

Different Motivations and Goals in the Historical Development of the Theory of Systems of Linear Inequalities

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1. Introduction

The theory of systems of linear inequalities has a very fragmented history. It seems to have been developed in bits and pieces. Singular as well as more comprehensive independent contributions pop up at different times, in different places, and for totally different reasons.

In the big picture of the history of mathematics the history of linear inequalities matters because it shows how diverse and complex the development of one small piece of mathematics can be. The theory of linear inequalities has been developed through interactions between different branches of mathematics as well as between mathematics and other scientific disciplines on the one hand and through different kinds of motivations and goals on the other hand. Its development has not only been shaped by the different scientific contexts in which it took place but also by social contexts. The history of linear inequalities demonstrates the significance of the context and it is well suited to give some insight into the practice of mathematical research.¹ It shows the historical development of a piece of modern mathematics as a human activity that takes place in a scientific as well as a public space.

In the sort of messy picture that maps the various contributions to a theory of linear inequalities, it is possible to distinguish three different contexts in which a theory of linear inequalities was developed: analytical mechanics, convexity, and the context of American mathematics in the twenties. The history of the theory of linear inequalities in these three contexts is discussed in Sects. 2, 3, and 4. It is shown how the different motivations and goals, as well as the contexts in which these works took place, influenced the kind of questions investigated, the methods they used, and thereby also the results obtained.

There is no *single* coherent line of development and interestingly enough, in only one of the three contexts mentioned above, was the main object of interest at the outset, the development of a theory of linear inequalities. Unlike the developments that took place in the other two contexts this third development stimulated further investigations and developments which eventually led to the final establishment of linear inequalities within

¹ For a meta-historical discussion of the practice of mathematics, see [Epple, 2000].

the theory of convexity. In Sect. 5 this further development is discussed together with the history of the very important “transposition” theorem in the sense of Theodore Motzkin.

During World War II it was realized that linear inequalities were the mathematical foundation of game theory. This in turn led to new results in the theory of linear inequalities – again developed independently of earlier developments and for a different reason. When, in the late forties, the connection between the newly developed linear programming and game theory was recognized this also, through funding from the USA’s military establishments, sparked new interest and new developments in the theory of linear inequalities. The history of the theory of linear inequalities in this fourth and last context and the significance of the scientific community in the USA in the post war period is discussed in Sect. 6, as well as the history of the important “duality” theorem in linear programming.²

2. A theory of linear inequalities in the context of analytical mechanics: The work of Julius Farkas

A theory of systems of linear inequalities was first developed at the end of the 19th century within the context of analytical mechanics by the Hungarian professor Julius Farkas (1847–1930).³ He proved the important theorem to day referred to as Farkas’s lemma and developed a solution method for a finite system of linear inequalities.

Before Farkas only Fourier seems to have had the idea of constructing a mathematical theory for systems of linear inequalities, but Fourier did not get very far. In his book “Analyse des Équations Déterminées” [Fourier, 1831], printed posthumously, Fourier referred to the work he intended to do on systems of linear inequalities, and which he had planned to present in the seventh book of the entire work:

The principle of the theory of inequalities will be expounded in the seventh and last of the books. This part of our work is concerned with a new kind of question, which offers varied applications to geometry, to algebraic analysis, to mechanics, and to the theory of probability. [Fourier, 1831 (1902, p. 71)]⁴

From this quote it seems that Fourier was motivated by a variety of applications belonging to the realm of mathematics proper as well as to the realm of applied mathematics. Fourier did not complete his work on inequalities before he died but he did publish a short paper “Solution d’une question particulière du calcul des inégalités” [Fourier, 1826] and he inserted two summaries on the subject in “l’Histoire de l’Académie Royale des Sciences”, one for the year 1823 and the other one for 1824 [Fourier, 1823, 1824].⁵ From these publications it is clear that Fourier had a geometrical understanding of the

² I would like to thank Jeremy Gray for correcting some of my worst grammatical errors.

³ Julius is the German translation of his Hungarian name Gyula. See also [Prékopa, 1980].

⁴ See also [Grattan-Guinness, 1970, p. 361] from which most of the translation is taken.

⁵ According to Grattan-Guinness, besides the published notes, which reflect how far Fourier got regarding the theoretical aspects, Fourier left “several hundred folios on problems connected with “analysis of inequalities”” [Grattan-Guinness, 1970, p. 362], see note 7 in [Grattan-Guinness, 1970, p. 363] and [Grattan-Guinness, 1972, p. 485, note 64] for precise references.

solution set to a system of linear inequalities in three variables as a polyhedron in three dimensional space [Fourier, 1824, (1890, p. 326)].⁶ While Fourier did not publish any substantial contribution to the development of a theory of linear inequalities he had, according to Darboux, been very much occupied by this question of inequalities; an occupation which provoked Darboux, writing sixty years after Fourier's death in the "Avertissement" of the *Oeuvres* of Fourier, to the following remark:

Nous avons aussi, par quelques emprunts à l'*Histoire de l'Académie* pour les années 1823 et 1824, pu faire connaître d'une manière assez précise certaines idées sur la théorie des inégalités auxquelles l'illustre géomètre attachait une importance qu'il est permis, aujourd'hui, de trouver un peu exagérée.⁷ [Darboux, 1890, p. v–vi]

Five years after this dismissive evaluation of the importance of a theory of linear inequalities by Darboux, Farkas began his work on the so-called Fourier Inequality Principle in analytical mechanics, which led Farkas to develop precisely such a theory.

In 1895 Farkas published the paper "Über die Anwendung des Mechanischen Princips von Fourier" [Farkas, 1895] in which the important theorem in linear inequality theory – referred to as Farkas's lemma – appeared for the first time. This paper was to become the first of a series of papers by Farkas where the focus gradually shifted from analytical mechanics towards the mathematical theory of linear inequalities, culminating with his well known 1901 paper "Theorie der einfachen Ungleichungen" where Farkas had almost completely detached the subject of linear inequalities from the original context of analytical mechanics. In this section it will be demonstrated how Farkas's main theorem arose from analytical mechanics and how this context shaped the content and the outlook of his theory of linear inequalities.

2.1. The motivating factor behind Farkas's theory of linear inequalities

In both the title of his paper and its lengthy introduction Farkas placed his first work containing his main result in systems of homogenous linear inequalities explicitly in the framework of Fourier's Inequality Principle in analytical mechanics [Farkas, 1895]. To understand how this context motivated, influenced and guided Farkas' work in linear inequalities I will briefly explain the Inequality Principle of Fourier as it was known to Farkas.

Fourier introduced his inequality principle in the paper "Mémoire sur la statique contenant la démonstration du principe des vitesses virtuelles et la théorie des moments" published in 1798, as an extension of Lagrange's work on the principle of virtual work

⁶ In [Grattan-Guinness, 1970, 1994] I. Grattan-Guinness interprets Fourier's work on linear inequalities in the framework of linear programming: "one of his [Fourier's] achievements was to create singlehanded a basic theory of linear programming" [Grattan-Guinness, 1970, p. 361]. Unfortunately Grattan-Guinness only presents the published work of Fourier as evidence for this conclusion, which to me seems to be an overestimation of Fourier's published work on linear inequalities.

⁷ "We have also included some papers from l'*Histoire de l'Académie* of 1823 and 1824 to show in a rather precise way the ideas on the theory of inequalities to which the illustrious geometer attached an importance that it is permitted, nowadays, to find a little exaggerated."

[Fourier, 1798]. The latter principle is one of the fundamental principles in analytical mechanics and it states that a given mechanical system is in equilibrium if and only if the virtual work of the applied forces equals zero. Lagrange treated it as an axiom in his work “Mécanique analytique” from 1788 [Lagrange, 1788]. This principle is formulated for reversible virtual displacements. A virtual displacement of a point of a system is not necessarily a “natural” displacement of the system but a possible displacement in the sense that a displacement is possible as long as it does not contradict the constraints of the system. In Lagrange’s “Mécanique analytique” the principle is formulated for systems where the constraints upon the system is of such a character that if a virtual displacement δr of a point P of the system is possible then so is the reverse displacement $-\delta r$.

Lagrange also formulated the so-called Lagrange Multiplier Method, which is a method for examining problems of stable equilibrium of mechanical systems. He considered a system of points with coordinates (x, y, z) , (x', y', z') , \dots constrained by a finite number of conditions represented by equations $L = 0$, $M = 0$, $N = 0$, \dots where $L, M, N \dots$ are functions of the variables $x, y, z, x', y', z', \dots$ [Lagrange, 1788, (1965, p. 69)]. To illustrate Lagrange’s idea we can consider the equilibrium of a system consisting of just one point $p = (x, y, z)$ with one force $P = (X_p, Y_p, Z_p)$ acting on it, and with one constraint $L = 0$, which means that the particle is constrained to lie on the surface $L = 0$.⁸ The force exerted on the particle by the constraint is normal to the surface, that is

$$\frac{\partial L}{\partial x} \delta x + \frac{\partial L}{\partial y} \delta y + \frac{\partial L}{\partial z} \delta z = 0 \quad (1)$$

and a virtual displacement $\delta r = (\delta x, \delta y, \delta z)$ of this system is one which satisfy Eq. (1), which is a linear equation in the displacements $\delta x, \delta y, \delta z$. The equilibrium condition states that

$$X_p \delta x + Y_p \delta y + Z_p \delta z = 0$$

where $\delta x, \delta y, \delta z$ are virtual displacements fulfilling the constraints on the system. Since the system is constrained by the surface $L = 0$ the virtual displacements $\delta x, \delta y, \delta z$ are not independent. To overcome this difficulty Lagrange multiplied the Eq. (1) by a constant λ and added the result to the condition of equilibrium:

$$X_p \delta x + \lambda \frac{\partial L}{\partial x} \delta x + Y_p \delta y + \lambda \frac{\partial L}{\partial y} \delta y + Z_p \delta z + \lambda \frac{\partial L}{\partial z} \delta z = 0 \quad (2)$$

By introducing the multiplier Lagrange got a free system where the virtual displacements $\delta x, \delta y, \delta z$ are independent reaching the following equilibrium conditions

$$X_p + \lambda \frac{\partial L}{\partial x} = 0, \quad Y_p + \lambda \frac{\partial L}{\partial y} = 0, \quad Z_p + \lambda \frac{\partial L}{\partial z} = 0, \quad L = 0 \quad (3)$$

This illustrates Lagrange’s method of multipliers.

Lagrange did not give a mathematical proof for his multiplier rule but it has an intuitive physical interpretation. The magnitude $\lambda dL = \lambda (\frac{\partial L}{\partial x}, \frac{\partial L}{\partial y}, \frac{\partial L}{\partial z})$ has a physical interpretation as a force of constraint. From a physical point of view it is clear that in equilibrium the constraint acts normally to the surface $L = 0$ and cancels out the applied force P .

⁸ This rephrasing of Lagrange’s text is build on [Fraser, 1992].

Ten years later Fourier extended the underlying principle to irreversible displacements. Instead of the virtual work Fourier considered “le moment de la force” [Fourier, 1798, (1890, p. 479)], which results in a change of sign such that Fourier’s original formulation of his inequality principle for systems constrained by inequalities was that the moment of the forces has to be zero or positive [Fourier, 1798, (1890, p. 494)].⁹ Using the virtual work of the applied forces the principle of Fourier states that a system is in equilibrium if and only if the virtual work of the applied forces is non-positive, which leads to a homogenous linear inequality in the virtual displacements.

In 1838 Ostrogradsky derived the equations of equilibrium for systems where the displacements are not necessarily reversible. Using the Fourier Inequality Principle he wrote the equilibrium condition in the following way: The total work $Pdp + Qdq + Rdr + \dots$ has to be non-positive for every possible displacement. Here P, Q, R, \dots denote the applied forces acting on the system, the constraints were called L, M, \dots [Ostrogradsky, 1838].¹⁰ Since the displacements were not necessarily reversible Ostrogradsky did not have $dL = 0, dM = 0, \dots$, but could conclude that dL, dM, \dots only change sign when the displacements move from possible displacements to displacements that contradict the constraints on the system,¹¹ which means that a virtual displacement is one that satisfies a system of linear inequalities.

In order to derive the necessary conditions for equilibrium Ostrogradsky introduced what he called generalized coordinates. He replaced the displacements $dp, dq, dr \dots$ by some other variables $d\xi, d\eta, d\psi, \dots$ and he let dL, dM, \dots , which are functions of $dp, dq, dr \dots$, be the first of these generalized coordinates [Ostrogradsky, 1838, p. 131]. This procedure of course means that it can only be used for mechanical systems where the number of constraints does not exceed the number of variables. This limitation was, as will be seen below, noticed by Farkas. Ostrogradsky then derived the equilibrium condition that the total work could be written as a linear combination of dL, dM, \dots where the multipliers are restricted in sign [Ostrogradsky, 1838, p. 132], that is, there exist multipliers λ, μ, \dots such that

$$Pdp + Qdq + Rdr + \dots = \lambda dL + \mu dM + \dots \quad (4)$$

and the sign of the multipliers λ, μ, \dots is the opposite of the sign of the corresponding dM, dL, \dots

In his 1895 paper on the application of Fourier’s principle in mechanics Farkas clarified the difference between Fourier’s principle and the principle of the virtual work, and to underline the main point he referred to the first one as the inequality principle and to the second as the equality principle. About the equality principle Farkas wrote:

... ist die Anwendung des Gleichheits-Princips seit Lagrange bloss eine Sache der reinen Analysis und insbesondere ein Problem der Auflösung von Gleichungen, welche

⁹ See for example [Prékopa, 1980, p. 530] and [Franken, 1985a, p. 143].

¹⁰ This work of Ostrogradsky was presented to the Academy in St. Petersburg in 1834, but it was not published until 1838.

¹¹ See [Ostrogradsky, 1838, p. 131].

jedesmal nach bereits aufliegenden Verfahrensmethoden dargestellt werden können.¹²
[Farkas, 1895, p. 264]

And he continued:

Mit dem Ungleichheits-Princip ist es nicht so weit gekommen.¹³ [Farkas, 1895, p. 264]

The goal for Farkas was to transform the application of the Fourier Inequality Principle into the question of solutions of inequalities. As was discussed above, Ostrogradsky tried to apply this inequality principle in 1838 and Farkas who knew the relevant literature gave the following evaluation of Ostrogradsky's work:

... er [Ostrogradsky] hatte dessen Anwendungen in Angriff genommen, er hat aber das Princip nicht in jener ganzen Allgemeinheit aufgefasst, welche demselben beigelegt werden kann, da er nur eine gewisse Klasse der Zwangs-Verhältnisse vor Augen hielt, jene, in welcher die Anzahl der Zwangs-Ausdrücke diejenige der virtuellen Verrückungs-Componenten nicht übertrifft; auf diese Weise haben sich seine Betrachtungen auf einen verhältnissmässig sehr kleinen Gültigkeits-Bereich beschränkt. Wie es scheint hat sich sonst Niemand mit der Anwendung des Principis abgegeben, und es scheint sogar auch die Meinung vor zu walten, dass das Princip nicht nutzbar sei.¹⁴ [Farkas, 1895, p. 265]¹⁵

Farkas explicitly stated in the paper that its purpose was to prove that Lagrange's method of multipliers can be adapted to the inequality principle. In order to do so, he needed some tools about systems of linear homogenous inequalities as a mathematical foundation for the application of the inequality principle and he derived that foundation in a series of papers published between 1895 and 1901.¹⁶

2.2. *Farkas's theory of systems of homogenous linear inequalities: Farkas's lemma and the parametric method of solution*

Farkas considered the following system of inequalities

$$\begin{aligned} R_1 &= A_1u + B_1v + \dots \geq 0 \\ R_2 &= A_2u + B_2v + \dots \geq 0 \\ &\dots \end{aligned} \quad (5)$$

¹² "Since Lagrange the application of the Equality Principle is just a matter of pure analysis and in particular a problem of the solution of equations which can always be described using already established methods."

¹³ "It has not come that far with the Inequality Principle."

¹⁴ "Ostrogradsky did tackle this application but he did not consider the principle in its most general form. He only considered a certain class of constraints, namely those for which the number of constraints does not exceed the number of virtual displacements. In this way his considerations were limited to a relatively very small domain of validity. As it seems nobody else have worked on the application of the principle and it even seems to be the opinion that the principle is not useful."

¹⁵ For an opinion on the uselessness of the inequality principle see, for example, [Rausenberger, 1888, p. 146], where Rausenberger wrote that it was interesting but not applicable in practice.

¹⁶ See [Farkas, 1895, 1897, 1899, 1901], most of these papers were first published in Hungarian, for a complete list of reference see [Prékopa, 1980].

where u, v, \dots are the variables [Farkas, 1895, p. 266–269]. He did not specify the number of inequalities, because he wanted a theory that covered all the cases, that is the number of variables and the number of inequalities should be entirely free. It is clear though, from the content of his paper, that he considered both numbers to be finite.

Farkas then added the extra inequality

$$R_0 = A_0u + B_0v + \dots \geq 0. \quad (6)$$

In the 1895 paper he called this extra inequality *new* in relation to the system (5) if the set of solutions to (5) changes, when (6) is included in the system otherwise the extra inequality was said to be *not new* [Farkas, 1895, p. 267]. Later, Farkas called an inequality which was not new in relation to a system a *consequence* of that system [Farkas, 1901].

At first, it could seem strange to introduce an extra inequality instead of just dealing with the system itself but it becomes quite natural when considered in relation to applications. If we take the displacements to be the variables, the system (5) represents the constraints on the displacements of a mechanical system and the “extra” inequality (6) represents the inequality principle. The necessary condition for equilibrium is that the inequality (6) is not new, or is a consequence, of the system (5).

Having Lagrange’s multiplier rule for the equality case in mind and knowing about the result of Ostrogradsky, it is not difficult to guess what kind of relationship is likely to hold between the inequalities in the system and an inequality, which is a consequence of the system. As phrased by Farkas:

ist eine Ungleichheit [(6)] nicht neu in Bezug auf das System [(5)], so kann dieselbe immer als eine Summe der positiven Vielfachen dieser dargestellt werden.¹⁷ [Farkas, 1895, p. 268]

that is, if an inequality (6) is a consequence of the system of inequalities (5), then there exist non-negative multipliers $\lambda_1, \lambda_2, \dots$ such that

$$R_0 = \lambda_1 R_1 + \lambda_2 R_2 + \dots \quad (7)$$

As noted by Farkas, it is quite obvious that if the inequality (6) can be written as a positive linear combination of the inequalities in (5), the inequality is not new in relation to the system (5). The difficult task was to prove the converse and here Farkas did not fully succeed in his first paper, where he gave an incomplete proof.¹⁸

The main point for Farkas in the 1895 paper was the application of the mechanical principle of Fourier and in the second part of the paper he discussed the application of his theorem (Farkas’ lemma) to the question of the equilibrium of systems constrained by inequality conditions. It was not until his second paper, published in 1897, that he

¹⁷ “if an inequality [(6)] is not new with respect to the system [(5)] it can always be written as a sum of positive multiples of it.”

¹⁸ See also [Brentjes, 1976a] and [Prékopa, 1980]. In [Franksen, 1985a, 1985b, 1985c] and [Prékopa, 1980] Farkas’s lemma is interpreted as what later became known as the Kuhn-Tucker theorem in nonlinear programming, for a discussion of which see [Kjeldsen, 2000].

considered the question of how to solve a system of linear inequalities. Besides giving a different proof for his theorem, he also developed in the 1897 paper what he called the parametric method for finding the solutions of a system of linear homogenous inequalities and equations in terms of parameters.

He considered the system

$$\begin{aligned}
 A_{11}u_1 + A_{12}u_2 + \dots + A_{1n}u_n &= 0 \\
 A_{21}u_1 + A_{22}u_2 + \dots + A_{2n}u_n &= 0 \\
 &\dots\dots\dots \\
 A_{i1}u_1 + A_{i2}u_2 + \dots + A_{in}u_n &= 0 \\
 B_{11}u_1 + B_{12}u_2 + \dots + B_{1n}u_n &\geq 0 \\
 B_{21}u_1 + B_{22}u_2 + \dots + B_{2n}u_n &\geq 0 \\
 &\dots\dots\dots
 \end{aligned}
 \tag{8}$$

where the number of independent equations is at most $(n - 1)$. He assumed that in (8) only the linearly independent forms are represented, that is $i < n$ [Farkas, 1897, p. 32].

