УДК 514.172:517.982:519.81:519.855

ORDER ANALYSIS AND DECISION MAKING

Kutateladze S. S.

The interaction between order analysis and decision making is illustrated with the multiple criteria problems of convex geometry over the space of Minkowski balls.

1. Introduction

Analysis is a very old term of science with a long history stemming from Ancient Hellas. Morris Kline stated in [1, p. 279] that the term was introduced by Theon of Alexandria (355–405 CE). Francois Viete used the term "analytic art" for algebra in 1591 in *In Artem Analyticem Isagoge* [2]. David Hilbert wrote in [3, p. 373] that analysis is "the most aesthetic and delicately erected structure of mathematics" and called it "a symphony of the infinite."

Prevalence of one magnitude over the other is one of the earliest abstractions of humankind. In the modern mathematical parlance, the idea of transitive antisymmetric relation had preceded the concept of order.

Order and analysis were combined in the first third of the twentieth century which marked an important twist in the content of mathematics. Mathematical ideas imbued the humanitarian sphere and, primarily, politics, sociology, and economics. Social events are principally volatile and possess a high degree of uncertainty. Economic processes utilize a wide range of the admissible ways of production, organization, and management. The nature of nonunicity in economics transpires: The genuine interests of human beings cannot fail to be contradictory. The unique solution is an oxymoron in any nontrivial problem of economics which refers to the distribution of goods between a few agents. It is not by chance that the social sciences and instances of humanitarian mentality invoke the numerous hypotheses of the best organization of production and consumption, the most just and equitable social structure, the codices of rational behavior and moral conduct, etc.

The twentieth century became the age of freedom. Plurality and unicity were confronted as collectivism and individualism. Many particular phenomena of life and culture reflect their distinction. The dissolution of monarchism and tyranny was accompanied by the rise of parliamentarism and democracy. Quantum mechanics and Heisenberg's uncertainty incorporated plurality in physics. The waves of modernism in poetry and artistry should be also listed. Mankind had changed all valleys of residence and dream.

In mathematics the quest for plurality led to the abandonment of the overwhelming pressure of unicity and categoricity. The latter ideas were practically absent, at least minor, in Ancient Greece and sprang to life in the epoch of absolutism and christianity. G. Cantor (1845–1918) was a harbinger of mighty changes, claiming that "*Wesen* der *Mathematik* liegt gerade in ihrer *Freiheit*."

Decision making has become a science in the twentieth century. The presence of many contradictory conditions and conflicting interests is the main particularity of the social situations under control of today. Management by objectives is an exceptional instance of the stock of rather complicated humanitarian problems of goal agreement which has no candidates for a unique solution.

The extremal problems of optimizing several parameters simultaneously are collected nowadays under the auspices of *vector* or *multiobjective optimization*. Search for control in these circumstances is *multiple criteria decision making*. The mathematical apparatus of these areas of research is not rather sophisticated at present (see [4] — [6] and the references therein). The overview of the history of multiple criteria decision making is presented in [7].

The today's research deals mostly with the concept of *Pareto optimality*. Consider a bunch of economic agents each of which intends to maximize his own income. The Pareto efficiency principle asserts that as an effective agreement of the conflicting goals it is reasonable to take any state in which nobody can increase his income in any way other than diminishing the income of at least one of the other fellow members. Formally speaking, this implies the search of maximal elements of the set comprising the tuples of incomes of the agents at every state; i.e., some vectors of a finite-dimensional arithmetic space endowed with the coordinatewise order. Clearly, the concept of Pareto optimality was already abstracted to arbitrary ordered vector spaces.

The variational principles of mechanics, precursors of variational cal-

culus, served at least partly to justifying the Christian belief in the unicity and beauty of the act of creation. The extremal problems, generously populating all branches of mathematics, use only scalar targets. Problems with many objectives have become the topic of research rather recently and noticeably beyond mathematics, which explains the substantial gap between the levels of complexity and power of the mathematical tools available for single objective and multiple objective problems. This challenges the task of enriching the stock of vector optimization problems within the theoretical core of mathematics.

