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ON COMBINED NONSTANDARD METHODS IN THE THEORY OF POSITIVE OPERATORS

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To Professor V. È. Lyantse on his 75th birthday

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Two applications of combined nonstandard methods to positive operators and vector measures are presented.

Introduction

Nonstandard methods of analysis split presently so as to form two main disciplines: *infinitesimal analysis* (*Robinson's nonstandard analysis*) and *Boolean-valued analysis*. The common feature of the disciplines is as follows: either executes a comparative study of two interpretations of a mathematical claim or construction considered as a formal symbolic expression by means of two different set-theoretic models, one standard and the other nonstandard, see [1] and the literature cited therein.

In studying the problems of functional analysis which are motivated by the theory of vector lattices, it is necessary to combine theoretical and technical methods that are offered by the two nonstandard disciplines. It is worth emphasizing that the Boolean-valued and infinitesimal versions of nonstandard analysis differ principally in their content and tools. Therefore, many ways are open to their simultaneous application. One of the simplest approaches consists in studying a standard Boolean-valued model in the universe of Nelson's internal set theory (or in the universe of Kawai's external sets). In this case specific infinitesimal methods are used in a world external to the Boolean-valued universe, see [2]. In some cases it is better to use a different approach, i.e., to apply infinitesimal methods to objects within a Boolean-valued universe, see [3].

In the present note we briefly outline two applications of combined nonstandard methods to positive operators and vector measures which are intended as invitation to further research.

We seize the opportunity to acknowledge insightful contributions of Professor V. È. Lyantse to use of nonstandard methods in operator theory.

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1. Boolean-valued models

The Boolean-valued approach is less popular than its infinitesimal counterpart. Thus, it stands to reason to sketch the former. Details can be found in [1, 4]. A Boolean-valued model $V^{(\mathbb{B})}$ is constructed from a complete Boolean algebra $\mathbb{B}$. The elements of $V^{(\mathbb{B})}$ are thought as $\mathbb{B}$-valued sets. Given a proposition $\mathcal{S} := \mathcal{S}(x_1, \dots, x_n)$ with parameters $x_1, \dots, x_n \in V^{\mathbb{B}}$ one defines its Boolean *truth value* $[\![\mathcal{S}]\!] \in V^{(\mathbb{B})}$. The proposition $\mathcal{S}$ is said to be *true in* $V^{(\mathbb{B})}$ if $[\![\mathcal{S}]\!] = 1$.

1.1. Boolean-valued transfer principle. *For every theorem $\mathcal{S}(x_1, \dots, x_n)$ of ZFC, the equality*

$$[\![\mathcal{S}(x_1, \dots, x_n)]\!] = 1$$

holds, i.e., $\mathcal{S}(x_1, \dots, x_n)$ is valid inside $V^{(\mathbb{B})}$.

1.2. Maximum principle. *For every formula $\mathcal{S}(v, x_1, \dots, x_n)$ $(x_1, \dots, x_n \in V^{(\mathbb{B})})$ of ZFC with one free variable v there exists an element $x \in V^{(\mathbb{B})}$ for which*

$$[\![(\exists v)\mathcal{S}(v, x_1, \dots, x_n)]\!] = [\![\mathcal{S}(x, x_1, \dots, x_n)]\!].$$

In particular, $V^{(\mathbb{B})}$ contains some object $\mathcal{R}$ that plays the role of the reals $\mathbb{R}$ inside $V^{(\mathbb{B})}$.

