UPPER SEMILATTICES OF FINITE-DIMENSIONAL GAUGES

S. S. KUTATELADZE

In Memory of Alex Rubinov (1940–2006)

Abstract. This is a brief overview of some applications of the ideas of abstract convexity to the upper semilattices of gauges in finite dimensions.

Introduction

Duality in convexity is a simile of reversal in positivity. The ghosts of this similarity underlay the research on abstract convexity we were engrossed in with Alex Rubinov in the early 1970s. Our efforts led to the survey [1] and its expansion in the namesake book [2]. We always cherished a hope to revisit this area and shed light on a few obscurities. However, the fate was against us.

Inspecting the archive of our drafts of these years, I encountered several items on the cones of Minkowski functionals or, equivalently, gauges. The results on the Minkowski duality in finite dimensions are practically unavailable in full form, whereas they rest on the technique that is still uncommon and unpopular but definitely profitable. The theorems on gauges appeared mostly in some mimeographed local sources that had disappeared two decades ago. We hoped and planned to expatiate on these matters when time will come.

Alex Rubinov was my friend up to his terminal day. He shared his inspiration and impetus with me. So does and will do his memory...

An abstract convex function is the upper envelope of a family of simple functions [1]–[3]. The cone of abstract convex elements is an upper semilattice. We describe the bipolar of such a semilattice through majorization generated by its polar. Polyhedral approximation simplifies the generators of the polar in finite dimensions to discrete measures. Decomposition reduces the matter to Jensen-type inequalities, which opens a possibility of linear programming and we are done. These ideas characterize our approach.

This article is organized as follows: Section 1 is a short discussion of majorization and decomposition in the spaces of continuous functions. Section 2 addresses the space of convex sets in finite dimensions and the influence of polyhedral approximation on the structure of dual cones. Section 3 illustrates the use of linear programming for revealing continuous linear selections over convex figures. Section 4 collects some dual representations for the members of upper semilattices of gauges. In Section 5 we deal with some upper lattices of gauges that are closed under intersection.

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1. Majorization and Decomposition

It was long ago in 1954 that Reshetnyak suggested in his unpublished thesis [4] to compare (positive) measures on the Euclidean unit sphere S_{N-1} as follows:

1.1. A measure μ *(linearly) majorizes* or *dominates* a measure ν provided that to each decomposition of S_{N-1} into finitely many disjoint Borel sets $U_1, \dots, U_m$ there are measures $\mu_1, \dots, \mu_m$ with sum μ such that every difference $\mu_k - \nu|_{U_k}$ annihilates all restrictions to S_{N-1} of linear functionals over $\mathbb{R}^N$. In symbols, we write $\mu \gg_{\mathbb{R}^N} \nu$.

Reshetnyak proved that

$$\int\limits_{S_{N-1}} p d\mu \geq \int\limits_{S_{N-1}} p d\nu$$

for every sublinear functional p on $\mathbb{R}^N$ if $\mu \gg_{\mathbb{R}^N} \nu$. This gave an important trick for generating positive linear functionals over various classes of convex surfaces and functions.

1.2. A similar idea was suggested by Loomis [5] in 1962 within Choquet theory. A measure μ *affinely majorizes* a measure ν, both given on a compact convex subset Q of a locally convex space X, provided that to each decomposition of ν into finitely many summands $\nu_1, \dots, \nu_m$ there are measures $\mu_1, \dots, \mu_m$ with μ such that every difference $\mu_k - \nu_k$ annihilates all restrictions to Q of the affine functions over X. In symbols, $\mu \gg_{\mathrm{Aff}(Q)} \nu$. Many applications of affine majorization are set forth in [6].

Cartier, Fell, and Meyer proved in [7] that

$$\int\limits_{Q} f d\mu \geq \int\limits_{Q} f d\nu$$

for every continuous convex function f on Q if and only if $\mu \gg_{\mathrm{Aff}(Q)} \nu$.

An analogous necessity part for linear majorization was published in [8]. In applications we use a more detailed version of majorization [9]:

1.3. Decomposition Theorem. *Assume that $H_1, \dots, H_n$ are cones in a vector lattice X. Assume further that f and g are positive linear functionals on X. The inequality*

$$f(h_1 \vee \dots \vee h_n) \geq g(h_1 \vee \dots \vee h_n)$$

holds for all $h_k \in H_k$ $(k := 1, \dots, n)$ if and only if to each decomposition of g into a sum of n positive terms $g = g_1 + \dots + g_N$ there is a decomposition of f into a sum of n positive terms $f = f_1 + \dots + f_n$ such that

$$f_k(h_k) \geq g_k(h_k) \quad (h_k \in H_k;\ k := 1, \dots, n).$$

2. The Space of Convex Figures

We will proceed in the Euclidean space $\mathbb{R}^N$.

2.1. A *convex figure* is a compact convex set. A *convex body* is a solid convex figure. The *Minkowski duality* identifies a convex figure S in $\mathbb{R}^N$ with its *support function* $S(z) := \sup\{(x, z) \mid x \in S\}$ for $z \in \mathbb{R}^N$. Considering the members of $\mathbb{R}^N$ as singletons, we assume that $\mathbb{R}^N$ lies in the set $\mathscr{V}_N$ of all compact convex subsets of $\mathbb{R}^N$.

