Math. Japonica **43,** No. 2(1996), 391–410

NONSTANDARD TOOLS FOR CONVEX ANALYSIS

S. S. Kutateladze

Abstract. Modern tools for local analysis of optimization problems are based on interplay between convexity, order and nonstandard models of set theory. This survey exposes some of the pertinent background ideas and principal results based on using vector-lattice theory and Boolean-valued analysis.

Nonsmooth analysis of today lays theoretical grounds for optimization theory with its numerous and versatile applications. It is innate in mathematics to adjust the shape of its own esoteric core so as to assimilate new problems and ideas. During the last two decades this feature transpires in regard to optimization theory, giving an impetus to rapid progress in nonlinear functional analysis.

In the present article an attempt is made at exposing some instances of the interplay between optimization, ordered vector spaces and nonstandard methods of analysis. The emphasis is placed primarily on those aspects of convex analysis which are principally developed at the Sobolev Institute of Mathematics in Novosibirsk under the influence of the late L. V. Kantorovich, a renowned analyst and a Nobel-prize winner in economics. The vast territory of optimization theory and nonsmooth analysis is left uncharted which is abundant with epsilons and infinitesimals. This challenge deserves another attempt.

1. Order and Convexity. Prolog

1.1. Enter the Reals.

To choose, we use preferences. To optimize, we use infima and suprema (for bounded subsets) which is practically the least upper bound property. So optimization needs ordered sets and primarily (boundedly) complete lattices.

To operate with preferences, we use group structure. To aggregate and scale, we use linear structure.

All these are happily provided by the *reals* $\mathbb{R}$, a one-dimensional (boundedly) order complete vector lattice. An order complete vector lattice is a *Kantorovich space* or a *K-space* for short.

1.2. Enter Convexity and Order.

Consider an abstract minimization problem formulated as follows:

$$x \in X, \quad f(x) \to \inf .$$

Key words and phrases. Optimization problem, Hahn-Banach theorem, Kreĭn-Milman theorem, subdifferential, Kantorovich space, Boolean-valued model.

This research was partially supported by grants of the Russian Foundation for Basic Research, the International Science Foundation and the American Mathematical Society.

Here X is a vector space and $f : X \to \overline{\mathbb{R}}$ is a numeric function taking possibly infinite values. In these circumstances, we are usually interested in the quantity $\inf f(x)$, the *value of the problem*, and in a *solution* of the problem (i.e., such an $\bar{x}$ that $f(\bar{x}) = \inf f(X)$), if the latter exists. It is a rare occurrence to solve an arbitrary problem explicitly. The sociological trick characteristic of functional analysis prompts us to include the problem into a parametric family by introducing the *Young-Fenchel transform* of the function f as follows

$$f^*(l) := \sup_{x\in X} (l(x) - f(x)),$$

with l a linear functional over X, i.e. a member of the *algebraic dual* $X^{\#}$. Observe that the quantity $-f^*(0)$ presents the value of the initial extremal problem and f^* is a convex function in its argument.

A convex function is locally a positively homogeneous convex function, a *sublinear functional.* A sublinear functional is a function $p : X \to \mathbb{R}$ whose *epigraph*

$$\operatorname{epi} p := \{(x,\ t) \in X \times \mathbb{R} : p(x) \le t\}$$

is a (convex) cone. Recall that a numeric function is uniquely determined from its epigraph.

By analogy, to every set $C \subset X$ we can assign a special correspondence $H(C)$ from X to $\mathbb{R}$, called the *Hörmander transform* of C, namely,

$$H(C) := \{(x,\ t) \in X \times \mathbb{R}^+ : x \in tC\}.$$

A set is convex if and only if its Hörmander transform is a cone.

A space with a cone is a *(pre)ordered vector space.*

1.3. Enter Hahn-Banach.

Introducing

$$\partial f(\bar{x}) := \{l \in X^{\#} : (\forall x \in X)\ l(x) - l(\bar{x}) \le f(x) - f(\bar{x})\},$$

the *subdifferential of f at $\bar{x}$*, we derive

1.3.1. Fermat Criterion. *A point $\bar{x}$ is a solution to the minimization problem of* 1.2 *if and only if*

$$0 \in \partial f(\bar{x}).$$

The Fermat criterion is clearly of little avail if we lack effective tools for calculating the subdifferential $\partial f(\bar{x})$. Recalling that the only general law of differential calculus is the "chain rule," we arrive at the question of deriving rules for calculation of the subdifferential of a composite mapping $\partial(f \circ G)(\bar{y})$. In optimization, an adequate understanding of G as a convex mapping requires that some structure of an ordered vector space be present in X.

Miraculously, the celebrated *Hahn-Banach Dominated Extension Theorem* in the above notation reads:

$$\partial(p \circ \iota)(0) = (\partial p)(0) \circ \iota,$$

with p a sublinear functional over X and ι the identical embedding of a subspace of X into X.

In fact, the situation is even more spectacular:

1.3.2. Theorem. *A preordered vector space admits dominated extension of linear operators if and only if it has the least upper bound property.*

Historically this theorem was established in two steps.

1.3.3. Hahn-Banach-Kantorovich Theorem. *Each Kantorovich space admits dominated extension of linear operators.*

This theorem proven by L. V. Kantorovich is the first theorem of the theory of K-spaces.

1.3.4. Bonnice-Silvermann-To Theorem. *Every ordered vector space admitting dominated extension of linear operators is a K-space.*

2. K-Spaces in Action

The prolog drives us to considering convex operators (in particular, sublinear operators) acting into a K-space as inevitable generalization of convex functions inherent in optimization theory. Convex sets result from intersecting half-spaces, the idea underlying linear programming and the like. In this section we expose the idea in analytical disguise.

2.1. Canonical Operator Requisites.

In the class of sublinear operators we distinguish canonical operators with comparatively simple structure so that only one canonical operator is assigned to each K-space and each cardinality. An arbitrary total sublinear operator is obtained as composition of a canonical operator and a linear operator. Thus there arises a possibility of reducing general questions of the theory of sublinear operators to the analysis of a canonical operator and a linear change of variables in it. This constitutes generally the main idea of the canonical operator method. Proceed to exact formulations.

Consider a K-space E and an arbitrary nonempty set $\mathfrak{A}$. Denote by $l_\infty(\mathfrak{A},\ E)$ the set of all (order) bounded mappings from $\mathfrak{A}$ into E; i.e., $f \in l_\infty(\mathfrak{A},\ E)$ if and only if $f : \mathfrak{A} \to E$ and the set $\{f(\alpha) : \alpha \in \mathfrak{A}\}$ is order bounded in E. It is easy to verify that $l_\infty(\mathfrak{A},\ E)$, endowed with the coordinatewise algebraic operations and order, is a K-space. The operator $\varepsilon_{\mathfrak{A},E}$ acting from $l_\infty(\mathfrak{A},\ E)$ into E by the rule

$$\varepsilon_{\mathfrak{A},E} : f \mapsto \sup\{f(\alpha) : \alpha \in \mathfrak{A}\} \quad (f \in l_\infty(\mathfrak{A},\ E))$$

is called the *canonical sublinear operator* given $\mathfrak{A}$ and E. We often write $\varepsilon_{\mathfrak{A}}$ instead of $\varepsilon_{\mathfrak{A},E}$ when it is clear from the context what K-space is meant. The notation ε_n is used when the cardinality of the set $\mathfrak{A}$ equals n and the operator ε_n is called *finitely-generated.*

Let X and E be ordered vector spaces. An operator $p : X \to E$ is called *increasing* or *isotonic* if for all $x_1, x_2 \in X$ from $x_1 \le x_2$ it follows that $p(x_1) \le p(x_2)$. An increasing linear operator is also called *positive.* As usual, the collection of all positive linear operators in the space $L(X,\ E)$ of all linear operators is denoted by $L^+(X,\ E)$. Obviously, the positivity of a linear operator T amounts to the inclusion $T(X^+) \subset E^+$, where $X^+ := \{x \in X : x \ge 0\}$ and $E^+ := \{e \in E : e \ge 0\}$ are the *positive cones* in X and E respectively.

Recall that the set $\partial p := \partial p(0) = \{T \in L(X,\ E) : (\forall x \in X)\ Tx \le p(x)\}$ is the *subdifferential* (at zero) or the *support set* of a sublinear operator p.

2.1.1. *A sublinear operator p from an ordered vector space X into a K-space E is increasing if and only if its support set ∂p consists of positive operators, i.e. $\partial p \subset L^+(X,\ E)$.*

2.1.2. *A canonical operator is increasing and sublinear. A finitely-generated canonical operator is order continuous.*

Consider a set $\mathfrak{A}$ of linear operators acting from a vector space X into a K-space E. The set $\mathfrak{A}$ is *weakly* (*order*) *bounded* if the set $\{\alpha x : \alpha \in \mathfrak{A}\}$ is order bounded for every $x \in X$. We denote by $\langle\mathfrak{A}\rangle x$ the mapping that assigns the element $\alpha x \in E$ to each $\alpha \in \mathfrak{A}$, i.e. $\langle\mathfrak{A}\rangle x : \alpha \mapsto \alpha x$. If $\mathfrak{A}$ is weakly order bounded, then $\langle\mathfrak{A}\rangle x \in l_\infty(\mathfrak{A},\ E)$ for every fixed $x \in X$. Consequently, we obtain the linear operator $\langle\mathfrak{A}\rangle : X \to l_\infty(\mathfrak{A},\ E)$ that acts as $\langle\mathfrak{A}\rangle : x \mapsto \langle\mathfrak{A}\rangle x$. Associate with $\mathfrak{A}$ one more operator

$$p_{\mathfrak{A}} : x \mapsto \sup\{\alpha x : \alpha \in \mathfrak{A}\} \quad (x \in X).$$

The operator $p_{\mathfrak{A}}$ is sublinear. The support set $\partial p_{\mathfrak{A}}$ is denoted by $\operatorname{cop}(\mathfrak{A})$ and is called the *support hull* of $\mathfrak{A}$. These definitions entail the following statement:

2.1.3. *If p is a sublinear operator with $\partial p = \operatorname{cop}(\mathfrak{A})$, then the representation holds*

$$p = \varepsilon_{\mathfrak{A}} \circ \langle\mathfrak{A}\rangle.$$

Clearly, $\partial p = \operatorname{cop}(\partial p)$. Consequently, every sublinear operator $p : X \to E$ admits the above representation with $\mathfrak{A} := \partial p$. Due to this fact the canonical sublinear operator is very useful in various problems connected with sublinear operators and, particularly, in calculating support sets and support hulls.

