Boolean Models and Simultaneous Inequalities

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August 27, 2009 / Maltsev-100 Meeting

Agenda

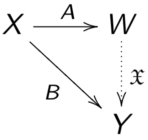
- The Farkas Lemma, also known as the Farkas–Minkowski Lemma, plays a key role in linear programming and the relevant areas of optimization.
- The aim of this talk is to demonstrate how Boolean valued analysis may be applied to simultaneous linear inequalities with operators.
- This particular theme is another illustration of the deep and powerful technique of "stratified validity" which is characteristic of Boolean valued analysis.

Environment

- Assume that X is a real vector space, Y is a *Kantorovich space* also known as a complete vector lattice or a complete Riesz space. Let $\mathbb{B} := \mathbb{B}(Y)$ be the *base* of Y, i.e., the complete Boolean algebras of positive projections in Y; and let $m(Y)$ be the universal completion of Y. Denote by $L(X, Y)$ the space of linear operators from X to Y.
- In case X is furnished with some Y-seminorm on X, by $L^{(m)}(X, Y)$ we mean the *space of dominated operators* from X to Y. As usual, $\{T \le y\} := \{T(\cdot) \le y\} := \{x \in X \mid Tx \le y\}$ and $\ker(T) = T^{-1}(0)$ for $T : X \to Y$.
- $\mathrm{Orth}(Y)$ is the commutant of $\mathbb{B}$ in $L^{(r)}(Y)$.

Inequalities: Explicit Dominance

- Find $\mathfrak{X}$ satisfying



- $(\exists \mathfrak{X})\ \mathfrak{X}A = B \leftrightarrow \ker(A) \subset \ker(B)$.
- *If W is ordered by W_+ and $A(X) - W_+ = W_+ - A(X) = W$, then*[1]

$$(\exists \mathfrak{X} \geq 0)\ \mathfrak{X}A = B \leftrightarrow \{A \leq 0\} \subset \{B \leq 0\}.$$

[1] The Kantorovich Theorem.

Farkas: Hidden Dominance

Lemma 1. *Let X be a vector space over some subfield R of the reals $\mathbb{R}$. Assume that f and g are R-linear functionals on X; in symbols, $f, g \in X^{\#} := L(X, \mathbb{R})$.*
For the inclusion

$$\{g \leq 0\} \supset \{f \leq 0\}$$

to hold it is necessary and sufficient that there be $\alpha \in \mathbb{R}_+$ satisfying $g = \alpha f$.

Proof.

- **Sufficiency** is obvious.
- **Necessity:** The case of $f = 0$ is trivial. If $f \neq 0$ then there is some $x \in X$ such that $f(x) \in \mathbb{R}$ and $f(x) > 0$. Denote the image $f(X)$ of X under f by R_0. Put $h := g \circ f^{-1}$, i .e. $h \in R_0^{\#}$ is the only solution for $h \circ f = g$. By hypothesis, h is a positive R-linear functional on R_0. By the Bigard Theorem$h$ can be extended to a positive homomorphism $\bar{h} : \mathbb{R} \to \mathbb{R}$, since $R_0 - \mathbb{R}_+ = \mathbb{R}_+ - R_0 = \mathbb{R}$. Each positive automorphism of $\mathbb{R}$ is multiplication by a positive real. As the sought α we may take $\bar{h}(1)$.
- The proof of the lemma is complete.

Reals: Explicit Dominance

- **Lemma 2.** *Let* X *be an* $\mathbb{R}$*-seminormed vector space over some subfield* R *of* $\mathbb{R}$*. Assume that* $f_1, \dots, f_N$ *and* g *are bounded* R*-linear functionals on* X*; in symbols,* $f_1, \dots, f_N, g \in X^* := L^{(m)}(X, \mathbb{R})$.
- *For the inclusion*

$$\{g \leq 0\} \supset \bigcap_{k=1}^{N} \{f_k \leq 0\}$$

to hold it is necessary and sufficient that there be $\alpha_1, \dots, \alpha_N \in \mathbb{R}_+$ *satisfying*