Farkas wanted to represent the solution, that is the variables u_1, u_2, \dots, u_n , as homogenous linear functions of new variables some of which can be chosen completely arbitrarily whereas the rest can be chosen arbitrarily but are non-negative. He proceeded in the following way: he considered a subsystem which consisted of the equations and as many of the inequalities as possible, provided that all the linear forms in the subsystem are linearly independent. This means that he obtained a subsystem of i equations and some, say k , inequalities with $i + k \leq n$. He rewrote the subsystem in the following form:

$$\begin{aligned}
 A_{11}u_1 + A_{12}u_2 + \dots + A_{1n}u_n &= 0 \\
 A_{21}u_1 + A_{22}u_2 + \dots + A_{2n}u_n &= 0 \\
 &\dots\dots\dots \\
 A_{i1}u_1 + A_{i2}u_2 + \dots + A_{in}u_n &= 0 \\
 B_{11}u_1 + B_{12}u_2 + \dots + B_{1n}u_n &= s_1 \\
 &\dots\dots\dots \\
 B_{k1}u_1 + B_{k2}u_2 + \dots + B_{kn}u_n &= s_k \\
 s_1 \geq 0, s_2 \geq 0, \dots, s_k \geq 0 &.
 \end{aligned}
 \tag{9}$$

Farkas then derived the first $i + k$ of the variables as homogeneous linear functions of the remaining $(n - (k + i))$ variables and the non-negative quantities, s_1, \dots, s_k [Farkas, 1897, p. 34].

inequalities was completed. This paper has the title “Theorie der einfachen Ungleichungen” and in the introduction Farkas wrote:

Während der letzten sieben Jahre habe ich mich wiederholt mit Ungleichungen beschäftigt und aus dem Gesichtspunkte der Anwendungen eine vorläufig abgeschlossene Theorie derselben zu Stande gebracht. Meine vereinzelt erschienenen Publicationen sind aber nicht geeignet, ein klares und zusammenhängendes Bild des Ganzen zu liefern, und davon abgesehen, ist ihre Zugänglichkeit auch beschränkt. Darum nehme ich mir die Freiheit, hier eine systematische Darstellung, nebs einigen Ergänzungen, dieser Arbeiten vorzulegen.¹⁹ [Farkas, 1901, p. 2]

In this paper, which was to become a classic in the field, Farkas presented his theory of linear inequalities as he had developed it in the context of the Fourier Inequality Principle in mechanics during the last half of the 1890s, but here, the presentation is almost completely detached from the equilibrium question in mechanics. This question is only mentioned in the introduction, where Farkas explained his motivation and justified the need for a comprehensive mathematical theory of homogeneous systems of linear inequalities and he gave only a very short indication of how it could be used in equilibrium questions in analytical mechanics. He extended his results to systems that included linear homogenous equations and he ended the paper with a discussion of the application of his results to “infinitesimale Systeme” that is to systems where the variables are differentiable functions of (x, y, z) in a three-dimensional space T , and the “extra” inequality is an integral inequality [Farkas, 1901, p. 17].

Farkas was motivated by equilibrium conditions for constrained mechanical systems and his goal was to make systems of linear inequalities the mathematical foundation for that subject. In this context, the main question for Farkas became the question of when an inequality is a consequence of a given system of inequalities. There are, of course, as we shall see in the following, other mathematical questions one can ask about such systems, but viewing the subject matter from the point of view of analytical mechanics, the question Farkas set out to investigate is an almost self-proposing question, the ‘natural’ one to ask. The same is the case with the approach he used in dealing with the solutions to such systems. The expressions of the variables u_1, \dots, u_n as linear homogeneous functions of arbitrary and partly arbitrary parameters could be applied immediately to the Fourier Inequality Principle, making it very easy to determine whether the necessary conditions for equilibrium were satisfied for any given constrained system of inequalities. Thus, the context of analytical mechanics influenced the kind of questions Farkas set out to investigate and thereby shaped the content and the outlook of the theory of linear inequalities as he developed it.

¹⁹ “During the last seven years I have worked repeatedly on inequalities and from the point of view of applications I have developed a provisional self-contained theory. However my scattered publications are not suited to give a clear and coherent picture of the whole, and apart from that their accessibility are limited. Therefore I here take the liberty to present a systematic presentation of these works and some additions to them.”

3. The emergence of a theory of linear inequalities in the context of convexity: The work of Hermann Minkowski

In 1896 Hermann Minkowski published the book “Geometrie der Zahlen”, which was announced as the first part of a work on a new geometrical theory of numbers. Minkowski’s geometrical number theory became very influential for several reasons. It provided a new tool to solve outstanding problems in number theory, and the geometry he developed in the book laid the foundation for an analytical theory of convexity. In this book, which came out at the same time as Farkas’s first papers on the Fourier Inequality Principle, Minkowski also derived a theory of linear inequalities independently of Farkas. In this section I will discuss on the one hand how the need for a theory of systems of linear inequalities arose within Minkowski’s investigations of properties of convex bodies and their applications to number theoretical problems about quadratic forms, and on the other hand how this context shaped the form of Minkowski’s theory of linear inequalities.

While the final establishment of linear inequalities within the theory of convexity, which took place in the thirties, is dealt with in Sect. 5, Alfred Haar’s treatment in 1917 of the theories developed by Farkas and Minkowski, which is the first attempt to found the theory of linear inequalities on the theory of convexity, is discussed in this section because it was motivated directly by Minkowski’s work.

3.1. *The need for a theory of systems of linear inequalities in Minkowski’s work*

Hermann Minkowski (1864–1909) was a professor at Göttingen from 1902 until he died in 1909 at the age of 44. At the time of the publication of “Geometrie der Zahlen” he was working in Königsberg. Minkowski is primarily known for his work on number theory and mathematical physics.²⁰ He worked on number theory, especially on quadratic forms, during most of his mathematical career and his development of a geometrical number theory is considered to be one of his most original achievements.²¹ Besides being a very strong tool for solving number theoretical problems the “Geometrie der Zahlen” initiated the study of convex bodies, and it is here, in an appendix to the first chapter, that we find the first work on the theory of linear inequalities in this context.

Minkowski himself gave the following two reasons for the inclusion of such an appendix in the book:

Es möge hier noch die Auflösung eines beliebigen Systems von linearen Ungleichungen in endlicher Anzahl auseinandergesetzt werden, ein Gegenstand, der mit den Untersuchungen dieses Kapitels in engem Zusammenhange steht und dessen Kenntniss späterhin

²⁰ See [Schwermer, 1991] and [Corry, 1997].

²¹ For a discussion of geometrical ideas and spatial intuition in the development of the theory of positive quadratic forms see [Schwermer, 1991].

mehrfach, namentlich im siebenten Kapitel erfordert werden wird.²² [Minkowski, 1896, (1910, p. 39)]

First of all, systems of linear inequalities are claimed to be closely related to the content of the first chapter of the book. In this chapter Minkowski introduced the concept of surfaces which are nowhere concave with respect to some given point e not on the surface, or “nirgends concave Flächen um den Punkt e ”, as he named them. He also introduced a geometric tool for measuring distances, so called “strahlendistances”, with which he laid the foundation for an analytical treatment of these surfaces. The connection to the system of linear inequalities treated in the appendix is furnished by hyperplanes, which, as will be shown below, entered naturally in Minkowski’s study of the properties of nowhere concave surfaces. Secondly, according to Minkowski the reader needs some knowledge of systems of linear inequalities to understand the subject matter of chapter seven. Chapter seven was included in neither the first nor the second, posthumous, publication of the book, which appeared in 1910.²³ David Hilbert and Andreas Speiser wrote in their preface to the second edition that the publication of a second part, a sequel to the 1896 publication, was delayed due to some unexpected difficulties that Minkowski encountered [Hilbert and Speiser, 1910].²⁴ Instead Minkowski published most of the results intended for the second part of the book in several papers. According to Hilbert and Speiser, these papers are the papers numbered XIII–XXI in Minkowski’s collected works [Hilbert and Speiser, 1910]. The material intended for chapter seven is probably the paper “Diskontinuitätsbereich für arithmetische Äquivalenz” in which Minkowski explicitly referred to his treatment of linear inequalities in “Geometrie der Zahlen” [Minkowski, 1905, (1911, vol. 2, p. 69)].

The geometry of numbers

The fundamental concept in the geometry Minkowski developed to treat number theoretical problems is the “Aichkörper” of arbitrary “Strahldistanzen”. Minkowski explained the basic ideas very neatly in a talk he gave in Chicago in 1893:

Die tieferen Eigenschaften des Zahlengitters [here ‘Zahlengitter’ is taken to be all the points with integral coordinates in the ordinary rectangular n -dimensional coordinate system] nun hängen mit einer Verallgemeinerung des Begriffs der *Länge einer geraden Linie* zusammen, bei der allein der Satz, dass in einem Dreiecke die Summe zweier Seiten niemals kleiner als die dritte ist, erhalten bleibt.

²² “Here I would like to discuss the solution of an arbitrary system of finitely many linear inequalities, a subject that is closely connected to the investigations of this chapter, and the knowledge of which is required several times in the following, particularly in chapter seven.”

²³ The 1896 and the 1910 editions of the book are identical except that in the 1910 version there is a preface by Hilbert and Speiser, and they included an advertisement from the publisher, written by Minkowski, dated 1893, as well as an announcement also written by Minkowski for the first part of the book published in 1896, and finally they included the last fourteen pages of the last chapter.

²⁴ See also the letter from Minkowski to Hilbert, on February 10, 1896 in [Rüdenberg and Zassenhaus, 1973, p. 77].

Man denke sich eine Funktion $S(ab)$ von zwei beliebig variablen Punkten a und b zunächst nur mit folgenden Eigenschaften: (1) Es soll $S(ab)$ immer positiv sein, wenn b von a verschieden ist, und Null, wenn b und a identisch sind; (2) sind a, b, c, d vier Punkte und darunter b von a verschieden, und besteht zwischen ihnen eine Beziehung $d - c = t(b - a)$ mit positivem t , so soll immer $S(cd) = tS(ab)$ sein; die genannte Beziehung ist im Sinne des baryzentrischen Kalküls aufzufassen und bedeutet, dass cd und ab Strecken von gleicher Richtung und mit Längen (im gewöhnlichen Sinne) im Verhältnisse $t:1$ sind. Zum Unterschiede von der gewöhnlichen Länge möge $S(ab)$ *Strahldistanz von a nach b* heissen.

Es sei o der Nullpunkt; offenbar werden alle Werte $S(ab)$ festgelegt sein, sowie die Menge der Punkte u gegeben ist, für welche $S(ou) \leq 1$ ist; diese Punktmenge heisse der *Aichkörper* der Strahldistanzen, es wird zu ihm in jeder Richtung von o aus eine Strecke von o aus mit endlicher, nichtverschwindender Länge gehören müssen.

Wenn nun ferner für irgend drei Punkte a, b, c immer

$$S(ac) \leq S(ab) + S(bc)$$

ist, sollen die Strahldistanzen *einhellig* heissen. Dann besitzt ihr Aichkörper die Eigenschaft, dass mit irgend zwei Punkten u, v in ihm immer die ganze Strecke uv zu diesem Körper gehört, und andererseits ist jeder *nirgends concave Körper* mit dem Nullpunkt im Inneren Aichkörper für ganz bestimmte einhellige Strahldistanzen.²⁵ [Minkowski, 1893, (1911, vol. 1, p. 272–273)]

Minkowski's main theorem in "Geometrie der Zahlen" is the so-called Lattice Point Theorem, which states that

Ein nirgends concaver Körper mit einem Mittelpunkt in einem Punkte des Zahlengitters und von einem Volumen $= 2^n$ enthält immer noch mindestens zwei weitere Punkte des Zahlengitters, sei es im Inneren, sei es auf der Begrenzung.²⁶ [Minkowski, 1896, (1910, p. 76)]

²⁵ "The deeper properties of the lattice is connected with a generalization of the concept of the *length of a straight line* by which only the theorem that the sum of two of the sides in a triangle is never less than the third side is maintained. Consider a function $S(ab)$ of two arbitrary variable points a and b at first only with the following property: (1) $S(ab)$ is positive when b is not equal to a , and equal to zero when b is equal to a ; (2) if a, b, c , and d are four points with b different from a , and if the relationship $d - c = t(b - a)$ holds for t positive then $S(cd) = tS(ab)$. The relationship should be understood in the sense of the Barycentric Calculus and means that cd and ab are line segments in the same direction and with length (in the usual sense) in the proportion $t : 1$. In contrast to the usual length, $S(ab)$ is called the *radial distance from a to b* .

Let 0 be the origin; obviously, all the values of $S(ab)$ are determined when the set of points u for which $S(0u) \leq 1$ are given. This set of points is called the *Aichkörper* of the radial distance. In any given direction from 0 there exist a line segment from 0 in this direction with non-vanishing length and belonging to the "Aichkörper". If moreover $S(ac) \leq S(ab) + S(bc)$ for arbitrary points a, b , and c the radial distance is called *einhellig*. Its "Aichkörper" then has the property that whenever two points u and v belong to the "Aichkörper" then the whole line segment uv will also belong to the "Aichkörper". On the other hand every nowhere concave body, which have the origin as an inner point, is the "Aichkörper" of a certain "einhellig" radial distance."

²⁶ "A nowhere concave body with a center in one of the lattice points and with volume $= 2^n$ contains always at least two additional lattice points either in its interior or on its boundary."

Here n is the dimension of the space. This is a very powerful theorem with which Minkowski was able to prove fundamental theorems in number theory without cumbersome calculations. In his “Nachrichten” for Minkowski Hilbert called this theorem “eine Perle Minkowskischer Erfindungskunst” [Hilbert, 1911, p. XI]. The theorem combines the geometric properties of the “Aichkörper” with the number theoretic property of the existence of lattice points, that is points with integral coordinates, in the “Aichkörper”.

In “Geometrie der Zahlen” Minkowski developed this new geometry and he devoted the first chapter to a thorough investigation of the properties of these “Strahldistanzen” and the associated “Aichkörper” – or unit balls, as we would call the “Aichkörper” today. In this chapter he studied the properties of the “Aichfläche”, which is the boundary of the “Aichkörper” associated with a “Strahldistanz”. In the course of these investigations Minkowski constructed what he called “Zellen”. A “Zelle” is a kind of hyper-tetrahedron with an apex in the origin 0 and a base – the so called “Flächenzelle” – with n “Basis-ecken”, that is n points a_1, \dots, a_n belonging to the “Aichfläche”, and for which the n directions $0a_1, \dots, 0a_n$ are linearly independent. Minkowski then filled up, so to speak, the “Aichkörper” with “Zellen”, whose inner points are mutually disjoint. Minkowski distinguished between “innerer” points and “inwendiger” points. The inner points in the “Zellen” are the points we would call inner points today. The “inwendiger” points are the inner points of the walls of the “Zelle” regarded as a point set in a hyperplane. That is the “inwendiger” points of the “Zelle” belong to the boundary of the “Zelle” but they do not lie on the edge or the rim of the boundary [Minkowski, 1896, (1910, p. 19–20.)]. He called the (finite) union of these “Zellen” P . Minkowski wanted to approximate the “Aichfläche” with the “Flächenzellen”. In order to do so he proved that only points belonging to the “Flächenzellen” belong to the boundary of P . In proving that, Minkowski used a “Hilfssatz” which states that if a point a belongs to the boundary of P then, among the walls in the individual “Zelle” in which a is contained, there will be at least one wall which contains an “inwendiger” point b of P , that is, b does not lie on the rim of the wall. The existence of such a point b , Minkowski argued, is the question of whether certain systems of linear inequalities have a solution or not [Minkowski 1896, (1910, p. 24–25)]. So systems of linear inequalities and their solutions are in fact closely related to the material developed in the first chapter. This becomes even clearer on considering that one of the main results in the first chapter is that through each point on the boundary of the “Aichkörper” there passes at least one hyperplane, which is a supporting hyperplane for the “Aichkörper” [Minkowski, 1896, (1910, p. 33–34)]. Minkowski presented his geometry of numbers not in what we would call synthetic geometry but in analytic geometry. The analytical expression of a hyperplane is given by a linear equation and to be a supporting hyperplane for an “Aichkörper” means that all the points of the “Aichkörper” belongs to the same one of the two half spaces defined by the hyperplane and the analytical expression of a half space is given by a linear inequality. Minkowski’s proof of his theorem of supporting hyperplanes for the “Aichkörper” depends on the property of the “Aichkörper”, that if two points belong to the “Aichkörper” then so do all points in the line segment joining the two points, which is exactly the property of convexity.

Minkowski's appendix: A theory of linear inequalities

In the appendix to Chapter one entitled “Anhang über lineare Ungleichungen” Minkowski treated the following system of linear inequalities

$$\xi_1 \geq 0, \quad \xi_2 \geq 0, \quad \dots \quad (11)$$

where ξ_1, ξ_2, \dots represent a finite number of linear forms:

$$U_1x_1 + \dots + U_nx_n \quad (12)$$

in n variables x_1, \dots, x_n and with constant coefficients U_1, \dots, U_n [Minkowski, 1896, (1910, p. 40)]. He argued that one only needs to consider situations where among the linear forms ξ_1, ξ_2, \dots there are n that are linearly independent, because if there were only $h < n$ linearly independent forms Minkowski reduced the number of variables to h by a change of variables [Minkowski, 1896, (1910, p. 40)].

Minkowski distinguished between different kinds of solutions to (11). By a *real* (“wirkliche”) solution, Minkowski understood a solution different from the trivial solution $x_1 = 0, \dots, x_n = 0$. If $A = (a_1, \dots, a_n)$ is a real solution then so is $\tau A = (\tau a_1, \dots, \tau a_n)$ for every positive τ . Minkowski called the solution τA a *multiple* (“Vielfaches”) of A . He did not consider real – that is non-trivial – solutions, which are multiples of each other as *essentially different* (“wesentlich verschieden”). It is obvious, that if $A = (a_1, \dots, a_n)$ and $B = (b_1, \dots, b_n)$ are real solutions to (11) so is $A + B = (a_1 + b_1, \dots, a_n + b_n)$ because, as Minkowski argued, the occurrence of n linear independent forms among ξ_1, ξ_2, \dots means, that at least one of the forms ξ_j takes a value different from zero in A , that is, it takes a positive value. With this, Minkowski was ready to characterize the last – and most important – kind of solutions, namely those he called “äusserste” solutions and which, in the English literature, have been called *extreme* or *fundamental* solutions.²⁷ By such a solution Minkowski understood a real solution which could not be written as the sum of two essentially different solutions [Minkowski, 1896, (1910, p. 41)]. The concept of “äusserste” or fundamental solutions is the keystone in Minkowski’s investigation of solutions to systems of linear inequalities because they function, as we shall see below, as building-blocks for the general solution.

First Minkowski realized that if, for a non-trivial solution A , there are among the forms $\xi_1, \xi_2, \dots, (n - 1)$ linearly independent ones that take the value zero for A , then A is a fundamental solution. If in the system (11) we replace the inequality sign with an equality sign for the $n - 1$ linearly independent forms that takes the value zero for a non-trivial solution A , then we can think of these $n - 1$ equations as hyperplanes in n -space. A then lies on the boundary of the solution set to (11), which is bounded by the hyperplanes $\xi_1 = 0, \xi_2 = 0, \dots$. The fact that A lies in each of these $n - 1$ hyperplanes means, because they are linearly independent, that A lies on the edge, so to speak, where these $n - 1$ independent hyperplanes intersect. Minkowski did not give this geometric picture of fundamental solutions but there is no doubt that his intuition came from such

²⁷ See [Hancock, 1939], [Dines and McCoy, 1933].

considerations. In an advertisement from the publisher from 1893, Minkowski wrote about the forthcoming book:

Geometrie der Zahlen habe ich diese Schrift betitelt, weil ich zu den Methoden, die in ihr arithmetische Sätze liefern, durch räumliche Anschauung geführt bin. Doch ist die Darstellung durchweg analytisch, wie dies schon durch den Umstand geboten war, dass ich von Anfang an eine Mannigfaltigkeit beliebiger Ordnung betrachte.²⁸ [Minkowski, 1910, p. V]

and in the announcement written for the 1896 edition he elaborated further on this:

Ich bin zu meinen Sätzen durch räumliche Anschauungen gekommen [...]. Weil aber die Beschränkung auf eine Mannigfaltigkeit von drei Dimensionen unthunlich erschien, so habe ich die Darstellung hier rein analytisch gefasst, nur befehle ich mich des Gebrauchs solcher Ausdrücke, die geeignet sind, geometrische Vorstellungen wachzurufen.²⁹ [Minkowski, 1910, VI]

This is very well illustrated by the German word “äusserste” that he chose to name these special solutions, the word means ‘farthest out’ or ‘uttermost’. As will be seen in Sect. 3.2, Minkowski’s application of his theory of linear inequalities, which is believed to have been the material intended for Chapter seven of his “Geometrie der Zahlen”, supports the suggestion that he had this geometrical picture of fundamental solutions lying on the edge of the solution set.

