Contents

Foreword

Nonstandard methods of analysis consist generally in comparative study of two interpretations of a mathematical claim or construction given as a formal symbolic expression by means of two different set-theoretic models: one, a "standard" model and the other, a "nonstandard" model.

The second half of the 20th century is a period of significant progress in these methods and their rapid development in a few directions.

The first of the latter appears often under the name minted by its inventor, A. Robinson. This memorable term *nonstandard analysis* often swaps places with its synonymous versions like *robinsonian* or *classical nonstandard analysis*, remaining slightly presumptuous and defiant.

The characteristic feature of robinsonian analysis is a frequent usage of many controversial concepts appealing to the actual infinitely small and infinitely large quantities that have resided happily in natural sciences from ancient times but were strictly forbidden in mathematics for many decades of the 20th century. The present-day achievements revive the forgotten term *infinitesimal analysis* which reminds us expressively of the heroic bygones of the *Calculus*.

Infinitesimal analysis expands rapidly, bringing about radical reconsideration of the general conceptual system of mathematics. The principal reasons for this progress are twofold. Firstly, infinitesimal analysis provides us with a novel understanding for the method of indivisibles rooted deeply in the mathematical classics. Secondly, it synthesizes both classical approaches to differential and integral calculus which belong to the noble inventors of the latter. Infinitesimal analysis finds newer and newest applications and merges into every section of contemporary mathematics. Sweeping changes are on the march in nonsmooth analysis, measure theory, probability, the qualitative theory of differential equations, and mathematical economics.

The second direction, *Boolean valued analysis*, distinguishes itself by ample usage of such terms as the technique of ascending and descending, cyclic envelopes and mixings, B-sets and representation of objects in $\mathbf{V}^{(B)}$. Boolean valued analysis originated with the famous works by P. J. Cohen on the continuum hypothesis.

Progress in this direction has evoked radically new ideas and results in many sections of functional analysis. Among them we list Kantorovich space theory, the theory of von Neumann algebras, convex analysis, and the theory of vector measures.

The book [1], printed by the Siberian Division of the Nauka Publishers in 1990 and translated into English by Kluwer Academic Publishers in 1994 (see [2]), gave a first unified treatment of the two disciplines forming the core of the present-day nonstandard methods of analysis.

The reader's interest as well as successful research into the field assigns us the task of updating the book and surveying the state of the art. Implementation of the task has shown soon that it is impossible to compile new topics and results in a single book. Therefore, the Sobolev Institute Press decided to launch the series *Nonstandard Methods of Analysis* which will consist of monographs on various aspects of this direction in mathematical research.

The series started with the book [3] whose English edition [4] appeared quite simultaneously. The second in the series was the collection [5] and its English counterpart [6]. This book continues the series and addresses infinitesimal analysis.

The antique treasure-trove contains the idea of an infinitesimal or an infinitely small quantity. Infinitesimals have proliferated for two millennia, enchanting scientists and philosophers but always raising controversy and sometimes despise. After about half a century of willful neglect, contemporary mathematics starts paying rapt attention to infinitesimals and related topics.

Infinitely large and infinitely small numbers, alongside the atoms of mathematics, "indivisibles" or "monads," resurrect in various publications, becoming part and parcel of everyday mathematical practice. A turning point in the evolution of infinitesimal concepts is associated with an outstanding achievement of A. Robinson, the creation of nonstandard analysis now called *robinsonian* and *infinitesimal.*

Robinsonian analysis was ranked long enough as a rather sophisticated, if not exotic, logical technique for corroborating the possibility of use of actual infinites and infinitesimals. This technique has also been evaluated as hardly applicable and never involving any significant reconsideration of the state-of-the-art.

By the end of the 1970s, the views of the place and role of infinitesimal analysis had been drastically changed and enriched after publication of the so-called internal set theory IST by E. Nelson and the external set theories propounded soon after IST by K. Hrbáček and T. Kawai.

In the light of the new discoveries it became possible to consider nonstandard elements as indispensable members of all routine mathematical objects rather than some "imaginary, ideal, or surd entities" we attach to conventional sets by ad hoc reasons of formal convenience.

