Mathematics and Its Applications

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Nonstandard Methods of Analysis

Kluwer Academic Publishers

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Volume 291

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SPRINGER SCIENCE+BUSINESS MEDIA, B.V.

A C.I.P. Catalogue record for this book is available from the Library of Congress

ISBN 978-94-010-4497-4 ISBN 978-94-011-1136-2 (eBook)

DOI 10.1007/978-94-011-1136-2

This is the translation of the original work
Nonstandard Methods of Analysis, Novosibirsk, Nauka © 1990
Translated from the Russian and typeset by L. Iouzina-Tarvainen.

Printed on acid-free paper

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INTRODUCTION

Nonstandard methods of analysis in the modern sense of this term consist in using two different models, i.e., 'standard' and 'nonstandard', of the theory of sets for studying concrete mathematical objects and problems. Such methods have been substantially developed in the last thirty years and have resulted in several trends, the principal ones of which, named *infinitesimal* analysis and *Boolean-valued* analysis, are the subject of the present monograph.

The former of these trends is often referred to, following its founder A.Robinson, by an expressive though slightly awkward term, *nonstandard* analysis (nowadays, however, they often say *classical* or *Robinson nonstandard* analysis). Infinitesimal analysis is characterized by a wide use of long-known in natural sciences but long-prohibited in the mathematics of the 20th century concepts pertaining to the notions of actually infinitely large and actually infinitely small magnitudes. The Robinson nonstandard analysis has been rapidly developing and has already introduced dramatic changes in the system of general mathematical notions. This is first of all associated with the fact that this method has both offered a new understanding of infinitesimal methods of indivisibles which stem from the ancient time and implemented a synthetic approach to the differential and integral calculuses originating from its founders. Nowadays infinitesimal analysis is being widely used and has penetrated all branches of modern mathematics, especially such as nonsmooth analysis, probability theory, the qualitative theory of differential equations and mathematical economics.

Boolean-valued analysis is characterized by extensive use of such terms as descents and ascents, cyclic envelopes and mixings, and B-sets and presentation of objects as models. The development of this trend which originated from the famous P.J.Cohen elaborations of the continuum hypothesis, has resulted in principally new ideas and data in a number of branches of functional analysis, and primarily in the theory of Kantorovich spaces, in the theory of von Neumann algebras, in convex analysis and the theory of vector measures.

When writing this monograph, we basically oriented ourselves to the reader who is striving, like ourselves, to acquire new methods for solving analytical problems, which has resulted in some peculiarities both in the structure and way of presenting the material. Taking into consideration the difference in the level of formal requirements for acquiring infinitesimal and Boolean-valued ideas, we found it necessary to divide the monograph into two parts, having ensured their relative independence, which fact is stressed by the independent enumeration of both parts. Our primary intention was to embrace synthetic nonstandard methods and to essentially increase the number of concrete analytical applications. The rapidly increasing volume of the monograph made us, however, shelve these ideas for the time being. In the reference section, which in no way claims completeness, we, however, preserved the references that pertain to the previous intentions. We also did our best to include all the references known to us which contain both historical and reference data of importance.

Studying nonstandard methods of analysis has given us plenty of positive emotions, which accounts for the fact that when presenting the material we sometimes try to avoid traditional dry formalized way of doing it. We bring our apologies beforehand to those who might be irritated by such passages. As an excuse, however, we would like to point out that emotional presentation not only leaves the mathematical facet of the subject intact, but sometimes, in our opinion, even facilitates its better understanding.

We wish to express gratitude to all our colleagues and friends whose interest and valuable advice helped us in our work on this monograph.

A.G.Kusraev
S.S.Kutateladze

PART I

INFINITESIMAL ANALYSIS

The idea of an infinitesimal, i.e., of an infinitely small magnitude, can be traced as far back as the ancient time. Nowadays, after approximately half a century of neglect, modern mathematics has started paying rapt attention to infinitesimal notions. Infinitely big or infinitely small numbers, mathematical atoms, i.e., 'indivisible' monads, are oftener encountered in various publications, becoming part and parcel of mathematical practice. A turning point in the evolution of infinitesimal concepts is associated with an outstanding achievement of A.Robinson who created nonstandard analysis.

For a prolonged period of time nonstandard analysis has been viewed as a quite subtle and even exotic logical technique designed for substantiating the method of actually infinitely large and infinitely small numbers. This technique has also been considered to be of restricted applicability and, in any case, to be principally unable to result in any serious reconsiderations of general mathematical notions. By the end of the 1970s, after the E.Nelson internal set theory (and, some time later, the external set theories propounded by K.Hrbaček and T.Kawai) was made public, the views on the place and role of nonstandard analysis had been dramatically enriched and changed. In the light of new discoveries it became possible to view nonstandard elements not as 'imaginary, ideal or surd entities' appended to common sets by considerations of formal convenience, but as indispensable parts of any routine mathematical objects. There arose a doctrine the essence of which was that every set is composed of both standard and nonstandard elements. Standard sets, in turn, form a kind of a frame of reference net densely located in the totality of all objects investigated by mathematics. It was discovered that in this case such objects of nonstandard mathematical analysis as monads of filters, standard parts of numbers and vectors, operator shadows, etc., form 'Cantorian' sets, which are not described by any of the canonized pictures drawn by known formal set theories. *The von Neumann universe does not exhaust the world of classical mathematics*, that was one of the obvious conclusions of the new approaches. Therefore, traditional views on nonstandard analysis started requiring at least a revision, a reconsideration of the infinitesimal concepts.

An important advantage of the new approaches originated was the axiomatic approach which made it possible to master the apparatus of nonstandard mathematical analysis without preliminary studying either the techniques of ultraproducts or Boolean-valued models, or their analogues. The axioms put forward are simple and precisely motivated at the semantic level within the framework of the 'naive' set-theoretical stances analysis commonly used. At the same time, they essentially broaden the number of mathematical objects, open up possibilities for developing a new formal apparatus, and enable one to significantly decrease dangerous gaps between the presentations, methodical doctrines and levels of rigour adopted in mathematics and its applications to natural and social sciences. In other words, the axiomatic set-theoretic foundation is of a general scientific significance.

In 1947 K.Gödel wrote: "There might exist axioms so abundant in their verifiable consequences, shedding so much light upon a whole discipline, and furnishing such powerful methods for solving given problems (and even solving them, as far as that is possible, in a constructive way) that quite irrespective of their intrinsic necessity they would have to be assumed at least in the same sense as any well established physical theory." [61, p.521]. This prediction by K.Gödel is becoming true.

The purpose of the present work is to make new roads to nonstandard analysis more accessible. To this end, we start with presenting the semantic qualitative views on standard and nonstandard objects, on the apparatus of nonstandard analysis at the 'naive' level of rigour, which is absolutely sufficient for effective applications without appealing to a logical formalism. Then we give a brief but sufficiently complete reference material pertaining to modern axiomatic constructions of nonstandard analysis within the Cantor doctrine. In this case we found it appropriate to pay special attention to the ideological and historic facets of the matter, which determined the peculiarities of our working plan. The historic data, the qualitative motivations of the principles of nonstandard analysis and the discussion of their simplest corollaries for differential and integral calculuses presented in the first and second chapters form the 'naive' foundation of infinitesimal analysis. Formal details of the corresponding apparatus of nonstandard set theory are given in the third chapter. A weighty argument in favour of a certain concentricity in our presenting the material are the remarkable words by N.N.Luzin: "Mathematical analysis is far from being an absolutely completed science, as it is sometimes inclined to be viewed, with once and forever found principles which are good only for further corollaries to be drawn from... Mathematical analysis is not different from any other science, and it has its own motion of ideas which is not only translational but also rotational, constantly returning to a group of former ideas which are always, however, newly lit" [160, p.389].

In the fourth and fifth chapters concluding the first part of the monograph we present infinitesimal methods under general topology and differential calculus. The choice of these topics from the variety of modern applications of nonstandard analysis has been basically determined by the personal preferences of the authors.

CHAPTER 1

EXCURSUS INTO THE HISTORY OF MATHEMATICAL ANALYSIS

The ideas of differential and integral calculuses can be traced as far back as the ancient time and are associated with the most fundamental mathematical concepts. Any detailed presentation of the history of the evolution of interpreting mathematical objects, of calculation and measurement processes determining modern views on infinitesimals, would require special efforts which are beyond our possibilities and intentions. The situation is significantly complicated by the fact that the history of mathematics has been subjected to well-known negative processes which arise under constant attempts to apologize some up-to-date views. In particular, the evolution of the apparatus of a mathematical analysis is far from always being presented sufficiently completely and objectively. One-sided views on the essence of a differential and an integral, hypertrophying the role played by the notion of limit, and neglect of actually infinitely large and infinitely small numbers have been so widely spread during the last fifty years that their existence cannot be shammed.

It has become a truism to say that "the very foundations of analysis have been for a long time surrounded with mystery as a result of unwillingness to admit that the notion of a limit enjoys an exclusive right to be a source of new methods." [30, p.562]. However, as has been justly noted by L.S.Pontryagin: "In a historical sense, integral and differential calculuses had already been well-developed fields of mathematics before the theory of limits appeared. The latter originated as a superstructure over the existing theory. Many physicists opine that the so-called rigorous definition of derivatives and integrals is not at all necessary for satisfactory comprehension of differential and integral calculuses. I share their viewpoint." [213, pp. 64-65].

Taking into account all these facts, we found it necessary to brief the reader about some turning points of the history of analysis, as well as about the ideas expressed by classics in the evolution of modern views. The choice of the corresponding fragments is certain to be subjective. We nevertheless hope that it will be sufficient to develop a critical attitude to one-sided and distorted delineations of the evolution of infinitesimal methods.

1.1 G.W.Leibniz and I.Newton

The ancient name for differential and integral calculuses is 'infinitesimal analysis'. It is this title that was used for the first textbook on mathematical analysis published as far back as 1696. The textbook was written by G. de l'Hôpital as a result of his contacts with I.Bernoulli (senior), one of the most famous disciples of G.W.Leibniz.

"Of all theoretical achievements of cognition hardly any other but the invention of calculus of the infinitely small in the second half of the XVII century can be considered the highest triumph of human spirit. If we are to have any pure and exclusive creation of human spirit somewhere, it is in this sphere", that was a high estimate given by F.Engels [44, p.582] to this new branch of mathematics.

The scientific legacy, the creative work and personal relations between the founders of mathematical analysis, G.W.Leibniz and I.Newton, have been subjected to a detailed, one may say, thorough examination. An attempt to restore the train of thought of the men of genius, to elucidate the ways resulted in the discovery of new truths, is both justified and understandable. One, however, should bear in mind that there is a principal difference between draft papers and notes, personal letters to colleagues and the works especially designed for publications. It is, therefore, first of all necessary to consider the 'official' presentations of the G.W.Leibniz's and I.Newton's views on infinitely small we are interested in.

The first work on differential calculus ever published was the paper by G.W.Leibniz "A new method of maxima and minima, as well as of tangents which is hampered by neither fractional or irrational quantities, and a special type of calculus for it" (see [152]). In this paper published in the Leipzig magazine 'Acta Eruditorum' in 1684, more than three centuries ago, Leibniz gives the following definition of differential. Considering a curve YY and an interval of the tangent passing through a fixed point of the curve Y, which corresponds to a chosen coordinate X on the axis AX, and denoting the point of intersection of the tangent with the axis considered by D, he writes: "Now some straight line selected arbitrarily is called dx, while another one whose ratio to dx is the same as that of ...y... to XD, will be called...dy, or difference (differentia) of ...y... ." This text is accompanied by a drawing whose essential features (with Leibniz's written explanations) are reproduced here (Fig.1).

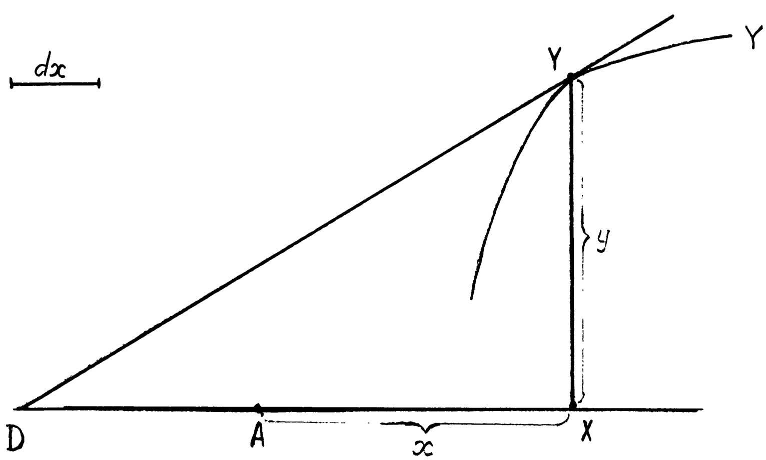


Fig.1

Therefore, according to Leibniz, at a point x, given an arbitrary dx, for the function $x \to y(x)$, we obtain

$$dy := \frac{YX}{XD} dx$$

In other words, the differential is determined as a corresponding linear mapping, i.e., in the manner which is readily recognized by the majority of modern specialists.

G.W.Leibniz is a thorough thinker who considered the "invention of the syllogistic form to be one the most beautiful and even important discoveries of the human spirit. This is a sort of *universal mathematics* the significance of which has not yet been completely realized. It can be said to contain the art of being faultless..." [151, pp. 492-493]. Realizing undoubtedly that the description and substantiation of the algorithm of differential calculus (in that way G.W.Leibniz referred to the rules of differentiation) suggested by him required the notion of a tangential to be refined, he further explains: "...to find a tangent means to draw a line that connects two points of the curve at the infinitely small distance, or the continual side of a polygon with an infinite number of angles which for us takes the place of the curve." Hence, Leibniz rests his calculus on the appeal to the structure of curves 'in the small'.

At that time there were practically two points of view as regards the status of infinitesimals. According to one of them, which seemed to be shared by G.W.Leibniz, an infinitely small number was thought to be less that any other 'arbitrarily given quantity'. Actually existing 'indivisible' elements which form quantities and figures are the images corresponding to the concept of the infinitely small in question. For G.W.Leibniz the existence of "simple substances incorporated into the structure of complex ones", i.e., monads, is doubtless. "It is these monads that are true atoms of nature, or, to put it short, elements of things" [150, p.413].

For another founder of the analysis, I.Newton, infinitesimals are related to the idea of vanishing quantities [194, 277]. He viewed indeterminate quantities "not as made up of indivisible particles but as described by a continuous motion", "... as increasing or decreasing by a perpetual motion, in their nascent or evanescent state". The famous 'method of prime and ultimate ratios' in his classical treatise "Mathematical Principles of Natural Philosophy" (1687) is formulated as follows.

"Quantities, and the ratios of quantities, which in any finite time converge continuously to equality, and before the end of that time approach nearer to each other than by any given difference, become ultimately equal" [277, p.101].

Pursuing the ideas which are nowadays closely associated with the theory of limits, I.Newton exhibited insight, prudence and wisdom in estimating the concurrent views inherent in true scientific workers. He wrote: "... to institute an analysis after this manner in finite quantities and investigate the prime or ultimate ratios of these finite quantities when in their nascent or evanascent state is consonant to the geometry of the ancients, and I was willing to show that in the method of fluxions there is no necessity of introducing figures infinitely small into geometry. Yet the analysis may be performed in any kind of figure, whether finite or infinitly small, which are imagined similar to the evanescent figures, as likewise in the figures, which, by the method of indivisibles, use to be reckoned as infinitely small provided you proceed with due caution" [194, p.169].

G.W.Leibniz's views were as much pliable and in-depth dialectic. In his well-known letter to P.Varignion of February 2, 1702 [277], stressing the idea that "... it is unnecessary to make mathematical analysis depend on metaphysical controversies", he points out to the entity of concurrent views on the objects of the new apparatus: "... if any opponent tries to contradict this proposition, it follows from our calculus that the error will be less than any possible assignable error, since it is in our power to make this incomparably small magnitude small enough for this purpose, inasmuch as we can always take a magnitude as small as we wish. Perhaps this is what you mean, Sir, when you speak on the inexhaustible, and the rigorous demonstration of the infinitesimal calculus which we use undoubtedly is to be found here. ... So it can also be said that infinites and infinitesimals are grounded in such a way that everything in geometry, and even in nature, takes place as if they were perfect realities. Witness not only our geometrical analysis of transcendental curves but also my law of continuity, by virtue of which we may consider rest as infinitely small motion (that is, as equivalent to a particular instance of its own contradictory), coincidence as infinitely small distance, equality as the limit of inequalities, etc."

Similar views were expressed by G.W.Leibniz in the quotation to follow, the italicized end of which is, according to A.Robinson [222, pp. 260-261], often cited in works on nonstandard analysis: "... it is not necessary here to view the infinite in the strict sense of the word, but only in the sense in which in optics they say that the rays of the sun are emitted from an infinitely distant point and can therefore be considered parallel. When there are different orders of the infinite and infinitely small, they are understood in the same sense in which Earth is considered to be a dot as compared with the distance to the fixed stars, while a ball in our hands is viewed as a dot as compared with the semi-diameter of our planet, so that the distance from the fixed stars is infinitely infinite or the infinity of the infinity as regards the ball diameter. The point is that, instead of the infinite and infinitely small, the values are chosen to be as big or as small as it is necessary for the mistake to be less than the given one and, therefore, *the difference from Archimedes style is only in expressions which in our method are more straightforward and more applicable to the art of invention*" [152, p.190].

1.2 Karl Marx on Mysterious Differential Calculus

The high level of the requirements on the soundness and rigour of their methods characteristic of the works by G.Leibniz and I.Newton was not, unfortunately, taken up by their followers who contributed a lot to make the mysterious mist surrounding the nontrivial as they were abstract ideas even thicker. It suffices to say that in the textbook by de l'Hôpital mentioned above the definition II says: "... an infinitely small part, which a variable value increases or decreases by is called its differential."

One can see that a colossal step back from the initial definition by G.Leibniz was made here. It is not incidentally that K.Marx called the mathematical analysis of the XVII century, while making first acquaintance with it, the 'mysterious differential calculus'.

Of interest is the fact that when interpreting the draft papers by K.Marx [176], a somewhat dramatized criticism of the actually infinitely small is sometimes derived from them. For instance, in one of the textbooks on the fundamentals of nonstandard analysis a known statement is quoted: "The consolation some rationalizing mathematicians are tightly holding at, that in the qualitative sense dy and dx are in fact only infinitely small, that their ratio] is only approaching $\frac{0}{0}$, is a chimera..." [176, p.33]. In this case omitted is the following principal observation: "...in fact $\frac{dy}{dx}$ does not denote the extravagant $\frac{0}{0}$, but, on the contrary, it is a festive uniform for $\frac{dy}{dx}$ when the latter is functioning as a ratio of infinitely small differences ..." [176, p. 71].

Therefore, in order to elucidate the details, let us give a complete quotation of one of the abstracts of K.Marx's manuscripts often cited but with blanks distorting the sense.

"We therefore have nothing else to do but to view increments of the variable h as infinitely small and to ascribe to them, as such, *independent existence*, in, for instance, symbols $\dot{x}$, $\dot{y}$, etc., or dx, dy [etc.]. Infinitely small magnitudes, however, are as much magnitudes as infinitely large ones are (the word infinitely [small] denotes in fact only indefinitely small); these dx, dy, etc., or $\dot{x}$, $\dot{y}$, [etc.] are also functioning in calculus as common algebraic magnitudes, and in the equation given above,

$$(y+k)-y \text{ or } k = 2xdx + dxdx,$$

the term $dxdx$ has the same right to exist as $2xdx$ does. But the most astonishing statement is the one by virtue of which this term is forced to be omitted due to relativity of the notion of the infinitely small; $dxdx$ is omitted since it is infinitely small as compared with dx and, hence, with $2xdx$ or $2x\dot{x}$. Or, if in

$$\dot{y} = \dot{u}z + \dot{z}u + \dot{u}\dot{z}$$

[the term] $\dot{u}\dot{z}$ is omitted since it is infinitely small as compared with $\dot{u}z$ or $\dot{z}u$, then mathematically this can be justified only by referring to the fact that $\dot{u}z + \dot{z}u$ has for us an approximate value which can be thought to be any close to the exact one. A maneuver of the kind can be encountered in common algebra as well. In this case, however, we face a greater wonder: due to this method for an arbitrary function [in] x we obtain not approximate but absolutely exact values (though, as above, only symbolically correct), as in the example $\dot{y} = 2x\dot{x} + \dot{x}\dot{x}$. Neglecting here $\dot{x}\dot{x}$, we obtain

$$\dot{y} = 2x\dot{x},$$
$$\frac{\dot{y}}{\dot{x}} = 2x,$$

which is the correct first derivative function of x^2 which has already been proved by the binom[ial theorem].

But this miracle is far from being a miracle. On the contrary, it would be a miracle if the forced omission of $\dot{x}\dot{x}$ *did not give an exact result. It is a certain mistake in calculations that is omitted*, which is an inevitable consequence of the method making it possible to introduce an indefinite increment, for instance, h, of the variable immediately as the differential dx or x, as a ready-made operation symbol and, therefore, differential calculus also immediately appears to be an independent way of calculus, other than conventional algebra" [176, pp.151-153].

1.3 L.Euler

In the history of mathematical analysis the eighteenth century is rightfully called L.Euler's century. Everyone who will get acquainted with his textbooks will be staggered by his subtle technique and in-depth penetration into the essence of the matter. One can recall that an outstanding engineer-scientist A.N.Krylov was in rapture seeing in the famous Euler formula $e^{i\pi} = -1$ a symbol of the entity of the whole of mathematics. He, in particular, noted that "... here -1 presents arithmetics, i algebra, π geometry, and e analysis".

L.Euler is characterized by a many-sided approach, a 'systemic' one, as might be put today, to studying mathematical problems, he widely used the whole of the apparatus developed by that time. It should be stressed that he was constantly making effective and productive use of the infinitesimal concepts, and, first of all, infinitely large and infinitely small numbers. L.Euler gave a sufficiently detailed explanations of the methodological foundation of his ideas, called 'calculus of zeros'. There is a tendency to look for spots on the sun (other than the existing ones) and, analogously, for weak points in men of genius. For many years L.Euler had been incriminated 'incorrect' treatment of diverging series, until his ideas were accepted. At present you can encounter such a phrase: "As to the problem of diverging series, Euler was sharing quite an up-to-date point of view... ." It would be more correct to topsy-turvy the phrase and say that modern mathematicians have finally caught it up with Euler's ideas. As will be shown in the sections to follow (see **2.2**, **2.3**), the opinion that "... we cannot admire the way Euler corroborates his analysis by introducing zeros of various orders" is as self-assured as the statement that "... the giants of science, mainly, Euler and Lagrange, have laid false foundations of analysis." It must be admitted once and for ever that Euler was in full possession of analysis and completely aware what he had created.

1.4 Statement by G.Berkeley

In their general form the ideas of analysis greatly affected the character of the ideological outlook in the XYIII century. Vivid examples of the depth of penetration of the notions of infinitely large and infinitely small quantities into the cultural media of that time are, in particular, "Gulliver's Travels" by Jonathan Swift published in 1726 (Lilliput and Brobdingnag!) and the famous "Micromegas 1752" written by bright and venomous F.M.Arouer, i.e., Voltaire. Of interest is the fact that as an epigraph for his classical treatise [222], A.Robinson chose the beginning of the following speech of Micromegas:

"Now I see clearer than ever that nothing can be judged by its visible magnitude. Oh, my God, who granted reason to creatures of such tiny sizes! An infinitely small thing is equal to an infinitely large one when facing you; if living beings still smaller than those were possible, they could have reason exceeding the intellect of those magnificent creatures of yours which I can see in the sky, and one foot of which could cover the earth" [266, p.154].

A serious 'reverse' impact on the development of mathematical analysis was made in 1734 by a publication of a great church figure and theologian, bishop G.Berkeley, a pamphlet "The Analyst, or a Discourse Addressed to an Infidel Mathematician, wherein it is examined whether the object, principles and inferences of the modern analysis are more deduced than religious mysteries and points of faith" [11, pp.396-422]. The antihumanistic spirit of the paper by G.Berkeley is combined with aphoristicity, subtle observations and killing precision of expression. "... By inconsistent supposition truth may be obtained, but not science though", this is the basic idea of his criticism of analysis. G.Berkeley's challenge was addressed to the whole of natural science: "I have no controversy about your conclusions, but only about your logic and method. How do you demonstrate? What objects are you conversant with, and whether you conceive them clearly? What principles you proceed upon; how sound they may be; and how you apply them?"

V.I.Lenin, exposing the venomous plot of G.Berkeley, wrote: "Assume the external world, nature to be a 'combination of perceptions' brought to our minds by a god. Admit it, refuse the idea of looking for 'the basis' of these perceptions outside your consciousness, outside the human being, and I will accept all natural sciences, all the significance and validity of its conclusions within the framework of my idealistic cognition theory. It is this framework and only this framework that I need for my deductions in favour of 'peace and religion'." This is Berkeley's idea" [153, p.22].

G.Berkeley's challenge could not but be answered by advanced representatives of the scientific thought of the XVIII century, the encyclopedians.

1.5 J.D'Alambert and L.Carnot

A turning point in the history of formulating the basic notions of analysis is associated with the ideas and activities of J.D'Alambert, one of the initiators and leading authors of the immortal masterpiece of the thought of the Age of Enlightment, “Encyclopedia or Explanatory Dictionary of Sciences, Arts and Crafts”. In the article “Differential” he wrote: “Newton never considered differential calculus to be calculus of the infinitely small, but he viewed it as the method of the first and last relations” [277, p.157]. D'Alambert was the first mathematician who declared that he had found the proof that the infinitely small “... do exist neither in the nature nor in the assumptions of geometricians” (a quotation from his article “Infinitesimal” of 1759). It is the J.D'Alambert's viewpoint presented in “Encyclopedia ...” that contributed much to the formulation by the end of the XVIII century of the understanding of the infinitely small as a magnitude tending to zero. In all probability, one should recall in this respect a paper by L.Carnot “Considerations on metaphysics of the infinitely small” where he remarks “... the notion of infinitesimal is less clear than that of limit since it is nothing else but the difference between such a limit and the quantity the ultimate value of which it is.”

1.6 B.Bolzano, O.Cauchy and K.Weierstrass

The XIX century was the century of the foundation of analysis by means of the theory of limits. An outstanding contribution to this process was made by B.Bolzano, O.Cauchy and K.Weierstrass, their achievements mirrored in any traditional textbook on differential and integral calculus. The new canon of rigour forwarded by B.Bolzano, the definition given by O.Cauchy to an infinitely small quantity as a variable with zero limit and, finally, the $\varepsilon - \delta$-technique suggested by K.Weierstrass are indispensable part of the history of mathematical thought, part and parcel of modern culture. It should be emphasized (see 2 [76]) that, while giving a verbal characteristic of continuity, both O.Cauchy and K.Weierstrass used practically the same expressions:

> “an infinitely small increment of a variable always results in that of the function itself”,
>
> O.Cauchy;
>
> “infinitely small variations in the argument correspond to those of the function”,
>
> K.Weierstrass.

The coincidence under discussion emphasizes the necessity shared by the cited authors and worthy of respect to interrelate the new ideas with the viewpoints of their great predecessors.

Speculating about significance of the reconsiderations of analytical views in the XIX century, one should bear in mind the important observation made by F.Severi [229, p.113] in this respect: "The reconsideration that has been mostly completed nowadays has, however, no definite value most scientists believe in. Rigour itself is, in fact, a function of the total volume of knowledge at every historic period which corresponds to the way of the scientific treatment of the truth."

1.7 N.N.Luzin

The beginning of the XX century in mathematics was marked by a further growth of distrust to the concept of infinitesimal. This tendency grew stronger as mathematics was being reconsidered on the basis of the set theoretical credo which gained the key strongholds in the thirties.