Minkowski used the fundamental solutions as buildings-blocks for the general solution in the following way. He started out with a real, that is non-trivial, solution $A = (a_1, \dots, a_n)$ of the system of inequalities (11) for which there are $m < n - 1$ linearly independent ones among the forms ξ_1, ξ_2, \dots that takes the value zero for A . Minkowski then considered a system of n linear independent forms among the forms ξ_1, ξ_2, \dots , where the first m are those that takes the value zero for A . Those he re-named η_1, \dots, η_m and the remaining $n - m$ he denoted $\zeta_1, \dots, \zeta_{n-m}$. He regarded these forms as new variables, and all the forms ξ_i , which are zero for A , will depend only on η_1, \dots, η_m . Since $n - m \geq 2$ it is possible, he argued, to find a solution $D = (d_1, \dots, d_n)$ to (11), which is not a multiple of A and for which all the forms ξ_i that vanish for A vanish for D . In geometric terms, what Minkowski did here was to construct a new solution D , which lies in the same hyperplanes $\xi_i = 0$ as A without being a multiple of A . Since A and D solve (11), so will $A + lD$ for $l > 0$. Minkowski then argued that since the forms that vanish for A also vanish for D , they will vanish for $A + lD$, and since at least one of the forms ξ_1, ξ_2, \dots will be positive for D , otherwise D would not be a real solution, it is possible to choose l , so that at least one of the forms $\zeta_1, \dots, \zeta_{n-m}$ is zero. Minkowski then fixed such an l , and for that l , $B = A + lD$ will

²⁸ “I have chosen the title Geometry of Numbers for this work because I reached the methods, that gives the arithmetical theorems, by spatial intuition. Yet the presentation is throughout analytic which was necessary for the reason that I consider manifolds of arbitrary order right from the beginning.”

²⁹ “I have reached my theorems through spatial intuition [...]. But here I have prepared a purely analytical presentation because the limitation to a three-dimensional manifold seemed impossible. Only, I aim at using expressions that are suitable for evoking geometrical imaginations.”

be a real solution to (11) for which at least $m + 1$ linearly independent forms among the ξ_1, ξ_2, \dots will vanish. It is clear, as Minkowski stated, that by continuing this procedure one will eventually reach a non-trivial solution Ψ for which there are $n - 1$ linear independent forms among ξ_1, ξ_2, \dots that will vanish for Ψ , which makes Ψ a fundamental solution, and all the forms that vanished for A will also vanish for Ψ . Minkowski then considered, for a positive number μ , the sum $A - \mu\Psi$, which will also be a solution for which all the forms that vanished for A will vanish. As before, since at least one of the forms ξ_i will not vanish for Ψ it is possible, Minkowski argued, to choose the number μ , such that one more of the forms ξ_i will vanish for $A - \mu\Psi$. For this particular μ , Minkowski defined $A_1 = A - \mu\Psi$. Now, A_1 is a solution for which there are $m + 1$ linearly independent forms among ξ_1, ξ_2, \dots that will vanish. Following the same procedure Minkowski constructed a fundamental solution Ψ_1 for which all the forms that vanished for A_1 will vanish. Using this solution it is possible, as before, to choose a positive number ν such that the solution $A_2 = A_1 - \nu\Psi_1$ will be a solution for which there are $m + 2$ linearly independent forms among ξ_1, ξ_2, \dots that will vanish including those that vanished for A_1 . If $m + 2 = n - 1$, A_2 will be a fundamental solution and Minkowski has then proved, that the solution A can be written as $A = A_2 + \mu\Psi + \nu\Psi_1$ that is as a positive linear combination of fundamental solutions. If $m + 2$ is not equal to n he just continued creating new fundamental solutions, and since this process is finite, Minkowski showed that every non-trivial solution to (11) can be written as a positive linear combination of fundamental solutions [Minkowski, 1896, (1910, p. 43)].

He also showed that there can be at most finitely many essentially different fundamental solutions and he called a system of fundamental solutions complete (“vollständig”), if every solution of (11) can be expressed as a non-negative linear combination of the system.

Minkowski ended the appendix by examining the question of when some of the inequalities in the system

$$\xi_1 \geq 0, \xi_2 \geq 0, \dots$$

can be said to be consequences of the others. For this he introduced the concept of independent solutions. He called $n - 1$ non-trivial solutions to the system (11) independent if only one plane can be determined that passes through the origin and the $n - 1$ points represented by these solutions. He proved that if ϕ is a linear form in x_1, \dots, x_n that takes a non-negative value for all fundamental solutions to (11), and such that among these fundamental solutions there are $n - 1$ independent ones for which ϕ vanish, then ϕ will necessarily have to be a multiple of one of the forms ξ_1, ξ_2, \dots

3.2. *Minkowski's use of systems of linear inequalities in the applications of his geometry of numbers*

Minkowski's second and probably most important reason for developing a theory of linear inequalities in “Geometrie der Zahlen” was that this theory was apparently needed in the missing Chapter seven, which is believed to be the paper “Diskontinuitätsbereich für arithmetische Äquivalenz” [Minkowski, 1905]. In this paper Minkowski represented a quadratic form

$$f(x_1, \dots, x_n) = \sum_{h,k=1}^n a_{hk} x_h x_k$$

in n variables with coefficients $a_{hk} \in \mathbf{R}$ as a point in an $\frac{n(n+1)}{2}$ -dimensional manifold A . The purpose of the investigation was, as put by Minkowski, to find:

in der Mannigfaltigkeit A einen Bereich B , in dem jede Klasse positiver quadratischer Formen durch einen Punkt, und wenn der Punkt in das Innere von B fällt, auch nur durch einen einzigen Punkt repräsentiert wird.³⁰ [Minkowski, 1905, (1911, vol. 2, p. 54)]

Two quadratic forms are considered to be equivalent or be in the same class if they can be turned into one another, by a linear transformation of the x 's with integral coefficients and determinant ± 1 . Minkowski introduced an ordering in equivalent classes of positive definite quadratic forms. If $f = \sum a_{hk} x_h x_k$ and $g = \sum b_{hk} y_h y_k$ are two equivalent positive quadratic forms, Minkowski called them equally placed ("gleichgestellt") if

$$a_{11} = b_{11}, a_{22} = b_{22}, \dots, a_{nn} = b_{nn} .$$

Otherwise, if for some l

$$a_{11} = b_{11}, \dots, a_{l-1,l-1} = b_{l-1,l-1}, a_{ll} > b_{ll}$$

he called f higher ("höher") than g and g lower ("niedriger") than f [Minkowski, 1905, (1911, vol. 2, p. 57)]. Minkowski was able to prove that in every class of positive quadratic forms f a form g could be found such that no other form in the class would be lower than g , that is a 'lowest' form exists in each such class. All equally placed forms equivalent with g are lowest forms of the class. In essence, the lowest forms of a class are the reduced forms as they had been defined by Hermite.³¹ Minkowski introduced a simpler definition of reduced forms, which are now called Minkowski-reduced forms:

A quadratic form $f = \sum a_{hk} x_h x_k$ is said to be a reduced form if the inequality

$$f(s_1^{(l)}, s_2^{(l)}, \dots, s_n^{(l)}) \geq a_{ll} \quad (13)$$

is fulfilled for every $l = 1, 2, \dots, n$ and all integral numbers $s_1^{(l)}, s_2^{(l)}, \dots, s_n^{(l)}$ for which the greatest common divisor of the numbers $s_l^{(l)}, s_{l+1}^{(l)}, \dots, s_n^{(l)}$ equals 1, and if

$$a_{12} \geq 0, a_{23} \geq 0, \dots, a_{n-1,n} \geq 0. \quad (14)$$

In the manifold A , Minkowski considered the subset B of points f , that satisfy all the inequalities (13) and (14) and he called B the reduced space ("reduzierten Raum") [Minkowski, 1905, (1911, vol. 2, p. 59–61)].

³⁰ "in the manifold A a domain B in which every class of positive quadratic forms is represented by a point. If this point is an inner point of B the form is represented by one and only one point."

³¹ See [Hermite, 1850] and [van der Waerden, 1968].

The conditions for being a Minkowski reduced form are that the coefficients of the form are a solution to a system of linear inequalities, and Minkowski was able to prove that among the infinitely many inequalities in (13) there are finitely many such that all the rest are consequences of those. In geometrical terms he proved that the reduced space is a convex cone with apex at the origin bounded by a finite number of hyperplanes through the origin [Minkowski, 1905, (1911, vol. 2, p. 68)]. This also means that Minkowski, as he explicitly pointed out, could use his theory of linear inequalities developed in “Geometrie der Zahlen”. So, if f is a reduced form, so too is cf for every positive factor c , and if g is another reduced form so is $(1 - t)f + tg$ for $0 < t < 1$. Adopting the notation from his theory of linear inequalities, Minkowski called a form ϕ which does not vanish identically, an edge- (“kanten”)-form if it is impossible to write ϕ as a sum of two reduced forms which do not vanish identically and which are not a positive multiple of each other. These edge-forms represent the “äusserste” solutions of the system of inequalities, and such a form is, as Minkowski noticed, completely characterized as a reduced form f for which there are $\frac{n(n+1)}{2} - 1$ linearly independent ones among the inequalities (13), (14) for which the equality sign holds. He then concluded that there can only be finitely many essentially different edge-forms and the rays from the origin towards the edge-forms make up the edges of the reduced space B . From his theory of linear inequalities Minkowski was also able to characterize the inequalities in (13) and (14) that are necessary for the definition of the reduced space, namely those for which the hyperplane that one gets if the inequality sign is substituted with the equality sign contains $\frac{n(n+1)}{2} - 1$ linearly independent edges of the reduced space. There exist finitely many edge-forms $\phi_1, \phi_2, \dots, \phi_r$ such that every reduced form f can be written as a non-negative linear combination of those [Minkowski, 1905, (1911, vol. 2, p. 70)].

Minkowski presented his theory of linear inequalities in such a way that it could be applied directly to the investigation of the set B representing the reduced forms of positive quadratic forms. The main results are the characterization of B as a convex cone bounded by finitely many hyperplanes and the representation of reduced forms as positive linear combinations of finitely many “Kantenformen”, that is the main feature is the concept of the “äusserte Lösungen”, which is the concept around which Minkowski developed his theory of linear inequalities. Minkowski explicitly stated that his main source of inspiration behind his development of geometry of numbers was Hermite’s letter to Jacobi published in Crelle’s Journal in 1850,³² in which Hermite proved the fundamental theorem of reduction of positive quadratic forms. This theorem states that one can always find integer values, not all zero, for the variables such that the value of the quadratic form for these integer values of the variables does not exceed a limit which only depends on the determinant of the form [Hermite, 1850]. In the same volume of Crelle’s Journal there is a paper by Dirichlet [Dirichlet, 1850] in which Dirichlet developed a new geometrical foundation for the theory of the reduction of quadratic forms in three variables. This led Minkowski to interpret Hermite’s result geometrically as properties of an ellipsoid and he realized that the essential property used in the proof was that ellipsoids are what we to day would call bounded convex bodies with the origin

³² See [Minkowski, 1910, p. IV]. See also [Minkowski, 1891, 1893], Minkowski’s letter to Hilbert dated November 6, 1889 in [Rüdenberg and Zassenhaus, 1973, p. 38], [Schwermer, 1991].

as center. This motivation shaped Minkowski's geometrical theory of numbers and the purpose – the applications he had in mind – guided the problems he investigated for systems of linear inequalities and shaped the outlook of his theory of such systems. But they did not, as we have seen, influence the approach he used in proving his results about linear inequalities. About twenty years later Alfred Haar was the first to prove the results of Farkas and Minkowski using the theory of convexity.

3.3. *A first theory of linear inequalities based on the theory of convexity: Alfred Haar's contribution*

In a paper presented to the Hungarian Academy of Science in 1917 and published in 1918 [Haar, 1918] Alfred Haar (1885–1933) pointed out that he found it a bit strange that Minkowski did not base his theory of linear inequalities on the geometrical interpretations of linear inequalities:³³

Merkwürdigerweise verlässt Minkowski in seinen genannten Werke [Geometrie der Zahlen] eben an dieser Stelle die geometrische Interpretation, obwohl das ganze Buch sonst dieser Denkungsart gewidmet ist.³⁴ [Haar, 1918 (1924, p. 1–2)]

Haar had studied in Göttingen from 1904 to 1909, where he received his doctoral under Hilbert with a dissertation on orthogonal systems of functions. Haar was familiar with Minkowski's work on geometrical number theory; in 1917 he published the paper "Die Minkowskische Geometrie und die Annäherung an stetige Funktionen" [Haar, 1917] and in his paper on linear inequalities of 1918 he referred to both Farkas's and Minkowski's work on linear inequalities [Farkas, 1918, p. 279].

In the introduction to his paper on linear inequalities, Haar explained the goal of his work:

Die Theorie der linearen Ungleichungen wurde von H. Minkowski und J. Farkas zuerst entwickelt; beide Verfasser gelangen auf gänzlich verschiedene Weise zu den beiden Hauptsätzen dieser Theorie . . .

Das Ziel der vorliegenden Arbeit . . . ist, diese Theorie in neuer Weise zu begründen.³⁵ [Haar, 1918, (1924, p. 1)]

Haar considered the system

$$a_{k1}u_1 + a_{k2}u_2 + \dots + a_{kn}u_n + a_{kn+1} \geq 0 \quad (k = 1, 2, \dots) \quad (15)$$

³³ Haar's paper was published in Hungarian but it was later translated into German. The German version was published in 1924 and in the German version Haar referred to some literature on linear inequalities written by the Americans L. L. Dines and W. B. Carver in the period between the Hungarian and the German version of his own paper.

³⁴ "Oddly enough Minkowski abandon the geometrical interpretation precisely at this point of his work [Geometrie der Zahlen] even though the whole book is dedicated to this way of thinking."

³⁵ "The theory of linear inequalities was first developed by H. Minkowski and J. Farkas. Both authors reached in very different ways the two main theorems of the theory . . .

The aim of the present work is to establish the theory in a new way."

and its homogenous version where $a_{kn+1} = 0$. His “goal” was to show how the theory of linear inequalities, via a suitable geometrical interpretation of the inequalities and the unknowns, can be based on simple theorems in multidimensional geometry, more precisely on theorems about convex bodies [Haar, 1918, (1924, p. 1)].

Haar gave a new proof of Farkas’s lemma for both homogenous and inhomogeneous systems of inequalities in the case where there are n unknowns. There could be finite or infinitely many inequalities but in the last case the set of points, whose coordinates are the coefficients $p_k = (a_{k1}, a_{k2}, \dots, a_{kn})$ in the respective inequalities (15), must be closed. Haar’s proof of Farkas’s lemma is based on simple analytical geometry and an important theorem of Carathéodory’s about the smallest convex region containing a closed set. In 1911 Carathéodory published a paper in which he gave a very clear exposition of the fundamental theorem about the smallest convex region containing a closed set, namely that each of its points can be considered as the center of gravity of a distribution of positive masses, with total mass equal 1, of at most $n + 1$ points from the set [Carathéodory, 1911, p. 200].

Haar was the first one who made the theory of convexity the foundation of the theory of systems of linear inequalities. Later, at the end of the twenties and the beginning of the thirties, mathematicians explored the subject of linear inequalities further within the realm of convexity. This is discussed in Sect. 5.

3.4. Discussion

In the above quotation (p. 489), Haar characterized the results of Farkas and Minkowski as being the same and the two main theorems of the theory. This is because they both dealt with the question of when an inequality is a consequence of the others and they both derived the general solution. According to Haar, the main difference between their theorems is the way they reached these results. As we have seen in Sect. 2 and Sect. 3 it is not only the approaches they used in deriving their results, but also the actual formulation of the results that differ from each other. It is true that the theorems follow from each other. This is probably the reason why Haar did not consider the results different from each other, which is indeed a very natural evaluation for a mathematician looking at the work twenty years later and extracting the basic mathematical ideas behind the theorems. But historically, evaluated with respect to the given time and context in which Farkas and Minkowski developed their respective theories of linear inequalities, their results are very different.

The main theorem for Farkas seemed to have been Farkas’s lemma. That was the first result he worked on, the solution method came later. For Minkowski the main result was probably the characterization of the extreme solutions and their central role in the general solution. That difference seems almost inevitable considering the context in which Farkas and Minkowski developed their theories and the reasons behind their developments. They were motivated by completely different factors, and the reasons that triggered their work were also very different and are reflected in the form of their theories. The main question in the context Farkas worked in was, as demonstrated in Sect. 2, the question to which Farkas’s lemma provides an answer. For Minkowski, the structure of the solution set was important, which is – again – reflected in his theory.

Both of them found the general solution to a system of linear inequalities but they presented it in very different forms. Farkas represented the general solution as homogeneous linear functions of new variables. He did not distinguish between different kind of solutions and he did not find a basis for the solution set. The fundamental solutions of Minkowski can be singled out from Farkas's "homogeneous linear functions" but Farkas did not do that, and it is not at all clear from his papers whether he was aware of the existence of a basis for the solution set. Minkowski's presentation of the general solution reveals a different kind of intuition and insight into the fundamental geometrical structure of the solution set, so even though they in some respect can be said to be "the same" result, they provide very different information about the solution set.

The above analysis of Farkas's and Minkowski's work reveals that the context played a significant role for the kind of questions Farkas and Minkowski investigated within the subject of linear inequalities, and that the context also, to a large degree, shaped the content and the outlook or form of the theories they developed.

4. The emergence of a theory of linear inequalities in the USA in the 1920's

The theories developed by Farkas and Minkowski have in common that neither of them were originally motivated by the subject of linear inequalities in itself. These theories were not developed because of an initial interest in such systems for their own sake. In contrast, this seems to have been the case in the twenties where a theory of linear inequalities was developed once again, this time in the USA and – again – independently of the earlier developments. The starting point was the publication of a small paper by William V. Lovitt titled "Preferential Voting"

4.1. Preferential voting: A nutcracker problem

William Vernon Lovitt (1881–1972) was an associate professor of mathematics at Purdue when he published a small paper on the solution to a problem which he called "preferential voting" for the *American Mathematical Monthly* in 1916 [Lovitt, 1916]. Lovitt considered a voting situation with S voters and three candidates A , B , and C . Each voter has a first, second, and third choice. The task that Lovitt set for himself was the following:

Suppose the voting to have been done. It is the object of this paper to determine the conditions under which it is possible to assign weights to the votes for first, second, and third choice so that any preassigned candidate may win. [Lovitt, 1916, p. 363]

Lovitt mathematized the problem in the following obvious way: A_i denoted the number of votes of choice i ($i = 1, 2, 3$) for candidate A , similar for B_i and C_i . Then

$$A_i + B_i + C_i = S \quad (i = 1, 2, 3)$$

and

$$A_1 + A_2 + A_3 = B_1 + B_2 + B_3 = C_1 + C_2 + C_3 = S .$$

He let x , y , and z denote the weights assigned to first, second, and third choice with the condition that $x > y > z$. The number of points received by A is then represented by $A_1x + A_2y + A_3z$, with similar expressions for B and C [Lovitt, 1916, p. 363].

He solved the problem geometrically by looking at the six planes

$$A_1x + A_2y + A_3z = 0$$

$$B_1x + B_2y + B_3z = 0$$

$$C_1x + C_2y + C_3z = 0$$

and

$$(A_1 - B_1)x + (A_2 - B_2)y + (A_3 - B_3)z = 0$$

$$(A_1 - C_1)x + (A_2 - C_2)y + (A_3 - C_3)z = 0$$

$$(B_1 - C_1)x + (B_2 - C_2)y + (B_3 - C_3)z = 0 .$$

The last three planes represent the loci of points where A and B , A and C , and B and C respectively, receive the same amount of points [Lovitt, 1916, p. 364].

