On the Farkas Lemma

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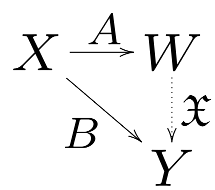
Agenda

The Farkas Lemma, also known as the Farkas–Minkowski Lemma, plays a key role in linear programming and the relevant areas of optimization. The aim of this talk is to demonstrate how Boolean valued analysis may be applied to simultaneous linear inequalities with operators. This particular theme is another illustration of the deep and powerful technique of "stratified validity" which is characteristic of Boolean valued analysis.

Environment

Assume that X is a real vector space, Y is a *Kantorovich space* also known as a complete vector lattice or a complete Riesz space. Let $\mathbb{B} := \mathbb{B}(Y)$ be the *base* of Y, i.e., the complete Boolean algebra of positive projections in Y; and let $m(Y)$ be the universal completion of Y. Denote by $L(X, Y)$ the space of linear operators from X to Y. In case X is furnished with some Y-seminorm on X, by $L^{(m)}(X, Y)$ we mean the *space of dominated operators* from X to Y. As usual, $\{T \le 0\} := \{x \in X : Tx \le 0\}$; $\ker(T) = T^{-1}(0)$ for $T \in L(X, Y)$.

Inequalities: Explicit Dominance



1: $(\exists \mathfrak{X})\ \mathfrak{X}A = B \leftrightarrow \ker(A) \subset \ker(B)$;

2: *If W is ordered by W_+ and $A(X) - W_+ = W_+ - A(X) = W$, then*

$$(\exists \mathfrak{X} \geq 0)\ \mathfrak{X}A = B \leftrightarrow \{A \leq 0\} \subset \{B \leq 0\}.$$

Farkas: Explicit Dominance

Theorem 1. *Assume that* $A_1, \dots, A_N$ *and* B *belong to* $L^{(m)}(X, Y)$.

The following are equivalent:

(1) *Given* $b \in \mathbb{B}$, *the operator inequality* $bBx \le 0$ *is a consequence of the simultaneous linear operator inequalities* $bA_1x \le 0, \dots, bA_Nx \le 0$, *i.e.*,

$$\{bB \le 0\} \supset \{bA_1 \le 0\} \cap \dots \cap \{bA_N \le 0\}.$$

(2) *There are positive orthomorphisms* $\alpha_1, \dots, \alpha_N \in \mathrm{Orth}(m(Y))$ *such that*

$$B = \sum_{k=1}^{N} \alpha_k A_k;$$

i.e., B *lies in the operator convex conic hull of* $A_1, \dots, A_N$.

Farkas: No (?) Dominance

Lemma 1. *Let X be a vector space over some subfield R of the reals $\mathbb{R}$. Assume that f and g are R-linear functionals on X; in symbols, $f, g \in X^{\#} := L(X, \mathbb{R})$.*

For the inclusion

$$\{g \leq 0\} \supset \{f \leq 0\}$$

to hold it is necessary and sufficient that there be $\alpha \in \mathbb{R}_+$ satisfying $g = \alpha f$.

Sufficiency is obvious.

Necessity: The case of $f = 0$ is trivial. If $f \neq 0$ then there is some $x \in X$ such that $f(x) \in \mathbb{R}$ and $f(x) > 0$. Denote the image $f(X)$ of X under f by R_0. Put $h := g \circ f^{-1}$, i.e. $h \in R_0^{\#}$ is the only solution for $h \circ f = g$. By hypothesis, h is a positive R-linear functional on R_0. By the Bigard Theorem h can be extended to a positive homomorphism $\bar{h} : \mathbb{R} \to \mathbb{R}$, since $R_0 - \mathbb{R}_+ = \mathbb{R}_+ - R_0 = \mathbb{R}$. Each positive automorphism of $\mathbb{R}$ is multiplication by a positive real. As the sought α we may take $\bar{h}(1)$.

The proof of the lemma is complete.

Reals: Explicit Dominance

Lemma 2. *Let X be an $\mathbb{R}$-seminormed vector space over some subfield R of $\mathbb{R}$. Assume that $f_1, \dots, f_N$ and g are bounded R-linear functionals on X; in symbols, $f_1, \dots, f_N, g \in X^* := L^{(m)}(X, \mathbb{R})$.*

For the inclusion

$$\{g \leq 0\} \supset \bigcap_{k=1}^{N} \{f_k \leq 0\}$$

to hold it is necessary and sufficient that there be $\alpha_1, \dots, \alpha_N \in \mathbb{R}_+$ satisfying

$$g = \sum_{k=1}^{N} \alpha_k f_k.$$

Origins

Cohen's final solution of the problem of the cardinality of the continuum within ZFC gave rise to the Boolean valued models by Vopěnka, Scott, and Solovay. Scott had forecasted the area in 1969:

"We must ask whether there is any interest in these nonstandard models aside from the independence proof; that is, do they have any mathematical interest? The answer must be yes, but we cannot yet give a really good argument."

Takeuti coined the term "Boolean valued analysis" for applications of the models to analysis.