This has given rise to a new doctrine claiming that every set is either standard or nonstandard. Moreover, the standard sets constitute some frame of reference "dense" everywhere in the universe of all objects of set-theoretic mathematics, which guarantees healthy translation of mathematical facts from the collection of standard sets to the whole universe.

At the same time many familiar objects of infinitesimal analysis turn out to be "cantorian" sets falling beyond any of the canonical universes in ample supply by formal set theories. Among these "external" sets we list the monads of filters, the standard part operations on numbers and vectors, the limited parts of spaces, etc.

The von Neumann universe fails to exhaust the world of classical mathematics: this motto is one of the most obvious consequences of the new stances of mathematics. Therefore, the traditional views of robinsonian analysis begin to undergo revision, requiring reconsideration of its backgrounds.

The crucial advantage of new ways to infinitesimals is the opportunity to pursue an axiomatic approach which makes it possible to master the apparatus of the modern infinitesimal analysis without learning prerequisites such as the technique and toolbox of ultraproducts, Boolean valued models, etc.

The suggested axioms are very simple to apply, while admitting comprehensible motivation at the semantic level within the framework of the "naive" set-theoretic stance current in analysis. At the same time, they essentially broaden the range of mathematical objects, open up possibilities of developing a new formal apparatus, and enable us to diminish significantly the existent dangerous gaps between the ideas, methodological credenda, and levels of rigor that are in common parlance in mathematics and its applications to the natural and social sciences.

In other words, the axiomatic set-theoretic foundation of infinitesimal analysis has a tremendous significance for science in general.

In 1947 K. Gödel wrote: "There might exist axioms so abundant in their verifiable consequences, shedding so much light upon the whole discipline and furnishing such powerful methods for solving given problems (and even solving them, as far as that is possible, in a constructivistic way), that quite irrespective of their intrinsic necessity they would have to be assumed at least in the same sense as any well established physical theory" [129, p. 521]. This prediction of K. Gödel turns out to be a prophecy.

The purpose of this book is to make new roads to infinitesimal analysis more accessible. To this end, we start with presenting the semantic qualitative views of standard and nonstandard objects as well as the relevant apparatus at the "naive" level of rigor which is absolutely sufficient for effective applications without appealing to any logical formalism.

We then give a concise reference material pertaining to the modern axiomatic expositions of infinitesimal analysis within the classical cantorian doctrine. We have found it appropriate to allot plenty of room to the ideological and historical facets of our topic, which has determined the plan and style of exposition.

Chapters 1 and 2 contain the historical signposts alongside the qualitative motivation of the principles of infinitesimal analysis and discussion of their simplest implications for differential and integral calculus. This lays the "naive" foundation of infinitesimal analysis. Formal details of the corresponding apparatus of nonstandard set theory are given in Chapter 3.

The following remarkable words by N. N. Luzin contains a weighty argument in favor of some concentricity of exposition:

"Mathematical analysis is a science far from the state of ultimate completion with unbending and immutable principles we are only left to apply, despite common inclination to view it so. Mathematical analysis differs in no way from any other science, having its own flux of ideas which is not only translational but also rotational, returning every now and then to various groups of former ideas and shedding new light on them" [335, p. 389].

Chapters 4 and 5 set forth the infinitesimal methods of general topology and subdifferential calculus.

Chapter 6 addresses the problem of approximating infinite-dimensional Banach spaces and operators between them by finite-dimensional spaces and finite-rank operators. Naturally, some infinitely large number plays the role of the dimension or such an approximate space.

The next of kin is Chapter 7 which provides the details of the nonstandard technique for "hyperapproximation" of locally compact abelian groups and Fourier transforms over them.

The choice of these topics from the variety of recent applications of infinitesimal analysis is basically due to the personal preferences of the authors.

Chapter 8 closes exposition, collecting some exercises for drill and better understanding as well as a few open questions whose complexity varies from nil to infinity.

We cannot bear residing in the two-element Boolean algebra and indulge occasionally in playing with general Boolean valued models of set theory. For the reader's convenience we give preliminaries to these models in the Appendix.

This book is in part intended to submit the authors' report about the problems we were deeply engrossed in during the last quarter of the 20th century. We happily recall the ups and downs of our joint venture full of inspiration and friendliness. It seems appropriate to list the latter among the pleasant manifestations and consequences of the nonstandard methods of analysis.

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