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BOOLEAN VALUED ANALYSIS: SOME NEW APPLICATIONS

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The term *Boolean valued analysis* signifies the technique of studying properties of an arbitrary mathematical object by comparison between its representations in two different set-theoretic models whose construction utilizes principally distinct Boolean algebras. As these models, the classical Cantorian paradise in the shape of the von Neumann universe and a specially-trimmed Boolean valued universe are usually taken. Comparison analysis is carried out by some interplay between these universes. We survey here some new applications of Boolean valued analysis that concern operator theory.

Key words: Boolean valued model, vector lattice, positive operator, rationally complete ring, injective module, JB-algebra, non-associative Radon–Nikodým theorem, harmonic analysis, Fourier transform, Bochner theorem.

1. Boolean Valued Requisites

The term "Boolean valued analysis" appeared within the realm of mathematical logic. It was Takeuti, a renowned expert in proof theory, who introduced the term. Takeuti defined Boolean valued analysis in [27, p. 1] as "an application of Scott–Solovay's Boolean valued models of set theory to analysis." Vopěnka invented similar models at the same time. The needed information on the theory of Boolean valued models is briefly presented in [15, Chapter 9] and [19, Chapter 1]; details may be found in [5, 17, 29]. A short survey of the Boolean machinery is also in [21].

Throughout the sequel $\mathbb{B}$ is a complete Boolean algebra with unit $\mathbb{1}$ and zero $\mathbb{O}$. A *partition of unity* in $\mathbb{B}$ is a family $(b_\xi)_{\xi\in\Xi} \subset \mathbb{B}$ such that $\bigvee_{\xi\in\Xi} b_\xi = \mathbb{1}$ and $b_\xi \wedge b_\eta = \mathbb{O}$ whenever $\xi \neq \eta$. We let := denote the assignment by definition, while $\mathbb{N}$, $\mathbb{Q}$, and $\mathbb{R}$ symbolize the naturals, the rationals, and the reals. Recall also that ZFC is an abbreviation for *Zermelo–Fraenkel set theory* with the axiom of choice.

1.1. Boolean valued universe and Boolean valued truth [19, § 1.2]. Given a complete Boolean algebra $\mathbb{B}$, we can define the universe $\mathbb{V}^{(\mathbb{B})}$, the class of $\mathbb{B}$-valued sets. For making statements about $\mathbb{V}^{(\mathbb{B})}$ take an arbitrary formula $\varphi = \varphi(u_1, \dots, u_n)$ of the language of set theory and replace the variables $u_1, \dots, u_n$ by elements $x_1, \dots, x_n \in \mathbb{V}^{(\mathbb{B})}$. Then we obtain some statement about the objects $x_1, \dots, x_n$. There is a natural way of assigning to each such statement an element $[\![\varphi(x_1, \dots, x_n)]\!] \in \mathbb{B}$ which acts as the *"Boolean truth-value"* of $\varphi(u_1, \dots, u_n)$ in the universe $\mathbb{V}^{(\mathbb{B})}$ and is defined by induction on the complexity of φ, assigning truth-values $[\![x \in y]\!] \in \mathbb{B}$ and $[\![x = y]\!] \in \mathbb{B}$, where $x, y \in \mathbb{V}^{(\mathbb{B})}$. We say that the statement $\varphi(x_1, \dots, x_n)$ *is valid within* $\mathbb{V}^{(\mathbb{B})}$ if $[\![\varphi(x_1, \dots, x_n)]\!] = \mathbb{1}$. In this event, we write also $\mathbb{V}^{(\mathbb{B})} \vDash \varphi(x_1, \dots, x_n)$.

1.2. Ascending–descending machinery [19, §§ 1.5, 1.6, and 2.2]. No comparison is feasible without some dialog between $\mathbb{V}$ and $\mathbb{V}^{(\mathbb{B})}$. The relevant *technique of ascending and descending* bases on the operations of the canonical embedding, descent, and ascent.

(1) The canonical embedding. There is a *canonical embedding* of the von Neumann universe $\mathbb{V}$ into the Boolean valued universe $\mathbb{V}^{(\mathbb{B})}$ which sends $x \in \mathbb{V}$ to its *standard name* $x^{\wedge} \in \mathbb{V}^{(\mathbb{B})}$. The standard name sends $\mathbb{V}$ onto $\mathbb{V}^{(\mathbb{2})}$, where $\mathbb{2} := \{\mathbb{O}, \mathbb{1}\} \subset \mathbb{B}$.

(2) Descent. Given a member x of a Boolean valued universe $\mathbb{V}^{(\mathbb{B})}$, define the *descent* $x{\downarrow}$ of x by $x{\downarrow} := \{y \in \mathbb{V}^{(\mathbb{B})} : [\![y \in x]\!] = \mathbb{1}\}$. The class $x{\downarrow}$ is a set; i. e., $x{\downarrow} \in \mathbb{V}$ for every $x \in \mathbb{V}^{(\mathbb{B})}$.

(3) Ascent. Assume that $x \in \mathbb{V}$ and $x \subset \mathbb{V}^{(\mathbb{B})}$. Then there exists a unique $x{\uparrow} \in \mathbb{V}^{(\mathbb{B})}$ such that $[\![u \in x{\uparrow}]\!] = \bigvee\{[\![u = y]\!] : y \in x\}$ for all $u \in \mathbb{V}^{(\mathbb{B})}$. The member $x{\uparrow}$ is called the *ascent* of x.

The operations of descent ascent and canonical embedding can be naturally extended to mappings and relations, [19, Chapter 1].

1.3. Principles of Boolean valued set theory [19, § 1.4]. The main properties of a Boolean valued universe $\mathbb{V}^{(\mathbb{B})}$ are collected in the following four propositions:

(1) Transfer Principle. All theorems of ZFC are valid within $\mathbb{V}^{(\mathbb{B})}$; i. e., $[\![\varphi]\!] = \mathbb{1}$ for every ZFC theorem φ, or, in symbols, $\mathbb{V}^{(\mathbb{B})} \models \text{ZFC}$.

(2) Maximum Principle. To each formula φ of ZFC there is a member x_0 of $\mathbb{V}^{(\mathbb{B})}$ satisfying $[\![(\exists x)\, \varphi(x)]\!] = [\![\varphi(x_0)]\!]$.

(3) Mixing Principle. For every family $(x_\xi)_{\xi \in \Xi}$ in $\mathbb{V}^{(\mathbb{B})}$ and every partition of unity $(b_\xi)_{\xi \in \Xi}$ in $\mathbb{B}$ there is a unique $x \in \mathbb{V}^{(\mathbb{B})}$ satisfying $b_\xi \leqslant [\![x = x_\xi]\!]$ for all $\xi \in \Xi$. This unique x is called the *mixing* of (x_ξ) by (b_ξ) and is denoted as follows: $x = \operatorname{mix}_{\xi \in \Xi}(b_\xi x_\xi) = \operatorname{mix}\{b_\xi x_\xi : \xi \in \Xi\}$.

(4) Restricted Transfer Principle. Given a restricted formula φ of ZFC and a collection $x_1, \dots, x_n \in \mathbb{V}$, the ZFC-equivalence holds:

$$\varphi(x_1, \dots, x_n) \leftrightarrow \mathbb{V}^{(\mathbb{B})} \models \varphi(x_1^{\wedge}, \dots, x_n^{\wedge}).$$

A formula is called *restricted* if each of its quantifiers occurs in the form $(\forall\, x \in y)$ or $(\exists\, x \in y)$ or if it can be proved equivalent in ZFC to a formula of this kind.

The transfer principle reads sometimes as '$\mathbb{V}^{(\mathbb{B})}$ is a Boolean valued model of ZFC'; the maximum principle guarantees the existence of various 'Boolean valued objects'; the mixing principle shows how these objects may be constructed.

1.4. Boolean valued technology. To prove the relative consistency of some set-theoretic propositions we use a Boolean valued universe $\mathbb{V}^{(\mathbb{B})}$ as follows: Let $\mathscr{T}$ and $\mathscr{S}$ be some enrichments of Zermelo–Fraenkel theory ZF (without choice). Assume that the consistency of ZF implies the consistency of $\mathscr{S}$. Assume further that we can define $\mathbb{B}$ so that $\mathscr{S} \models$ "B is a complete Boolean algebra" and $\mathscr{S} \models [\![\varphi]\!] = \mathbb{1}$ for every axiom φ of $\mathscr{T}$. Then the consistency of ZF implies the consistency of $\mathscr{T}$. That is how we use $\mathbb{V}^{(\mathbb{B})}$ in foundational studies.

Other possibilities for applying $\mathbb{V}^{(\mathbb{B})}$ base on the fact that irrespective of the choice of a Boolean algebra B, the universe is an arena for testing an arbitrary mathematical event. By the principles of transfer and maximum, every $\mathbb{V}^{(\mathbb{B})}$ has the objects that play the roles of numbers, groups, Banach spaces, manifolds, and whatever constructs of mathematics that are already introduced into practice or still remain undiscovered. These objects may be viewed as some nonstandard realizations of the relevant originals.

All ZFC theorems acquire interpretations for the members of $\mathbb{V}^{(\mathbb{B})}$, attaining the top truth value. We thus obtain a new technology of comparison between the interpretations of mathematical facts in the universes over various complete Boolean algebras. Developing the relevant tools is the crux of Boolean valued analysis.

A general scheme of the method is as follows; see [18, 19]. Assume that $\mathbf{X} \subset \mathbb{V}$ and $\mathbb{X} \subset \mathbb{V}^{(\mathbb{B})}$ are two classes of mathematical objects. Suppose that we are able to prove

The *Boolean Valued Representation:* Each $X \in \mathbf{X}$ embeds into a Boolean valued model, becoming an object $\mathscr{X} \in \mathbb{X}$ within $\mathbb{V}^{(\mathbb{B})}$.