Let $\Delta_{\mathfrak{A}} := \Delta_{\mathfrak{A},E}$ be the embedding of E into $l_\infty(\mathfrak{A},\ E)$ which assigns the constant mapping $\alpha \mapsto e$ $(\alpha \in \mathfrak{A})$ to every element $e \in E$ so that $(\Delta_{\mathfrak{A}} e)(\alpha) = e$ for all $\alpha \in \mathfrak{A}$.

2.1.4. *The following relations are true*:

$$\varepsilon_{\mathfrak{A},E} \circ \Delta_{\mathfrak{A},E} = I_E, \quad \Delta_{\mathfrak{A},E} \circ \varepsilon_{\mathfrak{A},E}(f) \geq f \quad (f \in l_\infty(\mathfrak{A},\ E)),$$

where I_E is the identity mapping in E.

2.1.5. *Let F be another K-space and $p : E \to F$ be an increasing sublinear operator. Then*

$$\partial\left(p \circ \varepsilon_{\mathfrak{A},E}\right) = \left\{T \in L^+(l_\infty(\mathfrak{A},\ E),\ F) : T \circ \Delta_{\mathfrak{A}} \in \partial p\right\}.$$

2.1.6. *For the support set of a canonical sublinear operator the following representation holds*:

$$\partial\varepsilon_{\mathfrak{A},E} = \left\{\alpha \in L^+\left(l_\infty(\mathfrak{A},\ E),\ E\right) : \alpha \circ \Delta_{\mathfrak{A},E} = I_E\right\}.$$

2.1.7. *For each weakly order bounded set $\mathfrak{A}$ of linear operators the equality holds*:

$$\operatorname{cop}(\mathfrak{A}) = \partial\varepsilon_{\mathfrak{A}} \circ \langle\mathfrak{A}\rangle.$$

The time has come to demonstrate how the canonical operator technique applies to subdifferential calculus. For instance, we calculate the support set of a composite sublinear operator.

2.1.8. Theorem. *Let $p_1 : X \to E$ be a sublinear operator and let $p_2 : E \to F$ be an increasing sublinear operator. Then*

$$\partial(p_2 \circ p_1) = \left\{T \circ \langle \partial p_1 \rangle : T \in L^+(l_\infty(\partial p_1,\ E),\ F)\ \&\ T \circ \Delta_{\partial p_1} \in \partial p_2\right\}.$$

Furthermore, if $\partial p_1 = \operatorname{cop}(\mathfrak{A}_1)$ and $\partial p_2 = \operatorname{cop}(\mathfrak{A}_2)$, then

$$\partial(p_2 \circ p_1) = \left\{T \circ \langle \mathfrak{A}_1 \rangle : T \in L^+(l_\infty(\mathfrak{A}_1,\ E),\ F)\ \&\ \left(\exists \alpha \in \partial\varepsilon_{\mathfrak{A}_2}\right)\ T \circ \Delta_{\mathfrak{A}_1} = \alpha \circ \langle \mathfrak{A}_2 \rangle\right\}.$$

$\lhd$ By 2.1.3, the representation $p_2 \circ p_1 = p_2 \circ \varepsilon_{\mathfrak{A}_1} \circ \langle \mathfrak{A}_1 \rangle$ holds. Applying 2.1.7 and 1.3.3, we successively deduce:

$$\begin{aligned}\partial(p_2 \circ p_1) &= \partial\left(p_2 \circ \varepsilon_{\mathfrak{A}_1} \circ \langle \mathfrak{A}_1 \rangle\right)\\ &= \partial(p_2 \circ \varepsilon_{\mathfrak{A}_1}) \circ \langle \mathfrak{A}_1 \rangle\\ &= \left\{T \in L^+(l_\infty(\mathfrak{A}_1,\ E),\ F) : T \circ \Delta_{\mathfrak{A}_1} \in \partial p_2\right\} \circ \langle \mathfrak{A}_1 \rangle\\ &= \{T \circ \langle \mathfrak{A}_1 \rangle : T \geq 0\ \&\ (\exists \alpha \in \partial\varepsilon_{\mathfrak{A}_2})\, T \circ \Delta_{\mathfrak{A}_1} = \alpha \circ \langle \mathfrak{A}_2 \rangle\},\end{aligned}$$

whence the claim follows. $\rhd$

2.1.9. Theorem. *For each band projection π in a K-space E (i.e. for an idempotent π in $L(E)$ satisfying $0 \leq \pi \leq I_E$) we have*

$$\partial(p_2 \circ p_1) = \bigcup_{T \in \partial p_2} (\partial(T \circ \pi \circ p_1) + \partial(T \circ \pi^\perp \circ p_1)),$$

with $\pi^\perp := I_E - \pi$ the complement of π.

2.1.10. *If $p_1 : X \to E$ is a sublinear operator and $p_2 : E \to F$ is an increasing sublinear operator, then*

$$\partial(p_2 \circ p_1) = \bigcup_{T \in \partial p_2} \partial(T \circ p_1).$$

2.2. Enter Orthomorphisms.

Several facts about specific operators in vector lattices are needed in the sequel.

Consider a K-space E. A linear operator $\alpha : E \to E$ with $0 \leq \alpha e \leq e$ for all $e \in E^+$, i.e. $0 \leq \alpha \leq I_E$, is called a *multiplicator* in E. The collection of all multiplicators in E is denoted by $M(E)$, so that $M(E) = [0,\ I_E]$ is an order interval in the space of *regular operators* $L^r(E) := L^+(E) - L^+(E)$. Multiplicators in particular possess the following useful properties.

2.2.1. *Every multiplicator preserves suprema and infima of arbitrary nonempty order bounded sets.*

2.2.2. *Every two multiplicators commute.*

2.2.3. *If a multiplicator α is a monomorphism, then $\alpha(E)$ is an order dense ideal in E; moreover, α is an order isomorphism between E and $\alpha(E)$.*

2.2.4. *For every multiplicator α and every number $\varepsilon > 0$ there exist finite-valued elements π_ε and ρ_ε such that*

$$0 \le \alpha - \pi_\varepsilon \le \varepsilon I_E, \quad 0 \le \rho_\varepsilon - \alpha \le \varepsilon I_E.$$

Recall that an operator $\pi \in L(E)$ is said to be a *finite-valued element* if there exist band projections $\pi_1, \dots, \pi_n$ and reals $t_1, \dots, t_n$ such that $\pi = t_1\pi_1 + \dots + t_n\pi_n$.

Let E be a K-space, and let I_E as usual be the identity operator in E. The band generated by I_E in the K-space $L^r(E)$ is denoted by $\mathrm{Orth}(E)$. A member of $\mathrm{Orth}(E)$ is called *orthomorphism*. It is proved in K-space theory that an orthomorphism π can be characterized as a regular operator commuting with band projections (the elements of the *base* of E) or with multiplicators (the elements of $M(E)$). Another characteristic property of an orthomorphism π, reflected in its title, is the following: if $e_1 \wedge e_2 = 0$ then $\pi e_1 \wedge e_2 = 0$. Let $\mathcal{Z}(E)$ denote the order ideal in $L^r(E)$ which is generated by I_E. This subspace is called the *ideal center* of E. It is obvious that $\mathrm{Orth}(E)$ and $\mathcal{Z}(E)$ are lattice-ordered algebras with respect to the natural ring structure and order relation induced from $L^r(E)$. In addition, $\mathcal{Z}(E)$ serves as an order dense ideal of $\mathrm{Orth}(E)$, i.e. in $\mathrm{Orth}(E) \setminus \{0\}$ there are no elements disjoint from $\mathcal{Z}(E)$. In turn, as it was already noted, $\mathrm{Orth}(E)$ is the centralizer of $\mathcal{Z}(E)$ in the algebra $L^r(E)$. Observe also that in the algebra $\mathrm{Orth}(E)$ the product $\pi_1 \circ \pi_2$ is often denoted simply by $\pi_1\pi_2$.

2.2.5. *For a positive operator $T \in L^r(E)$ the following statements are equivalent*:

(1) $T \in \mathrm{Orth}(E)$;

(2) *$T + I_E$ is a lattice homomorphism*;

(3) *$T + I_E$ possesses the Maharam property, i.e. it preserves order intervals.*

Given a vector space X, a set $\mathcal{U} \subset L(X,\ E)$ is called *operator-convex* or $\mathrm{Orth}(E)$*-convex* provided that for all elements $S, T \in \mathcal{U}$ and orthomorphisms $\alpha, \beta \in A^+$ such that $\alpha + \beta = I_E$, the relation $\alpha \circ S + \beta \circ T \in \mathcal{U}$ holds.

2.2.6. *Each support set is operator-convex.*

Observe that for every family $(S_\xi)_{\xi\in\Xi}$ of elements of ∂p and every family of multiplicators $(\alpha_\xi)_{\xi\in\Xi}$ such that $\sum_{\xi\in\Xi} \alpha_\xi = I_E$ we have $\sum_{\xi\in\Xi} \alpha_\xi \circ S_\xi \in \partial p$. Here the operators are summed with respect to pointwise o-convergence. In other words,

$$S = \sum_{\xi\in\Xi} \alpha_\xi \circ S_\xi \leftrightarrow (\forall x \in X)\ Sx = o\text{-}\sum_{\xi\in\Xi} \alpha_\xi \circ S_\xi x,$$

where in turn

$$y = o\text{-}\sum_{\xi\in\Xi} x_\xi \leftrightarrow y = o\text{-}\lim_{\theta\in\Theta} s_\theta, \quad s_\theta := \sum_{\xi\in\theta} x_\xi,$$

with Θ standing for the set of all finite subsets of Ξ. Finally, note that $y = o\text{-}\lim x_\xi$ means that there are an increasing family (a_ξ) and a decreasing family (b_ξ) such that $a_\xi \le x_\xi \le b_\xi$ and $\sup(a_\xi) = \inf(b_\xi) = y$.