For the sake of simplicity, it stands to reason to start with the problems using the concept of Pareto optimality. The point is that each problem of the sort is in fact equivalent to a parametric family of single objective problems that can be inspected by the classical methods. For instance, there is a curve joining the Legendre and Chebyshev polynomials which consists of the polynomials "Pareto-optimal" with respect to the uniform and mean square metrics. Clearly, some physical processes admit description in terms of vector optimization. For instance, we may treat the Leidenfrost effect of evaporation of a liquid drop in the spheroidal state as the problem of simultaneous minimization of the surface area and the width of a drop of a given volume.

Under study in this article is the class of geometrically meaningful vector optimization problems whose solutions can be found explicitly to some extend in terms of conditions on surface area measures. As model examples we give explicit solutions of the Urysohn-type problems aggravated by the flattening condition or the requirement to optimize the convex hull of a few figures. Technically speaking, everything reduces to the parametric programming of isoperimetric type problems with many subsidiary constraints along the lines of the approach developed in [8]. We will pay a special attention to vector problems of the space of Minkowski balls as presented in [9].

2. Convex Bodies, Balls, and Dual Cones

A *convex figure* is a compact convex set. A *convex body* is a solid convex figure. The *Minkowski duality* identifies a convex figure S in $\mathbb{R}^N$ and its *support function* $S(z) := \sup\{(x,z) \mid x \in S\}$ for $z \in \mathbb{R}^N$. Considering the members of $\mathbb{R}^N$ as singletons, we assume that $\mathbb{R}^N$ lies in the set $\mathscr{V}_N$ of all compact convex subsets of $\mathbb{R}^N$.

The Minkowski duality makes $\mathscr{V}_N$ into a cone in the space $C(S_{N-1})$ of

continuous functions on the Euclidean unit sphere S_{N-1}, the boundary of the unit ball $\mathfrak{z}_N$. The *linear span* $[\mathscr{V}_N]$ of $\mathscr{V}_N$ is dense in $C(S_{N-1})$, bears a natural structure of a vector lattice and is usually referred to as the *space of convex sets*.

The study of this space stems from the pioneering breakthrough of Alexandrov in 1937and the further insights of Radström, Hörmander, and Pinsker (see [9]).

A measure μ *linearly majorizes* or *dominates* a measure ν on S_{N-1} provided that to each decomposition of S_{N-1} into finitely many disjoint Borel sets $U_1, \dots, U_m$ there are measures $\mu_1, \dots, \mu_m$ with sum μ such that every difference $\mu_k - \nu|_{U_k}$ annihilates all restrictions to S_{N-1} of linear functionals over $\mathbb{R}^N$. In symbols, we write $\mu \gg_{\mathbb{R}^N} \nu$.

Reshetnyak proved in 1954 that

$$\int\limits_{S_N - 1} p d\mu \geq \int\limits_{S_N - 1} p d\nu$$

for each sublinear functional p on $\mathbb{R}^N$ if $\mu \gg_{\mathbb{R}^N} \nu$. This gave an important trick for generating positive linear functionals over various classes of convex surfaces and functions. The converse of the Reshetnyak result was appeared in [10] and [11].

Alexandrov proved the unique existence of a translate of a convex body given its surface area function, thus completing the solution of the Minkowski problem. Each surface area function is an *Alexandrov measure*. So we call a positive measure on the unit sphere which is supported by no great hypersphere and which annihilates singletons.

Each Alexandrov measure is a translation-invariant additive functional over the cone $\mathscr{V}_N$. The cone of positive translation-invariant measures in the dual $C'(S_{N-1})$ of $C(S_{N-1})$ is denoted by $\mathscr{A}_N$.

Given $\mathfrak{x}, \mathfrak{y} \in \mathscr{V}_N$, the record $\mathfrak{x} =_{\mathbb{R}^N} \mathfrak{y}$ means that $\mathfrak{x}$ and $\mathfrak{y}$ are equal up to translation or, in other words, are translates of one another. So, $=_{\mathbb{R}^N}$ is the associate equivalence of the preorder $\geq_{\mathbb{R}^N}$ on $\mathscr{V}_N$ of the possibility of inserting one figure into the other by translation.