2.2. The Minkowski duality makes $\mathscr{V}_N$ into a cone in the space $C(S_{N-1})$ of continuous functions on the Euclidean unit sphere S_{N-1}, the boundary of the unit ball $\mathfrak{z}_N$. This yields is the so-called *Minkowski structure* on $\mathscr{V}_N$. Addition of the support functions of convex figures amounts to taking their algebraic sum, also called the *Minkowski addition*. It is worth observing that the *linear span* $[\mathscr{V}_N]$ of $\mathscr{V}_N$ is dense in $C(S_{N-1})$, bears a natural structure of a vector lattice and is usually referred to as the *space of convex sets*. The study of this space stems from the pioneering breakthrough of Alexandrov [10] in 1937 and the further insights of Radström [11] and Hörmander [12].

2.3. A *gauge* p is a positive sublinear functional on a real vector space X viewed as the Minkowski functional of the conic segment $S_p := \{p \le 1\} := \{x \in X \mid p(x) \le 1\}$. The latter is also referred to as a *gauge* or *caliber*. A gauge p is a *norm* provided that its *ball* S_p is symmetric and absorbing. Recall that the *subdifferential* or *support set* ∂p of p is the *dual ball* or *polar* of S_p. The polar of a ball S is denoted by S° and the dual norm of $\|\cdot\|_S$ is $\|\cdot\|_{S^\circ}$. The "donkey bridge" of functional analysis consists in the *duality rules*:

$$\|\cdot\|_S = S^\circ(\cdot), \quad \|\cdot\|_{S^\circ} = S(\cdot).$$

We will restrict exposition to the norms and balls of $\mathbb{R}^N$ by way of tradition.

2.4. Approximation Lemma. *If H is a subcone of $\mathscr{V}_N$ then the signed measures with finite support are sequentially weakly* closed in the dual cone H^*.*

PROOF. Let $\mu \in H^*$. The mappings

$$z \mapsto \mu_+(z); \quad z \mapsto \mu_-(z),$$

with $z \in \mathbb{R}^N$, are linear functionals on $\mathbb{R}^N$. Therefore, there are $u, v \in \mathbb{R}^N$ such that $\mu_+(z) = (u, z)$ and $\mu_-(z) = (v, z)$. Put

$$\overline{\mu}_1; = \mu_+ + \mathrm{mes} + |u|\varepsilon_{-u/|u|};$$
$$\overline{\mu}_2 := \mu_- + \mathrm{mes} + |v|\varepsilon_{-v/|v|};$$
$$\mu_1 := \overline{\mu}_1 + |v|\varepsilon_{-v/|v|}; \quad \mu_2 := \overline{\mu}_2 + |u|\varepsilon_{-u/|u|}.$$

As usual, ε_z is the Dirac measure at $z \in \mathbb{R}^N$, while $|\cdot|$ is the Euclidean norm on $\mathbb{R}^N$, and mes is the Lebesgue measure on S_{N-1}: i. e. the surface area function of the Euclidean ball $\mathfrak{z}_N := \{x \in \mathbb{R}^N \mid |x| \le 1\}$. Note that $\mu = \mu_1 - \mu_2$. Moreover, the measures $\overline{\mu}_1$ and $\overline{\mu}_2$ are nondegenerate and translation-invariant. Indeed, check that so is $\overline{\mu}_1$. This signed measure is clearly positive and not supported by any great hypersphere. We are left with validating translation-invariance. If $k := 1, \dots, N$ then

$$\int\limits_{S_{N-1}} e_j d\overline{\mu}_1 = \int\limits_{S_{N-1}} e_j d\mu_+ + \int\limits_{S_{N-1}} e_j d\mu(\mathfrak{z}_N) - (u, e_k) = (u, e_k) - (u, e_k) = 0.$$

Consider a convex figure $\mathfrak{x}$ whose surface area function $\mu(\mathfrak{x})$ equals $\overline{\mu}_1$. The existence of this figure is guaranteed by the celebrated Alexandrov Theorem [10, p.108].

Let $(\mathfrak{x}_m)$ be a sequence of polyhedra including $\mathfrak{x}$ and converging to $\mathfrak{x}$ in the Hausdorff metric on $[\mathscr{V}_N]$which is induced by the Chebyshev norm on $C(S_{N-1})$.

Then the measures $\overline{\mu}^1_m = \mu(\mathfrak{x}_m)$ converge weakly* to $\overline{\mu}_1$ and $\overline{\mu}^1_m \gg_{\mathbb{R}^n} \overline{\mu}_1$. Indeed, given a convex figure $\mathfrak{z}$, we have

$$\int\limits_{S_{N-1}} \mathfrak{z}\, d\overline{\mu}^1_m = \int\limits_{S_{N-1}} \mathfrak{z}\, d\mu(\mathfrak{x}_m) = nV(\mathfrak{z}, \mathfrak{x}_m, \dots, \mathfrak{x}_m)$$

$$\geq nV(\mathfrak{z}, \mathfrak{x}, \dots, \mathfrak{x}) = \int\limits_{S_{N-1}} \mathfrak{z}\, d\mu(\mathfrak{x}) = \int\limits_{S_{N-1}} \mathfrak{z}\, d\overline{\mu}_1$$

by the inclusion monotonicity of the mixed volume $V(\cdot, \dots, \cdot)$ in every argument.. By analogy, there is a sequence $(\overline{\mu}^2_m)$, converging weakly* to $\overline{\mu}_2$ and such that $\overline{\mu}^2_m \gg_{\mathbb{R}^n} \overline{\mu}_2$. Putting

$$\mu^1_m := \overline{\mu}^1_m + |v|\varepsilon_{-v/|v|};$$
$$\mu^2_m := \overline{\mu}^2_m + |u|\varepsilon_{-u/|u|},$$

we see that $\mu^1_m - \mu^2_m$ converges weakly* to μ. The proof is complete.