In the first edition of the "Great Soviet Encyclopedia" in 1934, N.N.Luzin wrote: "As to a constant infinitely small quantity distinct from zero, modern mathematical analysis, without discarding the formal possibility of determining the idea of a constant infinitesimal (for instance, as corresponding length in non-Archimedean geometry), views this idea as absolutely fruitless, since it has proved to be impossible to introduce such an infinitesimal in calculus" [160, pp. 293-294]. At that time the publication in the USSR of a textbook by M.Ya.Vygodskii "Fundamentals of Infinitesimal Calculus" became a noticeable event and gave rise to a serious and sharp criticism. M.Ya.Vygodskii tried to preserve the concept of infinitesimals by appealing to the historical practice. He, in particular, remarked: "If it were only the problem of creating a logical apparatus which could work by itself then, having eliminated infinitesimals from considerations and having driven differentials out of mathematics, one could celebrate a victory over the difficulties that have been hampering the way of mathematicians and philosophers during the last two centuries. Infinitesimal analysis, however, originated from practical needs, its relations with natural sciences and technology (and, later, with social sciences) becoming increasingly strong and fruitful in the course of time. Complete neglect of infinitesimals would incidentally make these relations extremely difficult, not to say impossible" [271, p. 160].

Giving his estimate to the textbook by M.Ya.Vygodskii, N.N.Luzin wrote in the forties: "This course, being internally both integrated and lit by the great idea the author remains faithful to, goes beyond the framework of the style in which modern mathematical analysis has been developed for 150 years and which is now nearing its completion" [160, p.398].

The attitude of N.N.Luzin to infinitesimals deserves a special attention as an important manifestation of a hidden drama the history of any profound idea inspiring people is usually filled with. N.N.Luzin had a unique gift of penetration into the essence of the most intricate mathematical problems, and he might be said to possess a remarkable gift of foresight [147, 148, 259]. The idea of actual infinitesimals seemed to be extremely appealing to him psychologically, as he emphasized: "... the idea about them has never been successfully driven out of mind. There are, obviously, some deeply hidden reasons still not completely revealed that make our minds to be predisposed to treat infinitesimals seriously" [160, p.396]. In another of his publications N.N.Luzin remarks sorrowfully: "When the mind starts getting acquainted with analysis, i.e., during its spring season, it is the actually small, which can be termed the 'elements' of quantity, that it starts with. Gradually, however, step by step, as it is accumulating knowledge, theories, surfeit of abstraction and tiredness, the mind starts forgetting its primary intentions, it starts smiling at their 'childishness'. To put it short, when the mind is in its autumn season, it allows one to be convinced in the uniqueness of the correct foundation by means of limits" [264]. The latter point of view was energetically developed by N.N.Luzin in his textbook "Differential calculus", where he in particular emphasized the idea that "to understand correctly the very *essence of the matter*, the student should first of all understand that an infinitesimal is always, *by its very definition*, a variable quantity and, therefore, no constant number, however small it be, is *ever* infinitely small. The student should beware of using comparisons or resemblances of such, for instance, kind as 'One centimeter is a magnitude infinitely small as compared with the diameter of the sun'. This phrase is absolutely incorrect. Both magnitudes, i.e., a centimeter and the diameter of the sun, are constant values and, hence, they are *finite*, one being much smaller than the other, though. Incidentally, a centimeter is not a small value at all when, for instance, compared with the 'thickness of a hair', while for a moving microbe a centimeter is a vast space. In order to get rid of any risky comparisons and random subjective resemblances, the student *must remember that no constant value is infinitesimal, as well as no number however small it might be*. Therefore, it would be more correct to use not the term "*an infinitesimal*", but the term '*an infinitely decreasing variable*', as it expresses the *idea of variability* most vividly" [161, p. 61].

1.8 A.Robinson

The seventh (posthumous) edition of the textbook by N.N.Luzin under discussion was published in 1961 simultaneously with the A.Robinson's "Nonstandard Analysis" which laid an up-to-date foundation for the method of the actually infinitely small. A.Robinson based his work on the local theorem by A.I.Mal'tsev, stressing its significance as that of the 'fundamental importance for our theory' [222, p. 13] and giving direct references to an A.I.Mal'tsev's work dated as far back as 1936. The Robinson's discovery both elucidates the ideas of the founders of differential and integral calculus and gives a further confirmation of the dialectical character of the evolution of mathematics.

CHAPTER 2

NAIVE FUNDAMENTALS OF INFINITESIMAL METHODS

One of the most widely spread prejudices as regards the infinitely small and the constructions related to them through is the opinion that the apparatus of nonstandard analysis is extremely difficult to master. Moreover, it is usually emphasized that nonstandard analysis is based on the advanced sections of the modern formalized set theory and mathematical logic. Actually, the existence of this relation, although irrefutable, in no way hampers either understanding or handling infinitesimals. The purpose of the present chapter is to corroborate the above statement by way of presenting the methodology of nonstandard analysis at the level of rigour accepted in the modern system of mathematical education which is based on the ideas of the naive set-theoretic stance proposed by G.Cantor. Alongside with elucidating the essence of the concepts of nonstandard set theory and the principles of transfer, idealization and standardization adopted in it, certain attention will be also paid to paralleling quite recent ideas on the objects of elementary mathematical analysis and the approaches of classics. In doing so, we would like to confirm the continuity in the evolution of the ideas of differential and integral calculuses which nonstandard analysis sheds new light upon.

2.1. The Notion of Set in Nonstandard Analysis

In this section we will expose a fragment of the foundations of nonstandard methods at the level of rigour close to that adopted in introducing the elements of mathematical analysis.

2.1.1. Modern courses of mathematical analysis are often based on the notion of set.

2.1.2. Examples

(1) *L.Schwartz* "*Analysis*": "A set is the collection of certain objects.

Examples:

the set of graduates in one class,

the set of points on a plane,

the set of nondegenerate second-order surfaces in a three-dimensional space,
the set **N** of nonnegative integers,
the set **Z** of arbitrary integers,
the set **Q** of rational numbers,
the set **R** of real numbers,
the set **C** of complex numbers" [228, p. 9].

(2) *V.A.Il'yin, V.A.Sadovnitchii, Bl.Kh.Sendov "Mathematical Analysis"*:

"...for studying real numbers the notion of set is of importance. It should be emphasized that we view a set a an initial notion not determined from other ones.

In this section we shall study sets of an arbitrary nature. They are often called abstract sets, which implies that the objects comprising the set in question or, as they say, elements of the given set, do not necessarily have to be real numbers. Elements of an abstract set can, for instance, be functions, letters of the alphabet, figures on a plane, etc." [89, p.69].

(3) *Yu.G.Reshetnyak "Introduction to Mathematical Analysis"*:

"The notion of a set belongs to primary mathematical notions and cannot be determined by reduction to any other simpler notions.

A set is any collection of subjects considered as a whole entity. The word 'set' is a mathematical term used to denote some objects denoted in everyday speech by such words as a totality, a pack, a system, a cortege, an assembly, a family, etc.. For instance, we can speak about the set of solutions to an equation, about the set of pictures in a museum, the set of points of an interval, etc.

The objects comprising one set or another are called its elements. A set is considered to be given if for any object it is possible to determine whether it is an element of the set or not" [217, p.3].

(4) *V.A.Zorich "Mathematical Analysis"*:

"The basic suppositions of the Cantor (or, conditionally speaking, 'naive') theory of sets are as follows:

1^* a set can be composed of any different objects;

2^* a set is uniquely determined by the collection of objects comprising it;

3^* any property defines the set of objects that have this property.

If x is an object, P is a property, $P(x)$ is the notation that x has the property P, by $\{x \mid P(x)\}$ we shall denote the whole class of objects with the property P.

The objects comprising a class or set are termed *elements* of the class or set.

In the framework of naive set theory the terms 'class', 'family', 'union', 'collection' are treated as synonyms of the word 'set'.

The following examples demonstrate the application of this terminology:

the set of letters 'a' in the word 'I';
the set of Adam's wives;
the set of ten digits;
the legume set;
the set of grains of sand on Earth;
the totality of the points on a plane equidistant from its two given points;
a family of sets;
the set of all sets.

The difference in the possible degree of definiteness in constituting sets leads one to the conclusion that the notion of a set is not as simple and harmless as it might seem.

Indeed, the notion of the set of all sets is, for instance, just contradictory" [284, pp.17-18].

2.1.3. Nonstandard analysis or, in more detail, nonstandard mathematical analysis is a branch of mathematical analysis, and, therefore, it obviously adopts the accepted views on sets. In other words, *nonstandard analysis views as sets only those collections that the classical 'standard' theory operates with.* It should be emphasized that this statement is also valid when reformulated in the following way: nonstandard analysis does not view as sets only those unions that conventional mathematics does not accept as such. At the same time, *nonstandard analysis is related to refined views on sets*, i.e., it is, as they often say, constructed within the framework of nonstandard set theory.

2.1.4. Naive set theory is based on the classical G.Cantor's formulation: "A set is any many which can be thought of as one, that is every totality of definite elements which can be united to a whole through a law", and a set is "every collection to a whole of definite, well-differentiated objects of our intuition or our thought" [102, p.173]. Such concepts are well-known to be far too broad, this drawback by-passed by a certain detalization of the difference between sets and nonsets. For instance, in order to define inappropriate, 'too big' collections of sets the term 'class' is being used, just to denote the fact that the class does not have to be a set. In other words, when formalizing the notions of naive set theory, the procedures allowing one to introduce any 'Cantorian' set into mathematical operations are more fully and thoroughly regulated. All the sets allowed into mathematics enjoy the same rights. This in no case implies that they all are equal or have no distinctions. This implies that the sets are of one type, that they share a common status since they are elements of the 'class of all sets'.

2.1.5. The cornerstone of nonstandard set theory, its underlying principle is extremely simple: *sets can be different: standard and nonstandard.* It would therefore be more correct to speak not of the nonstandard theory of sets, but of the theory of sets, standard and nonstandard. The intuitive concept revealed by the phrase 'a set A is standard' implies that A has been described in plain and definite terms, has become an 'artefact' of the cognition activity of human beings. The term 'standard' draws a borderline between the objects determined from explicit mathematical constructions, using, for instance, theorems of existence and uniqueness, and called standard sets, and the objects originating in the course of investigation in an implicit, indirect way, those called nonstandard sets.

Such objects as the numbers π, e, $\sin 81$ that are given unambiguously as the sets of natural and real numbers that have been described in detail, are standard objects. Within the framework of the set-theoretic stance, however, an arbitrary 'abstract' real number arises in an indirect manner, being introduced as an element of the set of all real numbers mentioned earlier. A similar method of introducing objects into consideration is widely spread: a vector is an element of a vector space, a filter is a set of subsets of a given set with, incidentally, specific properties, etc.. Hence, among real numbers there are standard and nonstandard ones, there are standard and nonstandard vectors and filters, and, generally speaking, there are standard and nonstandard sets.

As an example, let us consider the multitude of sand grains on Earth. In his classical work "Psammiths" Archimedes wrote: "... of the numbers named by me and given in the work which I sent to Zeuxippes, some exceed not only the numbers of the mass of sand equal in magnitude to Earth filled up in the way described but also that of a mass equal in magnitude to the universe" [5, p.358]. Therefore, the number of the sand grains on Earth is a concrete natural number. It is, however, impossible either to give a direct definition of this number or to determine it. A sequential count of the sand grains is obvious to be impossible for implementation. Hence, the number of the sand grains on Earth is expressed by a '*nonrealizable*', '*nondefinable*' nonstandard natural number and, thus, the multitude of the sand grains is nonstandard.

It goes without saying that the presented *views on the difference between standard and nonstandard sets are of an auxiliary value* for mastering the rules of applying them into practice. One can find here an analogy with the situation in geometry where the intuitive visual presentation of spatial forms helps in elaborating the skills of using axioms which, in the long run, result in strict definitions of points, straight lines, planes, etc.. According to A.D.Alexandrov, one should remark that "axioms by themselves need no substantiation , they only sum up other data and give rise to a logical construction of a theory" [3, p. 51]. Therefore, the formal introduction of the axioms of nonstandard set theory should be preceded by their qualitative discussion.

As we already know, the nonstandard theory of sets begins with the primary observation: sets can be different, i.e., standard and nonstandard. Alongside with this postulate, the following ones are adopted (or, to be more exact, variations of the following postulates).

2.1.6. Transfer principle: *a conventional mathematical statement claiming the existence of a certain set simultaneously determines a standard set.*

In other words, the theorems of existence and uniqueness accepted in classical mathematics are considered to be direct explicit definitions of mathematical objects. An equivalent reformulation of this principle, elucidating the essence of its name, is as follows: *in order to prove a statement on all sets, it suffices to prove it only for standard ones.* An intuitive substantiation of the transfer principle is the evident fact that the statements pertaining to arbitrary sets are made when operating with the sets which have been in fact described, i.e., with standard sets.

If you think about the essence of the transfer principle, you will see that it deals with two aspects of presenting standard objects. The first is that *new standard objects arise from existing ones in a deductive way,* using descriptions of the kind of the existence and uniqueness theorems. This peculiarity can be expressed by the assertion that in a standard nonempty set there is a standard element, and that the object which is being constructed from or determined by existing standard elements is standard itself. The second aspect of the idea of being standard expressed by the transfer principle is continuously related with the former one and implies *presentability of the world of standard objects in the universe of all sets under investigation.* One can say that here postulated is the possibility of induction, i.e., of the cognition of ideal constructions by studying really accessible standard objects.

2.1.7. Idealization principle: *in every infinite set there is a nonstandard element.*

This statement certainly conforms with general ideas about infinity. The idealization principle is often given below in stronger forms which reflect the inexhaustible variety of ideal objects. For instance, the idea that *all standard sets are elements of a finite set* is often accepted. The number of the elements of such a 'universal' set is huge and, which is most important, it is not 'realizable', i.e., nonstandard, the nonstandardness of the universal set itself being, therefore, no surprise.

It should be emphasized that when working with the first two postulates presented above (not only with them though) one should be careful. Thus, for instance, by virtue of the transfer principle, a standard set is uniquely determined with its standard elements in the medium of its family, i.e., standard sets. The set under study is not however reduced to, generally speaking, the standard elements belonging to it. There can exist other nonstandard sets containing all the stock of the standard elements of the initial set and having no other standard elements. There is another peculiarity worth mentioning: *the notion 'statement' should be used with caution,* as though is the case in the conventional set theory as well. The transfer principle should deal with common mathematical propositions not appealing to the new property of sets which has been described at the semantic level, i.e., to be or not to be standard. In the opposite case we would, stemming from the fact that all standard sets are standard, come to the conclusion that an arbitrary set is standard which contradicts the idealization principle. Hence, the statement that a certain set is standard is not a conventional one.

2.1.8. Standardization principle: *any standard set and any property determine a new standard set which is a subset of the initial set, with its standard elements possessing the property under study.*

In more detail: let A be a standard set in question, and let P be its property. The standardization principle states that there is a standard set commonly denoted by ${}^*\{x \in A : P(x)\}$ and obeying the relation

$$y \in {}^*\{x \in A : P(x)\} \leftrightarrow y \in \{x \in A : P(x)\}$$

for any standard y. The set ${}^*\{x \in A : P(x)\}$ is often referred to as *standardization*, omitting the parameters that participate in its definition. An intuitive substantiation of the standardization principle lies in the fact that, having at our disposal implicit descriptions of mathematical objects, we can operate with new quite concrete sets composed of them by some definite laws. *Standardization extends the conventional method of formation of subsets by selecting elements with a given property* When thinking over the standardization principle you would reasonably pay attention to the fact that it says nothing about the nonstandard elements of the arising set. It is not by chance since such elements can possess or cannot possess the property under discussion. It should also be emphasized that the standardization principle must be used with due caution. Attempts to standardize a 'universal' set containing all standard sets would result in an immediate contradiction.

2.1.9. The postulates considered above give grounds for the axiomatic presentation of set theory. They will be discussed in detail below, and meanwhile we would like to share the V.A.Zorich point of view: "As a whole, any of the existing axiomatics is such that, on the one hand, it eliminates the known contradictions of the naive theory, and, on the other hand, it ensures freedom of operation with concrete sets arising in various sections of mathematics, and, before all, in mathematical analysis understood in the broad sense of the word" [284, pp. 18-19].

2.2. Simplest Properties of Standard and Nonstandard Real Numbers

Let us now go over to getting acquainted with new properties of the classical real axis which can be studied with the principles of nonstandard analysis.

2.2.1. For a set A we shall write $a \in {}^{\circ}A$ instead of the expression 'a is a standard element of A'.

2.2.2. *The following statements are valid:*

(1) *the principle of induction over standard natural numbers is valid, i.e., if A is a set for which $1 \in A$ and for $n \in {}^{\circ}\mathbf{N}$ $n \in A \to n+1 \in A$, then A contains all standard natural numbers*: ${}^{\circ}\mathbf{N} \subset A$;

(2) *a finite set (i.e.,one not allowing one-to-one mappings on its own subsets) composed of standard elements is standard*;

(3) *a standard finite set has only standard elements*;

(4) *if a set has only standard elements, it is finite*;

(5) *for an infinite standard set A (i.e., not finite, (see* **(2)***)) the union ${}^{\circ}A$ is not a set.*

⊲ (1) Using the standardization principle, let us form the following (standard) subset of the set of natural numbers:

$$B := {}^{*}\{n \in \mathbf{N} : n \notin A\}.$$

Let us assume $B \neq \varnothing$. Then B has the least element m standard by transfer. By condition, $m \neq 1$ (since $1 \in A$). Moreover, $m \notin A$, and hence, $m-1 \notin A$. According to the transfer principle, $m-1 \in {}^{\circ}\mathbf{N}$, i.e., $m-1 \in B$. We get $m-1 \geq m$, a contradiction. Hence, $B = \varnothing$, i.e., $(\forall n \in {}^{\circ}\mathbf{N})\, n \in A$, which implies the inclusion ${}^{\circ}\mathbf{N} \subset A$.

(2) An obvious corollary to the transfer principle.

(3) A singleton standard set has the only (and, hence, standard) element. The number of the elements of a finite standard set A is standard. Moreover, $A = (A - \{a\}) \cup \{a\}$ for every $a \in A$. Making use of the fact that the number of elements $A - \{a\}$ is also standard, we can apply the induction principle (1).

(4) A direct corollary to the idealization principle.

(5) Let us assume that $^{\circ}A$ is a set. By virtue of (4) we conclude that $^{\circ}A$ is finite. According to (2) $^{\circ}A$ is a standard set. In line with the transfer principle $A = {^{\circ}A}$ and, hence, A is finite, which is a contradiction. ⊳

2.2.3. *A natural number N is nonstandard (i.e., nonrealizable) iff N is greater than any standard natural number. Or, in symbols,*

$$N \in \mathbf{N} - {^{\circ}\mathbf{N}} \leftrightarrow (\forall n \in {^{\circ}\mathbf{N}}) \quad N > n.$$

⊲ It suffices to remark that by virtue of, for instance, **2.2.2**, the condition $n > N$ yields for a realilizable number n that N is realizable: $N \in {^{\circ}\mathbf{N}}$. ⊳

2.2.4. According to **2.2.3**, nonstandard natural numbers are termed *actually infinitely large* or, to put it shortly, *infinite.*

Despite the following widely spread statement: "Euler claimed quite light-mindedly that 1/0 stands for infinity, though he did not find it necessary to define what the infinity is, but only introduced its notation, ∞", in fact L.Euler directly pointed out [48, p. 89]: "...an infinite number and a number greater than any arbitrarily given one, are synonyms."

The fact that the number N is infinite is expressed by a symbol $N \approx \infty$ or, in more detail, $N \approx +\infty$. It should be emphasized that the use of the term 'an infinite number' can result in a contradiction. Indeed, if we strictly follow the set-theoretic stance, we will see that in the set-theoretic sense the corresponding set N is finite (compare with 2.2.2 (2)). The use of the phrase 'N is an infinite number' must, therefore, be in no case associated with the infiniteness of N as a set. In fact, N is a finite set whose number of elements is nonstandard. It is this and only this sense that is inferred in the notion of an infinitely large natural number N (within set-theoretic stance).

2.2.5. *The following statements are valid:*

(1) $N \approx +\infty, M \approx +\infty \rightarrow N + M \approx +\infty, NM \approx +\infty$;

(2) $N \approx +\infty, n \in {^{\circ}\mathbf{N}} \rightarrow N + n \approx +\infty, N - n \approx +\infty, nN \approx +\infty$;

(3) *for every* $n \in {^{\circ}\mathbf{N}}$,

$$N \approx +\infty \leftrightarrow N^n \approx +\infty \;;$$

(4) *an infinite large composite number has infinitely large divisors*;

(5) $N \approx +\infty, M \geq N \rightarrow M \approx +\infty$;

(6) "*...if* $\frac{1}{0}$ *denotes an infinitely large number, then, since* $\frac{2}{0}$ *is undoubtedly a doubled* $\frac{1}{0}$, *it is clear that a number, be it even infinitely large, can become still two or several times greater*" (L.Euler [176, p.620]);

(7) *let* t *be a real positive number. The integral part of* t *is infinitely small iff so is* t (i.e., $(\forall_r \in {}^{\circ}\mathbf{R})\, t \geq |r|$);

(8) *let* $\psi:\mathbf{N} \rightarrow \mathbf{N}$ *be a strictly increasing standard function. Then for* $N \in \mathbf{N}$ *we have*

$$N \approx +\infty \leftrightarrow \psi(N) \approx +\infty .$$

$\triangleleft$ Let us prove only (7) and (8), since the other statements are easier to check.

(7) If the integral part s of a number t is infinitely small and $(\exists_r \in \mathbf{R})\, t \leq r$, $t \leq n$ for a certain $n \in {}^{\circ}\mathbf{N}$. Hence, we get $n+2 \leq s \leq t \leq n$, which is ridiculous. Therefore, $t \approx +\infty$. But if $t \approx +\infty$, then $s+1 \geq t$, where s is the integral part of t. Hence, $s+1 \approx +\infty$, which yields, by virtue of **2.2.5** (2), $s \approx +\infty$.

(8) $\leftarrow$ Let first $\psi(N) \approx +\infty$ and $n \in {}^{\circ}\mathbf{N}$. Then the number $\psi(n)$ is realizable, i.e., $\psi(n) \in {}^{\circ}\mathbf{N}$ and, hence, $\psi(N) > \psi(n)$. Since ψ is strictly monotonic, we deduce: $N > n$, i.e., $N \approx +\infty$.

$\rightarrow$ Let us assume that $N \approx +\infty$. Then for $n \in {}^{\circ}\mathbf{N}$ we get $N > n$ and, hence, $\psi(N) > \psi(n) \geq n$. Thus, $\psi(N) \approx +\infty$. $\triangleright$

2.2.6. Let $\overline{\mathbf{R}}$ be the *extended numerical straight line*, i.e., $\overline{\mathbf{R}} := \mathbf{R} \cup \{-\infty, +\infty\}$, where $-\infty, +\infty$ are the largest and the least elements adjoined to R. It would be convenient to term the set $\infty := \{+\infty, -\infty\}$ the (symbolic) *potential infinity*, or to speak about $+\infty$ (or $-\infty$) as about the positive (or negative), respectively, (symbolic) infinity.

The number $t \in \mathbf{R}$ is called *finite*, *limited*, or *accessible* provided there is a standard number $n \in {}^{\circ}\mathbf{N}$, for which $|t| \leq n$. The condition that t is finite is presented in symbols as follows: $t \in {}^{\approx}\mathbf{R}$. The elements of $\mathbf{R}$ which are not finite are called *infinite*, or, in more detail, *actually infinite numbers*. For $t \notin {}^{\approx}\mathbf{R}$ and $t > 0$ we write $t \approx +\infty$. The presentation $t \approx -\infty$ is understood in an analogous way. The notation $t \approx \infty$ means that t is infinite.

Frequently used are also the convention $t \approx +\infty \leftrightarrow t \in \mu(+\infty)$ and the phrases like 'the number belongs to the *monad* of an infinitely distant point (in the monad of plus-infinity)'.

The number $t \in \mathbf{R}$ is termed *infinitesimal*, or *infinitely small* or, in more detail, *actually infinite small* if for any realizable $n \in {}^\circ\mathbf{N}$ we have $|t| \le 1/n$. In this case we write $t \approx 0$, or $t \in \mu(\mathbf{R})$ and say that t is in the *monad* of zero. (The symbol $\mu(\mathbf{R})$ is used alongside with the notation $\mu(0)$ to stress an obvious relation with the only separated vector topology on $\mathbf{R}$.) Positive infinitely small numbers are often called *infinitesimals*, the unsuccessful term *differentials* being of a more restricted use.

2.2.7. The term *monad* ($\mu o \nu \alpha \zeta$) goes as far back as the ancient times and has been traditionally translated, with no sufficient ground for that, as a unit. By the primary definition of Book VII of the Euclid's "Elements", a monad "...is <that> by virtue of which each of the things that exist is called one" [47, p.9].

Let us recall here some quantitative elucidation for views of the structure of a monad expressed by Sextus Empiricus:

"...Pythagoras used to say that the origin of the existing is a monad, by relation to which each of the existing is referred to as a unit" [230, p.361];

"...a point is structured as a monad, since as a monad is a certain origin of numbers, so a point is a certain origin of lines" [230, p.364];

"...a whole, as it is whole, is indivisible, and a monad, since it is a monad, is not divisible. Or, if it is partitioned into many parts, it becomes a union of many monads, not just a [simple] monad" [230. p.367].

Below we shall study monads, their status and structure in detail. Let us start with considering the elementary properties of the infinitesimals or, which is equivalent, the monad of the infinitely small.

2.2.8. *The following statements are valid*:

(1) $s \approx 0, t \approx 0 \rightarrow s + t \approx 0$;

(2) $t \approx 0, s \in {}^{\text{■}}\mathbf{R} \rightarrow st \approx 0$;

(3) $z \approx 0 \leftrightarrow 1/z \approx \infty$ (for $z \ne 0$):

"*...if z becomes a quantity less than any quantity that might be given, i.e., infinitely small, then the value of the fraction $\frac{1}{z}$ must become greater than any quantity that might be given, i.e., an infinitely large quantity*" (L.Euler [48, p.93]).

(4) ($t \approx 0$ *and t is standard*) $\rightarrow t = 0$.

◁ (1) Let $n \in {}^{\circ}\mathbf{N}$. Obviously, $|s| \le (2n)^{-1}$ and $|t| \le (2n)^{-1}$. Hence, $|s+t| \le |s| + |t| \le (2n)^{-1} + (2n)^{-1} = n^{-1}$, i.e., $s+t$ is infinitely small.

(2) Let $s \in {}^{\blacksquare}\mathbf{R}$ and $s \ne 0$ (otherwise there is nothing to prove). Let us assume $n \in {}^{\circ}\mathrm{N}$. By condition, for some $m \in {}^{\circ}\mathrm{N}$ we have $|s| \le m$. Therefore, $|t| \le (nm)^{-1}$, and, hence, $|st| \le |s||t| \le m(nm)^{-1} = n^{-1}$, i.e., $st \approx 0$.

(3) Let z be a finite nonzero number, i.e., $0 < |z| \le |n|$, where $n \in {}^{\circ}\mathbf{N}$. Obviously, we get $|1/z| \ge 1/n$, i.e., $1/z$ is not an infinitely small number. On the contrary, if $z \approx \infty$, then for any finite n we get $|z| \ge n$, which implies $z^{-1} \approx 0$.

(4) We have $|t| \le 2^{-1}|t|$ iff t is standard, which is impossible for $|t| \ge n$. Hence, $t = 0$.
▷

2.2.9. *The monad* $\mu(\mathbf{R})$ *is not a set.*

◁ Let us assume that the opposite is valid. Then $\mu(\mathbf{R})$ is a subset of $\mathbf{R}$. For every $t > 0$, $t \in {}^{\circ}\mathbf{R}$ we get $t \ge \mu(\mathbf{R})$. Hence, $t \ge s := \sup \mu(\mathbf{R})$. The number s is, obviously, infinitely small. Moreover, $2s \ge s \to s = 0$, but this contradicts the existence of nonstandard (actual) infinitely small numbers. ▷

2.2.10. When we work with real numbers, it is convenient to single out various cases of their mutual location.

For $s,t,r \in \mathbf{R}$ we write $s =_s t$, or $s \approx t (\mathrm{mod}\, r)$ provided $(s-t)/r \approx 0$ (here $r \ne 0$). In this case the numbers s and r are called r-close, or *infinitely close modulo* r. When $r = 1$, we simply write $s \approx t$ and say that s and t are *infinitely close.*

The founders of infinitesimal analysis often made no distinction between the numbers infinitely close to a certain number and the number itself. L.Euler expressed this in the following way: "...an infinitely small quantity is exactly zero" [48, p.92]. That was why for $x \in \mathbf{R}$, $y \in {}^{\circ}\mathbf{R}$ the expression $x = y$ used to be employed instead of $x \approx y$. In this respect G.W.Leibniz remarked: "...I consider equal not only those quantities the difference between which is absolutely nothing, but also those the difference between which is incomparably small" [152, p.188], emphasizing that "...the error is undetectable and cannot be found by means of whatever construction" [278, p.195].