The stated property of ∂p is referred to as *strong operator convexity*.

Consider now a mapping $f : X \to E$. As the *Young-Fenchel transform* of f or the *conjugate* operator to f we refer to the operator $f^* : L(X,\ E) \to E^{\cdot}$ defined by the relation

$$f^*(T) := \sup\{Tx - f(x) : x \in X\} \quad (T \in L(X,\ E)),$$

with $E^{\cdot} := E \cup \{+\infty\}$ and $+\infty$ a greatest element adjoint properly to E.

Clearly, this operator meets the version of the definition of convex function with orthomorphisms substituted for reals. Recalling that the space $L(X,\ E)$ is a module over $\mathrm{Orth}(E)$, we see that

2.2.7. *The Young-Fenchel transform f^* of an operator f with range in E is a module-convex operator over* $\operatorname{Orth}(E)$.

2.3. Enter Kreĭn-Milman.

Of profound import is the intrinsic structure of subdifferentials as is clear from considering their counterparts in finite dimensions, convex compact sets.

Let $p : X \to E$ be a sublinear operator. Denote by the symbol $\operatorname{ext}(p)$ the collection of all *extreme points* of the support set ∂p. Take another K-space F and let $T \in L^+(E,\ F)$. We call an operator $S \in \partial p$ a T-*extreme point* of ∂p (or a T-extreme point of p) and write $S \in \mathcal{E}(T,\ p)$ if $T \circ S \in \operatorname{ext}(T \circ p)$. If $\mathfrak{L}$ is a family of positive operators, then we put

$$\mathcal{E}(\mathfrak{L},\ p) := \bigcap_{T\in\mathfrak{L}} \mathcal{E}(T,\ p).$$

Recall that an operator T is said to be *order-continuous* or *o-continuous* if $T(\inf U) = \inf T(U)$ for every bounded below and downward-filtered set U in E. Let $\mathfrak{L}_0$ be the class of all o-continuous operators defined on a K-space E and taking their values in arbitrary K-spaces. The set $\mathcal{E}(\mathfrak{L}_0,\ p)$ is denoted by the symbol $\mathcal{E}_0(p)$ and its elements are called *o-extreme points* of ∂p (or p).

2.3.1. Kreĭn-Milman Theorem for *o*-Extreme Points. *Every support set is the support hull of the set of its o-extreme points. Symbolically,*

$$\partial p = \partial(\varepsilon_{\mathcal{E}_0(p)}) \circ \langle \mathcal{E}_0(p) \rangle.$$

2.3.2. Theorem. *The following statements are equivalent*:

(1) *an operator S belongs to* $\operatorname{ext}(p)$;

(2) *for all operators $S_1, S_2 \in \partial p$ and for multiplicators $\alpha_1, \alpha_2 \in [0,\ I_E]$, with $\alpha_1 + \alpha_2 = I_E$ and $\alpha_1 \circ S_1 + \alpha_2 \circ S_2 = S$ there is a band projection π in E for which $\pi \circ S = \pi \circ S_1$ and $\pi^\perp \circ S = \pi^\perp \circ S_2$*;

(3) *if for some operators $S_1, \dots, S_n \in \partial p$ and multiplicators $\alpha_1, \dots, \alpha_n \in [0,\ I_E]$ the equalities*

$$\sum_{k=1}^{n} \alpha_k = I_E, \quad \sum_{k=1}^{n} \alpha_k \circ S_k = S_1$$

hold then $\alpha_k \circ S = \alpha_k \circ S_k$ for every $k := 1, \dots, n$.

These facts allow us to reveal one of the most important peculiarities of the set of extreme points of a subdifferential, namely, the possibility of their mixing, i.e., the *cyclicity* of $\operatorname{ext}(p)$.

2.3.3. *Let $S_1, S_2 \in \operatorname{ext}(p)$ and let π be an arbitrary band projection in E. Then*

$$\pi \circ S_1 + \pi^\perp \circ S_2 \in \operatorname{ext}(p).$$

The following statement relates to strong operator convexity.

2.3.4. *Let $(S_\xi)_{\xi\in\Xi}$ be a family of elements of $\operatorname{ext}(p)$ and let $(\pi_\xi)_{\xi\in\Xi}$ be a family of band projections forming a partition of unity, i.e., such that*

$$\xi_1 \neq \xi_2 \to \pi_{\xi_1} \circ \pi_{\xi_2} = 0; \quad \sum_{\xi\in\Xi} \pi_\xi = I_E.$$

Then the operator $\sum_{\xi\in\Xi} \pi_\xi \circ S_\xi$ also belongs to $\operatorname{ext}(p)$.

The last fact allows us to call a (weakly order bounded) set $\mathfrak{A}$ in $L(X,\ E)$ *strongly cyclic* if for every family $(S_\xi)_{\xi\in\Xi}$ of elements of $\mathfrak{A}$ and for an arbitrary partition of unity $(\pi_\xi)_{\xi\in\Xi}$ we have $\sum_{\xi\in\Xi} \pi_\xi \circ S_\xi \in \mathfrak{A}$. The smallest strongly cyclic set containing the given set $\mathfrak{A}$ is said to be the *strong cyclic hull* of $\mathfrak{A}$ and is denoted by $\operatorname{scyc}(\mathfrak{A})$ or $\mathfrak{A}{\uparrow\downarrow}$ (cf. 4.2.1).

2.3.5. $\operatorname{scyc}(\mathcal{E}_0(p)) \subset \operatorname{ext}(p)$.

Now we study the connection between *o*-extreme and extreme points in more detail.

2.3.6. Theorem. *The set of extreme points of the support set of a canonical operator* $\varepsilon_{\mathfrak{A}}$ *coincides with the set of lattice homomorphisms from* $l_\infty(\mathfrak{A},\ E)$ *into* E *which are contained in* $\partial\varepsilon_{\mathfrak{A}}$. *Moreover,* $\operatorname{ext}(\varepsilon_{\mathfrak{A}}) = \mathcal{E}_0(\varepsilon_{\mathfrak{A}})$.

2.3.7. Milman Theorem. *Let* $p : Y \to E$ *be a sublinear operator acting from a vector space* Y *into a* K*-space* E. *Further, let* $T \in L(X,\ Y)$. *Then the following inclusion holds*:

$$\operatorname{ext}(p \circ T) \subset \operatorname{ext}(p) \circ T.$$

Some important corollaries of Milman's Theorem are now at hand.

2.3.8. *If a subdifferential* ∂p *is the support hull of a set* $\mathfrak{A}$, *then*

$$\operatorname{ext}(p) \subset \operatorname{ext}(\varepsilon_{\mathfrak{A}}) \circ \langle\mathfrak{A}\rangle.$$

$\lhd$ $\operatorname{ext}(p) = \operatorname{ext}(\operatorname{cop}(\mathfrak{A})) = \operatorname{ext}\left(\varepsilon_{\mathfrak{A}} \circ \langle\mathfrak{A}\rangle\right) \subset \operatorname{ext}\left(\varepsilon_{\mathfrak{A}}\right) \circ \langle\mathfrak{A}\rangle$ $\rhd$

2.3.9. *For each sublinear operator* p *the next inclusion is true*:

$$\operatorname{ext}(p) \subset \operatorname{ext}\left(\varepsilon_{\mathcal{E}_0(p)}\right) \circ \langle\mathcal{E}_0(p)\rangle.$$

Milman's Theorem and its corollaries give a complete characterization of the intrinsic structure of each support set "modulo" that of the subdifferential of a canonical operator and its extreme points.

3. The Scene of Hahn-Banach

The above consideration makes it incumbent upon us to consider abstract convexity so as to find the scope of possible applications of the Hanh-Banach technique.

3.1. Module Requisites.

We thus let A be an arbitrary lattice-ordered ring with positive unity $\mathbf{1}_A$. This implies that A is a unital ring and there is an order relation $\leq$ in A with respect to which A is a lattice. Moreover, addition and multiplication are compatible with the order relation in the conventional (and quite natural) fashion. In particular, all positive elements of the ring A constitute the semigroup A^+ with respect to addition in A. Now consider a *module* X *over the ring* A, or in short an A*-module* X. This module (as well as all the following) are always considered *unitary*, i.e. $\mathbf{1}_A x = x$ for all $x \in X$. Consider an operator $p : X \to E^{\cdot}$, where $E^{\cdot} := E \cup \{+\infty\}$ as above and E is an ordered A-module (a little thought about this notion prompts its natural definition). An operator is called A*-sublinear* or *module-sublinear* when the ring A is understood from the context, if for all $x, y \in X$ and $\pi, \rho \in A^+$ the inequality holds

$$p(\pi x + \rho y) \leq \pi p(x) + \rho p(y).$$

As a rule, in the sequel we restrict ourselves to the study of total A-sublinear operators $p : X \to E$. It should be observed that $p(0) = 0$. Indeed, $p(0) \leq 0p(0) = 0$ and in addition $p(0) = p(0 + 0) \leq 2p(0)$. At the same time it is easy to see that not for all $x \in X$ and $\pi \in A^+$ with $\pi \neq 0$ the equality $p(\pi x) = \pi p(x)$ is true (this makes an essential difference with $\mathbb{R}$-sublinear operators, i.e. the usual sublinear operators studied above). If $p(\pi x) = \pi p(x)$ for all $x \in X$ and $\pi \in A^+$, then p is called an A^+-*homogeneous operator*. Now consider the set $\mathrm{Hom}_A(X,\ E)$ which is also denoted by $L_A(X,\ E)$ or even $L(X,\ E)$ in case it causes no confusion. This set consists of all A-*linear operators* acting from X into E or, as they are also called, of A-homomorphisms. Thus

$$T \in \mathrm{Hom}_A(X,\ E)$$
$$\leftrightarrow (\forall x, y \in X)\ (\forall \pi, \rho \in A)\ T(\pi x + \rho y) = \pi Tx + \rho Ty.$$

For an A-sublinear operator $p : X \to E$ the *subdifferential* at zero or the *support set* of p and the *subdifferential at a point* x in X are defined by the relations

$$\partial^A p := \{T \in \mathrm{Hom}_A(X,\ E) : (\forall x \in X)\ Tx \leq p(x)\};$$
$$\partial^A p(x) := \{T \in \partial^A p : Tx = p(x)\}.$$

Consequently the representation holds

$$\partial^A p(x) = \{T \in \mathrm{Hom}_A(X,\ E) : (\forall y \in X)\ T(y - x) \leq p(y) - p(x)\}.$$

If $\mathbb{Z}$ is the group of integers then since X and E are $\mathbb{Z}$-modules (= abelian groups), the subdifferentials $\partial^{\mathbb{Z}} p$ and $\partial^{\mathbb{Z}} p(x)$ are defined which are denoted simply by ∂p and $\partial p(x)$. Experience shows that this agreement leads to no collision of notations.

