ON COMBINED NONSTANDARD METHODS IN FUNCTIONAL ANALYSIS

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Introduction

The main nonstandard tool-kits are known as *infinitesimal analysis* (Robinson's nonstandard analysis) and *Boolean-valued analysis* (see [1] and the references therein). Sharp distinctions between these two versions of nonstandard analysis in content and technique notwithstanding, many ways are open to their simultaneous application [2, 3]. One of the simplest approaches consists in successive application of different nonstandard methods [4, 5]. A few examples below demonstrate that combining is often useful in settling the problems of functional analysis which stem mainly from the theory of vector lattices.

We believe firmly in importance of constructing a synthetic theory that utilizes both nonstandard ways of mathematical reasoning.

1. Boolean-valued models

The Boolean-valued approach is less popular than its infinitesimal counterpart. It so stands to reason to sketch the former. The reader may find the relevant details in [1, 6]. A Boolean-valued model $V^{(\mathbb{B})}$ is constructed from a complete Boolean algebra $\mathbb{B}$. The class $V^{(\mathbb{B})}$ consists of $\mathbb{B}$-valued functions and its elements are thought as $\mathbb{B}$-valued sets. To a set-theoretic proposition $\mathcal{S} := \mathcal{S}(x_1, \dots, x_n)$ with parameters $x_1, \dots, x_n \in V^{\mathbb{B}}$ there is assigned its Boolean *truth value* $[\![\mathcal{S}]\!] \in \mathbb{B}$. The proposition $\mathcal{S}$ holds *true in* $V^{(\mathbb{B})}$ if $[\![\mathcal{S}]\!] = 1$. The following result asserts that $V^{(\mathbb{B})}$ is a Boolean-valued model of ZFC, the *Zermelo–Fraenkel Set Theory*.

1.1. Boolean-valued transfer principle. *For every theorem $\mathcal{S}(x_1, \dots, x_n)$ of ZFC, the equality $[\![\mathcal{S}(x_1, \dots, x_n)]\!] = 1$ holds, i.e., $\mathcal{S}(x_1, \dots, x_n)$ is valid inside $V^{(\mathbb{B})}$.*

1.2. Maximum principle. *For every formula $\mathcal{S}(v, x_1, \dots, x_n)$ ($x_1, \dots, x_n \in V^{(\mathbb{B})}$) of ZFC with one free variable v there exists an element $x \in V^{(\mathbb{B})}$ for which*

$$[\![(\exists v)\mathcal{S}(v, x_1, \dots, x_n)]\!] = [\![\mathcal{S}(x, x_1, \dots, x_n)]\!].$$

In particular, $V^{(\mathbb{B})}$ contains some object $\mathcal{R}$ that plays the role of the reals $\mathbb{R}$ inside $V^{(\mathbb{B})}$.

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1.3. Restricted transfer principle. There is an important procedure for passing from the conventional von Neumann universe V to $V^{(\mathbb{B})}$. Proceed as follows: Given $x \in V$, denote by the symbol ${}^\wedge x$ the *standard name* of x in $V^{(B)}$; i.e., the element determined from the following recursion schema:

$$ {}^\wedge\varnothing := \varnothing, \quad \operatorname{dom}({}^\wedge x) := \{{}^\wedge y \mid y \in x\}, \quad \operatorname{im}({}^\wedge x) := 1. $$

For every restricted formula $\mathcal{S}(v_1, \dots, v_n)$ *and elements* $x_1, \dots, x_n \in V^{(\mathbb{B})}$ *the equivalence holds*:

$$ \mathcal{S}(x_1, \dots, x_n) \leftrightarrow [\![\mathcal{S}({}^\wedge x_1, \dots, {}^\wedge x_n)]\!] = 1. $$

1.4. Descent functor. There is a functorial procedure, called *descent* and denoted by $(\cdot){\downarrow}$, which assigns to each mathematical object inside a Boolean-valued model a standard object of a similar type. Given an element $x \in V^{(\mathbb{B})}$, its descent is defined by $x{\downarrow} := \{t \in V^{(\mathbb{B})} \mid [\![t \in x]\!] = 1\}$. The descent $x{\downarrow}$ of x is a set. If $X, Y, f \in V^{(\mathbb{B})}$ and f is a mapping from X to Y inside $V^{(\mathbb{B})}$ then $f{\downarrow}$ is a mapping from $X{\downarrow}$ to $Y{\downarrow}$. The same is true for a relation inside $V^{(\mathbb{B})}$. Therefore, the descent of an algebraic structure is again an algebraic structure of a similar type.