The *Boolean Valued Transfer Principle* tells us then that every theorem about $\mathscr{X}$ within Zermelo–Fraenkel set theory has its counterpart for the original object X interpreted as a Boolean valued object $\mathscr{X}$.

The *Boolean Valued Machinery* enables us to perform some translation of theorems from $\mathscr{X} \in \mathbb{V}^{(\mathbb{B})}$ to $X \in \mathbb{V}$ by using the appropriate general operations and the principles of Boolean valued analysis.

2. Vector Lattices and Positive Operators

The reader can find the relevant information on the theory of vector lattices and order bounded operators in [4, 15, 24, 31, 32].

DEFINITION 2.1. A *vector lattice* is a vector space over the reals that is equipped with a partial order $\leqslant$ for which the *join* $x \vee y$ and the *meet* $x \wedge y$ exist for all vectors $x, y \in X$, and such that the positive cone $X_+ := \{x \in X : 0 \leqslant x\}$ is closed under addition and multiplication by non negative reals. A subset $U \subset X$ is *order bounded* if U is contained in an *order interval* $[a, b] := \{x \in X : \ a \leqslant x \leqslant b\}$ for some $a, b \in X$.

In the sequel, we assume that all vector lattices X are *Archimedean*, i. e., for every pair $x, y \in X$ it follows from $(\forall n \in \mathbb{N})\ nx \leqslant y$ that $x \leqslant 0$.

Most of the vector spaces that appear naturally in analysis (L^p, l^p, $C(K)$, c, c_0, etc.) are Archimedean vector lattices with respect to pointwise or coordinate-wise ordering.

DEFINITION 2.2. Say that two elements $x, y \in X$ are *disjoint* and write $x \perp y$ if $|x| \wedge |y| = 0$ where the module $|x|$ of x is defined as $|x| := x \vee (-x)$. A *band* in a vector lattice X is a subset of the form $B := A^{\perp} := \{x \in X : (\forall a \in A)\, |x| \wedge |a| = 0\}$ for a nonempty $A \subset X$. Clearly, $B \cap B^{\perp} = \{0\}$. A band B in X whith satisfies $X = B \oplus B^{\perp}$ is referred to as a *projection band,* while the associated projection (onto B parallel to $B^{\perp}$) is called a *band projection.* The inclusion-ordered sets of all bands and all band projections in a vector lattice X form complete Boolean algebras, which are denoted by $\mathbb{B}(X)$ and $\mathbb{P}(X)$, respectively.

DEFINITION 2.3. A vector lattice X is said to be *Dedekind complete* (resp. *laterally complete*) if every nonempty order bounded set (resp. every non-empty set of pairwise disjoint positive vectors) U in X has the least upper bound $\sup(U) \in X$. A vector lattice that is at the same time laterally complete and Dedekind complete is referred to as a *universally complete* vector lattice.

EXAMPLE 2.1. Assume that a measure space (Ω, Σ, μ) is semi-finite, that is, if $A \in \Sigma$ and $\mu(A) = \infty$ then there exists $B \in \Sigma$ with $B \subset A$ and $0 < \mu(A) < \infty$. The vector lattice $L^0(\Omega, \Sigma, \mu)$ (of μ-equivalence classes) of μ-measurable functions on Ω is universally complete if and only if (Ω, Σ, μ) is localizable. In this event $L^p(\Omega, \Sigma, \mu)$ is Dedekind complete, see [8, 241G]. Observe that $\mathbb{P}(L^0(\Omega, \Sigma, \mu)) \simeq \Sigma/\mu^{-1}(0)$.

EXAMPLE 2.2. Given a complete Boolean algebra $\mathbb{B}$ of projections in a Hilbert space H, denote by $\langle\mathbb{B}\rangle$ the space of all selfadjoint operators on H whose spectral resolutions are in $\mathbb{B}$; i. e., $A \in \langle\mathbb{B}\rangle$ if and only if $A = \int_{\mathbb{R}} \lambda\, dE_\lambda$ and $E_\lambda \in \mathbb{B}$ for all $\lambda \in \mathbb{R}$. Then $\langle\mathbb{B}\rangle$ is a universally complete vector lattice and $\mathbb{P}(\langle\mathbb{B}\rangle) \simeq \mathbb{B}$.

The fundamental result of Boolean valued analysis is the Gordon Theorem which describes an interplay between $\mathbb{R}$, $\mathbb{R}^\wedge$, $\mathscr{R}$, and $\mathbf{R} = \mathscr{R}{\downarrow}$: *Each universally complete vector lattice is an interpretation of the reals in an appropriate Boolean valued model,* see, [19, §§ 2.2–2.4].

Theorem 2.1 (Gordon Theorem)**.** *Let $\mathbb{B}$ be a complete Boolean algebra, $\mathscr{R}$ be a field of reals within $\mathbb{V}^{(\mathbb{B})}$. Then*

(1) *The algebraic structure $\mathbf{R}$ (with the descended operations and order) is an universally complete vector lattice.*

(2) *The internal field $\mathscr{R} \in \mathbb{V}^{(\mathbb{B})}$ can be chosen so that*

$$[\![\, \mathbb{R}^\wedge \text{ is a dense subfield of the field } \mathscr{R} \,]\!] = \mathbb{1}.$$

(3) *There is a Boolean isomorphism χ from $\mathbb{B}$ onto $\mathbb{P}(\mathbf{R})$ such that*

$$\chi(b)x = \chi(b)y \iff b \leqslant [\![\, x = y \,]\!],$$
$$\chi(b)x \leqslant \chi(b)y \iff b \leqslant [\![\, x \leqslant y \,]\!]$$
$$(x, y \in \mathbf{R};\ b \in \mathbb{B}).$$

Definition 2.4. A *complex vector lattice* is the complexification $X_{\mathbb{C}} := X \oplus iX$ (with i standing for the imaginary unity) of a real vector lattice X; i. e., the additive group of $X \times X$ is endowed additionally with the scalar multiplication $(\alpha + i\beta)(x, y) = (\alpha x - \beta y, \alpha y + \beta x)$ for all $\alpha, \beta \in \mathbb{R}$ and $x, y \in X$. Identifying $x \in X$ with $(x, 0) \in X$ and iy with $(0, y)$, we will write $x + iy$ instead of (x, y).

Often it is additionally required that the modulus

$$|z| := \sup \big\{ (\cos\theta)\, x + (\sin\theta)\, y :\ 0 \leqslant \theta < 2\pi \big\}$$

exists for every element $z := x + iy \in X \oplus iX$. This requirement is automatically satisfied in a uniformly complete vector lattice.

The version of the Gordon Theorem for complexes is also true.

Theorem 2.2. *Let $\mathscr{C}$ be the field of complex numbers in the model $\mathbb{V}^{(\mathbb{B})}$. Then the algebraic system $\mathscr{C}{\downarrow}$ is a universally complete complex f-algebra. Moreover, $\mathscr{C}{\downarrow}$ the complexification of the universally complete real f-algebra $\mathscr{R}{\downarrow}$; i. e., $\mathscr{C}{\downarrow} = \mathscr{R}{\downarrow} \oplus i\mathscr{R}{\downarrow}$.*

The two particular cases of Gordon's Theorem, corresponding to Examples 2.1 and 2.2, were studied independently by Takeuti [27].

Definition 2.5. Let X and Y be vector lattices. By $L(X, Y)$ we denote the space of all linear operators from X to Y. An operator $T \in L(X, Y)$ is *positive* provided that $T(X_+) \subset Y_+$, while T is *regular* provided that T can be presented as a difference of two positive linear operators, and T is *order bounded* or shortly *o-bounded* provided that T sends each order bounded subset of X to an order bounded subset of Y.

The sets of all regular, order bounded, and positive operators from X to Y are denoted by $L^r(X, Y)$, $L^\sim(X, Y)$, and $L_+(X, Y) := L^\sim(X, Y)_+$, respectively. Clearly, $L^r(X, Y)$ and $L^\sim(X, Y)$ are vector subspaces of $L(X, Y)$. The ordering in the spaces of regular and order bounded operators is defined by the following relations:

$$T \geqslant 0 \iff T \in L_+(X, Y), \quad S \geqslant T \iff S - T \geqslant 0.$$

The celebrated Riesz–Kantorovich Theorem tells us that $L^\sim(X, Y)$ is a Dedekind complete vector lattice and, in particular, $L^\sim(X, Y) = L^r(X, Y)$, whenever Y is Dedekind complete.

The fact that X is a vector lattice over the ordered field $\mathbb{R}$ may be rewritten as a restricted formula, say, $\varphi(X, \mathbb{R})$. Hence, recalling the restricted transfer principle, we come to the identity $[\![\, \varphi(X^\wedge, \mathbb{R}^\wedge) \,]\!] = \mathbb{1}$ which amounts to saying that $X^\wedge$ is a vector lattice over the ordered field $\mathbb{R}^\wedge$ within $\mathbb{V}^{(\mathbb{B})}$. The positive cone X_+ is defined by a restricted formula $\varphi(X, X_+) \equiv (\forall\, x \in X_+)(x \in X) \wedge (\forall\, x \in X)(x \in X_+ \leftrightarrow x \geqslant 0)$. Hence $(X^\wedge)_+ = (X_+)^\wedge$ by restricted transfer. By the same reason

$$|x^\wedge| = |x|^\wedge, \quad (x \vee y)^\wedge = x^\wedge \vee y^\wedge, \quad (x \wedge y)^\wedge = x^\wedge \wedge y^\wedge$$

for all $x, y \in X$, since the lattice operations $\vee$, $\wedge$, and $|\cdot|$ in X are defined by restricted formulas.