3. Labels and Decompositions

The Approximation Lemma allows us to reduce consideration to signed measures with finite support. These measures decompose easily. We will exhibit a typical application.

3.1. A family $(\mu_1, \dots, \mu_n)$ of regular Borel measures on the sphere S_{N-1} is a *labeling* on $\mathbb{R}^N$ provided that $(\mu_1(\mathfrak{x}), \dots, \mu_n(\mathfrak{x})) \in \mathfrak{x}$ for all $\mathfrak{x} \in \mathscr{V}_N$. The vector $(\mu_1(\mathfrak{x}), \dots, \mu_n(\mathfrak{x}))$ is a *label* of $\mathfrak{x}$.

3.2. Proposition. *A family $(\mu_1, \dots, \mu_n)$ is a labeling on $\mathbb{R}^N$ if and only if*

$$\varepsilon_x - \sum_{k=1}^{n} x_k \mu_k \in \mathscr{V}^*_N.$$

for all $x \in S_{N-1}$.

Proof. The Minkowski duality is an isomorphism of the relevant structures. Hence, the definition of labeling can be rephrased as follows:

$$\sum_{k=1}^{n} x_k \mu_k(\mathfrak{x}) \leq \mathfrak{x}(x) \quad (x \in \mathbb{R}^n,\ \mathfrak{x} \in \mathscr{V}_N).$$

3.3. Using linear majorization for describing $\mathscr{V}^*_N$, we can write down some criteria for labeling in terms of decompositions. For simplicity, we will argue in the planar case.

Consider the conditions:

$$(++) \qquad \varepsilon_{(\triangle_1, \triangle_2)} + \triangle_1\, \mu_1^- + \triangle_2\, \mu_2^- \underset{\mathbb{R}^2}{\gg} \triangle_1\, \mu_1^+ + \triangle_2\, \mu_2^+;$$

$$(+-) \qquad \varepsilon_{(\triangle_1, -\triangle_2)} + \triangle_1\, \mu_1^- + \triangle_2\, \mu_2^+ \underset{\mathbb{R}^2}{\gg} \triangle_1\, \mu_1^+ + \triangle_2\, \mu_2^-;$$

$$(-+) \qquad \varepsilon_{(-\triangle_1, \triangle_2)} + \triangle_1\, \mu_1^+ + \triangle_2\, \mu_2^- \underset{\mathbb{R}^2}{\gg} \triangle_1\, \mu_1^- + \triangle_2\, \mu_2^+;$$

$$(--) \qquad \varepsilon_{(-\triangle_1, -\triangle_2)} + \triangle_1\, \mu_1^+ + \triangle_2\, \mu_2^+ \underset{\mathbb{R}^2}{\gg} \triangle_1\, \mu_1^- + \triangle_2\, \mu_2^-;$$

with $(\triangle_1, \triangle_2) \in S_1 \cap \mathbb{R}^2_+$. Clearly, the requirement of 4.1 amounts to the four conditions simultaneously. By way of example, we will elaborate the relevant criterion only in the case of $(+-)$.

3.4. Proposition. *For $(+-)$ to hold it is necessary and sufficient that to all $(\triangle_1, \triangle_2)$ in $S_1 \cap \mathbb{R}^2_+$ and all decompositions $\{(\mu_1^+)_1, \dots, (\mu_1^+)_m\}$ of μ_1^+ and al decompositions $\{(\mu_2^-)_1, \dots, (\mu_2^-)_m\}$ of μ_2^- there exist a decomposition $\{(\mu_1^-)_1, \dots, (\mu_1^-)_m\}$ of μ_1^-, a decomposition $\{(\mu_2^+)_1, \dots, (\mu_2^+)_m\}$ of μ_2^+, and reals $\alpha_1, \dots, \alpha_m$ that make compatible the simultaneous inequalities:*

$$\alpha_1 \geq 0; \dots; \alpha_m \geq 0; \alpha_1 + \dots + \alpha_m = 1;$$
$$\triangle_1 \, (x_{(\mu_1^-)_k} - x_{(\mu_1^+)_k} + \alpha_k e_1)$$
$$= \triangle_2 \, (x_{(\mu_2^-)_k} - x_{(\mu_2^+)_k} + \alpha_k e_2) \quad (k := 1, \dots, m),$$

where x_μ is the representing point of μ; i. e., $\mu(u) = (u, x_\mu)$ for all $u \in \mathbb{R}^2$.

Proof. $\Longleftarrow$: Let $(\triangle_1, \triangle_2) \in S_1 \cap \mathbb{R}^2_+$ and let $\{\nu_1, \dots, \nu_m\}$ be an arbitrary decomposition of $\triangle_1 \, \mu_1^+ + \triangle_2 \, \mu_2^-$. By the Riesz Decomposition Lemma there are a decomposition $\{(\mu_1^+)_1, \dots, (\mu_1^+)_m\}$ of μ_1^+ and a decomposition $\{(\mu_2^-)_1, \dots, (\mu_2^-)_m\}$ of μ_2^- such that $\triangle_1 \, (\mu_1^+)_k + \triangle_2 \, (\mu_2^-)_k = \nu_k$. Find some parameters satisfying the simultaneous inequalities and put

$$\mu_k := \triangle_1 \, (\mu_1^-)_k + \triangle_2 \, (\mu_2^+)_k + \alpha_k \varepsilon_{(\triangle_1, -\triangle_2)}.$$