There is no indication of why Lovitt worked on this problem. He did not say why in the paper, and there are no references. In "American Men of Science" Lovitt is characterized as a mathematician interested in analysis, statistics, mathematics for students of agriculture and general science, mathematics for business, and integral equations [Cattell and Cattell, 1933, p. 691]. He seems to have been a mathematician with very broad interests and this problem could have come up in his teaching. The problem is an artificial one with no real application and it is to my knowledge the only paper he wrote on the subject. My guess is that it was just a "nutcracker" problem he stumbled over and solved.

Lovitt did not consider systems of linear inequalities in this small paper, in fact he never published anything on that subject, but it is quite obvious that the problem can be formulated as a problem of finding a solution to such a system. This was pointed out half a year later by Lloyd Lynes Dines in a paper which was a direct answer to Lovitt's, and which was to become the first in a series of papers dealing directly with the building of a theory of systems of linear inequalities.

4.2. *Linear inequalities enter the discussion*

In 1917 Lloyd L. Dines, at that time an assistant professor at the University of Saskatchewan in Canada, took up Lovitt's problem of preferential voting but with a very different approach.

Dines published his solution in the paper "Concerning Preferential Voting" in the *American Mathematical Monthly*. He began the paper with a reference to Lovitt:

In a recent number of the MONTHLY, Professor W. V. Lovitt has given an interesting discussion of a problem which may be stated as follows: . . . [Dines, 1917, p. 321]

Even though Dines in the opening sentence referred to Lovitt's "interesting discussion" of the problem it is clear, that it was not Lovitt's *discussion* Dines was interested in rather he wanted to give a "purely algebraic discussion" of the problem [Dines, 1917, p. 321]. Even though he did acknowledge that his algebraic discussion would be "less vivid" he gave no less than three reasons for the superiority of this algebraic approach compared with the geometrical discussion given by Lovitt: it had the advantage of leading "more directly to the necessary conditions for the existence of a solution", it would give "explicit ranges of possible values for the weights in term of the data A_i , B_i , and C_i ," and last but not least the algebraic method was much more powerful because it could "be extended to a treatment of the more general problem in which there are n candidates, and each voter expresses his first, second, . . . , and m^{th} choice" [Dines, 1917, p. 321–322].

In the following I will give a detailed exposition of Dines's way of solving the problem because the ideas behind his later development of a theory of systems of linear inequalities was extracted directly from the method he used in solving the voting problem.

Dines wrote the necessary conditions that candidate A wins in the following way

$$A_1x_1 + A_2x_2 + A_3x_3 > B_1x_1 + B_2x_2 + B_3x_3$$

$$A_1x_1 + A_2x_2 + A_3x_3 > C_1x_1 + C_2x_2 + C_3x_3$$

where A_i , B_i , and C_i denotes the number of votes with priority i , received by candidate A , B , and C respectively, and x_i denotes the weight of choice i ($i = 1, 2, 3$), with $x_1 > x_2 > x_3 > 0$.

By introducing new variables $\xi_1 = x_1 - x_2$, $\xi_2 = x_2 - x_3$, and $\xi_3 = x_3$ and using the relations

$$A_1 + A_2 + A_3 = B_1 + B_2 + B_3 =$$

$$C_1 + C_2 + C_3 = \text{number of voters}$$

Dines rewrote the necessary conditions that A wins in the following way

$$(A_1 - B_1)\xi_1 > (A_3 - B_3)\xi_2 \tag{16}$$

$$(A_1 - C_1)\xi_1 > (A_3 - C_3)\xi_2 \tag{17}$$

$$\xi_i > 0 \quad \text{for } i = 1, 2, 3 . \tag{18}$$

This is a system of linear inequalities, and in order to solve it, Dines noticed, that since ξ_3 does not appear in the system, it can have any positive value and since (16) and (17) only restrict the ratio $\frac{\xi_1}{\xi_2}$, also ξ_2 can have any positive value. The only thing left is to determine the range of positive values that ξ_1 may have [Dines, 1917, p. 322]. For each of the inequalities (16) and (17) he distinguished between three cases according to the sign of the coefficient of ξ_1 :

a) if $A_1 - B_1 > 0$ then $\xi_1 > \frac{A_3 - B_3}{A_1 - B_1} \xi_2$

b) if $A_1 - B_1 = 0$ then $(A_3 - B_3)\xi_2 < 0$

c) if $A_1 - B_1 < 0$ then $\xi_1 < \frac{A_3 - B_3}{A_1 - B_1} \xi_2$

and similarly for (17). From this Dines could determine the range for ξ_1 for all cases [Dines, 1917, p. 323].

This procedure can be extended directly to cases with more than three candidates, and Dines did so in the second half of the paper. He considered the case where there are $n + 1$ candidates, A and $B^{(i)}$ ($i = 1, 2, \dots, n$), and m weights x_1, \dots, x_m with the condition $x_1 > x_2 > \dots > x_{m-1} > x_m > 0$. Here $A_j, B_j^{(i)}$ still denotes the number of votes of choice j given to candidate A and $B^{(i)}$, respectively. As before, Dines formulated the necessary conditions that candidate A wins as a system of linear inequalities

$$\sum_{j=1}^m (A_j - B_j^{(i)})x_j > 0 \quad i = 1, 2, \dots, n \tag{19}$$

$$x_1 > x_2 > \dots > x_{m-1} > x_m > 0 . \tag{20}$$

Again he made the substitution

$$x_j = \sum_{h=j}^m \xi_h \quad j = 1, 2, \dots, m$$

and rewrote the necessary conditions that A wins in the following form [Dines, 1917, p. 324]:

$$\sum_{h=1}^m \sum_{j=1}^h (A_j - B_j^{(i)})\xi_h > 0 \quad i = 1, 2, \dots, n \tag{21}$$

$$\xi_j > 0 \quad j = 1, 2, \dots, m . \tag{22}$$

Dines then divided the n inequalities in the system (21) into three groups:

Type I: Inequalities where $A_1 - B_1^{(i)} > 0$, supposing these are the first n_1 inequalities.

By dividing all of them by $A_1 - B_1^{(i)}$, Dines obtained n_1 lower bounds for ξ_1 :

$$\xi_1 > - \sum_{h=2}^m \sum_{j=1}^h \frac{A_j - B_j^{(i)}}{A_1 - B_1^{(i)}} \xi_h \tag{23}$$

$$i = 1, 2, \dots, n_1 .$$

Type II: Inequalities where $A_1 - B_1^{(i)} = 0$, supposing these are the next n_2 inequalities. In these inequalities the coefficients of ξ_1 vanish, which means that these inequalities place no restrictions on ξ_1 .

$$\sum_{h=2}^m \sum_{j=1}^h (A_j - B_j^{(i)}) \xi_h > 0 \tag{24}$$

$$i = n_1 + 1, \dots, n_1 + n_2 .$$

Type III: Inequalities where $A_1 - B_1^{(i)} < 0$, supposing these are the last n_3 inequalities. By dividing all of them by $A_1 - B_1^{(i)}$, Dines obtained n_3 upper bounds for ξ_1 :

$$\xi_1 < - \sum_{h=2}^m \sum_{j=1}^h \frac{A_j - B_j^{(i)}}{A_1 - B_1^{(i)}} \xi_h \tag{25}$$

$$i = n_1 + n_2 + 1, \dots, n_1 + n_2 + n_3 .$$

Since $\xi_1 > 0$, the upper bounds given by (25) necessarily have to be positive, which implies that ξ_2, \dots, ξ_m are restricted by the n_3 inequalities:

$$\sum_{h=2}^m \sum_{j=1}^h (A_j - B_j^{(i)}) \xi_h > 0 \tag{26}$$

$$i = n_1 + n_2 + 1, \dots, n_1 + n_2 + n_3 .$$

In order that there can be an interval in which ξ_1 can take values, the differences between the upper bounds and the lower bounds defined by (25) and (23) must be positive, that is

$$\sum_{h=2}^m \sum_{j=1}^h \left(\frac{A_j - B_j^{(i)}}{A_1 - B_1^{(i)}} - \frac{A_j - B_j^{(k)}}{A_1 - B_1^{(k)}} \right) \xi_h > 0 \tag{27}$$

$$i = 1, \dots, n_1$$

$$k = n_1 + n_2 + 1, \dots, n_1 + n_2 + n_3 .$$

By arranging the inequalities into the three types above Dines was able to eliminate the variable ξ_1 because, as we have seen, ξ_1 is only restricted by (23) and (25); and the existence of a positive range for ξ_1 satisfying the conditions (23) and (25) is ensured by the conditions (26) and (27) which only involve the variables ξ_2, \dots, ξ_m .

The next step in Dines's process of elimination is to go through the same procedure for the equivalent system

$$\sum_{h=2}^m \sum_{j=1}^h (A_j - B_j^{(i)}) \xi_h > 0 \tag{28}$$

$$i = n_1 + 1, n_1 + 2, \dots, n_1 + n_2$$

$$\sum_{h=2}^m \sum_{j=1}^h (A_j - B_j^{(i)}) \xi_h > 0 \tag{29}$$

$$i = n_1 + n_2 + 1, \dots, n_1 + n_2 + n_3$$

$$\sum_{h=2}^m \sum_{j=1}^h \left(\frac{A_j - B_j^{(i)}}{A_1 - B_1^{(i)}} - \frac{A_j - B_j^{(k)}}{A_1 - B_1^{(k)}} \right) \xi_h > 0 \tag{30}$$

$$i = 1, \dots, n_1$$

$$k = n_1 + n_2 + 1, \dots, n_1 + n_2 + n_3$$

which is a homogeneous linear system of $(n_2 + n_3 + n_1 n_3)$ inequalities in the $m - 1$ variables ξ_2, \dots, ξ_m . Dines then grouped these inequalities into the three types above according to the sign of the coefficient of ξ_2 . By continuing this process, ranges for the variables are determined successively.

This elimination process also contains the necessary ingredients for the proof of the following theorem, which was Dines’s main result in the paper

A necessary and sufficient condition that there exist a system of weighting under which A wins, is that at some stage of the process of successive elimination of the variables ξ_i described above, the inequalities of the system [the system of inequalities that results at that stage] presenting itself shall be all of Type I. [Dines, 1917, p. 325]

The proof given by Dines is a straightforward consequence of the elimination process.

4.3. Towards a theory of systems of linear inequalities

Two years later, in March 1919, Dines published another paper in which he disengaged the above result from the question of preferential voting and discussed it explicitly within the context of systems of linear inequalities, which was also the title of the paper [Dines, 1919a]. The main agenda for Dines in this paper was to examine the relation between the matrix of the coefficients of a system of linear inequalities and on the one hand the existence of a solution and on the other hand the character of such a solution when it exists. In order to do this, Dines invented a vocabulary for dealing with such systems because, as he phrased it in the introduction of the paper

There seems however to be no terminology available for the characterization of the solution in terms of this matrix, and no well-recognized algorithm for the actual determination of the solution. This is in marked contrast to the analogous situation in the case of the system of equations ... [Dines, 1919a, p. 191]

This indicates that Dines was motivated by the matrix theory for systems of linear equations and his development, as will become clear in the following, was also “modeled” on this theory. The key concept introduced by Dines for dealing with questions about the existence and the characterization of solutions of systems of linear inequalities in terms of the matrix of the coefficients was the *I*-rank of the matrix. Here the *I* was short for *inequality* and this concept is, as pointed out by Dines, analogous to the rank of a matrix, which plays a central role in the theory of systems of linear equations.

The main object of investigation for Dines was the following system of inequalities

$$\begin{aligned}
 a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &> 0 \\
 a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &> 0 \\
 &\dots\dots\dots \\
 a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &> 0
 \end{aligned}
 \tag{31}$$

with the corresponding matrix

$$M = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}.$$

The coefficients listed in the *j*th column of the matrix represent the coefficients of the variable *x_j*. In the elimination procedure described in Sect. 4.2 the inequalities were grouped according to the sign of the coefficient of the variable to be eliminated. Dines’s main result in his previous paper, that a necessary and sufficient condition for the existence of a solution is that at some stage of the process all the coefficients of the variable to be eliminated are positive, led Dines to introduce his first definition in the context of linear inequality systems, namely the *I*-positivity (-negativity) of a matrix. Dines called the matrix *M* “*I*-positive (or *I*-negative) with respect to a given one of its columns if all elements of that column are positive (or negative). In either case the matrix will be said to be *I*-definite with respect to that column” [Dines, 1919a, p. 191].

In order to introduce the central notion of the *I*-rank of a matrix Dines introduced the concept of *I*-complement and *I*-minors of a matrix. If the matrix *M* of the coefficient is not *I*-definite with respect to the *r*th column, Dines grouped the elements of that column into three groups: the positive elements *a_{ir}* (*i* = *i*₁, *i*₂, ..., *i_P*); the negative elements *a_{jr}* (*j* = *j*₁, *j*₂, ..., *j_N*); and the zero ones *a_{kr}* (*k* = *k*₁, *k*₂, ..., *k_Z*) where *P*, *N*, and *Z* denote the number of positive, negative, and zero elements respectively. He then constructed a new matrix *M*₁^(*r*) with *PN* + *Z* rows and *n* – 1 columns. Dines derived this matrix from the original matrix *M* in the following way:

To each pair of elements a_{ir}, a_{jr} , the first positive and the second negative, corresponds one row of the derived matrix, the elements of which are second order determinants

$$\begin{vmatrix} a_{ir} & a_{i1} \\ a_{jr} & a_{j1} \end{vmatrix}, \begin{vmatrix} a_{ir} & a_{i2} \\ a_{jr} & a_{j2} \end{vmatrix}, \dots, \begin{vmatrix} a_{ir} & a_{i(r-1)} \\ a_{jr} & a_{j(r-1)} \end{vmatrix},$$

$$\begin{vmatrix} a_{ir} & a_{i(r+1)} \\ a_{jr} & a_{j(r+1)} \end{vmatrix}, \dots, \begin{vmatrix} a_{ir} & a_{in} \\ a_{jr} & a_{jn} \end{vmatrix}.$$

To each zero element a_{kr} corresponds one row of the derived matrix

$$a_{k1}, a_{k2}, \dots, a_{k(r-1)}, a_{k(r+1)}, \dots, a_{kn}.$$

[Dines, 1919a, p. 192]

If the original matrix M is I -definite with respect to the r th column, Dines let $M_1^{(r)}$ be “the matrix of one row and $n - 1$ columns, all elements of which are $+1$ or -1 according as M is I -positive or I -negative with respect to the r th column” [Dines, 1919a, p. 192]. Dines named this derived matrix the I -complement of the r th column of M , and the n matrices $M_1^{(1)}, \dots, M_1^{(n)}$ he called the I -minors of $n - 1$ columns of the matrix M [Dines, 1919a, p. 192]. Dines repeated this procedure and formed the I -complement of any column of the matrix $M_1^{(r)}$. In this way Dines recursively constructed a system of I -minors. He then defined the notion of I -rank in terms of this system of minors:

A matrix will be said to be of I -rank k if it possesses at least one I -minor of k columns which is I -definite, but does not possess any I -minor of $k + 1$ columns which is I -definite.
[Dines, 1919a, p. 193]

Dines’s main theorems for systems of homogeneous linear inequalities are first that “a necessary and sufficient condition for the existence of a solution of the system [(31) of inequalities with coefficient matrix M] is that the I -rank of the matrix M be greater than zero” and second “if the I -rank of the matrix M is $k (> 0)$, then the system [(31)] possesses a solution in which $k - 1$ of the unknowns may be assigned values at pleasure” [Dines, 1919a, p. 193].

At a first glance it seems quite tedious with all these I -complements and I -minors but keeping the elimination process from Dines’s paper on preferential voting in mind, the matrix $M_1^{(1)}$ is just the matrix of coefficients corresponding to the system of inequalities that arises after elimination of the first variable x_1 . The proofs that Dines gave for his two theorems is just like the one he gave in 1917 except that in 1919 he rewrote the elimination process from the earlier paper in terms of minors and I -rank of the corresponding matrix of coefficients. This also means that the theorems are the same as the theorem from the paper on voting except that the notion of I -rank gives the extra information about how many of the variables can be assigned “values at pleasure”. Dines also derived two similar theorems for inhomogeneous systems of linear inequalities.

Comparing these two papers by Dines it becomes clear that if one is looking at the mathematical results, that is the theorems proven, there is nothing really new in the second paper. The main tool is the process of elimination, which has the advantage of being constructive in the sense that it gives a method for actually finding the general solution of the system under consideration. What is new in his second paper is the development

of a terminology that incorporated the subject of systems of linear inequalities in the framework of the theory of matrices just as was the case for systems of linear equations. Dines's 1919 paper can be seen as an attempt to build a theory for systems of linear inequalities with its own terminology, concepts, and methods. The main difference between these two contributions of Dines is that in the second paper he singled out the system of linear inequalities as the main focus of attention. In the first paper it was a tool to solve the problem of preferential voting; in the second paper it was no longer just a tool but the main subject under investigation. In that respect, the character or the nature of the two papers is very different, as is also reflected in the places of publication. The first one was published in *American Mathematical Monthly*, a journal devoted to a broad audience whereas the second paper was published in the *Annals of Mathematics* which had recently relaunched itself as a journal for research papers in mathematics proper. The purpose of the second paper was, as Dines also stated himself, to study systems of linear inequalities "from the point of view of the matrix of coefficients" [Dines, 1919a, p. 191], that is to explore the underlying nature of solutions of such systems for their own sake.

4.4. *New questions and approaches: Carver's contribution*

It could have been left there and Dines's paper on system of linear inequalities could have joined the ranks of the isolated contributions we have been looking at up to now, had it not been for Walter B. Carver, who took up the discussion after Dines's 1919 paper. Carver was an assistant professor at Cornell when in 1922 he published the paper "Systems of Linear Inequalities" [Cattell and Cattell, 1933, p. 183]. Carver gave the paper the same title as Dines's and it was published in the same journal. This was, of course, on purpose and Carver also began his paper by referring back to Dines's 1919 paper [Carver, 1922, p. 212]. As Carver stated in the introductory paragraph, Dines found a necessary and sufficient condition for the existence of solution to a system of linear inequalities in terms of the I -rank of the corresponding matrix. According to Carver his paper distinguishes itself from Dines's in two ways: First of all Carver was looking for necessary and sufficient conditions for the *non-existence* of a solution to such a system and second he wanted to do it "in a quite different form" [Carver, 1922, p. 212]. Carver also wanted to discuss two aspects of the subject not included in Dines's treatment, namely "the questions of the independence of a system and the equivalence of two systems" [Carver, 1922, p. 212].

Like Dines, Carver introduced his own terminology for dealing with the system of linear inequalities

$$\sum_{j=1}^n \alpha_{ij} x_j + \beta_i > 0, \quad i = 1, 2, \dots, m. \quad (32)$$

He introduced terms like *consistent*, *inconsistent*, and *irreducible inconsistent*. The system (32) was called consistent if it had a solution, otherwise Carver called it inconsistent. He used irreducible inconsistent to denote a system (32), which is inconsistent but is such that every subsystem that arrives from removing one inequality from the system is consistent.

Carver's main theorem in the first part of the paper is the following where he used $L_i(x)$ as short for $\sum_{j=1}^n \alpha_{ij}x_j + \beta_i$:

A necessary and sufficient condition that a given system S [(32)] be inconsistent is that there should exist a set of $m + 1$ constants, k_1, k_2, \dots, k_{m+1} , such that

$$\sum_{i=1}^m k_i L_i(x) + k_{m+1} \equiv 0,$$

at least one of the k 's being positive, and none of them being negative. [Carver, 1922, p. 217]

Carver's proof technique is quite different from Dines's. First of all he gave a non-constructive existence proof, in the sense that the proof does not give any hints about how to find the constants k_i . Their existence is secured by the fact that the matrix M of coefficients of an inconsistent system necessarily must have a rank less than m [Carver, 1922, p. 216–217].

In order to treat the question of equivalence between systems of linear inequalities Carver introduced the notion of *independent* systems, which means a system (32) that has no superfluous inequalities, and a *superfluous* inequality is an inequality which "is satisfied by every point which satisfies all the other inequalities of the system" [Carver, 1922, p. 218]. With these notions and some results on necessary and sufficient conditions for an inequality to be superfluous in an inconsistent as well as in a consistent system, Carver proved that

If two systems S [(32)] and Σ , each of which is independent and consistent, are equivalent, the number of inequalities in the two systems is the same, and each inequality of one system is equivalent to one and only one inequality of the other system; i.e., the inequalities of the two systems are identical except for possible positive constant factors. [Carver, 1922, p. 219]

Here Carver had defined two systems to be equivalent, if every point that satisfies one of the systems also satisfy the other one and vice versa.