The sum of the surface area measures of $\mathfrak{x}$ and $\mathfrak{y}$ generates the unique class $\mathfrak{x} \# \mathfrak{y}$ of translates which is referred to as the *Blaschke sum* of $\mathfrak{x}$ and $\mathfrak{y}$. There is no need in discriminating between a convex figure, the coset of its translates in $\mathscr{V}_N / \mathbb{R}^N$, and the corresponding measure in $\mathscr{A}_N$.

Let $C(S_{N-1})/\mathbb{R}^N$ stand for the factor space of $C(S_{N-1})$ by the subspace of all restrictions of linear functionals on $\mathbb{R}^N$ to S_{N-1}. Let $[\mathscr{A}_N]$ be

the space $\mathscr{A}_N - \mathscr{A}_N$ of translation-invariant measures, in fact, the linear span of the set of Alexandrov measures.

$C(S_{N-1})/\mathbb{R}^N$ and $[\mathscr{A}_N]$ are made dual by the canonical bilinear form

$$\langle f, \mu \rangle = \frac{1}{N} \int\limits_{S_{N-1}} f d\mu$$
$$(f \in C(S_{N-1})/\mathbb{R}^N,\ \mu \in [\mathscr{A}_N]).$$

For $\mathfrak{x} \in \mathscr{V}_N/\mathbb{R}^N$ and $\mathfrak{y} \in \mathscr{A}_N$, the quantity $\langle \mathfrak{x}, \mathfrak{y} \rangle$ coincides with the *mixed volume* $V_1(\mathfrak{y}, \mathfrak{x})$.

Consider the set $\operatorname{Sym} \mathscr{V}_N$ of centrally symmetric cosets of convex compact sets. Clearly, a translation-invariant linear functional f is positive over $\operatorname{Sym} \mathscr{V}_N$ if and only if the *symmetrization* $\operatorname{Sym}(f)$ is positive over $\mathscr{V}_N$. Here $\operatorname{Sym}(f)$ is the dual of the descent of the even part operator on the factor-space, since the symmetrization of a measure is the dual of the even part operator over $C(S_{N-1})$. We will denote the even part operator, its descent and dual by the same symbol $\operatorname{Sym}(\cdot)$.

Given a cone K in a vector space X in duality with another vector space Y, the *dual* of K is

$$K^* := \{y \in Y \mid (\forall x \in K)\ \langle x, y \rangle \geq 0\}.$$

To a convex subset U of X and $\bar{x} \in U$ there corresponds

$$U_{\bar{x}} := \operatorname{Fd}(U, \bar{x}) := \{h \in X \mid (\exists \alpha \geq 0)\ \bar{x} + \alpha h \in U\},$$

the *cone of feasible directions* of U at $\bar{x}$.

Let $\bar{\mathfrak{x}} \in \mathscr{A}_N$. Then the dual $\mathscr{A}^*_{N,\bar{\mathfrak{x}}}$ of the cone of feasible directions of $\mathscr{A}_N$ at $\bar{\mathfrak{x}}$ may be represented as follows

$$\mathscr{A}^*_{N,\bar{\mathfrak{x}}} = \{f \in \mathscr{A}^*_N \mid \langle \bar{\mathfrak{x}}, f \rangle = 0\}.$$

The description of the dual of the feasible cones are well known:

Let $\mathfrak{x}$ and $\mathfrak{y}$ be convex figures. Then

(1) $\mu(\mathfrak{x}) - \mu(\mathfrak{y}) \in \mathscr{V}^*_N \leftrightarrow \mu(\mathfrak{x}) \gg_{\mathbb{R}^N} \mu(\mathfrak{y})$;

(2) *If* $\mathfrak{x} \geq_{\mathbb{R}^N} \mathfrak{y}$ *then* $\mu(\mathfrak{x}) \gg_{\mathbb{R}^N} \mu(\mathfrak{y})$;

(3) $\mathfrak{x} \geq_{\mathbb{R}^2} \mathfrak{y} \leftrightarrow \mu(\mathfrak{x}) \gg_{\mathbb{R}^2} \mu(\mathfrak{y})$;

(4) *If* $\mu(\mathfrak{y}) - \mu(\bar{\mathfrak{x}}) \in \mathscr{V}^*_{N,\bar{\mathfrak{x}}}$ *then* $\mathfrak{y} =_{\mathbb{R}^N} \bar{\mathfrak{x}}$ *for* $\bar{\mathfrak{x}} \in \mathscr{V}_N$.