Let $X^{\wedge\sim} := L^\sim_{\mathbb{R}^\wedge}(X^\wedge, \mathscr{R})$ be the space of regular $\mathbb{R}^\wedge$-linear functionals from $X^\wedge$ to $\mathscr{R}$. More precisely, $\mathscr{R}$ is considered as a vector space over the field $\mathbb{R}^\wedge$ and by the maximum principle there exists $X^{\wedge\sim} \in \mathbb{V}^{(\mathbb{B})}$ such that $[\![X^{\wedge\sim}$ is a vector space over $\mathscr{R}$ of $\mathbb{R}^\wedge$-linear order bounded functionals from $X^\wedge$ to $\mathscr{R}$ which is ordered by the cone of positive functionals$\,]\!] = \mathbb{1}$. A functional $\tau \in X^{\wedge\sim}$ is positive if $[\![(\forall\, x \in X^\wedge)\tau(x) \geqslant 0]\!] = \mathbb{1}$.

Definition 2.6. Let $X \in \mathbb{V}$ and $Y \in \mathbb{V}^{(\mathbb{B})}$ be such that $X \neq \varnothing$ and $[\![Y \neq \varnothing]\!] = \mathbb{1}$. Given an operator $T : X \to Y{\downarrow}$, there exists a unique $T{\Uparrow} \in \mathbb{V}^{(\mathbb{B})}$ (called the *modified ascent* of T) such that $[\![T{\Uparrow} : X^\wedge \to Y]\!] = \mathbb{1}$

and $[\![T\!\upharpoonleft\!\!\uparrow(x^\wedge) = T(x)]\!] = \mathbb{1}$ for all $x \in X$. Given a member $\tau \in \mathbb{V}^{(\mathbb{B})}$, there exists a unique operator $\tau\!\downarrow : X \to Y\!\downarrow$ (called the *modified descent* of τ) such that $[\![\tau(x^\wedge) = \tau\!\downarrow\!(x)]\!] = \mathbb{1}$ for all $x \in X$.

Theorem 2.3. *Let X and Y be vector lattices with Y universally complete and represented as $Y = \mathscr{R}\!\downarrow$. Given $T \in L^\sim(X, Y)$, the modified ascent $T\!\uparrow$ is an order bounded $\mathbb{R}^\wedge$-linear functional on $X^\wedge$ within $\mathbb{V}^{(\mathbb{B})}$; i. e., $[\![\, T\!\uparrow \in X^{\wedge\sim} \,]\!] = \mathbb{1}$. The mapping $T \mapsto T\!\uparrow$ is a lattice isomorphism between the Dedekind complete vector lattices $L^\sim(X, Y)$ and $X^{\wedge\sim}\!\downarrow$.*

Definition 2.7. A linear operator T from X to Y is a *lattice homomorphism* whenever $T(x_1 \vee x_2) = Tx_1 \vee Tx_2$ for all $x_1, x_2 \in X$. Say that T is *disjointness preserving* if $|x| \wedge |y| = 0$ implies $|T(x)| \wedge |T(y)| = 0$ for all $x, y \in X$. Two vector lattices X and Y are said to be *lattice isomorphic* if there is a lattice isomorphism from X onto Y.

It can be easily seen that a linear operator is disjointness preserving if and only if $|T(x)| = |T(|x|)|$ for all $x \in E$, while T is a lattice homomorphism if and only if $|T(x)| = T(|x|)$ for all $x \in E$.

Let $\mathrm{Hom}(X, Y)$ and $L^\sim_{dp}(X, Y)$ stand for the sets of all lattice homomorphisms and all disjointness preserving operators from X to Y, respectively. Put $X^\sim := L^\sim(X, \mathbb{R})$ and $X^\sim_a := L^\sim_a(X, \mathbb{R})$.

Corollary 2.1. *Consider $S, T \in L^\sim(X, Y)$ and put $\tau := T\!\uparrow$, $\sigma := S\!\uparrow$. The following equivalences are fulfilled:*

(1) $T \in \mathrm{Hom}(X, Y) \iff [\![\, \tau \in \mathrm{Hom}(X^\wedge, \mathscr{R}) \,]\!] = \mathbb{1}$;

(2) $T \in L^\sim_{dp}(X, Y) \iff [\![\, \tau \in (X^{\wedge\sim})_{dp} \,]\!] = \mathbb{1}$.

3. Sums of Disjointness Preserving Operators

In this section we examine the problem of finding conditions for a sum of finite collections of order bounded disjointness preserving operators to be n-disjoint. In the case of functionals, the problem is easily settled.

Definition 3.1. A linear operator $T : X \to Y$ is said to be n*-disjoint* if, for every collection of $n+1$ pairwise disjoint elements $x_0, \dots, x_n \in E$, the meet of $\{|Tx_1|, \dots, |Tx_n|\}$ equals zero:

$$(\forall\, x_0, x_1 \dots, x_n \in X)\ x_k \perp x_l\ (k \neq l) \implies |Tx_0| \wedge \dots \wedge |Tx_n| = 0.$$

Thus 1-disjoint is the same as disjointness preserving; see Definition 2.7.

The sum of n order bounded disjointness preserving operators acting between vector lattices is n-disjoint, see [6, Proposition 2]. The following result states that the converse is also true.

Theorem 3.1. *Let X and Y be vector lattices with Y Dedekind complete and let $T : X \to Y$ be an order bounded n-disjoint operator. Then there exist n lattice homomorphisms $T_1, \dots, T_n$ from X to Y with $T = T_1 + \dots + T_n$. Moreover, $T_1, \dots, T_n$ can be chosen pairwise disjoint.*

$\triangleleft$ The first assertion is obtained in Bernau, Huijsmans, and de Pagter [6, Theorem 6]; the second one in Radnaev [25]. $\triangleright$

It is easy to see that the representation of the n-disjoint operator as a sum of n disjointness preserving operators is not unique, see [6, Example 7]. So, the question arises: In what exact sense should uniqueness be understood? To answer this question, we need the following concept.

Definition 3.2. A linear operator $T : X \to Y$ is called *purely n-disjoint* if n is the least natural for which πT is n-disjoint for all nonzero $\pi \in \mathbb{P}\big(T(X)^{\perp\perp}\big)$. Two linear operators $S_1, S_2 : X \to Y$ are said to be *purely disjoint* if $S_1 \perp S_2$ and $S_1(X)^{\perp\perp} = S_2(X)^{\perp\perp}$.

Clearly, if an operator is purely n-disjoint and purely m-disjoint then $n = m$. Moreover, an order bounded n-disjoint operator uniquely decomposes into a sum of purely k-disjoint components, $k \leqslant n$.

Theorem 3.2. *Let X and Y be vector lattices with Y having the projection property, $n \in \mathbb{N}$, and T an n-disjoint linear operator from X to Y with $Y = T(X)^{\perp\perp}$. Then there exist a unique collection $(k_1, \pi_1), \dots (k_l, \pi_l)$ with naturals $1 \leqslant k_1 < \dots < k_l \leqslant n$ and a partition of unity $\{\pi_1, \dots, \pi_l\}$ in $\mathbb{P}(F)$ with nonzero terms such that $\pi_i T$ is purely k_i-disjoint for all $i := 1, \dots, l$.*

$\triangleleft$ According to Theorem 2.2 $\tau := T\!\uparrow$ is a linear functional on $X^\wedge$ within $\mathbb{V}^{(\mathbb{B})}$ where $\mathbb{B} = \mathbb{P}(Y)$. Moreover, τ is $n^\wedge$-disjoint as can be seen from Theorem 3.1. It is clear from Definition 3.1 that there is the least natural $k \leqslant n^\wedge$ for which the functional τ is k-disjoint, that is, the following set-theoretic formulae holds:

$$\phi(\tau, n^\wedge) := (\exists k \leqslant n)(\tau \in \mathscr{D}_k) \wedge (\forall l \leqslant n^\wedge)(\tau \in \mathscr{D}_l \to k \leqslant l),$$

where $\tau \in \mathscr{D}_k$ means that τ is k-disjoint. By the transfer principle we have $[\![\phi(\tau, n^\wedge)]\!] = \mathbb{1}$. An elementary computation with truth values yields a partition of unity $\rho_1, \dots, \rho_n$ such that $[\![\tau \text{ is purely } k^\wedge\text{-disjoint}]\!] \geqslant \rho_k$ for all $k \leqslant n$ or, equivalently, $\rho_k T$ is purely k-disjoint for all $k \leqslant n$. It remains to separate nonzero terms $\{\rho_{k_1}, \dots, \rho_{k_l}\}$ of the partition of unity $\{\pi_1, \dots, \pi_l\}$ with $1 \leqslant k_1 < \dots < k_l \leqslant n$ and denote $\pi_j := \rho_{k_j}$. $\triangleright$

To describe the structure of purely n-disjoint operators we need one more concept which, in case $F = \mathbb{R}$, is just the usual permutation.

Definition 3.3. Given two collections $\mathscr{T} := (T_1, \dots, T_N)$ and $\mathscr{S} := (S_1, \dots, S_N)$ of linear operators from X to Y, say that $\mathscr{S}$ is a $\mathbb{P}(Y)$-*permutation* of $\mathscr{T}$ whenever there exists an $N \times N$ matrix $(\pi_{i,l})$ with entries from $\mathbb{P}(Y)$, whose rows and columns are partitions of unity in $\mathbb{P}(Y)$, such that $S_i = \sum_{l=1}^{N} \pi_{i,l} T_l$ for all $i := 1, \dots, N$ (and so $T_l = \sum_{i=1}^{N} \pi_{i,l} S_i$ for all $l := 1, \dots, N$).

