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On Nonstandard Methods in Functional Analysis

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INTRODUCTION

Nonstandard methods consist in simultaneously using two different set-theoretic models, one standard and the other nonstandard, for study of concrete mathematical objects and problems. This approach involves comparison of two interpretations of a mathematical claim or construction, considered as a formal symbolic expression. Such comparative study enables one to reveal new interconnections, sometimes leading to conceptual clarity and technical simplification even for elaborated theories. The main development of such methods dates back to the last thirty years, and they have by now crystallized in several directions. The main developments are now known as *infinitesimal analysis* and *Boolean-valued analysis*, see [1].

Roughly speaking, the essence of these applications can be expressed as follows: Infinitesimal analysis allows one to somehow consider an operator as a matrix, i.e., to carry out a *discretization* of an operator under study. Boolean-valued analysis makes it possible to consider the elements of some functional classes as numbers, and enables one, in particular, to work with an operator as with a functional, i.e., to implement a *scalarization* of the operator. The two methods are based on a well-developed machinery that makes a part of the theory algorithmic. Nevertheless, the "discretization" and "scalarization" procedures are by no means automatic, depending essentially on the operators and the problems in question.

In this short survey we shall briefly outline some new applications of nonstandard methods to different problems of functional analysis and point out some directions of further research which seem to be promising.

1 BOOLEAN-VALUED INTEGRALS [2]

The first example concerns a certain class of positive operators. Let X and Y be vector lattices. A linear operator $\Phi : X \to Y$ is said to be a *Maharam operator* if it is positive ($x \geq 0 \Rightarrow \Phi x \geq 0$), order-interval preserving ($\Phi([0,x]) = [0, \Phi(x)]$), and order continuous ($x_\alpha \downarrow 0 \Rightarrow Tx_\alpha \downarrow 0$).

The Boolean-valued model $V^{\mathbf{B}}$ is constructed from a complete Boolean algebra $\mathbf{B}$. The elements of $V^{\mathbf{B}}$ are thought of as $\mathbf{B}$-valued sets. Given a statement $\mathcal{S} := \mathcal{S}(x_1, \dots, x_n)$ with parameters $x_1, \dots, x_n \in V^{\mathbf{B}}$ one defines its Boolean truth value $[\![\mathcal{S}]\!] \in V^{\mathbf{B}}$. The statement $\mathcal{S}$ is said to be true in $V^{\mathbf{B}}$ if $[\![\mathcal{S}]\!] = 1$. There is a functorial procedure, called *descent* and denoted by $(\cdot)\downarrow$, that assigns to each mathematical object inside the Boolean-valued model a standard object of a similar type.

THEOREM 1 Let Φ be a strictly positive Maharam operator. There exist an order complete vector lattice $\mathcal{X}$, a strictly positive order continuous linear functional $\phi : \mathcal{X} \to \mathcal{R}$ in a suitable Boolean-valued model $V^{\mathbf{B}}$, and an isomorphism $\iota : X \to \mathcal{X}\downarrow$ such that $\Phi = \phi\downarrow \circ \iota$.

This result reduces study of the class of Maharam operators to analysis of the class of order continuous positive functionals. Since any order continuous linear functional is representable as Lebesgue integral over some measure space, we conclude that the essential part of the theory of Maharam operators is simply a model-theoretic modification of the classical integration theory, being deducible from it by means of some Boolean-valued machinery.

2 HYPERFINITE APPROXIMATION OF LCA GROUPS [3, 4]

The second example concerns the "discretization" of the Fourier transform. It is well known that a standard Radon measure can often be represented as the Loeb measure with respect to a nonstandard counting measure. For the Haar measure it is important that the corresponding counting measure is defined on a hyperfinite subgroup of the LCA group under consideration. If this is the case, then the Fourier transform can be approximated by the discrete Fourier transform. For certain classes of locally compact groups this was done by E. Gordon, see [3, 4]. Let G be an internal hyperfinite group, G_0 be one of its external subgroups, and consider the quotient group $G^\sharp := G/G_0$.