An A-module E is said to possess the A-*extension property* if for all A-modules X and Y, given an A-sublinear operator $p : Y \to E$, and a homomorphism $T \in \mathrm{Hom}_A(X,\ Y)$, the *Hahn-Banach formula*

$$\partial^A(p \circ T) = \partial^A p \circ T$$

is valid. If in addition the subdifferential $\partial p(y)$ is nonempty for every y in Y, then E is said to *admit convex analysis.*

3.1.1. *Let an A-module E possess the A-extension property and let $p : X \to E$ be an A-sublinear operator. The following statements are true*:

(1) *there exists $T \in \partial^A p$ such that $Tx = y$ if and only if $\pi y \leq p(\pi x)$ for all $\pi \in A$*;

(2) *an operator p is A^+-homogeneous if and only if its subdifferential at every point is nonempty, i.e. $\partial^A p(x) \neq \varnothing$ for all $x \in X$.*

3.2. Enter an Erased *K*-Space.

Let E be an ordered abelian group (i.e. an ordered $\mathbb{Z}$-module). Put $E_b := E^+ - E^+$ and assume that E_b is an erased K-space. Recall that by an *erased K-space* we mean each group that results from a K-space on ignoring multiplication by reals, i.e. by partly forgetting information about the space.

3.2.1. Bigard Theorem. *An ordered $\mathbb{Z}$-module E possesses the $\mathbb{Z}$-extension property if and only if E_b is an erased K-space.*

We also need some properties of $\mathbb{Z}$-sublinear operators which follow from the Bigard Theorem.

3.2.2. *Let $p : X \to E$ be a $\mathbb{Z}$-sublinear operator. For every $n \in \mathbb{N}$ we have $\partial(np) = n\partial p$.*

3.2.3. *For every $n \in \mathbb{N}$ the equality holds*

$$\sum_{k=1}^{n} \partial p = n\partial p.$$

3.2.4. *Let T_1, $T_2 \in \partial p$ and $nT_1 = nT_2$ for some $n \in \mathbb{N}$. Then $T_1 = T_2$.*

3.2.5. *Let $p : X \to E$ be a $\mathbb{Z}$-sublinear $\mathbb{Z}^+$-homogeneous operator and let $x \in X$. Then for every $h \in X$ there exists an o-limit*

$$\begin{aligned} p'(x)(h) &:= o\text{-}\lim_{n\in\mathbb{N}}(p(nx + h) - p(nx)) \\ &= \inf\{p(nx + h) - p(nx) : n \in \mathbb{N}\}. \end{aligned}$$

Moreover, $\partial(p'(x)) = \partial p(x)$.

3.2.6. *For every $n \in \mathbb{N}$ we have $(np)'(x) = np'(x)$.*

3.2.7. *Given a $\mathbb{Z}$-sublinear $p : X \to E$, put $h_p(x) := \sup\{Tx : T \in \partial p\}$. Then h_p is the greatest $\mathbb{Z}$-sublinear $\mathbb{Z}^+$-homogeneous operator dominated by p. Moreover, $\partial h_p = \partial p$.*

3.3. Reenter Kreĭn-Milman.

Now we turn to the Kreĭn-Milman Theorem in groups. To begin with, agree that an operator $T \in \partial p$ is called *extreme* if the conditions $T_1, T_2 \in \partial p$ and $T_1 + T_2 = 2T$ imply that $T = T_1 = T_2$. The set of extreme operators in the subdifferential ∂p is denoted by $\operatorname{ext}(p)$. It is seen that this notation agrees with the conventional one.

We also extend the notion of canonical operator and the corresponding formalism to the case of groups. Namely, for a nonempty set $\mathfrak{A}$ we denote by $l_\infty(\mathfrak{A},\ E)$ the $\mathbb{Z}$-module of order bounded E-valued functions on $\mathfrak{A}$. This set is endowed with the natural structure of an ordered $\mathbb{Z}$-module (a submodule of the usual product $E^{\mathfrak{A}}$). Let the symbol $\varepsilon_{\mathfrak{A}}$ denote the *canonical $\mathbb{Z}$-sublinear operator*

$$\begin{gathered} \varepsilon_{\mathfrak{A}} : l_\infty(\mathfrak{A},\ E) \to E, \\ \varepsilon_{\mathfrak{A}}(f) := \sup f(\mathfrak{A}) \quad (f \in l_\infty(\mathfrak{A},\ E)). \end{gathered}$$

In addition, if $\mathfrak{A}$ is a weakly order bounded set of the homomorphisms from X into E, then we define the homomorphism $\langle\mathfrak{A}\rangle : X \to l_\infty(\mathfrak{A},\ E)$ as in 2.1: $\langle\mathfrak{A}\rangle : x \mapsto (Tx)_{T\in\mathfrak{A}}$, i.e. $\langle\mathfrak{A}\rangle x : \mathfrak{A} \ni T \mapsto Tx \in E$.

Following the classical pattern, we can prove

3.3.1. Kreĭn-Milman Theorem for Groups. *For every $\mathbb{Z}$-sublinear operator p the representation holds*

$$\partial p = \partial\varepsilon_{\operatorname{ext}(p)} \circ \langle\operatorname{ext}(p)\rangle.$$

One more property of extreme points is beneficial below.

3.3.2. *For every $\mathbb{Z}$-sublinear operator $p : X \to E$ and $n \in \mathbb{N}$ the equality holds*

$$\operatorname{ext}(np) = n \operatorname{ext}(p).$$

3.3.3. *Let A be a semiring and sublattice in* $\operatorname{Orth}(E)$. *Given elements $\pi, \gamma \in A^+$ with $\pi \geq I_E$, put*

$$[\pi^{-1}](\gamma) := \inf\{\delta \in A^+ : \delta\pi \geq \gamma\}.$$

Then $[\pi^{-1}] : A \to \operatorname{Orth}(E)^{\cdot}$ is an increasing A-sublinear operator and $\gamma = [\pi^{-1}](\pi\gamma)$ for all $\gamma \in A^+$.

Now we are ready to present the main result of the current subsection which states that the additive minorants of a A-sublinear operator are automatically homomorphisms provided that we deal with a subring and sublattice A of the orthomorphism ring $\operatorname{Orth}(E_b)$ which acts naturally in E_b. Before launching into formalities, we sketch out the proof whose idea is rather conspicuous.

In fact, it is almost obvious that extreme points of subdifferentials must commute with multiplicators. Moreover, according to the Kreĭn-Milman Theorem for Groups, each element of a subdifferential, called also a *subgradient*, is obtained by "integrating" extreme points. It remains to observe that the corresponding "dispersed" integrals, i.e. the elements of the subdifferential of a canonical operator, commute with orthomorphisms. Now some formal details follow.

3.3.4. *Let $E = E_b$ and let $\mathfrak{A}$ be an arbitrary set. If the group $l_\infty(\mathfrak{A},\ E)$ is endowed with the natural structure of a $\mathcal{Z}(E)$-module, then the inclusion holds*

$$\partial\varepsilon_{\mathfrak{A}} \subset \operatorname{Hom}_{\mathcal{Z}(E)}(l_\infty(\mathfrak{A},\ E),\ E).$$

3.3.5. *If p is an A-sublinear operator then*

$$\partial^{A\cap\mathcal{Z}(E_b)}p \subset \partial^A p.$$

3.3.6. Theorem. *An additive subgradient of a module-sublinear operator is a module homomorphism.*

$\lhd$ Thus for an A-sublinear operator $p : X \to E$ we need to prove the equality

$$\partial p = \partial^A p.$$

First of all, we establish that $T \in \partial^{A\cap\mathcal{Z}(E_b)}p$ for every $T \in \operatorname{ext}(p)$. Take $\pi \in A^+ \cap \mathcal{Z}(E_b)$. Observe that $\pi \leq n\mathbf{1}_A$ for some $n \in \mathbb{N}$, since $\mathbf{1}_A = I_{E_b}$. Since multiplication by $\mathbf{1}_A$ acts as identity in X and in E, we obtain

$$\begin{gathered} nT = n\mathbf{1}_A \circ T = \pi \circ T + (n\mathbf{1}_A - \pi) \circ T; \\ nT = T \circ \pi + T \circ (n\mathbf{1}_A - \pi); \\ 2nT = \pi \circ T + T \circ (n\mathbf{1}_A - \pi) + (T \circ \pi - (n\mathbf{1}_A - \pi) \circ T). \end{gathered}$$

Taking into consideration the obvious inclusions

$$\begin{gathered} \pi \circ T + T \circ (n\mathbf{1}_A - \pi) \in \partial(np), \\ T \circ \pi + (n\mathbf{1}_A - \pi) \circ T \in \partial(np) \end{gathered}$$

and 3.2.3 by which $nT \in \partial(np)$, we infer

$$nT = \pi \circ T + T \circ (n\mathbf{1}_A - \pi).$$

Thus $T \circ \pi = \pi \circ T$.