1.3. Restricted transfer principle. There is an important procedure for passing to $V^{(\mathbb{B})}$ from the ordinary von Neumann universe V. It is defined as follows: Given $x \in V$, denote by the symbol $x^{\wedge}$ the *standard name* of x in $V^{(B)}$; i.e., the element defined by the following recursion schema:

$$\varnothing^{\wedge} := \varnothing, \quad \mathrm{dom}\,(x^{\wedge}) := \{ y^{\wedge} \mid y \in x\}, \quad \mathrm{im}(x^{\wedge}) := 1.$$

For every $\mathcal{S}(v_1, \dots, v_n)$ and elements $x_1, \dots, x_n \in V^{(\mathbb{B})}$ the equivalence holds:

$$\mathcal{S}(x_1, \dots, x_n) \leftrightarrow [\![\mathcal{S}(x_1^{\wedge}, \dots, x_n^{\wedge})]\!] = 1.$$

1.4. Descent functor. There is a functorial procedure, called *descent* and denoted by $(\cdot)\!\downarrow$, which assigns to each mathematical object inside a Boolean-valued model a standard object of a similar type. Given an element $x \in V^{(\mathbb{B})}$, its descent is defined by $x\!\downarrow := \{t \in V^{(\mathbb{B})} \mid [\![t \in x]\!]$ and is a set. If $X, Y, f \in V^{(\mathbb{B})}$ and f is a mapping from X into Y inside $V^{(\mathbb{B})}$, then $f\!\downarrow$ is a mapping from $X\!\downarrow$ into $Y\!\downarrow$. The same is true for a relation inside $V^{(\mathbb{B})}$. Therefore, the descent of an algebraic system is again an algebraic system of a similar type. The following result is basic for what follows, see [1, 5].

1.5. Gordon's Theorem. *Let $\mathcal{R}$ be the field of reals inside $V^{(\mathbb{B})}$. The algebraic system $\mathcal{R}\!\downarrow$ is a universally complete Kantorovich space. Moreover, there exists an isomorphism χ of $\mathbb{B}$ onto the Boolean algebra $\mathfrak{P}(\mathcal{R}\!\downarrow)$ of band projections of $\mathcal{R}\!\downarrow$ such that*

$$b \leq [\![x = y]\!] \leftrightarrow \chi(b)x = \chi(b)y,$$
$$b \leq [\![x \leq y]\!] \leftrightarrow \chi(b)x \leq \chi(b)y$$

for all $x, y \in \mathcal{R}\!\downarrow$ and $b \in \mathbb{B}$.

2. The fragments of a positive operator

The first example concerns a class of positive operators, see [6]. Preliminary definitions and facts can be found in [7–10]. Let E and F be vector lattices and let $S : E \to F$ be a positive linear operator. A positive operator $R : E \to F$ is called a *fragment* or *component* of S if $R \wedge (S - R) = 0$. If F is a *Kantorovich space*, i.e. an order complete vector lattice; then every fragment of S can be obtained in the form $R = \mathrm{P}_T S$ by applying the band projection P_T of S onto the principal band $T^{\perp\perp}$ generated by a positive operator $T : E \to F$. Now, the problem consists in finding some explicit representation for the band projection P_T.

Infinitesimal approach: First we consider the case of functionals and derive a formula for P_T.

2.1. Theorem. *Let φ and ψ be positive functionals on a vector lattice E and e be a positive element of E. The following representation holds for the band projection P_φ onto the principal band $\varphi^{\perp\perp}$:*

$$\mathrm{P}_\varphi \psi(x) \mathrel{\underset{\rightharpoondown}{=}} \inf{}^*\{\psi(f) \mid \varphi(e - f) \approx 0, \quad 0 \le f \le e\},$$

where the sign $\underset{\rightharpoondown}{=}$ symbolizes the exactness of the formula from the right, i.e., the attainability of equality at some element of the set on the right-hand side.

Proof. Put $\varrho := \mathrm{P}_\varphi \psi$. If $0 \le f \le e$ and $\varphi(e - f) \approx 0$, then, since

$$\varrho(e) = \varrho(f) + \varrho(e - f) \le \psi(f) + \varrho(e - f)$$

and $\varrho \in \varphi^{\perp\perp}$, we see that $\varrho(e - f) \approx 0$. Therefore, $\varrho(e) \le {}^\circ\psi(f)$. To establish exactness, note the equality $\varphi \wedge (\psi - \varrho)$. This equality ensures the validity of the following claims: $\varphi(e - f) \approx 0$ and $\varrho(f) \approx \psi(f)$ for certain f from the order interval $[0, e]$. Since $\varrho \in \varphi^{\perp\perp}$; therefore, $\varrho(e) \approx \varrho(f)$ and we come to desired equality $\varrho(e) = {}^\circ\psi(f)$.□