In this paper Carver asked new questions to systems of linear inequalities and he also got new results not included in the treatment given by Dines. What we can see here is that the development of the subject continued. From questions regarding necessary and sufficient conditions for existence of solutions it broadened to involve necessary and sufficient conditions for the non-existence of solutions and also the structure of "different" systems was being explored, new terminology was added and new techniques were used in dealing with the subject.

4.5. Continuing work on a theory of linear inequalities

Inspired by Carver's paper Dines published a series of three papers dealing with systems of linear inequalities in one way or another. The papers were published in the *Annals of Mathematics* in 1925, 1926, and 1927 respectively. In these papers Dines took up Carver's discussion of the dependency/independency of systems of linear inequalities, he further developed the theorems on the non-existence of a solution, and finally

he applied the techniques developed to questions about positive solutions of a system of linear equations.

The main purpose of the first of these papers, sparked by Carver's results, was two-fold: First, to emphasize the notion of definite linear dependence, which was – again – a new term introduced by Dines and derived from Carver's notion of independent systems, and second, to compare his own previous paper from 1919 with Carver's. Dines defined m sets of n real constants

$$\begin{array}{cccc} a_{11}, & a_{12}, & \dots, & a_{1n}, \\ a_{21}, & a_{22}, & \dots, & a_{2n}, \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}, & a_{m2}, & \dots, & a_{mn}, \end{array} \quad (33)$$

to be *definitely linearly dependent* if there exist n identities of the form

$$c_1 a_{1r} + c_2 a_{2r} + \dots + c_m a_{mr} = 0$$

in which not all the c 's are zero, and all the nonzero ones have the same sign [Dines, 1925, p. 57]. Rewritten in terms of this definition Carver's main theorem states that a necessary and sufficient condition for the non-existence of solutions of the system (32) is that the m sets of the n coefficients in the system are definitely independent. Dines's main agenda in this paper was to show the equivalence of the result obtained by Carver and his own result from 1919. He set up the three conditions:

1. The m sets of n constants (33) is definitely linearly dependent.
2. The system of inequalities $a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n > 0$, ($i = 1, 2, \dots, m$) admit no solution.
3. The I -rank of the matrix of (33) is zero.

As Dines wrote himself:

The object of the present paper, in addition to emphasizing the notion of definite linear dependence, is to give a somewhat more direct approach to the equivalence of the three conditions 1, 2, 3, than is furnished by the two papers cited [(Dines, 1919a) and (Carver, 1922)]. [Dines, 1925, p. 57–58]

In this paper, Dines did not come forward with anything that was really new. He elaborated his previous results and compared them with Carver's. He did introduce a geometrical interpretation, where he wrote the system of inequalities as m scalar products between two n -dimensional vectors, but it led to nothing essentially new. Dines just redefined his notions of I -definite and I -rank in terms of vectors instead of matrices but basically he proceeded as he had done in the paper of 1919. As it was later noticed by Ruth W. Stokes even though Dines interpreted the coefficients of each inequality as a vector he did not "represent the solution geometrically" and he did not "employ a geometric method of reasoning" [Stokes, 1931, p. 784]. The paper's justification lies in its explanation of the relation between Carver's and his own work so far, and in the fact that Dines pointed towards the possibility of generalizations by replacing the coefficient matrix by a function $A(p, q)$ of two variables on general ranges [Dines, 1925, p. 58].

The following year Dines reformulated the equivalence between 1. and 2. above and gave it the following form:

A necessary and sufficient condition that the system of linear inequalities

$$\sum_{j=1}^m a_{ij}x_j > 0, \quad (i = 1, 2, \dots, n) \quad (34)$$

admit no solution (x_1, x_2, \dots, x_m) is that the associated system of linear equalities

$$\sum_{i=1}^n y_i a_{ij} = 0 \quad (j = 1, 2, \dots, m) \quad (35)$$

admit a solution (y_1, y_2, \dots, y_n) in which not every y_i is zero, and the non-zero ones are positive. [Dines, 1926a, p. 41]

With this reformulation, Dines proved a similar result where instead of the inequality sign in (34) he introduced a new sign \geq , which meant “is not less than, and for at least one value of the range of indices is definitely greater than” [Dines, 1926a, p. 41]. The effect of this is that the system of inequalities (34) can have equalities also, but for at least one of the forms strict inequality must hold. Dines showed that a necessary and sufficient condition that such a system admit no solution is that the associated system (35) of equations admits a solution (y_1, y_2, \dots, y_n) in which every y_i is positive. Dines also gave a property of the matrix of coefficients that is sufficient to ensure the condition in the theorem. As he pointed out, the property is very restrictive and not necessary, but as he phrased it “it has the merit of being easily detected when present” [Dines, 1926a, p. 42]. This shows Dines’s constructive approach once again. This property gives a tool which can sometimes be used to conclude that the system of inequalities admits no solution and that the associated system of equations has a solution where all the variables are positive.

Dines’s reformulation of Carver’s theorem clearly shows the duality aspect between systems of linear inequalities and associated systems of equations. It is an example of the type of theorems later named ‘transposition theorems’ by Motzkin because the matrices of coefficients that are involved are transposed to one another. For a discussion of this type of results in the history of linear inequalities see Sect. 5.4. This kind of theorems was re-derived in the beginning of the thirties and in the forties and played a significant role in the theory of mathematical programming and game theory, which will be explored in Sect. 6.

In his third paper, which was published in 1927, Dines developed an algorithm for determining positive solutions to systems of linear equations which can function as a tool for determining whether the corresponding system of inequalities has a solution or not [Dines, 1927a]. The way he wrote the main theorem in the previous paper clearly showed this connection between positive solutions to systems of equations and systems of inequalities, and as Dines wrote “This question [the sign of the components x_s of the solution (x_1, x_2, \dots, x_n) to a system of equations], conceivably important in certain types of applications, has found its way into several recent papers” [Dines, 1927a, p. 386]. The recent papers are of course the one by Carver and the earlier papers by Dines himself. It is interesting to note that there also is a reference to an abstract of a paper presented to the Mathematical Association in 1926 by Professor C. F. Gummer

from Queen's University in Canada [Dines, 1927a, p. 386]. At the tenth summer meeting of the Mathematical Association of America Gummer gave a talk entitled "Sets of linear equations in positive unknowns". He never published the paper, but the abstract was published in *American Mathematical Monthly* in a report of the meeting. Apparently Gummer gave a necessary and sufficient condition for the existence of positive solutions to a system of homogeneous equations. He also "extended the result to finite systems of equations with a denumerably infinite set of unknowns, and also to the continuously infinite case where the sums of terms are replaced by definite integrals" [Gummer, 1926, p. 488]. Dines's reference to Gummer occurs in a footnote where he explained that "This abstract appears in the Amer. Math. Monthly of December 1926, and came to the author's notice after the present paper was written" [Dines, 1927a, p. 386]. The main result in Dines's paper is an algorithm for determining a positive solution when one exists to a system of linear equations.

4.6. *Motivation and goal: The context of American mathematics in the Twenties*

Dines's first impulse for studying linear inequalities was provoked by Lovitt's paper on preferential voting but it seems that he was mainly inspired by the *lack* of a theory dealing with those kind of systems. He repeatedly compared the (lacking) theory of linear inequalities with the theory of linear equations about which he expressed the following opinion:

Among the most elegant chapters of algebra is undoubtedly that one which treats the solution of a system of linear equations . . . from the point of view of the matrix of coefficients." [Dines, 1927a, p. 386]

And, as was pointed out above, in his first paper devoted exclusively to systems of linear inequalities Dines wrote about the missing terminology to characterize solutions and the lack of an algorithm, that

This is in marked contrast to the analogous situation in the case of the system of equations formed by replacing the symbol $>$ by the symbol $=$ " [Dines, 1919a, p. 191]

This impression is reinforced in the invited talk "Linear Inequalities and some Related Properties of Functions" that Dines gave at the Joint Meeting of the Mathematical Association of America and the American Mathematical Society in December 1929 [Dines, 1930]. In the introductory paragraph of this talk Dines emphasized the difference between the stage of the theoretical development in systems of linear equations and systems of linear inequalities:

If one attempts such a discussion [of systems of linear inequalities in more than three unknowns], he will almost certainly contrast the lack of available formal machinery, with the elegant theory of the system of *equations* which results from [a system of linear inequalities] if the inequality signs be replaced by equality signs. Yet it is obvious that for inequalities as for equations, the entire body of facts is inherited in the matrix of coefficients. [Dines, 1930, p. 393]

And, as we have seen above, by defining the concept of I -rank he proceeded in the spirit of the theory of linear equations.

This work of Dines and Carver on linear inequalities apparently took place without any applications in mind, either inside or outside mathematics proper, as was not the case with the work of Minkowski and Farkas. In itself, the subject does not seem to have had any significance in the broader context of mathematics at that time and it could hardly have been considered to be extremely important for its own sake. It seems to me that one cannot avoid the obvious question: Why did Dines pursue this subject of linear inequalities?

The second part of his invited talk mentioned above gives a hint of a possible answer to that question. Dines devoted that part of the talk to discussions of possible extensions and generalizations and explained his “hopes” for the theory of linear inequalities

The theory of systems of linear algebraic equations has led in various ways to theories of systems of linear equations in infinitely many unknowns, and to theories of linear integral equations. It is natural to inquire whether the theory of algebraic *inequalities* can be extended in similar fashion. To put the question, for preliminary examination, in a very general form suggested by the work of Professor E. H. Moore, one may ask what can be said of an inequality of the form

$$J_q \alpha(p, q) \xi(q) > 0, \quad (p \text{ on range } P), \quad (36)$$

[...] where p and q are variables whose ranges are classes of elements P and Q respectively, $\alpha(p, q)$ is a given real-valued function, $\xi(q)$ is the unknown function to be determined, and J_q is a linear operator of more or less general nature. [Dines, 1930, p. 399]

In fact, Dines had worked on this generalization since at least 1926 and his efforts can be followed in the reports of various meetings published in the *Bulletin of the American Mathematical Society*.³⁶

It seems that in the long run the ultimate goal for Dines was this generalization of the finite theory, and considered in relation to the current trend in the mathematical research at the time and place this was not odd at all. In 1916 Edward Van Vleck wrote a paper for the *Bulletin* on “Current Tendencies of Mathematical Research” where he posed the question: “What is the dominant problem or central thought in the research of to-day, if there be one?” For pure mathematics, which was the only kind of mathematics in the USA at the time really appreciated³⁷ he gave the following answer:

In pure mathematics, to which I shall confine my attention, the number of conspicuous problems is legion, but above them all there looms, I think, in manifold aspects the problem of the *infinite set*. [Van Vleck, 1916, p. 2]

And in his guidance to the mathematicians in selecting a worthy problem to work on he emphasized the value of generalizing, as he wrote: “Mann muss immer generalisieren” by which he did not “mean cheap generalization to n dimensions or variables of that which has been already done for two or three. Not infrequently, however, generalization

³⁶ See [Dines, 1926b, 1927b, 1928a].

³⁷ See [Parshall, 2000], [Prager, 1972].

even to n variables is a problem of importance and difficulty, inasmuch as the solution for the special case of two or three variables may possess a distinctive structure and character and hence be in no way typical. . . . But whether or not extension to n dimensions be trivial, generalization to a countably infinite number of dimensions becomes sublime!" [Van Vleck, 1916, p. 3]. He elaborated further on this through examples and concluded:

I trust that you see that the Fischer-Riesz theorem affords an elegant example of modern mathematical research – characteristic in its generalized sweep, . . . [Van Vleck, 1916, p. 13]

And once more in the same paragraph: “. . . and the combination of sweeping generalization with rigor is astonishing.” To rule out any doubt of this doctrine in pure mathematics van Vleck explicitly referred to Eliakim H. Moore:

An excellent program for work could be found in extension of almost any finite theory. It is your own Professor Moore whom I have heard glowingly preach that to every finite theory there must correspond, under proper limitations, a general transcendental theory with an infinite number of variables. [Van Vleck, 1916, p. 4]

As an example van Vleck gave the generalization of system of linear equations in n variables to the theory of integral equations. As the quotation above (p. 504) showed this was exactly the program or “hope” that Dines, following Moore, expressed for the theory of inequalities.

Eliakim H. Moore (1862–1932) was an extremely influential person in modern American mathematics.³⁸ In [Parshall, 2000, p. 388] he is characterized as having been “instrumental in spearheading America’s move into pure mathematics.” Dines’s work on linear inequality theory seems to be a prime example of the above characterization of the trends in mathematical research at that time and Dines was also fully familiar with the ideas of Moore. He had himself been a student at Chicago, where Moore was the leading mathematician who, together with Bolza and Maschke, created the mathematics department at Chicago and turned it into one of the best in the USA. Dines wrote his thesis “The Highest Common Factor of a System of Polynomials in One Variable” [Dines, 1913]³⁹ at Chicago under the supervision of Gilbert A. Bliss, one of Moore’s many students. Moore’s influence on Dines can be followed in Dines’s publications. In December 1916 and September 1917 Dines presented some results to the American Mathematical Society which he published in the paper “Projective Transformations in Function Space” [Dines, 1919b] where he considered a general projective transformation which has as a special case an analog of the Fredholm transformation

$$\phi'(x) = \phi(x) + \int_0^1 \gamma(x, y)\phi(y)dy,$$

considered as a geometric transformation of a point $\phi(x)$ of function space into a point $\phi'(x)$ of the same space. In the paper Dines studied an analogue of this transformation in

³⁸ See [Parshall and Rowe, 1994], [Parshall, 2000], [Duren, 1989].

³⁹ The thesis was read before the American Mathematical Society on October 28 in 1911, but was not published until 1913.

function space [Dines, 1919b, p. 45]. He finished the paper in the spirit of Moore with a section titled “A Suggested Generalization” where he referred to a paper by Moore “On the Foundations of the Theory of Linear Integral Equations” [Moore, 1911–12]. In this paper, which is a part of what is called Moore’s “General Analysis”,⁴⁰ Moore developed a generalized Fredholm transformation. In 1927 Dines published another paper “On Sets of Functions of a General Variable” which begins with a reference to Moore’s “General Analysis”: “A theory of Functions of a general variable is due to E. H. Moore” [Dines, 1927c, p. 463]. In the paper Dines investigated functions which have the property that they are somewhere positive and nowhere negative on D or the property that they are somewhere negative and nowhere positive on D , where D is the range of the general variable, which again means that D is a class of elements entirely unconditioned [Dines, 1927c, p. 463]. This paper is a generalization in the spirit of Moore of the theory of linear inequalities.⁴¹

Dines’s work on linear inequalities was a part of this tendency towards generalizations, first as a generalization from systems of linear *equations* to systems of linear *inequalities* and second, as a generalization from finitely many unknowns to the continuum. Hence, the motivation and goals behind the development of a theory of systems of linear inequalities for their own sake in USA in the twenties can be understood in this context of the self-understanding of mathematical research at that time and place.

5. The circles are spreading – The threads are gathered The final establishment within the theory of convexity and the work of Theodore Motzkin

In contrast to the work of Farkas and Minkowski, none of which, except from the paper by Haar, resulted directly in further investigations on linear inequalities, Dines’s work reached a broad audience. His work was published in the *Annals of Mathematics* and most of his results were also announced in the *Bulletin of the American Mathematical Society*. Especially the invited talk at the Joint AMS-MAA meeting indicates that mathematicians were becoming aware of Dines’s work and that the subject was considered interesting enough. The talk also had the effect that a lot of mathematicians became exposed to the subject and to the theory that had been developed so far by Dines and Carver. Their work slowly began to inspire other primarily younger people in the USA who began to work on the subject of systems of linear inequalities. And, as will be seen in this section, other people also from outside the USA began to comment on the work in the field, and to draw attention to connections with other kinds of work. As the older literature was discovered connections were realized, and the final establishment of the subject within the theory of convexity took place.

⁴⁰ See [Siegmond-Schultze, 1998] for a discussion of Moore’s “General Analysis” and its failure.

⁴¹ For further evidence of Dines’s effort to generalize his theory of systems of linear inequalities in n variables see [Dines, 1926/27, 1928b, 1928c].

5.1. The Japanese

The first reaction to the theory developed by Dines and Carver came from outside the USA. In two papers published in 1928 and 1930 by the Japanese mathematician Matsusaburō Fujiwara the subject of linear inequalities was connected with some earlier works by S. Kakeya, another Japanese mathematician, and with a paper by A. Huber from Freiburg.

The purpose of the first of these papers was not so much to add new results to the theory already developed by Dines and Carver as to direct attention to the work done by Kakeya on the problem of linear differential inequalities, and to express the opinion that the three problems: Systems of linear inequalities, linear integral inequalities, and linear differential inequalities “belong to the same category” [Fujiwara, 1928, p. 330].

Even though Fujiwara did not refer to Minkowski in his paper his list of publications reveals that he knew Minkowski’s work on the geometry of numbers, and it also shows that he was quite familiar with the theory of convexity.⁴² Whether he was aware of Haar’s work or not is unclear but, like Haar, he used the theory of convexity and especially the idea of center of mass to “solve the problem of linear inequalities” [Fujiwara, 1928, p. 330]. In order to clarify the connections between the three problems mentioned above Fujiwara rewrote the system of linear inequalities as a problem about systems of linear equations.

He considered the system of equations

$$l_j(u) = a_{j1}u_1 + a_{j2}u_2 + \dots + a_{jn}u_n = b_j \quad j = 1, \dots, m \quad (37)$$

where $a_{ij}, b_j \in \mathbf{R}$ for $i = 1, \dots, n, j = 1, \dots, m$, together with the adjoint system of linear equations

$$M_i(v) = a_{i1}v_1 + a_{i2}v_2 + \dots + a_{in}v_n = 0 \quad i = 1, \dots, n . \quad (38)$$

He proved that if $v^{(\lambda)} = (v^{(\lambda)}_1, v^{(\lambda)}_2, \dots, v^{(\lambda)}_n)$, $\lambda = 1, 2, \dots, s$, where $s = m - r$, r being the rank of the matrix (a_{ik}) , are linearly independent solutions of the adjoint system (38) then (37) has a solution if and only if (b_1, b_2, \dots, b_m) satisfies the conditions

$$\begin{aligned}
 &b_1 v^{(1)}_1 + b_2 v^{(1)}_2 + \dots + b_m v^{(1)}_m = 0 \\
 &b_1 v^{(2)}_1 + b_2 v^{(2)}_2 + \dots + b_m v^{(2)}_m = 0 \\
 &\dots\dots\dots\dots\dots\dots\dots \\
 &b_1 v^{(s)}_1 + b_2 v^{(s)}_2 + \dots + b_m v^{(s)}_m = 0 .
 \end{aligned} \quad (39)$$

He then applied this result to the system

⁴² See for example [Fujiwara, 1915a, 1915b, 1915c].

$$\begin{aligned}
 a_{11}u_1 + a_{12}u_2 + \dots + a_{1n}u_n &\geq 0 \\
 a_{21}u_1 + a_{22}u_2 + \dots + a_{2n}u_n &\geq 0 \\
 &\dots\dots\dots \\
 a_{m1}u_1 + a_{m2}u_2 + \dots + a_{mn}u_n &\geq 0
 \end{aligned}
 \tag{40}$$

of linear inequalities and obtained the following result: (40) has a solution if and only if (39) has the nonnegative solution $b_1, b_2, \dots, b_m \geq 0$ [Fujiwara, 1928, p. 331].

Fujiwara made a geometrical translation of this theorem in the framework of convexity:

A necessary and sufficient condition for the existence of a solution of [(40)] is that the origin lies in or on the boundary of the least convex polyhedron in the s -dimensional space, which contains the m points $v^{(\lambda)}, \lambda = 1, 2, \dots, s$. [Fujiwara, 1928, p. 331]

In a similar way Fujiwara also formulated the corresponding result concerning the non-existence of solutions to (40) in the framework of convex sets and got the following result

For the inconsistency of [(40)] ... it is necessary and sufficient that the origin lies within or on the boundary ... of the least convex polyhedron K' which contains m points Q_1, Q_2, \dots, Q_m whose Cartesian coordinates are respectively

$$(a_{i1}, a_{i2}, \dots, a_{in}).$$

[Fujiwara, 1928, p. 331]

As pointed out by Fujiwara, the condition given by Dines for the inconsistency of (40) is that (38) has positive solutions $v_1, v_2, \dots, v_m > 0$. Fujiwara also interpreted the work of Eric Stiemke in this framework as a “treatment of the inverse problem” [Fujiwara, 1928, p. 331]. This is the first occurrence of a reference to Stiemke’s result in the theory of linear inequalities. I will return to the work of Stiemke in Sect. 5.4.