Clearly, $\mu_k \geq 0$ and, moreover,

$$\sum_{k=1}^{m} \mu_k = \triangle_1 \, \mu_1^- + \triangle_2 \, \mu_2^+ + \varepsilon_{(\triangle_1, -\triangle_2)}.$$

Furthermore,

$$x_{\mu_k} - x_{\nu_k} = \triangle_1 \, x_{(\mu_1^-)_k} + \triangle_2 \, x_{(\mu_2^+)_k} + \alpha_k \, \triangle_1 \, e_1$$
$$-\alpha_k \, \triangle_2 \, e_2 - \triangle_1 \, x_{(\mu_1^+)_k} - \triangle_2 \, x_{(\mu_2^-)_k} = 0,$$

and so $\mu_k - \nu_k$ belongs to the polar of $\mathbb{R}^2$ in $C(S_1)$.

$\Longrightarrow$: Assume $(+-)$ valid.

Given decompositions $\{(\mu_1^+)_1, \dots, (\mu_1^+)_m\}$ and $\{(\mu_2^-)_1, \dots, (\mu_2^-)_m\}$ there is a decomposition $\{\nu_1, \dots, \nu_{2m}\}$ of $\varepsilon_{(\triangle_1, -\triangle_2)} + \triangle_1 \, \mu_1^- + \triangle_2 \, \mu_2^+$ such that

$$x_{\nu_k} = x_{(\mu_1^+)_k}; \quad x_{\nu_{m+k}} = x_{(\mu_2^-)_k} \quad (k := 1, \dots, m).$$

We are left with appealing to the Riesz Decomposition Lemma and representing the decomposition $\{\nu_1, \dots, \nu_{2m}\}$ through the corresponding decompositions of $\varepsilon_{(\triangle_1, -\triangle_2)}$, $\triangle_1 \, \mu_1^-$, and $\triangle_2 \, \mu_2^+$. The proof is complete.

3.5. If it is possible to chose decompositions in 3.4 independently of $(\triangle_1, \triangle_2)$, then we come to a sufficient condition for labeling. Let us illustrate this by exhibiting an example of one of the simplest labelings.

We will seek a labeling of the form

$$\mu_1 := |\mu^+| \varepsilon_{\mu^+/|\mu^+|} - |\mu^-| \varepsilon_{\mu^-/|\mu^-|};$$

$$\mu_2 := |\nu^+| \varepsilon_{\nu^+/|\nu^+|} - |\nu^-| \varepsilon_{\nu^-/|\nu^-|},$$

with μ^+, μ^-, ν^+, and ν^- some points on the plane. The sufficient condition we have just suggested paraphrases as follows:

$$\alpha_k, \beta_k, a_k, b_k, \gamma_k, c_k \geq 0;$$

$$\alpha_k + a_k = 1; \quad \beta_k + b_k = 1; \quad \gamma_k + c_k = 1 \quad (k := 1, \dots, 4);$$

$$\mu^+ = \alpha_1 \mu^- + \gamma_1 e_1; \quad \beta_1 \nu^- + \gamma_1 e_2 = 0;$$

$$\nu^+ = b_1 \nu^- + c_1 e_2; \quad a_1 \mu^- + c_1 e_1 = 0;$$

$$\mu^- = \alpha_2 \mu^+ - \gamma_2 e_1; \quad \beta_2 \nu^- + \gamma_2 e_2 = 0;$$

$$\nu^+ = b_2 \nu^- + c_2 e_2; \quad a_2 \mu^+ - c_2 e_1 = 0;$$

$$\mu^- = \alpha_3 \mu^+ - \gamma_3 e_1; \quad \beta_3 \nu^+ - \gamma_3 e_2 = 0;$$

$$\nu^- = b_3 \nu^+ - c_3 e_2; \quad a_3 \mu^+ - c_3 e_1 = 0;$$

$$\mu^+ = \alpha_4 \mu_- + \gamma_4 e_1; \quad \beta_4 \nu^+ - \gamma_4 e_2 = 0;$$

$$\nu^- = b_4 \nu^+ - c_4 e_2; \quad a_4 \mu^- + \gamma_4 e_1 = 0.$$

The solution of the last system is given by the parameters:

$$\alpha_k = b_k = 0; \quad \beta_k = a_k = 1;$$

$$\gamma_k = c_k = \frac{1}{2} \quad (k := 1, \dots, 4).$$

Moreover,

$$\mu^+ = \frac{1}{2} e_1; \quad \nu^+ = \frac{1}{2} e_2; \quad \mu^- = -\frac{1}{2} e_1; \quad \nu^- = -\frac{1}{2} e_2.$$

Therefore, the simplest labeling of $\mathfrak{x}$ is the point $\frac{1}{2}(\mathfrak{x}(e_1) - \mathfrak{x}(-e_1), \mathfrak{x}(e_2) - \mathfrak{x}(-e_2))$. It is worth emphasizing that the validation of the above conditions belongs to linear programming which enables us to seek for arbitrary labelings by signed measures with finite support.