$$g = \sum_{k=1}^{N} \alpha_k f_k.$$

Farkas: Explicit Dominance

- **Theorem 1.** *Assume that* $A_1, \dots, A_N$ *and* B *belong to* $L^{(m)}(X, Y)$. *The following are equivalent:*
  (1) *Given* $b \in \mathbb{B}$, *the operator inequality* $bBx \le 0$ *is a consequence of the simultaneous linear operator inequalities* $bA_1x \le 0, \dots, bA_Nx \le 0$, *i.e.,*

$$\{bB \le 0\} \supset \{bA_1 \le 0\} \cap \dots \cap \{bA_N \le 0\}.$$

  (2) *There are positive orthomorphisms* $\alpha_1, \dots, \alpha_N \in \mathrm{Orth}(m(Y))$ *such that*

$$B = \sum_{k=1}^{N} \alpha_k A_k;$$

  i.e., B *lies in the operator convex conic hull of* $A_1, \dots, A_N$.

Origins

- Cohen's final solution of the problem of the cardinality of the continuum within ZFC gave rise to Boolean models.
- Scott forecasted in 1969:

  We must ask whether there is any interest in these nonstandard models aside from the independence proof; that is, do they have any mathematical interest? The answer must be yes, but we cannot yet give a really good argument.
- Takeuti coined the term "Boolean valued analysis" for applications of the models to analysis.

Boolean Valued Universe

- Let $\mathbb{B}$ be a complete Boolean algebra. Given an ordinal α, put

$$V_\alpha^{(\mathbb{B})} := \{x \mid (\exists \beta \in \alpha)\ x : \mathrm{dom}(x) \to \mathbb{B}\ \&\ \mathrm{dom}(x) \subset V_\beta^{(\mathbb{B})}\}.$$

- The *Boolean valued universe* $\mathbb{V}^{(\mathbb{B})}$ is

$$\mathbb{V}^{(\mathbb{B})} := \bigcup_{\alpha \in \mathrm{On}} V_\alpha^{(\mathbb{B})},$$

  with On the class of all ordinals.
- The truth value $[\![\varphi]\!] \in \mathbb{B}$ is assigned to each formula φ of ZFC relativized to $\mathbb{V}^{(\mathbb{B})}$.

Descending and Ascending

- Given φ, a formula of ZFC, and y, a member of $\mathbb{V}^{\mathbb{B}}$; put $A_\varphi := A_{\varphi(\cdot,\ y)} := \{x \mid \varphi(x,\ y)\}$.
- The *descent* $A_\varphi{\downarrow}$ of a class A_φ is

$$A_\varphi{\downarrow} := \{t \mid t \in \mathbb{V}^{(\mathbb{B})}\ \&\ [\![\varphi(t,\ y)]\!] = \mathbb{1}\}.$$

- If $t \in A_\varphi{\downarrow}$, then it is said that t *satisfies* $\varphi(\cdot,\ y)$ *inside* $\mathbb{V}^{(\mathbb{B})}$.
- The *descent* $x{\downarrow}$ of $x \in \mathbb{V}^{(\mathbb{B})}$ is defined as

$$x{\downarrow} := \{t \mid t \in \mathbb{V}^{(\mathbb{B})}\ \&\ [\![t \in x]\!] = \mathbb{1}\},$$

  i.e. $x{\downarrow} = A_{\cdot \in x}{\downarrow}$. The class $x{\downarrow}$ is a set.
- If x is a nonempty set inside $\mathbb{V}^{(\mathbb{B})}$ then

$$(\exists z \in x{\downarrow})[\![(\exists t \in x)\ \varphi(t)]\!] = [\![\varphi(z)]\!].$$

- The *ascent* functor acts in the opposite direction.

The Reals Within

- There is an object $\mathscr{R}$ inside $\mathbb{V}^{(\mathbb{B})}$ modeling $\mathbb{R}$, i.e.,

$$[\![\mathscr{R} \text{ is the reals}]\!] = \mathbb{1}.$$

- Let $\mathscr{R}{\downarrow}$ be the descent of the carrier $|\mathscr{R}|$ of the algebraic system $\mathscr{R} := (|\mathscr{R}|, +, \,\cdot\,, 0, 1, \leq)$ inside $\mathbb{V}^{(\mathbb{B})}$.
- Implement the descent of the structures on $|\mathscr{R}|$ to $\mathscr{R}{\downarrow}$ as follows:

$$x + y = z \leftrightarrow [\![x + y = z]\!] = \mathbb{1};$$
$$xy = z \leftrightarrow [\![xy = z]\!] = \mathbb{1};$$
$$x \leq y \leftrightarrow [\![x \leq y]\!] = \mathbb{1};$$
$$\lambda x = y \leftrightarrow [\![\lambda^{\wedge} x = y]\!] = \mathbb{1} \ (x, y, z \in \mathscr{R}{\downarrow},\ \lambda \in \mathbb{R}).$$