For $s,t \in \mathbf{R}$ we write $s = O(t)$ for $s/t \in {}^{\blacksquare}\mathbf{R}$; if $s = O(t)$ and $t = O(s)$, then we say that s and t have *the same order*; if $s/t \approx 0$, then we write $s = o(t)$ and say that s has *higher order of smallness* than t; and, finally, if $s - t = o(t)$ and $s - t = o(s)$, then s and t are called *equivalent* and we write $s \sim t$.

Presenting his views on higher order infinitesimals, G.W.Leibniz wrote: "I would like to add one more remark in order to prevent all arguments against the reality of differences of any orders, and, namely, that they can always be represented as conventional intervals of a straight line proportional to them... . I have already explained how to present first-order differences with conventional straight intervals of a straight line, when first presenting the elements of this calculus in 'Acta' in October 1684" (see [278, pp.188-190, cf. **1.1**).

2.2.11. *Let us introduce for* $s,t \in \mathbf{R}$ *the following natural abbreviations*:

$$s \in O := O(t) \leftrightarrow s = O(t); \; s \in o := o(t) \leftrightarrow s = o(t).$$

The E.Landau rules hold:

$$O + O \subset o; \quad O + o \subset O; \quad o + o \subset O;$$
$$Oo \subset O; \quad OO \subset O; \quad oo \subset o.$$

◁ Let us check, for definiteness, the relation $O + o \subset O$. So, denote $s = O(t)$ and $r = o(t)$. Then $s / t \in {}^{\approx}\mathbf{R}$ and $r / t \approx 0$. Hence, $(s + r) / t \in {}^{\approx}\mathbf{R}$, i.e., $(s + t) = O(t)$. ▷

2.2.12. *For the numbers* $s,t \in \mathbf{R}$ *the following statements are equivalent*:

(1) s *and* t *are equivalent*;

(2) $s - t = o(t)$ *or* $t - s = o(s)$;

(3) $s / t \approx 1$ *or* $t / s \approx 1$;

(4) $s / t \approx 1$ *and* $t / s \approx 1$.

◁ It is clear that (1) → (2). If, for instance, $t - s = o(s)$, $(t - s) / s \approx 0$, i.e., $t / s - 1 \approx 0$. Hence, for $\varepsilon > 0$ and $\varepsilon \in {}^{\circ}\mathbf{R}$ we have $1 - \varepsilon \leq t / s \leq 1 + \varepsilon$. Therefore, $(1 - \varepsilon)^{-1} \geq s / t \geq (1 + \varepsilon)^{-1}$ and $\varepsilon/(1 - \varepsilon) \geq s / t - 1 \geq \varepsilon/(1 + \varepsilon)$, i.e., $s / t \approx 1$. Hence, (2) → (3) → (4), and the implication (4) → (1) is obvious. ▷

2.2.13. *Let* $N \in \mathbf{N}$ *and* $\alpha_k, \beta_k \in o(1)$ *be infinitely small, and* $\alpha_k \sim \beta_k$ *for* $k := 1, \dots, N$. *The following statements are valid*:

(1) $\sum_{k=1}^{N} \alpha_k \sim \sum_{k=1}^{N} \beta_k$ *for* $\alpha_k, \beta_k \geq 0$;

(2) $\sum_{k=1}^{N} = |\alpha_k| O(1)$, *(i.e., if the sum under discussion is in* $^{\text{▀}}\mathbf{R}$*), then*

$$\sum_{k=1}^{N} \alpha_k \approx \sum_{k=1}^{N} \beta_k .$$

◁ To prove this, let us remark that by virtue of **2.2.12**, for every standard $\varepsilon > 0$ we get $\varepsilon\alpha_k + \alpha_k \leq \beta_k \leq \alpha_k + \varepsilon\alpha_k$, which yields (1). Moreover, if $t := \sum_{k=1}^{N} |\alpha_k| \in {}^{\text{▀}}\mathbf{R}$, then

$$\left|\sum_{k=1}^{N} (\alpha_k - \beta_k)\right| \leq \sum_{k=1}^{N} |\alpha_k - \beta_k| \leq \frac{\varepsilon}{n} \sum_{k=1}^{N} |\alpha_k| \leq \varepsilon ,$$

as soon as a standard $n \in {}^{\circ}\mathbf{N}$ is such that $|t| \leq n$. ▷

2.2.14. *There is such a natural number N that for any standard number t in* **R** *the product Nt is infinitely close to a certain natural number.*

◁ Let us choose in **R** a finite subset $\{x_1, \ldots, x_n\}$ containing all standard real numbers, and an infinitely small positive number $\varepsilon > 0$, $\varepsilon \approx 0$. In the theory of numbers there is a theorem, 'the Dirichlet principle for sets', which states: for any $\varepsilon > 0$ and arbitrary $x_1, \ldots, x_n \in \mathbf{R}$ there is such an integer $N \in \mathbf{N}$ that the numbers $N_{x_1}, \ldots, N_{x_n}$ differ from integers by for most ε. Now one has to apply this theorem to the parameters in question. ▷

2.2.15. It would be useful to remark that infinite closeness (as well as equivalence) of numbers cannot be called a subset of the product $\mathbf{R} \times \mathbf{R}$. Indeed, in the opposite case the image of the element zero under this relation, i.e., the monad $\mu(\mathbf{R})$, would become a set. We, however, have already established the monad $\mu(\mathbf{R})$ not to be a set. It should be also emphasized that the monad $\mu(\mathbf{R})$ is *indivisible* in the following implicit sense: for every standard n we have: $n^{-1}\mu(\mathbf{R}) = \mu(\mathbf{R})$.

When thinking over the role of the monad $\mu(\mathbf{R})$ in constructing the system of integers, for us it would be advisable to address Definition **2** of Book VII of the Euclid's "Elements": "A number is a set composed of monads" [47, p.9]. Analogously, the whole 'nonstandard' extended number straight line $\overline{\mathbf{R}}$ and, which is most nontrivial, its accessible part $^{\text{▀}}\mathbf{R}$ are ensembles of monads located at standard points. A more rigorous formulation of this statement rests on the following fundamental fact, the proof of which is essentially based on the standardization principle.

2.2.16. *For every finite number there is a standard (and, moreover, unique) number infinitely close to it.*

◁ By virtue of the standardization principle one can, given $t \in {}^{*}\mathbf{R}$, organize a standard set $A := {}^{*}\{a \in \mathbf{R} : a \leq t\}$. Obviously, $A \neq \varnothing$ and $A \leq n$, where the standard number $n \in {}^{\circ}\mathbf{N}$ is such that $-n \leq t \leq n$. Indeed, for every standard $a \in A$ we have $a \leq t \leq n$. By the transfer principle, we deduce $A \leq n$. Since $\mathbf{R}$ is complete, we have $s := \sup A \in \mathbf{R}$. Obviously, s is a standard number. Let us show that $s \approx t$. In the opposite case we get, for a certain standard $\varepsilon > 0$, $|s - t| > \varepsilon$. If $s \geq t$, then for every standard $a \in A$ we get $s \geq t + \varepsilon$, i.e., $s \geq a + \varepsilon$. In this case, however, we would get $s \geq s + \varepsilon$, which is impossible. The remaining possibility, $s < t$ also results in a contradiction, since then we would get $t > s + \varepsilon$, and, again, $s \geq s + \varepsilon$. ▷

2.2.17. The standard number infinitely close to a finite number $t \in {}^{*}\mathbf{R}$ is called the *standard part*, or the *shadow* of t, and is denoted by $\mathrm{st}(t)$ or ${}^{\circ}t$. For the sake of convenience it is also assumed that ${}^{\circ}t = \mathrm{st}(t) = +\infty$ if $t \approx +\infty$ and, respectively, ${}^{\circ}t = \mathrm{st}(t) = -\infty$ for $t \approx -\infty$ (under the obligatory supposition that $+\infty \approx +\infty$ and $-\infty \approx -\infty$). Therefore, every (standard) $t \in \overline{\mathbf{R}}$ is put into correspondence with its monad $\mu(t)$, i.e., all the elements s of $\mathbf{R}$ for which $s \approx t$.

Thus, in nonstandard analysis an extended straight line should be visualized in connection with the scheme presented in Figure 2. Choosing a standard number ${}^{\circ}t$ on the axis $\mathbf{R}$, we draw a big dot, a blob, a monad $\mu({}^{\circ}t)$, which is an 'indivisible implicit presentation of ${}^{\circ}t$'. If we view the region of the point ${}^{\circ}t$ with a powerful microscope, we will see a smeared small cloud with a blurred boundary, which is the image of $\mu({}^{\circ}t)$.

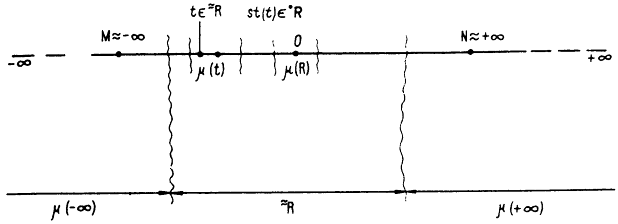


Fig. 2

When using a more powerful microscope, the portion of the 'point-monad' under observation would enlarge, reveal more details and partially disappear from the view. In this case, however, we again would deal with the same standard real number, which is, if you please, described by the process of 'studying the microstructure of a physical straight line' presented above.

2.2.18. *The following statements are valid:*

(1) *for* $s \in \mathbf{R}$, $t \in {}^{\approx}\mathbf{R}$,

$$\mathrm{st}(s+t) = \mathrm{st}(s) + \mathrm{st}(t); \quad \mathrm{st}(st) = \mathrm{st}(s)\mathrm{st}(t);$$

(2) *if* $s,t \in \mathbf{R}$ *and* $s \le t$, *then* ${}^{\circ}s \le {}^{\circ}t$;

(3) *for* $s,t \in \overline{\mathbf{R}}$ *we have*

$$(\exists t' \approx t)\ t' \ge s \leftrightarrow {}^{\circ}s \le {}^{\circ}t \leftrightarrow (\forall \varepsilon > 0,\ \varepsilon \in {}^{\circ}\mathbf{R})\quad s \le {}^{\circ}t + \varepsilon;$$
$$(\forall t' \approx t)\ t' \ge s \leftrightarrow {}^{\circ}s < {}^{\circ}t \quad (t \in {}^{\approx}\mathbf{R});$$

(4) *the transition from a real number to its standard part is not a set (nor a function, in particular).*

◁ (1) Let us, for instance, prove that the transition to the standard part is multiplicative. We have $s \approx \mathrm{st}(s) \to ts \approx t\mathrm{st}(s)$. Besides, $t \approx \mathrm{st}(t) \to \mathrm{st}(s)t \approx \mathrm{st}(t)\mathrm{st}(s)$. Finally, $st \approx \mathrm{st}(s)\mathrm{st}(t)$. It now remains to recall that the product of standard numbers is standard.

(2) Let $s < t$ (otherwise everything is obvious). If $s \approx t$, then $\mathrm{st}(s) = \mathrm{st}(t)$. In the opposite case the monads $\mu(s)$ and $\mu(t)$ do not intersect. Hence, ${}^{\circ}s < {}^{\circ}t$.

(3) In the initial equivalence the implication to the right is obvious, while the reverse one is ensured by the fact that for $s \le {}^{\circ}t$ we get $s \le {}^{\circ}t + s - {}^{\circ}s$. Besides, $s < t + \varepsilon \to \mathrm{st}(s) \le \mathrm{st}(t) + \mathrm{st}(\varepsilon) = {}^{\circ}t + \varepsilon$ for every $\varepsilon > 0$, $\varepsilon \in {}^{\circ}\mathbf{R}$. By transfer this implies that for an arbitrary positive ε we get ${}^{\circ}s \le {}^{\circ}t + \varepsilon$, and, hence, ${}^{\circ}s \le {}^{\circ}t$. In turn, if ${}^{\circ}s < {}^{\circ}t$, then, making use of the fact that the monads $\mu\,{}^{\circ}(s)$ and $\mu({}^{\circ}t)$ are disjoint, we deduce $s < {}^{\circ}t + \varepsilon$ for any $\varepsilon > 0$, $\varepsilon \in {}^{\circ}\mathbf{R}$.

In order to check the arrow to the right in the lower equivalence, let us note that s does not lie in the monad $\mu(t)$ of the number t. Hence, the whole of the monad of s lies to the left from the monad of t, i.e., $\mu(s) < \mu(t)$. Therefore, ${}^{\circ}s < {}^{\circ}t$. And, finally, to prove the remaining implication let us remark that for ${}^{\circ}s = -\infty$ we get either $\mu(t) > {}^{\circ}s$ or $t \in {}^{\approx}\mathbf{R}$. But if ${}^{\circ}s \in {}^{\circ}\mathbf{R}$, then $\mu({}^{\circ}s) < {}^{\circ}t$. Hence, for $t' \approx t$ the condition $t' \ge s$ is met.

(4) If the law $t \to \mathrm{st}(t)$ were a set, then the monad $\mu(\mathbf{R})$ would also be a set (since $t \in \mu(\mathbf{R}) \leftrightarrow {}^{\circ}t = 0$). Now we have to make use of **2.2.9.** $\triangleright$

2.3. Primary Notions of Mathematical Analysis on a Real Line

Let us now discuss the fundamental notions pertaining to differential and integral calculus of functions in a single real variable.

2.3.1. Nonstandard criteria of limits. *For a standard sequence* (a_n) *and a standard number* $a \in \mathbf{R}$ *the following statements are valid:*

(1) *the number* a *is a partial limit of* (a_n) *iff for a certain infinitely large* N *the condition* $a = {}^{\circ}a_N$ *is met;*

(2) *the number* a *is the limit of* (a_n) *iff for all infinitely large numbers* N *the term* a_N *is infinitely close to* a, *i.e.,*

$$a = \lim a_n \leftrightarrow (\forall N \approx +\infty)\, a_N \approx a.$$

$\triangleleft$ These statements are checked in an analogous way. Therefore, let us prove one of them, for instance, (2). So, let $a_n \to a$, and let, for the sake of definiteness, assume $a \in \mathbf{R}$ (the cases $a = +\infty$ and $a = -\infty$ are proved by the same scheme). By condition, for an arbitrary positive number $\varepsilon > 0$ and a certain $n \in \mathbf{N}$ we have $|a_N - a| \le \varepsilon$ as soon as $N \in \mathbf{N}$ and $n \le N$. Hence, by the transfer principle, for a standard $\varepsilon > 0$ there is a standard n with the same property Every infinitely large N majorizes the obtained n, i.e., $|a_N - a| \le \varepsilon$. Since ε is arbitrary, this implies $a_N \approx a$.

Let it, in turn, be known that for all $N \approx +\infty$ we have ${}^{\circ}a_N = a$. For the sake of definiteness and diversity, let us assume $a := -\infty$. Let us choose an arbitrary standard number $n \in {}^{\circ}\mathbf{N}$. Obviously, for all $N \ge M$, where M is an infinitely large number, we have $a_N \le -n$. Hence, for every standard n we have proved 'something' (namely, 'something': $= (\exists M)(\forall N \ge M)\ (a_N \le -n)$). By the transfer principle, this 'something' is valid for every $n \in \mathbf{N}$. It is the latest fact that, as is well known, signifies $a_n \to -\infty$. $\triangleright$

2.3.2. Let us emphasize the merits of the criteria obtained. We have seen that the partial limits of a standard sequence are exactly those 'realizable' numbers which correspond to infinitely large indices. In other words, the partial limit is an 'observable' value of a certain infinitely far term of the sequence. The statements under consideration have an implicit intuitive foundation and differ drastically from conventional definitions of partial limit as a number to which a subsequence of the initial sequence is tending, or as such an element of a straight line that every interval containing it intersects with any remainder, i.e., 'any tail', of the sequence in question.

It would be useful to get acquainted with a clarification of the notion of partial limit for a [generalized] sequence with which N.N.Luzin furnished the formulation of the conventional definition (see [162], [169]). "At first the reader will undoubtedly view this formulation as cumbersome and abstract. But the sense of ambiguity would disappear if the reader recalls the conventional notions of a 'variable' and 'time'. Indeed, what is the intention of this formulation when translated into the language of a 'variable' and 'time'? In order to understand this, let us consider a variable x which runs through a given numerical sequence M, going over from preceding numbers to succeeding ones ... in the language of a variable and time this formulation means that the ([partial]) *limit* of a numerical sequence M is such a number a that the variable x cannot get completely detached from, since 'at certain periods of time' the value of the variable x gets 'any close' to a."

Using the same images, in nonstandard analysis we can still be more presentable and clear: 'if at a certain infinitely distant moment of time a variable x is infinitely small different from a, then a is the [partial] limit of M'.

Going now over to considering the nonstandard criterion of the limit of a sequence, let us recall the following statements by R.Courant.

"**Motivation of the implicit definition of limit.** It is no wonder that nobody who hears the abstract definition of the limit of a sequence for the first time can at once understand it completely. The definition of limit mimics a game between two persons, A and B: A demands that a constant value a be approximately presented by a value a_n in such a way that the deviation be less than an arbitrary bound $\varepsilon = \varepsilon_1$, set by A himself. B fulfills his requirements by proving the existence of such an integer number $N = N_1$, that all a_n, starting from the element a_N, obey the requirement of ε_1. Then A wants to set a new, smaller bound, $\varepsilon = \varepsilon_2$, while B, in turn, fulfills this requirement by finding a new integer number $N = N_2$ (possibly, much greater), etc.. If B is always ready to meet the requirements set by A, whatever small bound A could set, then we have the situation that is symbolically expressed as $a_n \to a$.

There is, no doubt, a psychological difficulty in mastering this exact definition of passage to the limit. Our visual presentation imposes upon us a 'dynamical' idea of passage to the limit as a result of motion, i.e., we 'run' through the sequence of numbers $1,2,3,\ldots,n,\ldots$ observing on our way the behavior of the sequence a_n. We have a feeling that while 'running' we must observe this approximation. This 'natural' formulation, however, does not allow a rigorous mathematical one. In order to get an exact definition, the order of consideration should be reversed: instead of first watching the argument n followed by considering the dependent variable a_n associated with it, we base our definition on steps which make it possible to subsequently check the statement $a_n \to a$. Under such an approach one has to first choose whatever small interval surrounding a, and then to check if the condition is fulfilled by choosing the variable n to be sufficiently large. In this way we come to the exact definition of limit denoting the expressions 'whatever small boundary' and 'a sufficiently large n' by ε and N" [31, pp.66-67].

It goes without saying that the criterion formulated in **2.3.1** (2): "if for all infinitely large N the general term a_N is impossible to be distinguished from a standard number a, then a is declared (and, in fact, is) the limit of (a_n)" successfully expresses the dynamic idea of passage to the limit.

In this case one should always bear in mind that the nonstandard criterion of limit is applicable only to standard sequences and is not, generally speaking, valid for nonstandard, ill-described sequences. Thus, if $a_n : N / n$, where $N \approx +\infty$, then $a_n \to 0$ and, at the same time, $a_N = 1$. In other words, criterion **2.3.1** supplements the up-to-date views on limit without rejecting or abolishing them. Or, more exactly, when determining the limit only for standard sequences, we therefore automatically form the standard set of all converging sequences by employing the standardization principle. To state it otherwise, both the conventional $\varepsilon - N$-definition and the unusual definition with the actual infinitely large and infinitely small are closely interrelated, tightly interwoven.

It would be useful to put special emphasis on the fact that in concrete applications (in particular, in physics) one often has to face real, explicitly described, i.e., standard sequences. Moreover, under such conditions 'the infinitely large' has an exact (physical) sense, i.e., the horizon, or the boundary beyond which the numbers are declared indiscernible, is given overtly. Bearing in mind that in practice the problems of existence are also solved on the ground of semantic considerations (if there is no physical speed, it is not worth while looking for), there arises a problem of recognizing the limit of a standard sequence known beforehand. Nonstandard analysis gives a simple receipt: "Take a general term of your sequence with any (no matter which) infinitely large index; it is the number determined by this term (up to infinitesimal) that is the sought limit". In this respect more understandable becomes the background of the methods of the founders of differential and integral calculuses who were seeking the answers to the problems on the exact values of concrete 'standard' objects: areas of figures, equations of tangents to 'named' curves, integrals of implicitly written analytical expressions, etc..

2.3.3. An important new contribution of nonstandard analysis is the formulation of the notion of limit for a *finite sequence* $a[N] := (a_1,\dots,a_N)$, where N is an infinitely large natural number. The intuitive idea melted in the foundation of of the definition to follow, well reflects practical techniques of finding numerical characteristics of indiscernible discrete unions, such as thermodynamic parameters of fluid media, estimates for the population demand, etc..

2.3.4. The number a is called *the microlimit* or *the nearlimit value* of a sequence $a[N]$ provided, for all infinitely large M less than N, we have $a_M \approx a$. In this case we also say that $a[N]$ *converges nearly* to a. When a is a finite number, the standard part $^\circ a$ is termed the *limit* (or the S-*limit*) of the sequence $a[N]$ and we write either $^\circ a = {}^{\approx}\lim a[N]$ or $^\circ a = S - \lim\limits_{n<N} a_n$. Therefore,

$$^\circ a = {}^{\approx}\lim a[N] \leftrightarrow a \in {}^{\approx}\mathbf{R} \wedge (\forall M \approx +\infty, M \le N)\ a_M \approx a.$$

2.3.5. *Let* (a_n) *be a standard sequence,* $N \approx +\infty$ *and* $a \in {}^{\approx}\mathbf{R}$. *The following statements are equivalent:*

(1) a *is the microlimit of* $a[N]$;

(2) *the sequence* (a_n) *converges to* $^\circ a$.

◁ The implication (2) → (1) is contained in **2.3.1** (2). To prove (1) → (2), let us choose an arbitrary standard $\varepsilon > 0$ and consider the set

$$A := \{m \in \mathrm{N} : (\forall n)(m \le n \le N) \rightarrow |a_n - {}^\circ a| \le \varepsilon\}.$$

The set A is nonempty as $N \in A$. Hence, A contains the least element m. If $m \approx +\infty$, then $m - 1 \approx +\infty$, and, by condition, $m - 1 \in A$. Therefore, m is standard. Moreover, if $n \ge m$ and n is standard, then $n \le N$ and $|a_n - {}^\circ a| \le \varepsilon$. Hence, $(\forall \varepsilon \in {}^\circ\mathbf{R}, \varepsilon > 0)$ $(\exists m \in {}^\circ\mathbf{N})$ $(\forall n \in {}^\circ\mathbf{N})$ $n \ge m \rightarrow |a_n - {}^\circ a| \le \varepsilon$. Using the transfer principle, we deduce that (a_n) converges to $^\circ a$. ▷

2.3.6. The criterion just stated gives an exact foundation for the *principle of a granted horizon*, which states that in concrete investigations one indicates a ‘physical’ or ‘economic’ actually infinitely large number which serves both as a measure of presentability of the collection under study and its natural bound from above.

2.3.7. Examples

(1) $\lim \dfrac{n-1}{n} = 1$.

◁ Let us choose an infinitely large i. We have ${}^\circ\left(\frac{i-1}{i}\right) = {}^\circ\left(1 - \frac{1}{i}\right) = 1$. In more detail (L.Euler): "Since i is an infinitely large number, $\frac{i-1}{i} = 1$; and indeed, obviously, the greater number is put instead of i, the closer the value of $\frac{i-1}{i}$ will be to 1; if i becomes greater than any given number, then the fraction $\frac{i-1}{i}$ will become equal to 1" [50, p.116]. ▷

(2) $\lim \frac{n}{2^n} = 0$.

◁ For every infinitely large N we have $2^N = (1+1)^N \geq N(N-1)/2$, i.e., $0 \leq N/2^N \leq 2/(N-1) \approx 0$. Hence, $N/2^N \approx 0$. ▷

(3) $\lim \sin(2\pi n! e) = 0$.

◁ For any natural n, we have

$$0 < e - \sum_{k=1}^{n} \frac{1}{k!} < \frac{3}{(k+1)!}.$$

From here we deduce, for an infinitely large N;

$$0 \leq N!\left(e - \sum_{k=1}^{N} \frac{1}{k!}\right) \leq \frac{3N!}{(N+1)!} = \frac{3}{N+1} \approx 0.$$

Let $x := 2\pi N! n$ and $y := 2\pi N! \sum_{k=1}^{N} 1/k!$. Then $x \approx y$, in which case $y = 0$. Obviously,

$$|\sin x - \sin y| = 2\left|\cos\frac{x+y}{2}\sin\frac{x-y}{2}\right| \leq |x - y|,$$

i.e., $\sin x \approx 0$. ▷

(4) *Let (a_n) be such that the sequences (a_{2n}), (a_{2n+1}) and (a_{3n}) converge. Then (a_n) converges, too.*

◁ One can consider (a_n) to be a standard sequence. For an infinitely large N we have $2N \approx +\infty$, $2N+1 \approx +\infty$ and $3N \approx +\infty$, i.e., $a_{2N} \approx a$, $a_{2N+1} \approx b$, $a_{3N} \approx c$ for some standard numbers a, b, c, respectively. In particular, $a_{6N} \approx a \approx c$ and $a_{6N+1} \approx b \approx c$. Hence, $a = b = c$, which was required. ▷

(5) *Let* (a_n) *vanish. Then*

$$\lim \frac{a_1+\ldots+a_n}{n} = 0.$$

◁ By transfer, the sequence (a_n) can be considered standard. Let us choose $N \approx +\infty$. Let M be the integral part of $\sqrt{N}$. It is obvious that M is an infinitely large number. In this case for every standard $n \in {}^\circ\mathbf{N}$ we have $|a_N| \le n^{-1}$, and, hence,

$$s_N := \left|\frac{a_1+\ldots+a_N}{N}\right| \le \left|\frac{a_1+\ldots+a_M}{N}\right| + \left|\frac{a_{M+1}+\ldots+a_N}{N}\right|$$
$$\le \frac{M}{N} \sup_{n\in N} |a_n| + \frac{1}{n}\,\frac{N-M-1}{N}.$$

Since $1/N \approx 0$ and $\sup_{n\in N} |a_n| \in {}^\circ\mathbf{R}$, we come to the conclusion that the number s_N is infinitely small. ▷

(6) *There is a Banach limit, i.e., such a continuous linear functional* l *in the space* $l_\infty := l_\infty(\mathbf{N},\mathbf{R})$, *that for any sequence* $a := (a_n)$ *of* l_∞ *we have*:

$$(\exists \lim a_n) \to l(a) = \lim a_n;$$
$$\liminf a_n \le l(a) \le \limsup a_n;$$
$$('a)(n) := a_{n+1} \to l(a) = l('a).$$

◁ To prove this statement, choose an infinitely large natural number N. For every standard sequence a of l_∞ the value

$$f(a) := \frac{1}{N} \sum_{k=N}^{2N-1} a_k$$

is finite. Indeed, as a is standard, the value $\|a\|_\infty := \sup_{n\in N} |a_n|$ is standard. Moreover,

$$|f(a)| \le \frac{1}{N} \sum_{k=N}^{2N-1} |a_k| \sum_{k=N}^{2N-1} \|a\|_\infty \le \|a\|_\infty.$$

Let us now consider (making use of the fact that the set $l_\infty \times \mathrm{R}$ is standard) the standardization

$$l := {}^*\{(a,t) \in l_\infty \times \mathbf{R} : t = {}^\circ f(a)\}.$$

First we prove that l is the sought object, starting with the statement that l is a function. We have to show that

$$(\forall a \in l_\infty)(\forall t_1,t_2 \in \mathbf{R})\,(a,t_1) \in l \wedge (a,t_2) \in l \rightarrow t_1 = t_2.$$

It suffices, according to the transfer principle, to check this property for standard a,t_1,t_2. In this case, however, by the definition of standardization, we have $t_1 = {}^\circ f(a)$ and $t_2 = {}^\circ f(a)$. Now we recall (see **2.2.16**) that the standard part is unique. The linearity of l is checked by a similar consideration. It is also obvious that $a \geq 0 \rightarrow l(a) \geq 0$, i.e., that l is positive.