2.2. The formula obtained in 2.1 can be easily expanded by application of the Nelson algorithm. Thus, one may conclude that $\varrho = \mathrm{P}_\varphi \psi$ if and only if

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall 0 < f < e)\ \varphi(e - f) \le \delta \to \varrho(e) \le \psi(f) + \varepsilon,$$
$$(\forall \varepsilon > 0)(\forall \delta > 0)(\exists 0 < f < e)\ \varphi(e - f) \le \delta\ \&\ \psi(f) \le \varrho(e) + \varepsilon.$$

Boolean-valued approach: Now, we are in a position to interpret the results for functionals in a Boolean-valued model. In the sequel we assume a vector lattice F to be a universally complete Kantorovich space. In addition, we agree that its filter of order units is denoted by $\mathcal{E}$ and the Boolean algebra of band projections in it is denoted by $\mathbb{B} := \mathbb{B}(F)$. Further, we may presume in virtue of 1.5 that $F = \mathcal{R}\!\downarrow$. Let $L_r(E, F)$ denote the *space of regular operators*, with a linear operator called *regular* whenever it is representable as difference of positive operators. The following theorem enables us to scalarize an operator under study.

2.3. Theorem. *For each $T \in L_r(E, F)$ there is a unique element $\varphi \in V^{(\mathbb{B})}$ such that*

$$[\![\varphi \in L_r(\,\widehat{}E, \mathbb{R})]\!] = 1 \quad \text{and} \quad [\![\varphi(\,\widehat{}x) = Tx]\!] = 1$$

for all $x \in X$. The mapping $T \mapsto \varphi$ implements a linear and latticial isomorphism between $L_r(E, F)$ and $L_r(\,\widehat{}E, \mathbb{R})\!\downarrow$.

The functional φ inside $V^{(\mathbb{B})}$ is often denoted by $T\!\uparrow$.

2.4. Theorem. *Let S and T be positive operators from E into F and let R be the band projection of S onto the principal band $T^{\perp\perp}$. The following representation is valid for a positive $e \in E$:*

$$Re = \sup_{\varepsilon\in\mathcal{E}} \inf\{\pi Sf + \pi^{\perp}Se \mid 0 \leq f \leq e, \quad \pi \in \mathbb{B}(F), \quad \pi T(e-f) \leq \varepsilon\}.$$

Proof. We pass to the Boolean-valued universe over the Boolean algebra $\mathbb{B} := \mathbb{B}(E)$, while putting $\varphi := T\!\uparrow$, $\psi := S\!\uparrow$, and $\varrho := R\!\uparrow$. Then $\varrho = \mathrm{P}_{\varphi}\psi$ inside $V^{(\mathbb{B})}$ and we may apply 2.1. Computing the truth values, we obtain

$$(\forall\varepsilon \in \mathcal{E})(\exists\delta \in \mathcal{E})(\forall 0 < f < e)(\forall \pi \in \mathbb{B})$$
$$\pi T(e-f) \leq \delta \to Re \leq \pi Sf + \pi^{\perp}Se + \varepsilon,$$
$$(\forall\varepsilon \in \mathcal{E})(\forall\delta \in \mathcal{E})(\exists(f_{\xi}))(\exists(\pi_{\xi}))(\forall\xi)$$
$$\pi_{\xi}T(e-f) \leq \delta \ \ \& \ \ \pi_{\xi}S(f_{\xi}) \leq \pi_{\xi}Re + \varepsilon$$

for a certain family (f_{ξ}) of elements of the order interval $[0, e]$ and a certain partition of unity (π_{ξ}) in the algebra $\mathbb{B}$. Whence the sought representation is immediate.□