2. Boolean-valued integrals

Necessary information from the theory of vector lattices can be found in [10]. The following result is basic for what follows, see [1, 7].

2.1. Gordon's Theorem. *Let* $\mathcal{R}$ *be the reals inside* $V^{(\mathbb{B})}$. *The algebraic system* $\mathcal{R}{\downarrow}$ *is a universally complete vector lattice. Moreover, there exists an isomorphism* χ *of* $\mathbb{B}$ *onto the Boolean algebra* $\mathfrak{P}(\mathcal{R}{\downarrow})$ *of band projections in* $\mathcal{R}{\downarrow}$ *such that*

$$ b \le [\![x = y]\!] \leftrightarrow \chi(b)x = \chi(b)y, $$
$$ b \le [\![x \le y]\!] \leftrightarrow \chi(b)x \le \chi(b)y $$

for all $x, y \in \mathcal{R}{\downarrow}$ *and* $b \in \mathbb{B}$.

Moreover, every order complete vector lattice F is isomorphic to an order dense ideal of the universally complete vector lattice $\mathcal{R}{\downarrow}$. Denote such isomorphism by ι_F.

We now consider the class of the so-called Maharam operators. Let E and F be vector lattices. A linear operator $\Phi : E \to F$ is a *Maharam operator* if Φ is positive ($0 \le x \in E \Rightarrow 0 \le \Phi x$), order-interval preserving ($\Phi([0, x] = [0, \Phi x],\ 0 \le x \in E$), and order continuous ($x_\alpha \searrow 0 \Rightarrow \Phi x_\alpha \searrow 0$). The following result gives a Boolean-valued interpretation for a Maharam operator [10].

2.2. Theorem. *Given order complete vector lattices* E *and* F, *let* $\Phi : E \to F$ *be a strictly positive Maharam operator. In a suitable Boolean-valued model* $V^{(\mathbb{B})}$ *there are an order complete vector lattice* $\mathcal{E}$ *and a strictly positive order continuous functional* $\varphi : \mathcal{E} \to \mathcal{R}$ *such that the diagram commutes*

$$ \begin{array}{ccc} E & \xrightarrow{\ \Phi\ } & F \\ {\scriptstyle \iota_E}\big\downarrow & & \big\downarrow{\scriptstyle \iota_F} \\ \mathcal{E}{\downarrow} & \xrightarrow{\ \varphi\ } & \mathcal{R}{\downarrow} \end{array} $$

i.e., $\iota_F \circ \Phi = \varphi\!\downarrow \circ \iota_E$ *for a suitable order continuous lattice monomorphism* $\iota_E : E \to \mathcal{E}\!\downarrow$.

This result reduces the study of the class of Maharam operators to that of the class of order continuous positive functionals. At the same time every order continuous linear functional is representable as Lebesgue integral over some measure space. We may thus conclude that the essential part of the theory of Maharam operators is simply a model-theoretic modification of the classical integration theory. The former emerges from the latter on using the Boolean-valued machinery.

The Loeb measure is proved to be a very useful tool of nonstandard analysis. This construction extends without difficulties to measures with values in a Banach space. Extension becomes more involved for measures with values in a vector lattices without a distinguished norm. Further progress in this direction requires developing infinitesimal constructions in a Boolean-valued model.