Fujiwara treated the linear integral inequality

$$L(\phi) = \phi(x) - \lambda \int_a^b K(x, y)\phi(y)dy \geq 0$$

in a similar way, and reached the conclusion that the necessary and sufficient condition for the existence of a solution to the integral inequality is a direct consequence of a theorem concerning a system of integral equations proved by Takeya in the paper “On Some Integral Equations” published in 1913–14 [Takeya, 1913–14, p. 189–190].

In the second paper Fujiwara took up the discussion of a result that A. Huber had used in the paper “Eine Erweiterung der Dirichletschen Methode des Diskontinuitätsfaktors und ihre Anwendung auf eine Aufgabe der Wahrscheinlichkeitsrechnung” to characterize a region of integration given by a system of linear inequalities. Huber gave a proof due to his teacher Furtwangler, of the following theorem: If at least one of r linear forms

$$L_i(x) = \sum_{k=1}^n a_{ik}x_k, \quad (i = 1, 2, \dots, r)$$

becomes positive for any system of non-negative $x_1, \dots, x_n \geq 0$, then a positive solution $p_1, \dots, p_r > 0$ to the system of inequalities

$$M_i(p) = \sum_{k=1}^r a_{ki} p_k > 0, \quad (i = 1, 2, \dots, n)$$

exists [Huber, 1930, p. 58–59]. Fujiwara proved that the condition given by Hubert and Furtwangler is not only sufficient but also necessary for the existence of positive numbers p_1, p_2, \dots, p_r such that $M_i(p) \geq 0$ for $i = 1, 2, \dots, n$ [Fujiwara, 1930, p. 297–298]. Again, Fujiwara reached the result quite easily using the technique from his former paper about the relation between the origin and the smallest convex set containing the points of coefficients.

The new thing with Fujiwara is not so much the treatment of the generalization, which also had been touched by Dines, but more that he reformulated the results in the framework of convex sets in a more direct way than Haar had done, and which clearly showed that the existence of a solution to a system of linear inequalities depends solely on the question whether or not the origin lies in the smallest convex set containing the points of coefficients of the system.

5.2. The Americans

The next contribution to the theory of linear inequalities came from Ruth W. Stokes, who took up the subject for her Ph.D. thesis at Duke University. She published an edited version of her thesis in *Transactions of the American Mathematical Society* in the summer of 1931 [Stokes, 1931]. Stokes's agenda was to present a geometric theory of solution of linear inequalities. She referred to Dines, Carver and Lovitt as her background literature indicating quite clearly that she was working in a direct line from Dines. In addition to that she also has a reference to the work of Minkowski, which is the first time his name appears in the line of research on the subject initiated by Dines. To make the list of references complete Stokes also mentioned the two papers by Fujiwara but emphasized that she was not familiar with the work of Fujiwara [Stokes, 1931, p. 784]. In her introduction Stokes briefly discussed the work of Dines and Minkowski in order to justify and distinguish her own contribution from theirs. The object of Stokes's investigation is the system

$$\sum_{i=1}^n a_{ij} x_i \geq 0 \quad j = 1, \dots, N \quad (41)$$

where the a_{ij} 's are real numbers and the x_i 's are the unknowns.⁴³ She developed her version of the theory in such a way that without further complications it gives a solution to any kind of system that can be made by combining (41), systems of the same form where strict inequality holds, and systems of the form $\sum_{i=1}^n a_{ij} x_i = 0$, $j = 1, \dots, N$.

⁴³ Stokes used a different notation for the coefficients and the unknowns, x_i^j and λ^i respectively. I have changed the notation so it coincides with the notation used by Dines and Minkowski in order to avoid confusion.

In that way she obtained a theory where Minkowski's work, Dines's work, and the theory of linear homogeneous equations can be considered as three extreme cases.

Even though Stokes in this way can be said to encompass the work of Dines and Minkowski in this slightly more general theory brought about by the unification of the three systems mentioned above this is not her main contribution to the theory of linear inequalities. The innovative element in Stokes's contribution to the literature she is building on is the *method* she invoked in developing the theory and the numerical solution method she derived.

She represented the system of inequalities (41) by the number N of points $(a_{i1}, a_{i2}, \dots, a_{in})$, $i = 1, \dots, N$, that is, she represented each inequality by a point in n -space, whose coordinates are the coefficients of the inequality. The set of points representing the given system was named S_n . With this interpretation Stokes reformulated the solution concept to (41) in terms of oriented hyperplanes through the origin, since to any non-trivial solution x of (41) there corresponds an oriented $(n - 1)$ -flat

$$\sum_{i=1}^n a_i x_i = 0$$

such that none of the points of S_n lies on the negative side of it, that is such that the corresponding linear form is non-negative for every point in S_n . Conversely, the coefficients (a_1, \dots, a_n) of any oriented $(n - 1)$ -flat with this property is a non-trivial solution to (41) [Stokes, 1931, p. 785]. Stokes's geometrical interpretation is the same as Haar's, but she was probably not aware of Haar's work, which was published in the Hungarian journal *Acta Litterarum Ac Scientiarum*. Stokes developed most of her results without using the notion of convexity and only at the end of her paper did she state her necessary and sufficient conditions for a solution in the context of convex sets and derive results similar to those of Fujiwara [Stokes, 1931, 803–805].

On the basis of her theory Stokes also developed a numerical method for determining the fundamental solutions of a given system. She constructed an array in which "each column is headed by a combination of the indices of the points of the set $[S_n]$ taken $(n - 1)$ at a time, each combination being written once and only once . . . The rows are numbered with the indices of the points of the set $[S_n]$." The entries in the array are the signs of the determinants having the coordinates of the involved points as its rows. From this array the fundamental solutions can be read off immediately, since the combination of points for which the corresponding column does not contain any variation in sign will determine a fundamental solution as long as these points do not lie in an $(n - 2)$ -flat [Stokes, 1931, 795].

The last work directly inspired by the work of Dines was published in the *American Mathematical Monthly* in 1932 and was done by Helen M. Schlauch, who was at Hunter College at the time [Schlauch, 1932]. The object of Schlauch's work was to extend the theorems by Dines and Carver on necessary and sufficient conditions for the existence and non-existence of solutions to a mixed system containing both equations and inequalities. Apparently Schlauch was not aware of the work of Stokes.

By then, the theory of systems of finitely many linear inequalities in finitely many unknowns had reached a kind of mature level due to the development initiated and continued by Dines, and in 1933 Dines wrapped it all up in a paper written together with N. H. McCoy, which can be classified as a "state of the art" paper [Dines and McCoy,

1933]. They listed the different kinds of systems that had been treated by the different authors and discussed the respective methods used, and all of the people who have been discussed so far in this paper are included among the authors. The last 14 pages they devoted to a discussion of generalizations and here again the influence from Moore's General Analysis is evident.

*5.3. The final establishment within the theory of convexity
and the work of Theodore Motzkin*

Dines and McCoy briefly discussed the significance of the smallest convex region containing a given set of points in their 1933 paper but most of their discussion was still concentrated around the analytical method developed by Minkowski, and Dines's notion of the I -rank of the corresponding matrix of coefficients. Stokes has also placed her results, formulated in the context of convexity, at the end of her paper under the subheading "Another form of the necessary and sufficient condition for a solution" [Stokes, 1931, p. 803]. The first presentation after Haar's of a theory of linear inequalities completely imbedded in the theory of convexity appeared in 1935 in Hermann Weyl's (1885–1955) paper "Elementare Theorie der konvexen Polyeder" [Weyl, 1935]. The object of Weyl's paper is the fundamental theorem concerning convex closures of closed bounded point sets S in n -dimensional space. The points of the convex closure of such a set S can be characterized either as the centers of gravity of at most $n + 1$ points from S or as the points that belongs to all of the supports of S , where by a support of S Weyl understood a halfspace

$$\alpha_1 x_1 + \dots + \alpha_n x_n + \alpha \geq 0$$

containing all the points of S [Weyl, 1935, (1950, p. 3)]. According to the fundamental theorem these two characterizations are equivalent. Weyl was interested in the case where the set S consists of a finite number of points and he wanted to derive the fundamental theorem for this case not by set-theoretic methods, as it is done for the general case, but with finite methods [Weyl, 1935, (1950, p. 3)]. As he wrote in the introduction to the paper:

It must be possible to derive the fundamental theorem for this case by finite methods. . . . There appears to be a gap here in the literature which should be filled; it is for this reason that I publish this little note which I had occasion to write down in the summer of 1933 for my last seminar in Göttingen, which had as its subject convex bodies. [Weyl, 1935, (1950, p. 3)]

In this note Weyl gave the following characterization of the theory of systems of linear inequalities:

What we will consider could also be called an elementary theory of finite systems of linear inequalities. [Weyl, 1935, (1950, p. 3) Weyl's emphasis]

Here Weyl clearly understood the subject of finite systems of linear inequalities as the study of the convex closure of a set containing finitely many points and he derived results like Farkas's lemma, the presentation of solutions as non-negative linear combinations of extreme solutions, and the duality between a system of linear inequalities and the system of inequalities obtained by considering the extreme solutions of the first system

as coefficients in the second (dual) system as consequences of the fundamental theorem for convex closures of closed bounded point sets [Weyl, 1935, (1950, p. 9–13)].

A year later, in 1936, Dines published his last paper – a talk given at an AMS meeting in November 1935 – solely devoted to systems of linear inequalities [Dines, 1936]. In the introduction Dines wrote:

Today my purpose is to focus attention on geometric aspects of the theory [of systems of linear inequalities], and in particular to show its close relationship to a certain geometric notion which in recent years has been useful in many investigations in analysis. [Dines, 1936, p. 353]

The notion to which he is referring is, of course, the notion of convex sets. He used the name “convex extension” about the convex hull of a set. Dines was at this point familiar with Weyl’s paper and with Fenchel and Bonnesen’s “Theorie der Konvexen Körper” from 1934, so after indicating how the theory of linear inequalities can be derived from the fundamental theorem about convex extensions he proceeded to give a discussion of the generalizations in this framework of convex extensions. He finished the paper by deriving the theorem of Carver and his own “transposition-version” of it from 1926 (see p. 502) as a simple consequence of the equivalency between the two ways of characterizing the convex extension, and he concluded

... unfortunately unknown to either of us [Carver and himself], Stiemke had obtained the essence of both [for systems of strict inequalities and systems with ≥ 0] in the *Mathematische Annalen* of 1915. All of the purely analytical proofs were quite complicated. [Dines, 1936, p. 364]

This is the second time a reference to Stiemke appeared in the literature. In Sect. 5.4 I will return to a discussion of these earlier versions of what, after the work of Motzkin, became known as the transposition theorem.

The last contribution to the theory of systems of linear inequalities to be discussed in this section is the thesis of Theodore Motzkin (1908–1970), which was published in 1936. Historically, Motzkin’s thesis is important for at least two different reasons: On the one hand, it can be said to be a kind of culmination of the whole development of a theory of systems of linear inequalities up to 1935, and on the other hand, it became a main inspiration for the development that took place in a completely different context after the second world war.

Motzkin was born in Germany and he mainly studied in Göttingen and Berlin. He was Jewish and he wrote his thesis in Basel under the supervision of Ostrowski.⁴⁴ He finished his thesis in 1934 but it didn’t get published until 1936. At that time Motzkin held a position at the Hebrew University in Jerusalem where he published the thesis. Motzkin came to the USA after the war and from 1950 he held a position at UCLA.

The title of Motzkin’s thesis is “Beiträge zur Theorie der linearen Ungleichungen” and in the introduction he indicated the motivation behind this choice of subject [Motzkin, 1936]. According to Motzkin it goes back to Descartes’s rule of signs which

⁴⁴ For biographical information about Motzkin see the introduction in [Cantor et al., 1983] and the Obituary in *Journal of Combinatorial Theory*, (A), 14, 1973, pp. 271–272.

states that the number of positive roots in a polynomial can never exceed the number of alterations in sign of the coefficients. The central question is the number of times the sign changes in a sequence of numbers. If the numbers are considered as variables one can ask how the number of alterations in sign changes under linear transformations of the variables. This question was discussed by Fekete and Polya in a series of letters [Fekete and Polya, 1912]⁴⁵ and in 1930 I. Schönberg solved the central problem of determining all the linear transformations which do not increase the number of alterations in sign for the situation where the rank of the matrix of coefficients corresponding to the transformation equals the number of independent variables [Schönberg, 1930]. In his thesis Motzkin solved this problem for the general case, and in order to decide whether a linear transformation is a so-called variation diminishing transformation it is enough – as pointed out by Motzkin – to know the sign of all the minors of the matrix of coefficients. Systems of linear inequalities entered the picture because knowledge about these signs of the minors also is sufficient to answer a much more general question about conditions for the solvability of such systems [Motzkin, 1936, p. 1].⁴⁶ The major part of Motzkin's thesis is devoted to a thorough investigation of systems of linear inequalities and in the last part he turned his attention to the variation diminishing transformations.

I gave a dual characterization of Motzkin's work above as both a final culmination of the developments until Motzkin, and a source of inspiration for new developments. While the latter will be discussed in Sect. 6 the former will be dealt with here. First of all, Motzkin did an extensive and I think almost complete search of literature. He found all the older sources directly dealing with a theory of systems of linear inequalities, the literature containing isolated results about linear inequalities as well as the literature in themes connected with his own investigations.⁴⁷ Motzkin was aware of all the literature that has been discussed so far including a reference to the paper by Weyl. Motzkin dated Weyl's paper wrongly as having been published in 1934 but apparently the paper was not received until March 1935, that is after Motzkin had finished his thesis;⁴⁸ whether Motzkin knew the work of Weyl before he finished his thesis is unclear.

Motzkin's list of literature encompassed all the previous literature on systems of linear inequalities and in his introduction he gave a brief commentary on some of the earlier works including a comparison of the work of Fourier with a contribution made by Paul Gordan (1837–1912). This was the first time Gordan's name appeared in the context of linear inequalities and Motzkin wrote about it, that:

Von solcher Anschaulichkeit [as Fourier's] war Gordan, der nach fünfzig Jahren durch invariantentheoretische Untersuchungen ziemlich nebenbei auf arithmetische Fragen verwandter Natur kam, weit entfernt.

⁴⁵ The paper in *Rendiconti del circolo Matematico di Palermo* displays the correspondance.

⁴⁶ Schönberg himself also investigated linear inequality systems but with infinitely many variables, connecting the subject to work of Hausdorff and the theory of the moment problem. It is beyond the scope of this paper to treat these contributions to this extension of systems of linear inequalities in finitely many variables. For references see [Schönberg, 1932, 1933, 1934].

⁴⁷ See Motzkin's list of references in [Motzkin, 1936, p. 7–11].

⁴⁸ See [Weyl, 1935, p. 306].

Gordan sprach den eleganten Transpositionssatz in verbrämter Form aus und bewies ihn ziemlich umständlich, specialisierte sich dann aber auf diophantische Probleme ...⁴⁹ [Motzkin, 1936, p. 5]

Before I discuss the earlier versions of the theorem, which Motzkin named the transposition theorem, including his evaluation of Gordan's work, I will describe Motzkin's own contribution.

Motzkin himself pointed out that the idea of looking at systems of inequalities for all combinations of sign at the same time is new. He wrote the general inhomogeneous system in the form

$$xA + c \in V$$

where A is a coefficient matrix, c is a column of constants, and V is a m -dimensional "sign range" (Vorzeichenbereich), which is uniquely determined by the given combinations of sign [Motzkin, 1936, p. 19]. By a "sign range" Motzkin understood a region that is formed by restricting some how or other the coordinates of \mathbf{R}^m through the conditions $=, > 0, < 0, \geq 0,$ and ≤ 0 .

Motzkin proved the existence of a basis for the solution of a system of inequalities for all combinations of sign, and he also proved the very important "Transpositionssatz" – as he named it – in the most general form:

A system σ and its transposed system σ' are complementary in the sense that precisely one of them is solvable. [Motzkin, 1936, p. 51]

Here the system σ , and the transposed system σ' , have the following forms

$$\sigma : \begin{aligned} \sum_{v=1}^n a_{jv}x_v &> 0 \quad (j = 1, \dots, m_1) \\ \sum_{v=1}^n a_{iv}x_v &\geq 0 \quad (i = m_1 + 1, \dots, m_2) \\ \sum_{v=1}^n a_{kv}x_v &= 0 \quad (k = m_2 + 1, \dots, m) \end{aligned}$$

and

$$\sigma' : \begin{aligned} \sum_j a_{jv}y_j + \sum_i a_{iv}y_i + \sum_k a_{kv}y_k &= 0 \quad (v = 1, \dots, n) \\ y_j \geq 0, \quad y_i \geq 0, \quad (y_1, y_2, \dots, y_{m_1}) &\neq (0, 0, \dots, 0) \end{aligned} \quad (42)$$

Besides this very general form of the transposition theorem, whose full significance became clear only at the end of the 1940s (see Sect. 6), Motzkin's thesis is also important

⁴⁹ "Gordan, who fifty years later was rather incidentally led to similar arithmetic questions through his invariant theoretic investigations, was far removed from such intuition [as Fourier's]. He stated the elegant transposition theorem in disguised form and proved it in a roundabout way, but then confined himself to Diophantine problems." The last part of the translation is taken from D. R. Fulkerson's translation of Motzkin's thesis in [Cantor et al., 1983, p. 7].

in the history of linear inequalities because it was – as also pointed out by the editors of Motzkin’s collected works – the first coherent synthesis of all the work previously done in this field.⁵⁰ Motzkin’s thesis is important historically for yet another reason. With his work the establishment of the theory of linear inequalities within the theory of convexity which began with Haar and continued with Weyl was fully completed. In Motzkin’s work the theory of convexity runs underneath the whole presentation and he very carefully stated all the results in both the analytical form and the corresponding geometrical form. All the way through he offered the geometrical interpretations in the framework of the theory of convexity.

Before the discussion in the next section of the development that took place in the post war period and the significance of Motzkin’s thesis in that context, especially his transposition theorem, I will briefly – to render this account more complete – discuss the earlier versions of the important transposition theorem, especially Motzkin’s commentary about Gordan’s “disguised” transposition theorem.

5.4. The “disguised” transposition theorem

As we saw above (p. 502) in 1926 Dines reformulated Carver’s theorem about the conditions for the non-existence of a solution to a system of linear inequalities in such a way that the duality between the system of inequalities and an adjoint system of equations became transparent. Ten years later Dines realized that this theorem can be derived as a simple consequence of the fundamental theorem concerning convex closures – or extensions – of closed bounded point sets (see p. 512). Stiemke and Gordan have also been given credit for deriving theorems of this type.⁵¹ In the following I will discuss these different versions of “transposition” theorems in their original context.

In [Tucker, 1956, p. 9] Tucker wrote about Gordan’s result that “it seems to be the earliest known instance of a “transposition theorem”,” and as we saw above (p. 514) Motzkin called Gordan’s result the transposition theorem in “disguised” form. The result they are referring to is a theorem concerning the existence of positive solutions to a system of linear equations, which Gordan had proved in the paper “Ueber die Auflösung linearer Gleichungen mit reellen Coefficienten ” published in 1873 [Gordan, 1873]. As the title indicates Gordan was interested in properties regarding systems of linear equations – not inequalities. Gordan considered the following system

$$\begin{aligned} X_1 &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1,r}x_r = 0 \\ X_2 &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2,r}x_r = 0 \\ &\dots \\ X_s &= a_{s1}x_1 + a_{s2}x_2 + \cdots + a_{s,r}x_r = 0 \end{aligned} \tag{43}$$

⁵⁰ See [Cantor et al., 1983, p. xiv].
⁵¹ See [Fenchel, 1983, p. 122], [Motzkin, 1936, p. 5], and [Tucker, 1956, p. 3].

where $a_{ik} \in \mathbf{R}$ for all i and k , and the task he set himself was to examine under which circumstances a system of linear equations has a positive solution [Gordan, 1873, p. 23]. He proved that the system (43) does not have a positive solution if and only if it is possible, by forming linear combinations of the left hand sides of the equations in (43), to make an equation of the form

$$F = A_1x_1 + \dots + A_rx_r = 0 \tag{44}$$

where $A_i > 0$ for all i . It is, as Gordan also implied, obviously true that if it is possible to make an equation like (44) then the system (43) has no positive solution. The “if” part of the theorem is not so obvious and the main part of the paper is devoted to the task of proving that part.