4. The Case of Joining Gauges

We now apply the above ideas to studying the classes of N-dimensional convex surfaces which comprise upper semilattices in $\mathscr{V}_N$. To simplify notation we will discuss only balls, denoting the set of balls in $\mathscr{V}_N$ by $\mathscr{V} S_N$. It is convenient formally to add the apex to $\mathscr{V} S_N$. If $S \in \mathscr{V}_N S$ differs from the origin then we use the symbol $\|\cdot\|_S$ not only for the gauge of S but also for the *operator norm* corresponding to S in the endomorphism space $\mathscr{L}(\mathbb{R}^N)$ of $\mathbb{R}^N$. In other words,

$$\|x\|_S := \inf\{\alpha > 0 \mid x/\alpha \in S\} \quad (x \in \mathbb{R}^N);$$

$$\|A\|_S := \sup\{\|Ax\|_S \mid x \in S\} \quad (A \in \mathscr{L}(\mathbb{R}^N)).$$

Recall that

$$S^\circ = \{x \in \mathbb{R}^N \mid |(x, y)| \leq 1 \ (y \in S)\},$$

where $(\cdot, \cdot)$ is the standard inner product of $\mathbb{R}^N$.

Observe that $\mathscr{V}_N S$ is a lattice and simultaneously a cone. However, $\mathscr{V}_N S$ is not closed in $\mathscr{V}_N$. This circumstance notwithstanding, given a family $(S_\xi)_{\xi \in \Xi}$ in $\mathscr{V}_N S$, sometimes we may soundly speak of the *upper hull* $\pi^\uparrow(\Xi)$, *lower hull* $\pi_\downarrow(\Xi)$, and *hull* $\pi(\Xi)$ of this family, implying the least closed cones that lie in $\mathscr{V}_N S$, include S_ξ for all $\xi \in \Xi$, and are closed under the join, the meet, and both operations in the lattice

of convex figures $\mathscr{V}_N$. An example is provided by any instance of *nondegenerate family.* The latter is by definition any family of nonzero sets $(S_\xi)_{\xi\in\Xi}$ such that,

$$\sup_{\xi\in\Xi}\|A\|_{S_\xi}<+\infty\quad (A\in\mathscr{L}(\mathbb{R}^N)).$$

Indeed, put

$$\mathscr{A}(\Xi):=\{A\in\mathscr{L}(\mathbb{R}^N)\mid AS_\xi\subset S_\xi\ (\xi\in\Xi)\},$$

and let $\mathrm{M}(\Xi)$ be the set of the symmetric elements of $\mathscr{V}_N$ such that $AS\subset S$ for all $A\in\mathscr{A}(\Xi)$. Since $(S_\xi)_{\xi\in\Xi}$ is nondegenerate, all members of $\mathrm{M}(\Xi)$ but the zero singleton are absorbing. Moreover, $\mathrm{M}(\Xi)$ is clearly a closed sublattice of $\mathscr{V}_N$.

We will need the helpful property of a nondegenerate family: If $y\in\mathbb{R}^N$ differs from the zero of $\mathbb{R}^N$ then

$$S_y:=\bigwedge_{\xi\in\Xi}\frac{S_\xi}{S_\xi(y)}$$

is absorbing. Indeed, given $z\in\mathbb{R}^N$ we infer that

$$\sup_{\xi\in\Xi}S_\xi(y)S_\xi^\circ(z)=\sup_{\xi\in\Xi}\|y\|_{S_\xi^\circ}\|z\|_{S_\xi}=\sup_{\xi\in\Xi}\|y\otimes z\|_{S_\xi}<+\infty,$$

where $y\otimes z: x\mapsto (y,x)z$ for all $x\in\mathbb{R}^N$. Hence, the polar of S_y is compact, which implies that S_y is absorbing. Without further specification, we will address only nondegenerate families of balls in the sequel.

4.1. Theorem. *A gauge S belongs to $\pi^\uparrow(\Xi)$ if and only if*

$$\frac{S}{\sum\limits_{k=1}^{n}\|x_k\|_{S^\circ}}\le\bigvee_{\xi\in\Xi}\frac{S_\xi}{\sum\limits_{k=1}^{n}\|x_k\|_{S_\xi^\circ}}$$

for any collection of the vectors $x_1,\dots,x_p\in\mathbb{R}^N$ that are not all zero simultaneously.

Proof. It is obvious that $\pi^\uparrow(\Xi)$ is the closure of the upper semilattice of all H-convex functions with H the conic hull of the family $(S_\xi)_{\xi\in\Xi}$. The polar of $\pi^\uparrow(\Xi)$ may be approximated with finitely supported signed measures by the Approximation Lemma. Using the Bipolar Theorem, we see that $S\in\pi^\uparrow(\Xi)$ if and only if $\sum_{k=1}^n S(x_k)\ge S(y)$ whenever $y,x_1,\dots,x_n\in\mathbb{R}^N$ satisfy $\sum_{k=1}^n S_\xi(x_k)\ge S_\xi(y)$ for all $\xi\in\Xi$. By duality, $S\in\pi^\uparrow(\Xi)$ if and only if

$$\bigwedge_{\xi\in\Xi}\sum_{k=1}^{n}\|x_k\|_{S_\xi^\circ}S_\xi^\circ\subset\sum_{k=1}^{n}\|x_k\|_{S^\circ}S^\circ.$$

Taking polars, we complete the proof of the theorem.

4.2. Corollary. *A nonzero gauge S belongs to $\pi^\uparrow(\Xi)$ if and only if*

$$S=\bigwedge_{(x_1,\dots,x_n)}\sum_{k=1}^{n}S(x_k)\bigvee_{\xi\in\Xi}\frac{S_\xi}{\sum\limits_{k=1}^{n}S_\xi(x_k)},\tag{4.2.1}$$

where the intersection ranges over all nonzero tuples $(x_1,\dots,x_n)\in\mathbb{R}^N$.