2.3. Problem. *Suggest a Boolean-valued version of the Loeb measure and integration technique. Study the corresponding class of linear operators.*

3. Atomic decomposition of vector measures

We now demonstrate that successive application of the two nonstandard approaches is effective sometimes. We consider the problem of atomic decomposition of a finitely additive measure. Let $\mathcal{A}$ be a Boolean algebra and let F be a vector lattice. By a *vector measure* we mean an arbitrary mapping $\mu : \mathcal{A} \to F$ such that $\mu(a_1 \vee a_2) = \mu(a_1) + \mu(a_2)$ for all disjoint $a_1, a_2 \in \mathcal{A}$. Denote by $\mathrm{ba}\,(\mathcal{A},\ F)$ the space of all bounded F-valued measures and put $\mathrm{ba}\,(\mathcal{A}) := \mathrm{ba}\,(\mathcal{A}, \mathcal{R})$. A measure $\mu \in \mathrm{ba}\,(\mathcal{A}, F)$ is *positive* if $\mu(a) \geq 0$ for all $a \in \mathcal{A}$. It is well known that $\mathrm{ba}\,(\mathcal{A}, F)$ is a Kantorovich space whose positive cone coincides with the set of positive measures. An *atom* of a measure $\mu \in \mathrm{ba}\,(\mathcal{A})$ is an element $a_0 \in \mathcal{A}$ such that $\mu(a_0) \neq 0$ and for every $a \in \mathcal{A}$, $a \leq a_0$, either $\mu(a) = 0$, or $\mu(a_0 \setminus a) = 0$. We say that a measure μ is *atomic* if μ belongs to the band in $\mathrm{ba}\,(\mathcal{A})$ generated by discrete elements, and μ is *diffuse* if μ belongs to the complementary band.

The *infinitesimal approach* uses Loeb's [11] concept of infinitely fine partition. By a *finite partition* in a Boolean algebra $\mathcal{A}$, we mean a finite collection $\{a_1, \dots, a_n\} \subset \mathcal{A}$ of nonzero elements with $\bigvee_{k=1}^{n} a_k = 1$ and $a_k \wedge a_l = 0$ for $k \neq l$. Denote by $\mathcal{P} := \mathcal{P}(\mathcal{A})$ the collection of all finite partitions in $\mathcal{A}$. Take $p_1, p_2 \in \mathcal{P}$. We say that p_1 is *finer* than p_2 if $a = \bigvee\{b \in p_1 \mid b \leq a\}$ for each $a \in p_2$. The idealization principle (in Nelson's credo) or the saturation principle (in Robinson's credo) guarantees that in every standard Boolean algebra we may find a hyperfinite partition which is finer than each standard finite partition. We call this object an *infinitely fine partition.*

V. Troitskiĭ noticed that an infinitely fine partition possesses some additional properties that can be used for a nonstandard approach to atomic decomposition.

3.1. Hammer–Sobczyk Decomposition Theorem. *Let* μ *be a finite finitely additive positive measure on a Boolean algebra* $\mathcal{A}$. *Then there exist a sequence* $(\delta_n)_{n\in\mathbb{N}}$ *of distinct zero-one measures on* $\mathcal{A}$, *a sequence* $(r_n)_{n\in\mathbb{N}}$ *of positive reals, and a strongly continuous positive measure* μ_0 *on* $\mathcal{A}$, *such that* $\sum_{n=1}^{\infty} r_n < \infty$ *and* $\mu = \mu_0 + \sum_{n=1}^{\infty} r_n\delta_n$. *Furthermore, this decomposition is unique.*

$\lhd$ Let $\mathcal{P}$ be an infinitely fine partition of $\mathcal{A}$. Take as p_1 an element of $\mathcal{P}$ of greatest measure. Let $r_1 = {}^{\circ}\mu(p_1)$ and $\delta_1 = \delta_{p_1}$. If μ is strongly continuous then

$r_1 = 0$. It follows that $\mu_1 = \mu - r_1\delta_1$ is a positive standard measure. Now we may apply this procedure to μ_1 and obtain $\mu_2 = \mu_1 - r_2\delta_2$, etc. Iterating the process, we obtain the decreasing sequence $(r_n)_{n\in{}^\circ\mathbb{N}}$ of standard positive reals, the sequence $(\delta_n)_{n\in{}^\circ\mathbb{N}}$ of standard distinct zero-one measures. By the transfer principle we may extend these sequences to some standard sequences indexed by $\mathbb{N}$. It is easy that μ_0 defined by $\mu_0 = \mu - \sum_{n=1}^{\infty} r_n\delta_n$ is a standard strongly continuous positive measure. $\triangleright$

The *Boolean-valued approach* deals with vector-valued measures. We introduce a new notion of atomicity, and prove a corresponding result on atomic decomposition. As above, we denote by $\mathbb{B} := \mathfrak{P}(F)$ the Boolean algebra of all band projections in F.