Theorem 3.3. *Given $\tau_i, \sigma_i \in \mathbb{V}^{(\mathbb{B})}$ with $[\![\tau_i, \sigma_i \in (X^\wedge)^\sim]\!] = \mathbb{1}$, $i \in \{1, \dots, N\}^\wedge$, put $T_l := \tau_{l^\wedge}\!\downarrow$ and $S_l := \sigma_{l^\wedge}\!\downarrow$. Then $(\sigma_1, \dots, \sigma_{N^\wedge})$ is a permutation of $(\tau_1, \dots, \tau_{N^\wedge})$ within $\mathbb{V}^{(\mathbb{B})}$ if and only if $(S_1, \dots, S_N)$ is a $\mathbb{P}(Y)$-permutation of $(T_1, \dots, T_N)$.*

$\lhd$ See [19, Proposition 3.8.5]. $\rhd$

We now present a characterization of the collections of order bounded disjointness preserving operators $S_1, \dots, S_N$ with purely n-disjoint sum $|S_1| + \dots + |S_N|$. It can be easily observed that if $S_1, \dots, S_N$ are functionals then $|S_1| + \dots + |S_N|$ is n-disjoint if and only if there exists a permutation $(T_1, \dots, T_N)$ of $(S_1, \dots, S_N)$ such that $T_1, \dots, T_n$ are pairwise disjoint and, whenever $n > N$, each of $T_{n+1}, \dots, T_N$ is representable as $T_i = \lambda_i S_{\kappa(i)}$ with some $\kappa(i) \in \{1, \dots, n\}$ and $\lambda_i \in \mathbb{R}$, $|\lambda_i| \leqslant 1$. Having settled the scalar case, the desired characterization is the matter of Boolean valued technology.

Theorem 3.4. *Let X and Y be vector lattices with Y Dedekind complete and $n, N \in \mathbb{N}$ with $n \leqslant N$. For a collection of order bounded disjointness preserving operator $T_1, \dots, T_N$ from X to Y the operator $|T_1| + \dots + |T_N|$ is purely n-disjoint if and only if there exists a $\mathbb{P}(Y)$-permutation $S_1, \dots, S_N$ of $T_1, \dots, T_N$ such that $S_1, \dots, S_n$ are pairwise purely disjoint and, whenever $n < N$, each of $S_{n+1}, \dots, S_N$ is representable as $S_j = \sum_{k=1}^{n} \alpha_{j,k} S_k$ for some pairwise disjoint $0 \leqslant \alpha_{j,1}, \dots, \alpha_{j,n} \in \mathscr{Z}(Y)$ $(j := n+1, \dots, N)$.*

$\lhd$ We can assume that $Y = \mathscr{R}\!\downarrow$ and put $\tau_l := T_l\!\uparrow$. Moreover, there is no loss of generality in assuming that $T_1, \dots, T_N$ are lattice homomorphisms so that $\tau_1, \dots, \tau_N$ are also assumed to be lattice homomorphisms within $\mathbb{V}^{(\mathbb{B})}$. We confine ourselves to the “only if” part.

Supposing $T_1 + \dots + T_N$ is purely n-disjoint and working within $\mathbb{V}^{(\mathbb{B})}$, observe that $\tau_1 + \ldots + \tau_N$ is $n^\wedge$-disjoint and so there exists a permutation $\nu : \{1, \dots, N\}^\wedge \to \{1, \dots, N\}^\wedge$ such that $\tau_{\nu(1)}, \dots, \tau_{\nu(n)}$ are nonzero pairwise disjoint lattice homomorphisms and, whenever $n < N$, each of the homomorphisms $\tau_{\nu(n+1)}, \dots, \tau_{\nu(N)}$ is proportional to some of $\tau_{\nu(1)}, \dots, \tau_{\nu(n)}$ with a constant of modulus $\leqslant 1$. The latter is formalized as follows:

$$\Phi \equiv (\forall\, i \in \{n+1, \dots, N\}^{\wedge})(\exists\, j \in \{1, \dots, n\}^{\wedge}) \\ (\exists\, \beta \in \mathscr{R})(|\beta| \leqslant 1 \wedge \tau_{\nu(i)} = \beta\tau_{\nu(j)}).$$

Put $S_i := \tau_{\nu(i^{\wedge})}{\downarrow}$ ($i := 1, \dots, N$). Then $(S_1, \dots, S_N)$ is a $\mathbb{P}(Y)$-permutation of $(T_1, \dots, T_N)$ by Theorem 3.3 and $(S_1, \dots, S_n)$ are pairwise disjoint by Corollary 2.1. Moreover, $[\![\Phi]\!] = \mathbb{1}$ by transfer. Hence,

$$\mathbb{1} = \bigwedge_{i=n+1}^{N} \bigvee_{j=1}^{n} [\![(\exists\, \beta)(\beta \in \mathscr{R})(|\beta| \leqslant 1 \wedge \tau_{\nu(i^{\wedge})} = \beta\tau_{\nu(j^{\wedge})})]\!].$$

It follows that for each $n+1 \leqslant i \leqslant N$ there is a partition of unity $\{b_{i,1}, \dots, b_{i,n}\}$ in $\mathbb{B}$ such that $b_{i,j} \leqslant [\![(\exists\, \beta)(\beta \in \mathscr{R})(|\beta| \leqslant 1 \wedge \tau_{\nu(i^{\wedge})} = \beta\tau_{\nu(j^{\wedge})})]\!]$. According to the maximum principle there exists $\beta_{i,j} \in \mathscr{R}{\downarrow}$ with $b_{i,j} \leqslant [\![|\beta_{i,j}| \leqslant 1]\!] \wedge [\![\tau_{\nu(i^{\wedge})} = \beta_{i,j}\tau_{\nu(j^{\wedge})}]\!]$. Observe that for each $x \in X$ we have

$$b_{i,j} \leqslant [\![\tau_{\nu(i^{\wedge})} = \beta_{i,j}\tau_{\nu(j^{\wedge})})]\!] \leqslant [\![\tau_{\nu(i^{\wedge})}(x^{\wedge}) = \beta_{i,j}\tau_{\nu(j^{\wedge})}(x^{\wedge})]\!] \\ \wedge [\![\tau_{\nu(i^{\wedge})}(x^{\wedge}) = S_i x]\!] \wedge [\![\tau_{\nu(j^{\wedge})}(x^{\wedge}) = S_j x]\!] \leqslant [\![S_i x = \beta_{i,j} S_j x]\!].$$

Putting $\pi_{i,j} := \chi(b_{i,j})$ and $\alpha_{i,j} := \pi_{i,j}\beta_{i,j}$ and using the Gordon Theorem, we see that $\pi_{i,j} S_i x = \alpha_{i,j} S_j x$, whence $S_i = \sum_{j=1}^{n} \alpha_{i,j} S_j$ as required. ▷

Definition 3.4. A collection $(k_1, \pi_1), \dots, (k_l, \pi_l)$ is said to be a *decomposition series* in $\mathbb{P}(Y)$ whenever $1 \leqslant k_1 < \dots < k_l$ are naturals and $\{\pi_1, \dots, \pi_l\}$ is a partition of unity in $\mathbb{P}(Y)$ with nonzero terms. We say that an n-disjoint operator $T : X \to Y$ *has a decomposition series* $(k_1, \pi_1), \dots, (k_l, \pi_l)$ in $\mathbb{P}(Y)$ if, in addition to the above, $k_l \leqslant n$ and there exist order bounded disjointness preserving operators $T_1, \dots, T_{k_l}$ from E to F such that $T = T_1 + \dots + T_{k_l}$ and $\pi_i T_1, \dots, \pi_i T_{k_i}$ are pairwise purely disjoint for every $i = 1, \dots, l$.

Putting together Theorems 3.2 and 3.4 we get the following result.

Theorem 3.5. *Every order bounded n-disjoint operator T from a vector lattice X to a Dedekind complete vector lattice Y has a unique decomposition series $(k_1, \pi_1), \dots, (k_l, \pi_l)$ in $\mathbb{P}(T(X)^{\perp\perp})$.*

4. Classification of Injective Modules

In this section, K is a commutative semiprime ring with unit, and X be a module over K. Boolean valued analysis provides a transfer principle from the theory of vector spaces over fields to the theory of modules over rings. Below we describe the class of modules for which this transfer works perfectly. Further details can be found in the book by Lambek [22]. Theorems 4.1-4.4 below are due to Gordon; see [10, 11].

Definition 4.1. An *annihilator ideal* of K is a subset of the form $S^{\perp} := \{k \in K : (\forall\, s \in S) ks = 0\}$ with a nonempty subset $S \subset K$. A subset S of K is called *dense* provided that $S^{\perp} = \{0\}$; i. e., the equality $k \cdot S := \{k \cdot s : s \in S\} = \{0\}$ implies $k = 0$ for all $k \in K$.

Definition 4.2. A ring K is said to be *rationally complete* whenever, to each dense ideal $J \subset K$ and each group homomorphism $h : J \to K$ such that $h(kx) = kh(x)$ for all $k \in K$ and $x \in J$, there is an element r in K satisfying $h(x) = rx$ for all $x \in J$.

A ring K is rationally complete if and only if K is selfinjective (cp. [18, Theorem 8.2.7 (3)]). The inclusion ordered set $\mathbb{A}(K)$ of all annihilator ideal of a commutative semiprime ring K is a complete Boolean algebra.

