by Broyden's theorem, there exists a sign matrix D and a positive vector z such that

$$(I + K)^{-1}(I - K)z = Dz.$$

This can be rewritten as

$$(I - K)z = (I + K)Dz,$$

whence

$$z - Kz = Dz + KDz,$$

or

$$z - Dz = K(z + Dz).$$

Defining $x = z + Dz$ one has $x \geq 0$, $Kx \geq 0$ and $x + Kx = 2z > 0$, proving Tucker's theorem.

See also

- ▶ [Farkas Lemma: Generalizations](#)
- ▶ [Linear Optimization: Theorems of the Alternative](#)
- ▶ [Linear Programming](#)
- ▶ [Motzkin Transposition Theorem](#)
- ▶ [Theorems of the Alternative and Optimization](#)
- ▶ [Tucker Homogeneous Systems of Linear Relations](#)

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Farkas Lemma: Generalizations

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Global optimization; Convex inequality systems;
Convex-like systems; Nonsmooth optimization;
Semidefinite programming; Alternative theorem

The key to identifying optimal solutions of constrained nonlinear optimization problems is the Lagrange multiplier conditions. One of the main approaches to establishing such multiplier conditions for inequality constrained problems is based on the dual solvability characterizations of systems involving inequalities. J. Farkas [7] initially established such a dual characterization for linear inequalities which was used in [23] to derive necessary conditions for optimality for nonlinear programming problems. This dual characterization is popularly known as Farkas' lemma, which states that given any vectors a_1, \dots, a_m and c in \mathbf{R}^n , the linear inequality $c^\top x \geq 0$ is a consequence of the linear system $a_i^\top x \geq 0, i = 1, \dots, m$, if and only if there exist multipliers $\lambda_i \geq 0$ such that $c = \sum_{i=1}^m \lambda_i a_i$. This result can also be expressed as

a so-called *alternative theorem*: Exactly one of the following alternatives is true:

i) $\exists x \in \mathbf{R}^n, a_i^\top x \geq 0, c^\top x < 0$,

ii) $\exists \lambda_i \geq 0, c = \sum_{i=1}^m \lambda_i a_i$.

This lemma is the key result underpinning the linear programming duality and has played a central role in the development of nonlinear optimization theory. A large variety of proofs of the lemma can be found in the literature (see [5,25,26]). The proof [3,5] that relies on the separation theorems has led to various extensions. These extensions cover wide range of systems including systems involving infinite-dimensional linear inequalities, *convex inequalities* and matrix inequalities. Applications range from classical nonlinear programming to modern areas of optimization such as nonsmooth optimization and semidefinite programming. Let us now describe certain main generalizations of Farkas' lemma and their applications to problems in various areas of optimization.

Infinite-Dimensional Optimization

The Farkas lemma for a finite system of linear inequalities has been generalized to systems involving arbitrary convex cones and continuous linear mappings between spaces of arbitrary dimensions. In this case the lemma holds under a crucial closure condition. In symbolic terms, the main version of such extension to arbitrary dual pairs of vector spaces states that the following equivalence holds [6]:

$$[A(x) \in S \Rightarrow c(x) \geq 0] \Leftrightarrow c \in A^\top(S^*), \quad (1)$$

provided the cone $A^\top(S^*)$ is closed in some appropriate topology. Here A is a continuous linear mapping between two Banach spaces, S is a closed convex cone having the dual cone S^* [5]. The closure condition holds when S is a polyhedral cone in some finite-dimensional space. For simple examples of nonpolyhedral convex cones in finite dimensions where the closure condition does not hold, see [1,5]. However, the following asymptotic version of Farkas' lemma holds without a closure condition:

$$[A(x) \in S \Rightarrow c(x) \geq 0] \Leftrightarrow c \in \text{cl}(A^\top(S^*)), \quad (2)$$

where $\text{cl}(A^\top(S^*))$ is the closure of $A^\top(S^*)$ in the appropriate topology. These extensions resulted in the development of asymptotic and nonasymptotic first

order necessary optimality conditions for infinite-dimensional smooth constrained optimization problems involving convex cones and duality theory for infinite-dimensional linear programming problems (see e. g. [12]). Smooth optimization refers to the optimization of a differentiable function. A nonasymptotic form of an extension of Farkas' lemma that is different from the one in (1) is given in [24] without the usual closure condition. For related results see [4]. An approach to the study of semi-infinite programming, which is based on generalized Farkas' lemma for infinite linear inequalities is given in [12].