From this the dual cones are available in the case of Minkowski balls.

Let $\mathfrak{x}$ and $\mathfrak{y}$ be convex figures. Then

(1) $\mu(\mathfrak{x}) - \mu(\mathfrak{y}) \in \operatorname{Sym} \mathscr{V}_N^* \leftrightarrow \operatorname{Sym}(\mu(\mathfrak{x})) \gg_{\mathbb{R}^N} \operatorname{Sym}(\mu(\mathfrak{y}))$;

(2) *If* $\mathfrak{x} \geq_{\mathbb{R}^N} \mathfrak{y}$ *then* $\operatorname{Sym}(\mu(\mathfrak{x})) \gg_{\mathbb{R}^N} \operatorname{Sym}(\mu(\mathfrak{y}))$;

(3) $\operatorname{Sym}(\mathfrak{x}) \geq_{\mathbb{R}^2} \operatorname{Sym}(\mathfrak{y}) \leftrightarrow \operatorname{Sym}(\mu(\mathfrak{x})) \gg_{\mathbb{R}^2} \operatorname{Sym}(\mu(\mathfrak{y}))$;

(4) *If* $\mu(\mathfrak{y}) - \mu(\bar{\mathfrak{x}}) \in \operatorname{Sym} \mathscr{V}_{N,\bar{\mathfrak{x}}}^*$ *then* $\operatorname{Sym}(\mathfrak{y}) =_{\mathbb{R}^N} \bar{\mathfrak{x}}$ *for* $\bar{\mathfrak{x}} \in \operatorname{Sym} \mathscr{V}_N$.

Alexandrov observed that the gradient of $V(\cdot)$ at $\mathfrak{x}$ is proportional to $\mu(\mathfrak{x})$ and so minimizing $\langle \cdot, \mu \rangle$ over $\{V = 1\}$ will yield the equality $\mu = \mu(\mathfrak{x})$ by the Lagrange multiplier rule. But this idea fails since the interior of $\mathscr{V}_N$ is empty. The fact that DC-functions are dense in $C(S_{N-1})$ is not helpful at all.

Alexandrov extended the volume to the positive cone of $C(S_{N-1})$ by the formula $V(f) := \langle f, \mu(\operatorname{co}(f)) \rangle$ with $\operatorname{co}(f)$ the envelope of support functions below f. He also observed that $V(f) = V(\operatorname{co}(f))$. The ingenious trick settled all for the Minkowski problem. This was done in 1938 but still is one of the summits of convexity.

In fact, Alexandrov suggested a functional analytical approach to extremal problems for convex surfaces. To follow it directly in the general setting is impossible without the above description of the dual cones. The obvious limitations of the Lagrange multiplier rule are immaterial in the case of convex programs. It should be emphasized that the classical isoperimetric problem is not a Minkowski convex program in dimensions greater than 2. The convex counterpart is the Urysohn problem of maximizing volume given integral breadth [12]. The constraints of inclusion type are convex in the Minkowski structure, which opens way to complete solution of new classes of Urysohn-type problems.

External Urysohn Problem: Among the convex figures, circumscribing $\mathfrak{x}_0$ and having integral breadth fixed, find a convex body of greatest volume.