Boolean Valued Universe

Let $\mathbb{B}$ be a complete Boolean algebra. Given an ordinal α, put

$$V_\alpha^{(\mathbb{B})} := \{x : \\ (\exists \beta \in \alpha)\ x : \mathrm{dom}(x) \to \mathbb{B}\ \&\ \mathrm{dom}(x) \subset V_\beta^{(\mathbb{B})}\}.$$

The *Boolean valued universe* $\mathbb{V}^{(\mathbb{B})}$ is

$$\mathbb{V}^{(\mathbb{B})} := \bigcup_{\alpha \in \mathrm{On}} V_\alpha^{(\mathbb{B})},$$

with On the class of all ordinals. The truth value $[\![\varphi]\!] \in \mathbb{B}$ is assigned to each formula φ of ZFC relativized to $\mathbb{V}^{(\mathbb{B})}$.

Descending and Ascending

Given φ, a formula of ZFC, and y, a member of $\mathbb{V}^{\mathbb{B}}$; put $A_\varphi := A_{\varphi(\cdot,\ y)} := \{x : \varphi(x,\ y)\}$. The *descent* $A_\varphi{\downarrow}$ of a class A_φ is

$$A_\varphi{\downarrow} := \{t : t \in \mathbb{V}^{(\mathbb{B})} \ \&\ [\![\varphi(t,\ y)]\!] = \mathbb{1}\}.$$

If $t \in A_\varphi{\downarrow}$, then it is said that t *satisfies* $\varphi(\cdot,\ y)$ *inside* $\mathbb{V}^{(\mathbb{B})}$. The *descent* $x{\downarrow}$ of $x \in \mathbb{V}^{(\mathbb{B})}$ is defined as

$$x{\downarrow} := \{t : t \in \mathbb{V}^{(\mathbb{B})} \ \&\ [\![t \in x]\!] = \mathbb{1}\},$$

i.e. $x{\downarrow} = A_{\cdot \in x}{\downarrow}$. The class $x{\downarrow}$ is a set. If x is a nonempty set inside $\mathbb{V}^{(\mathbb{B})}$ then

$$(\exists z \in x{\downarrow})[\![(\exists z \in x)\ \varphi(z)]\!] = [\![\varphi(z)]\!].$$

The *ascent* functor acts in the opposite direction.

The Reals Within

There is an object $\mathscr{R}$ inside $\mathbb{V}^{(\mathbb{B})}$ modeling $\mathbb{R}$, i.e.,

$$[\![\mathscr{R} \text{ is the reals}]\!] = \mathbb{1}.$$

Let $\mathscr{R}{\downarrow}$ be the descent of the carrier $|\mathscr{R}|$ of the algebraic system $\mathscr{R} := (|\mathscr{R}|, +, \,\cdot\,, 0, 1, \leq)$ inside $\mathbb{V}^{(\mathbb{B})}$. Implement the descent of the structures on $|\mathscr{R}|$ to $\mathscr{R}{\downarrow}$ as follows:

$$x + y = z \leftrightarrow [\![x + y = z]\!] = \mathbb{1};$$
$$xy = z \leftrightarrow [\![xy = z]\!] = \mathbb{1};$$
$$x \leq y \leftrightarrow [\![x \leq y]\!] = \mathbb{1};$$
$$\lambda x = y \leftrightarrow [\![\lambda^{\wedge} x = y]\!] = \mathbb{1}$$
$$(x, y, z \in \mathscr{R}{\downarrow},\ \lambda \in \mathbb{R}).$$

Gordon Theorem. *$\mathscr{R}{\downarrow}$ with the descended structures is a universally complete vector lattice with base $\mathbb{B}(\mathscr{R}{\downarrow})$ isomorphic to $\mathbb{B}$.*

Proof of Theorem 1.

(1)$\to$ (2): If $B = \sum_{k=1}^{N} \alpha_k A_k$ for some positive $\alpha_1, \ldots, \alpha_N$ in Orth($m(Y)$) while $bA_k x \leq 0$ for $b \in \mathbb{B}$ and $x \in X$, then

$$bBx = b \sum_{k=1}^{N} \alpha_k A_k x = \sum_{k=1}^{N} \alpha_k b A_k x \leq 0$$

since orthomorphisms commute and projections are orthomorphisms of $m(Y)$.

(2)→(1): Consider the separated Boolean valued universe $\mathbb{V}^{(\mathbb{B})}$ over the base $\mathbb{B}$ of Y. By the Gordon Theorem the ascent $Y{\uparrow}$ of Y is $\mathscr{R}$, the field of reals inside $\mathbb{V}^{(\mathbb{B})}$.

Using the canonical embedding, we see that $X^{\wedge}$ is a vector space over the standard name $\mathbb{R}^{\wedge}$ of the reals $\mathbb{R}$. Moreover, $\mathbb{R}^{\wedge}$ is a subfield and sublattice of $\mathscr{R} = Y{\uparrow}$ inside $\mathbb{V}^{(\mathbb{B})}$.

