Contents

Foreword **ix**

Chapter 1. Nonstandard Methods and Kantorovich Spaces
(A. G. Kusraev and S. S. Kutateladze) **1**

§ 1.1. Zermelo–Fraenkel Set Theory **5**

§ 1.2. Boolean Valued Set Theory **7**

§ 1.3. Internal and External Set Theories **12**

§ 1.4. Relative Internal Set Theory **18**

§ 1.5. Kantorovich Spaces **23**

§ 1.6. Reals Inside Boolean Valued Models **26**

§ 1.7. Functional Calculus in Kantorovich Spaces **30**

§ 1.8. Lattice Normed Spaces **34**

§ 1.9. Nonstandard Hulls **40**

§ 1.10. The Loeb Measure **44**

§ 1.11. Boolean Valued Modeling in a Nonstandard Universe **48**

§ 1.12. Infinitesimal Modeling in a Boolean Valued Universe **53**

§ 1.13. Extension and Decomposition of Positive Operators **57**

§ 1.14. Fragments of Positive Operators **61**

§ 1.15. Order Continuous Operators **65**

§ 1.16. Cyclically Compact Operators **69**

References **73**

Chapter 2. Functional Representation of a Boolean Valued Universe (A. E. Gutman and G. A. Losenkov) **81**

§ 2.1. Preliminaries **83**

§ 2.2. The Concept of Continuous Bundle **89**

§ 2.3. A Continuous Polyverse **92**

§ 2.4. Functional Representation **100**

References **104**

Chapter 3. Dual Banach Bundles (A. E. Gutman and A. V. Koptev) **105**

§ 3.1. Auxiliary Results **109**

§ 3.2. Homomorphisms of Banach Bundles **119**

§ 3.3. An Operator Bundle **130**

§ 3.4. The Dual of a Banach Bundle **138**

§ 3.5. Weakly Continuous Sections **150**

References **158**

Chapter 4. Infinitesimals in Vector Lattices (È. Yu. Emel′yanov) **161**

§ 4.0. Preliminaries **164**

§ 4.1. Saturated Sets of Indivisibles **172**

§ 4.2. Representation of Archimedean Vector Lattices **178**

§ 4.3. Order, Relative Uniform Convergence, and the Archimedes Principle **184**

§ 4.4. Conditional Completion and Atomicity **189**

§ 4.5. Normed Vector Lattices **194**

§ 4.6. Linear Operators Between Vector Lattices **198**

§ 4.7. $*$-Invariant Homomorphisms **201**

§ 4.8. Order Hulls of Vector Lattices **206**

§ 4.9. Regular Hulls of Vector Lattices **211**

§ 4.10. Order and Regular Hulls of Lattice Normed Spaces **214**

§ 4.11. Associated Banach–Kantorovich Spaces **220**

References **228**

Chapter 5. Vector Measures and Dominated Mappings
(A. G. Kusraev and S. A. Malyugin) **231**

§ 5.1. Vector Measures **234**

§ 5.2. Quasi-Radon and Quasiregular Measures **236**

§ 5.3. Integral Representations and Extension of Measures **241**

§ 5.4. The Fubini Theorem **247**

§ 5.5. The Hausdorff Moment Problem **256**

§ 5.6. The Hamburger Moment Problem **260**

§ 5.7. The Hamburger Moment Problem for Dominant Moment Sequences **272**

§ 5.8. Dominated Mappings **278**

§ 5.9. The Bochner Theorem for Dominated Mappings **286**

§ 5.10. Convolution **290**

§ 5.11. Boolean Valued Interpretation of the Wiener Lemma **294**

References **296**

Notation Index **300**

Subject Index **303**

Foreword

Nonstandard methods of analysis consist generally in comparative study of two interpretations of a mathematical claim or construction given as a formal symbolic expression by means of two different set-theoretic models: one, a "standard" model and the other, a "nonstandard" model. The second half of the twentieth century is a period of significant progress in these methods and their rapid development in a few directions.

The first of the latter appears often under the name coined by its inventor, A. Robinson. This memorable but slightly presumptuous and defiant term, *nonstandard analysis*, often swaps places with the term *Robinsonian* or *classical nonstandard analysis*. The characteristic feature of Robinsonian analysis is a frequent usage of many controversial concepts appealing to the actual infinitely small and infinitely large quantities that have resided happily in natural sciences from ancient times but were strictly forbidden in modern mathematics for many decades. The present-day achievements revive the forgotten term *infinitesimal analysis* which reminds us expressively of the heroic bygones of Calculus.

Infinitesimal analysis expands rapidly, bringing about radical reconsideration of the general conceptual system of mathematics. The principal reasons for this progress are twofold. Firstly, infinitesimal analysis provides us with a novel understanding for the method of indivisibles rooted deeply in the mathematical classics. Secondly, it synthesizes both classical approaches to differential and integral calculuses which belong to the noble inventors of the latter. Infinitesimal analysis finds newer and newest applications and merges into every section of contemporary mathematics. Sweeping changes are on the march in nonsmooth analysis, measure theory, probability, the qualitative theory of differential equations, and mathematical economics.