- **Gordon Theorem.** *$\mathscr{R}{\downarrow}$ with the descended structures is a universally complete vector lattice with base $\mathbb{B}(\mathscr{R}{\downarrow})$ isomorphic to $\mathbb{B}$.*

Proof of Theorem 1.

(2)$\rightarrow$ (1): If $B = \sum_{k=1}^{N} \alpha_k A_k$ for some positive $\alpha_1, \dots, \alpha_N$ in $\mathrm{Orth}(m(Y))$ while $bA_k x \leq 0$ for $b \in \mathbb{B}$ and $x \in X$, then

$$bBx = b \sum_{k=1}^{N} \alpha_k A_k x = \sum_{k=1}^{N} \alpha_k b A_k x \leq 0$$

since orthomorphisms commute and projections are orthomorphisms of $m(Y)$.

Proof of Theorem 1.

(1)$\to$ (2):

- Consider the separated Boolean valued universe $\mathbb{V}^{(\mathbb{B})}$ over the base $\mathbb{B}$ of Y. By the Gordon Theorem the ascent $Y\!\uparrow$ of Y is $\mathscr{R}$, the reals inside $\mathbb{V}^{(\mathbb{B})}$.
- Using the canonical embedding, we see that $X^\wedge$ is an $\mathscr{R}$-seminormed vector space over the standard name $\mathbb{R}^\wedge$ of the reals $\mathbb{R}$.
- Moreover, $\mathbb{R}^\wedge$ is a subfield and sublattice of $\mathscr{R} = Y\!\uparrow$ inside $\mathbb{V}^{(\mathbb{B})}$.

Proof of Theorem 1.

(1)$\to$ (2):

- Put $f_k := A_k\uparrow$ for all $k := 1, \dots, N$ and $g := B\uparrow$. Clearly, all $f_1, \dots, f_N, g$ belong to $(X^\wedge)^*$ inside $\mathbb{V}^{\mathbb{B}}$.
- Define the finite sequence

$$f : \{1, \dots, N\}^\wedge \to (X^\wedge)^*$$

as the ascent of $(f_1, \dots, f_N)$. In other words, the truth values are as follows:

$$[\![f_{k^\wedge}(x^\wedge) = A_k x]\!] = \mathbb{1}, \quad [\![g(x^\wedge) = Bx]\!] = \mathbb{1}$$

for all $x \in X$ and $k := 1, \dots, N$.

Proof of Theorem 1.

(1)→ (2):

Put

$$b := [\![A_1 x \le 0^\wedge]\!] \wedge \cdots \wedge [\![A_N x \le 0^\wedge]\!].$$

Then $bA_k x \le 0$ for all $k := 1, \dots, N$ and $bBx \le 0$ by (1).

Therefore,

$$[\![A_1 x \le 0^\wedge]\!] \wedge \cdots \wedge [\![A_N x \le 0^\wedge]\!] \le [\![Bx \le 0^\wedge]\!].$$

In other words,

$$[\![(\forall k := 1^\wedge, \dots, N^\wedge) f_k(x^\wedge) \le 0^\wedge]\!]$$

$$= \bigwedge_{k:=1,\dots,N} [\![f_{k^\wedge}(x^\wedge) \le 0^\wedge]\!] \le [\![g(x^\wedge) \le 0^\wedge]\!].$$

Proof of Theorem 1.

(1)→ (2):

By Lemma 2 inside $\mathbb{V}^{(\mathbb{B})}$ and the maximum principle of Boolean valued analysis, there is a finite sequence $\alpha : \{1^\wedge, \dots, N^\wedge\} \to \mathscr{R}_+$ inside $\mathbb{V}^{(\mathbb{B})}$ satisfying

$$[\![(\forall x \in X^\wedge) g(x) = \sum_{k=1^\wedge}^{N^\wedge} \alpha(k) f_k(x)]\!] = \mathbb{1}.$$

Put $\alpha_k := \alpha(k^\wedge) \in \mathscr{R}_+{\downarrow}$ for $k := 1, \dots, N$.