Denote by $E(G)$ the internal finite dimensional Hilbert space consisting of all complex-valued functions on G with the scalar product

$$(f_1, f_2) := \frac{1}{|G|} \sum_{g \in G} f_1(g)\overline{f_2(g)}.$$

If $E^\sharp$ denotes the nonstandard hull of a Banach space E (in the sense of W. A. J. Luxemburg) then $E(G)^\sharp$ is a nonseparable Hilbert space. Given a microcontinuous function

$f : G^\sharp \to \mathbf{C}$ one can choose another microcontinuous function $f_0 : G \to \mathbf{C}$ such that $f_0 = F \circ \iota$, where $\iota : G \to G^\sharp$ is the canonical homomorphism. Putting $\phi(f) := f_0$ and extending the operator ϕ to the whole of $L_2(G^\sharp)$ we obtain an embedding $\phi : L_2(G^\sharp) \to E(G)^\sharp$. Analogously , we can define an embedding $\psi : l_2(G^{\sharp *}) \to E(G^*)^\sharp$. Finally, let $\mathcal{F} : L_2(G^\sharp) \to l_2(G^{\sharp *})$ denotes the Fourier transform and consider the discrete Fourier transform $\Phi : E(G) \to E(G^*)$ by letting

$$\Phi(f)(\chi) := \frac{1}{|G|} \sum_{g \in G} f(g)\overline{\chi(g)}.$$

THEOREM 2 Let G be a hyperfinite group. Then the discrete Fourier transform Φ serves as the nonstandard finite dimensional analog of the Fourier transform $\mathcal{F}$. i.e., the following diagram is commutative:

$$\begin{array}{ccc} L_2(G^\sharp) & \xrightarrow{\ \mathcal{F}\ } & l_2(G^{\sharp *}) \\ \phi \big\downarrow & & \psi \big\downarrow \\ E(G)^\sharp & \xrightarrow{\ \Phi^\sharp\ } & E(G^*)^\sharp \end{array}$$

It should be noted that for any compact abelian group $\mathcal{G}$ there exist an internal hyperfinite group G and an external subgroup G_0 of G such that the quotient group $G^\sharp := G/G_0$ is topologically isomorphic to $\mathcal{G}$. We can thus say that any compact group admits a "good" hyperfinite approximation. A complete description for the class of locally compact groups with the same property is unknown.

3 BOOLEAN-VALUED ANALYSIS AND BANACH ALGEBRAS [5]

In the last decade Boolean-valued models of set theory were applied to operator algebras, C^*-algebras, and JB-algebras. This direction was initiated by G. Takeuti. All Banach algebras that were studied by means of Boolean-valued analysis are special cases of the so-called *cyclic Banach spaces*. A Banach space X is said to be $\mathbf{B}$-*cyclic* if there exists a complete Boolean algebra of norm one projections $\mathbf{B} \subset \mathcal{L}(X)$ such that the following conditions hold:

(1) for any partition of unity $(b_\xi)_{\xi \in \Xi} \subset \mathbf{B}$ and any bounded family $(x_\xi)_{\xi \in \Xi}$ there is a unique element $x \in X$, called the *mixing* of (x_ξ) with probabilities b_ξ, for which $b_\xi x_\xi = b_\xi x$ ($\xi \in \Xi$);

(2) the unit ball of X is closed with respect to mixing.

THEOREM 3 A restricted descent of a Banach space in the model $V^{(\mathbf{B})}$ is a $\mathbf{B}$-cyclic Banach space. Conversely, if X is a $\mathbf{B}$-cyclic Banach space, then there exists a unique (up to isometric isomorphism) Banach space $\mathcal{X}$ in the model $V^{(\mathbf{B})}$ whose restricted descent is isometrically $\mathbf{B}$-isomorphic to X.

Next, it can be easily verified that if X is a Banach algebra of the above-mentioned type then $\mathcal{X}$ too is a Banach algebra of the same type. Moreover, we can choose a Boolean algebra $\mathbf{B}$ so that $\mathcal{X}$ becomes a factor (i.e., it has one-dimensional center). Combining these results with standard arguments we can prove the following:

THEOREM 4 Let A be a type IAW^*-algebra. There exits an unique (up to a congruence) family $(Q_\gamma)_{\gamma\in\Gamma}$ of nonempty extremally disconnected compacta such that the following conditions are fulfilled:

(1) Γ is a set of cardinals and the compactum Q_γ is γ-stable for each $\gamma \in \Gamma$;

(2) if $\simeq$ denotes an isomorphism of AW^*-algebras then

$$A \simeq \sum_{\gamma\in\Gamma}^{\oplus} \mathrm{SC}_{\sharp}(Q_\gamma, \mathcal{L}(l_2(\gamma))),$$

where $\mathrm{SC}_{\sharp}$ denotes the space of operator-valued functions defined on comeager subsets and continuous in the strong operator topology.