Theorem 4.1. *If $\mathscr{K}$ is a field within $\mathbb{V}^{(\mathbb{B})}$ then $\mathscr{K}{\downarrow}$ is a rationally complete commutative semiprime ring, and there is an isomorphism χ of $\mathbb{B}$ onto the Boolean algebra $\mathbb{A}(\mathscr{K}{\downarrow})$ such that*

$$b \leqslant [\![x = 0]\!] \iff x \in \chi(b^*) \quad (x \in K,\ b \in \mathbb{B}).$$

$\lhd$ See [18, Theorems 8.3.1]. $\rhd$

Theorem 4.2. *Assume that K is a rationally complete commutative semiprime ring and $\mathbb{B} = \mathbb{A}(K)$. Then there is an element $\mathscr{K} \in \mathbb{V}^{(\mathbb{B})}$ such that $[\![\, \mathscr{K}$ is a field $]\!] = \mathbb{1}$ and the rings K and $\mathscr{K}{\downarrow}$ are isomorphic.*

$\lhd$ See [18, Theorems 8.3.2]. $\rhd$

Definition 4.3. A K-module X is *separated* provided that for every dense ideal $J \subset K$ the identity $xJ = \{0\}$ implies $x = 0$. Recall that a K-module X is *injective* whenever, given a K-module Y, a K-submodule $Y_0 \subset Y$, and a K-homomorphism $h_0 : Y_0 \to X$, there exists a K-homomorphism $h : Y \to X$ extending h_0.

The *Baer criterion* states that a K-module X is injective if and only if for each ideal $J \subset K$ and each K-homomorphism $h : J \to X$ there exists $x \in X$ with $h(a) = xa$ for all $a \in J$; see Lambek [22, § 4.2, Lemma 1]. All modules under consideration are assumed to be *faithful*, i. e., $Xk \neq \{0\}$ for any $0 \neq k \in K$, or equivalently, the canonical representation of K by endomorphisms of the additive group X is one-to-one.

Theorem 4.3. *Let $\mathscr{X}$ be a vector space over a field $\mathscr{K}$ within $\mathbb{V}^{(\mathbb{B})}$, and let $\chi : \mathbb{B} \to \mathbb{B}(\mathscr{K}{\downarrow})$ be a Boolean isomorphism in Theorem 4.1. Then $\mathscr{X}{\downarrow}$ is a separated unital injective module over $\mathscr{K}{\downarrow}$ such that $b \leqslant [\![x = 0]\!]$ and $\chi(b)x = \{0\}$ are equivalent for all $x \in \mathscr{X}{\downarrow}$ and $b \in \mathbb{B}$.*

$\lhd$ See [18, Theorems 8.3.12]. $\rhd$

Theorem 4.4. *Let K be a rationally complete semiprime ring, let $\mathscr{K}$ be as in Theorem 4.1, and $\mathbb{B} := \mathbb{A}(K)$. Then for every unital separated injective K-module X there exists an internal vector space $\mathscr{X} \in \mathbb{V}^{(\mathbb{B})}$ over $\mathscr{K}$ such that the K-module X is isomorphic to $\mathscr{X}{\downarrow}$. Moreover if $\jmath : K \to \mathscr{K}{\downarrow}$ is an isomorphism in Theorem 4.2, then one can choose an isomorphism $\imath : X \to \mathscr{X}{\downarrow}$ such that $\imath(ax) = \jmath(a)\imath(x)$ $(a \in K,\ x \in X)$.*

$\lhd$ See [18, 8.3.13]. $\rhd$

Theorem 4.4 enables us to apply Boolean valued approach to unital separated injective modules over the rings described in Theorem 4.3.

Definition 4.3. A family $\mathscr{E}$ in a K-module X is called *K-linearly independent* or symply *linearly independent* whenever, for all $n \in \mathbb{N}$, $\alpha_1, \dots, \alpha_n \in K$, and $e_1, \dots, e_n \in \mathscr{E}$, the equality $\sum_{k=1}^{n} \alpha_k e_k = 0$ implies $\alpha_1 = \ldots = \alpha_n = 0$. An inclusion maximal K-linearly independent subset of X is called a *Hamel K-basis* for X.

Every unital separated injective K-module X has a Hamel K-basis. A K-linearly independent set $\mathscr{E}$ in X is a Hamel K-basis if and only if for every $x \in X$ there exist a partition of unity $(\pi_k)_{k\in\mathbb{N}}$ in $\mathbb{P}(K)$ and a family $(\lambda_{k,e})_{k\in\mathbb{N}, e\in\mathscr{E}}$ in K such that

$$\pi_k x = \sum\nolimits_{e\in\mathscr{E}} \lambda_{k,e}\pi_k e \quad (k \in \mathbb{N})$$

and for every $k \in \mathbb{N}$ the set $\{e \in \mathscr{E} : \lambda_{k,e} \neq 0\}$ is finite.

Definition 4.4. Let γ be a cardinal. A K-module X is said to be *Hamel γ-homogeneous* whenever there exists a Hamel K-basis of cardinality γ in X. For $\pi \in \mathbb{P}(X)$ denote by $\varkappa(\pi)$ the least cardinal γ for which πX is Hamel γ-homogeneous. Say that X is *strictly Hamel γ-homogeneous* whenever X is Hamel γ-homogeneous and $\varkappa(\pi) = \gamma$ for all nonzero $\pi \in \mathbb{P}(X)$.

Theorem 4.5. *Let K be a semiprime rationally complete commutative ring and let X be a separated injective module over K. There exists* *a unique partition of unity* $(e_\gamma)_{\gamma\in\Gamma}$ *in* $\mathbb{P}(K)$ *with* Γ *a set of cardinals such that* $e_\gamma X$ *is strictly Hamel* γ*-homogeneous for all* $\gamma \in \Gamma$*. Moreover,* X *is isomorphic to* $\prod_{\gamma\in\Gamma} e_\gamma X$ *and the partition of unity* $(e_\gamma)_{\gamma\in\Gamma}$ *is unique up to permutation.*

$\lhd$ According to Theorems 4.2 and 4.4 we may assume that $K = \mathscr{K}{\downarrow}$ and $X = \mathscr{X}{\downarrow}$, where $\mathscr{X}$ is a vector space over the field $\mathscr{K}$ within $\mathbb{V}^{(\mathbb{B})}$. Moreover, $\dim(\mathscr{X}) \in \mathbb{V}^{(\mathbb{B})}$, the algebraic dimension of $\mathscr{X}$, is an internal cardinal and, since each Boolean valued cardinal is a mixture of some set of relatively standard cardinals [19, 1.9.11], we have $\dim(\mathscr{X}) = \mathrm{mix}_{\gamma\in\Gamma} b_\gamma \gamma^\wedge$ where Γ is a set of cardinals and $(b_\gamma)_{\gamma\in\Gamma}$ is a partition of unity in $\mathbb{B}$. Thus, for all $\gamma \in \Gamma$ we have $e_\gamma \leqslant [\![\dim(\mathscr{X}) = \gamma^\wedge]\!]$, whence $e_\gamma X$ is strictly Hamel γ-homogeneous. The remaining details are elementary. $\rhd$

Definition 4.5. The *passport* for X is $(e_\gamma)_{\gamma\in\Gamma}$ in Theorem 4.1.

Theorem 4.6. *Two separated injective modules over a semiprime rationally complete commutative ring are isomorphic if and only if their passports coincide.*

$\lhd$ The passport $\Gamma(X)$ is the interpretation of the algebraic dimension $\dim(\mathscr{X})$ in $\mathbb{V}^{(\mathbb{B})}$ with $\mathbb{B} = \mathbb{A}(X)$. Therefore, the required result follows from the fact that two vector spaces are isomorphic if and only if their algebraic dimensions coincide. $\rhd$

Remark 4.1. Recently, Chilin and Karimov [7, Theorem 4.2], without using the Boolean valued approach, obtained that particular case of Theorem 4.5 when $K = L^0(\Omega, \Sigma, \mu)$ is a real or complex universally complete f-algebra of measurable functions, see Example 2.1. Another particular case of Theorem 16 when $\mathscr{X}$ is a vector subspace of $\mathscr{R}$ (considered as a vector space over $\mathbb{R}^\wedge$) was examined by Kusraev and Kutateladze [19, Chap. 4]. Chilin and Karimov [7, Theorem 4.3] also proved Theorem 4.6 for $K = L^0(\Omega, \Sigma, \mu)$.

5. The Radon–Nikodým Theorem for *JB*-Algebras

In this section we sketch some further applications of the Boolean value approach to nonassociative Radon–Nikodým type theorems.

Definition 5.1. Let A be a vector space over some field $\mathbb{F}$. Say that A is a *Jordan algebra,* if there is given a (generally) nonassociative binary operation $A \times A \ni (x, y) \mapsto xy \in A$ on A, called *multiplication* and satisfying the following for all $x, y, z \in A$ and $\alpha \in \mathbb{F}$:

$$xy = yx, \quad (x + y)z = xz + yz,$$
$$\alpha(xy) = (\alpha x)y, \quad (x^2 y)x = x^2(yx).$$

An element e of a Jordan algebra A is a *unit element* or a *unit* of A, if $e \neq 0$ and $ea = a$ for all $a \in A$.

Definition 5.2. Recall that a *JB-algebra* is simultaneously a real Banach space A and a unital Jordan algebra with unit $\mathbb{1}$ such that

(1) $\|xy\| \leqslant \|x\| \cdot \|y\|$, $x, y \in A$,

(2) $\|x^2\| = \|x\|^2$, $x \in A$,

(3) $\|x^2\| \leqslant \|x^2 + y^2\|$, $x, y \in A$.

The set $A_+ := \{x^2 : x \in A\}$, presenting a proper convex cone, determines the structure of an ordered vector space on A so that the unity $\mathbb{1}$ of the algebra A serves as a strong order unit, and the order interval $[-\mathbb{1}, \mathbb{1}] := \{x \in A : -\mathbb{1} \leqslant x \leqslant \mathbb{1}\}$ serves as the unit ball. Moreover, the inequalities $-\mathbb{1} \leqslant x \leqslant \mathbb{1}$ and $0 \leqslant x^2 \leqslant \mathbb{1}$ are equivalent.