Multiplication by an element in $\mathscr{R}{\downarrow}$ is an orthomorphism of $m(Y)$. Moreover,

$$B = \sum_{k=1}^{N} \alpha_k A_k,$$

which completes the proof.

Counterexample: No Dominance

- Lemma 1, describing the consequences of a single inequality, does not restrict the class of functionals under consideration.
- The analogous version of the Farkas Lemma simply fails for two simultaneous inequalities in general.
- The inclusion $\{f = 0\} \subset \{g \leq 0\}$ equivalent to the inclusion $\{f = 0\} \subset \{g = 0\}$ does not imply that f and g are proportional in the case of an arbitrary subfield of $\mathbb{R}$. It suffices to look at $\mathbb{R}$ over the rationals $\mathbb{Q}$, take some discontinuous $\mathbb{Q}$-linear functional on $\mathbb{Q}$ and the identity automorphism of $\mathbb{Q}$.

Reconstruction: No Dominance

- **Theorem 2.**
  Take A and B in $L(X, Y)$. The following are equivalent:
  (1) $(\exists \alpha \in \mathrm{Orth}(m(Y)))\ B = \alpha A$;
  (2) *There is a projection $\varkappa \in \mathbb{B}$ such that*

$$\{\varkappa bB \leq 0\} \supset \{\varkappa bA \leq 0\}; \quad \{\neg\varkappa bB \leq 0\} \supset \{\neg\varkappa bA \geq 0\}$$

  for all $b \in \mathbb{B}$.

- **Proof.** Boolean valued analysis reduces the claim to the scalar case. Applying Lemma 1 twice and writing down the truth values, complete the proof.

Interval Operators

- Let X be a vector lattice. An *interval operator* $\mathbf{T}$ from X to Y is an order interval $[\underline{T}, \overline{T}]$ in $L^{(r)}(X, Y)$, with $\underline{T} \le \overline{T}$.
- The interval equation $\mathbf{B} = \mathfrak{X}\mathbf{A}$ has a *weak interval solution* provided that $(\exists \mathfrak{X})(\exists A \in \mathbf{A})(\exists \mathbf{B} \in \mathbf{B})\ \mathbf{B} = \mathfrak{X}\mathbf{A}$.
- Given an interval operator $\mathbf{T}$ and $x \in X$, put

$$P_{\mathbf{T}}(x) = \overline{T}x_{+} - \underline{T}x_{-}.$$

- Call $\mathbf{T}$ *adapted* in case $\overline{T} - \underline{T}$ is the sum of finitely many disjoint addends.
- Put $\sim(x) := -x$ for all $x \in X$.

Interval Equations

- **Theorem 3.** *Let X be a vector lattice, and let Y be a Kantorovich space. Assume that $\mathbf{A}_1, \dots, \mathbf{A}_N$ are adapted interval operators and $\mathbf{B}$ is an arbitrary interval operator in the space of order bounded operators $L^{(r)}(X, Y)$.*
  The following are equivalent:
  (1) *The interval equation*

$$\mathbf{B} = \sum_{k=1}^{N} \alpha_k \mathbf{A}_k$$

  has a weak interval solution $\alpha_1, \dots, \alpha_N \in \mathrm{Orth}(Y)_+$.
  (2) *For all $b \in \mathbb{B}$ we have*

$$\{b\mathfrak{B} \geq 0\} \supset \{b\mathfrak{A}_1^{\sim} \leq 0\} \cap \dots \cap \{b\mathfrak{A}_N^{\sim} \leq 0\},$$

  where $\mathfrak{A}_k^{\sim} := P_{\mathbf{A}_k} \circ \sim$ for $k := 1, \dots, N$ and $\mathfrak{B} := P_{\mathbf{B}}$.