Proof. Clearly, (4.2.1) guarantees the inclusion of 4.1 and so $S\in\pi^\uparrow(\Xi)$. The last containment in turn implies the simple representation:

$$S=\bigwedge_{x\ne0}S(x)\bigvee_{\xi\in\Xi}\frac{S_\xi}{S_\xi(x)}.\tag{4.2.2}$$

Indeed, denote by $\widetilde{S}$ the right-hand side of (4.2.2). By 4.1, $S \le \widetilde{S}$. If $z \in \mathbb{R}^n$ then

$$\widetilde{S}(z) = \Big(\bigwedge_{x\ne 0} S(x) \bigvee_{\xi\in\Xi} \frac{S_\xi}{S_\xi(x)}\Big)(z)$$
$$\le S(z)\Big(\bigvee_{\xi\in\Xi} \frac{S_\xi}{S_\xi(z)}\Big)(z) = S(z) \bigvee_{\xi\in\Xi} \frac{S_\xi(z)}{S_\xi(z)} = S(z).$$

By the Minkowski duality $\widetilde{S} \le S$. Denote by $\widetilde{\widetilde{S}}$ the right-hand side of (4.2.1). Since $S \le \widetilde{\widetilde{S}} \le \widetilde{S} \le S$; therefore, $S = \widetilde{\widetilde{S}}$ and we are done.

4.3. From 4.2 it follows that if each closed subset of $\mathscr{V}_n S$ is a cone provided that it contains the convex hull and intersection of any pair of its elements as well as the dilation $\alpha\mathfrak{x}$, with $\alpha \ge 0$, of its every member $\mathfrak{x}$.

4.4. The proof of Theorem 4.1 shows that a positively homogeneous continuous function f on $\mathbb{R}^N$ is the support function of a member of $\pi^\uparrow(\Xi)$ if and only if $\sum_{k=1}^n f(x_k) \ge f(y)$ provided that $\sum_{k=1}^n S_\xi(x_k) \ge S_\xi(y)$ for all $\xi \in \Xi$. Observe that we may restrict the range of the index to $n = 1$ only on condition that the balls S_ξ are dilations of one another. Indeed, in this event the polar $\pi^\uparrow(\Xi)$ is the weakly* closed conic hull of two-points relations and so the functions of the form $x \mapsto \alpha S_{\xi_1}(x) \wedge \beta S_{\xi_2}(x)$ turn out sublinear for positive α and β.

5. The Case of Meeting Gauges

We now address some properties of gauges which are tied with intersection. This operation involves some peculiarities since the intersection of balls differs in general from the pointwise infimum of their support functions. However, the idea of decomposition applies partially to this case.

5.1. Theorem. *Let H be a cone in $\mathscr{V}_N S$ and $H = \pi_\downarrow(H)$. Assume given a nonzero vector y in $\mathbb{R}^N$ such that*

$$S_y := \bigvee_{S\in H;\, S\ne\{0\}} \frac{S}{S(y)}$$

is absorbing. Take $x_1, \dots, x_n$ in $\mathbb{R}^N$. The inequality

$$\sum_{k=1}^n S(x_k) \ge S(y)$$

holds for every gauge $S \in H$ if and only if there are vectors $z_1, \dots, z_n$ in $\mathbb{R}^N$ such that $\sum_{k=1}^n z_k = y$ and, moreover, $S(x_k) \ge S(z_k)$ for all $S \in H$.

Proof. $\Longleftarrow$: Since S is a gauge, the support function of S is a sublinear functional and

$$\sum_{k=1}^n S(x_k) \ge \sum_{k=1}^n S(z_k) \ge S\Big(\sum_{k=1}^n z_k\Big) = S(y).$$

$\Longrightarrow$: For simplicity we restrict exposition to the case when S_y is absorbing for every nonzero $y \in \mathbb{R}^N$. Put

$$K := \sup_{x\in S_y^\circ} |x|.$$

By hypotheses, $K < +\infty$. We further put

$$U := \Big\{(\nu_1, \nu_2) \in C'(S_{N-1}) \times C'(S_{N-1}) \mid \nu_1 \geq 0, \nu_2 \geq 0;$$

$$\|\nu_1\| \vee \|\nu_2\| \leq K; \quad \int\limits_{S_{N-1}} (l, \cdot) d(\nu_1 + \nu_2) = (l, y) \quad (l \in \mathbb{R}^N)\Big\};$$

$$\widetilde{U} := U + H^* \times H^*;$$

$$\mu_1 := |x_1| \varepsilon_{x_1/|x_1|}; \quad \mu_2 := \sum_{k=2}^{n} |x_k| \varepsilon_{x_k/|x_k|}.$$

As usual, we agree that the symbol $|0|\varepsilon_{0/|0|}0$ stands for the zero vector.

Assume that the pair (μ_1, μ_2) does not belong to $\widetilde{U}$. Since U is a weakly* compact convex set; therefore, $\widetilde{U}$ is weakly* closed and convex. By the Separation Theorem there are nonzero functions S_1' and S_2' in H such that

$$\mu_1(S_1') + \mu_2(S_2') < \nu_1(S_1') + \nu_2(S_2') \tag{5.1.1}$$

for all $(\nu_1, \nu_2) \in U$. Put

$$S_1 := \frac{S_1'}{S_1' \wedge S_2'(y)}; \quad S_2 := \frac{S_2'}{S_1' \wedge S_2'(y)}.$$

