Two Giants from St. Petersburg

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Abstract This is a short tribute to Aleksandr Alexandrov and Leonid Kantorovich for their contribution to the field of nonsmooth analysis, convexity and optimization.

1 Introduction

Nonsmooth analysis and nondifferentiable optimization are next of kin to convexity and linear programming. This year we celebrate not only the fifty years of nonsmooth analysis, but also the centenary of Aleksandr Alexandrov and Leonid Kantorovich, the mathematical giants whose contributions are indispensable for the fields of convexity and optimization.

Alexandrov was a fast-rising star in mathematics who had maintained his second DSc thesis by the age of 30 and became the youngest rector of Leningrad (now St. Petersburg) State University a decade later. The Mathematics Subject Classification, produced jointly by the editorial staffs of *Mathematical Reviews* and *Zentralblatt für Mathematik* in 2010, has Section 53C45 "Global surface theory (convex surfaces à la A. D. Aleksandrov)". None of the other Russian geometers, Lobachevsky inclusively, has this type of acknowledgement. Alexandrov became the first and foremost Russian geometer of the twentieth century.

Kantorovich was a prodigy who graduated from St. Petersburg University at the age of 18, became a professor at the age of 20, was elected a full member of the Department of Mathematics of the Academy of Sciences of the USSR, and was awarded with a Nobel Prize in economics.

These extraordinary events of the lives of Alexandrov and Kantorovich deserve some attention in their own right. But they hardly lead to any useful conclusions for the general audience in view of their extremely low probability. This is not so with

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their creative legacy, for what is done for the others remains unless it is forgotten, ruined, or libeled. Recollecting the paths and contributions of persons to culture, we preserve their spiritual worlds for the future.

2 Aleksandr D. Alexandrov (1912–1999)

Life's Signposts. Aleksandr Danilovich Alexandrov was born in the Volyn village of the Ryazan province on August 4, 1912. His parents were high school teachers. He entered the Physics Faculty of Leningrad State University in 1929 and graduated in 1933. His supervisors were Boris Delauney, a prominent geometer and algebraist, and Vladimir Fok, one of the outstanding theoretical physicists of the last century. The first articles by Alexandrov dealt with some problems of theoretical physics and mathematics. But geometry soon became his main speciality.

Alexandrov defended his PhD thesis in 1935 and his second doctorate thesis in 1937. He was elected to a vacancy of corresponding member of the Academy of Sciences of the USSR in 1946 and was promoted to full membership in 1964.

From 1952 to 1964 Alexandrov was Rector of Leningrad State University. These years he actively and effectively supported the struggle of biologists with Lysenkoism. The name of Rector Alexandrov is connected with the uprise of the new areas of science such as sociology and mathematical economics which he backed up in the grim years. Alexandrov was greatly respected by established scholars as well as academic youth. "He led the University by moral authority rather than the force of direct order," so wrote Vladimir Smirnov in the letter of commendation on the occasion of Alexandrov's retirement from the position of Rector.

In 1964 Mikhail Lavrentyev invited Alexandrov to join the Siberian Division of the Academy of Sciences of the USSR. Alexandrov moved with his family to Novosibirsk where he found many faithful friends and students. By 1986 he headed a department of the Institute of Mathematics (now, the Sobolev Institute), lectured in Novosibirsk State University, and wrote new versions of geometry textbooks at the secondary school level. Alexandrov opened his soul and heart to Siberia, but was infected with tick-borne encephalitis which undermined his health seriously. From April of 1986 up to his death on July 27,1999, Alexandrov was on the staff of the St. Petersburg Department of the Steklov Mathematical Institute.

Contribution to Science. Alexandrov's life business was geometry. The works of Alexandrov made tremendous progress in the theory of mixed volumes of convex figures. He proved some fundamental theorems on convex polyhedra that are celebrated alongside the theorems of Euler and Minkowski. While discovering a solution of the Weyl problem, Alexandrov suggested a new synthetic method for proving the theorems of existence. The results of this research ranked the name of Alexandrov alongside the names of Euclid and Cauchy.