A feasible convex body $\bar{\mathfrak{x}}$ is a solution to the external Urysohn problem if and only if there are a positive measure μ and a positive real $\bar{\alpha} \in \mathbb{R}_+$ satisfying

(1) $\bar{\alpha}\mu(\mathfrak{z}_N) \gg_{\mathbb{R}^N} \mu(\bar{\mathfrak{x}}) + \mu$;

(2) $V(\bar{\mathfrak{x}}) + \frac{1}{N} \int\limits_{S_{N-1}} \bar{\mathfrak{x}} d\mu = \bar{\alpha} V_1(\mathfrak{z}_N, \bar{\mathfrak{x}})$;

(3) $\bar{\mathfrak{x}}(z) = \mathfrak{x}_0(z)$ *for all z in the support of μ, i. e. $z \in \operatorname{spt}(\mu)$.*

If $\mathfrak{x}_0 = \mathfrak{z}_{N-1}$ then $\bar{\mathfrak{x}}$ is a *spherical lens* and μ is the restriction of the surface area function of the ball of radius $\bar{\alpha}^{1/(N-1)}$ to the complement of the support of the lens to S_{N-1}.

3. Order Analysis over Minkowski Balls

Consider a bunch of economic agents each of which intends to maximize his own income. The *Pareto efficiency principle* asserts that as an effective agreement of the conflicting goals it is reasonable to take any state in which nobody can increase his income in any way other than diminishing the income of at least one of the other fellow members. Formally speaking, this implies the search of the maximal elements of the set comprising the tuples of incomes of the agents at every state; i.e., some vectors of a finite-dimensional arithmetic space endowed with the coordinatewise order. Clearly, the concept of Pareto optimality was already abstracted to arbitrary ordered vector spaces.

By way of example, consider a few multiple criteria problems of isoperimetric type. For more detail, see [8].

Vector Isoperimetric Problem over Minkowski Balls: Given are some convex bodies $\mathfrak{y}_1, \dots, \mathfrak{y}_M$. Find a symmetric convex body $\mathfrak{x}$ encompassing a given volume and minimizing each of the mixed volumes $V_1(\mathfrak{x}, \mathfrak{y}_1), \dots, V_1(\mathfrak{x}, \mathfrak{y}_M)$. In symbols,

$$\mathfrak{x} \in \operatorname{Sym}(\mathscr{A}_N);\ \widehat{p}(\mathfrak{x}) \geq \widehat{p}(\bar{\mathfrak{x}});\ (\langle \mathfrak{y}_1, \mathfrak{x} \rangle, \dots, \langle \mathfrak{y}_M, \mathfrak{x} \rangle) \to \inf.$$

Clearly, this is a Slater regular convex program in the Blaschke structure.

Each Pareto-optimal solution $\bar{\mathfrak{x}}$ of the vector isoperimetric problem has the form

$$\bar{\mathfrak{x}} = \alpha_1 \operatorname{Sym}(\mathfrak{y}_1) + \dots + \alpha_m \operatorname{Sym}(\mathfrak{y}_m),$$

where $\alpha_1, \dots, \alpha_m$ are positive reals.

Internal Urysohn Problem with Flattening over Minkowski Balls: Given are some convex body $\mathfrak{x}_0 \in \operatorname{Sym} \mathscr{V}_N$ and some flattening direction $\bar{z} \in S_{N-1}$. Considering $\mathfrak{x} \subset \mathfrak{x}_0$ of fixed integral breadth, maximize the volume of $\mathfrak{x}$ and minimize the breadth of $\mathfrak{x}$ in the flattening direction: $\mathfrak{x} \in \operatorname{Sym} \mathscr{V}_N;\ \mathfrak{x} \subset \mathfrak{x}_0;\ \langle \mathfrak{x}, \mathfrak{z}_N \rangle \geq \langle \bar{\mathfrak{x}}, \mathfrak{z}_N \rangle;\ (-p(\mathfrak{x}), b_{\bar{z}}(\mathfrak{x})) \to \inf.$