Let a be a standard sequence converging to $\bar{a}$. Then, by virtue of **2.3.1** (2) for any standard $\varepsilon > 0$ we have $|a_N - \bar{a}| \leq \varepsilon, \ldots, |a_{2N-1} - \bar{a}| \leq \varepsilon$, as for $M \geq N$ all a_M are infinitely close to $\bar{a}$. Hence,

$$|f(a) - \bar{a}| = \Big|\frac{1}{N}\sum_{k=N}^{2N-1}(a_k - \bar{a})\Big| \leq \varepsilon,$$

i.e., $\bar{a} = {}^\circ f(a)$. It is the obtained property combined with the positiveness of l that ensures the sought estimates.

We now have to establish that the functional constructed is invariant under shifts, i.e., that $l('a) = l(a)$ for all $a \in l_\infty$. And again, the sequence a can be considered standard, in which case the element $'a$ is also standard and, hence,

$$l('a) = {}^\circ\!\left(\frac{1}{N}\sum_{k=N}^{2N-1} a_{k+1}\right) = \mathrm{st}(N^{-1}(a_{N+1} + a_{N+2} + \ldots + a_{2N}))$$
$$= \mathrm{st}\Big(\frac{1}{N}\sum_{k=N}^{2N-1} a_k + \frac{1}{N}a_2N - \frac{1}{N}a_N\Big) = {}^\circ(f(a) + N^{-1}a_{2N}) - N^{-1}a_N)$$
$$= {}^\circ f(a) + (N^{-1}a_{2N}) - {}^\circ (N^{-1}a_N) = {}^\circ f(a) = l(a).$$

Here account has been made of the finiteness of the numbers a_{2N} / N and a_N / N, as well as of **2.2.18**. ▷

2.3.8. Nonstandard criterion for continuity. *Let f be a standard real function and x be a standard point of its standard domain of definition,* dom(f). *Then the following statements are equivalent:*

(**1**) *f is continuous at the point x;*

(**2**) *f maps the points infinitely close to x into the points infinitely close to $f(x)$, i.e.,*

$$x' \approx x,\ x' \in \mathrm{dom}(f) \rightarrow f(x') \approx f(x).$$

◁ (1) → (2). Let $\varepsilon > 0$ be a standard number. There is a $\delta > 0$, such that for $|x' - x| \leq \delta$ and $x' \in \mathrm{dom}(f)$ we have $|f(x') - f(x)| \leq \varepsilon$. By transfer, there also is a standard δ with the same property. If $x' \approx x$ and $x' \in \mathrm{dom}(f)$, then, obviously, $|x' - x| \leq \delta$ (since $\delta \in {}^\circ\mathbf{R}$ and, hence, $|f(x) - f(x')| \leq \varepsilon$). Since $\varepsilon \in {}^\circ\mathbf{R}$ is arbitrary, this implies $f(x') \approx f(x)$.

(2) → (1) Let us choose an arbitrary $\varepsilon > 0$. Now we have to find a δ which is also participating in the '$\varepsilon - \delta$-definition'. By transfer, it suffices to find such a δ only for the standard ε. In the latest case, however, instead of δ we can take any actually small positive number. ▷

2.3.9. According to **2.3.8** (2), the function $f:\mathrm{dom}(f) \to \mathbf{R}$ is called *microcontinuous at the point* x *of* dom$f$ provided for $x' \in \mathrm{dom}\, f$ and $x' \approx x$, we have $f(x') \approx f(x)$.

2.3.10. When discussing the nonstandard criterion obtained, i.e. that 'for standard functions at a standard point both continuity and microcontinuity coincide', we can repeat the argumentation that was given in **2.3.2**. One should emphasize, following R.Courant, that "as was the case for the limit of a sequence, the Cauchy definition rests, so to say, on the reversion of the intuitively acceptable order in which we would like to consider variables. Instead of first considering the independent and then dependent variable, we first direct out attention to 'the boundary of accuracy' of ε, and then try to to restrict the corresponding 'arena' of δ for the independent variable" [31, p.73]. The nonstandard criterion makes us free from the unpleasant reversal of quantifiers for all functions and points accessible for us, i.e., standard. At the same time, the $\varepsilon - \delta$-definition in its full scope is only indirectly restored through microcontinuity at a point, using the standardization procedure. In this way the standard and nonstandard approaches again, as has to be expected, demonstrate their intricate but genuine unity. A new mathematical property, microcontinuity of the function at a point, seems to be an interesting acquisition. The following statements will facilitate our understanding of microcontinuity on a greater scale.

2.3.11. Examples

(1) *The function* $x \to x^2$ *is not microcontinuous at every infinitely large point* $t \in \mathbf{R}$.

◁ Indeed, $t + t^{-1} \approx t$ and, at the same time, $(t + t^{-1})^2 - t^2 \approx 2$. ▷

(2) *Let* δ *be a strictly positive infinitely small number. Let us consider the function*

$x \to \delta \sin x^{-1}$ *which is additionally defined to be zero at zero . This function is discontinuous at zero and microcontinuous.*

◁ It suffices to remark that $\sin x \in {}^{*}\mathbf{R}$ for $x \in \mathbf{R}$ and refer to the properties of infinitely small numbers in **2.2.8**. ▷

2.3.12. Nonstandard criterion of uniform continuity. *For a standard numerical function f determined on a standard set* $\mathrm{dom}(f)$ *the following statements are valid:*

(1) *f is microcontinuous, i.e., f is microcontinuos at every point of* $\mathrm{dom}(f)$ *or, in symbols,*

$$(\forall x, x' \in \mathrm{dom}(f))\, x' \approx x \to f(x') \approx f(x);$$

(2) *f is uniformly continuous.*

◁ (1) → (2). Let $\varepsilon > 0$ be a standard number, and let $\delta > 0$ be infinitely small. Obviously, for $|x - x'| \le \delta$ we have $x \approx x'$. Therefore,

$$(\forall \varepsilon \in {}^{\circ}\mathrm{R}, \varepsilon > 0)\, (\exists \delta > 0)\, (\forall x, x' \in \mathrm{dom}(f)) \quad |x - x'| \le \delta$$
$$\to |f(x) - f(x')| \le \varepsilon .$$

Applying the transfer principle, we see that f is uniformly continuous.

(2) → (1) By the transfer principle, for every standard $\varepsilon > 0$ and some standard $\delta > 0$ we get $|x - x'| \le \delta \to |f(x) - f(x')| \le \varepsilon$ for all $x, x' \in \mathrm{dom}(f)$. By remarking that $x \approx x' \to |x - x'| \le \delta$, we obtain the required result. ▷

2.3.13. Nonstandard criterion of a derivative. *Let f be a standard function given in a standard neighbourhood of a standard point x of ${}^{\circ}\mathbf{R}$. The following statements are equivalent:*

(1) *f is differentiable at the point x and $f'(x) = t$;*

(2) *for every nonzero infinitely small number h we have:*

$$t = \mathrm{st}((f(x + h) - f(x)) / h).$$

◁ The result required is a direct corollary to **2.3.8.** ▷

2.3.14. *Let* y *be a standard function given in a neighbourhood of a standard point* x *and differentiable at this point. Let, then,* dx *be an arbitrary nonzero infinitesimal. Let us denote (following G.W.Leibniz) by the symbol* dy *the differential of the function* y *at the point* x *applied to the element* dx. *In this case*

$$dy \approx y(x+dx) - y(x), \quad \frac{dy}{dx} = y'(x).$$

◁ By the Leibniz definition, making use of **2.3.9**, we get:

$$dy = y'(x)dx, \quad y'(x) = \operatorname{st}\left(\frac{y(x+dx) - y(x)}{dx}\right).$$

Hence,

$$dy \approx \frac{y(x+dx) - y(x)}{dx} dx = y(x+dx) - y(x),$$

which proves the first part of the statement. The second part follows from **2.3.10.** ▷

2.3.15. The nonstandard considerations of the role of infinitesimals when determining derivatives, differentials and increments given in **2.3.13** and **2.3.14** complement the following statement by L.Euler.

"I have already remarked that in differential calculus the problem of finding differentials should be understood in relative rather than absolute sense; this means that if y is a function of x, then it is not its differential but rather its relation to the differential dx that should be determined . Indeed, since all differentials are exactly equal to zero, then, whatever the function y of the quantity x might be, dy is always equal to zero; therefore, in the absolute sense there is nothing here distinct than that to be sought for. The correct formulation of the problem is as follows: x gets an increasingly small, i.e., vanishing [=evanescent, the actual number that 'is exactly zero'] increment dx; the task is to determine what is the relation to dx of the increment that the function y finally gets. Though both increments are zero, there is a certain relation between them which is found out duely by differential calculus. Since if $y = x^2$, then, as is proved in differential calculus, $\frac{dy}{dx} = 2x$, and this ratio of the increments is valid only if the increment dx which dy is generated by, is considered to be equal zero. Nevertheless, after this warning on the true notion of differential has been made, it is allowed to make use of conventional expressions which treat the differential as in the absolute sense, provided, though, the truth is constantly borne in mind. For instance, we have the right to say: if $y = x^2$, then $dy = 2xdx$. In fact, if somebody said that $dy = 3xdx$ or that $dy = 4xdx$, then it would also be true, since even these equalities are valid as $dx = 0$ and $dy = 0$. Only the first equality, however, agrees with the true ratio $\frac{dy}{dx} = 2x$" [49, p.9].

It would be useful to remark that L.Euler used the sign '=' at the place where we write '$\approx$' (see **2.2.10**). Moreover, it should be emphasized that he was seeking for the differential that was considered as existing, while working with concrete (differentiable) functions. Therefore, it would be quite legitimate to use for finding the differential any infinitely small dx chosen in any way,.

Therefore, L.Euler had no doubts that the differential dy (calculated for an infinitely small dx) "is exactly zero", the differential dy is exactly the increment, i.e., "the absolute differential", and at the same time the differential dy is "the fourth proportional" for infinitely small increments, i.e., in up-to-date notation:

$$
{}^{\circ}dy = 0, \quad {}^{\circ}(dy - (y(x + dx) - y(x)) = 0;
$$
$$
{}^{\circ}\left(\frac{dy}{dx} - \frac{y(x + dx) - y(x)}{dx}\right) = 0.
$$

The analysis carried out demonstrates the soundness of the ideas and techniques used by L.Euler when working with explicitly set, i.e., standard, objects. Repeating the aphoristic expression by F.Engels, one can say that "... dx is infinitely small but is, nonetheless, active and productive of everything" [44, p.580].

In the limelight of the above-said the following statements by R.Courant should be treated at a certain critical angle: "...if we would like to understand the essence of differential calculus, we should beware of viewing a derivative as a ratio of two actually existing (actual) 'infinitely small values'. The point is that first we must always form a ratio of the increments $\Delta y / \Delta x$, where the difference Δx is not equal to zero. Then we should imagine that either by way of transforming this ratio, or by some other way passage to the limit has been accomplished. But in no case you should imagine that *at first* there is a transition from Δx to an infinitely small value dx, which is in any case is other than zero, and from Δy to dy, followed by dividing these 'infinitely small values' one by the other. Such a viewpoint on the derivative is absolutely incompatible with the requirement of the mathematical clarity of the notions, and it has hardly any sense at all" [31, p.126-128]. The excessive rigidity of the last phrase is but partially smoothed down by the clarification that follows: "A physicist, a biologist, a technician or any other specialist who has to deal with these notions in practice, has, therefore, the right to identify, within the accuracy required, the derivative with the ratio of the increments...

...'physically infinitely small' values have an exact sense. They are, undoubtedly, finite and distinct from zero values, chosen though to be sufficiently small in the problem under discussion, less, for instance, than a certain portion of the wavelength, or smaller than the distance between two electrons in an atom, etc., generally speaking, less than a certain required degree of accuracy" [31, p.135].

2.3.16. Nonstandard presentation of the Riemann integral. *Let $f:[a,b]\to \mathbf{R}$ be a standard continuous function, and let $a = x_1 < x_2 < \ldots < x_N < x_{N+1} = b$ be a partition of $[a,b]$, such that $\xi_k \in [x_k, x_{k+1}]$ and $x_k \approx x_{k+1}$ for $k := 1,\ldots,N$. In this case the following equality is valid:*

$$\int_a^b f(x)dx = {}^{\circ}\left(\sum_{k=1}^{N} f(\xi_k)(x_{k+1} - x_k)\right).$$

◁ It should be first of all observed that N is infinitely large, and use should be made of the definition of the integral, as well as of the nonstandard criteria on **2.3.1** and on that of uniform continuity **2.3.12.** ▷

2.3.17. Basic principle of integral calculus. "*...When calculating the sum of an infinitely large number of infinitely small addends (of the same sign) it is possible to subtract from every addent a higher-order infinitesimal.*"

◁ Consider a sum ${}^{\circ}\sum_{k=1}^{N} \alpha_k = t$, where $\alpha_k \approx 0$. By condition, we have $\beta_k := \alpha_k - o(\alpha_k)$. By **2.2.13** (2), we deduce $\beta_k \sim \alpha_k$, and, hence,

$$t = {}^{\circ}\left(\sum_{k=1}^{N} \alpha_k\right) = {}^{\circ}\left(\sum_{k=1}^{N} \beta_k\right). \ ▷$$

2.3.18. The above statements give a formal foundation of presenting the integral as a finite sum of infinitely small elements, i.e., it justifies the position of viewing integration as a specific process of summation, which dates back to the ancient time. In this respect it would be useful to quote here the following definition of the integral ('with a variable upper limit') suggested by L.Euler:

"Integration is usually defined in the following way. It is said to be a summation of all the values of the differential expression Xdx provided the variable x sequentially assumes all the values that differ by the difference dx, starting from a certain given value up to x, this difference being considered infinitely small... . From the method presented it is, in any case, clear that integration can be obtained, within any accuracy, from summation; it is also obvious that integration can be accomplished in no other way but by setting these differences to be infinitesimals, i.e., zeros" [49, p.163].

It is worthwhile again to emphasize that in order to find the integral of a standard continuous function one should, by virtue of the facts presented above, calculate the exact value (= the standard part) of only one finite sum of an infinitely large number of infinitely small addends, in which the higher-order small values can be neglected. For nonstandard functions this technique does not work in general. In other words, we again discover, as has been repeatedly the case in the material presented above, that nonstandard ideas on the objects of mathematical analysis are supplementing, refining and developing (but in no case abolishing) their classical analogues.

2.3.19. All these facts manifest the nonstandard analysis in its contemporary forms to be a direct descendent of calculus of infinitesimals. That is why the term 'infinitesimal analysis' has been lately gaining in popularity; it is more exact in presenting the essence of the matter than a somewhat extravagant term 'nonstandard analysis' which often causes an irritation quite understandable in the long run.

Special attention should be paid to the fact that the concept of actually infinitely large and infinitely small quantities has never been abandoned as a working tool in natural sciences but was just absent from mathematics for nearly thirty years. This makes it possible for us not to go into details as regards the importance of nonstandard methods.

CHAPTER 3

SET-THEORETIC FORMALISMS OF NONSTANDARD ANALYSIS

The 'naive'-level discussion of differences between the standard, i.e., realizable, and nonstandard, i.e., indirect, methods of introducing objects conducted in Chapter **2**, has enriched our notions of actually infinitely large and actually infinitely small numbers with a sense agreeable with intuition. A remarkable acquisition is that the understanding of the methods of consideration adopted when formulating mathematical analysis has been extended. At the same time, we face serious complications even in simple examples. First of all, the method of distinguishing standard objects from nonstandard ones is still open to argument, which makes us admit the possibility of incorrectly applying the principles of nonstandard analysis. Growing alarm has been caused by the appearance of objects formed with mathematical constructions which, at first sight, seem to be quite legitimite but cannot unconditionally be given the status of the 'naive' sets. Such are, for instance, all kinds of monads, families of standard elements, and objects like $\approx$, $^{\sim}\mathbf{R}, O, o$, etc.. Still more unpleasant is the fact that the 'mathematical law' $x \to {}^{\circ}x$, acting from $\overline{\mathbf{R}}$ to $\overline{\mathbf{R}}$, is not a function. The point is that the notion of function had been formed in mathematics long before the set-theoretic stance was developed. Thus, as far back as in 1775 L.Euler wrote: "When certain quantities depend on other ones in such a way that when the latter are changed the former are also subject to changes, then the former are called functions of the latter. The application of this term is extremely wide; it embraces all the methods by which one quantity can be determined with the help of others. Therefore, ... all quantities which are dependent on x in one or another way, i.e., determined by x, are called functions of x" [48, p.38]. The dynamic idea of transforming some objects into other ones is not embraced completely by the 'stationary' set theoretical view of a function as a set reigning now. This presentation is a "formal set-theoretic model of the intuitive idea of a function, *a model*, which embraces only one aspect of the idea but not its importance as a whole" [63, p.32]. It should be recalled in this respect that for $s,t \in [0,1]$ we have

$$^{\circ}(s+t) = {}^{\circ}s + {}^{\circ}t, \quad {}^{\circ}0 = 0, \quad {}^{\circ}1 = 1,$$

and, moreover, ${}^{\circ}t = 0$ in a certain subinterval $t \in [0,h]$, where h is a strictly positive number (any actually infinitely small number). The presence of such a 'numerical' function is a sure sign of a contradiction or, to put it delicately, implies the presence of antinomes.

All these circumstances require immediate and implicit elucidation of the concepts and means we are using, outlining the grounds which they are built on.

As has been earlier remarked, nonstandard analysis is substantiated within the set-theoretic stance. To be more exact, it appears that the ideas of the 'naive' nonstandard set theory developed above can be placed on the same solid foundation on which the Cantor theory or, strictly speaking, axiomatic set theories on the same foundation, rest.

In order to bring into focus the relations between mathematical analysis and set theory, the following statements are worth comparing.

> "... analysis ... is the science of infinity by itself"
>
> (G.W.Leibniz),

> "Leibniz is the founder of mathematics of the infinite"
>
> (F.Engels),

> "... mathematical analysis is just the science of the infinite. This old definition has been living through ages ..."
>
> (N.N.Luzin),

> "SETS, THEORY OF, a branch of mathematics studying general properties of sets, basically of infinite ones"
>
> (The Soviet Encyclopedic Dictionary).

The very notion, 'the infinity', is seen to be tightly associated with set theory. One should not forget, however, that the classical works by G.Cantor appeared two centuries after the invention of mathematical analysis. The laying of the set-theoretic foundation under mathematics could be compared with a modern method of erecting buildings, rack mounting, when a house is assembled starting with upper stores, 'from attic to cellar'. Of interest is the fact that in this case the foundation of the building is laid beforehand. The initial foundation of mathematical analysis was likewise laid by practical activities of people.

The present day mathematics is essentially resting on set theory, or, to be more exact, the set-theoretic foundation has been floated under the main stores of modern mathematics. What is going to happen next will be shown by the future. By now we can only state that the process of erecting the building of the future mathematics is going on, and that this process is fraught with dramatic changes. This accelerated development has been revealing itself through aggravation of the situation, collision of opinions, a fierce struggle of ideas. The set of citations to follow (far from claiming for completeness) will illustrate the process of polarization now in progress.

Pro:

"After the initial period of incredulity there began a triumphal march of the created theory of sets in all branches of mathematics. Its influence on the mathematics of our century manifests itself both in the choice of modern problems and in the methods of their solution. Set theory is being applied everywhere."

K.Kuratowski and A.Mostowski, [112, p. 7]

"One of the creations of George Cantor is the theory of sets, some of whose elements are nowadays taught in senior grades of high schools, and even earlier. It is one more field of mathematics which used to be thought of as having no practical application whatsoever. What a fallacy! Elements of set theory are now being used even by the authors of detective novels. Well known is the use of set theory in making programs for computers, the latter servicing an innumerable host of practical projects."

L.Young [276, p.154]

Contra:

"... it is claimed that the theory of sets is important for scientific-technological progress and is one of the newest achievements in mathematics. In fact the theory of sets has got nothing to do with scientific-technological progress and is not one of the newest achievements of mathematics."

L.S.Pontryagin [212, p.6]

"The mathematics based on Cantor set theory has turned into that of Cantor set theory... . Therefore, modern mathematics is studying a construction whose relation to the real world is at least problematic... . This questions the validity of mathematics as a scientific and useful method. The role of mathematics can be reduced to a simple game played in a certain specific artificial world. This is not a danger facing mathematics in the future but an outright crisis of modern mathematics".

P.Vopenka [268. p.14]

By way of concluding the preliminary discussion it should be emphasized that only now, having dispelled the illusion that it is possible to provide some final 'absolute' foundation of nonstandard analysis (as well as the whole of mathematics) by set-theoretic stance, we can get down to implementing this project.

3.1. The Language of Set Theory

Axiomatic set theories regulate legitimate ways of set formation in exact terms. Figuratively speaking, axiomatics describe worlds, i.e., universes, of sets which are to serve adequate presentations of our intuitive ideas about the 'Cantor paradise', the universe of the naive theory of sets. The axiomatics of interest for us are built and studied as formal theories. It should be specially emphasized that, despite obvious limitations (mathematics cannot be reduced to the syntax of its texts) and in many respects thanks to them (singling out semiotic aspects clarifies the problem of meaning), the formal approach has proved ito be exclusively fruitful (the Gödel theorem, independence of the continuum-hypothesis and of the axiom of choice, Boolean-valued analysis, etc.).

The cornerstone of a formal theory is its language. The exact description and study of such a language is, if required, carried out by means of a certain, generally speaking, different language which is usually called a metalanguage. A metalanguage is commonly a collection of fragments of natural languages limited and regulated in a certain way and enriched with various technical terms. The means allowed in a metalanguage are important from the mathematical point of view. Taking into account the fact that we are interested not in mathematical but in applied set-theoretic aspects of formal set theory, we do not impose extremely rigid constraints on the metalanguage. In particular, conventional expressive means and the level of rigour employed by conventional meaningful mathematics will be widely used further on.

3.1.1. Any axiomatic set theory is a *formal system*. The components of any system of the kind are its alphabet, formulas, axioms and rules of inference. As an alphabet, a fixed set A of symbols of arbitrary nature, i.e., a Cantor set, taken. Finite sequences of elements of A are called expressions and, sometimes, texts. If in some way (by prescriptions, algorithms, etc.) a certain set of 'well-formed' expressions $\Phi(A)$ has been chosen, then we say that a language with alphabet A is given, the chosen expressions called formulas. After that fixed are certain finite (or infinite) families of formulas called axioms, and implicitly described are the admissible rules of inference, i.e., relations in $\Phi(A)$. Formulas obtained from axioms in a finite number of steps by rules of inference are called theorems. There is a more free and convenient way of expression which is often used (we will do the same). Namely, they say that the theorems of a formal system comprise the least set of formulas which contains all the axioms and is closed relative to the rules of inference.

3.1.2. Of interest for us will be a special type of formal language, i.e., *a first-order language* (of the predicate logic) (with equality). The signature σ is a triplet (F,P,a), where F and P are some sets called the set of operation symbols and that of predicate symbols, respectively, while a is a mapping of $F \cup P$ into a set of natural numbers. They say that $u \in F \cup P$ is an *n-ary symbol* provided $a(u) = n$. The alphabet of a first-order language of signature σ consists of the following terms:

(1) *a set of symbols* of signature σ, i.e., the set $F \cup P$;

(**2**) *a set of variables*: small or capital Latin letters; possibly, with indices;

(**3**) *propositional connectives*: $\wedge$ is conjunction, $\vee$ is disjunction, $\rightarrow$ is implication, $\neg$ is negation;

(**4**) *quantifiers*: $\forall$ is the universal quantifier, $\exists$ is the existential quantifier;

(**5**) *equality sign*, $=$;

(**6**) *auxiliary symbols*: (is the opening parenthesis,) is the closing parenthesis, , is a comma.

3.1.3.

(1) *A term* of signature σ is an element of the least set of expressions of the language (of the same signature) obeying the following conditions:

(**a**) any variable is a term;
(**b**) any nullary operation symbol is a term;
(**c**) if $f \in F, a(f) = n$ and $t_1,\ldots,t_n$ are terms, then $f(t_1,\ldots,t_n)$ is a term.

(**2**) *Atomic (= atom) formulas* of signature σ are all expressions of the kind

$$t_1 = t_2, \quad p(y_1,\ldots,y_n), q,$$

where $t_1, t_2, y_1,\ldots,y_n$ are terms of signature σ, p is a certain n-ary predicate symbol, and q is a nullary predicate symbol.

(3) *Formulas of signature* σ constitute the least set of expressions obeying the following conditions:

(**a**) the atomic formulas of signature σ are formulas of signature σ;
(**b**) if φ and ψ are formulas of the signature σ, then $(\varphi \wedge \psi), (\varphi \vee \psi), (\varphi \rightarrow \psi), \neg\varphi$ are formulas of signature σ, too;
(**c**) if φ is a formula of signature σ, and x is a variable, then $(\forall x)\varphi$ and $(\exists x)\varphi$ are formulas of signature σ, too

The occurrence of the variable x in the formula φ is *bound* in φ, or belongs to the domain of a quantifier, provided x is incorporated into a subformula of φ of the kind $(\forall x)\varphi$ or $(\exists x)\varphi$. In the opposite case the occurrence of x into φ is free. They say that x *is free* (bound) in φ if all occurrences of x in φ are free. When we like to stress that only the variables $x_1,\ldots,x_n$ are free in the formula φ, we write $\varphi = \varphi(x_1,\ldots,x_n)$, or simply $\varphi(x_1,\ldots,x_n)$. A formula with no free variables is called a *sentence*.

3.1.4. *The language of set theory* is a first-order language whose signature contains but one binary predicate symbol $\in$ and has neither other predicate nor functional symbols. We usually write $x \in y$ instead of $\in(x,y)$, and say that x is an *element of* y. Therefore, the formulas of set theory are formal texts composed of atomic formulas $x \in y$ and $x = y$ by way of propositional connectives and quantifiers.

Set theory is built on the basis of the laws of classical logics. In other words, it accepts conventional logical axioms and rules of inference of the propositional calculus with equality, which can be found in nearly every manual on mathematical logic (see, for instance, [27, 45, 232]). Besides, accepted are some nonlogical and special axioms reflecting the adopted presentation of sets and classes. When being varied within reasonable limits, special axioms result in axiomatic systems for set theory, different in their expressive forces. In the present chapter we shall describe three systems of the kind: the set theory of Zermelo-Fraenkel, the theory of internal sets by Nelson, and the theory of external sets by Kawai-Hrbacek. The theory of classes by von Neumann-Gödel-Bernays will be presented for discussion in the first chapter of the second part of the present monograph.

3.1.5. One of the most important functions of a metalanguage is the introduction of new abbreviating symbols and the establishment of new corresponding syntax. The point is that formalization of even simple fragments of meaningful mathematics results in cumbersome texts, the writing and reading of which are problematic by both physical and psychological reasons. That is why we have to introduce a great number of abbreviations and, in fact, build a more convenient abridged variation of the initial symbolic language. A necessary requirement is in this case the principal possibility of a one-to-one translation of a reduced presentation into a formalized language. In accordance with our intentions, we will not expatiate on the ways of introducing reductions, exact descriptions, functional expressions, etc.. For instance, henceforth, as before, we use the term *the assignment operator* or *definor* $:=$, without going into corresponding subtleties.

3.1.6. We will give some examples of abbreviating formal texts in set theoretic language, semantic expressions of such texts appealing to intuitive naive presentations of sets. First of all we recall the following conventional abbreviations:

$$(\exists!\, x)\ \varphi(x) := (\exists x)\ \varphi(x) \wedge (\forall x)(\forall y)(\varphi(x) \wedge \varphi(y) \to x = y);$$
$$(\exists x) \in y)\ \varphi := (\exists x)\,(x \in y \wedge \varphi);$$
$$(\forall x) \in y)\ \varphi := (\forall x)\,(x \in y \to \varphi),$$

where φ is a formula. We also set $x \neq y := \neg\ (x = y)$ and $x \notin y := \neg\ (x \in y)$. For the simplest set-theoretic operations the following conventional abbreviation are adopted:

$$x \subset y := (\forall z)\ (z \in x \rightarrow z \in y);$$
$$u = \cup x = \cup(x) := (\forall z)\ (z \in u \leftrightarrow (\exists y \in x)\ z \in y);$$
$$u = \cap x = \cap(x) := (\forall z)\ (z \in u \leftrightarrow (\forall y \in x)\ z \in y);$$
$$u = y - x = y \setminus x := (\forall z)\ (z \in u \leftrightarrow (z \in y \wedge z \notin x)).$$

If φ is a formula, then a family $\mathcal{P}_\varphi(x)$ of all x subsets obeying the condition φ is described by the expression

$$u = \mathcal{P}_\varphi(x) := (\forall z)\ (z \in u \leftrightarrow (z \subset x) \wedge (\varphi(z)).$$

The empty set $\varnothing$ contains no elements, so

$$u = \varnothing := (\forall x)\ (x \in u \leftrightarrow x \neq x).$$

In the texts presented above use has been made of one of the most wide-spread methods of abbreviation, the removal of part of the parentheses.