Definition 5.3. The intersection of all maximal associative subalgebras of A is called the *center* of A and denoted by $\mathscr{Z}(A)$. The element a belongs to $\mathscr{Z}(A)$ if and only if $(ax)y = a(xy)$ for all $x, y \in A$. If $\mathscr{Z}(A) = \mathbb{R} \cdot \mathbb{1}$, then A is said to be a *JB-factor*.

The center $\mathscr{Z}(A)$ is an associative *JB*-algebra, and hence is isometrically isomorphic to the real Banach algebra $C(Q)$ of continuous functions on some compact space Q.

Definition 5.4. The idempotents of a *JB*-algebra are also called *projections*. The set of all projections $\mathbb{P}(A)$ forms a complete lattice with the order defined as $\pi \leqslant \rho \iff \pi \circ \rho = \pi$. The sublattice of *central projections* $\mathbb{P}_c(A) := \mathbb{P}(A) \cap \mathscr{Z}(A)$ is a Boolean algebra. Assume that $\mathbb{B}$ is a subalgebra of the Boolean algebra $\mathbb{P}_c(A)$. Then we say that A is a $\mathbb{B}$-*JB-algebra* if, for every partition of unity $(e_\xi)_{\xi \in \Xi}$ in $\mathbb{B}$ and every family $(x_\xi)_{\xi \in \Xi}$ in A, there exists a unique $\mathbb{B}$-mixing $x := \operatorname{mix}_{\xi \in \Xi}(e_\xi x_\xi)$, i. e., a unique element $x \in A$ such that $e_\xi x_\xi = e_\xi x$ for all $\xi \in \Xi$. If $\mathbb{B} = \mathbb{P}_c(A)$, then a $\mathbb{B}$-*JB*-algebra is also referred to as *centrally extended JB-algebra.*

Theorem 5.1. *The restricted descent of a JB-algebra in the model $\mathbb{V}^{(\mathbb{B})}$ is a $\mathbb{B}$-JB-algebra. Conversely, for every $\mathbb{B}$-JB-algebra A there exists a unique (up to isomorphism) JB-algebra $\mathscr{A}$ within $\mathbb{V}^{\mathbb{B}}$ whose restricted descent is isometrically $\mathbb{B}$-isomorphic to A. Moreover, A is centrally extended if and only if $[\![\mathscr{A}$ is a JB-factor$]\!] = \mathbb{1}$.*

$\triangleleft$ See [18, Theorem 12.7.6] and [16, Theorem 3.1]. $\triangleright$

Now we give two applications of the above Boolean valued representation result to $\mathbb{B}$-*JB*-algebras. Theorems 5.3 and 5.5 below appear by transfer of the corresponding facts from the theory of *JB*-algebras. Let A be a $\mathbb{B}$-*JB*-algebra and $\Lambda := \Lambda(\mathbb{B})$ stand for a Dedekind complete unital AM-space with $\mathbb{B} \simeq \mathbb{P}(\Lambda)$. Denote by $A^{\#}$ the Banach space of all bounded linear operators from A to Λ commuting with projections in $\mathbb{B}$.

Definition 5.5. An operator $\Phi \in A^{\#}$ is called a Λ-*valued state* if Φ is positive ($\Phi(A_+) \subset \Lambda_+$) and $\Phi(\mathbb{1}) = \mathbb{1}$. A state (or a weight) Φ is said to be *normal* if, for every increasing net (x_α) in A with the least upper bound $x := \sup x_\alpha$, we have $\Phi(x) = o\text{-}\lim \Phi(x_\alpha)$.

If $\mathscr{A}$ and A are as in Theorem 5.1, then the ascent $\varphi := \Phi\uparrow$ is a bounded linear functional on $\mathscr{A}$ by [19, Theorem 5.8.12]. Moreover, φ is positive and order continuous; i. e., φ is a normal state on $\mathscr{A}$. The converse is also true: if $[\![\varphi \text{ is a normal state on } \mathscr{A}]\!] = \mathbb{1}$, then the restriction of the operator $\varphi\downarrow$ to A is a Λ-valued normal state. Now we will characterize $\mathbb{B}$-*JB*-algebras that are $\mathbb{B}$-dual spaces. To this end, it suffices to give Boolean valued interpretation for the following result.

Theorem 5.2. *A JB-algebra is a dual Banach space if and only if it is monotone complete and has a separating set of normal states.*

$\lhd$ See [26, Theorem 2.3]. $\rhd$

Theorem 5.3. *Let $\mathbb{B}$ be a complete Boolean algebra and let Λ be a Dedekind complete unital AM-space with $\mathbb{B} \simeq \mathbb{P}(\Lambda)$. A $\mathbb{B}$-JB-algebra A is a $\mathbb{B}$-dual space if and only if A is monotone complete and admits a separating set of Λ-valued normal states. If one of these equivalent conditions holds, then the part of $A^{\#}$ consisting of order continuous operators serves as a $\mathbb{B}$-predual space of A.*

$\lhd$ See [18, Theorem 12.8.5] and [16, Theorem 4.2]. $\rhd$

Definition 5.6. An algebra A satisfying one of the equivalent conditions 5.3 is called a $\mathbb{B}$-JBW-*algebra*. If, moreover, $\mathbb{B}$ coincides with the set of all central projections, then A is said to be a $\mathbb{B}$-JBW-*factor*.

It follows from Theorems 5.1 and 5.3 that A is a $\mathbb{B}$-JBW-algebra ($\mathbb{B}$-JBW-factor) if and only if its Boolean valued representation $\mathscr{A} \in \mathbb{V}^{(\mathbb{B})}$ is a JBW-algebra (JBW-factor).

Definition 5.7. A mapping $\Phi : A_+ \to \Lambda \cup \{+\infty\}$ is a (Λ-valued) *weight* if the conditions are satisfied (under the assumptions that $\lambda + (+\infty) := +\infty + \lambda := +\infty$, $\lambda \cdot (+\infty) =: \lambda$ for all $\lambda \in \Lambda$, while $0 \cdot (+\infty) := 0$ and $+\infty + (+\infty) := +\infty$):

(1) $\Phi(x+y) = \phi(x) + \Phi(y)$ for all $x, y \in A_+$.

(2) $\Phi(\lambda x) = \lambda\Phi(x)$ for all $x \in A_+$ and $\lambda \in \Lambda_+$.

A weight Φ is said to be a *trace* if the additional condition is satisfied

(3) $\Phi(x) = \Phi(U_s x)$ for all $x \in A_+$ and $s \in A$ with $s^2 = \mathbb{1}$.

Here U_a is the operator from A to A defined for a given $a \in A$ as $U_a : x \mapsto 2a(ax) - a^2$ $(x \in A)$. This operator is positive, i. e., $U_a(A_+) \subset A_+$. If $a \in \mathscr{Z}(A)$, then $U_a x = a^2 x$ $(x \in A)$.

Definition 5.8. A weight (trace) Φ is called: *semifinite* if there exists an increasing net (a_α) in A_+ with $\sup_\alpha a_\alpha = \mathbb{1}$ and $\Phi(a_\alpha) \in \Lambda$ for all α; *bounded* if $\Phi(\mathbb{1}) \in \Lambda$. Given two Λ-valued weights Φ and Ψ on A, say that Φ is dominated by Ψ if there exists $\lambda \in \Lambda_+$ such that $\Phi(x) \leqslant \lambda\Psi(x)$ for all $x \in A_+$.

We need a few additional remarks about descents and ascents. Fix $+\infty \in \mathbb{V}^{(\mathbb{B})}$ and put $\Lambda^\star := (\mathscr{R} \cup \{+\infty\})\!\downarrow = \operatorname{mix}(\mathscr{R}\!\downarrow \cup \{+\infty\})$. Clearly, $\Lambda = \mathscr{R}\!\Downarrow$ and $\Lambda^u = \mathscr{R}\!\downarrow$, while $\Lambda^\star$ consists of all elements of the form $\lambda_\pi := \operatorname{mix}(\pi\lambda, \pi^\perp(+\infty))$ with $\lambda \in \Lambda^u$ and $\pi \in \mathbb{P}(\Lambda)$. At the same time,

$$(\Lambda^u \cup \{+\infty\})\!\uparrow = (\Lambda \cup \{+\infty\})\!\uparrow = \Lambda\!\uparrow \cup \{+\infty\}\!\uparrow = \mathscr{R} \cup \{+\infty\}.$$

Thus, $\Lambda^u \cup \{+\infty\}$ is a proper subset of $\Lambda^\star$, since $x_\pi \in \Lambda \cup \{+\infty\}$ if and only if $\pi = 0$ or $\pi = I_\Lambda$. Assume now that $A = \mathscr{A}\!\downarrow$ with $\mathscr{A}$ a *JB*-algebra within $\mathbb{V}^{(\mathbb{B})}$ and $\mathbb{B}$ equal to $\mathbb{P}(A)$. Every bounded weight $\Phi : A \to \Lambda$ is evidently extensional, that is $[\![x = y]\!] \leqslant [\![\Phi(x) = \Phi(y)]\!]$ for all $x, y \in A$. But an unbounded weight may fail to be extensional: if $\Phi(x_0) = +\infty$ and $\Phi(x) \in \Lambda$ for some $x_0, x \in A$ and $b \in \mathbb{P}(A)$ then

$$\Phi(\operatorname{mix}(bx, b^\perp x_0)) = \operatorname{mix}(b\Phi(x), b^\perp(+\infty)) \notin \Lambda \cup \{+\infty\}.$$

Definition 5.9. Given a semifinite weight Φ on A, we define its *extensional modification* $\widehat{\Phi} : A \to \Lambda^\star$ as follows: If $\Phi(x) \in \Lambda$ then $\widehat{\Phi}(x) := \Phi(x)$. Otherwise $x = \sup D$ with $D := \{a \in A : 0 \leqslant a \leqslant x,\ \Phi(a) \in \Lambda\}$. Let b stand for the greatest element of $\mathbb{P}(\Lambda)$ such that $\Phi(bD)$ is order bounded in Λ^u and put $\lambda := \sup \Phi(bD)$. Define $\widehat{\Phi}(x) := \lambda_b = \operatorname{mix}(b\lambda, b^\perp(+\infty))$; i. e., $b\widehat{\Phi}(x) = \lambda$ and $b^\perp\widehat{\Phi}(x) = b^\perp(+\infty)$.