For a feasible symmetric convex body $\bar{\mathfrak{x}}$ to be Pareto-optimal in the internal Urysohn problem with the flattening direction $\bar{z}$ over Minkowski balls it is necessary and sufficient that there be positive reals α and β together with a convex figure $\mathfrak{x}$ satisfying

$$\begin{aligned} \mu(\bar{\mathfrak{x}}) &= \operatorname{Sym}(\mu(\mathfrak{x})) + \alpha\mu(\mathfrak{z}_N) + \beta(\varepsilon_{\bar{z}} + \varepsilon_{-\bar{z}}); \\ \bar{\mathfrak{x}}(z) &= \mathfrak{x}_0(z) \quad (z \in \operatorname{spt}(\mu(\mathfrak{x})). \end{aligned}$$

Rotational Symmetry: Assume that a plane convex figure $\mathfrak{x}_0 \in \mathscr{V}_2$ has the symmetry axis $A_{\bar{z}}$ with generator $\bar{z}$. Assume further that $\mathfrak{x}_{00}$ is

the result of rotating $\mathfrak{x}_0$ around the symmetry axis $A_{\bar{z}}$ in $\mathbb{R}^3$. Consider the problem:

$$\begin{gathered}
\mathfrak{x} \in \mathscr{V}_3; \\
\mathfrak{x} \text{ is a convex body of rotation around } A_{\bar{z}}; \\
\mathfrak{x} \supset \mathfrak{x}_{00};\ \langle \mathfrak{z}_N, \mathfrak{x} \rangle \geq \langle \mathfrak{z}_N, \bar{\mathfrak{x}} \rangle; \\
(-p(\mathfrak{x}), b_{\bar{z}}(\mathfrak{x})) \to \inf.
\end{gathered}$$

Each Pareto-optimal solution is the result of rotating around the symmetry axis a Pareto-optimal solution of the plane internal Urysohn problem with flattening in the direction of the axis.

External Urysohn Problem with Flattening over Minkowski Balls: Given are some convex body $\mathfrak{x}_0 \in \mathscr{V}_N$ and flattening direction $\bar{z} \in S_{N-1}$. Considering Minkowski balls $\mathfrak{x} \supset \mathfrak{x}_0$ of fixed integral breadth, maximize volume and minimize breadth in the flattening direction: $\mathfrak{x} \in \operatorname{Sym} \mathscr{V}_N;\ \mathfrak{x} \supset \mathfrak{x}_0;\ \langle \mathfrak{x}, \mathfrak{z}_N \rangle \geq \langle \bar{\mathfrak{x}}, \mathfrak{z}_N \rangle;\ (-p(\mathfrak{x}), b_{\bar{z}}(\mathfrak{x})) \to \inf$.

For a feasible convex body $\bar{\mathfrak{x}}$ to be a Pareto-optimal solution of the external Urysohn problem with flattening over Minkowski balls it is necessary and sufficient that there be positive reals α and β together with a convex figure $\mathfrak{x}$ satisfying

$$\begin{gathered}
\mu(\bar{\mathfrak{x}}) + \operatorname{Sym}(\mu(\mathfrak{x})) \gg_{\mathbb{R}^N} \alpha\mu(\mathfrak{z}_N) + \beta(\varepsilon_{\bar{z}} + \varepsilon_{-\bar{z}}); \\
V(\bar{\mathfrak{x}}) + V_1(\operatorname{Sym}(\mathfrak{x}), \bar{\mathfrak{x}}) = \alpha V_1(\mathfrak{z}_N, \bar{\mathfrak{x}}) + 2N\beta b_{\bar{z}}(\bar{\mathfrak{x}}); \\
\bar{\mathfrak{x}}(z) = \mathfrak{x}_0(z) \quad (z \in \operatorname{spt}(\mu(\mathfrak{x}))).
\end{gathered}$$

4. Vistas in Order, Modeling, and Decision Making

Vector optimization is a very challenging area of science since it serves as one of the theoretical cornerstones of multiple criteria decision making, MCDM. A single criterion decision making, SCDM, makes a pattern for MCDM and there are techniques that enables us to study MCDM using SCDM. These techniques are instances of scalarization.

We have touched here only the simplest version of scalarization which is *parametric programming*. Much more general and and powerful technique of scalarization is Boolean valued analysis which consists in using some nonstandard models of set theory. Nonstandard models of set theory provide the tools that transform the vector optimization problems with target in a Dedekind complete vector lattice into scalar optimization problems. In fact, each formal mathematical result of SCDM is an

encoded result of MSDM. This aspect of interaction between order analysis and decision making deserves a special attention but is left uncharted by limitations of space-time.