3.1.7. The statement that x is an *unordered pair* of elements y and z is formalized as follows:

$$(\forall u)\ (u \in x \leftrightarrow u = y \vee u = z).$$

In this case we set $\{y, z\} := x$. It should be remarked that braces do not belong to the initial alphabet and thus they are metasymbols.

An ordered pair and an ordered n-tuple are introduced by the Kuratowski trick:

$$(x, y) := \langle x, y \rangle := \{x\{x\}, \{x, y\}\};$$
$$(x_1, \ldots, x_n) := \langle x_1, \ldots, x_n \rangle := \langle\langle x_1, \ldots, x_{n-1} \rangle, x_n \rangle,$$

where $\{x\} := \{x, x\}$. The overuse of round brackets is inevitable and should not be regarded as a pretext for introducing new symbols.

Using the agreements made, a formalized sense can be attributed to the expression 'X is the Cartesian product of $Y \times Z$'. Namely, according to the definition, we assume that $X := \{(y, z) : y \in Y, z \in Z\}$.

3.1.8. Let us consider the following statement:

(1) $\mathrm{Rel}(X)$, i.e., X is *a correspondence* (= *relation*);

(2) $Y = \mathrm{dom}(X)$;

(3) $Z = \mathrm{im}(X)$.

Therefore, in (1) - (3) we state that elements of X are ordered pairs, Y being the domain of definition of X, Z being the range of values or the image of X.

The corresponding formal texts are as follows:

(1) $(\forall u)(u \in X \rightarrow (\exists v)(\exists \omega)\, u = (v,\omega)$;

(2) $(\forall u)(u \in Y \leftrightarrow (\exists v)(\exists \omega)\, \omega = (u,v) \wedge \omega \in X)$;

(3) $(\forall u)((u \in Z \leftrightarrow (\exists v)(\exists \omega)\, \omega = (v,u) \wedge \omega \in X)$.

The functionality of X, or $\mathrm{Un}(X)$, is expressed by the formula

$$\mathrm{Un}(X) := (\forall u)(\forall v_1)(\forall v_2)((u,v_1) \in X \wedge (u,v_2) \in X \rightarrow v_1 = v_2).$$

A single-valued relation is termed *functional.* We set $\mathrm{Fnc}(X) := \mathrm{Func}(X) := \mathrm{Un}(x) \wedge \mathrm{Rel}(X)$. In this case, in order to express $(u,v) \in X$, we write $v = X(u)$, $X: u \rightarrow v$, etc.. Then the phrase F *is a mapping* (=*function*) from X in Y implies that F is a subclass of $X \times Y$, F is functional, and the domain of F coincides with X:

$$F: X \rightarrow Y := F \subset X \times Y \wedge \mathrm{Func}(F) \wedge \mathrm{dom}(F) = X.$$

The *restriction* of X to U is, by the definition, $X \cap (U \times Z)$, and is denoted by $X|U$.

If there is, and the only one, Z for which $(Y,Z) \in Z$, then we set $X'Y := Z$. In all the remaining cases we set $X'Y := \varnothing$. And, finally, by definition, $X''Y := \mathrm{im}(X|Y)$. Instead of $X''\{z\}$ we write $X(x)$, or even Xx if it does not result in misunderstanding. It is worth emphasizing that henceforth we adopt a free point of view on placing and removing parentheses. In other words, both their introduction and elimination are as a rule governed by considerations of convenience and by requirements on the level of formalization of the fragment of the text under discussion.

3.1.9. *The superposition* (or *composition*) of relations X and Y, denoted by the symbol $Y \circ X$, consists exactly of ordered pairs (z,x) such that $(x,y) \in X$ and $(y,z) \in Y$ for a certain y:

$$(\forall u)\,(u \in Y \circ X \leftrightarrow (\exists x)\,(\exists y)\,(\exists z) \quad (x,y) \in X \wedge (y,z) \in Y \wedge u = (x,z))\,.$$

The relation X^{-1}, inverse to X, is defined as

$$(\forall u) \quad (u \in X^{-1} \leftrightarrow (\exists x)\,(\exists y) \quad (x,y) \in X \wedge u = (y,z))\,.$$

The symbol I_X denotes *the identity relation* on X, i.e.,

$$(\forall u) \quad (u \in I_X \leftrightarrow (\exists x) \quad (x \in X \wedge u = (x,x)))\,.$$

3.1.10. When $\mathrm{Rel}(X) \wedge ((X \cap Y^2) \circ (X \cap Y^2) \subset X)$, we say that X is a *transitive* relation on Y. This X is called *reflexive* (over Y) if $\mathrm{Rel}(X) \wedge (I_Y \subset X)$. And, finally, if $\mathrm{Rel}(X) \wedge ((X \cap X^{-1}) \cap Y^2 \subset I_Y)$ the term "X is an *antisymmetric* relation on Y' is used. Here, of course, a standard abbreviation $Y^2 := Y \times Y$ is used. An antisymmetric, reflexive and transitive relation X on Y is termed an *order* (or order relation). Another terminology standard under these conditions is also used. It should, in particular, be recalled that an order X on Y is called *linear*, while Y itself is called a *chain* (relative to X) provided $Y^2 \subset X \cup X^{-1}$. If any nonempty subset of the set Y has the least (relative to the order of X) element, then we say that X *well-orders* Y, or that Y *is well-ordered* (by the understood order of X).

3.1.11. Quantifiers are termed *restricted* or *bounded*, if they appear in the text as $(\forall x \in y)$ or $(\exists x \in y)$. There is a classification of the formulas of set theory (and, generally speaking, of any first-order theory) based on the way of using restricted or unrestricted (i.e., those not restricted) quantifiers. In our further discussions of special importance will be two classes of formulas: restricted or bounded formulas (= Σ_0-formulas) and Σ_1-formulas. We say that the formula φ is *restricted* if any quantifier encountered in it either has the form $(\forall x \in y)$ or the form $(\exists x \in y)$ (see abbreviations **3.1.6**). The formula φ is of class Σ_1 and is called an Σ_1-*formula*, if it is constructed from atomic formulas and their negations, using only logical operations $\wedge$, $\vee$, $(\forall x \in y)$ and $(\exists x)$. Clearly, every restricted formula belongs to the class Σ_1. Not every Σ_1-formula is, however, restricted, and there are formulas not belonging to the class Σ_1. Let us consider corresponding examples, starting with restricted formulas.

3.1.12. The proposition $z = \{x, y\}$ is equivalent to the restricted formula

$$x \in z \wedge y \in z \wedge (\forall u \in z) \quad (u = x \vee u = y)\,.$$

An ordered pair is seen from here to be introduced by a restricted formula. The same is true as regards a Cartesian product, since $Z = X \times Y$ can be written as

$$(\forall z \in Z)(\exists x \in X)(\exists y \in Y) \quad (z = (x,y))$$
$$\wedge (\forall z \in X)(\forall y \in Y)((\exists z \in Z) \quad (z = (x,y)).$$

One more restricted formula results from the notion 'a mapping F from X to Y' (see **3.1.8**). Indeed, it follows from the above that $F \subset X \times Y$ is a restricted formula, and that, moreover, the expressions $\mathrm{dom}(F) = X$ and $\mathrm{Un}(F)$, which are equivalent to the respective formulas

$$(\forall x \in X)(\exists y \in Y)(\exists z \in F) \quad z = (x,y),$$
$$(\forall z_1 \in F)(\forall z_2 \in F)(\forall x \in X)(\forall y_1 \in Y)(\forall y_2 \in Y)$$
$$z_1 = (x,y_1) \wedge z_2 = (x,y_2) \rightarrow y_1 = y_2,$$

are restricted, too.

3.1.13. The statement 'the sets x and y *are equipotent* or have the same cardinality or, in other words, 'there is a bijection between x and y', or, symbolically, $x \simeq y$, is presented by the following $\sum_1$-formula:

$$(\exists f)(f\colon x \rightarrow y \wedge \mathrm{im}(f) = y \wedge \mathrm{Un}(f^{-1})).$$

This peculiarity is not, however, expressed by a restricted formula. One more $\sum_1$-formula can be given by the notion of a relation:

$$\mathrm{Rel}(X) := (\forall u \in X)(\exists v)(\exists \omega) \quad u = (v,\omega).$$

The following formula, stating that the set y is equal to none of its elements, is not of class Σ_1:

$$(\forall x \in y) \quad \neg(x \simeq y).$$

Remarks.

(1) It goes without saying that not only special axioms of a first-order theory (see **3.1.4**) can be varied but also its logical part, i.e., logical axioms and rules of inference. A great number of theorems obtained in this way can essentially differ from each other. Thus, for instance, eliminating the law of the excluded middle from the axioms of propositional calculus, we obtain the intuitionistic calculus of propositions. The intuitive calculus of predicates (see [63,97]) is constructed in an analogous way.

(2) The modern formal logic has been with great difficulties formed in the course of the evolution of philosophical and mathematical thought. The classical calculus of predicates stems from the Aristotle syllogistic, while the origin of the intuitionistic logic is related to other philosophical ideas. Logical systems essentially different from both systems mentioned were created in various periods of times for various purposes. Thus, the ancient Indian logic had three types of negation: something has never been and cannot be, something has been but is now absent, something that is now but will soon disappear.

(3) As is seen from **3.1.6** and **3.1.7**, abbreviations can be encountered in formulas, in abbreviations, in abbreviations of abbreviations, etc.. The invention and introduction of abbreviating symbols is an art of its own, and as any art, it cannot be completely formalized. Nevertheless, systematization and codification of the rules of determining abbreviations is necessary from both theoretical and practical points of view. Some of such systems of rules (exact descriptions, introduction of functional letters, etc.) can be found in literature [26, 27, 60].

3.2. Zermelo-Fraenkel Set Theory

As has been noted in **3.1.4**, the axioms of set theory include the general-logical axioms of first-order theories with equality, which fix the classical rules of logical inference. Below we shall enumerate special axioms of set theory ZF_1-ZF_6 and AC. If ZF_1-ZF_6 are assumed as special axioms, then the arising axiomatic system is called the Zermelo-Fraenkel set theorem (or system) and is denoted by ZF. When ZF is modified with the axiom of choice AC then there arises a wider theory which is conventionally denoted by ZFC. It should be remarked that the parallel verbal formulations of the axioms are motivated by the Cantor ideas on sets.

3.2.1. In studying ZFC the terms *'a property'* and *'a class'* are often used. Let us elucidate their formal status. Consider a formula $\varphi = \varphi(\mathrm{x})$ of ZFC (in symbols, $\varphi \in$ (ZFC)). Instead of the text $\varphi(y)$ we write $y \in \{x : \varphi(x)\}$. Therefore, the so-called *Church scheme* for classification works:

$$y \in \{x\colon \varphi(x)\} := \varphi(y).$$

In the language of ZFC the expression $y \in \{x\colon \varphi(x)\}$ implies that y has the property φ, or that y lies in the class $\{x: \varphi(x)\}$. In this sense the property, the formula and the class mean the same in ZFC. The Church schema has been already practically used in **3.1.6** and **3.1.7**. When working with ZFC, it would be convenient for us to make use of many widely-spread abbreviations and, in particular, of the following:

$\mathrm{V} := \{x: x = x\}$ is the universe of discourse, or the class of all sets;

$\{x: \varphi(x)\} \in \mathrm{V} := (\exists z)\ (\forall y)\ \varphi(y) \leftrightarrow y \in z$;

$x \cup y := \cup\{x, y\}, \quad x \cap y \cap z := \cap\{x,y,z\} \dots$.

Let us now go over to formulating special axioms of ZFC.

3.2.2. The Axiom of Extensionality, ZF_1: *two sets are equal iff they consist of the same elements:*

$$(\forall x)(\forall y)(\forall z) \quad (z \in x \leftrightarrow x \in y) \leftrightarrow x = y.$$

It should be remarked that the last equivalence can be replaced by $\rightarrow$ without loss of scope, since the reverse implication is the theorem of predicate calculus with equality.

3.2.3. The Axiom of Union, ZF_2: *a union of a set of sets is also a set:*

$$(\forall x)(\exists y)(\forall z)(\exists u)\ (u \in z \wedge z \in x) \leftrightarrow z \in y.$$

Using the abbreviations of **3.1.6** and **3.2.1**, the axiom ZF_2 can be presented as

$$(\forall x) \ \cup x \in \mathrm{V}.$$

3.2.4. The Axiom of Powers (of Powersets), ZF_3: *all subsets of a given set constitute a certain set, i.e.,*

$$(\forall x)(\exists y)(\forall z)\ (z \in y \leftrightarrow (\forall u)\ (u \in z \rightarrow u \in x)),$$

or, in an abbreviated form,

$$(\forall x)\ \mathcal{P}(x) \in \mathrm{V}.$$

3.2.5. The Axiom of Replacement, ZF_4^{φ} : *an arbitrary one-to-one image of a set is a set, too*:

$$(\forall x)(\forall y)(\forall z)\,(\varphi(x,y)\wedge\varphi(x,z)\rightarrow y=z)\rightarrow$$
$$\rightarrow(\forall a)(\exists b)\,((\exists s\in x)(\exists t)\,\varphi(s,t)\leftrightarrow t\in y).$$

And now in a contracted form:

$$(\forall z)(\forall y)(\forall z)\ \varphi(x,y)\wedge\varphi(x,z)\rightarrow y=z)\rightarrow$$
$$\rightarrow(\forall a)\,(\{v\colon(\exists u\in a)\,\varphi(u,v)\}\in \mathrm{V}).$$

Here φ is a formula of ZFC containing no free occurrences of a. It should be remarked that ZF_4^{φ} is a schema for an infinite set of axioms, since for any appropriate $\varphi\in$(ZFC) its own axiom is formed. Nevertheless, for the sake of brevity and uniformity, we speak about the axiom of replacement, bearing in mind the peculiarity mentioned above.

Let us now formulate some useful corollaries to ZF_4^{φ}.

3.2.6. Let $\psi=\psi(z)$ be a formula of ZFC. Given any set x, we can compose its subset, by choosing the elements of x with the property ψ, i.e.,

$$(\forall x)\,\{z\in x\colon\psi(z)\}\in\mathbf{V}.$$

This statement is the axiom ZF_4^{φ}, where the formula $\psi(u)\wedge(u=v)$ is used instead of φ. The situation under discussion is often called *the axiom of separation*, or *comprehension.*

3.2.7. Applying the axiom ZF_4^{φ} to the formula

$$\varphi(u,v):=(u=\varnothing\rightarrow v=x)\wedge(u\neq\varnothing\rightarrow v=y)$$

of a set $z:=\mathcal{P}(\mathcal{P}(\varnothing))$, we make sure that the unordered pair $\{x,y\}$ of two sets (cf. **3.1.7**) is also a set. The preceding statement is often referred to as *the axiom of pairing.*

3.2.8. The Axiom of Infinity, ZF_5: *there is at least one infinite set*:

$$(\exists x)\,(\varnothing\in x\wedge(\forall y)\,(y\in x\rightarrow y\cup\{y\}\in x)).$$

Therefore, there is such a set x, that $\varnothing \in x$, $\{\varnothing\} \in x$, $\{\varnothing,\{\varnothing\}\} \in x$, $\{\varnothing,\{\varnothing\},\{\varnothing,\{\varnothing\}\}\} \in x$, etc..

3.2.9. The Axiom of Foundation (of Regularity), ZF_6: *any nonempty set has an element non-intersecting with it*

$$(\forall x)\,(x \neq \varnothing \to (\exists y)(y \in x \wedge y \cap x = \varnothing)).$$

Applying the axiom ZF_6 to a unielement set $x := \{y\}$, we get $y \notin y$. It should be remarked, speaking a bit beforehand, that by an analogous reason (if we take $x := \{x_1, \dots, x_n\}$) there are no infinitely decreasing $\in$-sequences $x_1 \ni x_2 \ni \dots \ni x_n \ni \dots$.

3.2.10. The Axiom of Choice (of Multiplication), AC: *the product of a set of nonempty sets is nonempty:*

$$(\forall x)\,(\exists f)\,(\mathrm{Func}(f) \wedge x \subset \mathrm{dom}(f)) \wedge (\forall y \in x)\; y \neq \varnothing \to f(y) \in y.$$

The function f is, under the circumstances, called a *selector* for x.

There is a great number of mathematical statements equivalent, within the framework of the theory under consideration, to the axiom of choice (see [95]). Let us recall the formulations of the two most popular among them.

Zermelo theorem (the well-ordering principle). *Any set can be well-ordered.*

Kuratowski-Zorn lemma (the maximality principle). *Let M be a (partially) ordered set whose every chain has an upper bound. Then for every $x \in M$ there is a maximal element $m \in M$ such that $m \geq x$.*

3.2.11. On the basis of the axiomatics discussed above we get an exact presentation for the class of all sets as the 'von Neumann universe'. An initial object of construction is the empty set. An elementary step of introducing new sets from those already constructed consists in forming the union of sets of the subsets of available sets. Transfinite repetition of such steps exhausts the class of all sets. Classes (in the 'Platonic' sense) can be viewed as external objects relative to the elements of the von Neumann universe. Within this approach a class is a family of sets obeying a set-theoretic property described by a formula of Zermelo-Fraenkel theory. Therefore, the class consisting of elements of a certain set is (according to the axiom of replacement) also a set. A formally correct definition of the von Neumann universe requires preliminary acquaintance with the notions of ordinal and cumulative hierarchy. Below we give the minimum of information on these objects necessary for a 'naive' definition, a more detailed presentation can be found in the first chapter of the second part of the book.

3.2.12. A set x is called *transitive* if every element of x is a subset of x. A set x is called *an ordinal* if x is transitive and linearly ordered by the relation $\in$. These definitions in symbolic form are as follows:

$$\mathrm{Tr}(x) := (\forall y \in x)\,(y \subset x) := \text{'}x \text{ is a transitive set'};$$
$$\mathrm{Ord}(x) := \mathrm{Tr}(x) \wedge (\forall y \in x)\,(\forall z \in x)$$
$$(y \in z \vee z \in y \vee z = y) := \text{'}x \text{ is an ordinal'}.$$

Ordinals are commonly denoted by lower-case Greek letters. Every ordinal is considered with a natural order: for $\beta, \gamma \in \alpha$ we designate

$$\gamma \leq \beta \leftrightarrow \gamma \in \beta \vee \gamma = \beta.$$

The class of all ordinals is denoted by the symbol On , and, thus, $\mathrm{On} := \{\alpha : \mathrm{Ord}(\alpha)\}$.

The ordinal is a well-ordered set; i.e., it is linearly ordered and its every subset has the least element (which is ensured by the axiom of foundation). We can easily see that

$$\alpha \in \mathrm{On} \wedge \beta \in \mathrm{On} \rightarrow \alpha \in \beta \vee \alpha = \beta \vee \beta \in \alpha;$$
$$a \in \mathrm{On} \wedge \beta \in \alpha \rightarrow \beta \in \mathrm{On};$$
$$a \in \mathrm{On} \rightarrow \alpha \cup \{\alpha\} \in \mathrm{On};$$
$$\mathrm{Ord}(\varnothing).$$

The ordinal $\alpha + 1 := \alpha \cup \{\alpha\}$ is called *the successor of* α. An ordinal which is not equal to zero and not a successor is termed a *limit* ordinal. The following notation is used:

$$K_I := \{\alpha \in \mathrm{On} : (\exists \beta)\ \mathrm{Ord}(\beta) \wedge a = \beta + 1 \vee \alpha = \varnothing\};$$
$$K_{II} := \{\alpha \in \mathrm{On} : \alpha \text{ is a limit ordinal}\};$$
$$0 := \varnothing, \quad 1 := 0 + 1, \quad 2 := 1 + 1, \ldots,$$
$$\omega := \{0, 1, 2, \ldots\}.$$

3.2.13. It should be remarked that within ZFC one can prove the possibility of using well-known (at a 'naive' level) properties of ordinals, and, in particular, the validity of transfinite induction and recursive definitions. Let us define the von Neumann universe, so far purposefully omitting formal validation for such definitions. For every ordinal α,

$$V_\alpha := \bigcup_{\beta < \alpha} \mathcal{P}(V_\beta),$$

i.e., $V_\alpha = \{x : (\exists\beta)(\beta \in \alpha \wedge x \subset V_\beta)\}$. Or, in more detail,

$$V_0 := \varnothing;$$
$$V_{\alpha+1} := \mathcal{P}(V_\alpha);$$
$$V_\beta := \bigcup_{\alpha<\beta} V_\alpha, \text{ provided } \beta \in K_{II}.$$

Of principal importance is the following theorem ensured by the axiom of foundation:

$$(\forall x)(\exists\alpha)(\mathrm{Ord}(\alpha) \wedge x \in V_\alpha),$$

which can be presented in the form

$$\mathbf{V} = \bigcup_{\alpha\in\mathrm{On}} V_\alpha.$$

In words this can be expressed as *'the class of all sets is the von Neumann universe'*, or, using the terminology stemming from Mirimanoff, *'any set is well-founded'*.

Graphically the von Neumann universe can be depicted as shown in Figure **3**, the 'lower' levels of the universe being as follows:

$$V_0 = \varnothing, \quad V_1 = \{\varnothing\}, \quad V_2 = \{\varnothing, \{\varnothing\}\}, \ldots,$$
$$V_\omega = \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}, \ldots\}, \ldots .$$

3.2.14. The representation of the universe $\mathbf{V}$ as the *'cumulative hierarchy'* of sets $(\mathrm{V}_\alpha)_{\alpha\in\mathrm{On}}$ makes it possible to relate rank to any set:

$$\mathrm{rank}\,(x) := \text{a least ordinal } \alpha \text{ such that } x \in V_{\alpha+1}.$$

We can readily prove that

$$a \in b \to \mathrm{rank}\,(a) < \mathrm{rank}\,(b);$$
$$\mathrm{Ord}(\alpha) \to \mathrm{rank}\,(\alpha) = \alpha;$$
$$(\forall x)(\forall y)\ \mathrm{rank}\,(x) \to (\varphi(y) \to \varphi(x)) \to (\forall x)\ \varphi(x),$$

where φ is a formula of ZFC. The preceding theorem (or, more precisely, the schema of theorems) is called *the principle of induction on rank.*

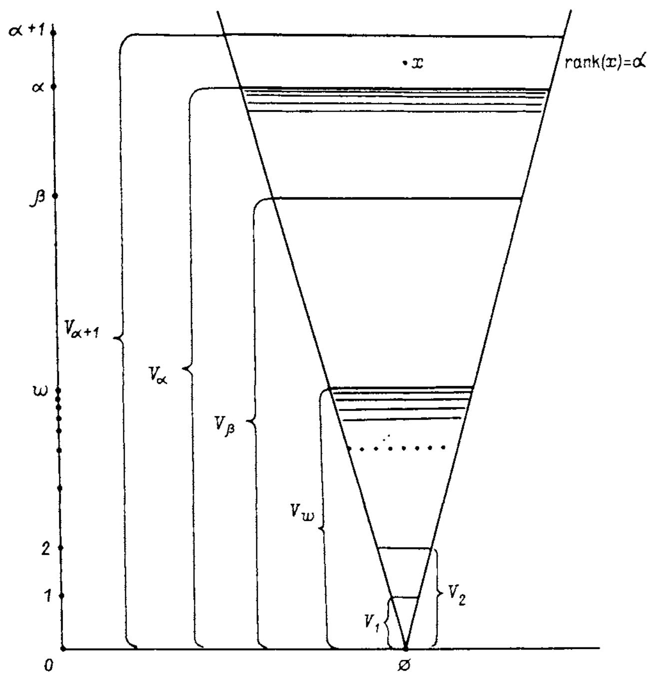


Fig. 3

3.2.15. Remarks.

(1) The first system of axioms for set theory (together with the B.Russel theory of types) suggested by E.Zermelo in 1908, essentially coincides with ZF_1 - ZF_3, ZF_5, **3.2.5**, **3.2.6**. The axiom of extensionality ZF_1 and union ZF_2 were proposed earlier by G.Freget (1883) and G.Cantor (1899), respectively. The idea of the axiom of infinity ZF_5 belongs to R.Dedekind.

(2) The axiom of choice AC had, in all probability, been implicitly used for a long time before it was noticed by G.Peano in 1890 and B.Levy in 1902. This axiom was introduced

by E.Zermelo in 1904 and for many years remained most disputable. The axiom of choice is the cornerstone of many important fragments of modern mathematics and there is no wonder that it is being accepted by the majority of the mathematicians working at present. Discussions of the place and role of the axiom of choice can be found elsewhere [28, 56, 61, 95, 98].

(3) The system of ZFC was completely elaborated at the beginning of the 1920s. By that time the formalization of the set theoretic language had been completed, which made it possible to clarify the vague description of the type of properties admissible in the axiom of comprehension. On the other hand, Zermelo axioms do not yield as a corollary the Cantor statement that each one-to-one image of a set is a set. This drawback was eliminated by A.Fraenkel in 1922 and T.Scolem in 1923, who suggested variations of the axiom of replacement.

(4) The axiom of foundation ZF_6 was in fact suggested by von Neumann in 1925. This axiom is independent of the other axioms of ZFC.

(5) The system of the axioms of ZFC is not infinite, as noted in **3.2.4**. Nonfinite axiomatizability for ZFC was proved by R.Montague in 1960 (see [75, 56, 155]).

3.3. Nelson Internal Set Theory

The preliminary analysis of the properties of standard and nonstandard sets carried out above showed that in the von Neumann universe there is a place for infinitely small numbers but there is no place for the whole of their union. In other words, nonstandard analysis points out that Zermelo-Fraenkel theory, while describing the classical world of 'standard' mathematics, singles out a proper internal part of the universe of 'naive' sets. In order to emphasize this peculiarity, in the nonstandard theory of sets, the elements of the von Neumann universe are called *internal sets*. Therefore, a set in the sense of Zermelo-Fraenkel theory and an internal set are synonyms. A convenient foundation for the nonstandard analysis is given by internal set theory IST suggested by E.Nelson.

3.3.1. The alphabet of formal IST is obtained by adjoining the only one new symbol to that of ZFC, the symbol of the unary predicate St that expresses the property of being a *standard set*. In other words, the number of admissible fragments of the texts of IST is enlarged by expressions of the type $\mathrm{St}(x)$, or, in more detail, 'x is standard', or, finally, 'x is a standard set'. Therefore, the semantic domain of definition for variables of IST is the Zermelo-

Fraenkel universe, i.e., the von Neumann universe, in which standard and nonstandard sets are now distinguished.

3.3.2. The formulas of IST are defined by a conventional procedure. In this case the number of atom formulas is modified by the texts $\mathrm{St}(x)$, where x is a variable. Each of the formulas of ZFC is a formula of IST, the converse statement being obviously not valid. In order to distinguish the formulas, the following terminology is used: the formulas of ZFC are called *internal*, the formulas of IST that are not ZFC are called *external*. Hence, the text 'x is standard' is an external formula of IST.

Sometimes, in our further discussions it would be convenient to use the following abbreviations: we shall write $\varphi \in$(IST) instead of φ is a formula of IST and, respectively, $\varphi \in$(ZFC) instead of φ is a formula of ZFC, i.e., φ is an internal formula of IST.

3.3.3. The difference between the formulas of IST results in singling out external and internal classes. If φ is an external formula of IST, then the text $\varphi(y)$ is described with the following words: '*y is an element of the external class* $\{x\colon \varphi(x)\}$'. The term *internal class* is used in the same sense as the term *class* in Zermelo-Fraenkel theory. In the cases in which this does not result in misunderstanding, both external and internal classes are called simply classes.