Using the infinitesimal technique within a Boolean-valued universe and descending Theorem 2.1, in the natural notation we arrive at

2.5. Theorem. *The following exact formula holds:*

$$Re \rightleftharpoons \inf\{\pi^{\circ}Sf + \pi^{\perp}Se \mid \pi T(e-f) \approx 0, \quad 0 \leq f \leq e\}$$

3. Atomic decomposition of vector measures

Now we apply the above-mentioned nonstandard methods to the problem of atomic decomposition of a finitely additive measure. Let $\mathcal{A}$ be a Boolean algebra and let F be a vector lattice. By a *vector measure* we mean an arbitrary mapping $\mu : \mathcal{A} \to F$ such that

$$\mu(a_1 \vee a_2) = \mu(a_1) + \mu(a_2)$$

for all disjoint $a_1, a_2 \in \mathcal{A}$. Denote by $\mathrm{ba}(\mathcal{A}, F)$ the space of all bounded F-valued measures and put $\mathrm{ba}(\mathcal{A}) := \mathrm{ba}(\mathcal{A}, \mathcal{R})$. A measure $\mu \in \mathrm{ba}(\mathcal{A}, F)$ is said to be *positive* if $\mu(a) \geq 0$ for all $a \in \mathcal{A}$. It is well known that $\mathrm{ba}(\mathcal{A}, F)$ is a Kantorovich space whose positive cone coincides with the set of positive measures. An *atom* of a measure $\mu \in \mathrm{ba}(\mathcal{A})$ is an element $a_0 \in \mathcal{A}$ such that $\mu(a_0) \neq 0$ and for every $a \in \mathcal{A}$, $a \leq a_0$, either $\mu(a) = 0$, or $\mu(a_0 \setminus a) = 0$. We say that a measure μ is *atomic* if it belongs to the band in $\mathrm{ba}(\mathcal{A})$ generated by discrete elements, and is *diffuse* if it belongs to the complementary band.

Infinitesimal approach: We shall use in this section Loeb's [11] concept of infinitely fine partition.

3.1. By a *finite partition* in a Boolean algebra $\mathcal{A}$, we mean a finite collection $\{a_1, \dots, a_n\} \subset \mathcal{A}$ of nonzero elements with $\bigvee_{k=1}^{n} a_k = 1$ and $a_k \wedge a_l = 0$ for $k \neq l$. Denote by $\mathcal{P} := \mathcal{P}(\mathcal{A})$ the collection of all finite partitions in $\mathcal{A}$. Take $p_1, p_2 \in \mathcal{P}$. We say that p_1 is *finer* than p_2 if $a = \bigvee\{b \in p_1 \mid b \leq a\}$ for each $a \in p_2$. The idealization principle (in Nelson's credo) or the saturation principle (in Robinson's credo) guarantees that in every standard Boolean algebra we can find a hyperfinite partition which is finer than each standard finite partition. We call such a partition an *infinitely fine partition* (i.f.p.).

3.2. Troitskiĭ (in his Master thesis) noticed that an i.f.p. possesses some additional properties that can be used for a nonstandard approach to atomic decomposition. Two elements of an i.f.p. are said to be *equivalent* if the partition remains infinitely fine after substituting the join of the two elements for each of them. Two elements are equivalent if and only if they cannot be separated by a standard element of $\mathcal{A}$. Call each collection of mutually equivalent elements a *monadic* i.f.p.

Among the elements of a monadic i.f.p., one is distinguished; we call it *central.* Each element of an i.f.p. generates some standard zero-one measure that assigns 1 to every standard set containing this element. Equivalent elements generate the same zero-one measure. This measure takes value 1 on the central element of the corresponding monad and vanishes on all the other elements of the i.f.p.

Lemma. *For the central element $p \in \mathcal{P}$ of a monadic i.f.p. the inequality $\mu(p) \geq {}^\circ\mu(p)$ holds.*

Now we are ready to give a nonstandard proof of the Hammer-Sobczik Decomposition Theorem.