Another outstanding contribution of Alexandrov to science is the creation of the intrinsic geometry of irregular surfaces. He suggested his amazingly visual and pow-

erful method of cutting and gluing. This method enabled him to solve many extremal problems of the theory of manifolds of bounded curvature.

Alexandrov developed the theory of metric spaces with one-sided constraints on curvature. This gave rise to the class of metric spaces generalizing the Riemann spaces in the sense that these spaces are furnished with some curvature, the basic concept of Riemannian geometry. The research of Alexandrov into the theory of manifolds with bounded curvature prolongates and continues the traditions of Gauss, Lobachevsky, Poincaré, and Cartan.

Retreat to Euclid. Alexandrov accomplished the turnround to the ancient synthetic geometry in a much deeper and subtler sense than it is generally acknowledged today. The matter is not simply in transition from smooth local geometry to geometry in the large without diffentiability restrictions. In fact Alexandrov enriched the methods of differential geometry by the tools of functional analysis and measure theory, driving mathematics to its universal status of the epoch of Euclid. The mathematics of the ancients was geometry (there were no other instances of mathematics at all). Synthesizing geometry with the remaining areas of the today's mathematics, Alexandrov climbed to the antique ideal of the universal science incarnated in mathematics.

Alexandrov contributed to nonsmooth analysis, developing the theory of DC-surfaces and inspecting the first and second order differentiability of convex functions in the classical and distribution senses. He also was a pioneer of using the functional-analytical technique for studying the spaces of compact convex bodies.

Alexandrov overcame many local obstacles and shortcomings of the differential geometry based on the infinitesimal methods and ideas by Newton, Leibniz, and Gauss. Moreover, he enriched geometry with the technique of functional analysis, measure theory, and partial differential equations. Return to the synthetic methods of *mathesis universalis* was inevitable and unavoidable as illustrated in geometry with the beautiful results of the students and descendants of Alexandrov like Misha Gromov, Grisha Perelman, Alexei Pogorelov, Yuri Reshetnyak, and Victor Zalgaller.

Alexandrov and the Present Day. Alexandrov emphasized the criticism of science and its never-failing loyalty to truth. Science explains "how the thingummy's actually going on" with greatness and modesty, using experience, facts, and logic. The universal humanism of the geometer Alexandrov, stemming from the heroes of antiquity, will always remain in the treasure-trove of the best memes of humankind.

3 Leonid V. Kantorovich (1912–1986)

Life's Signposts. Kantorovich was born in the family of a venereologist at St. Petersburg on January 19, 1912. The boy's talent was revealed very early. In 1926, just at the age of 14, he entered Leningrad State University. After graduation from the university in 1930, Kantorovich started teaching, combining it with intensive scientific research. Already in 1932 he became a full professor at the Leningrad Institute of Civil Engineering. From 1934 Kantorovich was a full professor at his *alma mater*.

The main achievements in mathematics belong to the “Leningrad” period of Kantorovich’s life. In the 1930s he published more papers in pure mathematics whereas his 1940s are devoted to computational mathematics in which he was soon appreciated as a leader in this country.

In 1935 Kantorovich made his major mathematical discovery—he defined *K-spaces*, i. e., vector lattices whose every nonempty order bounded subset had an infimum and supremum. The Kantorovich spaces have provided the natural framework for developing the theory of linear inequalities which was a practically uncharted area of research those days. The concept of inequality is obviously relevant to approximate calculations where we are always interested in various estimates of the accuracy of results. Another challenging source of interest in linear inequalities was the stock of problems of economics. The language of partial comparison is rather natural in dealing with what is reasonable and optimal in human behavior when means and opportunities are scarce. Finally, the concept of linear inequality is inseparable from the key idea of a convex set. Functional analysis implies the existence of nontrivial continuous linear functional over the space under consideration, while the presence of a functional of this type amounts to the existence of nonempty proper open convex subset of the ambient space. Moreover,each convex set is generically the solution set of an appropriate system of simultaneous linear inequalities.