It is easy to check that $\widehat{\Phi}$ is an extensional mapping. Thus, for $\varphi := \widehat{\Phi}\!\uparrow$ we have $[\![\varphi : \mathscr{A} \to \mathscr{R} \cup \{+\infty\}]\!] = \mathbb{1}$ and, according to [19, 1.6.6], $\widehat{\Phi} = \varphi\!\downarrow \neq \Phi$. But if we define $\varphi\!\Downarrow$ as $\varphi\!\Downarrow\!(x) = \varphi\!\downarrow\!(x)$ whenever $\varphi\!\downarrow\!(x) \in \Lambda$ and $\varphi\!\Downarrow\!(x) = +\infty$ otherwise, then $\Phi = (\widehat{\Phi}\!\uparrow)\!\Downarrow$.

Theorem 5.4. *Let A be a JBW-algebra and let τ be a normal semifinite real-valued trace on A. For each real-valued weight φ on A dominated by τ there exists a unique positive element $h \in A$ such that $\varphi(a) = \tau(U_{h^{1/2}}a)$ for all $a \in A_+$. Moreover, φ is bounded if and only if $\tau(h)$ is finite and φ is a trace if and only if h is a central element of A.*

$\lhd$ This fact was proved in [14]. $\rhd$

Theorem 5.5. *Let A be a $\mathbb{B}$-JBW-algebra and let* T *be a normal semifinite Λ-valued trace on A. For each weight Φ on A dominated by* T *there exists a unique positive $h \in A$ such that $\Phi(x) = \mathrm{T}(U_{h^{1/2}}x)$ for all $x \in A_+$. Moreover, Φ is bounded if and only if $\mathrm{T}(h) \in \Lambda$ and Φ is a trace if and only if h is a central element of A.*

$\lhd$ We present a sketch of the proof. Taking into consideration Definition 5.9 we define $\varphi = \widehat{\Phi}\!\uparrow$ and $\tau = \widehat{\mathrm{T}}\!\uparrow$. Then within $\mathbb{V}^{(\mathbb{B})}$ the following hold: τ is a semifinite normal real-valued trace on $\mathscr{A}$ and φ is real-valued weight on $\mathscr{A}$ dominated by τ. By transfer we may apply Theorem 5.4 and find $h \in \mathscr{A}$ such that $\varphi(x) = \tau(U_{h^{1/2}}x)$ for all $x \in \mathscr{A}_+$. Actually, $h \in A$ and $\varphi{\Downarrow}(x) = \tau{\Downarrow}(U_{h^{1/2}}x)$ for all $x \in A_+$. It remains to note that $\Phi = \varphi{\Downarrow}$ and $\mathrm{T} = \tau{\Downarrow}$. The details are left to the reader. $\rhd$

Remark 5.1. *JB*-algebras are nonassociative real analogs of C^*-algebras and von Neumann operator algebras. The theory of these algebras stems from Jordan, von Neumann, and Wigner [13] and exists as a branch of functional analysis since the mid 1960s. The class of *AJW*-algebras was firstly mentioned by Topping in [30].The main areas of research are reflected in the works by Ajupov [1, 2]; Hanshe-Olsen and Störmer [12]. The Boolean valued approach to *JB*-algebras was charted by Kusraev in the article [16] which contains Theorems 5.1 and 5.3 (also see [16]).

6. Transfer in Harmonic Analysis

In what follows, G is a locally compact abelian group and τ is the topology of G, while $\tau(0)$ is a neighborhood base of 0 in G and G' stands for the dual group of G. Note that G is also the dual group of G' and we write $\langle x, \gamma\rangle := \gamma(x)$ ($x \in G$, $\gamma \in G'$).

By restricted transfer, $G^\wedge$ is a group within $\mathbb{V}^{(\mathbb{B})}$. At the same time $\tau(0)^\wedge$ may fail to be a topology on $G^\wedge$. But $G^\wedge$ becomes a topological group on taking $\tau(0)^\wedge$ as a neighborhood base of $0 := 0^\wedge$. This topological group is again denoted by $G^\wedge$ itself. Clearly, $G^\wedge$ may not be locally compact. Let U be a neighborhood of 0 such that U is compact.

Then U is totally bounded. It follows by restricted transfer that $U^{\wedge}$ is totally bounded as well, since total boundedness can be expressed by a restricted formula. Therefore the completion of $G^{\wedge}$ is locally compact. The completion of $G^{\wedge}$ is denoted by $\mathscr{G}$, and by the above observation $\mathscr{G}$ is a locally compact abelian group within $\mathbb{V}^{(\mathbb{B})}$.

Definition 6.1. Let Y be a Dedekind complete vector lattice and let $Y_{\mathbb{C}}$ be the complexification of Y. A vector-function $\varphi : G \to Y$ is said to be *uniformly order continuous* on a set K if

$$\inf_{U\in\tau(0)} \sup\big\{|\varphi(g_1)-\varphi(g_2)| : g_1, g_2 \in K,\ g_1 - g_2 \in U\big\} = 0.$$

This amounts to saying that φ is order bounded on K and, if $e \in Y$ is an upper bound of $\varphi(K)$, then for each $0 < \varepsilon \in \mathbb{R}$ there exists a partition of unity $(\pi_\alpha)_{\alpha\in\tau(0)}$ in $\mathbb{P}(Y)$ such that $\pi_\alpha|\varphi(g_1)-\varphi(g_2)| \leqslant \varepsilon e$ for all $\alpha \in \tau(0)$ and $g_1, g_2 \in K$, $g_1 - g_2 \in \alpha$. If, in this definition we put $g_2 = 0$, then we arrive at the definition of mapping *order continuous at zero*.

Definition 6.2. A mapping $\psi : G \to Y_{\mathbb{C}}$ is called *positive definite* if

$$\sum_{j,k=1}^{n} \psi(g_j - g_k)\ c_j\overline{c}_k \geqslant 0$$

for all finite collections $g_1, \dots, g_n \in G$ and $c_1, \dots, c_n \in \mathbb{C}$ $(n \in \mathbb{N})$.

For $n = 1$, the definition implies readily that $\psi(0) \in Y_+$. For $n = 2$, we have $|\psi(g)| \leqslant \psi(0)$ $(g \in G)$. If we introduce the structure of an f-algebra with unit $\psi(0)$ in the order ideal of Y generated by $\psi(0)$ then, for $n = 3$, from the above definition we can deduce one more inequality

$$\big|\psi(g_1) - \psi(g_2)\big|^2 \leqslant 2\psi(0)(\psi(0) - \operatorname{Re}\psi(g_1 - g_2)) \quad (g_1, g_2 \in G).$$

It follows that every positive definite mapping $\psi : G \to Y_{\mathbb{C}}$ o-continuous at zero is order-bounded (by $\psi(0)$) and uniformly o-continuous.

Definition 6.3. A mapping $\varphi : G \to Y$ is called *dominated* if there exists a positive definite mapping $\psi : G \to Y_{\mathbb{C}}$ such that

$$\left|\sum_{j,k=1}^{n} \varphi(g_j - g_k)c_j\overline{c}_k\right| \leqslant \sum_{j,k=1}^{n} \psi(g_j - g_k)c_j\overline{c}_k$$

for all $g_1, \dots, g_n \in G$, $c_1, \dots, c_n \in \mathbb{C}$ $(n \in \mathbb{N})$. In this case we also say that ψ is a *dominant* of φ. It can be easily shown that if $\varphi : G \to Y_{\mathbb{C}}$

has dominant order continuous at zero then φ is order bounded and uniformly order continuous.

We denote by $\mathfrak{D}(G, Y_{\mathbb{C}})$ the vector space of all dominated mappings from G into $Y_{\mathbb{C}}$ whose dominants are order continuous at zero. We also consider the set $\mathfrak{D}(G, Y_{\mathbb{C}})_+$ of all positive definite mappings from G into $Y_{\mathbb{C}}$. This set is a proper cone in $\mathfrak{D}(G, Y_{\mathbb{C}})$ and defines the order compatible with the structure of a vector space on $\mathfrak{D}(G, Y_{\mathbb{C}})$. Actually, $\mathfrak{D}(G, Y_{\mathbb{C}})$ is a Dedekind complete complex vector lattice; cp. 6.3 below. Also, define $\mathfrak{D}(\mathscr{G}, \mathscr{C}) \in \mathbb{V}^{(\mathbb{B})}$ to be the set of functions $\varphi : \mathscr{G} \to \mathscr{C}$ with the property that $[\![\varphi$ has dominant continuous at zero$]\!] = \mathbb{1}$.