Gordan proved the theorem by induction on the number of equations in the system (43). If the system consists of only one equation, that is if $s = 1$, the theorem is obviously true. Under the assumption that the theorem is true for systems with fewer equations than s , Gordan first proved that if it is possible to make an equation with positive coefficients where only some of the x_i 's are present, that is an equation of the form

$$G = \alpha_{\nu+1}x_{\nu+1} + \alpha_{\nu+2}x_{\nu+2} + \dots + \alpha_r x_r = 0, \tag{45}$$

then it is also possible to make an equation of the desired form (44), where all the x_i 's are present and all the coefficients are positive. Gordan's proof of this is straightforward using the assumption of the induction. In order to prove that an equation of the type (45) can be made from (43) Gordan distinguished between three cases: 1) One of the equations can be made from the others; 2) The equations are linearly independent and $s \geq r$; and 3) The equations are linearly independent and $r - s = k > 0$. Only for the third case Gordan needed a longer argument: He did not use any sophisticated mathematics though, the only result he used is from the theory of determinants from which he knew that because of the linear independence there is at least one determinant of the form

$$\begin{vmatrix} a_{1,k+1} & a_{1,k+2} & \dots & a_{1,r} \\ a_{2,k+1} & a_{2,k+2} & \dots & a_{2,r} \\ \vdots & \vdots & \ddots & \vdots \\ a_{s,k+1} & a_{s,k+2} & \dots & a_{s,r} \end{vmatrix} \tag{46}$$

which will not vanish. Gordan used this fact to solve the system of equations (43) with respect to

$$x_{k+1}, x_{k+2}, \dots, x_r.$$

Letting y_1, y_2, \dots, y_s denote this solution Gordan reached a new system

$$\begin{aligned} Y_1 &= y_1 + b_{11}x_1 + b_{12}x_2 + \dots + b_{1,k}x_k = 0 \\ Y_2 &= y_2 + b_{21}x_1 + b_{22}x_2 + \dots + b_{2,k}x_k = 0 \\ &\dots\dots\dots \\ Y_s &= y_s + b_{s1}x_1 + b_{s2}x_2 + \dots + b_{s,k}x_k = 0 \end{aligned} \tag{47}$$

which is equivalent to the system (43) [Gordan, 1873, p. 25]. In order to prove the theorem for the system (47) he divided the system into classes in the following way: Class 0 consists of systems for which all the coefficients

$$b_{11}, b_{12}, \dots, b_{1,k}$$

in the first equation Y_1 , vanishes; class 1 consists of those systems for which only one of these coefficients does not vanish etc. Gordan then proved the theorem by induction on classes, going through the different cases considering if all the nonzero coefficients b_{1i} are positive, if they are all negative, or if they have different sign.

Gordan did not give any motivation for this investigation of the circumstances under which a system of linear equations have positive solutions, but he ended the paper with a short discussion of positive (integer) solutions of a homogenous system of linear Diophantine equations, showing how to proceed in order to find such a solution given that the system allows positive solutions.

The content of Gordan's paper suggests that the main issue for Gordan in this paper was positive solutions to systems of linear Diophantine equations. Gordan's theorem gives a tool for determining under which circumstances a given system of this kind actually has a positive solution. This interpretation of Gordan's paper gains further support when it is viewed in the context of Gordan's other mathematical work at the time especially his well known theorem – first proved in 1868 – on the construction of a finite basis for the invariants of binary forms.

Gordan's main interest, and the field in which his main contributions fell, was the theory of invariants. This theory is traditionally said to have begun in the mid 19th century in England. In his lectures on invariant theory held at the University of Göttingen in the summer of 1897, Hilbert described invariant theory as a theory concerned with the properties of a given form f ,⁵² that remain unchanged under linear transformations. Geometrically, a form set equal to zero represents an algebraic hypersurface in an n dimensional space and the theory of invariants studies the properties of the figure which are preserved during linear transformations. To quote Hilbert, the question is:

Do there exist properties that are common to all these “*equivalent*” forms, which are derived from f through all possible linear transformations? [Hilbert, 1897 (1993, p. 17)]

These properties can be represented by a polynomial function of the coefficients of the form, and such a polynomial function is an invariant of the form if it only changes by a factor equal to a power of the determinant of the linear transformation when – in the polynomial – the coefficients of the form is substituted by the coefficients of the transformed form [Hilbert, 1897 (1993, p. 17)].

The main question of invariant theory during the second half of the 19th century was the Finite Basis Problem. The question was whether it was possible for a given form to find a complete system of finitely many invariants, i.e. a finite basis, such that all other invariants of the form can be expressed as an integral rational function of this

⁵² By the term form is understood a rational, integral, and homogeneous algebraic function of a finite number of variables.

basis with numerical coefficients. The first major breakthrough was due to Gordan.⁵³ In a paper published in 1868 he gave a complete finite system of invariants for binary forms [Gordan, 1868]. A year later Gordan extended his result to forms with three variables but because of the method's calculating nature he only did it for forms of degree 3 [Gordan, 1869].⁵⁴ Gordan's proofs were based on the German so-called symbolic method. They were constructive in the sense that they provided a method for calculating the complete system of invariants, and even though this can be seen as a very strong way of proving things it also made it very technical and difficult to follow.⁵⁵ The English had a hard time understanding Gordan's original proofs, so a search for simpler proofs was a natural way to proceed and indeed Gordan continued to explore and simplify his proofs. The question that is interesting for us is, of course, what relevance all this has for Gordan's 1873 paper on positive solutions to linear systems of equations. Gordan's method of calculating the generating system of the invariants also gave the finiteness of the system. Later, in his "Vorlesungen über Invariantentheorie" from 1885 he gave a simpler proof of the finiteness of the generating system based on the theorem that a system of Diophantine equations has only a finite number of irreducible positive solutions [Gordan, 1885, p. 223–233]. By 1872 Gordan was aware of the connection between the generating system of invariants and positive integral solutions to a system of Diophantine equations,⁵⁶ and the question discussed by Gordan a year later in the paper where the "disguised" transposition theorem is proved is precisely how to find such positive integral solutions and demonstrate their reducibility [Gordan, 1873].

The fact that the number of irreducible positive solutions to a system of Diophantine equations is finite can be used to prove directly the finiteness of the complete systems of invariants and covariants, and Gordan did that in his "Vorlesungen über Invariantentheorie" [Gordan, 1885, vol. 2, p. 223]. Viewed in this context, there is a clear motivation to examine under which circumstances a system of linear equations has a positive solution, and it is in this context Gordan's 1873 paper should be evaluated. There is nothing that indicates that Gordan was interested in systems of linear inequalities and his theorem, as it is presented in his paper, does not exhibit the relationship between two systems of equations having a matrix and its transpose as the matrices of coefficients. Evaluated with respect to the later development it is clear though that a transposition theorem can be derived fairly easily from Gordan's theorem and it is probably in that light the credit to Gordan for the first transposition theorem should be understood.

A paper by Erich Stiemke (1892–1915) also figures in the literature on linear inequalities and like Gordan, he has been getting credit for a transposition theorem (see p. 515). Stiemke handed in his doctoral thesis "Über unendliche algebraische Zahlkörper" [Stiemke, 1925] in the summer of 1914 but he died in the First World War and the thesis

⁵³ For the "race" between the English invariant theorists and the German ones in proving this theorem, which in the literature often is referred to as Gordan's Theorem, see [Crilly, 1986, 1988] and [Parshall, 1989].

⁵⁴ In [Parshall, 1989] there is a discussion of the problems involved.

⁵⁵ See [Parshall, 1989] and [Crilly, 1986, 1988] for a discussion of the history of invariant theory, especially the differences between the English and the German approach.

⁵⁶ See [Gordan, 1872].

was not printed until 1925. In 1915 *Mathematische Annalen* published a small paper “Über positive Lösungen homogener linearer Gleichungen” in which Stiemke proved that

- I. Wenn in keiner linearen Verbindung von gegebenen homogenen linearen Gleichungen die Koeffizienten positiv sind, so haben die Gleichungen eine durchweg positive Lösung.
- II. Wenn in keiner linearen Verbindung von gegebenen homogenen linearen Gleichungen die Koeffizienten durchweg positiv sind, so haben die Gleichungen eine positive Lösung.⁵⁷ [Stiemke, 1915, p. 340]

By a “durchweg positive” solution Stiemke understood a solution for which all the unknowns are positive; if all the unknowns are non-negative and at least one of them is positive he called the solution positive. We recognize Gordan’s result in the second part of Stiemke’s theorem and as was the case with Gordan’s theorem Stiemke’s does not display the relation between dual systems with a matrix and its transpose as matrices of coefficients.

Stiemke proved the theorems algebraically by induction on the number of unknowns. He explicitly wrote that he was led to the results of I. and II. during his investigations on infinite modules [Stiemke, 1915, p. 340]. In the first chapter of his thesis Stiemke studied that subject and he found criteria for the existence of a basis for infinite modules [Stiemke, 1925, p.10], but he gave no indication of how his study of infinite modules brought him to consider the question of positive solutions to systems of linear equations.

6. The context of game theory and linear programming in the USA: The significance of the Post War Period

In the years following the Second World War new areas of applied mathematics were developed and new disciplines were established. Financed by the military a lot of research in these subjects was promoted in the USA, covering not only investigations into actual applications but also into the underlying mathematical structures. Linear programming was one of these subjects. Game theory was another subject that, even though it did not emerge in the course of the war, received extensive funding in the postwar period in the USA especially from the RAND Corporation and the Office of Naval Research (ONR). In the 1930s and 1940s it was gradually realized that the theory of linear inequalities constitutes the mathematical foundation for zero-sum two-person games and at the end of the 1940s the relationship between game theory and linear programming was recognized. The history of linear inequalities – and the very important duality theorem – in the context of game theory and linear programming in the postwar period in the USA is discussed in this section.

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- I. If in no linear relation between given homogeneous linear equations the coefficients are positive then the equations have a “durchweg” positive solution.
- II. If in no linear relation between given homogeneous linear equations the coefficients are “durchweg” positive then the equations have a positive solution.

6.1. *Linear inequalities and von Neumann's minimax theorem*

The first attempt to build a mathematical theory of games is due to Émile Borel (1871–1956), who published several papers on the subject in the twenties,⁵⁸ but the first major result is due to John von Neumann (1903–1957) who in 1928 proved the so-called minimax theorem for zero-sum two-person games, which constitutes a solution for such games [von Neumann, 1928].⁵⁹ The minimax theorem states that to every finite, zero-sum two-person game a value V can be assigned, which is the average gain that player I can expect to win from player II, if both players choose “sensible” strategies, which are called the optimal strategies. The proof of the minimax theorem has quite an interesting history, from von Neumann’s very complicated first proof in 1928 to 1944 where von Neumann was able to derive the theorem as a fairly simple consequence of a transposition theorem – in the sense of Motzkin – called “Alternatives for Matrices”.⁶⁰ This last proof appeared in the first coherent book on game theory, the “Theory of Games and Economic Behavior” written by von Neumann and the Austrian-born economist Oscar Morgenstern (1902–1976), in which von Neumann based the theory of zero-sum two-person games on what he himself characterized as “the mathematico-geometrical theory of *linearity* and *convexity*” [von Neumann and Morgenstern, 1944, p. 128].⁶¹

The first proof of the minimax theorem from 1928 was analytical in nature and as von Neumann’s biographer Steven Heims once wrote “A tour de force” [Heims, 1980, p. 91]. Contrary to what is indicated in some of the secondary literature, von Neumann did not make any connections with systems of linear inequalities in that proof.⁶² The first indication that von Neumann was aware that the Minimax theorem had a relation to systems of linear inequalities appeared in von Neumann’s paper “Über ein ökonomisches Gleichungssystem und eine Verallgemeinerung des Brouwerschen Fixpunktsatzes” published in 1937. Here von Neumann discussed a linear production model whose mathematical formulation resulted in a system of linear inequalities that had to be solved [von Neumann, 1937, p. 76]. The following quotation from the paper shows that von Neumann had realized by that time that there is a connection between the minimax-solution of zero-sum two-person games and the existence of a solution to a system of linear inequalities:

The question whether our problem [the existence of a solution to the system of linear inequalities representing the linear production model] has a solution is oddly connected with that of a problem occurring in the Theory of Games. . . [von Neumann, 1937, (1945, p. 5, note 1)]

The bridge between the existence of optimal solutions to zero-sum two-person games and the existence of solutions to a system of linear inequalities was established through the existence of a saddle point for a real valued function on an $m \times n$ dimensional region.

⁵⁸ See [Borel, 1921, 1923, 1924, 1927].

⁵⁹ For the history of game theory see [Weintraub, 1992].

⁶⁰ For the history of von Neumann’s conception of the theorem see [Kjeldsen, 2001].

⁶¹ For the collaboration of von Neumann and Morgenstern in “Theory of Games and Economic Behavior” see [Leonard, 1995], [Rellstab, 1992].

⁶² For a discussion of that see [Kjeldsen, 2001, p. 53].

Von Neumann's proof of the existence of such a saddle point, however, was not based on the algebra of linear inequalities but on an extension of Brouwer's Fix-point theorem, which he also proved in that paper [von Neumann, 1937, p. 80–81].

In the beginning of the forties – during the process of writing the book on game theory – von Neumann realized how the minimax theorem could be proved using the algebra of linear inequalities; an insight that resulted in the proof presented in the “Theory of Games and Economic Behavior” based on von Neumann's transposition theorem “Alternative for matrices”. In Morgenstern's diary one can get a glimpse of the excitement of this recognition

Both [a book of Borel and a proof of the minimax theorem by Jean Ville] are unknown to Johnny. Now he has discovered additional proofs that are becoming increasingly simple and are purely algebraic!! It necessitates some modification in the text, but we can print it. (quoted in [Rellstab, 1992, p. 87])

The proof of the minimax theorem by Jean Ville that Morgenstern is referring to was published as a note in Borel's book “Traité du Calcul des probabilités et de ses applications” from 1938 and it is generally acknowledged to be the first algebraic proof of the minimax theorem [Ville, 1938]. Ville derived the minimax theorem as a consequence of a theorem about linear forms, which on the other hand was derived as a corollary to the following theorem about linear inequalities given by Ville in the note:

Let p linear forms in n variables be given:

$$f_j(x) = \sum a_{ij}x_i \quad (j = 1, \dots, p; i = 1, \dots, n)$$

where $x = (x_1, \dots, x_n)$.

Suppose they have the following property:

For all $x \geq 0$ there exists a j in $\{1, \dots, p\}$, such that $f_j(x) \geq 0$.

Then there exists at least one set of nonnegative coefficients

$$X_1, \dots, X_p \text{ with } X_1 + \dots + X_p = 1,$$

such that

$$\sum_{j=1}^p X_j f_j(x) \geq 0 \text{ for all } x \geq 0.$$

[Ville, 1938, p. 105]

Ville's proof inspired von Neumann to base the proof of the minimax theorem given in “Theory of Games and Economic Behavior” on the theory of “linearity and convexity”. Using the theory of convexity, especially the theorem of supporting hyperplanes, von Neumann proved his transposition theorem, which he himself called “the Theorem of the Alternative for Matrices”, and which states that

If A is a $(n \times m)$ matrix then exactly one of the following two systems of inequalities has a solution:

$$Ax \leq 0, \quad x \geq 0, \quad \sum_{j=1}^m x_j = 1,$$

$$wA > 0, \quad w > 0, \quad \sum_{i=1}^n w_i = 1.$$

[von Neumann and Morgenstern, 1944, p. 138–141]

Von Neumann then reduced the proof of the minimax theorem to a simple consequence of this Theorem of Alternatives for Matrices, and thereby based the theory of zero-sum two-person games on the theory of linear inequalities.

Von Neumann did not develop a full theory of systems of linear inequalities in the “Theory of Games and Economic Behavior” but his Theorem of Alternatives for Matrices is a transposition theorem which was – again – developed almost independently from the earlier works discussed so far. The only reference to earlier work that von Neumann gave in the book was to Weyl’s paper on convex polyhedrons, and as we have seen in this section, von Neumann’s transposition theorem emerged in the context of game theory, inspired directly by the proof of Ville. Not only was the theorem developed independently it was also – once again – developed for a completely different reason. In their essay on John von Neumann’s work in the theory of games and mathematical economics Kuhn and Tucker tell that

Oskar Morgenstern told us that he drew Ville’s article to von Neumann’s attention after seeing it quite by chance while browsing in the library of the Institute for Advanced Study. They decided at once to adopt a similar elementary procedure, trying to make it as pictorial and simple to grasp as possible. [Kuhn and Tucker, 1958, p. 116]

The goal was to make the proof of the minimax theorem as simple as possible, avoiding the non-elementary topological proof using the fix-point theorem. Having in mind that this book by von Neumann and Morgenstern was written mostly for economists they had good reasons to make the mathematics as “pictorial and simple” as they could. The fact that game theory could be based on the theory of linear inequalities sparked new interests and new developments in that subject after the Second World War, when game theory became an important area of research funded by different military establishments in the USA.

6.2. Linear programming

Another context in which linear inequalities came to play an important role was in what is nowadays called, linear programming.⁶³ This development grew out of the Second World War and took place – in the beginning – within the United States Air Force, who in the spring of 1947 set up the project named SCOOP, which stood for Scientific Computing of Optimum Programs [Brentjes, Ph.D-thesis]. The advent of the computer was the main reason behind the establishment of the project.

Originally, the Air Force had rehired the mathematician George B. Dantzig in 1946 to

⁶³ The history of the development of linear programming, in which George B. Dantzig was one of the main actors, has been reported at several places, see e.g. [Dantzig, 1982, 1991], [Lenstra et al., 1991], [Grattan-Guinness, 1970, 1994]. For a history of the Russian contribution see [Brentjes, 1976b], [Charnes and Cooper, 1961].

[...] develop some kind of analog devise which would accept, as input, equations of all types, basic data, and ground rules, and use these to generate as output a consistent Air Force plan. [Dantzig, 1988, p. 12]

An Air Force plan or program was a plan for activities within the Air Force in which a huge amount of different kinds of supplies were involved. During the Second World War Dantzig had worked in the Air Force with these programs teaching the Air Force staff how to compute them. The methods they used were slow and ineffective and it took more than seven months to set up a program [Geisler and Wood, 1951, p. 189]. The main concern during the war was to construct a *consistent* program but when it became clear that the computer could be used in calculating Air Force programs the possibility of building some kind of objective into the programs appeared. This resulted in a very intensive working period in May/June of 1947 where an objective function was included. The group that later became project SCOOP was formed at first with Dantzig and Marshall Wood, who was an expert on programming procedures, as the main persons, later John Norton and Murray Geisler joined [Dantzig, 1991]. In a paper published in 1949 Dantzig and Wood defined a program as

the construction of a schedule of actions by means of which an economy, organization, or other complex of activities, may move from one defined state to another or from a defined state toward some specifically defined objective. [Dantzig and Wood, 1949, p. 193–194]

and they gave the following formulation of the problem:

we seek to determine that program which will, in some sense, most nearly accomplish objectives without exceeding stated resource limitations. So far as is known, there is so far no satisfactory procedure for solution of the type of problem. [Dantzig and Wood, 1949, p. 195]

Linear inequalities entered the picture because the model for the Air Force program resulted in the mathematical problem of maximizing or minimizing a linear function subject to linear inequalities [Dantzig, 1949].

6.3. *The duality theorem in linear programming*

The connection between linear programming and game theory was recognized in October 1947 when Dantzig visited John von Neumann at the Institute for Advanced Study in Princeton [Dantzig, 1982, 1988]. Von Neumann played a significant role in the mobilization and development of military related science during the war and he held various consulting positions within the military after the war. In addition he chaired several high profile committees.⁶⁴

In [Dantzig, 1982] Dantzig described how he remembered this meeting, recollecting the words of von Neumann:

⁶⁴ See [Ulam, 1958] for a list of some of these.