From the end of the 1930s the research of Kantorovich acquired new traits in his audacious breakthrough to economics. The novelty of the extremal problems arising in social sciences is connected with the presence of multidimensional contradictory utility functions. This raises the major problem of agreeing conflicting aims.

Kantorovich’s booklet *Mathematical Methods in the Organization and Planning of Production* which appeared in 1939 is a material evidence of the birth of linear programming. Linear programming is a technique of maximizing a linear functional over the positive solutions of a system of linear inequalities. It is no wonder that the discovery of linear programming was immediate after the foundation of the theory of Kantorovich spaces.

At the end of the 1940s Kantorovich formulated and explicated the thesis of interdependence between functional analysis and applied mathematics and suggested three new techniques: the Cauchy method of majorants, the method of finite-dimensional approximations, and the Lagrange method for general optimization problems in topologicasl vector spaces. Kantorovich based his study of the Banach space versions of the Newton method on domination in general ordered vector spaces. His analysis had been so profound that the term “Newton–Kantorovich method” was often used in common parlance. Kantorovich had applied his general approach to the Monge problem, which led to the modern trasport theory.

The economic works of Kantorovich were hardly visible at the surface of the scientific information flow in the 1940s. But the problems of economics prevailed in his creative studies. During the Second World War he completed the first version of his book *The Best Use of Economic Resources* which led to the Nobel Prize awarded to him and Tjalling C. Koopmans in 1975.

In 1957 Kantorovich accepted the invitation to join the newly founded Siberian Division of the Academy of Sciences of the USSR. He moved to Novosibirsk and soon became a corresponding member of the Department of Economics in the first elections to the Siberian Division. Since then his major publications were devoted to economics with the exception of the celebrated course of functional analysis, "Kantorovich and Akilov" in the students' jargon.

The 1960s became the decade of recognition. In 1964 Kantorovich was elected a full member of the Department of Mathematics of the Academy of Sciences of the USSR, and in 1965 he was awarded the Lenin Prize. In these years he vigorously propounded and maintained his views of interplay between mathematics and economics and exerted great efforts to instill the ideas and methods of modern science into the top economic management of the Soviet Union, which was almost in vain.

At the beginning of the 1970s Kantorovich left Novosibirsk for Moscow where he was deeply engaged in economic analysis, not ceasing his efforts to influence the everyday economic practice and decision making in the national economy. His activities were mainly waste of time and stamina in view of the misunderstanding and hindrance of the governing retrogradists of this country. Cancer terminated his path in science on April 7, 1986. He was buried at Novodevichy Cemetery in Moscow.

Linear Programming. The principal discovery of Kantorovich at the junction of mathematics and economics is linear programming which is now studied by hundreds of thousands of people throughout the world. The term signifies the colossal area of science which is allotted to linear optimization models. In other words, linear programming is the science of the theoretical and numerical analysis of the problems in which we seek for an optimal (i. e., maximum or minimum) value of some system of indices of a process whose behavior is described by simultaneous linear inequalities. It was in 1939 that Kantorovich formulated the basic ideas of the new area of science.

The term "linear programming" was minted in 1951 by Tjalling C. Koopmans, an American economist with whom Kantorovich shared in 1975 the Nobel Prize for research of optimal use of resources. The most commendable contribution of Koopmans was the ardent promotion of the methods of linear programming and the strong defence of Kantorovich's priority in the invention of these methods.

In the USA the independent research into linear optimization models was started only in 1947 by George B. Dantzig who also noted the priority of Kantorovich.

It is worth observing that to an optimal plan of every linear program there corresponds some optimal prices or "objectively determined estimators." Kantorovich invented this bulky term by tactical reasons in order to enhance the "criticism endurability" of the concept. The conception of optimal prices as well as the interdependence of optimal solutions and optimal prices is the crux of the economic discovery of Kantorovich.

Mathematics and economics. Mathematics studies the forms of reasoning. The subject of economics is the circumstances of human behavior. Mathematics is abstract and substantive, and the professional decision of mathematicians do not interfere with the life routine of individuals. Economics is concrete and declarative, and the practical exercises of economists change the life of individuals substantially. The

aim of mathematics consists in impeccable truths and methods for acquiring them. The aim of economics is the well-being of an individual and the way of achieving it. Mathematics never intervenes into the private life of an individual. Economics touches his purse and bag. Immense is the list of striking differences between mathematics and economics.