Next, consider an operator $p_1 := p - T$, where $T \in \operatorname{ext}(p)$. Clearly, $\operatorname{im}(p_1 - T) \subset E_b$. According to this, we can write

$$\operatorname{ext}(p_1) \subset \partial^{A \cap \mathscr{Z}(E_b)} p_1.$$

Moreover, by the Kreĭn-Milman Theorem for Groups and 3.3.4, the following relations hold:

$$\partial p_1 = \partial \varepsilon_{\operatorname{ext}(p_1)} \circ \langle \operatorname{ext}(p_1) \rangle;$$
$$\partial \varepsilon_{\operatorname{ext}(p_1)} \subset \operatorname{Hom}_{A \cap \mathscr{Z}(E_b)}(l_\infty(\operatorname{ext}(p_1),\ E_b),\ E_b).$$

From this it is immediate that

$$\partial p_1 = \partial^{A \cap \mathscr{Z}(E_b)} p_1.$$

Now if $S \in \partial p$ then $S - T \in \partial p_1$. Consequently the operator $S - T$ is in fact an $A \cap \mathscr{Z}(E_b)$-homomorphism. The same is true for the operator T. Finally $S \in \partial^{A \cap \mathscr{Z}(E_b)} p$. The application of 3.3.5 completes the proof. $\triangleright$

3.3.7. *An ordered A-module E possesses the A-extension property.*

$\triangleleft$ It suffices to refer to the Bigard Theorem and 3.3.6. $\triangleright$

3.4. Conversion.

Now we attempt at conversing the previous statement. Namely, we wish to demonstrate that, under common stipulations, convex analysis comes into effect if and only if one deals with a Kantorovich space considered as module over the algebra of its orthomorphisms. According to Theorem 3.3.6, which automatically guarantee the commutation conditions, we thus arrive at a rather paradoxical conclusion that there is no special "module" convex analysis at all.

With this in mind, we start with an analog of the celebrated Ioffe Fan Theorem.

3.4.1. Theorem. *If an ordered A-module E possesses the A-extension property then E_b is an erased K-space.*

3.4.2. Theorem. *Let A be a d-ring, i.e. $(\pi_1\pi_2)^+ = \pi_1^+\pi_2$ and $(\pi_2\pi_1)^+ = \pi_2\pi_1^+$ for all $\pi_1 \in A$ and $\pi_2 \in A^+$. An ordered A-module E possesses the A-extension property if and only if E_b is an erased K-space and the natural linear representation of A in E_b is a ring and lattice homomorphism onto a subring and sublattice of the orthomorphism ring* $\operatorname{Orth}(E_b)$. *Moreover, $\partial^A p = \partial p$ for every A-sublinear operator p acting into E.*

The condition imposed on the ring A in Theorem 3.4.2 can be altered, although it is impossible to eliminate such an assumption in principle if we want to preserve A^+-homogeneity of a $\mathbb{Z}^+$-homogeneous A-sublinear operator. Observe here that by Theorem 3.3.6 the extension property holds in a stronger form, i.e. a group homomorphism defined on the subgroup and dominated by a module-sublinear operator admits an extension up to a module homomorphism preserving the domination.

To describe modules that admit convex analysis we need one more notion. A subring A of the orthomorphism ring is called *almost rational* if for every $n \in \mathbb{N}$ there exists a decreasing net of multiplicators $(\pi_\xi)_{\xi \in \Xi}$ in A such that for every $y \in E^+$ we have

$$\frac{1}{n} y = o\text{-}\lim_{\xi \in \Xi} \pi_\xi y = \inf_{\xi \in \Xi} \pi_\xi y.$$

3.4.3. *A ring A is almost rational if and only if each A-sublinear operator is A^+-homogeneous.*

3.4.4. Theorem. *An ordered A-module E admits convex analysis if and only if E_b is an erased K-space and the natural representation of A in E_b is a ring and lattice homomorphism onto an almost rational ring of orthomorphisms in E_b.*

$\lhd$ The operators $\pi \mapsto \pi^+ y$ and $z \mapsto z^+$, with $\pi \in A$, $y \in E^+$ and $z \in E_b$, are obviously A-sublinear. Therefore, if the A-module E admits convex analysis, then by 3.1.1 (2) these operators are A^+-homogeneous. According to 2.2.5, this means that the natural linear representation of A in E_b is a ring and lattice homomorphism onto a ring and sublattice of $\mathrm{Orth}(E_b)$. By 3.4.3, the image is almost rational. To complete the proof it suffices to implement the necessary factorization and to refer to Theorem 3.4.2 and 3.4.3. $\rhd$

4. Omnipresent Reals

Our previous discussion demonstrates that the thorough and high-principled analysis of subdifferential calculus is impossible without some esoteric technique of Kantorovich spaces. It is easy to succumb to temptation and to become proud of finding new formulas in fuller generality. Paradoxically, the scope of the new results is the same as that of those obtained in the scalar case.

4.1. Enter Boolean-Valued Analysis.

Boolean-valued analysis is a branch of functional analysis which uses a special model-theoretic technique, the Boolean-valued models of set theory. It is interesting to observe that the invention of the Boolean-valued models was connected neither with the theory of ordered vector spaces nor with optimization. The necessary language and technical tools have been available within mathematical logic since the early 1960s. However, there was no general idea to breathe life into the already-created mathematical apparatus and promote rapid progress in model theory. Such an idea appeared along with P. J. Cohen's discovery; in 1963 he established that the classical continuum hypothesis is undecidable (in a rigorous mathematical sense). It was the Cohen forcing method whose comprehension gave rise to the Boolean-valued models of set theory. Their appearance is commonly associated with the names of P. Vopěnka, D. Scott, and R. Solovay. It is a pleasure to emphasize the role of G. Takeuti who stood at the cradle of Boolean-valued analysis.

4.2. Boolean Requisites.

We start with auxiliary facts about the construction and rules of treating Boolean-valued models.

Let B be a complete Boolean algebra. Given an ordinal α, put

$$V_\alpha^{(B)} := \{x : (\exists \beta \in \alpha)\ x : \mathrm{dom}(x) \to B \ \&\ \mathrm{dom}(x) \subset V_\beta^{(B)}\}.$$

After this recursive definition the *Boolean-valued universe* $V^{(B)}$ or, in other words, the *class of B-sets* is introduced by

$$V^{(B)} := \bigcup_{\alpha \in \mathrm{On}} V_\alpha^{(B)},$$

with On standing for the class of all ordinals.

Let φ be an arbitrary formula of ZFC (= Zermelo-Fraenkel set theory with the axiom of choice).

The *Boolean truth-value* $[\![\varphi]\!] \in B$ is introduced by induction on the length of a formula φ by naturally interpreting the propositional connectives and quantifiers in the Boolean algebra B and taking into consideration the way in which this formula is built up from atomic formulas. The Boolean truth-values of the *atomic formulas* $x \in y$ and $x = y$, with $x, y \in V^{(B)}$, are defined by means of the following recursion schema:

$$[\![x \in y]\!] := \bigvee_{z \in \operatorname{dom}(y)} y(z) \wedge [\![z = x]\!];$$

$$[\![x = y]\!] := \bigwedge_{z \in \operatorname{dom}(x)} x(z) \Rightarrow [\![z \in y]\!] \wedge \bigwedge_{z \in \operatorname{dom}(y)} y(z) \Rightarrow [\![z \in x]\!].$$

(The sign $\Rightarrow$ symbolizes the implication in B: $a \Rightarrow b := a^{\perp} \vee b$ where $a^{\perp}$ is as usual the *complement* of a.)

The universe $V^{(B)}$ with the Boolean truth-value of a formula is a model of set theory in the sense that the following statement is fulfilled.

4.2.1. Transfer Principle. *For every theorem φ of* ZFC, *we have $[\![\varphi]\!] = \mathbf{1}$, i.e. φ is true inside $V^{(B)}$.*

Enter into the next agreement: If x is an element of $V^{(B)}$ and $\varphi(\cdot)$ is a formula of ZFC, then the phrase "x satisfies φ inside $V^{(B)}$" or, briefly, "$\varphi(x)$ is true inside $V^{(B)}$" means that $[\![\varphi(x)]\!] = \mathbf{1}$.

For an element $x \in V^{(B)}$ and for an arbitrary $b \in B$ the function

$$bx : z \mapsto bx(z) \quad (z \in \operatorname{dom}(x))$$

is defined. (Here we presume that $b\varnothing := \varnothing$ $(b \in B)$.)

There is a natural equivalence relation $x \sim y \leftrightarrow [\![x = y]\!] = \mathbf{1}$ in the class $V^{(B)}$. Choosing a representative of smallest rank in every equivalence class, or more exactly with the help of the so-called "Frege-Russel-Scott trick," we obtain the *separated Boolean-valued universe* $\overline{V}^{(B)}$ in which $x = y \leftrightarrow [\![x = y]\!] = \mathbf{1}$. It is easily seen that the Boolean truth-value of a formula remains unaltered if we replace in it each element of $V^{(B)}$ by one of its equivalents. In this connection from now on we take $V^{(B)} := \overline{V}^{(B)}$ without further specification. Observe that in $\overline{V}^{(B)}$ the element bx is defined correctly for $x \in \overline{V}^{(B)}$ and $b \in B$ since by (4) we have $[\![x_1 = x_2]\!] = \mathbf{1} \rightarrow [\![bx_1 = bx_2]\!] = b \Rightarrow [\![x_1 = x_2]\!] = \mathbf{1}$. For a similar reason, we often write $\mathbf{0} := \varnothing$, and in particular $\mathbf{0}\varnothing = \varnothing = \mathbf{0}x$ for $x \in V^{(B)}$.