Nonsmooth Optimization

The success of linear programming duality and the practical nature of the Lagrange multiplier conditions for smooth optimization have led to extensions of Farkas' lemma to systems involving nonlinear functions. Convex analysis allowed to obtain extensions in terms of *subdifferentials* replacing the linear systems by sublinear (convex and positively homogeneous) systems [8,31]. A simple form of such an extension states that the following statements are equivalent:

$$-g(x) \in S \Rightarrow f(x) \geq 0 \tag{3}$$

$$0 \in \text{cl} \left[\partial f(0) + \bigcup_{\lambda \in S^*} \partial(\lambda g)(0) \right], \tag{4}$$

where the real valued function f is sublinear and lower semicontinuous, and the vector function g is sublinear with respect to the cone S and vg is lower semicontinuous for each $v \in S^*$. When f is continuous the statement (4) collapses to the condition

$$0 \in \partial f(0) + \text{cl} \left[\bigcup_{\lambda \in S^*} \partial(\lambda g)(0) \right]. \tag{5}$$

This extension was used to obtain optimality conditions for convex optimization problems and *quasidifferentiable problems* in the sense of B.N. Pshenichnyi [27]. A review of results of Farkas type for systems involving sublinear functions is given in [13,14].

Difference of sublinear (DSL) functions which arise frequently in nonsmooth optimization provide useful approximations for many classes of nonconvex *nons-*

mooth functions. This has led to the investigation of results of Farkas type for systems involving DSL functions.

A mapping $g: X \rightarrow Y$ is said to be *difference sublinear* (DSL) (with respect to S) if, for each $v \in S^*$, there are (weak $*$) compact convex sets, here denoted $\underline{\partial}(vg)(0)$ and $\overline{\partial}(vg)(0)$, such that, for each $x \in X$,

$$vg(x) = \max_{u \in \underline{\partial}(vg)(0)} u(x) - \max_{w \in \overline{\partial}(vg)(0)} w(x),$$

where X and Y are Banach spaces. If $Y = \mathbf{R}$ and $S = \mathbf{R}_+$ then this definition coincides with the usual notion of a difference sublinear real-valued function. Thus a mapping g is DSL if and only if vg is a DSL function for each $v \in S^*$. The sets $\underline{\partial}(vg)(0)$ and $\overline{\partial}(vg)(0)$ are the *subdifferential* and *superdifferential* of vg , respectively. For a DSL mapping $g: X \rightarrow Y$ we shall often require a *selection* from the class of sets $\{\overline{\partial}(vg)(0): v \in S^*\}$. This is a set, denoted (w_v) , in which we select a single element $\overline{\partial}(vg)(0)$ for each $v \in S^*$. An extension of the Farkas lemma for DSL systems states that the following statements are equivalent [10,20]:

- i) $-g(x) \in S \Rightarrow f(x) \geq 0$;
- ii) for each selection (w_v) with $w_v \in \overline{\partial}(vg)(0)$, $v \in S^*$, $\overline{\partial}f(0) \subseteq \underline{\partial}f(0) + B$,

where $B = \text{cl cone co} \left[\bigcup_{v \in S^*} (\underline{\partial}(vg)(0) - w_v) \right]$. A unified approach to generalizing the Farkas lemma for sublinear systems which uses multivalued functions and convex process is given [2,17,18].

Global Nonlinear Optimization

Given that the optimality of a constrained *global optimization* problem can be viewed as the solvability of appropriate inequality systems, it is easy to see that an extension of Farkas' lemma again provides a mechanism for characterizing *global optimality* of a range of nonlinear optimization problems. The ϵ -*subdifferential* analysis here allowed to obtain a new version of the Farkas lemma replacing the linear inequality $c(x) \geq 0$ by a *reverse convex inequality* $h(x) \leq 0$, where h is a convex function with $h(0) = 0$. This extension for systems involving DSL functions states that the following conditions are equivalent.

- i) $-g(x) \in S \Rightarrow h(x) \leq 0$;

- ii) for each selection (w_v) with $w_v \in \bar{\partial}(vg)(0)$, $v \in S^*$ and for each $\epsilon \geq 0$,

$$\partial_\epsilon h(0) \subseteq \text{cl cone co} \left[\bigcup_{v \in S^*} (\partial(vg)(0) - w_v) \right].$$

Such an extension has led to the development of conditions which characterize optimal solutions of various classes of global optimization problems such as convex maximization problems and fractional programming problems (see [19,20]).