A positive measure $\mu \in \mathrm{ba}\,(\mathcal{A},\ F)$ is $\mathbb{B}$-*discrete* if for every measure $\nu \in \mathrm{ba}\,(\mathcal{A},\ F)$ such that $0 \le \nu \le \mu$ the representation $\nu = \alpha\mu$ holds for some operator $\alpha : F \to F$ satisfying $0 \le \alpha \le I_F$.

We say that a measure μ is $\mathbb{B}$-*atomic* if μ belongs to the band in $\mathrm{ba}\,(\mathcal{A},\ F)$ generated by $\mathbb{B}$-discrete elements, and μ is $\mathbb{B}$-*diffuse* if μ belongs to the complementary band.

The essential starting point for $\mathbb{B}$-atomic decomposition is the following result on Boolean-valued representation of vector measures, see [12].

3.2. Theorem. *For every measure $\mu \in \mathrm{ba}\,(\mathcal{A},\ F)$ there exists a unique element $m \in V^{(\mathbb{B})}$ such that $[\![m \in \mathrm{ba}\,(\mathcal{A},\ \mathcal{R})]\!] = 1$ and $[\![m(a) = m(\widehat{a})]\!] = 1$ $a \in \mathcal{A}$. The correspondence $\mu \mapsto m$ is a lattice isomorphism from* $\mathrm{ba}\,(\mathcal{A},\ F)$ *to* $\mathrm{ba}\,(\mathcal{A},\ \mathcal{R}){\downarrow}$. *If $F = \mathcal{R}{\downarrow}$ then the isomorphism is a bijection. Moreover,*

(1) *μ is $\mathbb{B}$-discrete if and only if m is discrete inside $V^{(\mathbb{B})}$;*
(2) *μ is $\mathbb{B}$-atomic if and only if m is atomic inside $V^{(\mathbb{B})}$;*
(3) *μ is $\mathbb{B}$-diffuse if and only if m is diffuse inside $V^{(\mathbb{B})}$.*

We are now in a position to state the $\mathbb{B}$-atomic decomposition theorem and the fact that the $\mathbb{B}$-atomic fragment of a vector measure is the sum of a disjoint sequence of "spectral measures," see [13].

3.3. Theorem. *Let $\mu : \mathcal{A} \to F$ be a positive bounded measure. Then there exist a positive $\mathbb{B}$-diffuse measure $\mu_0 : \mathcal{A} \to F$, a decreasing sequence of positive elements $(F_n)_{n\in\mathbb{N}}$ in F, and a sequence of pairwise disjoint Boolean homomorphisms $h_n : \mathcal{A} \to \mathbb{B}$, $n \in \mathbb{N}$, such that the series $\sum_{n=1}^{\infty} f_n$ is order convergent and*

$$\mu(a) = \mu_0(a) + \sum_{n=1}^{\infty} h_n(a) f_n \quad (a \in \mathcal{A}).$$

The representation of μ in this form is unique.

The problem of extending a finitely additive vector measure may be treated analogously by successive application of the two nonstandard approaches we discuss, see [14].

4. Boolean-valued Banach spaces

Here we expose the Boolean-valued approach to studying Banach spaces with complete Boolean algebras of projections. These spaces appear in various branches of analysis, most frequently in the theory of operator algebras [4, 15, 16].

Let X be a Banach space and let $\mathcal{L}(X)$ stand for the set of all bounded linear operators on X. Assume that a mapping $\varphi : \mathbb{B} \longrightarrow \mathcal{L}(X)$ is injective and satisfies the following conditions:

(1) $\varphi(b)$ is a projection with norm one for all $b \in \mathbb{B}$; moreover, $\varphi(\mathbf{1}) = I_X$ and $\varphi(\mathbf{0}) = 0$;

(2) the projections $\varphi(b)$ and $\varphi(b')$ commute for arbitrary $b, b' \in \mathbb{B}$;

(3) the equalities $\varphi(b \vee b') = \varphi(b) \circ \varphi(b')$ and $\varphi(b^*) = I_X - \varphi(b)$ hold for all b and b'.