3.3. Hammer-Sobczik Decomposition Theorem. *Let μ be a finite finitely additive positive measure on a Boolean algebra $\mathcal{A}$. Then there exist a sequence $(\delta_n)_{n\in\mathbb{N}}$ of distinct zero-one measures on $\mathcal{A}$, a sequence $(r_n)_{n\in\mathbb{N}}$ of positive reals, and a strongly continuous positive measure μ_0 on $\mathcal{A}$, such that $\sum_{n=1}^{\infty} r_n < \infty$ and*

$$\mu = \mu_0 + \sum\nolimits_{n=1}^{\infty} r_n\delta_n.$$

Furthermore, this decomposition is unique.

Proof. Let $\mathcal{P}$ be an i.f.p. of $\mathcal{A}$. Take as p_1 an element of $\mathcal{P}$ of greatest measure. Let $r_1 = {}^\circ\mu(p_1)$ and $\delta_1 = \delta_{p_1}$. If μ is strongly continuous, then $r_1 = 0$. Otherwise, p_1 is central and, by the Lemma, we have $r_1 \leq \mu(p_1)$. It follows that $\mu_1 = \mu - r_1\delta_1$ is a positive standard measure. Now we may apply this procedure to μ_1 and obtain $\mu_2 = \mu_1 - r_2\delta_2$, etc. Iterating the process, we obtain the decreasing sequence $(r_n)_{n\in{}^\circ\mathbb{N}}$ of standard positive reals, the sequence $(\delta_n)_{n\in{}^\circ\mathbb{N}}$ of standard distinct zero-one measures, and the sequence $(\mu_n)_{n\in{}^\circ\mathbb{N}}$ of standard measures such that

$$\mu_n = \mu - \sum\nolimits_{k=1}^{n} r_k\delta_i$$

for every $n \in {}^\circ\mathbb{N}$. By the transfer principle we may extend these sequences to some standard sequences indexed by $\mathbb{N}$ with the same properties. In particular,

$$\mu(1) \geq \sum\nolimits_{k=1}^{n} r_k\delta_k(1) = \sum\nolimits_{k=1}^{n} r_k$$

for every natural n; it follows that $\sum_{k=1}^{\infty} r_k < \infty$.

It is easily verified that μ_0 defined by

$$\mu_0 = \mu - \sum\nolimits_{n=1}^{\infty} r_n\delta_n$$

is a standard positive measure, and $\mu_0 \leq \mu_n$ for all $n \in \mathbb{N}$. Assume that μ_0 is not strongly continuous; then there exists $p \in P$ such that ${}^\circ\mu(p) > 0$. Since r_n converges to zero, we can find $n \in {}^\circ\mathbb{N}$ such that $r_n < {}^\circ\mu(p)$; by the definition of r_n it follows that ${}^\circ\mu_n(p) < {}^\circ\mu(p)$, but this contradicts the fact that $\mu \leq \mu_n$.

We see that the reals r_n and measures δ_n are determined by the values of μ at the central elements of $\mathcal{P}$ up to permutation. Thus, the constructed decomposition is unique.□

Boolean-valued approach: Now we introduce a new notion of atomicity, and prove a corresponding result on atomic decomposition. As above, we denote by $\mathbb{B} := \mathfrak{P}(F)$ the Boolean algebra of all band projections in F.

A positive measure $\mu \in \mathrm{ba}(\mathcal{A}, F)$ is called $\mathbb{B}$*-discrete* if for every measure $\nu \in \mathrm{ba}(\mathcal{A}, F)$, such that $0 \leq \nu \leq \mu$, we have the representation $\nu = \alpha\mu$ for some operator $\alpha : F \to F$, $0 \leq \alpha \leq I_F$.

We say that a measure μ is $\mathbb{B}$-*atomic* if it belongs to the band in $\mathrm{ba}(\mathcal{A}, F)$ generated by $\mathbb{B}$-discrete elements, and is $\mathbb{B}$-*diffuse* if it belongs to the complementary band.