However, simple examples show that the asymptotic forms of the above results of Farkas type do not hold if we replace the DSL (or sublinear) system by a convex system. Ch.-W. Ha [15] established a version of the Farkas lemma for convex systems in terms of *epigraphs of conjugate functions*. A simple form of such a result [29] states that the following statements are equivalent:

- i) $(\forall i \in I) g_i(x) \leq 0 \Rightarrow h(x) \leq 0$;
 ii) $\text{epi } h^* \subseteq \text{cl cone co} [\cup_{i \in I} \text{epi } g_i^*]$,

provided the system

$$i \in I, \quad g_i(x) \leq 0$$

has a solution. Here h and, for each $i \in I$, g_i are continuous convex functions, I is an arbitrary index set, and h^* and g_i^* are conjugate functions of h and g_i respectively. This result has also been employed to study infinite-dimensional nonsmooth nonconvex problems [30]. A basic general form of the Farkas lemma for convex system with application to *multi-objective convex optimization* problems is given in [11]. Extensions to systems involving the difference of convex functions are given in [21,29]. A more general result involving *H-convex functions* [29] with application to global nonlinear optimization is given in [29].

Nonconvex Optimization

The convexity requirement of the functions involved in the extended Farkas lemma above can be relaxed to obtain a form of Farkas' lemma for convex-like system. Let $F: X \times Y \rightarrow \mathbf{R}$ and let $f: X \rightarrow \mathbf{R}$, where X and Y are arbitrary nonempty sets. The pair (f, F) is *convex-like* on X if

$$(\exists \alpha \in (0, 1))(\forall x_1, x_2 \in X)(\exists x_3 \in X), \\ f(x_3) \leq \alpha f(x_1) + (1 - \alpha)f(x_2)$$

and $(\forall y \in Y)$:

$$F(x_3, y) \leq \alpha F(x_1, y) + (1 - \alpha)F(x_2, y).$$

If the pair (f, F) is convex-like on X , there is $x_0 \in X$ with $(\forall y \in Y) F(x_0, y) \leq 0$ and if a regularity condition holds then the following statements are equivalent [21]:

$$\forall y \in Y, F(x, y) \leq 0 \implies f(x) \geq 0,$$

$$(\forall \theta < 0)(\exists \lambda \in \Lambda)(\forall x \in X)$$

$$f(x) + \sum_{y \in Y} \lambda_y F(x, y) > \theta,$$

where Λ is the dual cone of the convex cone of all non-negative functions on Y . An asymptotic version of the above result holds if the regularity hypothesis is not fulfilled. This extension has been applied to develop Lagrange multiplier type results for minimax problems and constrained optimization problems involving convex-like functions. For related results see [16].

Semidefinite Programming

A useful corollary of the Farkas lemma, which is often used to characterize the *feasibility problem* for linear inequalities, states that exactly one of the following alternatives is true:

$$\text{i) } \exists x \in \mathbf{R}^n \ a_i^\top x \leq b_i, \ i = 1, \dots, m,$$

$$\text{ii) } \exists \lambda_i \geq 0 \ \sum_{i=1}^m \lambda_i a_i = 0, \ \sum_{i=1}^m b_i \lambda_i = -1.$$

This form of the Farkas lemma has also attracted various extensions to nonlinear systems, including *sublinear and DSL systems* [20] with the view to characterize the feasibility of such systems. The feasibility problem, which has been of great interest in *semidefinite programming*, is the problem of determining whether there exists an $x \in \mathbf{R}^n$ such that $Q(x) \succeq 0$, for real symmetric matrices Q_i , $i = 0, \dots, m$, where \succeq denotes the partial order, i. e. $B \succeq A$ if and only if $A - B$ is positive semidefinite, and $Q(x) = Q_0 - \sum_{i=1}^m x_i Q_i$. However, simple examples show that a direct analog of the alternative does not hold for the semidefinite inequality systems $Q(x) \succeq 0$ without additional hypothesis on Q . A modified dual conditions which characterize solvability of the system $Q(x) \succeq 0$ is given in [28].

See also

► [Farkas Lemma](#)

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Feasible Sequential Quadratic Programming

FSQP

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