Theorem 6.1. *Let $Y = \mathscr{R}{\downarrow}$. For every $\varphi \in \mathfrak{D}(G, Y_{\mathbb{C}})$ there exists a unique $\widetilde{\varphi} \in \mathbb{V}^{(\mathbb{B})}$ such that $[\![\widetilde{\varphi} \in \mathfrak{D}(\mathscr{G}, \mathscr{C})]\!] = \mathbb{1}$ and $[\![\widetilde{\varphi}(x^{\wedge}) = \varphi(x)]\!] = \mathbb{1}$ for all $x \in G$. The mapping $\varphi \mapsto \widetilde{\varphi}$ is an linear and order isomorphism from $\mathfrak{D}(G, Y)$ onto $\mathfrak{D}(\mathscr{G}, \mathscr{C}){\downarrow}$.*

Define $C_0(G)$ as the space of all continuous complex functions f on G vanishing at infinity. The latter means that for every $0 < \varepsilon \in \mathbb{R}$ there exists a compact set $K \subset G$ such that $|f(x)| < \varepsilon$ for all $x \in G \backslash K$. Denote by $C_c(G)$ the space of all continuous complex functions on G having compact support. Evidently, $C_c(G)$ is dense in $C_0(G)$ with respect to the norm $\|\cdot\|_\infty$. Introduce the class of dominated operators.

Definition 6.4. Let X be a complex normed space and let Y be a complex Banach lattice. A linear operator $T : X \to Y$ is said to be *majorizing* if T sends the unit ball of X into an order bounded subset of Y. This amounts to saying that there exists $c \in Y_+$ such that $|Tx| \leqslant c\|x\|_\infty$ for all $x \in C_0(Q)$. The set of all dominated operators from X to Y is denoted by $L_m(X, F)$. If Y is Dedekind complete then

$$|\!|T|\!| := \{|Tx| : \ x \in X, \ \|x\| \leqslant 1\}$$

exists and is called the *abstract norm* of T. If X is a vector lattice and Y is Dedekind complete then $L_m(X, Y)$ is a vector sublattice of $L^{\sim}(X, Y)$.

Given a positive $T \in L_m(C_0(G'), Y)$, we can define the mapping $\varphi : G \to Y$ by putting $\varphi(x) = T(\langle x, \cdot \rangle)$ for all $x \in G$, since $\gamma \mapsto \langle x, \gamma \rangle$ lies in $C_0(G')$ for every $x \in G$. It is not difficult to ensure that φ is order continuous at zero and positive definite.

Consider a metric space (M, r). The definition of metric space can be written as a bounded formula, say $\varphi(M, r, \mathbb{R})$, so that $[\![\varphi(M^{\wedge}, r^{\wedge}, \mathbb{R}^{\wedge})]\!] = \mathbb{1}$ by restricted transfer. In other words, $(M^{\wedge}, r^{\wedge})$ is a metric space within $\mathbb{V}^{(\mathbb{B})}$. Moreover we consider the internal function

$r^\wedge : M^\wedge \to \mathbb{R}^\wedge \subset \mathscr{R}$ as an $\mathscr{R}$-valued metric on $M^\wedge$. Denote by $(\mathscr{M}, \rho)$ the completion of $(M^\wedge, r^\wedge)$; i. e., $[\![(\mathscr{M}, \rho)$ is a complete metric space$]\!] = \mathbb{1}$, $[\![M^\wedge$ is a dense subset of $\mathscr{M}]\!] = \mathbb{1}$, and $[\![r(x^\wedge) = \rho(x^\wedge)]\!] = \mathbb{1}$ for all $x \in M$.

Now, if $(X, \|\cdot\|)$ is a real (or complex) normed space then $[\![X^\wedge$ is a vector space over the field $\mathbb{R}^\wedge$ (or $\mathbb{C}^\wedge$) and $\|\cdot\|^\wedge$ is a norm on $X^\wedge$ with values in $\mathbb{R}^\wedge \subset \mathscr{R}]\!] = \mathbb{1}$. So, we will consider $X^\wedge$ as an $\mathbb{R}^\wedge$-vector space with $\mathscr{R}$-valued norm within $\mathbb{V}^{(\mathbb{B})}$. Let $\mathscr{X} \in \mathbb{V}^{(\mathbb{B})}$ stand for the (metric) completion of $X^\wedge$ within $\mathbb{V}^{(\mathbb{B})}$. It is not difficult to see that $[\![\mathscr{X}$ is a real (complex) Banach space including $X^\wedge$ as a dense $\mathbb{R}^\wedge(\mathbb{C}^\wedge)$-linear subspace$]\!] = \mathbb{1}$, since the metric $(x, y) \mapsto \|x - y\|$ on $X^\wedge$ is translation invariant. Clearly, if X is a real (complex) Banach lattice then $[\![\mathscr{X}$ is a real (complex) Banach lattice including $X^\wedge$ as an $\mathbb{R}^\wedge(\mathbb{C}^\wedge)$-linear sublattice$]\!] = \mathbb{1}$.

Theorem 6.2. *Let $Y = \mathscr{C}{\downarrow}$ and $\mathscr{X}'$ be the topological dual of $\mathscr{X}$ within $\mathbb{V}^{(\mathbb{B})}$. For every $T \in L_m(X, Y)$ there exists a unique $\tau \in \mathscr{X}'{\downarrow}$ such that $[\![\tau(x^\wedge) = T(x)]\!] = \mathbb{1}$ for all $x \in X$. The mapping $T \mapsto \phi(T) := \tau$ defines an isomorphism between the $\mathscr{C}{\downarrow}$-modules $L_m(X, Y)$ and $\mathscr{X}'{\downarrow}$. Moreover, $|\!|T|\!| = |\!|\phi(T)|\!|$ for all $T \in L_m(X, Y)$. If X is a normed lattice then $[\![\mathscr{X}'$ is a Banach lattice $]\!] = \mathbb{1}$, while $\mathscr{X}'{\downarrow}$ is a vector lattice and ϕ is a lattice isomorphism.*

$\lhd$ Suffice it to consider the real case. Apply [15, Theorem 8.3.2] to the lattice normed space $X := (X, |\!|\cdot|\!|)$ with $|\!|x|\!| = \|x\|\mathbb{1}$. By [15, Theorem 8.3.4 (1) and Proposition 8.3.4 (2)] the spaces $\mathscr{X}'{\downarrow} := \mathscr{L}^{(\mathbb{B})}(\mathscr{X}, \mathscr{R}){\downarrow}$ and $L_m(X, Y)$ are linear isometric. We are left with referring to [15, Proposition 5.5.1 (1)]. $\rhd$

Now we are able to state an operator version of the Bochner Theorem describing the set of all inverse Fourier transforms of positive operators.

Theorem 6.3. *A mapping $\Phi : G \to Y_{\mathbb{C}}$ is order continuous at zero and positive definite if and only if there exists a unique positive operator $T \in L_m(C_0(G'), Y_{\mathbb{C}})$ such that $\Phi(x) = T(\langle x, \cdot\rangle)$ for all $(x \in G)$.*

$\lhd$ By transfer and Theorems 6.1 and 6.2, we can replace Φ and T by their Boolean valued representations φ and τ. The norm completion of $C_0(G')^\wedge$ within $\mathbb{V}^{(\mathbb{B})}$ coincides with $C_0(\mathscr{G}')$. (This can be proved by the reasoning similar to that in Takeuti [28, Proposition 3.2].) Application of the classical Bochner Theorem (see Loomis [23, Section 36A]) to φ and τ yields the desired result. $\rhd$

Denote by $\mathrm{qca}(Q, Y)$ the vector lattice of all σ-additive $Y_{\mathbb{C}}$-valued quasi-Radon measures on $\mathscr{B}(Q)$, see [15, 6.2.11]. Applying an integral representation result [15, Theorem 6.2.12 (2)] to T yields the following version of the Bochner theorem.

Theorem 6.4. *Let G be a locally compact abelian group, let G' be the dual group of G, and let Y be a Dedekind complete real vector lattice. For a mapping $\varphi : G \to Y_{\mathbb{C}}$ the following are equivalent:*

(1) *φ has dominant order continuous at zero.*

(2) *There exists a unique measure $\mu \in \mathrm{qca}(G', Y_{\mathbb{C}})$ such that*

$$\varphi(g) = \int\limits_{G'} \chi(g)\, d\mu(\chi) \quad (g \in G).$$

$\lhd$ This is immediate from Theorem 6.2 and [20, Theorem 2.5]. For more details see [19, 5.14.B and 5.14.C]. $\rhd$

Theorem 6.5. *The Fourier transform establishes an order and linear isomorphism between the space of measures $\mathrm{qca}(G', Y)$ and the space of dominated mappings $\mathfrak{D}(G, Y_{\mathbb{C}})$. In particular, $\mathfrak{D}(G, Y_{\mathbb{C}})$ is a Dedekind complete complex vector lattice.*

Remark 6.1. In [28] Takeuti introduced the Fourier transform for the mappings defined on a locally compact abelian group and having as values pairwise commuting normal operators in a Hilbert space. In case $Y := \langle \mathbb{B} \rangle$ (see Example 2.2) Theorem 6.2 is essentially Takeuti's result [28, Theorem 1.3]. Theorem 6.1 is due to Gordon [9, Theorem 2].

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БУЛЕВОЗНАЧНЫЙ АНАЛИЗ: НЕКОТОРЫЕ НОВЫЕ ПРИМЕНЕНИЯ

А. Г. Кусраев, С. С. Кутателадзе

Термин *булевозначный анализ* означает технику исследования свойств произвольного математического объекта путем сравнения его представлений в двух различных теоретико-множественных моделях, в построении которых используются принципиально различные булевы алгебры. В качестве таких моделей обычно берутся классический канторов рай в форме универсума фон Неймана и специально построенный булевозначный универсум. Сравнительный анализ проводится с помощью определенной технологии взаимодействия между этими универсумами. В настоящем миникурсе лекций мы рассмотрим некоторые новые применения булевозначного анализа в теории операторов.

Ключевые слова: булевозначная модель, булевозначные числа, векторная решетка, положительный оператор, рационально полное кольцо, инъективный модуль, JB-алгебра, неассоциативная теорема Радона — Никодима, гармонический анализ, преобразование Фурье, теорема Бохнера.