The second direction, *Boolean valued analysis*, distinguishes itself by ample usage of such terms as the technique of ascending and descending, cyclic envelopes and mixings, B-sets and representation of objects in $\mathbf{V}^{(B)}$. Boolean valued analysis originated with the famous works by P. J. Cohen on the continuum hypothesis.

Progress in this direction has evoked radically new ideas and results in many sections of functional analysis. Among them we list Kantorovich space theory, the theory of von Neumann algebras, convex analysis, and the theory of vector measures.

The book [3], printed by the Siberian Division of the Nauka Publishers in 1990 and translated into English by Kluwer Academic Publishers in 1994 (see [4]), gave a first unified treatment of the two disciplines forming the core of the present-day nonstandard methods of analysis.

The reader's interest as well as successful research into the field assigns a task of updating the book and surveying the state of the art. Implementation of the task has shown soon that it is impossible to compile new topics and results in a single book. Therefore, the Sobolev Institute Press decided to launch the series "Nonstandard Methods of Analysis" which will consist of monographs on various aspects of this direction in mathematical research.

The series started with the book [5] whose English edition [6] appeared quite simultaneously.

The present book continues the series and addresses applications to vector lattice theory. The latter stems from the early thirties and its rise is attributed primarily to the effort and contribution of H. Freudenthal, L. V. Kantorovich, and F. Riesz. Drifting in the general wake of functional analysis, the theory of vector lattices has studied those features of classical Banach spaces and operators between them which rest on the innate order relations.

The mid-seventies landmark a new stage of rapid progress in vector lattice theory. The reason behind this is an extraordinarily fruitful impact of the principal ideas of the theory on the mathematical research inspired by social sciences and, first of all, economics. The creative contribution of L. V Kantorovich has played a leading role in a merger between ordered vector spaces, optimization, and mathematical economics.

The next most important circumstance in the modern development of vector lattice theory is the discovery of a prominent place of Kantorovich spaces in Boolean valued models of set theory. Constructed by D. Scott, R. Solovay, and P. Vopěnka while interpreting the topical work of P. J. Cohen, these models turn out inseparable from vector lattices. The fundamental theorem by E. I. Gordon demonstrates that the members of each Dedekind complete vector lattice depict reals in an appropriate nonstandard model of set theory. This rigorously corroborates the *heuristic Kantorovich principle* which declares that the elements of every vector lattice are generalized numbers.

Some results of the development of vector lattice theory in the eighties were summarized in the book [1] published by the Siberian Division of the Nauka Publishers in 1992. It was in 1996 that Kluwer Academic Publishers printed a revised and enlarged English version of this book [2]. These articles, in particular, drafted

some new synthetic approaches to vector lattice theory that use the modern nonstandard methods of analysis. The aim of the present monograph is to reveal the most recent results that were obtained along these lines in the last decade.

This book consists of five chapters which are closely tied by the scope of the problems addressed and the common methods involved. For the reader's convenience, exposition proceeds so that the chapters can be studied independently of one another. To this end, each chapter contains its own introduction and list of references, whereas the subject and notation indexes are common for the entire book.

Chapter 1 is a general introduction to the nonstandard methods of analysis applicable to vector lattice theory. That is why to study its first sections will do no harm to the reader, his or her further intentions notwithstanding. This chapter gives quite a few diverse applications among which we mention the technique for combining nonstandard models and the theory of cyclically compact operators. Chapter 1 is written by A. G. Kusraev and S. S. Kutateladze.

Chapters 2 and 3 belong to Boolean valued analysis. The former studies a new concept of continuous polyuniverse which is a continuous bundle of set-theoretic models. The class of continuous sections of such a polyuniverse maintains all principles of Boolean valued analysis. Furthermore, each of the similar algebraic systems is realizable as the class of sections of a suitable continuous polyuniverse. Chapter 2 was prepared by A. E. Gutman jointly with G. A. Losenkov.

Chapter 3 suggests a new approach to the definition of dual bundle which is motivated by studying the realization of dual Banach spaces in Boolean valued models. Chapter 3 is written by A. E. Gutman jointly with A. V. Koptev.

Chapter 4 by È. Yu. Emel′yanov deals mainly with adapting the methods of infinitesimal analysis for study of intrinsic problems of vector lattice theory. Incidentally, the author explicates some properties of the infinite dimensional analogs of the standard part operation over the hyperreals that are unpredictable in advance.

Chapter 5 is written by A. G. Kusraev and S. A. Malyugin and belongs to vector measure theory. Study of Banach-space-valued measures is well known to involve other tools than those of Boolean valued measures. The bulk of the chapter sets forth a principally new unified approach to both directions of research in measure theory which rests upon the concept of lattice normed space. It is worth observing that locally convex spaces and vector lattices are just some particular instances of lattice normed space. Also important is the fact that these spaces depict Banach spaces inside Boolean valued models.

Among particular topics of this chapter, we mention a criterion for a dominated operator to admit integral representation with respect to some quasi-Radon measure, a new version of the celebrated Fubini Theorem, and analysis of various statements of the Hausdorff and Hamburger moment problems.

The authors of the chapters and the editor tried to ensure the unity of style and level exposition, striving to avoid nauseating repetitions and verbosity. As it usually happens, the ideal remains intact and unreachable. The editor is the sole person to be blamed for this and other shortcomings of the book.

S. Kutateladze

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