Despite the efforts that were put forth to tackle the problems of decision making in the twentieth century, much is left to be desired. Mathematics is still constricted by the ties of unicity and categoricity. Order analysis opens up a few possibilities to scalarize the problems of decision making but we are still at the very beginning of the epoch of new understanding and new methods of mathematical modeling and reasoning.

In 1947 Kurt Gödel wrote in [13, p. 521]: "There might exist axioms so abundant in their verifiable consequences, shedding so much light upon the whole discipline and furnishing such powerful methods for solving given problems (and even solving them, as far as that is possible, in a constructivistic way), that quite irrespective of their intrinsic necessity they would have to be assumed at least in the same sense as any well established physical theory." This prediction of Gödel was a prophetic foresight.

Now the abundance of new nonstandard models is available which open revolutionary new ways to determining the trends of various current processes and predict the future from trends. This aspect of interaction between order analysis and decision making traverses the course of the Kantorovich heuristic principle within Boolean valued models of set theory and relevant ideas of nonstandard analysis.

References

1. *Kline M.* Mathematical Thought From Ancient to Modern Times. Vol. 1.—New York and Oxford: Oxford University Press, 1990.—390+xxii P.
2. *Viéte F.* The Analytic Art.—Kent: Kent State University Press, 1983.—450 P.
3. *Hilbert D.* On the Infinite // In: Van Heijenoort J., Source Book in Mathematical Logic, 1879–1931.—Cambridge: Harvard University Press, 1967.—P. 367–392.
4. *Kusraev A. G. and Kutateladze S. S.* Subdifferentials: Theory and Applications.—Dordrecht: Kluwer Academic Publishers, 1995.—398 P.
5. *Figueira J., Greco S., and Ehrgott M.* Multiple Criteria Decision Analysis. State of the Art Surveys.—Boston: Springer Science + Business Media, Inc., 2005.—1085 P.
6. *Boţ R. I., Grad S-M., and Wanka G.* Duality in Vector Optimization.—Berlin–Heidelberg: Springer-Verlag, 2009.— 408 P.
7. *Köksals M., WAllenius J., and Zionts S.* Multiple Criteria Decision Making From Early History to 21st Century.—New Jersy, London, etc.: World Scientific, 2011.—198 P.

8. *Kutateladze S. S.* Multiobjective Problems of Convex Geometry // Siberian Math. J..—2009.—Vol. 50, No. 5.—P. 887–897.
9. *Kutateladze S. S.* Multiple Criteria Problems over Minkowski Balls // J. Appl. Indust. Math.—2013.—Vol. 7, No. 2, P. 208–214.
10. *Kutateladze S. S.* Positive Linear Functionals over Convex Surfaces // Optimal Planning.—1969.—No. 14.—P. 80–87.
11. *Kutateladze S. S.* Positive Minkowski-Linear Functionals over Convex Surfaces // Soviet Math. Dokl.—1970.—Vol. 11, No. 3.—P. 767–769.
12. *Urysohn P. S.* Interdependence Between the Integral Breadth and Volume of a Convex Body // Mat. Sb.—1924.—Vol. 31, No. 3.—P. 477–486.
13. *Gödel K.* What Is Cantor's Continuum Problem? // Amer. Math. Monthly.—1947.—Vol. 54, No. 9.—P. 515–525.

Кутателадзе С. С.
Институт математики им. С. Л. Соболева
главный научный сотрудник
пр. Коптюга 4, Новосибирск, 630090, Россия
E-mail: `sskut@matj.nsc.ru`

ПОРЯДКОВЫЙ АНАЛИЗ И ПРИНЯТИЕ РЕШЕНИЙ

КУТАТЕЛАДЗЕ С. С.

Взаимодействие между порядковым анализом и принятием решений иллюстрируется многкритериальными задачами выпуклой геометрии над пространством шаров Минковского.