Mathematical economics is an innovation of the twentieth century. It is then when the understanding appeared that the problems of economics need a completely new mathematical technique.

Homo sapiens has always been and will stay forever homo economicus. Practical economics for everyone as well as their ancestors is the arena of common sense. Common sense is a specific ability of a human to instantaneous moral judgement. Understanding is higher than common sense and reveals itself as the adaptability of behavior. Understanding is not inherited and so it does nor belong to the inborn traits of a person. The unique particularity of humans is the ability of sharing their understanding, transforming evaluations into material and ideal artefacts.

Culture is the treasure-trove of understanding. The inventory of culture is the essence of outlook. Common sense is subjective and affine to the divine revelation of faith that is the force surpassing the power of external proofs by fact and formal logic. The verification of statements with facts and by logic is a critical process liberating a human from the errors of subjectivity. Science is an unpaved road to objective understanding. The religious and scientific versions of outlook differ actually in the methods of codifying the artefacts of understanding.

The rise of science as an instrument of understanding is a long and complicated process. The birth of ordinal counting is fixed with the palaeolithic findings hat separated by hundreds of centuries from the appearance of a knowing and economic human. Economic practice precedes the prehistory of mathematics that became the science of provable calculations in Ancient Greece about 2500 years ago.

It was rather recently that the purposeful behavior of humans under the conditions of limited resources became the object of science. The generally accepted date of the birth of economics as a science is March 9, 1776— the day when there was published the famous book by Adam Smith *An Inquiry into the Nature and Causes of the Wealth of Nations*.

Consolidation of Mind. Ideas rule the world. John Maynard Keynes completed this banal statement with a touch of bitter irony. He finished his most acclaimed treatise *The General Theory of Employment, Interest, and Money* in a rather aphoristic manner: “Practical men, who believe themselves to be quite exempt from any intellectual influences, are usually the slaves of some defunct economist.”

Political ideas aim at power, whereas economic ideas aim at freedom from any power. Political economy is inseparable from not only the economic practice but also the practical policy. The political content of economic teachings implies their special location within the world science. Changes in epochs, including their technological achievements and political utilities, lead to the universal proliferation of spread of the emotional attitude to economic theories, which drives economics in the position unbelievable for the other sciences. Alongside noble reasons for that, there is one rather cynical: although the achievements of exact sciences drastically

change the life of the mankind, they never touch the common mentality of humans as vividly and sharply as any statement about their purses and limitations of freedom.

Georg Cantor, the creator of set theory, remarked as far back as in 1883 that “the essence of mathematics lies entirely in its freedom.” The freedom of mathematics does not reduce to the absence of exogenic restriction on the objects and methods of research. The freedom of mathematics reveals itself mostly in the new intellectual tools for conquering the ambient universe which are provided by mathematics for liberation of humans by widening the frontiers of their independence. Mathematization of economics is the unavoidable stage of the journey of the mankind into the realm of freedom.

The nineteenth century is marked with the first attempts at applying mathematical methods to economics in the research by Antoine Augustin Cournot, Carl Marx, William Stanley Jevons, Leon Walras, and his successor in Lausanne University Vilfredo Pareto.

John von Neumann and Leonid Kantorovich, mathematicians of the first calibre, addressed the economic problems in the twentieth century. The former developed game theory, making it an apparatus for the study of economic behavior. The latter invented linear programming for decision making in the problems of best allocation of scarce resources. These contributions of von Neumann and Kantorovich occupy an exceptional place in science. They demonstrated that the modern mathematics opens up broad opportunities for economic analysis of practical problems. Economics has been drifted closer to mathematics. Still remaining a humanitarian science, it mathematizes rapidly, demonstrating high self-criticism and an extraordinary ability of objective thinking.