4 NONSTANDARD METHODS FOR NONSMOOTH ANALYSIS [1, 6]

Nonstandard methods are also very useful in nonsmooth analysis evolved by optimization theory. In [1] two new notions of optimality for extremal problems were discussed: *generalized solution* and *infinitesimal solution.* A generalized solution arises, roughly speaking, in "pointwise scalarization" of a vector program, while an infinitesimal solution formalizes the intuitive notion of "approximate solution", which is accepted in practice.

Subdifferential is one of the central concepts of optimization theory. It is often used as a one-sided local approximation to a nonsmooth operator, playing in this respect the role of the classical differential. Moreover, a subdifferential as a closed convex set of linear operators has some very rich structure that deserves an independent study, see [6] . Here we will expose a Kuhn–Tucker type result in terms of *infinitesimal subdifferentials.*

Let X be a standard vector space and $f : X \to R \cup \{+\infty\}$ be a convex function. The infinitesimal subdifferential of f at $x_0 \in \mathrm{dom}(f)$, denoted by $Df(x_0))$, is a set of all linear functionals $l : X \to \mathbf{R}$ such that $lx - lx_0 \leq f(x) - f(x_0) + \varepsilon$ $(x \in X)$ for some infinitely small $\varepsilon > 0$.

Suppose now that we are given a convex function $x \in C$, $f(x) \to \inf$ and a convex subset $C \subset X$. The problem is to minimize f over C. A feasible point $x_0 \in C$ is said to be an *infinitesimal solution*, if $f(x_0) \leq f(x)+\varepsilon$ for any standard number $\varepsilon > 0$ and for any $x \in X$. We say that the Slayter regularity condition holds if $C = \{x \in X : Tx = Tx^*, g(x) \leq 0\}$, where $T : X \to Y$ and $g : X \to F$ are a linear operator and a convex operator respectively, and $-g(x_0)$ is a strong order unit for some $x_0 \in C$.

THEOREM 5 Assume that the Slayter regularity condition holds. A feasible point x^* is an infinitesimal solution of the considered problem if and only if the following conditions are consistent:

$$0 \leq \gamma \in L(F, E), \quad \mu \in L(Y, E), \quad \gamma g(x^*) \approx 0,$$
$$0 \in Df(x^*) + D(\gamma \circ g)(x^*) + \mu \circ T.$$

5 COMBINED NONSTANDARD METHODS [7]

For applications to the theory of operators, it is of essential importance to construct a synthetic theory in the framework of which both the nonstandard methods offered by Boolean-valued models and external set theories can be used. Only preliminary results have so far been achieved in this direction. There are at least two ways to combine nonstandard methods: first, to do infinitesimal constructions in a Boolean-valued model, and second, to look for the Boolean-valued interpretation of internal and external set theories. Each of these approaches involves its own technical difficulties and we are unable to overcome them now. At the same time there are several examples in which the successive application of different nonstandard methods has proved to be a powerful tool. This direction was first developed by the second author. One more example: a combination of nonstandard methods leads to a Hammer–Sobczyk type decomposition result for vector measures. In view of these facts one can suppose that the above-mentioned combinations of nonstandard methods are not only highly desirable but also manageable. In conclusion we shall list some unsolved problems.

5.1 Problem 1.

Develop some combined "discretization—scalarization" machinery in order to unify and extend the successive application of different nonstandard methods.

5.2 Problem 2.

Develop a Boolean-valued version of the Loeb measure technique. In particular, test the candidates for the Loeb measure with values in an order complete vector lattice.

(Some steps in this direction were taken by P. Loeb and H. Osswald.)

5.3 Problem 3.

Apply combined "discretization—scalarization" procedures to constructing hyperfinite approximations for the representations of LCA groups in a Hilbert space.

5.4 Problem 4.

Using different nonstandard methods, find a combined transfer principle from finite-dimensional normed algebras to general Banach algebras.

5.5 Problem 5.

Develop the concept of infinitesimal solution for variational and optimal control problems and, on using the Loeb measure technique, study the corresponding nonstandard Loeb-Sobolev type function spaces (cf. [8]).

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