Inhomogeneous Inequalities

- **Theorem 4.** *Let X be a Y-seminormed reaL vector space, with Y a Kantorovich space. Take $A_1, \dots, A_N, B \in L^{(m)}(X, Y)$ and $u_1, \dots, u_N, v \in Y$. Assume that the system $A_1 x \le u_1, \dots, A_N x \le u_N$ is consistent.*

  The following are equivalent:

  (1) $\{bB \le bv\} \supset \{bA_1 \le bu_1\} \cap \dots \cap \{bA_N \le bu_N\}$ *for all* $b \in \mathbb{B}$.

  (2) *There are* $\alpha_1, \dots, \alpha_N \in \mathrm{Orth}(m(Y))_+$ *satisfying*

$$B = \sum_{k=1}^{N} \alpha_k A_k; \quad v \ge \sum_{k=1}^{N} \alpha_k u_k.$$

Inhomogeneous Matrix Inequalities

- In applications we encounter inhomogeneous matrix inequalities over various finite-dimensional spaces.
- **Theorem 5.** *Let X be a Y-seminormed real vector space, with Y a Kantorovich space. Assume that $A \in L^{(m)}(X, Y^s)$, $B \in L^{(m)}(X, Y^t)$, $u \in Y^s$ and $v \in Y^t$, where s and t are some naturals.*

  The following are equivalent:

  (1) *For all $b \in \mathbb{B}$ the inhomogeneous operator inequality $bBx \leq bv$ is a consequence of the consistent inhomogeneous inequality $bAx \leq bu$, i.e., $\{bB \leq bv\} \supset \{bA \leq bu\}$.*

  (2) *There is some $s \times t$ matrix with entries positive orthomorphisms of $m(Y)$ such that $B = \mathfrak{X}A$ and $\mathfrak{X}u \leq v$ for the corresponding linear operator $\mathfrak{X} \in L_+(Y^s, Y^t)$.*

Complex Scalars

- **Theorem 6.** *Let X be a Y-seminormed complex vector space, with Y a Kantorovich space. Assume given some $u_1, \dots, u_N, v \in Y$ and dominated operators $A_1, \dots, A_N, B \in L^{(m)}(X, Y_{\mathbb{C}})$ from X into the complexification $Y_{\mathbb{C}} := Y \otimes iY$ of Y.*[2] *Assume further that the simultaneous inhomogeneous inequalities $|A_1 x| \le u_1, \dots, |A_N x| \le u_N$ are consistent. Then the following are equivalent:*
  (1) $\{b|B(\cdot)| \le bv\} \supset \{b|A_1(\cdot)| \le bu_1\} \cap \dots \cap \{b|A_N(\cdot)| \le bu_N\}$ *for all* $b \in \mathbb{B}$.
  (2) *There are complex orthomorphisms* $c_1, \dots, c_N \in \mathrm{Orth}(m(Y)_{\mathbb{C}})$ *satisfying*
$$B = \sum_{k=1}^{N} c_k A_k; \quad v \ge \sum_{k=1}^{N} |c_k| u_k.$$

[2] Cp. [3, p. 338].

Theorem of the Alternative

- **Theorem 7.** *Let X be a Y-seminormed real vector space, with Y a Kantorovich space. Assume that $A_1, \dots, A_N$ and B belong to $L^{(m)}(X, Y)$.*
  Then one and only one of the following holds:
  (1) *There are $x \in X$ and $b, b' \in \mathbb{B}$ such that $b' \le b$ and*

  $$b'Bx > 0, bA_1x \le 0, \dots, bA_Nx \le 0.$$

  (2) *There are $\alpha_1, \dots, \alpha_N \in \mathrm{Orth}(m(Y))_+$ such that*

  $$B = \sum_{k=1}^{N} \alpha_k A_k.$$

All Is Number

- The above curiosities are nothing more than simple illustrations of the powerful technique of model theory shedding new light at the *Pythagorean Thesis*.
- The theory of the reals enriches mathematics, demonstrating the liberating role of logic.

Pursuit of Truth

- We definitely feel truth, but we cannot define truth properly. That is what Tarski explained to us in the 1930s.
- We pursue truth by way of proof, as wittily phrased by Mac Lane. Model theory evaluates and counts truth and proof.
- The chase of truth not only leads us close to the truth we pursue but also enables us to nearly catch up with many other instances of truth which we were not aware nor even foresaw at the start of the rally pursuit. That is what we have learned from the Boolean models elaborated in the 1960s by Scott, Solovay, and Vopěnka.

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