Note that $S_1, S_2 \in H$. Consequently, the meet $S_1 \wedge S_2$ belongs to H. Moreover,

$$\|y\|_{S_1^\circ \vee S_2^\circ} = (S_1^\circ \vee S_2^\circ)^\circ(y) = S_1 \wedge S_2(y) = \frac{S_1' \wedge S_2'}{S_1' \wedge S_2'(y)}(y) = 1.$$

Since $S_1 \wedge S_2 \supset S_y$; therefore, $S_1^\circ \vee S_2^\circ \subset S_y^\circ$. In particular,

$$\sup_{x \in S_1^\circ \vee S_2^\circ} |x| \leq K \tag{5.1.2}$$

Let V be a face of $S_1^\circ \vee S_2^\circ$ that contains y; i. e., the intersection of $S_1^\circ \vee S_2^\circ$ with some supporting hyperplane to $S_1^\circ \vee S_2^\circ$ at y. Denote by $\operatorname{ext}(V)$ the set of extreme points of V. By the Choquet Theorem there is a probability measure $\overline{\nu}$ with support $\operatorname{ext}(V)$ and barycenter y. Put $V_1 := \operatorname{ext}(V) \cap S_1^\circ$ and $V_2 := \operatorname{ext}(V) \setminus V_1$. The set V_2 lies in S_2°. Let $\overline{\nu}_1 := \overline{\nu}|_{V_1}$ and $\overline{\nu}_2 := \overline{\nu}|_{V_2}$. Then $\overline{\nu} = \overline{\nu}_1 + \overline{\nu}_2$.

We will treat a continuous function f on S_{N-1} as the restriction to S_{N-1} of the unique positively homogeneous namesake function on $\mathbb{R}^N$ and put

$$\nu_1 : f \mapsto \int\limits_{V_1} f d\overline{\nu}_1;$$

$$\nu_2 : f \mapsto \int\limits_{V_2} f d\overline{\nu}_2 \quad (f \in C(S_{N-1}));$$

$$\nu := \nu_1 + \nu_2.$$

Using (5.1.2) and the estimate $\overline{\nu}_1(\mathbb{1}) \leq \overline{\nu}(\mathbb{1}) = 1$, with $\mathbb{1}$ the identically one function; we see that

$$\|\nu_1\| = \nu_1(\mathbb{1}) = \int\limits_{V_1} |\cdot| d\overline{\nu}_1 \leq \sup_{x \in S_1^\circ \vee S_2^\circ} |x| < K.$$

By analogy $\|\nu_2\| \le K$. Moreover,

$$\nu(l) = \int\limits_{V_1} (l,\cdot)d\overline{\nu}_1 + \int\limits_{V_2} (l,\cdot)d\overline{\nu}_2 = \int\limits_{\mathrm{ext}(V)} (l,\cdot)d\overline{\nu} = (l,y)$$

for all $l \in \mathbb{R}^N$. Hence, (ν_1,ν_2) belongs to U and

$$\nu_1(S_1) + \nu_2(S_2) = \int\limits_{V_1} S_1 d\overline{\nu}_1 + \int\limits_{V_2} S_2 d\overline{\nu}_2$$

$$= \int\limits_{V_1} \|\cdot\|_{S_1^\circ} d\overline{\nu}_1 + \int\limits_{V_2} \|\cdot\|_{S_2^\circ} d\overline{\nu}_2 = \overline{\nu}(\mathbb{1}) = 1 = S_1 \wedge S_2(y).$$

By (5.1.1)

$$\sum_{k=1}^{p} S_1 \wedge S_2(x_k) \le \mu_1(S_1) + \mu_2(S_2) < \nu_1(S_1) + \nu_2(S_2)$$

$$= S_1 \wedge S_2(y) \le \sum_{k=1}^{p} S_1 \wedge S_2(x_k).$$

We arrive at a contradiction, which means that (μ_1,μ_2) lies in $\widetilde{U}$; i. e. there are measures ν_1, ν_2 such that $\mu_1 - \nu_1 \in H^*$, $\mu_2 - \nu_2 \in H^*$, and $(\nu_1,\, \nu_2) \in U$. Consider the representing points

$$u_1 : z \mapsto \nu_1(z); \quad u_2 : z \mapsto \nu_2(z) \quad (z \in \mathbb{R}^N).$$

Then $u_1 + u_2 = y$, and for $S \in H$ we have

$$\mu_1(S) \ge \nu_1(S) \ge S(u_1); \quad \mu_2(S) \ge \nu_2(S) \ge S(u_2).$$

Proceed by induction and apply the above process to the measure μ_2 and the nonzero point u_2 (it is exactly the place where we invoke the simplification of the beginning of the proof). We thus come to what was desired. In case $u_2 = 0$, the sought decomposition may be composed of the copies of the zero vectors. The proof is complete.

By way of illustration of Theorem 5.1 we will provide a description for $\pi(\Xi)$.