“I [von Neumann] don’t want you [Dantzig] to think that I am pulling all this out of my sleeve on the spur of the moment like a magician. I have just recently completed a book with Oscar Morgenstern on the theory of games. What I am doing is conjecturing that the two problems are equivalent. The theory that I am outlining for your problem is an analogue to the one we have developed for games.” [Dantzig, 1982, p. 45]

As we saw in Sect. 6.1 game theory was at this point fully embedded in the theories of convexity and linear inequalities and the realization of this connection to game theory paved the way to the detection of the important duality theorem in linear programming. According to Dantzig he himself wrote down the first rigorous proof of the duality theorem in the note “A Theorem on Linear Inequalities” dated January 5, 1948 [Dantzig, 1982, p. 45]. To questions about why he did not publish the result he gave the full credit to von Neumann:

Because it was not my result – it was von Neumann’s. All I did was write it up, for internal circulation, my own proof of what von Neumann outlined. [Dantzig, 1982, p. 45–46]

Von Neumann also wrote a note that circulated privately. It has the title “Discussion of a Maximum Problem” and it is dated November 15–16, 1947. About the content of this note, Harold Kuhn wrote in 1991:

von Neumann circulated privately a short typewritten note that was first published fifteen years later. This note formulated the dual for a linear program and gave a flawed proof of the equality of optimal objective values based on an invalid inhomogeneous form of Farkas’ Lemma. [Kuhn, 1991, p. 85]

In her Ph.D. thesis from the 1970s Sonja Brentjes gave quite another interpretation of this note

Allerdings gibt es keine schriftliche Fixierung eines Dualitätssatzes durch von Neumann, denn selbst sein Manuskript vom November 1947 ‘On a Maximization Problem’ enthält keine Dualitätsaussagen, obwohl es nach Dantzig sonst den wesentlichen Inhalt des Gespräches zwischen ihm und von Neumann vom October 1947 wiedergibt.⁶⁵ [Brentjes, Ph.D-thesis, p. 171]

Von Neumann’s main concern in the note was to find the maximum

$$\max_x a \cdot x \quad (a \geq 0)$$

subject to the constraints $x \geq 0$, $xA \leq \alpha$, ($\alpha \geq 0$), where $x = (x_k : k = 1, \dots, n)$ and $\alpha = (\alpha_\kappa : \kappa = 1, \dots, \nu)$ is a n - resp. ν -dimensional vector, and $A = (A_{k\kappa} : k = 1, \dots, n; \kappa = 1, \dots, \nu)$ is a $(n \times \nu)$ - matrix. [Von Neumann, 1947, p. 90], which is a linear programming problem. In order to solve it he considered the case where a finite maximum \tilde{a} existed, that is he supposed the existence of a $x_0 \geq 0$, with $x_0 A \leq \alpha$, such that

⁶⁵ “Certainly there are no written record of a duality theorem of von Neumann, even his manuscript of November 1947 ‘On a Maximization Problem’ contains no duality statement, although according to Dantzig it records the essential content of the conversation between him and von Neumann.”

$$\max_x a \cdot x = a \cdot x_0 = \check{\alpha} < \infty.$$

Von Neumann analyzed the problem as follows: It is obvious that

$$\check{\alpha} = \max\{a' \mid a' = a \cdot x, x \geq 0, xA \leq \alpha\},$$

and it is also obvious that

$$\check{\alpha} = \min\{a'' \mid a'' - a \cdot x \geq 0, \text{ for all } x, x \geq 0, xA \leq \alpha\}.$$

Since $a' = a \cdot x$, and $a'' \geq a \cdot x$ for all feasible x , (that is, for all $x \geq 0$ satisfying the constraints $xA \leq \alpha$), von Neumann concluded that

$$a' \geq a''$$

if and only if

$$a' = a'' = \check{\alpha},$$

which makes the x in $a' = a \cdot x \geq a''$ the maximum point x_0 [von Neumann, 1947, p. 90].

That is, under the assumption of the existence of a finite maximum $\check{\alpha}$ the maximum problem can be solved by solving the following problem:

$$a' \geq a'' \tag{48}$$

for

$$a' = a \cdot x, \quad x \geq 0, \quad xA \leq \alpha, \tag{49}$$

together with the condition

$$a'' - a \cdot x \geq 0, \quad \text{for all } x \geq 0, \quad xA \leq \alpha. \tag{50}$$

Von Neumann showed that the conditions in (50) can be expressed as

$$a'' = \xi \cdot \alpha, \tag{51}$$

$$\xi \geq 0, \tag{52}$$

$$A\xi \geq a \tag{53}$$

[von Neumann, 1947, p. 91].⁶⁶ With this von Neumann was able to show that the original conditions $x \geq 0$, $xA \leq \alpha$ ($\alpha \geq 0$), (48), (49), and (50) are equivalent to the conditions

⁶⁶ In (51) it should have been \geq instead of $=$, but it has no significance for the results in the note. The flaw in the proof was corrected by Kuhn and Tucker in a footnote [von Neumann, 1947, p. 91].

$$\begin{aligned}
 x &\geq 0, \\
 xA &\leq \alpha, \\
 \xi &\geq 0, \\
 A\xi &\geq a, \\
 a \cdot x &\geq \alpha \cdot \xi
 \end{aligned} \tag{54}$$

and with that von Neumann finished his analysis of the problem. From (54) we can see that what von Neumann did in this note was to reformulate the original maximum problem subject to linear constraints into a solution problem of a system of linear inequalities.

The important duality theorem in linear programming is concerned with a linear programming problem (the primal problem) and its dual problem:

$$\begin{array}{ll}
 \textbf{primal:} & \text{maximize } a \cdot x \\
 & \text{subject to } x \geq 0 \\
 & \phantom{\text{subject to }} xA \leq \alpha \\
 \textbf{dual:} & \text{minimize } \alpha \cdot \xi \\
 & \text{subject to } \xi \geq 0 \\
 & \phantom{\text{subject to }} A\xi \geq a
 \end{array}$$

If von Neumann's result in the note is interpreted with regard to the modern theory of linear programming one will probably say that he had shown that if a finite maximum \check{a} , with a maximum point x_0 , satisfying the constraints $x_0 \geq 0$, $x_0A \leq \alpha$, exists, then a $\xi \geq 0$ exists such that $A\xi \geq a$ and $a \cdot x \geq \xi \cdot \alpha$. That is, there exists a ξ satisfying $A\xi \geq a$ and (since $a'' \geq a'$ is always true) minimizing $\xi \cdot \alpha$. The following comment by Kuhn and Tucker to the result of von Neumann should probably be understood in this framework:

The following Duality Theorem states this result as it is viewed today: Suppose x achieves the finite maximum of $a \cdot x$ subject to $[x \geq 0]$ and $[xA \leq \alpha]$. Then there exists a ξ minimizing $\alpha \cdot \xi$ subject to $[\xi \geq 0]$ and $[A\xi \geq a]$ and $a \cdot x = \alpha \cdot \xi$ for this x and ξ . [Von Neumann, 1947, p. 92, footnote by Kuhn and Tucker.]

However, as Brentjes wrote in her thesis, there is no direct duality statement in the note of von Neumann. He neither formulated the dual problem or the duality theorem, but he did introduce the dual variables even though he didn't call them such and he derived their region of feasibility. Judging from the note alone it cannot be said that von Neumann proved the duality theorem. This seems also to have been the opinion of Tucker at an earlier stage, before the publication of von Neumann's note, where Tucker, according to Dantzig, was surprised that Dantzig, in his first textbook on linear programming, ascribed the duality theorem to von Neumann

'Why', he [Tucker] asked 'do you [Dantzig] ascribe duality to von Neumann and not to my group?' 'Because he was the first to show it to me.' He said, 'that is strange for we have found nothing in writing about what von Neumann has done. What we have is his paper On a Maximizing Problem.' 'True', I said, 'but let me send you a paper I wrote as a result of my first meeting with von Neumann.' I sent him my report A Theorem on Linear

Inequalities, dated 5 January 1948, which contained (as far as I know) the first rigorous proof of duality. [Dantzig, 1982, p. 45]

Whether we ascribe a proof of the duality theorem to von Neumann or not, there is no doubt that von Neumann, his note, and his game theory, came to play a very important role for the further development and establishment of game theory and linear programming and thereby also promoted further research in the theory of linear inequalities, which as the mathematical foundation, was to become an important part of this enterprise.

6.4. An ONR project in game theory, linear programming, and the underlying mathematical structure of convexity and linearity

Words about the work done by the Air Force group on programming problems reached the Office of Naval Research, which had been established in 1946 and was the main sponsor for government supported research at the universities – applied as well as basic research – in the first four years of its existence.⁶⁷ The following quote from an ONR report from 1949 gives a hint about the significance of ONR for scientific research at the universities in the USA in the post war period:

the huge university research program of the Navy Department is the greatest peacetime cooperative undertaking in history between the academic world and the government. This significant educational and scientific experiment now embraces approximately 1200 projects in about 200 institutions with a total expenditure of approximately \$ 20,000,000 a year. Nearly 3,000 scientists and 2,500 college and university graduate students are actively engaged in basic research projects in the many fields of vital interest to the Navy. (Cited in [Schweber, 1988, p. 17])

One of these projects was established with the objective of clarifying the connection between linear programming and game theory, and to study the mathematical foundation further. Mina Rees, who was the head of the mathematics division of ONR, wrote down the following recollection in 1977:

... when, in the late 1940s the staff of our office became aware that some mathematical results obtained by George Dantzig, who was then working for the Air Force, could be used by the Navy to reduce the burdensome costs of their logistics operations, the possibilities were pointed out to the Deputy Chief of Naval Operations for Logistics. His enthusiasms for the possibilities presented by these results was so great that he called together all those senior officers who had anything to do with logistics, as well as their civilian counterparts, to hear what we always referred to as a “presentation”. The outcome of this meeting was the establishment in the Office of Naval Research of a separate Logistics Branch with a separate research program. This has proved to be a most successful activity of the Mathematics Division of ONR, both in its usefulness to the Navy, and in its impact on industry and the universities. [Rees, 1977a, p. 111]

⁶⁷ For literature on ONR see [Kjeldsen, 2000], [Old, 1961], [Sapolsky, 1979], [Rees, 1977b], [Schweber, 1988].

In May 1948 Dantzig went back to Princeton to discuss with von Neumann the possibility of setting up such a university based project. That resulted in the establishment, that summer, of a trial project with Albert W. Tucker (1905–1995), a mathematician from Princeton, as principal investigator [Albers and Alexanderson, 1985, p. 342–343]. Tucker got two graduate students to work with him on the project that summer, Harold W. Kuhn and David Gale. They began by studying von Neumann’s and Morgenstern’s book on game theory as well as von Neumann’s note “Discussion of a Maximum Problem”.⁶⁸ They presented their results at the first conference on linear programming, which was held a year later in June 1949 in Chicago. The papers were published in the proceeding “Activity Analysis of Production and Allocation”, which came out in 1951 [Koopmans, 1951].

Kuhn, Gale and Tucker considered the linear programming problem of optimizing a linear function subject to linear inequality constraints in a more general framework. Instead of looking at the original “scalar” problem, as they called it, they considered a more general “matrix” problem, which has the “scalar” problem as a special case. Their two main results are the duality theorem and an existence theorem. The latter states that a matrix D (scalar δ) that solves both the primal and the dual problem, exists, if a certain system of linear inequalities and a “transposed” system has a solution. They based the proof of both the duality theorem and the existence theorem on Farkas’s lemma. About the problem of existence they wrote

A problem of this symmetric sort was formulated by von Neumann [1947] for the case in which D reduces to a scalar δ . . . [Gale et al., 1951, p. 323]

indicating once more that at this time they did not think of von Neumann’s result in his note “Discussion of a Maximum Problem” as the duality theorem but rather as belonging to their existence theorem.

The first paper published on the mathematical model that Dantzig and his group developed for the Air Force programs appeared in *Econometrica* in 1949 [Dantzig, 1949]. In this paper Dantzig discussed the linear programming model of optimizing a linear form subject to linear inequality constraints. Going through the list of references it becomes clear that the inspiration came entirely from economics and von Neumann’s game theory; there are no references to the literature on systems of linear inequalities. From an economic point of view the following quotation from a paper by Wood and Geisler from the first conference on linear programming reveals some of the high hopes and the prospects they attached to linear programming:

These complexities [of the Air Force programming problem] have been spelled out to indicate a whole range of planning problems which, because of the present difficulties of computing alternative programs, receive little or no consideration. *So much time and effort is now devoted to working out the operational program that no attention can be given to the question whether there may not be some better program that is equally compatible with the given conditions.* It is perhaps too much to suppose that this difference between programs is as much as the difference between victory and defeat, but it is certainly a significant difference with respect to the tax dollar and the division of the total national product between military and civilian uses.

⁶⁸ Interview with Kuhn, Princeton, 23. April, 1998.

Consideration of the practical advantages to be gained by comparative programming, and particularly by the selection of “best” programs, leads to a requirement for a technique for handling all program elements simultaneously and for introducing the maximization process directly into the computation of programs. Such a technique is now in prospect. [Geisler and Wood, 1951, p. 194]

With this prospect for the usefulness of linear programming it is not surprising that the ONR decided to continue the project on the mathematical theory of linear programming and game theory. The duality theorem and the connection between linear inequality systems and the theory of convexity also made the subject of linear programming interesting from a mathematical point of view, and some of this work, especially Gale’s paper on “Convex Polyhedral Cones and Linear Inequalities” and Murray Gerstenhaber’s paper “Theory of Convex Polyhedral Cones”, were already included in the proceeding from the first conference on linear programming [Gale, 1951], [Gerstenhaber, 1951]. It is noteworthy that these two papers were not done under contracts with ONR or the RAND Corporation, indicating that linear inequality systems and the theory of convexity were in fact mathematically interesting and worth pursuing regardless of external funding. Tucker continued as principal investigator for the ONR project on game theory and linear programming. As was mentioned above Tucker, Kuhn and Gale based their proofs of the duality theorem and the existence theorem on Farkas’s lemma and they also referred to Weyl and Minkowski, signifying that they were becoming aware of the existing literature on the subject. It is not clear precisely when they became aware of the work of Motzkin. There are no references to Motzkin in Kuhn, Gale, and Tucker’s first work on linear programming and game theory, which was carried out during the summer of 1948. Kuhn clearly remembers Motzkin’s work as an important source of inspiration. He is certain that they knew about it in the spring of 1949, and that it was through Motzkin’s thesis they became aware of Farkas’s 1901 paper.⁶⁹ Motzkin’s work was important both for the research done in Tucker’s group and for the research done at the Rand Corporation, both places had the thesis translated into English about 1951 by S. Bargmann for Tucker’s ONR project and by D. R. Fulkerson for the Rand Corporation.⁷⁰

The importance attached to these subjects by the ONR and the RAND Corporation sparked new developments in the theory of systems of linear inequalities as well as in the theory of convexity. In 1956 Kuhn and Tucker published a collection of papers under the head title “Linear Inequalities and Related Systems” [Kuhn and Tucker, 1956]. Almost all the papers were supported in one way or another by the USA military offices and the research that led to those papers were motivated by the developments of game theory and linear programming, which is clearly expressed in the introduction:

The eighteen papers collected here explore various aspects of one mathematical theme, the theory of linear inequalities. Although they are related by this fact, the papers are bound together more closely by the areas of intended application. Without exception, the direction or technique of each is determined by recent developments in the subject of

⁶⁹ Personal communication, March 13, 2002.

⁷⁰ See the introduction to Motzkin’s collected works p. xiv in [Cantor et al., 1983], and [Billera and Lucas, 1978, p. 5].

linear programming, matrix game, and related or derivative economic models. [Kuhn and Tucker, 1956, p. v]

7. Discussion and conclusion

The theory of systems of linear inequalities was developed for several different reasons, in different mathematical, scientific, and social contexts, and it was influenced by many different factors. Only one of the developments was motivated by a primary interest in systems of linear inequalities, the others were motivated by other problems in pure and applied mathematics, as well as in other scientific disciplines.

The principal lines of developments were Farkas's, Minkowski's, Dines's, Motzkin's, and the work done by the scientific community of game theorists and linear programmers in the postwar period in the USA. Common to all of them is that their motivation and their goals were totally different from one another, which also had the effect that there was a diversity in the questions they investigated. The different works were done not only in different contexts but also in different kinds of contexts. Farkas was motivated by equilibrium conditions for constrained mechanical systems and his goal was to make systems of linear inequalities the mathematical foundation for that subject. In this case the reason for developing a theory of linear inequalities was problems in physics and this influenced the kind of questions Farkas set out to investigate in the subject of linear inequalities. As was discussed in Sect. 2 this context of analytical mechanics shaped the content and the outlook of Farkas's theory of linear inequalities. He viewed linear inequalities as the theoretical foundation for the Fourier Inequality Principle and it was for that reason that he was interested in the theory of linear inequalities not just for their own sake.

Minkowski developed his theory of linear inequalities in connection with investigations in pure mathematics. His goal was to obtain results in number theory. Motivated by the work of Hermite and especially the work of Dirichlet on quadratic forms, Minkowski created a new geometry of numbers as a tool to solve number-theoretical problems. Linear inequalities appeared as a natural element in Minkowski's new geometry in the algebraic expression of supporting hyperplanes. Minkowski's geometrical intuition led him to characterize different kind of solutions to systems of linear inequalities and the basic notion in his theory is the extreme ('*ausserste*') solutions. His theory and the questions he asked of such systems were primarily guided by the number-theoretical questions to which he wanted to apply his new geometry of numbers. Minkowski was interested in linear inequalities as a tool in his investigations on quadratic forms and, as was discussed in Sect. 3, the content and the outlook of his theory can be understood in this framework.

In contrast to Farkas and Minkowski, Dines developed his theory not because it was the theoretical foundation for other kinds of problems or because he needed it as a tool for investigations in other branches of mathematics but because he found an interest in linear inequalities for their own sake. He seems to have been motivated by the lack of a theory for such systems, which he compared with the matrix theory of systems of linear equations. As was discussed in Sect. 4 his theory for linear inequalities was modeled on the matrix theory for linear equations. Dines's goal was to extend the theory to the infinite case and to generalize it in the spirit of Moore's general analysis. His reasons

for developing a theory of linear inequalities can be understood in the context of the self-understanding of mathematical research in the USA at that time.

These three developments have in common that they obtained some of the same results – e.g. the general solution, and criteria for an inequality being a consequence of other inequalities – but they reached the results in very different ways, showing that their results can also be said to be very different, because they provide very different kind of knowledge and insights into the theory.

Motzkin was motivated by Schönberg's work on variation diminishing matrices, and his reasons for developing a theory of linear inequalities in this context was, as was explained in Sect. 5, guided by the fact that the only thing one need to know to solve the general problem about variation diminishing matrices is the sign of all the minors of the matrix of coefficients, and the deeper reason underneath this fact is that the same knowledge is sufficient for finding explicit conditions for the solvability of systems of linear inequalities. While the three developments by Farkas, Minkowski, and Dines were done independently, Motzkin was familiar with all the older sources and his work can be seen as a culmination of all the work done so far. With Motzkin's work the theory was definitively established within the theory of convexity.

Finely, during the thirties and the forties it was realized that systems of linear inequalities are the mathematical foundation for zero-sum two-person games leading – once again – to new results, which were – once again – reached independently of the earlier literature. In the late forties, when linear programming was being developed and the connection to game theory was realized, new interest in the theory of linear inequalities emerged, and especially the huge funding primarily by the ONR promoted new research in linear inequalities in which the work of Motzkin eventually became a main source of inspiration.

The history of linear inequalities shows that there are many different driving forces in the development of mathematics. If we look at it from a discipline point of view the history shows that the theory of linear inequalities was developed through interactions between different branches of pure and applied mathematics, through interactions between other scientific disciplines, and through interactions with the social sciences. These disciplines are different in nature and that is reflected in the theory of linear inequalities as it was developed in the respective disciplines. If we look at it from a more personal point of view mathematics can be seen as developing through tasks. In [Bos, 2002] Henk Bos suggests that we look at past mathematics with more emphasis on the tasks which led mathematicians to form their ideas. Bos understands the tasks as “self-imposed by the mathematicians in response to available knowledge, open questions and more diffuse challenges.” This leads Bos to conclude that the tasks are time-dependent, and one might add also context dependent. The history of linear inequalities illustrates this viewpoint. The analyses put forward in this paper of the various mathematical texts are performed with respect to the original context in which the texts appeared and with the focus on motivations and goals. As we have seen, parts of the history of linear inequalities can be understood in this framework of mathematician's self-imposed tasks. Some of these tasks seem to have been governed by the subject itself according to some (maybe time-dependent) “rules” for research in pure mathematics while others were driven by interactions between different subjects either within or outside of mathematics. In contrast to this, in other parts of the development the original task was not self-imposed,

but imposed on the mathematicians by authorities outside of mathematics, in a military context, meaning that the tasks were not originally worked on because they were important in mathematics as such or for the individual mathematician, but for other reasons which lay outside of mathematics.

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