4.2.2. Mixing Principle. *Let $(b_\xi)_{\xi \in \Xi}$ be a partition of unity in B, i.e. $\xi \neq \eta \rightarrow b_\xi \wedge b_\eta = 0$ and $\sup_{\xi \in \Xi} b_\xi = \sup B = \mathbf{1}$. For every family $(x_\xi)_{\xi \in \Xi}$ in the universe $V^{(B)}$ there exists a unique element x in the separated universe such that*

$$[\![x = x_\xi]\!] \geq b_\xi \quad (\xi \in \Xi).$$

This element is called the *mixing* of $(x_\xi)_{\xi \in \Xi}$ with probabilities $(b_\xi)_{\xi \in \Xi}$ and is denoted by $\sum_{\xi \in \Xi} b_\xi x_\xi$.

4.2.3. Maximum Principle. *For every formula* φ *of* ZFC *there exists a* B*-valued set* x_0 *such that*

$$[\![(\exists x)\varphi(x)]\!] = [\![\varphi(x_0)]\!].$$

Recall that the *von Neumann universe* V is defined by the recursion schema

$$V_\alpha := \{x : (\exists \beta \in \alpha)\ x \in \mathcal{P}(V_\beta)\};$$
$$V := \bigcup_{\alpha \in \mathrm{On}} V_\alpha.$$

In other words V is the class of all sets. For each member x of V, i.e. for every *set* x, we define $x^\wedge \in V^{(B)}$ by the recursion schema

$$\varnothing^\wedge := \varnothing; \quad \mathrm{dom}(x^\wedge) := \{y^\wedge : y \in x\}, \quad \mathrm{im}(x^\wedge) := \{\mathbf{1}\}.$$

(More precisely, we define a distinguished representative of the equivalence class $x^\wedge$.) The element $x^\wedge$ is called the *standard name* of x. Thus, we obtain the *canonical embedding* of V into $V^{(B)}$. In addition, for $x \in V$ and $y \in V^{(B)}$ we have

$$[\![y \in x^\wedge]\!] = \bigvee_{z \in x} [\![y = z^\wedge]\!].$$

A formula is said to be *restricted* or *bounded* if every quantifier in it has the form $\forall x \in y$ or $\exists x \in y$; i.e., if all its quantifiers range over specific sets.

4.2.4. Restricted Transfer Principle. *For all* $x, x_1, \dots, x_n \in V$ *the following equivalence*

$$\varphi(x_1, \dots, x_n) \leftrightarrow [\![\varphi(x_1^\wedge, \dots, x_n^\wedge)]\!] = \mathbf{1}$$

holds for each restricted formula φ *of* ZFC.

4.3. Enter Descents and Ascents.

Now we list the main facts about representations of simplest objects (sets, correspondences, etc.) in Boolean-valued models.

Let φ be a formula of ZFC and let y be a fixed collection of elements of a Boolean-valued universe. Further, let $A_\varphi := A_{\varphi(\cdot,\ y)} := \{x : \varphi(x,\ y)\}$ be the class of sets definable by y. The *descent* $A_\varphi{\downarrow}$ of a class A_φ is defined by the relation

$$A_\varphi{\downarrow} := \{t : t \in V^{(B)}\ \&\ [\![\varphi(t,\ y)]\!] = \mathbf{1}\}.$$

If $t \in A_\varphi{\downarrow}$, then it is said that t *satisfies* $\varphi(\cdot,\ y)$ *inside* $V^{(B)}$.

The descent of each class is *strongly cyclic*, i.e. it contains all mixings of its elements. Moreover, two classes inside $V^{(B)}$ coincide if and only if they consist of the same elements inside $V^{(B)}$.

The *descent* $x{\downarrow}$ of an element $x \in V^{(B)}$ is defined by the rule

$$x{\downarrow} := \{t : t \in V^{(B)}\ \&\ [\![t \in x]\!] = \mathbf{1}\},$$

i.e. $x{\downarrow} = A_{\cdot \in x}{\downarrow}$. The class $x{\downarrow}$ is a set. Moreover, $x{\downarrow} \subset \mathrm{scyc}(\mathrm{dom}(x))$, where scyc is the symbol of the taking of the *strong cyclic hull*. It is noteworthy that for a nonempty set x inside $V^{(B)}$ we have

$$(\exists z \in x{\downarrow}) [\![(\exists z \in x)\ \varphi(z)]\!] = [\![\varphi(z)]\!].$$

4.3.1. *Let F be a correspondence from X into Y inside $V^{(B)}$. There exists a unique correspondence $F\downarrow$ from $X\downarrow$ into $Y\downarrow$ such that for each (nonempty) subset A of the set X inside $V^{(B)}$ we have*

$$F\downarrow(A\downarrow) = F(A)\downarrow .$$

It is easily seen that $F\downarrow$ is defined by the rule

$$(x,\ y) \in F\downarrow \leftrightarrow [\![(x,\ y) \in F]\!] = \mathbf{1}.$$

Let $x \in \mathcal{P}(V^{(B)})$, i.e. x is a set composed of B-valued sets. Put $\varnothing\uparrow := \varnothing$ and

$$\operatorname{dom}(x\uparrow) := x, \quad \operatorname{im}(x\uparrow) := \{\mathbf{1}\}$$

for $x \neq \varnothing$. The element $x\uparrow$ (of the separated universe V^B, i.e. the distinguished representative of the corresponding class) is called the *ascent* of x.

Clearly, $x\uparrow\downarrow = \operatorname{scyc}(x)$ with x a subset of $V^{(B)}$, whereas $x\downarrow\uparrow = x$ with x a nonempty set inside $V^{(B)}$. Observe also that if $x \in V$ and $\widehat{x}$ is its *standard domain*, i.e.

$$\widehat{x} := \{z^\wedge : z \in V \ \&\ z \in x\},$$

then $\widehat{x}\uparrow = \widehat{x}^\wedge$.

4.3.2. *Let $X, Y \in \mathcal{P}(V^{(B)})$ and let F be a correspondence from X into Y with $\operatorname{dom} F = X$. There exists a unique correspondence $F\uparrow$ from $X\uparrow$ into $Y\uparrow$ inside $V^{(B)}$ such that the equality*

$$F\uparrow(A\uparrow) = F(A)\uparrow$$

holds for every subset A of X if and only if F is extensional. The last means that

$$y_1 \in F(x_1) \to [\![x_1 = x_2]\!] \leq \bigvee_{y_2 \in F(x_2)} [\![y_1 = y_2]\!].$$

Let x be a set and $f : x \to Y\downarrow$, where $Y \in V^{(B)}$. Taking note of the equality $Y = Y\downarrow\uparrow$, we can consider f as a mapping from $\widehat{x}$ into $\operatorname{dom}(Y)$. Obviously f is extensional; therefore, it is reasonable to speak about the element $f\uparrow$ in $V^{(B)}$. Observe that, according to the above, $[\![f\uparrow : x^\wedge \to Y]\!] = \mathbf{1}$ and in addition, for every $g \in V^{(B)}$ such that $[\![g : x^\wedge \to Y]\!] = \mathbf{1}$ there is a unique mapping $f : x \to Y\downarrow$ for which $g = f\uparrow$. Obviously, the descent $g\downarrow$ of the mapping g (translated from $\widehat{x}$ onto x) is a mapping of such kind.

4.4. Exeunt *K*-Spaces.

Now we consider several facts connected with translation of the notions arising in representing a canonical sublinear operator in an appropriate Boolean-valued model. We start with revealing the reals in a Boolean-valued universe.

According to the maximum principle there is an object $\mathcal{R}$ inside $V^{(B)}$ for which the statement

$$[\![\mathcal{R} \text{ is the reals}]\!] = \mathbf{1}$$

is true.

Here we mean that $\mathcal{R}$ is the carrier set of the field of the reals inside $V^{(B)}$. Note also that $\mathbb{R}^\wedge$ (= the standard name of the field $\mathbb{R}$ of reals), being an Archimedean ordered field inside $V^{(B)}$, is a dense subfield in $\mathcal{R}$ inside $V^{(B)}$ (up to isomorphism).

Implement the descent of structures from $\mathcal{R}$ to $\mathcal{R}\downarrow$ according to the general rules:

$$\begin{aligned} x + y = z &\leftrightarrow [\![x + y = z]\!] = \mathbf{1}; \\ xy = z &\leftrightarrow [\![xy = z]\!] = \mathbf{1}; \\ x \leq y &\leftrightarrow [\![x \leq y]\!] = \mathbf{1}; \\ \lambda x = y &\leftrightarrow [\![\lambda^\wedge x = y]\!] = \mathbf{1} \quad (x, y, z \in \mathcal{R}\downarrow,\ \lambda \in \mathbb{R}). \end{aligned}$$

4.4.1. Gordon Theorem. *The set $\mathcal{R}{\downarrow}$ with descended structures is a universally complete K-space with base $\mathcal{B}(\mathcal{R}{\downarrow})$ (= the Boolean algebra of band projections in $\mathcal{R}{\downarrow}$) isomorphic to B. Such isomorphism is implemented by identifying B with the descent of the field $\{0^\wedge, 1^\wedge\}$, i.e. with the mapping $\iota : B \to \mathcal{B}(\mathcal{R}{\downarrow})$ acting by the rule*

$$[\![\iota(b) = 1^\wedge]\!] = b, \quad [\![\iota(b) = 0^\wedge]\!] = b^\perp \quad (0, 1 \in \mathbb{R}).$$

Moreover, for every $x, y \in \mathcal{R}$ we have

$$[\![\iota(b)x = \iota(b)y]\!] = b \Rightarrow [\![x = y]\!];$$
$$b\iota(b)x = bx, \quad b^\perp\iota(b)x = \mathbf{0}.$$

In particular, the following equivalences are valid:

$$\iota(b)x = \iota(b)y \leftrightarrow [\![x = y]\!] \geq b;$$
$$\iota(b)x \geq \iota(b)y \leftrightarrow [\![x \geq y]\!] \geq b.$$

Recall that a K-space E is *universally complete* provided that every subset of E composed of pairwise disjoint elements is bounded above.

4.5. Exeunt Operators.

Our next aim is to demonstrate how to get rid of convex operators and their subdifferentials, reducing everything to functionals. It suffice to settle the case of canonical operators.

