CRITERIA FOR EXCELLENCE IN MATHEMATICS

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Dedicated to Professor Guy HIRSCH

(Research supported in part by NSF grant DMS 8420698)

Ever since Guy Hirsch and I met to discuss algebraic topology (in Zurich in 1948) we have also taken time in our discussions to explore the nature and character of mathematical research. One central such question, and one hard to answer, is that of telling when a given development in Mathematics is poor, good, or even excellent. In a recent seminar at the University of Chicago I endeavored to formulate a few criteria for excellence in Mathematics. They are hard to state exactly and harder still to apply in given cases. So here they are, hopefully as stimulus to discussion.

INEVITABLE: Excellent mathematics should be inevitable, so that one can say of it: "This result was sure to be found, sooner or later. Now that we have it, we cannot imagine getting along without it". For example the description of geometric symmetry by means of groups, the development of the calculus, and the theory of distributions are all inevitable. The opposite of "inevitable" might come when one says that a piece of mathematics is incidental (why bother) or even frivolous. That can happen.

ILLUMINATING: A mathematical result is illuminating when it helps to understand some item in science, something in daily life or some previously obscure mathematical result; for example when some simple new concept clarifies and orders a circle of ideas – as the Galois theory illuminated the solution of polynomial equations, or in the way in which algebraic functions, compared with algebraic numbers, gave rise to the p-adic numbers and their remarkable properties. In mechanics, the Lagrangian and Hamiltonian equations provide such illumination. There are also more modest examples. The result that three points determine a plane explains the stability of tripod stools. But the opposite of illumination does come too often, when there are highly elaborate technical results not apposite to any real concern, perhaps serving chiefly for display.

Bulletin de la Société Mathématique de Belgique, t. XXXVIII, 1986.

DEEP: This is a much admired quality. One might say that a piece of mathematics is deep when it makes a surprising application which exposes hidden connections which were previously impossible to understand. The development of an original idea may be deep, but a deep result is not necessarily a difficult one. Thus I count the discovery of the Heine-Borel theorem and the related notion of compactness as deep. The notion of adjoint functors is deep but not hard. Results that call on the full resources of a subject can be deep, as in the proof of the Mordell conjecture, which required the full use of arithmetic algebraic geometry, and also the proof of the Poincaré conjecture in dimension 4. The opposite of the quality deep is trivial; I refrain from giving examples.

RELEVANT; A mathematical development is relevant when it builds effectively on prior concepts or when it answers outstanding questions in mathematics or science. Thus topology is relevant to complex analysis, while complex analysis in its turn is relevant to integration, to series expansions of real functions, to electromagnetism and to fluid motion. But also group theory, spectral sequences and related algebraic devices are relevant to topology. Group representations are relevant to arithmetic and to theoretical physics. The opposite of relevance is artificiality – as in the notable case of fuzzy sets.

RESPONSIVE; This is the quality which appears for the solution of famous problems. The importance of such solutions is well recognized, but there are other examples of responsive results (Those which clear up confusions, for example); moreover solving big problems is not the only real measure of mathematical progress. New concepts may also be responsive. The opposite of "responsive" is "Off beat", but sometimes an off beat idea turns out later to be relevant, as in Dedekind's early discovery of lattices (His "dual groups").

TIMELY; A piece of mathematics is timely (one might say fashionable) if it seizes upon newly available techniques to settle active problems. A striking example is the recent classification of all finite simple groups. An earlier one is the introduction of cohomology groups and cohomology operations in topology – ideas which settled geometric questions. "Old fashioned" is an opposite of timely; however the old sometimes turns out to matter. Good mathematics isn't restricted to what the pundits say.

These qualities may well be hard to recognize in new cases; the same may well apply to other qualities, such as "original" or "fruitful" (how can one tell without waiting to see what develops?) So I offer this tentative list of measures of excellence as a small way of honoring the excellent work and influence of Guy Hirsch.

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(Received December 22, 1985)