3.3.4. The external classes composed of elements of an internal set are termed *external sets*, or, in more detail, external subsets of the given set. It would useful to recall again that the internal class composed of elements of an internal set is also an internal set. Alongside with the abbreviations assumed in ZFC, in internal set theory some additional agreements are concluded. A list of them follows:

$\mathbf{V}^{\mathrm{st}} := \{x : \mathrm{St}(x)\}$ is an external class of standard sets;
$x \in \mathbf{V}^{\mathrm{st}} := x$ is standard $= (\exists y)\ \mathrm{St}(y) \wedge y = x$;
$(\forall^{\mathrm{st}} x)\ \varphi := (\forall x)\ (x \text{ is standard} \to \varphi)$;
$(\exists^{\mathrm{st}} x)\ \varphi := (\exists x)\ (x \text{ is standard} \wedge \varphi)$;
$(\forall^{\mathrm{st\,fin}} x)\ \varphi := (\forall^{\mathrm{st}} x)\ (x \text{ is finite} \to \varphi)$;
$(\exists^{\mathrm{st\,fin}} x)\ \varphi := (\exists^{\mathrm{st}} x)\ (x \text{ is finite} \wedge \varphi)$;
${}^{\circ}x := \{y \in x\colon y \text{ is standard}\}$.

The external set ${}^{\circ}x$ is often called the *standard core of* x.

The collision of notation arising by virtue of the traditional terminology (for $x \in {}^{*}\mathbf{R}$ the symbol ${}^{\circ}x$ denotes the standard part, $\mathrm{st}(x)$, of this number as well) results in no significant controversy.

3.3.5. The axioms of IST are obtained by supplementing the list of axioms of ZFC with three new schemata, bearing, as we have remarked earlier, the name of the principles of nonstandard set theory:

(1) the transfer principle,

$$(\forall^{st} x_1)(\forall^{st} x_2)\dots(\forall^{st} x_n)((\forall^{st} x)\varphi(x,x_1,\dots,x_n) \rightarrow (\forall x)\,\varphi(x,x_1,\dots,x_n))$$

for every internal formula φ;

(2) the idealization principle,

$$(\forall x_1)(\forall x_2)\dots(\forall x_n)((\forall^{st\,fin} z)(\exists x)(\forall y \in z)\,\varphi(x,y,x_1,\dots,x_n) \leftrightarrow (\exists x)(\forall^{st} y)\,\varphi(x,y,x_1,\dots,x_n)),$$

where $\varphi \in$(ZFC) is an arbitrary internal formula;

(3) the standardization principle,

$$(\forall x_1)\dots(\forall x_n)((\forall^{st} x)(\exists^{st} y)(\forall^{st} z)\, z \in y \leftrightarrow z \in x \wedge \varphi(z,x_1,\dots,x_n))$$

for every formula φ.

3.3.6. Powell theorem. IST *is a conservative extension of* ZFC.

◁ A proof is given in [187]. ▷

3.3.7. The above theorem implies that the internal theorems of internal set theory are theorems of Zermelo-Fraenkel theory. In other words, when proving 'standard' theorems on sets we are rightful to use the formalism of IST to the same degree of validity we enjoy while working within ZFC. One should not, however, forget that ZFC is, in the long run, substantiated with its practical infallibility and semantical justification.

3.3.8. When thinking about the sense of the formal expression of the axioms of IST, one cannot help but notice that the presentation of the idealization principle is somewhat cumbersome. While the refined rules of transfer and standardization presented above quite adequately portray the naive concepts put forward earlier, the position of the idealization principle formulation is controversial. Therefore, let us first of all prove that the idealization principle **3.3.5** (2) guarantees the presence of nonstandard elements.

3.3.9. *There is a finite internal set, among whose elements every standard set is encountered.*

◁ Let us consider the following formula: $\varphi := (x \text{ is finite }) \wedge (y \in x)$. Note that $\varphi \in$ (ZFC). Obviously, for every standard finite z we can find an x such that for all $y \in z$ we have $\varphi(x,y)$. If we choose z itself as such an x and use the idealization principle, then the proof will be completed. ▷

3.3.10. When applying the idealization principle, it is useful for us to bear in mind that standard finite sets are exactly the sets whose every element is standard, which fact has been proved earlier (**2.2.2**). It would be instructive to consider its formal inference based on the idealization principle.

3.3.11. *For an internal set* A,

$$A = {}^{\circ}A \leftrightarrow (A \textit{ is standard}) \wedge (A \textit{ is finite}).$$

◁ Let us construct the formula $\varphi := x \in A \wedge x \neq y$. No doubt, $\varphi \in$(ZFC). Then, by virtue of the idealization principle,

$$(\forall^{\mathrm{st\,fin}} z)\,(\exists x)\,(\forall y \in z)\quad \varphi(x,y,A) \leftrightarrow (\exists x)(\forall^{\mathrm{st}} y)\, x \in A \wedge x \neq y$$
$$\leftrightarrow (\exists x \in A)\quad (x \text{ is nonstandard}) \leftrightarrow A \setminus {}^{\circ}A \neq \varnothing.$$

In other words, we get

$$A = {}^{\circ}A \leftrightarrow (\exists^{\mathrm{st\,fin}} z)(\forall x)(\exists y \in z)\, x \notin A \vee x = y$$
$$\leftrightarrow (\exists^{\mathrm{st\,fin}} z)(\forall x \in A)(\exists y \in z)\, x = y \leftrightarrow (\exists^{\mathrm{st\,fin}} z)\, A \in z.\ ▷$$

3.3.12. *Let* X, Y *be standard sets, and* $\varphi = \varphi(x,y,z)$ *be a formula of* IST. *The rule of introduction of standard functions* (= construction principle) *is valid:*

$$(\forall^{st}x)(\exists^{st}y)\quad (x\in X \to y\in Y \wedge \varphi(x,y,z))$$
$$\leftrightarrow (\exists^{st}y(\cdot))\,(\forall^{st}x)(y(\cdot)\ \textit{is a function from}\ X\ \text{to}\ Y \wedge (x\in X\to \varphi(x,y(x),z))\,.$$

◁ Let us consider the standardization $\overline{F}(x) :=^{*} \{y\in Y\colon \varphi(x,y,z)\}$. Applying **3.3.5** (3) again, we form a standard set

$$F :=^{*} \{(x,A)\in X\times \mathcal{P}(Y)\colon \overline{F}(x) = A\}$$

(here we use the fact that $\mathcal{P}(Y)$ is standard, ensured by the supposition that Y is standard). By hypothesis, $(\forall^{st}x\in X)\ \overline{F}(x)\neq\varnothing$. In this case, by the definition of F, $F(x) = \overline{F}(x)$. Hence, by virtue of the transfer principle, we get

$$(\forall^{st}x\in X)\ F(x)\neq\varnothing \to (\forall x\in X)\ F(x)\neq\varnothing\,.$$

Using now the axiom of choice, we can conclude:

$$(\exists y(\cdot))\ (y(\cdot)\ \text{is a function from}\ X\ \text{in}\ Y)\wedge(\forall x\in X)\,(y(x)\in F(x))\,.$$

Applying the transfer principle, we deduce that there is a standard function $y(\cdot)$, determined on X with the values in Y, for which $y(x)\in F(x)$ for all $x\in X$. If we recall the definition of F once again, we shall see that $y(\cdot)$ is the sought function. ▷

3.3.13. From now on, as above, it would be convenient to use some symbolic presentations of the rules deduced, making certain infringements to the agreements concluded earlier. Thus, the rules for introducing standard functions from **3.3.12** can be conveniently rewritten as follows:

(1) $(\forall^{st}x)(\exists^{st}y)\,\varphi(x,y)\leftrightarrow(\exists^{st}y(\,))(\forall^{st}x)\,\varphi(x,y(x))$,

(2) $(\exists^{st}x)\,(\forall^{st}y)\ \varphi(x,y)\leftrightarrow(\forall^{st}y(\cdot))(\exists^{st}x)\ \varphi(x,y(x))$,

where $\varphi\in$(IST), i.e., an arbitrary formula of IST. In other words, we neglect the requirements of possible presence of free variables in φ and of necessary assumption of the 'limitedness', which implies that x and y are considered ranging over given standard sets. In the same way, if $\varphi = \varphi(x_1,\dots,x_n)$ and $\psi = \psi(y_1,\dots,y_n)$, then we shall write $\varphi\to\psi$ when

$$(\forall^{st}x_1)\dots(\forall^{st}x_n)\,(\forall^{st}y_1)\dots(\forall^{st}y_n)\quad \varphi(x_1,\dots,x_n)\leftrightarrow \psi(y_1,\dots y_n),$$

and say that the formulas φ and ψ are equivalent (though if one of the formulas φ and ψ is external, the formulas $\varphi(x_1,\dots,x_n)$ and $\psi(y_1,\dots y_n)$ can be not equivalent for a certain choice of variables!) Using the new means, we can present the transfer principle in a reduced form by the symbols:

(3) $(\forall^{st}x)\ \varphi(x) \leftrightarrow (\forall x)\ \varphi(x)$,

(4) $(\exists^{st}x)\ \varphi(x) \leftrightarrow (\exists x)\ \varphi(x)$,

always keeping in mind that the formula φ must be internal in such a presentation: $\varphi \in$(ZFC). It would be useful to give here some elementary rules valid for any formula φ,

(5) $(\forall x)(\forall^{st}y)\ \varphi(x,y) \leftrightarrow (\forall^{st}y)(\forall x)\ \varphi(x,y)$,

(6) $(\exists x)(\exists^{st}y)\ \varphi(x,y) \leftrightarrow (\exists^{st}y)(\exists x)\ \varphi(x,y)$,

as well as new presentations of the idealization principle:

(7) $(\forall^{stfin}z)(\exists x)(\forall y \in z)\ \varphi(x,y) \leftrightarrow (\exists x)(\forall^{st}y)\ \varphi(x,y)$,

(8) $(\exists^{stfin}z)(\forall x)(\exists y \in z)\ \varphi(x,y) \leftrightarrow (\forall x)(\exists^{st}y)\ \varphi(x,y)$,

pertaining, obviously, only to internal formulas $\varphi \in$(ZFC).

3.3.14. The rules discussed above enable one to transfer many (though, obviously, not all) notions and suppositions of the nonstandard analysis into equivalent mathematical definitions and statements appealing to standardness. In other words, the formulas of IST expressing 'something unusual' about standard objects can be transformed into equivalent formulas of ZFC, which are conventional mathematical records of the propositions under consideration. The procedure by means of which we get the described result is called *the Nelson algorithm*, rules **3.3.13** (1) - **3.3.13** (8) being parts of this procedure. In qualitative terms the essence of the 'decoding' algorithm is as follows: introducing standard functions, applying idealization and permutations of quantifiers, we reduce the statement to a form adopted for the transfer. In the long run, the transfer is reducing a formula to the form convenient for eliminating the external notion of standardness. It should be emphasized that in all cases of practical application of any statements **3.3.13**, the requirements mentioned above which ensure the legitimacy of their application, must be set beforehand.

3.3.15. The Nelson algorithm consists of the following steps:

(**1**) a statement of the nonstandard analysis is presented as a formula of IST, i.e., all the abbreviations are decoded;

(**2**) the obtained formula of IST is reduced to a normal prenex form

$$(Q_1 x_1)\dots(Q_n x_n)\ \varphi(x_1,\dots,x_n),$$

where φ is a ZFC formula, while $Q_k := \forall \vee \exists \vee \forall^{st} \vee \exists^{st}$ for $k := 1,\dots,n$;

(**3**) if Q_n is *an 'internal' quantifier*, i.e., $\forall$ or $\exists$, then we set $\varphi := (Q_n x_n)\ \varphi(x_1,\dots,x_n)$ and go over to step (**2**);

(**4**) if Q_n is *an 'external' quantifier*, i.e., $\forall^{st}$ or $\exists^{st}$, then there can be found the first internal quantifier when looking through the quantifier prefix $(Q_1 x_1)\dots(Q_n x_n)$ from right to left;

(**5**) if there are no internal quantifiers in step (**4**), then, by virtue of **3.3.13** (3) and **3.3.13** (4), the quantifier Q_n is replaced by the corresponding internal quantifier, and then we go over to step (**2**) (i.e., going from right to left, we step by step 'erase' the super index st over each quantifier);

(**6**) let Q_m be the first internal quantifier encountered. Let us assume Q_{m+1} to be an external quantifier of the same type as Q_m (i.e., $Q_m = \forall$ and $Q_{m+1} = \forall^{st}$, or $Q_m = \exists$ and $Q_{m+1} = \exists^{st}$). Now we use rules **3.3.13** (5) and **3.3.13** (6) and then return to (**2**);

(**7**) if all the quantifiers $Q_{m+1},\dots,Q_n$ are of the same type, then we apply the idealization principle in the form **3.3.13** (7) or **3.3.13** (8), and go over to (**2**);

(**8**) if the quantifiers alternate, i.e., Q_{p+1} is of the same type as Q_m, while all quantifiers $Q_{m+1},\dots,Q_p$ are of the opposite type, then we can apply **3.3.13** (1) or **3.3.13** (2), under the assumption that $x := (x_{m+1},\dots,x_p)$, $y := x_{p+1}$. Then we go over to (**2**).

3.3.16. One should bear in mind that the same statement can be expressed in different ways, including the form absolutely blocking any comprehension. In this respect, when applying the Nelson algorithm, one should take into account concrete possibilities of reducing the procedure of 'dragging external quantifiers out'. In particular, it is not always expedient to consider the formulas reduced to a normal prenex form from the onset (i.e. to carry out step **(2)** of the algorithm to its completion).

3.3.17. Examples

(1) *In nonstandard analysis the principle of external induction is valid, i.e., for an arbitrary formula* $\varphi \in$ (IST),

$$(\varphi(1) \wedge ((\forall n \in {}^{\circ}\mathbf{N}) \quad \varphi(n)\varphi(n+1))) \rightarrow (\forall n \in {}^{\circ}\mathbf{N}) \quad \varphi(\mathrm{n}).$$

◁ We cannot directly apply the Nelson algorithm to the formal presentation of the principle under study, since the formula φ may be external. Therefore, let us consider the standardization $A:= \{n \in \mathbf{N}: \varphi(n)\}$. Obviously, we get $1 \in A$, and for every standard $n \in A$ we have $n+1 \in A$. The task is to prove that ${}^{\circ}\mathbf{N} \subset A$. Let us write out the required formula and apply the Nelson algorithm to it:

$$(1 \in A \wedge (\forall^{\mathrm{st}} n \in \mathbf{N})(n \in A \rightarrow (n+1) \in A)) \rightarrow {}^{\circ}\mathbf{N} \subset A$$
$$\leftrightarrow (\forall^{\mathrm{st}} m)\ (\forall^{\mathrm{st}} n)(m \in \mathbf{N} \wedge n \in \mathbf{N} \wedge 1 \in A \wedge n \in A \rightarrow (n+1) \in A)$$
$$\rightarrow m \in A \leftrightarrow (1 \in A \wedge (\forall n \in \mathbf{N})(n \in A \rightarrow (n+1) \in A)) \rightarrow \mathbf{N} \subset A. \ \triangleright$$

(2) *The sum of infinitesimals is an infinitesimal.*

$$\triangleleft \quad (\forall s \in \mathbf{R})\ (\forall t \in \mathbf{R})\ s \approx 0 \wedge t \approx 0 \rightarrow s+t \approx 0$$
$$\leftrightarrow (\forall s \in \mathbf{R})(\forall t \in \mathbf{R})\ (s \approx 0 \wedge t \approx 0 \rightarrow (\forall^{\mathrm{st}} \varepsilon > 0)\,|s+t| < \varepsilon)$$
$$\leftrightarrow (\forall^{\mathrm{st}} \varepsilon > 0)(\forall s \in \mathbf{R})(\forall t \in \mathbf{R})\ ((\forall^{\mathrm{st}} \delta_1 > 0)$$
$$\wedge (\forall^{\mathrm{st}} \delta_2 > 0)\ (|s| < \delta_1 \wedge |t| < \delta_2 \rightarrow |s+t| < \varepsilon)$$
$$\leftrightarrow (\forall^{\mathrm{st}} \varepsilon > 0)(\forall s \in \mathbf{R})(\forall t \in \mathbf{R})\ (\exists^{\mathrm{st}} \delta_1 > 0)\ (\exists^{\mathrm{st}} \delta_2 > 0)(|s| < \delta_1$$
$$\wedge |t| < \delta_2 \rightarrow |s+t| < \varepsilon) \leftrightarrow (\forall^{\mathrm{st}} \varepsilon)(\forall s)(\forall t)(\exists^{\mathrm{st}} \delta_1)(\exists^{\mathrm{st}} \delta_2)(\varepsilon > 0$$
$$\ldots \wedge \delta_2 > 0 \wedge |s| < \delta_1 \wedge |t| < \delta_2 \rightarrow |s+t| < \varepsilon)$$
$$\leftrightarrow (\forall^{\mathrm{st}} \varepsilon)(\forall s)(\forall t)(\exists^{\mathrm{st}} \delta_1)(\exists^{\mathrm{st}} \delta_2)(|s| < \delta_1 \wedge |t| < \delta_2 \rightarrow |s+t| < \varepsilon)$$
$$\leftrightarrow (\forall^{\mathrm{st}} \varepsilon)(\exists^{\mathrm{stfin}} \Delta_1)(\exists^{\mathrm{stfin}} \Delta_2)(\forall s)(\forall t)(\exists \delta_1 \in \Delta_1)(\exists \delta_2 \in \Delta_2)$$
$$(|s| < \delta_1 \wedge |t| < \delta_2 \rightarrow |s+t| < \varepsilon)$$
$$\leftrightarrow (\forall^{\mathrm{st}} \varepsilon)(\exists^{\mathrm{st}} \delta_1)\ (\exists^{\mathrm{st}} \delta_2)(\forall |s| < \delta_1)(\forall |t| < \delta_2)\ |s+t| \leq \varepsilon$$
$$\leftrightarrow (\forall \varepsilon > 0)(\exists \delta > 0)(\forall |s| < \delta)(\forall |t| < \delta)\ |s+t| \leq \varepsilon. \ \triangleright$$

(3) Robinson lemma. *Let* a_n *be an internal sequence of numbers, and* $a_n \approx 0$ *for all* $n \in {}^{\circ}\mathbf{N}$. *Then there is an* $N \approx +\infty$, *such that* $a_n \approx 0$ *for any* $n \leq N$.

◁ Let us apply the Nelson algorithm to the required conclusion:

$$(\exists N \approx +\infty)(\forall n \leq N)\ a_n \approx o \leftrightarrow$$

$$\leftrightarrow (\exists N \in \mathbf{N})((\forall^{st} m \in \mathbf{N})\ N \geq m) \wedge (\forall n \in \mathbf{N})\,(n \leq N$$
$$\rightarrow (\forall^{st} \varepsilon > 0)\,|a_n| < \varepsilon) \leftrightarrow (\exists N)(\forall^{st} m)(\forall^{st} \varepsilon)$$
$$(\forall n)(N \geq m \wedge (n < N \rightarrow |a_n| < \varepsilon_k))$$
$$\leftrightarrow (\forall^{st}\{m_1,\dots,m_p\})\,(\forall^{st}\{\varepsilon_1,\dots,\varepsilon_p\})\,(\exists N)\,(\forall k = 1,\dots,p)$$
$$N \geq m_k \wedge n \leq N \rightarrow |a_n| < \varepsilon_k)$$
$$\leftrightarrow (\forall^{st} m)\ (\forall^{st} \varepsilon)\,(\exists N)\ (N \geq m \wedge (n \leq N \rightarrow |a_n| < \varepsilon)) \leftrightarrow$$
$$\leftrightarrow (\forall^{st} m)\ (\forall^{st} \varepsilon)\,(m \in \mathbf{N} \wedge \varepsilon > 0 \rightarrow |a_m| < \varepsilon).$$

Let us now apply the Nelson algorithm to the premises of the statement in question:

$$(\forall n \in {}^{\circ}\mathbf{N})\ a_n \approx 0 \leftrightarrow (\forall^{st} n)\,(n \in \mathbf{N} \rightarrow (\forall^{st} \varepsilon > 0)\,|a_n| < \varepsilon)$$
$$\leftrightarrow (\forall^{st} n)\ (\forall^{st} \varepsilon)\,(n \in \mathbf{N} \wedge \varepsilon > 0 \rightarrow |a_n| < \varepsilon).$$

Therefore, both the premises and the conclusion are equivalent. ▷

3.4. External Set Theories

The basic statements of nonstandard analysis are adequately reflected in the formal apparatus of Nelson internal set theory. The Powell theorem makes it possible to view IST as a technique of studying the von Neumann universe. At the same time, the presence of external objects completely undermines the widely-spread opinion that Zermelo-Fraenkel formalism ensures a sufficient operational freedom from the viewpoint of the naive set theory. Remaining within the approach of IST, we cannot ask even such, for instance, a question as: 'Is it possible to single out such numbers that every element of **R** could have a one-to-one presentation in the form of their certain combination with standard coefficients, since **R** can be considered a vector space over $^{\circ}\mathbf{R}$?' The quantity of such inadmissible questions, undoubtedly mathematical from the semantic point of view, is great to the extent that the necessity of extending the limits of IST becomes vitally important. The *a priori* prohibition against formulating problems is nothing but imposing arbitrary restrictions on the reason. The introduction of the *ad hoc* dogma, "the explicitly expressed prohibition against thinking" (as was aphoristically remarked by L.Feuerbach) is the way *a afortiori* unacceptable when searching for the truth. A practical solution of the problem of returning to the 'Cantor paradise' is, in particular, in finding a formalism which enables one to work with external, as regards the von Neumann universe sets, with conventional mathematical means. Now we are going to get acquainted with axiomatic approximations to studying external sets. The first variation of the appropriate formalism, EXT, was suggested by K.Hrbacek. A close variation, NST, was constructed later by T.Kawai. The nonstandard set theories mentioned above demonstrate, semantically speaking, the universe of external sets to be, from the viewpoint of a mathematical pragmatist-Philistine, constructed in the same way as the universe of naive sets. In other words, it allows classical set-theoretic operations, including singling out subsets using the properties (the axiom of comprehension) and putting arbitrary sets into complete order (the choice axiom). At the same time, among external sets there is a whole set of standard and nonstandard internal sets, which obey variations of the principles of transfer, idealization and standardization close to their intuitive formulations. In more strict terms, one may say that internal sets are included into the number of external ones by the definition.

From the standpoint of practical requirements of conventional (standard and nonstandard) mathematical analysis, both EXT and NST furnish us with practically equal means which are more than enough for a substantiated use of common analytical constructions. It is, however, necessary to go through details of the axiomatics of the external set theory under discussion, armed with attention and a certain degree of criticism, in order to avoid illusions pertaining to the euphoria of omnipotence. For instance, it should be emphasized that the universe of external sets is not the von Neumann universe (the axiom of foundation is absent, which is of importance). Moreover, the exact formulations of the principles of nonstandard analysis in EXT have technical differences from their analogs in IST. Therefore, EXT is not an extension of Nelson's IST, though EXT is a conservative extension of ZFC. This gap was filled by T.Kawai, whose NST enriches the formal apparatus of IST and, alongside with IST and EXT, provides a reliable technique for studying ZFC.

3.4.1. The alphabet of formal EXT is obtained by adjoining to the of alphabet IST the only new symbol, the symbol of a unary predicate, Int, which expresses the property of being an internal set. In other words, allowable for consideration are texts containing records of the type $\mathrm{Int}(x)$, or, in more detail, "x is internal', and finally, 'x is an internal set'. It is intuitively considered that the contextual domain of changing the variables of EXT is *the universe of all external sets* $\mathbf{V}^{\mathrm{Ext}} := \{x : x = x\}$, which contains both *the universe of standard sets* $\mathbf{V}^{\mathrm{St}} := \{x \in \mathbf{V}^{\mathrm{Ext}} : \mathrm{St}(x)\}$ and the *universe of internal sets* $\mathbf{V}^{\mathrm{Int}} := \{x \in \mathbf{V}^{\mathrm{Ext}} : \mathrm{Int}(x)\}$ that includes the universe $\mathbf{V}^{\mathrm{St}}$.

3.4.2. The conventions of EXT are analogous to those of ZFC and IST. In particular, we, by all means, are going to use in EXT the 'classifiers', i.e., braces (see **3.3.3**) and conventional symbols for denoting the simplest operations with classes of external sets. Following the previous samples, for a formula φ of EXT (or, symbolically, $\varphi \in (\mathrm{EXT})$) we shall write:

$$(\forall^{\mathrm{st}})\,\varphi := (\forall x) \quad (\mathrm{St}(x) \to \varphi) := (\forall x \in \mathbf{V}^{\mathrm{St}})\,\varphi,$$

$$(\exists^{\mathrm{Int}})\ \varphi := (\exists x)\quad (\mathrm{Int}(x) \wedge \varphi) := (\exists x \in \mathbf{V}^{\mathrm{Int}})\ \varphi .$$

Rules of the kind, i.e., understandable from the context, will be further on used without further specifications. Besides, we will need a special new notion and a corresponding notation. We shall say that an external set A has *standard size* (symbolically, $A \in V^{size}$), if there is a standard set a and an external function f such that $(\forall X)\,(X \in A \leftrightarrow (\exists^{st} x \in a)\ X = f(x))$.

3.4.3. Let $\varphi \in$ (ZFC) be a formula of EXT which is also a formula of ZFC (i.e., φ contains no symbols St and Int). Let us replace every quantifier Q in the presentation of φ by Q^{st}. The formula obtained is denoted by φ^{St} and is termed *the standardization* of φ, or *the relativization* of φ to $\mathbf{V}^{\mathrm{St}}$. Analogously, replacing every quantifier Q with Q^{Int}, we obtain the formula φ^{Int} which is termed the *internalization* of φ or *relativization* of φ to $\mathbf{V}^{\mathrm{Int}}$. It should be emphasized that in this case nothing happens to free variables in φ. This rule is also valid for abbreviations. For instance, for external sets A and B we write:

$$A \subset^{\mathrm{St}} B := (\forall^{st} x)\,(x \in A \rightarrow x \in B)$$
$$:= ((\forall x)\,(x \in A \rightarrow x \in B))^{\mathrm{St}} := (A \subset B)^{\mathrm{St}};$$
$$A \in^{\mathrm{Int}} B := (A \in B)^{\mathrm{Int}} := A \in B := A \in^{\mathrm{St}} B := (A \in B)^{\mathrm{St}}.$$

3.4.4. Special axioms of EXT fall into three groups. The first group is *the rules of the formation of external sets*, the second group is *the axioms interrelating the universes* of sets $\mathbf{V}^{\mathrm{St}}$, $\mathbf{V}^{\mathrm{Int}}$ and $\mathbf{V}^{\mathrm{Ext}}$, and, finally, the third group is formed by *the principles of transfer, idealization and standardization.*

3.4.5. In EXT valid are the laws of *Zermelo set theory* (the theory Z), i.e., the following axioms of constructing external sets are accepted:

(1) the axiom of extensionality:

$$(\forall A)(\forall B)\,(A \subset B \wedge B \subset A) \leftrightarrow A = B;$$

(2) the axiom of pairing:

$$(\forall A)(\forall B)\,\{A, B\} \in \mathrm{V}^{\mathrm{Ext}};$$

(3) the axiom of union:

$(\forall A)\ \cup A \in \mathrm{V}^{\mathrm{Ext}};$

(4) the axiom of powersets:

$(\forall A)\ \mathcal{P}(A) \in \mathrm{V}^{\mathrm{Ext}};$

(5) the axiom schema of comprehension:

$(\forall A)\ (\forall X_1)\ldots(\forall X_n)\{X \in A\colon \varphi(X, X_1, \ldots, X_n)\} \in \mathbf{V}^{\mathrm{Ext}}$

for an arbitrary formula $\varphi \in$(EXT);

(6) the axiom of well-ordering: *every external set can be well-ordered.*

The last property, the Zermelo theorem, ensures, as is known (cf. (**3.2.10**)), the axiom of choice either in the conventional multiplicative form or in the form of the Kuratowski-Zorn lemma. It should be also remarked here that the axioms of Z commonly include the axiom of infinity, which in EXT will appear below.