The essential starting point for the $\mathbb{B}$-atomic decomposition is the following result on Boolean-valued representation of vector measures, see [12].

3.4. Theorem. *For every measure $\mu \in \mathrm{ba}(\mathcal{A}, F)$ there exists a unique element $m \in V^{(\mathbb{B})}$ such that*

$$[\![m \in \mathrm{ba}(\mathcal{A}, \mathcal{R})]\!] = 1 \quad \text{and} \quad [\![m(a) = m(\widehat{a})]\!] = 1$$

$a \in \mathcal{A}$. The correspondence $\mu \mapsto m$ represents a lattice isomorphism from $\mathrm{ba}(\mathcal{A}, F)$ to $\mathrm{ba}(\mathcal{A}, \mathcal{R}){\downarrow}$. If $F = \mathcal{R}{\downarrow}$ then the isomorphism is a bijection.

It can be easily proved that

(1) μ is $\mathbb{B}$-discrete if and only if m is discrete inside $V^{(\mathbb{B})}$;
(2) μ is $\mathbb{B}$-atomic if and only if m is atomic inside $V^{(\mathbb{B})}$;
(3) μ is $\mathbb{B}$-diffuse if and only if m is diffuse inside $V^{(\mathbb{B})}$.

Now we may prove the $\mathbb{B}$-atomic decomposition theorem and the fact that the $\mathbb{B}$-atomic fragment of a vector measure is the sum of a disjoint sequence of "spectral measures," see [13].

3.5. Theorem. *Let $\mu : \mathcal{A} \to F$ be a positive bounded measure. Then there exist a positive $\mathbb{B}$-diffuse measure $\mu_0 : \mathcal{A} \to F$, a decreasing sequence of positive elements $(F_n)_{n\in\mathbb{N}}$ in F, and a sequence of pairwise disjoint Boolean homomorphisms $h_n : \mathcal{A} \to \mathbb{B}$, $n \in \mathbb{N}$, such that the series $\sum_{n=1}^{\infty} f_n$ is order convergent and*

$$\mu(a) = \mu_0(a) + \sum_{n=1}^{\infty} h_n(a) f_n \quad (a \in \mathcal{A}).$$

The representation of μ in this form is unique.

The problem of extending a finitely additive vector measure can be treated analogously, see [14].

4. Concluding remarks

For applications in functional analysis and operator theory, it is highly desirable to expound a synthetic theory in the framework of which all the nonstandard methods offered by Boolean-valued models and external set theories are available. Only preliminary results have so far been achieved in this direction ([15], [16]).

In the previous sections we demonstrated that successive application of the two nonstandard approaches is a powerful tool sometimes. In view of these facts one may suppose that the above-presented combinations of nonstandard methods are not only highly desirable but also manageable.

It seems reasonable to develop and simplify the formal possibilities that appear along the described path of combining the procedures of descent and ascent with an intermediate application of the apparatus of Robinson's nonstandard analysis.

Roughly speaking, infinitesimal analysis allows one to somehow consider an operator as a matrix, i.e., to carry out *discretization* of an operator under study. Boolean-valued analysis makes it possible to treat the elements of some functional classes as reals, and enables one, in particular, to work with an operator as with a functional, i.e., to implement *scalarization* of the operator. Thus, it stands to reason to develop some combined "discretization-scalarization" machinery in order to unify and simplify successive application of different nonstandard methods.

As we already mentioned, there are at least two ways of combining nonstandard methods: first, to accomplish infinitesimal constructions in a Boolean-valued model, and second, to seek for the Boolean-valued interpretation of internal and external set theories. Each of these approaches involves its own technical difficulties, merits and demerits.

The Loeb measure is proved to be a very useful tool in nonstandard analysis. This construction can be generalized without difficulties to measures with values in a Banach space. As far as measures with values in a vector lattice without a norm are concerned such extension becomes more involved. Further progress requires developing a Boolean-valued version of the Loeb measure technique.

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