Put $f_k := A_k{\uparrow}$ for all $k := 1, \dots, N$ and $g := B{\uparrow}$. Clearly, all $f_1, \dots, f_N, g$ belong to $X^{\wedge *}$ inside $\mathbb{V}^{\mathbb{B}}$.

Define the finite sequence

$$f^{\wedge} : ((N+1) \backslash 0)^{\wedge} \to X^{\wedge *}$$

as the ascent of $f_1, \dots, f_N$. In other words, the truth values are as follows:

$$[\![f_{k^{\wedge}}(x^{\wedge}) = A_k x]\!] = \mathbb{1}, \quad [\![g(x^{\wedge}) = Bx]\!] = \mathbb{1}$$

for all $x \in X$ and $k := 1, \dots, N$.

Put

$$b := [\![A_1 x \leq 0^\wedge]\!] \wedge \dots, \wedge [\![A_N x \leq 0^\wedge]\!].$$

Then $bA_k x \leq 0$ for all $k := 1, \dots, N$ and $bBx \leq 0$ by (1).

Therefore,

$$[\![A_1 x \leq 0^\wedge]\!] \wedge \dots \wedge [\![A_N x \leq 0^\wedge]\!] \leq [\![Bx \leq 0^\wedge]\!].$$

In other words,

$$[\![(\forall k := 1^\wedge, \dots, N^\wedge) f_k(x) \le 0^\wedge]\!]$$

$$= \bigwedge_{k:=1,\dots,N} [\![f_{k^\wedge}(x^\wedge) \le 0^\wedge]\!] \le [\![g(x^\wedge) \le 0^\wedge]\!].$$

Using Lemma 2 inside $\mathbb{V}^{(\mathbb{B})}$ and appealing to the maximum principle of Boolean valued analysis, we infer that there are positive elements $\alpha_1, \dots, \alpha_N$ of $\mathscr{R}{\downarrow}$ satisfying

$$[\![(\forall x \in X^\wedge) g(x^\wedge) = \sum_{k=1^\wedge}^{N^\wedge} \alpha_k f_k(x^\wedge)]\!] = \mathbb{1}.$$

Multiplication by an element in $\mathscr{R}{\downarrow}$ is an orthomorphism of $m(Y)$. Moreover $B = \sum_{k=1}^N \alpha_k A_k$, which completes the proof.

Counterexample: No Dominance

Lemma 1, describing the consequences of a single inequality, does not restrict the class of functionals under consideration. The analogous version of the Farkas Lemma simply fails for two simultaneous inequalities in general. Indeed, the inclusion $\{f = 0\} \subset \{g \le 0\}$ equivalent to the inclusion $\{f = 0\} \subset \{g = 0\}$ does not imply that f and g are proportional in the case of an arbitrary subfield of $\mathbb{R}$. It suffices to look at $\mathbb{R}$ over the rationals $\mathbb{Q}$, take some discontinuous $\mathbb{Q}$-linear functional on $\mathbb{R}$ and the identity automorphism of $\mathbb{R}$. This gives grounds for the next result.

Reconstruction: No Dominance

Theorem 2. *Take A and B in $L(X, Y)$. The following are equivalent:*

(1) $(\exists \alpha \in m(Y))\ B = \alpha A$;

(2) *There is a projection $\varkappa \in \mathbb{B}$ such that*

$$\{\varkappa b B \leq 0\} \supset \{\varkappa b A \leq 0\};$$

$$\{\neg \varkappa b B \leq 0\} \supset \{\neg \varkappa b A \geq 0\}$$

for all $b \in \mathbb{B}$.

Proof. Boolean valued analysis reduces the claim to the scalar case. Applying Lemma 1 twice and writing down the truth values, complete the proof.

Difference of Riesz Homomorphisms

Let $T : X \to Y$ be an order bounded operator whose positive and negative parts are Riesz homomorphisms. Observe that for every band projection $b \in \mathbb{B}$ the operator bT, called a *stratum* of T, is the difference of some Riesz homomorphisms on X and so the kernel of bT is a Riesz subspace of X. In fact, the converse is valid too. In other words, we have the following

Theorem 3. *An order bounded operator from a Riesz space to a Kantorovich space is the difference of some Riesz homomorphisms if and only if the kernel of its every stratum is a Riesz subspace of the ambient Riesz space.*

Sum of Riesz Homomorphisms

A subspace H of a Riesz space is a G-*space* or *Grothendieck subspace* provided that H enjoys the following property:

$$(\forall x, y \in H)\ (x \vee y \vee 0 + x \wedge y \wedge 0 \in H).$$

Theorem 4. *The modulus of an order bounded operator $T : X \to Y$ is the sum of some pair of Riesz homomorphisms if and only if the kernel of each stratum bT of T with $b \in \mathbb{B}$ is a Grothendieck subspace of the ambient Riesz space X.*

Summary

The above results, although curious to some extent, are nothing more than simple illustrations of the powerful technique of model theory shedding new light at the *Pythagorean Thesis*: "All Is Number." The theory of the reals enriches the theory of Banach lattices, demonstrating the uncountable fruits of freedom which is the essence of mathematics.