3.4.6. The second group of the axioms of EXT contains the following statements:

(1) the modelling principle: *the universe of internal sets* $\mathbf{V}^{\mathrm{Int}}$ *is the von Neumann universe*, i.e., for every φ axiom of Zermelo-Fraenkel theory the internalization φ^{Int} is an axiom of EXT;

(2) the axiom of transitiveness:

$(\forall x \in \mathbf{V}^{\mathrm{Int}})\ x \subset \mathbf{V}^{\mathrm{Int}};$

i.e., internal sets are composed of only internal elements;

(3) the axiom of embedding:

$\mathbf{V}^{\mathrm{St}} \subset \mathbf{V}^{\mathrm{Int}},$

i.e., standard sets are internal.

3.4.7. The third group of the axioms of EXT includes the following statements:

(1) the transfer principle:

$$(\forall^{st}x_1)\ldots(\forall^{st}x_n)\ \varphi^{St}(x_1,\ldots,x_n) \leftrightarrow \varphi^{Int}(x_1,\ldots,x_n)$$

for every formula $\varphi \in$ (ZFC);

(2) the idealization principle:

$$\begin{aligned}&(\forall^{Int}x_1)\ldots(\forall^{Int}x_n)(\forall A \in V^{size})(((\forall^{fin}z)\ z \subset A\\ &\rightarrow (\exists^{Int}x)(\forall y \in z)\ \varphi^{Int}(x,y,x_1,\ldots x_n))\\ &\rightarrow (\exists^{Int}x)(\forall^{Int}y \in A)\ \varphi^{Int}(x,y,x_1,\ldots x_n))\end{aligned}$$

for an arbitrary $\varphi \in$ (ZFC);

(3) the standardization principle:

$$(\forall A)(\exists^{st}a)(\forall^{st}x)\ (x \in A \leftrightarrow x \in a),$$

i.e., for any external set A there is its standardization $^*A$.

3.4.8. A simplest useful corollary to the above axioms worth mentioning is that *the bounded formulas* ZFC *are absolute*. To be more precise, for $\varphi \in (\Sigma_0)$ we get

$$\begin{aligned}&(\forall^{Int}x_1)\ldots(\forall^{Int}x_n)\ \varphi(x_1,\ldots x_n) \leftrightarrow \varphi^{Int}(x_1,\ldots x_n),\\ &(\forall^{st}x_1)\ldots(\forall^{st}x_n)\ \varphi^{St}(x_1,\ldots x_n) \leftrightarrow \varphi^{Int}(x_1,\ldots x_n) \leftrightarrow \varphi(x_1,\ldots x_n).\end{aligned}$$

Hence, any 'bounded' property of standard sets can be safely expressed both in terms of external and internal or standard elements. For instance, $x \subset y \leftrightarrow x \subset^{St} y \leftrightarrow x \subset^{Int} y$ for standard sets x and y.

3.4.9. Hrbacek theorem. *EXT is conservative over* ZFC, *i.e., for every* $\varphi \in$ (ZFC) *we have*

(φ *is a theorem of* ZFC) $\leftrightarrow$ (φ^{Int} *is a theorem of* EXT) $\leftrightarrow$ (φ^{St} *is a theorem of* EXT).

$\triangleleft$ The proof of this theorem can be found in [85].

3.4.10. When thinking over the axiomatics presented above, it would be first of all useful to realize that EXT is not an extension of IST. In other words, the universe of internal sets $\mathbf{V}^{\mathrm{Int}}$ is not a model of the Nelson internal set theory, since the idealization and standardization principles in these theories are formulated in a different way. In the universe $\mathbf{V}^{\mathrm{Int}}$ standardization is allowed under essentially less strict propositions than that in IST. Thus, for any $\varphi \in$ (IST) and an arbitrary $A \in \mathbf{V}^{\mathrm{Int}}$ we can organize ${}^{*}\{x \in A\colon \varphi(x)\}$ since $\{x \in A\colon \varphi(x)\}$ is an external subset of A. In this case in EST an additional requirement that A be standard is, generally speaking, necessary since in IST one cannot standardize a set that contains all standard elements. In EXT, in turn, the union of all standard elements $\mathbf{V}^{\mathrm{St}}$ is not included in an external (to say nothing of an internal) set at all. Then, indeed, the following statement is valid.

3.4.11. *There is no external set, a element of* $\mathbf{V}^{\mathrm{Ext}}$, *which contains each standard set.*

◁ Let us, on the contrary, assume that for a certain $X \in \mathbf{V}^{\mathrm{Ext}}$ we have $\mathbf{V}^{\mathrm{St}} \subset X$. By virtue of the axiom of comprehension of **3.4.5** (5), for the formula $\varphi(x) = \mathrm{St}(x)$ we conclude that $\mathbf{V}^{\mathrm{St}}$ is an external set, i.e., $(\exists Y)(\forall Z)\ CZ \in Y \leftrightarrow \mathrm{St}(Z)$. If we consider the standardization ${}^{*}\mathbf{V}^{\mathrm{St}}$, it appears to be a standard finite set containing every standard set. The last is, obviously, impossible. ▷

3.4.12. The above proposition shows the idealization principle in EXT ('relativized' to $\mathbf{V}^{\mathrm{Int}}$) to be different from its analogue in IST not only in the form but also in the essence. At the same time, these differences should not be absolutized. The following facts will help to elucidate this statement.

3.4.13. *The following statements are valid:*

(1) *external natural numbers and standard natural numbers coincide*;

(2) *a finite external set is standard iff it consists of only standard elements*;

(3) *for an arbitrary external set A its standard core* ${}^{\circ}A := \{a \in A\colon \mathrm{St}(a)\}$ *is a set of standard size*;

(4) *every infinite internal set contains a nonstandard element.*

◁ (1) By the principle of induction on standard natural numbers (which is, obviously, valid in EXT (cf. **2.2.2** (1))), for a set $\mathbf{N}^{\mathrm{Ext}}$ of external natural numbers we have $\mathbf{N}^{\mathrm{Ext}} \supset {}^\circ\mathbf{N}$. Moreover, it is clear that ${}^*\varnothing = \varnothing$ and ${}^*1 = {}^*\{\varnothing\} = \{\varnothing\} = 1$. Hence, by virtue of induction on external natural numbers (a conventional theorem of Z), we get $\mathbf{N}^{\mathrm{Ext}} \supset {}^\circ\mathbf{N}$, and, finally, ${}^\circ\mathbf{N} = \mathbf{N}^{\mathrm{Ext}}$.

(2) A standard set is internal. Hence, making use of **3.4.6** (2), we can use the argumentation of the proof of **2.2.2** (3). According to **2.2.2** (2), a finite set composed of standard elements is standard.

(3) Let ${}^*A$ be the standardization of A. Let us set $f(a) := a$ for $a \in {}^\circ A$. Obviously, $(\forall X)(X \in {}^\circ A \leftrightarrow (\exists^{\mathrm{st}} x \in {}^*A)\, f(x) = X)$.

(4) Let us denote the internal set under discussion by A. By virtue of (3), ${}^\circ A$ is of standard size. Hence, we can apply the idealization principle for $\varphi(x,y) := y \neq x \wedge x \in A$. Since A is infinite, for every finite $z \subset {}^\circ A$ we, obviously, get $(\exists x \in A)(\forall y \in z)\ x \neq y$. And, finally, $(\exists x \in A)(\forall y \in {}^\circ A)\ x \neq y$. ▷

3.4.14. As regards **3.4.13** and **3.4.9**, it would be convenient a variation of INT which is a conservative extension of ZFC and such that EXT, in turn, is an extension of INT. The difference between INT and EXT in adopting the idealization and standardization principles is as follows:

$$(1)\ (\forall A)(\forall x_1)\ldots(\forall x_n)((\forall^{\mathrm{stfin}} z)\ z \subset A(\exists x)(\forall y \in z)\ \varphi(x,y,x_1,\ldots,x_n) \leftrightarrow (\exists x)(\forall^{\mathrm{st}} y \in A)\ \varphi(x,y,x_1,\ldots,x_n))$$

for every $\varphi \in$ (ZFC);

$$(2)\ (\forall A)(\exists^{\mathrm{st}}\ {}^*A)(\forall^{\mathrm{st}} x)\ (x \in A \leftrightarrow x \in {}^*A \wedge \varphi(x))$$

for an arbitrary $\varphi \in$ (INT).

It should be remarked that the Nelson algorithm is, in its essential features, operative in INT.

3.4.15. Let us now go over to the description of NST in a variation most close to EXT and IST (in fact, T.Kawai has constructed a somewhat different scheme, enabling one to consider the classes of the von Neumann-Gödel-Bernays theory as external sets).

3.4.16. The alphabet and conventions in formal NST coincide with those of EXT. Moreover, NST adopts all the axioms of constructing external sets, all the axioms interrelating the universes of sets, and the transfer principle of EXT. The differences between NST and EXT are in the ways of formulating the idealization and standardization principles, as well in the following supplementary postulate.

3.4.17. The axiom of acceptability, $\mathbf{V}^{\mathrm{St}} \in \mathbf{V}^{\mathrm{Ext}}$, *i.e., the universe of standard sets of the Kawai theory is an external set.*

In line with the axiom formulated, in NST the external set A is called *a set of appropriate size*, and we write $A \in \mathbf{V}^{\mathrm{a-size}}$ provided there is an external function f mapping $\mathbf{V}^{\mathrm{St}}$ on A. It should be emphasized that $\mathbf{V}^{\mathrm{St}}$ is of appropriate size. It should be also remarked that in the sequel the presentation $\mathrm{a\text{-}fin}(A)$ implies that there is a one-to-one external A mapping on a standard finite set.

3.4.18. The standardization principle in NST is as follows:

$$(\forall A)((\exists^{\mathrm{st}} X)\ A \subset X \to (\exists^{\mathrm{st}}\ {}^{*}\!A)(\forall^{\mathrm{st}} x)\,(x \in A \leftrightarrow x \in {}^{*}\!A)).$$

In other words, in NST only external subsets of standard sets can be standardized, not arbitrary external subsets, as is the case in EXT.

3.4.19. The idealization principle in NST is as follows:

$$\begin{aligned}&(\forall^{\mathrm{Int}} x_1)\ldots(\forall^{\mathrm{Int}} x_n)(\forall A \in \mathbf{V}^{\mathrm{a-size}})\,(((\forall z)\ z \subset A \wedge a = \mathrm{fin}(z)\\ &\to (\exists^{\mathrm{Int}} x)(\forall y \in z)\ \varphi^{\mathrm{Int}}(x, y, x_1, \ldots, x_n)\\ &\to (\exists^{\mathrm{Int}} x)(\forall^{\mathrm{Int}} y \in A)\ \varphi^{\mathrm{Int}}(x, y, x_1, \ldots, x_n))\end{aligned}$$

for an arbitrary formula $\varphi \in$ (ZFC).

3.4.20. Kawai theorem. NST *is a conservative extension of* ZFC.

◁ The proof can be found in [104]. ▷

3.4.21. Let us again pay attention to the fact that the world of the internal sets V^{Int} in the universe of NST with the standardization, idealization and transfer principles relativized, is a model for IST. In other words, the technical means employed by NST to work with external sets arising in IST can be surely used to obtain the statements of 'standard' mathematics. It should be also remarked that the proof of the Kawai theorem, as well as that of the Hrbacek and Powell theorems is essentially based on the application of appropriate analogues of the Mal'tsev local theorem, or, to be more exact, on the technique of ultraproducts and ultralimits. A detailed presentation of the aforementioned apparatus is beyond the scope of the present book, the second part of this work, however, dealing with nonstandard methods of analysis which consider every possibility of understanding and constructing the proofs of the theorem in question.

3.4.22. Taking liberties with notations, we denote the universe of external sets by $\mathbf{V}^E$ (irrespective of whether NST or EXT is in question). Analogously, let us use the notation $\mathbf{V}^I$ (and, respectively, $\mathbf{V}^S$) to denote the world of internal (and, respectively, standard) sets. Repeating the scheme of constructing the von Neumann universe, i.e., iterating step-by-step the operations of taking unions and powersets of all external subsets of a given set, we see that the empty set can give rise to the world $\mathbf{V}^C$, i.e., *the universe of 'classical' sets.* In more detail, we set

$$V_\beta^C := \{x: (\exists^{\mathrm{st}} \alpha \in \beta) \quad x \in \mathcal{P}^{\mathrm{Ext}}(V_\alpha^C)\},$$
$$\mathbf{V}^C := \bigcup_{\beta \in \mathrm{On}^{\mathrm{St}}} V_\beta^C,$$

where $\mathrm{On}^{\mathrm{St}}$ is the class of all standard ordinals. Therefore, an empty set is 'classical', and every 'classical' set is composed of 'classical' elements' only.

3.4.23. Using recursion, i.e., a walk about the stores of the universe of 'classical' sets, one can determine the Robinson standardization or the $*$-map.

A standard set $*A$ is called *the Robinson standardization* or *the $*$-image* of a 'classical' set A iff every standard element of $*A$ is a $*$-image of a certain element of A. Or, symbolically.

$$*\varnothing := \varnothing, \quad *A := {}^*\{*a : a \in A\}.$$

It should be remarked that within EXT the legitimacy of applying the conventional standardization gives rise to no ambiguity. In NST the admissibility of employing this operation in determining the Robinson standardization results from the method of constructing $\mathbf{V}^C$. Analogous considerations (cf. **3.2.12**) prove that that the $*$-map identifies, and does it in a one-to-one manner, the worlds $\mathbf{V}^C$ and $\mathbf{V}^S$. Moreover, the Robinson standardization ensures the validity of the *transfer principle*

$$(\forall A_1 \in \mathbf{V}^C) \ldots (\forall A_n \in \mathbf{V}^C)\ \varphi^C(A_1,\ldots,A_n) \leftrightarrow \varphi^S({}^*A_1,\ldots,{}^*A_n)$$

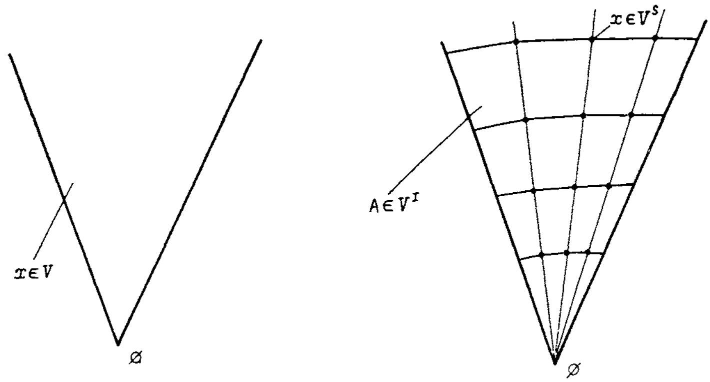
for an arbitrary formula φ of Zermelo-Fraenkel theory (as usual, φ^C and φ^S are φ relativizations on $\mathbf{V}^C$ and $\mathbf{V}^S$, respectively).

3.5. Set-Theoretic Stances on Nonstandard Analysis

The considerations carried out in the preceding paragraphs have enriched and extended the initial naive ideas of a set used in nonstandard analysis. We have passed from the conventional von Neumann universe, $\mathbf{V}$, to the world $\mathbf{V}^I$ of internal set theory with its points of reference, standard sets, comprising the class $\mathbf{V}^S$ (Fig. 4). Further analysis has shown $\mathbf{V}^I$ to belong to a new class, the universe $\mathbf{V}^E$ of external sets (comprising the Zermelo world). In $\mathbf{V}^E$ we have singled out the universe of 'classical' sets $\mathbf{V}^C$, which is another implementation of the world of standard sets, $\mathbf{V}^S$. To be more exact, we mean the Robinson $*$-standardization which identifies $\mathbf{V}^C$ and $\mathbf{V}^S$ in the elementwise manner. In this case, by virtue of the transfer principles, $\mathbf{V}^C$, $\mathbf{V}^S$ and $\mathbf{V}^I$ can be viewed as 'hypostases' of the von Neumann universe, $\mathbf{V}$ (Fig. 5).

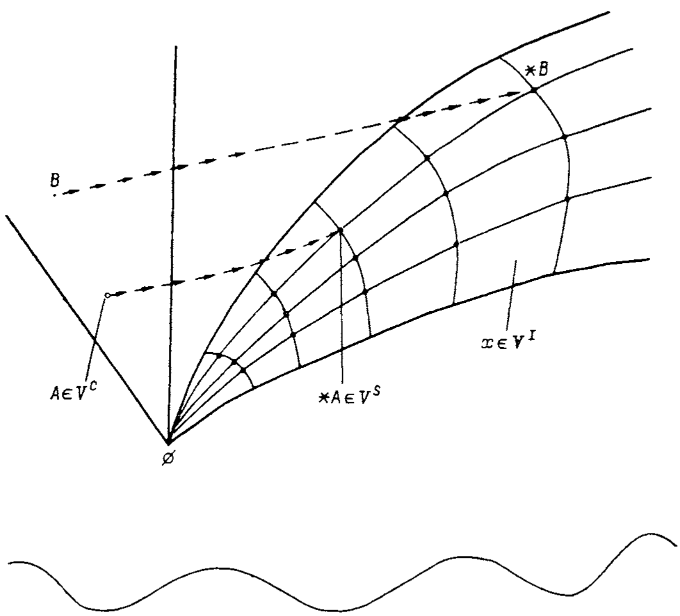
3.5.1. The presented picture of the location as well as the other known interrelations between the worlds $\mathbf{V}^E$, $\mathbf{V}^I$, $\mathbf{V}^S$ and $\mathbf{V}^C$ result in three general set-theoretic stances on nonstandard analysis. These stances, which are called classical, neoclassical and radical, fix the presentation of the subject and the means of its investigation. The acceptance of this or that stance determines, in particular, the way of presenting the mathematical results obtained with nonstandard methods. Therefore, acquaintance with the aforementioned stances are to be considered a must.

3.5.2. The classical stance on nonstandard analysis corresponds to the methods employed by its founder A.Robinson, the corresponding formalism being most widely-spread nowadays. Under this stance *the world of classical mathematics identified with the universe of 'classical' sets,* $\mathbf{V}^C$, *is declared to be the principal object to be studied.* The latter is considered to be 'a standard universe' (in practice, however, the object of investigation is the so-called 'superstructure', i.e., a sufficiently large fragment of $\mathbf{V}^C$ containing all necessary objects of study). As the technique of investigating the initial, i.e., standard, universe, viewed are the 'nonstandard universe' of internal sets, $\mathbf{V}^I$ (or its appropriate part) and the $*$-map, pasting conventional standard objects to their images in



Von Neumann universe Universe of internal sets

Fig. 4



Universe of external sets

Fig. 5

the 'nonstandard universe'. It would be expedient to note a peculiar use of the words 'standard' and 'nonstandard' in the approach under discussion. Robinson standardizations, i.e. elements of the universe $\mathbf{V}^S$, are viewed as 'nonstandard' objects. A 'standard' set which is, by definition, an arbitrary representative of the world of 'classical' sets $\mathbf{V}^C$, is a member of the 'standard universe'. The $*$-map adds, as a rule, new 'ideal' elements into the set, which implies that $*A = \{*a: a \in A\}$ iff a 'classical', i.e., 'standard', set A is finite. For instance, putting $\mathbf{R}$ in $\mathbf{V}^C$ and studying its $*$-image $*\mathbf{R}$ in accordance with what have been discussed above, we see that $*\mathbf{R}$ plays the role of the field of real numbers in the sense of the universe of internal sets, i.e., 'in the inner sense of the nonstandard universe'. At the same time, $*\mathbf{R}$ is not reduced to the set of its standard elements, $^\circ(*\mathbf{R}) = \{*t: t \in \mathbf{R}\}$. Making use of the fact that $*\mathbf{R}$ is the 'internal set of real numbers $\mathbf{R}$', while $^\circ(*\mathbf{R})$ is its standard core, it would be a certain excessive liberty to assume $^\circ\mathbf{R} := \{*t: t \in \mathbf{R}\}$, or even $\mathbf{R} := \{*t: t \in \mathbf{R}\}$. The presence of 'new' elements in $*\mathbf{R}$ is expressed by the symbol $*\mathbf{R} - \mathbf{R} \neq \varnothing$, and we say about the construction of the system of 'hyperreal' numbers $*\mathbf{R}$ which extends the conventional field of real numbers $\mathbf{R}$. An analogous policy is pursued when considering an arbitrary classical set X. Namely, it is assumed that $X = \{*x: x \in X\}$ and, hence, $X \subset *X$. If X is infinite, then $*X - X \neq \varnothing$. In other words, through the Robinson standardization all infinite sets are saturated with new elements. Moreover, a large quantity of 'ideal' objects is added, since in $\mathbf{V}^I$ operates the idealization principle which is often called *the technique of concurrence and saturation* within the stance under consideration.

3.5.3. Let U be an arbitrary correspondence, and A and B be sets. We say that *U is concurrent from A to B* if for every nonempty finite subset A_0 in A there is an element $b \in B$ such that $(a_0, b) \in U$ for all $a_0 \in A_0$.

3.5.4. Weak concurrence principle. *For any correspondence U concurrent from A to B there is an element $b \in *B$, maintaining the relation $(*a, b) \in *U$ for every $a \in A$.*

3.5.5. It is obvious that the validity of the concurrence principle ensures, in turn, a natural equivalent of the weak idealization principle, i.e., that which is 'relativized for standard sets'. In this respect in applications distinguished are conservative extensions of the classical set theory using both the possibility of weak idealization mentioned earlier and conventional formulations ensuring additional possibilities of introducing nonstandard elements which are more adequate to the contents of the idealization principle in its complete expression.

3.5.6. Strong concurrence principle. *Let a correspondence U be such that $^*U$ is directed from A to $^*B$. Then there is an element $b \in {}^*B$ for which $({}^*a, b) \in {}^*U$ for all $a \in A$.*

3.5.7. Saturation principle. *Let $A_1 \supset A_2 \supset \dots$ be a decreasingly nested sequence of nonempty internal sets. Then $\bigcap_{n \in \mathbb{N}} A_n \neq \varnothing$.*

3.5.8. It would be expedient to recall that in the 'extended', 'nonstandard' world, i.e., in the universe of internal sets $\mathbf{V}^I$, there operates the transfer principle. Thus, making use of the Robinson standardization we may write

$$(\forall x_1 \in \mathbf{V}^C)\dots(\forall x_n \in \mathbf{V}^C) \quad \varphi^C(x_1,\dots,x_n) \leftrightarrow \varphi^I({}^*x_1,\dots,x_n)$$

for every formula φ of Zermelo-Fraenkel set theory. The transfer principle in such a form is often referred to as *the Leibniz principle.*

3.5.9. When working with the 'nonstandard universe', special use is sometimes made of the *'technique of internal sets'*, i.e., the way of proving based on the fact that external sets defined in the 'set-theoretic fashion' are internal. Here is one of the possibilities of applying this technique.

3.5.10. *Let A be an infinite set. For every set-theoretic property of φ the following statement is invalid:*

$$\{x \colon \varphi^I(x)\} = {}^*A - A.$$

◁ Let us assume that the opposite statement is valid. Then the class $\{x \colon \varphi^I(x)\}$ is an internal set $^*A$. Hence, A is an internal set. For a finite A, however, the external set $^*A - A$ is not internal. ▷

3.5.11. Summarizing, one can say that under the classical stance two universes are being worked with, i.e., standard and nonstandard. There are formal possibilities of interrelating the properties of standard and nonstandard objects by the procedure of 'starring', i.e., using a $*$-map. In this case we enjoy the right of freely transferring the statements on objects from one world to another through the action of the Leibniz principle. The nonstandard world is abundant with ideal elements, and all possible transfinite constructions are actually implementable there since valid is the concurrence principle. The sets falling beyond the limits of the nonstandard universe are called external (here the peculiarity of the assumed terminology is revealing itself: internal sets are not external under the approach in question). The technique of internal sets is an expedient method of investigation.

The principal merit of the classical stance is the presence of the *-map, which makes it possible to apply the apparatus of nonstandard analysis to common arbitrary sets. For instance, one can claim that the function $f:[a,b] \to \mathbf{R}$ is uniformly continuous iff $^*f:{}^*[a,b] \to {}^*\mathbf{R}$ is microcontinuous, i.e., if $^*f$ does not loose infinite proximity of hyperreal numbers. The basic difficulty hampering the way to understanding the phenomena under discussion is related with the necessity to imagine a colossal host of new ideal objects joining common sets. Noticeable complications result from a natural desire to work (at least at initial stages) with two sets of variables pertaining, respectively, to the standard and nonstandard universes. (When constructing the internalization φ^I of the formula φ we, in fact, presuppose such a procedure). To put it short, *the bilingual basis and Robinson standardization that are indispensable attributes of the classical stance* determine all the peculiarities of the latter, as well as the merits and demerits of its apparatus.

3.5.12. The neoclassical stance on nonstandard analysis corresponds to the technique suggested by E.Nelson. Under this stance *the principal object of investigation is the world of mathematics viewed as a universe* $\mathbf{V}^I$ *lying in the medium of external sets that are elements of* $\mathbf{V}^E$. 'Classical' sets are not analysed separately. Standard and nonstandard elements are given in conventional mathematical objects comprising $\mathbf{V}^I$. For instance, as the field of real numbers used is $\mathbf{R}$ from the world $\mathbf{V}^I$ which, obviously, coincides with the field $^*\mathbf{R}$ of hyperreal numbers which is, in its turn, an 'ideal' object of the classical stance. The statements considered in Chapter **2** correspond to the neoclassical stance. Its basic advantage is the possibility of studying sets which are already well-known and to find new features in their construction using additional language means. As has been remarked by E.Nelson, "... really new in nonstandard analysis are not theorems or proofs but the notions, i.e., external predicates ..." [188, p. 134]. The drawbacks of the neoclassical stance are caused by necessity to implicitly transfer the definitions and properties from standard objects to internal ones, which peculiarity we have already observed.

3.5.13. The radical stance on nonstandard analysis is that *the object of mathematical investigation is the universe of external sets* in all the completeness and complexity of its structure. Under the radical approach the classical and neoclassical ideas of nonstandard analysis as of a technique for studying mathematics (based on the Zermelo-Fraenkel formalism) are declared to be 'narrow', 'shy' and brushed away. At first sight this approach seems to be very radical and not serious. Upon proper thinking, however, the ideas about the extremity of the radical stance on nonstandard analysis should be abandoned. This 'extremity' is just a seeming one. Both a widely-spread viewing mathematics as a science about forms and relations taken irrespective of their contents, and even less imposing classical set-theoretic stance stemming from G.Cantor, embrace, undoubtedly, the 'extreme' thoughts about the subject of the nonstandard analysis. Therefore, the most 'intrepid' views on sets arising as a result of rather laborious investigations have finally become a comprising part of the initial point, having enriched it with new contents. And the initial point for us was a modest statement that nonstandard analysis operates with exactly the same sets as the whole of mathematics (see **2.1.3**).

It would be expedient to recall here the aphoristic observations, made by V.I.Lenin, which pertain to the dynamics of the cognition process:

"Every shade of a thought is a circle on the great circle (spiral) of the development of the human thought in general" [154, p. 221].

"Cognition of a man is not (not goes along) a straight line, but a curve infinitely approaching a number of circles, a spiral" [154, p.322].

CHAPTER 4

MONADS IN GENERAL TOPOLOGY

Within the set-theoretic stance on mathematics, at the beginning of the XX century a universal approach was developed to study the structure of continuity and proximity which was formulated in general topology. When considering the microstructure of the numerical line we have already seen that from the viewpoint of nonstandard analysis a set of infinitesimals arises as a monad, i.e., the external intersection of standard elements of the filter of zero neighbourhoods of the only separated topology that agrees with the algebraic structure of the field of real numbers. One can say that through the notion of the monad of a filter a certain synthesis of general topological and infinitesimal ideas is implemented, the corresponding relations being the basic objects of investigation of the present chapter. We will focus our attention on the most elaborated ways of studying classical topological concepts and constructions that group around compactness which is allowed into the nonstandard set theory through idealization. The contribution of the new approach to the problem under discussion is basically associated with the elaboration of a new principally important notion, that of a nearstandard point. The corresponding criterion of compactness of a standard space, i.e., the nearstandardness of its every point, demonstrates the value and essence of the concept of nearstandardness which carries out a certain individualization for the points of the conventional notion of compactness pertaining to sets. Similar techniques of individualization comprise an important and characteristic part of the arsenal of the nonstandard methods of analysis.