The turn in the mentality of the mankind that was effected by von Neumann and Kantorovich is not always comprehended to full extent. There are principal distinctions between the exact and humanitarian styles of thinking. Humans are prone to reasoning by analogy and using incomplete induction, which invokes the illusion of the universal value of the tricks we are accustomed to. The differences in scientific technologies are not distinguished overtly, which in turn contributes to self-isolation and deterioration of the vast sections of science.

Universal Heuristics. The integrity of the outlook of Kantorovich was revealed in all instances of his versatile research. The ideas of linear programming were tightly interwoven with his methodological standpoints in the realm of mathematics. Kantorovich viewed as his main achievement in this area the distinguishing of Dedekind complete vector lattes, also called K-spaces or Kantorovich spaces in the literature of the Russian provenance, since Kantorovich wrote about “my spaces” in his personal memos.

The abstract theory of K-spaces, linear programming, and approximate methods of analysis were particular outputs of Kantorovich’s universal heuristics.

More recent research has corroborated that the ideas of linear programming are immanent in the theory of K-spaces. It was demonstrated that the validity of one of the various statements of the duality principle of linear programming in an abstract

mathematical structure implies with necessity that the structure under consideration is in fact a K-space.

The Kantorovich heuristics is connected with one of the most brilliant pages of the mathematics of the twentieth century the famous problem of the continuum. Recall that some set A has the cardinality of the continuum whenever A in equipollent with a segment of the real axis. The continuum hypothesis is that each subset of the segment is either countable of has the cardinality of the continuum. The continuum problem asks whether the continuum hypothesis is true or false.

The continuum hypothesis was first conjectured by Cantor in 1878. He was convinced that the hypothesis was a theorem and vainly attempted at proving it during his whole life. In 1900 the Second Congress of Mathematicians took place in Paris. In the opening session Hilbert delivered his epoch-making talk "Mathematical Problems." He raised 23 problems whose solution was the task of the nineteenth century bequeathed to the twentieth century. The first on the Hilbert list was open the continuum problem. Remaining unsolved for decades, it gave rise to deep foundational studies. The efforts of more than a half-century yielded the solution: we know now that the continuum hypothesis can neither be proved nor refuted.

The two stages led to the understanding that the continuum hypothesis is an independent axiom. Gödel showed in 1939 that the continuum hypothesis is consistent with the axioms of set theory, and Cohen demonstrated in 1963 that the negation of the continuum hypothesis does not contradict the axioms of set theory either. Both results were established by exhibiting appropriate models; i. e., constructing a universe and interpreting set theory in the universe.

Cohen's method of forcing was simplified in 1965 on using the tools of Boolean algebra and the new technique of mathematical modeling which is based on the nonstandard models of set theory. The progress of the so-invoked Boolean valued analysis has demonstrated the fundamental importance of the so-called universally complete K-spaces. Each of these spaces turns out to present one of the possible noble models of the real axis and so such a space plays a similar key role in mathematics. The spaces of Kantorovich implement new models of the reals, this earning their eternal immortality.

Kantorovich heuristics has received brilliant corroboration, this proving the integrity of science and inevitability of interpenetration of mathematics and economics.

Memes for the Future. Kantorovich's message was received as witnessed by the curricula and syllabi of every economics or mathematics department in any major university throughout the world. The gadgets of mathematics and the idea of optimality belong to the tool-kit of any practicing economist. The new methods erected an unsurmountable firewall against the traditionalists that view economics as a testing polygon for the technologies like Machiavellianism, flattery, common sense, or foresight.

Economics as an eternal boon companion of mathematics will avoid merging into any esoteric part of the humanities, or politics, or belles-lettres. The new generations of mathematicians will treat the puzzling problems of economics as an inexhaustible

source of inspiration and an attractive arena for applying and refining their impeccably rigorous methods. Calculation will supersede prophesy.

4 Conclusion

Alexandrov and Kantorovich were live-long friends and collaborators. They had mutual students and followers who made great contributions to pure and applied mathematics. This led to the extraordinary fusion of various ideas from geometry, functional analysis and optimization which resides now within the realm of nonsmooth analysis and nondifferentiable optimization.