In this case the set $\mathcal{B} := \varphi(\mathbb{B})$ is referred to as a *complete Boolean algebra of projections* in the space X. The above situation is symbolized by $\mathbb{B} \sqsubset \mathcal{L}(X)$ and the Boolean algebras $\mathbb{B}$ and $\mathcal{B}$ are identified. If $(e_\xi)_{\xi\in\Xi}$ is a partition of unity in $\mathbb{B}$ and $(x_\xi)_{\xi\in\Xi}$ is a family in X then an element $x \in X$ for which $e_\xi x_\xi = e_\xi x$ $(\xi \in \Xi)$ is called the *mixing* of (x_ξ) given (e_ξ). A Banach space X is $\mathbb{B}$-*cyclic* if $\mathbb{B} \subset \mathcal{L}(X)$ and the following conditions hold:

(4) the mixing of an arbitrary bounded family in X given a partition of unity in $\mathbb{B}$ (with the same index set) exists and is unique;

(5) the unit ball of X is closed under mixing.

The simplest example of a $\mathbb{B}$-cyclic Banach space is the space of continuous functions on the Stone space of $\mathbb{B}$. Clearly, an $AW^\star$-algebra and an JW-algebra are $\mathbb{B}$-cyclic Banach spaces.

Let X and Y be Banach spaces; moreover, assume that $\mathbb{B} \subset \mathcal{L}(X)$ and $\mathbb{B} \subset \mathcal{L}(Y)$. An operator $T : X \longrightarrow Y$ is $\mathbb{B}$-*linear* if T is linear and commutes with projections in $\mathbb{B}$. The set of all bounded $\mathbb{B}$-linear operators from X into Y is $\mathbb{B}$-cyclic if Y is a $\mathbb{B}$-cyclic Banach space. We call a bijective $\mathbb{B}$-linear operator a $\mathbb{B}$-*isomorphism*; if, in addition, the operator is norm-preserving, we speak about an *isometric* $\mathbb{B}$-*isomorphism.*

If X is a Banach space inside $V^{(\mathbb{B})}$ then the *restricted descent* of X consists of elements $x \in X{\downarrow}$ satisfying $[\![\|x\| \leq {}^\wedge c]\!] = 1$ for some $c \in \mathbb{R}$.

4.1. Theorem. *The restricted descent of a Banach space within* $V^{(\mathbb{B})}$ *is a* $\mathbb{B}$-*cyclic Banach space. Conversely, if* X *is a* $\mathbb{B}$-*cyclic Banach space, then there is a Banach space* $\mathcal{X}$ *in* $V^{(\mathbb{B})}$, *unique up to isometric isomorphism, such that the restricted descent of* $\mathcal{X}$ *is isometrically* $\mathbb{B}$-*isomorphic to* X.

The notions of *nonstandard hull and* ρ*-nonarchimedean hull* of a normed space are very useful patent tools of the geometric theory of Banach spaces. These notions were originally introduced by W. A. J. Luxemburg as a development of ideas of A. Robinson. It seems interesting to extend both notions to $\mathbb{B}$-cyclic Banach spaces. This could provide a unified view of many results on the nonstandard hulls of Banach spaces and Banach lattices.

4.2. Problem. *Develop a theory of nonstandard hulls in a Boolean-valued model. Study the corresponding construction of the "descended" nonstandard hull.*

To address the problem needs developing and simplifying the formal facilities that appear along the described path of combining descents and ascents with an intermediate application of Robinson's infinitesimal analysis.

In closing, we observe that there are at least three possibilities of combining nonstandard methods: the first consists in implementing infinitesimal constructions in a Boolean-valued model; the second, in applying nonstandard methods successively; and the third, in seeking for Boolean-valued interpretations of internal and external set theories. Each of these approaches involves its own technical advantages and disadvantages and should be judged upon its own merits and demerits.

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