Let $\mathfrak{A}$ be a nonempty set. By the maximum principle, there is an object $l_\infty(\mathfrak{A}^\wedge,\ \mathcal{R})$ in $V^{(B)}$ such that $[\![l_\infty(\mathfrak{A}^\wedge,\ \mathcal{R})$ is the K-space of bounded functions defined on $\mathfrak{A}^\wedge$ and taking values in $\mathcal{R}]\!] = \mathbf{1}$.

Consider the descent

$$l_\infty(\mathfrak{A}^\wedge,\ \mathcal{R}){\downarrow} := \{t \in V^{(B)} : [\![t \in l_\infty(\mathfrak{A}^\wedge,\ \mathcal{R})]\!] = \mathbf{1}\}.$$

Translate algebraic operations and order relations from $l_\infty(\mathfrak{A}^\wedge,\ \mathcal{R})$ to $(\mathfrak{A}^\wedge,\ \mathcal{R}){\downarrow}$ by descent. Obviously, $l_\infty(\mathfrak{A}^\wedge,\ \mathcal{R}){\downarrow}$ is a K-space and, moreover, a module over $\mathcal{R}{\downarrow}$.

4.5.1. *The mapping "ascent" assigning to each bounded $\mathcal{R}{\downarrow}$-valued function on $\mathfrak{A}$ its ascent, i.e. a bounded $\mathcal{R}$-valued function on $\mathfrak{A}^\wedge$ inside $V^{(B)}$, implements an algebraic and order isomorphism between $l_\infty(\mathfrak{A},\ \mathcal{R}{\downarrow})$ and $l_\infty(\mathfrak{A}^\wedge,\ \mathcal{R}){\downarrow}$.*

Consider the object $l_\infty(\mathfrak{A}^\wedge,\ \mathcal{R})^\#$ in $V^{(B)}$ for which the statement

$$[\![l_\infty(\mathfrak{A}^\wedge,\ \mathcal{R})^\# \text{ is the dual space of } l_\infty(\mathfrak{A}^\wedge,\ \mathcal{R})]\!] = \mathbf{1}$$

is true. The descent $l_\infty(\mathfrak{A}^\wedge,\ \mathcal{R})^\#{\downarrow}$ is endowed with the descended structures. In particular, there is no doubt that $l_\infty(\mathfrak{A}^\wedge,\ \mathcal{R})^\#{\downarrow}$ is an $\mathcal{R}{\downarrow}$-module.

Let $\mu \in l_\infty(\mathfrak{A}^\wedge,\ \mathcal{R})^\#{\downarrow}$, i.e.

$$[\![\mu \text{ is an } \mathcal{R}\text{-homomorphism from } l_\infty(\mathfrak{A}^\wedge,\ \mathcal{R}) \text{ to } \mathcal{R}]\!] = \mathbf{1}.$$

Further, let $\mu{\downarrow}: l_\infty(\mathfrak{A}^\wedge,\ \mathcal{R}){\downarrow} \to \mathcal{R}{\downarrow}$ be the descent of μ. Given $f \in l_\infty(\mathfrak{A},\ \mathcal{R}{\downarrow})$, put

$$\mu_{\downarrow}(f) := \mu{\downarrow}\,(f{\uparrow}).$$

4.5.2. *The mapping "descent" $\mu \mapsto \mu_{\downarrow}$ implements an isomorphism between the $\mathcal{R}{\downarrow}$-modules $l_\infty(\mathfrak{A}^\wedge,\ \mathcal{R})^{\#}{\downarrow}$ and $\mathrm{Hom}_{\mathcal{R}\downarrow}(l_\infty(\mathfrak{A},\ \mathcal{R}{\downarrow}),\ \mathcal{R}{\downarrow})$, the space of $\mathcal{R}{\downarrow}$-homomorphisms acting from $l_\infty(\mathfrak{A},\ \mathcal{R}{\downarrow})$ to $\mathcal{R}{\downarrow}$.*

We denote the mapping inverse to the mapping "descent" $\mu \mapsto \mu_{\downarrow}$ by $t \mapsto t^{\uparrow}$, with $t \in \mathrm{Hom}_{\mathcal{R}\downarrow}(l_\infty(\mathfrak{A},\ \mathcal{R}{\downarrow}),\ \mathcal{R}{\downarrow})$. Of course, it means that

$$t^{\uparrow}(f) := t(f{\downarrow}) \quad (f \in l_\infty(\mathfrak{A}^\wedge,\ \mathcal{R}){\downarrow}).$$

Let $\varepsilon_{\mathfrak{A}^\wedge}$ be some canonical sublinear operator inside $V^{(B)}$, i.e. an object in $V^{(B)}$ for which

$$[\![\varepsilon_{\mathfrak{A}^\wedge} : l_\infty(\mathfrak{A}^\wedge,\ \mathcal{R}) \to \mathcal{R}]\!] = \mathbf{1},$$
$$[\![(\forall f \in l_\infty(\mathfrak{A}^\wedge,\ \mathcal{R}))\ \varepsilon_{\mathfrak{A}^\wedge}(f) = \sup f(\mathfrak{A}^\wedge)]\!] = \mathbf{1}.$$

Straightforward calculation shows that for every element f in $l_\infty(\mathfrak{A},\ \mathcal{R}{\uparrow})$ the relation

$$[\![\varepsilon_{\mathfrak{A}^\wedge}(f{\uparrow}) = \varepsilon_{\mathfrak{A}}(f)]\!] = \mathbf{1}$$

holds.

4.5.3. *Let $\partial\varepsilon_{\mathfrak{A}^\wedge}$ be the subdifferential of $\varepsilon_{\mathfrak{A}^\wedge}$ inside $V^{(B)}$ and let $\mathrm{ext}(\varepsilon_{\mathfrak{A}^\wedge})$ be the set of extreme points of $\partial\varepsilon_{\mathfrak{A}^\wedge}$ inside $V^{(B)}$. Then for every element t in $\mathrm{Hom}_{\mathcal{R}\downarrow}(l_\infty(\mathfrak{A},\ \mathcal{R}{\downarrow}),\ \mathcal{R}{\downarrow})$ and every μ in $l_\infty(\mathfrak{A}^\wedge,\ \mathcal{R})^{\#}{\downarrow}$ the following equivalences hold:*

$$t^{\uparrow} \in (\partial\varepsilon_{\mathfrak{A}^\wedge}){\downarrow} \leftrightarrow t \in \partial\varepsilon_{\mathfrak{A}};$$
$$t^{\uparrow} \in \mathrm{ext}(\varepsilon_{\mathfrak{A}^\wedge}){\downarrow} \leftrightarrow t \in \mathrm{ext}(\varepsilon_{\mathfrak{A}});$$
$$\mu_{\downarrow} \in \partial\varepsilon_{\mathfrak{A}} \leftrightarrow \mu \in (\partial\varepsilon_{\mathfrak{A}^\wedge}){\downarrow};$$
$$\mu_{\downarrow} \in \mathrm{ext}(\varepsilon_{\mathfrak{A}}) \leftrightarrow \mu \in \mathrm{ext}(\varepsilon_{\mathfrak{A}^\wedge}){\downarrow}\ .$$

Clarify the terminology. Let $B := \mathcal{B}(E)$ be the base of a K-space E. Take a partition of unity in B. If $(T_\xi)_{\xi\in\Xi}$ is a family of operators in $L(X,\ E)$ and the operator $T \in L(X,\ E)$ is such that $Tx = \sum_{\xi\in\Xi} b_\xi T_\xi x$ for all $x \in X$, then T is called the *mixing of* $(T_\xi)_{\xi\in\Xi}$ *with probabilities* $(b_\xi)_{\xi\in\Xi}$. By 4.3.2 and Gordon's Theorem it is easy to see that in fact such use of the word "mixing" is sound.

Obviously, for a given A in $\mathfrak{A}$, the δ-*function* $\varepsilon_A : f \mapsto f(A)$ $(f \in l_\infty(\mathfrak{A},\ E))$ belongs to $\mathrm{ext}(\varepsilon_{\mathfrak{A}})$. A mixing of the family $(\varepsilon_A)_{A\in\mathfrak{A}}$ is called a *pure state* on $\mathfrak{A}$. It is easily seen that every pure state is an o-extreme point of the canonical sublinear operator in question.

4.5.4. *The mapping "descent" implements a bijection between the set of pure states on $\mathfrak{A}$ and the subset of $V^{(B)}$ composed of δ-functions on the standard name $\mathfrak{A}^\wedge$ inside $V^{(B)}$. In other words, $t \in \mathrm{Hom}_{\mathcal{R}\downarrow}(l_\infty(\mathfrak{A},\ \mathcal{R}{\downarrow}),\ \mathcal{R}{\downarrow})$ is a pure state on $\mathfrak{A}$ if and only if*

$$[\![(\exists A \in \mathfrak{A}^\wedge)\ t^{\uparrow} = \varepsilon_A]\!] = \mathbf{1}.$$

Now it is easy to describe the structure of extreme points and the elements of the subdifferential of a canonical operator. To this end, it suffices to interpret externally in an appropriate Boolean-valued model the Kreĭn-Milman Theorem and Milman Theorem in a scalar setting.

4.5.5. *The subdifferential of a canonical sublinear operator coincides with the pointwise relatively uniform closure of the strong operator-convex hull of the set of δ-functions.*

Now by applying the Milman Theorem we may assert for instance that

4.5.6. Theorem. *Every extreme point of each subdifferential $\partial(p)$ serves as the pointwise relatively uniform limit of a net in the strong cyclic hull of the set of o-extreme points of $\partial(p)$.*

5. Epilog

In the history of functional analysis, the rise of the theory of ordered vector spaces is commonly attributed to the efforts of G. Birkhoff, H. Freudenthal, L. V. Kantorovich, M. G. Kreĭn, H. Nakano, F. Riesz, et al. At present, the theory of ordered vector spaces and its applications constitute a vast field of mathematics representing, in fact, one of the main sections of contemporary functional analysis. The theory is well exposed in many monographs (see [2, 3, 25, 26, 33, 36, 42, 43]). The credit for finding the most important instance of ordered vector spaces, an order complete vector lattice or a K-space, is due to L. V. Kantorovich. The notion appeared in Kantorovich's first fundamental article on this topic [23], where he wrote: "In this note, I define a new type of space which I call a semiordered linear space. The introduction of such a space allows us to study linear operations of one abstract class (those with values in the space) as linear functionals."