5.2. Theorem. *Let H be a cone in $\mathscr{V}_N$ and $H = \pi_\downarrow(H)$. Assume that*

$$S_y := \bigwedge_{S\in H;S\ne\{0\}} \frac{S}{S(y)}$$

is absorbing for every nonzero $y \in \mathbb{R}^N$. Then $\pi^\uparrow(H)$ is closed with respect to $\wedge$. Moreover, and a nonzero S in $\mathscr{V}_N$ belongs to $\pi^\uparrow(H)$ if and only if

$$S = \bigwedge_{x\ne 0} S(x) \bigvee_{S_0\in H} \frac{S_0}{S_0(x)} \qquad (5.2.1)$$

Proof. We have already demonstrated that each $S \in \pi^\uparrow(H)$ may be written as in (5.2.1) (cp. (4.2.2)). Assume in turn that S has the shape (5.2.1). By Theorem 4.1 we have to validate the implication

$$\sum_{k=1}^{n} S_0(x_k) \ge S_0(y) \quad \text{for all } S_0 \in H \Longrightarrow \sum_{k=1}^{n} S(x_k) \ge S(y).$$

Since $H = \pi^{\uparrow}(H)$, by Theorem 4.1 there are vectors $z_1, \dots, z_n$ such that

$$\sum_{k=1}^{n} z_k = y;$$
$$S_0(x_k) \geq S_0(z_k) \quad (S_0 \in H)$$

Since S is represented as (5.2.1), $S(x_k) \geq S(z_k)$. Hence,

$$\sum_{k=1}^{n} S(x_k) \geq \sum_{k=1}^{n} S(z_k) \geq S\bigg(\sum_{k=1}^{n} z_k\bigg) = S(y).$$

Thus, $S \in^{\pi}\uparrow (H)$.

We are left with checking that $\pi^{\uparrow}(H)$ is closed under $\wedge$. By above, $S \in \pi^{\uparrow}(H)$ if and only if $S(x) \geq S(y)$ for all $x, y \in \mathbb{R}^N$ satisfying $S_0(x) \geq S_0(y)$ for all $S_0 \in H$.

So, take $S_1, S_2 \in \pi^{\uparrow}(H)$ and assume that $S_0(x) \geq S_0(y)$ for all $S_0 \in H$.

We are to compute $S_1 \wedge S_2(x)$. Arguing as in Theorem 5.1 and replacing the reference to the Choquet Theorem to the Carathéodory Theorem, find vectors $x_1, \dots, x_n$ such that $\sum\limits_{k=1}^{n} x_k = x$ and

$$S_1 \wedge S_2(x) = \sum_{k=1}^{t} S_1(x_k) + \sum_{k=t+1}^{n} S_2(x_k).$$

If $S_0 \in H$ then

$$\sum_{k=1}^{n} S_0(x_k) \geq S_0\bigg(\sum_{k=1}^{p} x_k\bigg) = S_0(x) \geq S_0(y).$$

Hence, by Theorem 5.1 there are vectors $z_1, \dots, z_n \in \mathbb{R}^N$ such that $\sum_{k=1}^{n} z_k = y$ and $S_0(x_k) \geq S_0(z_k)$ for all $S_0 \in H$ and $k := 1, \dots, n$. Thus, $S_1(x_k) \geq S_1(z_k)$ and $S_2(x_k) \geq S_2(z_k)$. Consequently,

$$S_1 \wedge S_2(x) = \sum_{k=1}^{t} S_1(x_k) + \sum_{k=t+1}^{n} S_2(x_k) \geq \sum_{k=1}^{t} S_1(z_k) + \sum_{k=t+1}^{n} S_2(z_k)$$
$$\geq \sum_{k=1}^{n} S_1 \wedge S_2(z_k) \geq S_1 \wedge S_2\bigg(\sum_{k=1}^{n} z_k\bigg) = S_1 \wedge S_2(y).$$

Therefore, $S_1 \wedge S_2$ belongs to $\pi^{\uparrow}(H)$, which completes the proof.

5.3. Corollary. *Let $(S_\xi)_{\xi \in \Xi}$ be a nondegenerate family of balls. Then*

$$\pi(\Xi) = \pi^{\uparrow}(\pi_{\downarrow}(\Xi)).$$

In this event a nonzero gauge S belongs to $\pi(\Xi)$ if and only if

$$S = \bigwedge_{x \neq 0} S(x) \bigvee_{S_0 \in \pi_{\downarrow}(\Xi)} \frac{S_0}{S_0(x)}.$$

Proof. Obviously, $\pi^{\uparrow}(\pi_{\downarrow}(\Xi))$ lies in $\pi(\Xi)$. Note now that

$$S_y := \bigwedge_{S_0 \in \pi_{\downarrow}(\Xi); S_0 \neq \{0\}} \frac{S_0}{S_0(y)} \supset \bigwedge_{S_0 \in \mathrm{M}(\Xi); S_0 \neq \{0\}} \frac{S_0}{S_0(y)}$$

The family $\mathrm{M}(\Xi)$ is nondegenerate since so is $(S_\xi)_{\xi \in \Xi}$. Hence, S_y is absorbing. By Theorem 5.2 $\pi^{\uparrow}(\pi_{\downarrow}(\Xi))$ is closed under $\wedge$, thus serving as a superset of $\pi(\Xi)$.

5.4. In study of the properties of gauges with are related to intersection, we have actually used the accompanying representation

$$\int_{S_{N-1}} S_1 \wedge S_2 d\mu = \inf_{\mu_1+\mu_2 \underset{\mathbb{R}^N}{\gg} \mu} \left(\int_{S_{N-1}} S_1 d\mu_1 + \int_{S_{N-1}} S_2 d\mu_2 \right), \tag{5.4.1}$$

which generalizes the standard formula for the *infimal convolution* $\square$, a routine operation of convex analysis:

$$S_1 \wedge S_2 = S_1 \square S_2.$$

It is an easy matter to see the lattice-theoretic provenance of (5.4.1). Some slightly annoying subtlety of the general case which was obviated by finite dimensionality is connected with the fact the infimum of abstract convex elements in the lattice of these elements is just a partial superlinear operator.

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Sobolev Institute of Mathematics
4 Koptyug Avenue
Novosibirsk, 630090
Russia
E-mail address: sskut@member.ams.org