Here L. V. Kantorovich stated an important methodological principle, the *heuristic transfer principle* for K-spaces. An exemplar application of this principle is Theorem 3 of [23] now referred to as the Hahn-Banach-Kantorovich Theorem. It claims that the *Kantorovich principle* is valid in relation to the classical Dominated Extension Theorem; i.e., we can replace the reals in the standard Hahn-Banach Theorem by elements of an arbitrary K-space and a linear functional by a linear operator with values in this K-space.

Consult [6, 29, 32, 37, 38, 41] about the generalities of Boolean-valued analysis. The Boolean-valued status of the concept of K-space is established in Gordon's Theorem obtained in [18]. This fact can be regarded as follows: a universally complete K-space is the interpretation of the reals in an appropriate Boolean-valued model. Moreover, it turns out that every theorem on reals (in the framework of ZFC) has an analog for the corresponding K-space. Theorems are transferred by means of precisely-defined procedures: ascent, descent, and canonical embedding, that is, algorithmically as a matter of fact. Thereby Kantorovich's assertion that "the elements of a K-space are generalized numbers" acquires a rigorous mathematical status. Consult [22, 28, 29] about the Boolean-valued approach to K-spaces.

Convex sets became the object of independent study at the turn of the twentieth century. At that time various interconnections were discovered between convex functions and convex sets. The latter were well known long before the notion of convex function was introduced. Preliminary study of these objects was connected mainly with finite-dimensional geometry. In the 1930s the interest in convexity was tied with the development of functional analysis. The formation of modern convex analysis began in the sixties under the strong influence of the theory of extremal problems, the elaboration of optimization methods and research into mathematical economics. The term "convex analysis" gains popularity after R. T. Rockafellar's book [35] in which Professor A. W. Tucker at Princeton University is indicated as the person who coined the term. The discipline itself is framed mainly by the contributions of W. Fenchel, J. J. Moreau and R. T. Rockafellar. About this consult [4, 15, 17, 19, 21, 30, 34, 39].

The Kreĭn-Milman Theorem was established in 1940 and since then has been developed in different directions as one of the most important general principles of geometric functional analysis.

The problem of dominated extension of linear operators originates with the Hahn-Banach Theorem (see [13] for its history). Theorem 1.3.3 as stated was discovered by L. V. Kantorovich in 1935 and was perceived as a generalization serving a bizarre purpose. Now it became a truism that convex analysis and the theory of ordered vector spaces are boon companions. The equivalence between the extension and least upper bound properties was first established in W. Bonnice and R. Silvermann [9] and T.-O. To [40]; a decisive simplification of its proof is due to A. D. Ioffe [20]. A nice but still incomplete survey of the Hahn-Banach Theorem is the article [13] by G. Buskes.

The origin of the Hahn-Banach-Kantorovich Theorem in subdifferential form is reflected in [2, 30, 31]. Consult [1, 10, 11, 16, 31] about subdifferential calculus and related vector optimization problems. The articles [7, 8, 12] relate to module versions discussed in 3.1.

A more systematic exposition of the topics treated in the present article is given in [29, 30] which contain extensive lists of references and detailed comments.

References

1. Achilles A., Elster K.-H., and Nehse R., *Bibliographie zur Vectoroptimierung*, Math. Operationsforsch. und Statist. Ser. Optimization **10** (1972), no. 2, 227–321.
2. Akilov G. P. and Kutateladze S. S., *Ordered Vector Spaces*, Nauka, Novosibirsk, 1978.
3. Aliprantis C. D. and Burkinshaw O., *Positive Operators*, Academic Press, New York, 1985.
4. Asimow L. and Ellis A. S., *Convexity Theory and Its Applications in Functional Analysis*, Academic Press, London, 1980.
5. Aubin J.-P. and Vinter R. B. (eds.), *Convex Analysis and Applications*, Pitman, London, 1980.
6. Bell T. L., *Boolean Valued Models and Independence Proofs in Set Theory*, Clarendon Press, Oxford, 1979.
7. Benko I. and Scheiber E., *Monotonic linear extensions for ordered modules, their extremality and uniqueness*, Mathematica **25** (1989), no. 2, 119–126.
8. Bigard A., *Modules ordonnes injectifs*, Mathematica **15** (1973), no. 1, 15–24.
9. Bonnice W. and Silvermann R., *The Hahn-Banach extension and the least upper bound properties are equivalent*, Proc. Amer. Math. Soc. **18** (1967), no. 5, 843–849.
10. Borwein J. M., *Subgradients of convex operators*, Math. Operationsforsch. Statist. Ser. Optimization **15** (1984), 179–191.
11. Borwein J. M., Penot J. P., and Thera M., *Conjugate convex operators*, J. Math. Anal. Appl. **102** (1989), 399–414.
12. Breckner W. W. and Scheiber E., *A Hahn-Banach type extension theorem for linear mappings into ordered modules*, Mathematica **19** (1977), no. 1, 13–27.
13. Buskes G., *The Hahn-Banach Theorem Surveyed*, Diss. Math. 132, Warszawa, 1993.
14. Cohen P. J., *Set Theory and the Continuum Hypothesis*, W. A. Benjamin Inc., New York, 1966.
15. Ekeland I. and Temam R., *Convex Analysis and Variational Problems*, North-Holland, Amsterdam, 1976.
16. Elster K.-H. and Nehse R., *Konjugierte operatoren und Subdifferentiale*, Math. Operationsforsch. Statist. Ser. Optimization **6** (1975), 641–657.
17. Fenchel W., *Convex Cones, Sets and Functions*, Princeton Univ. Press, Princeton, 1953.
18. Gordon E. I., *The reals in Boolean-valued models of set theory and K-spaces*, Dokl. Akad. Nauk SSSR **237** (1977), no. 4, 773–775.
19. Holmes R. B., *Geometric Functional Analysis and Its Applications*, Springer-Verlag, Berlin etc., 1975.
20. Ioffe A. D., *A new proof of the equivalence of the Hahn-Banach extension and the least upper bound properties*, Proc. Amer. Math. Soc. **82** (1981), no. 3, 385–389.
21. Ioffe A. D. and Tikhomirov V. M., *Theory of Extremal Problems*, North-Holland, Amsterdam, 1979.
22. Jech T. J., *Boolean-Linear Spaces*, Pennsylvania Univ., Pennsylvania, 1987.
23. Kantorovich L. V., *On semiordered linear spaces and their applications to the theory of linear operations*, Dokl. Akad. Nauk SSSR **4** (1935), no. 1–2, 11–14.

24. Kantorovich L. V., *Functional analysis: Main ideas*, Sibirsk. Mat. Zh. **28** (1987), 7–16.
25. Kantorovich L. V. and Akilov G. P., *Functional Analysis*, Pergamon Press, Oxford and New York, 1982.
26. Kantorovich L. V., Vulikh B. Z., and Pinsker A. G., *Functional Analysis in Semiordered Spaces*, Gostekhizdat, Moscow-Leningrad, 1950.
27. Kusraev A. G. and Kutateladze S. S., *Nonstandard methods in geometric functional analysis*, Amer. Math. Soc. Transl. **151** (1992), no. 2, 91–105.
28. Kusraev A. G. and Kutateladze S. S., *Nonstandard methods for Kantorovich spaces*, Siberian Adv. Math. **2** (1992), no. 2, 114–152.
29. Kusraev A. G. and Kutateladze S. S., *Nonstandard Methods of Analysis*, Kluwer, Dordrecht, 1994.
30. Kusraev A. G. and Kutateladze S. S., *Subdifferentials*: *Theory and Applications*, Kluwer, Dordrecht, 1995.
31. Kutateladze S. S., *Convex operators*, Uspekhi Mat. Nauk **34** (1979), no. 1, 167–196.
32. Kutateladze S. S., *Descents and ascents*, Dokl. Akad. Nauk SSSR **272** (1983), no. 3, 521–524.
33. Luxemburg W. A. J. and Zaanen A. C., *Riesz Spaces. Vol.* 1, North-Holland, Amsterdam-London, 1971.
34. Phelps R., *Convex Functions, Monotone Operators and Differentiability*, Springer-Verlag, Berlin etc., 1989.
35. Rockafellar R. T., *Convex Analysis*, Princeton Univ. Press, Princeton, 1970.
36. Schaefer H. H., *Banach Lattices and Positive Operators*, Springer-Verlag, Berlin etc., 1974.
37. Takeuti G., *Two Applications of Logic to Mathematics*, Iwanami and Princeton Univ. Press, Tokyo-Princeton, 1978.
38. Takeuti G. and Zaring W. M., *Axiomatic Set Theory*, Springer-Verlag, New York etc., 1973.
39. Tikhomirov V. M., *Convex analysis*, Analysis 2. Contemporary Problems of Mathematics. Fundamental Trends. Vol. 14, VINITI, Moscow, 1987, pp. 5–102.
40. To T.-O., *The equivalence of the least upper bound property and the Hahn-Banach extension property in ordered vector spaces*, Proc. Amer. Math. Soc. **30** (1971), no. 2, 287–295.
40. Vopěnka P. and Hajek P., *The Theory of Semisets*, North-Holland, Amsterdam, 1972.
41. Vulikh B. Z., *Introduction to the Theory of Partially Ordered Spaces*, Noordhoof, Groningen, 1967.
42. Zaanen A. C., *Riesz Spaces. Vol.* 2, North-Holland, Amsterdam etc., 1983.

Sobolev Institute of Mathematics
Universitetskiĭ pr. 4
Novosibirsk, 630090
RUSSIA
E-mail address: smj@math.nsk.su