4.1. Monads and Filters

A simplest example of a filter is, as is known, the family of supersets of a certain nonempty set. Nonstandard analysis makes it similarly possible to study an arbitrary standard filter as the standardization of the filter of external supersets of an appropriate external set, i.e., as the monad of this filter. The method of introducing such monads and their simplest properties are to be considered in the present section.

4.1.1. Let X be a standard set and $\mathcal{B}$ be a standard filter base in X. Therefore, $\mathcal{B} \neq \varnothing$, $\mathcal{B} \subset \mathcal{P}(X)$, $\varnothing \notin \mathcal{B}$ and $B_1, B_2 \in \mathcal{B} \to (\exists B \in \mathcal{B})\ B \subset B_1 \cap B_2$. The symbol $\mu(\mathcal{B})$ denotes the *monad* of $\mathcal{B}$, i.e., the external set determined by the relation

$$\mu(\mathcal{B}) := \bigcap\{B : B \in {}^{\circ}\mathcal{B}\}.$$

4.1.2. *An internal set is a superset of a certain standard element of a standard filter base $\mathcal{B}$ iff it contains the monad $\mu(\mathcal{B})$.*

◁ If $A \supset B$ and $B \in {}^{\circ}\mathcal{B}$, then $A \supset \mu(\mathcal{B})$ by definition. Conversely, if $A \supset \mu(\mathcal{B})$ then by virtue of the idealization principle there is an internal set $B \in \mathcal{B}$ for which $B \subset \mu(\mathcal{B})$, and we deduce $A \supset B$. ▷

4.1.3. *Every standard filter $\mathfrak{F}$ is the standardization of the principal external filter of supersets of the monad $\mu(\mathcal{B})$.*

◁ Symbolically, we have to establish

$$(\forall^{\mathrm{st}} A)\ A \in \mathfrak{F} \leftrightarrow A \supset \mu(\mathfrak{F}).$$

The preceding relation is obviously contained in **4.1.2.** ▷

4.1.4. *The monad of a filter $\mathfrak{F}$ is an internal set iff the former set is standard. In this case the initial standard $\mathfrak{F}$ is the filter of supersets of $\mu(\mathfrak{F})$.*

◁ If $\mu(\mathfrak{F})$ is an internal set, then, taking into account **4.1.3** and the idealization principle, we get

$$(\exists A)\,(\forall^{\mathrm{st}} F)\quad (F \in \mathfrak{F} \leftrightarrow F \supset A) \leftrightarrow (\forall^{\mathrm{st\,fin}} \mathfrak{U})\,(\exists A)$$
$$(\forall F \in \mathfrak{U})\quad (F \in \mathfrak{F} \leftrightarrow F \supset A) \leftrightarrow (\forall^{\mathrm{st}} U)\,(\exists A)\ (U \in \mathfrak{F} \leftrightarrow U \supset A).$$

Using the transfer principle, we come to the conclusion that $\mathfrak{F}$ is the filter of supersets of a certain set A. Since such a set A is unique, $A = \mu(\mathfrak{F})$ with A being standard here. ▷

4.1.5. For a standard filter base $\mathcal{B}$ the elements of $\mu(\mathcal{B})$ are termed *infinitesimal* or *distant*, or *remote*, or *astray* (relative to $\mathcal{B}$). Analogously, an element $B \in \mathcal{B}$ such that $B \subset \mu(\mathcal{B})$ is also called infinitesimal or distant, or remote, or astray. The union of all infinitesimal sets of $\mathcal{B}$ is denoted by ${}^a\mathcal{B}$.

4.1.6. Examples.

(1) The monad $\mu(\mathbf{R})$ is the monad of the filter of zero neighbourhoods for the natural topology on $\mathbf{R}$.

(2) Let $\mathcal{B}$ be a filter base, and $\tilde{\mathcal{B}}$ be the filter generated by $\mathcal{B}$, i.e., the collection of supersets of the elements of $\mathcal{B}$, or, symbolically,

$$\tilde{\mathcal{B}} := \{F \subset X: \ (\exists B \in \mathcal{B})\, B \subset F\}.$$

In line with the transfer principle, if $\mathcal{B}$ is a standard filter (in a standard set X), then $\tilde{\mathcal{B}}$ is also a standard (base of the) filter. In this case $\mu(\mathcal{B}) = \mu(\tilde{\mathcal{B}})$. It should be remarked that further on it would be more convenient to operate with the monad of an arbitrary internal filter $\mathfrak{F}$ which is determined in an obvious way: $\mu(\mathfrak{F}) := \bigcap {}^\circ\mathfrak{F}$. It should be emphasized that in a standard set X the monad of the filter $\mathfrak{F}$ is obligatory an external superset of an internal element of $\mathfrak{F}$.

(3) Let Ξ be a standard direction, i.e., a nonempty directed set. According to the idealization principle, in Ξ there are internal elements majorizing all standard points of Ξ. Such Ξ elements are called *distant*, *remote*, *infinitely large*, or *astray* in Ξ. Let us consider a standard base of the tail filter $\mathcal{B} := \{[\xi, \rightarrow) := \{\eta \in \Xi: \eta \geq \xi\}: \xi \in \Xi\}$. By definition, $\eta \in \mu(\mathcal{B}) \leftrightarrow (\forall^{\mathrm{st}} \xi \in \Xi)\ \eta \geq \xi$, i.e., the monad of the tail filter is, as was to be expected, composed of distant elements of the direction considered. We shall use the notation ${}^a\Xi := \mu(\mathcal{B})$.

(4) Let $\mathcal{E}$ be a standard cover of a standard set X, i.e., $X \subset \bigcup \mathcal{E}$. Let us consider the family $\Xi(\mathcal{E})$ of standard finite unions of elements of $\mathcal{E}$. Therefore, $\Xi(\mathcal{E}) := {}^*\{\bigcup \mathcal{E}_0: \mathcal{E}_0 \in \mathcal{P}_{\mathrm{stfin}}(\mathcal{E})\}$, where $\mathcal{P}_{\mathrm{stfin}}(\mathcal{E})$ is the set of standard finite subsets of $\mathcal{E}$. The external collection of distant elements of $\Xi(\mathcal{E})$ is termed the monad of $\mathcal{E}$ and is denoted by $\mu(\mathcal{E})$. Hence,

$$\mu(\mathcal{E}) = \bigcup \{E: E \in {}^\circ\mathcal{E}\}.$$

One can analogously determine the monad of any family of sets filtered upwards.

(5) Let $f \subset X \times Y$ and $\mathfrak{F}$ is a (base of a) filter in X such that f meets $\mathfrak{F}$, i.e.,

$(\forall F \in \mathfrak{F})\ \mathrm{dom} f \cap F \neq \varnothing$. Let us, as usual, assume

$$f(\mathfrak{F}) := \{B \subset Y : (\exists F \in \mathfrak{F})\ B \supset f(F)\}.$$

Therefore $f(\mathfrak{F})$, a filter in Y, is the image of $\mathfrak{F}$ under the correspondence f. In 'standard entourage', i.e., assuming that X, Y, f and $\mathfrak{F}$ are standard objects and employing the idealization principle, we get

$$\begin{aligned} & y \in \mu(f(\mathfrak{F})) \leftrightarrow (\forall^{\mathrm{st}} B \in f(\mathfrak{F}))\ y \in B \leftrightarrow (\forall^{\mathrm{st}} F \in \mathfrak{F})\ y \in f(F) \\ & \leftrightarrow (\forall^{\mathrm{st}} F \in \mathfrak{F})(\exists x)\ x \in F \wedge y \in f(x) \\ & \leftrightarrow (\forall^{\mathrm{st\,fin}} \mathfrak{F}_0 \subset \mathfrak{F})(\exists x)(\forall F \in \mathfrak{F}_0)\ x \in F \wedge y \in f(x) \\ & \leftrightarrow (\exists x)(\forall^{\mathrm{st}} F \in \mathfrak{F})\ x \in F \wedge y \in f(x) \\ & \leftrightarrow (\exists x \in \mu(\mathfrak{F}))\ y \in f(x) \leftrightarrow y \in f(\mu(\mathfrak{F})). \end{aligned}$$

Therefore, *the image of the monad of a filter is the monad of the image of this filter:*

$$\mu(f(\mathfrak{F})) = f(\mu(\mathfrak{F})).$$

Let now $\mathcal{B}$ be a base of the filter in Y, and let f^{-1} meet $\mathcal{B}$. Let us consider the preimage or inverse image $f^{-1}(\mathcal{B})$ of the filter $\mathcal{B}$ f (i.e., the image of this filter under the correspondence f^{-1}). Obviously, in line with the above-proved, $\mu(f^{-1}(\mathcal{B})) = f^{-1}(\mu(\mathcal{B}))$. It would be expedient to remark that the last relation can be proved without using 'saturation'. Indeed, strictly by definition, we deduce

$$\mu(f^{-1}(\mathcal{B})) = \bigcap_{G \in {}^{\circ}\mathcal{B}} f^{-1}(G) = f^{-1}\Big(\bigcap_{G \in {}^{\circ}\mathcal{B}} G\Big) = f^{-1}(\mu(\mathcal{B})),$$

i.e., the monad of the preimage of a filter is the preimage of the monad of the initial filter. It is worth to emphasize that when deducing this statement we have made use of the fact that the correspondence f allows one to define external preimages of external sets Y as well.

4.1.7. *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two standard filter bases in a certain standard set. In this case*

$$\tilde{\mathcal{B}}_1 \supset \tilde{\mathcal{B}}_2 \leftrightarrow \mu(\mathcal{B}_1) \subset \mu(\mathcal{B}_2).$$

$\triangleleft$ $\rightarrow$ If B_2 is standard and $B_2 \supset \mu(\mathcal{B}_2)$, then, by **4.1.2**, $B_2 \in \tilde{\mathcal{B}}_2$ and, hence, $B_2 \in \tilde{\mathcal{B}}_1$. Therefore, $B_2 \supset \mu(\mathcal{B}_1)$, and finally, $\mu(\mathcal{B}_1) \subset \mu(\mathcal{B}_2)$.

$\leftarrow$ Let F_2 be a standard element of $\tilde{\mathcal{B}}_2$, i.e., a superset of a certain standard $B_2 \in \mathcal{B}_2$.

By hypothesis, B_2 contains the monad $\mu(\mathcal{B}_1)$. Hence, by virtue of **4.1.2**, $B_2 \in \tilde{\mathcal{B}}_1$ and, thus, $F_2 \in \tilde{\mathcal{B}}_1$. Now we are to use the transfer principle. ▷

4.1.8. *Let $f: X \to Y$, and let $\mathfrak{A}$ be a base of a filter in X, while $\mathcal{B}$ be such of that in Y. If the parameters are standard, then the following statements are valid:*

(1) $f(\mathfrak{A}) \supset \tilde{\mathcal{B}}$;

(2) $f^{-1}(\mathcal{B}) \subset \tilde{\mathfrak{A}}$;

(3) $\mu(f(\mathfrak{A})) \subset \mu(\mathcal{B})$;

(4) $f(\mu(\mathfrak{A})) \subset \mu(\mathcal{B})$.

◁ The following argument proves the equivalence (1) ↔ (2):

$$f(\mathfrak{A}) \supset \tilde{\mathcal{B}} \leftrightarrow (\forall B \in \mathcal{B})(\exists A \in \mathfrak{A}) \quad f(A) \subset B$$
$$\leftrightarrow (\forall B \in \mathcal{B})(\exists A \in \mathfrak{A}) \quad A \subset f^{-1}(B) \leftrightarrow f^{-1}(\mathcal{B}) \subset \tilde{\mathfrak{A}}.$$

Equivalence between (1) and (3) is ensured by **4.1.7**. To complete the proof, it should be remarked that by **4.1.6** (5) we get

$$f(\mu(\mathfrak{A})) \subset \mu(\mathcal{B}) \leftrightarrow \mu(\mathfrak{A}) \subset f^{-1}(\mu(\mathcal{B}))$$
$$\leftrightarrow \mu(\mathfrak{A}) \subset \mu(f^{-1}(\mathcal{B})) \leftrightarrow f^{-1}(\mathcal{B}) \subset \tilde{\mathfrak{A}}. \triangleright$$

4.1.9. Assuming the classical stance, we can reduce the formulation of **4.1.8**. Namely, we can omit the words 'if the parameters are standard', presenting **4.1.8** (4) as $*f(\mu(\mathfrak{A})) \subset \mu(\mathcal{B})$, where $*$ is the Robinson standardization. Common practice is to silently accept $f := *f$, which results in the most presentable and easily memorizable formulation. The same formulation is also often used under neoclassical and radical doctrines. In other words, if nonstandard analysis is used as a technique of studying the von Neumann universe, the 'given' parameters are, if not otherwise stated, considered standard sets, while the term an 'internal' set is replaced with a more common one, 'a set'. This convenient agreement, obviously, correlates with the qualitative ideas on standard objects. From now on, we shall continue sharing this free point of view, omitting, wherever possible, indications as to the type of the sets arising in all cases when it should not result in any serious misunderstanding.

4.1.10. *The following statements are valid:*

(1) *filters* $\mathfrak{F}_1$ *and* $\mathfrak{F}_2$ *have the least upper bound iff* $\mu(\mathfrak{F}_1) \cap \mu(\mathfrak{F}_2) \neq \varnothing$;

(2) *for any set of filters* $\mathfrak{E}$ *bounded from above we have*

$$\mu(\sup \mathfrak{E}) = \cap\{\mu(\mathfrak{F}) : \mathfrak{F} \in {}^\circ\mathfrak{E}\},$$

i.e., the monad of the intersection of filters is the intersection of their monads.

◁ Statement (1) results immediately from **4.1.7**. To prove (2), let us first remark that for $\mathfrak{F} \in {}^\circ\mathfrak{E}$ we have $\mathfrak{F} \leq \sup \mathfrak{E}$ and, hence, $\mu(\sup \mathfrak{E}) \subset \mu(\mathfrak{F})$. This ensures the inclusion $\mu(\sup \mathfrak{E}) \subset \cap\{\mu(\mathfrak{F}) : \mathfrak{F} \in {}^\circ\mathfrak{E}\}$. Let now $F \in {}^\circ\sup \mathfrak{E}$. By virtue of the properties of the filter, there is a standard finite set $\mathfrak{E}_0 \subset \mathfrak{E}$ such that $F \in \sup \mathfrak{E}_0$. According to **4.1.3**, with (1) taken into account, we deduce $F \supset \mu(\sup \mathfrak{E}_0) = \cap\{\mu(\mathfrak{F}) : \mathfrak{F} \in \mathfrak{E}_0\}$. And, finally,

$$\mu(\sup \mathfrak{E}) \supset \cap\{\mu(\mathfrak{F}) : \mathfrak{F} \in \mathfrak{E}_0, \mathfrak{E}_0 \in \mathcal{P}_{\mathrm{stfin}}(\mathfrak{E})\} = \cap\{\mu(\mathfrak{F}) : \mathfrak{F} \in^\circ \mathfrak{E}\}. \ ▷$$

4.1.11. *Let* $\mathfrak{U}$ *be an ultrafilter, i.e., one maximal by inclusion in the set of filters* $\mathfrak{F}(X)$ *of the set* X *under discussion, and* $\mathfrak{F}$ *be a filter:* $\mathfrak{F} \in \mathfrak{F}(X)$. *Then either* $\mu(\mathfrak{U}) \cap \mu(\mathfrak{F}) = \varnothing$, *or* $\mu(\mathfrak{U}) \subset \mu(\mathfrak{F})$.

◁ If $\mu(\mathfrak{U}) \cap \mu(\mathfrak{F}) \neq \varnothing$, then, by **4.1.10** (1), there is a least upper bound $\mathfrak{U} \vee \mathfrak{F} = \mathfrak{U}$. Hence, $\mathfrak{F} \subset \mathfrak{U}$, and, by **4.1.7**, valid is $\mu(\mathfrak{U}) \subset \mu(\mathfrak{F})$. ▷

4.1.12. Nonstandard criterion for an ultrafilter. *A filter* $\mathfrak{F}$ *in* X *is an ultrafilter iff its monad is easy to catch, i.e., for any standard subsets* A *and* B *in* X *such that* $A \cup B = X$, *then we have either* $\mu(\mathfrak{F}) \subset A$ *or* $\mu(\mathfrak{F}) \subset B$.

◁ → As soon as $\mu(\mathfrak{F}) \subset A \cup B$, we can assume $\mu(\mathfrak{F}) \cap A \neq \varnothing$. Since $A = \mu(\{\tilde{A}\})$, therefore, by **4.1.11**, $\mu(\mathfrak{F}) \subset A$.

← Let $\mathfrak{G} \supset \mathfrak{F}$. Then, according to **4.1.7**, $\mu(\mathfrak{G}) \subset \mu(\mathfrak{F})$. If A is standard and $A \supset \mu(\mathfrak{G})$, then either $A \supset \mu(\mathfrak{F})$, or $A' := X - A \supset \mu(\mathfrak{F})$, by hypothesis. The case $A' \supset \mu(\mathfrak{F})$ is impossible, since in that case we would get $\mu(\mathfrak{F}) \cap \mu(\mathfrak{G}) \subset A \cap A' = \varnothing$.

Hence, $A \supset \mu(\mathfrak{F})$, i.e., $A \in \mathfrak{F}$ (by **4.1.2**). Therefore, for any standard $A \in \mathfrak{G}$ we have $A \in \mathfrak{F}$. In line with the transfer principle, $\mathfrak{G} \subset \mathfrak{F}$; i.e., $\mathfrak{F}$ is an ultrafilter. ▷

4.1.13. Standard criterion for an ultrafilter. *A filter $\mathfrak{F}$ is an ultrafilter iff $A \cup B \in \mathfrak{F} \to A \in \mathfrak{F} \vee B \in \mathfrak{F}$.*

◁ → If $A \cup B \in \mathfrak{F}$, then the monad is caught; $\mu(\mathfrak{F}) \subset A \cup B$. If $\mu(\mathfrak{F}) \cap A \neq \varnothing$, then $\mu(\mathfrak{F}) \subset A$ and $A \in \mathfrak{F}$. If $\mu(\mathfrak{F}) \cap B \neq \varnothing$, then $\mu(\mathfrak{F}) \subset B$ and $B \in \mathfrak{F}$.

← Let $A \cup B = X$. If $A \in \mathfrak{F}$, then $A \supset \mu(\mathfrak{F})$. If $B \in \mathfrak{F}$, then $B \supset \mu(\mathfrak{F})$, i.e., the monad is easily caught. ▷

4.1.14. *Every limit of a filter is one of its adherent points. An adherent point of an ultrafilter is one of its limits.*

◁ It suffices to work in 'standard entourage'. Obviously, $\mathfrak{F} \to x \leftrightarrow \mu(\mathfrak{F}) \subset \mu(x) := \mu(\tau(x))$. Besides, $x \in \mathrm{cl}\,\mathfrak{F} := \bigcap\{\mathrm{cl}\, F : F \in \mathfrak{F}\} \leftrightarrow (\forall F \in \mathfrak{F})\ \ (\forall U \in \tau(x))\ \ U \cap F \neq \varnothing \leftrightarrow \mu(\mathfrak{F}) \cap \mu(x) \neq \varnothing$, by **4.1.10** (1). Therefore, the first part of the statement has been proved. If now $\mathfrak{F}$ is an ultrafilter, and $x \in \mathrm{cl}\,\mathfrak{F}$, then $\mu(\mathfrak{F}) \cap \mu(x) \neq \varnothing$. Based on the alternative described in **4.1.11**, we deduce $\mu(\mathfrak{F}) \subset \mu(x)$, i.e., $\mathfrak{F} \to x$. ▷

4.1.15. *Let $\mathfrak{E}$ be the cover of X. The following statements are equivalent:*

(1) *there is a standard finite subcover $\mathfrak{E}_0$ in, such $\mathfrak{E}_0 \in \mathcal{P}_{\mathrm{stfin}}(\mathfrak{E})$ that $X \subset \bigcup \mathfrak{E}_0$;*

(2) *the monad $\mu(\mathfrak{E})$ coincides with X;*

(3) *the monad $\mu(\mathfrak{E})$ is a standard set;*

(4) *the monad $\mu(\mathfrak{E})$ is an internal set;*

(5) *for every standard ultrafilter $\mathfrak{F}$ in X there is an $E \in \mathfrak{E}$ lying in $\mathfrak{F}$.*

◁ Implications (1) → (2) → (3) → (4) are obvious. If $\mu(\mathfrak{E})$ is an internal set, then by **4.1.6** (4) and **4.1.4** we conclude that $\mu(\mathfrak{E})$ is standard, i.e., there is a standard finite $\mathfrak{E}_0 \subset \mathfrak{E}$ such that $\mu(\mathfrak{E}) = \bigcup \mathfrak{E}_0 \supset X$. Hence, (4) → (1). Implication (1) → (5) is obvious. To prove (5) → (1), let us assume that, on the contrary, $(\forall^{\mathrm{stfin}} \mathfrak{E}_0) \bigcup \mathfrak{E}_0 \neq X$.

Let us consider $\mathcal{E}' := \{E' := X - E : E \in \mathcal{E}\}$. The family $\mathcal{E}'$ can obviously be considered as generating a filter base in X. Let $\mathfrak{F}$ be an ultrafilter containing this base. In this case there is an $E \in \mathcal{E}$ such that $E \in \mathfrak{F}$. Besides, by construction, $E' \in \mathfrak{F}$. Thus, we come to a contradiction. ▷

4.1.16. By way of concluding the present section, let us give some useful characteristic features based on the 'technique of internal sets'.

4.1.17. Cauchy principle. *Let $\mathfrak{F}$ be a standard filter in a standard set. Let, then, $\varphi := \varphi(x)$ be a certain internal property (i.e., $\varphi = \varphi^I$ for a set-theoretic formula φ). If for every remote element x we have $\varphi(x)$, then there is a standard set such that $(\forall x \in F)\ \varphi(x)$.*

◁ There is an internal set F with the required property (such is any distant element of the filter $\mathfrak{F}$). Hence, in line with the transfer principle, there is a standard F sought. ▷

4.1.18. Principle of a granted horizon. *Let X and Y be standard sets, $\mathfrak{F}$ and $\mathcal{G}$ be standard filters in X and Y, respectively, in which case $\mu(\mathfrak{F}) \cap {}^{\circ}X \neq \varnothing$. Let us now fix a remote set, a 'horizon', F in ${}^{a}\mathfrak{F}$. For a standard correspondence $f \subset X \times Y$ meeting $\mathfrak{F}$, the following statements are equivalent:*

(1) $f(\mu(\mathfrak{F}) - F) \subset F) \subset \mu(\mathcal{G})$;

(2) $(\forall F' \in^{a} \mathfrak{F})\, f(F' - F) \subset \mu(\mathcal{G})$;

(3) $f(\mu(\mathfrak{F})) \subset \mu(\mathcal{G})$.

◁ Obviously, (3) → (1) → (2). Hence, we have only to establish the implication (2) → (3).

Choose a $G \in^{\circ} \mathcal{G}$. Assume that for every standard F'' of ${}^{\circ}\mathfrak{F}$ there is an x of $F'' - F$, for which $f(x) \notin G$. According to the idealization principle, in this case there is an $x' \in \mu(\mathfrak{F})$ such that $x' \notin F$ and, at the same time, $f(x') \notin G$. Now consider $F' := F \cup \{x'\}$. Obviously, $F' \in {}^{a}\mathfrak{F}$, which results in a contradiction implying that for a certain standard $F'' \in \mathfrak{F}$ we have $f(F'' - F) \subset G$. Making use of the fact that there no standard elements X in F, we deduce:

$$(\forall^{st} G \in \mathcal{G})\,(\exists^{st} F \in \mathfrak{F})\,(\forall^{st} x \in F)\quad f(x) \in G.$$

Now the transfer principle is to be used. ▷

4.2. Monads in Topological Spaces

In this paragraph we shall study the properties of the monads of the filters of neighbourhoods in topological spaces.

4.2.1. Let (X,τ) be a standard *pretopological space*. Therefore, to every (standard) x of X a (standard) filter $\tau(x)$ is assigned in X. Let us denote $\mu(x) := \mu_\tau(x) := \mu(\tau(x))$. The elements $\mu(x)$ are called *infinitely close* to x. Obviously, $\mu(x)$ is the monad of the neighbourhood filter $\tau(x)$ of the point x. The pretopological space (X,τ) is termed *topological* if every neighbourhood of a point in X contains an open neighbourhood of this point. In other words, any $x \in {}^\circ X$ has an infinitely small neighbourhood $U \in \tau(x)$, for which $\mu(x') \subset \mu(x)$ for all $x' \in U$.

4.2.2. Let G be a (external) set in a topological space (X,τ). Let us set $h(G) := \cup\{\mu(x) : x \in {}^\circ G\}$. The set $h(G)$ is called the *halo* of G in X. The set $G \cap h(G)$ is called the *autohalo* or *nearstandard part* of G and is denoted by $\operatorname{nst}(G)$. If $G \supset h(G)$, then G is called *saturated* or, in more detail, *τ-saturated.* If for any $x \in G$ we have $\mu(x) \subset G$, then G is called *well-saturated* (*well-τ-saturated*).

4.2.3. *A standard set is open iff it is saturated.*

◁ If G is open and $x \in {}^\circ G$, then $G \supset \mu(x)$. Hence, G contains its halo. On the contrary, if $G \supset h(G)$, then, choosing a distant element U_x of the filter $\tau(x)$ for $x \in {}^\circ G$, we see that $G \supset U_x$. Then by the transfer principle, G is open. ▷

4.2.4. A standard point x of X is called a *microlimit* point of U provided $\mu(X) \cap U \neq \varnothing$. A standard set formed by all microlimit points of U is termed the *microclosure* of U and is denoted by $\operatorname{cl}_{\approx} U$.

4.2.5. *The microclosure* $\operatorname{cl}_{\approx} U$ *of an arbitrary internal set* U *is closed. If* U *is a standard set, then the microclosure* $\operatorname{cl}_{\approx} U$ *coincides with the closure* $\operatorname{cl} U$ *of the set* U.

◁ Let $A := \mathrm{cl}_{\approx} U = {}^*\{x \in X : \mu(x) \cap U \neq \varnothing\}$ and $y \in \mathrm{cl}\, A$. The task is to establish that $y \in A$. According to the transfer principle, y can be considered as standard element. Let us choose a standard open neighbourhood V of the point y. By hypothesis, there is a standard point $x \in V$ such that $x \in A$. From the definitions of standardization and a monad, we deduce that $V \supset \mu(x)$ and $\mu(x) \cap U \neq \varnothing$. Hence, $(\forall^{\mathrm{st}} V \in \tau(y))\ V \cap U \neq \varnothing$. Then by the idealization principle we deduce $\mu(y) \cap U \neq \varnothing$, i.e., $y \in \mathrm{cl}_{\approx} U$.

Let now U be standard. Obviously, ${}^\circ U \subset \mathrm{cl}_{\approx} U$. Hence, in line with the above-proved, $U \subset \mathrm{cl}_{\approx} U$ and $\mathrm{cl}\, U \subset \mathrm{cl}_{\approx} U$. If we choose $y \in \mathrm{cl} U$, then $(\forall^{\mathrm{st}} V \in \tau(y))\ \ V \cap U \neq \varnothing$. Hence, by the idealization principle, $\mu(y) \cap U \neq \varnothing$, i.e., $y \in \mathrm{cl}_{\approx} U$. ▷

4.2.6. *For a point x and a nonempty set U the following statements are equivalent:*

(**1**) *x is an adherent point of U;*

(**2**) *x is a microlimit point of U;*

(**3**) *there is a standard filter $\mathfrak{F}$ whose monad $\mu(\mathfrak{F})$ lies in the monad $\mu(x)$;*

(**4**) *there is a standard net $(x_\xi)_{\xi \in \Xi}$ of the U point such that its elements with infinitely large indices are infinitely close to x, i.e., $x_\xi \in \mu(x)$ for all $\xi \in {}^a\Xi$.*

◁ (1) → (2). If $x \in \mathrm{cl}\, U$, then there is a least upper bound $\tau(x) \vee \tilde{\{U\}}$. By **4.1.10** (1), we get

$$\varnothing \neq \mu(\tau(x) \vee \tilde{\{U\}}) = \mu(\tau(x)) \cap \mu(\tilde{\{U\}}) = \mu(x) \cap U,$$

the last implying $x \in \mathrm{cl}_{\approx} U$.

(2) → (3). If $x \in \mathrm{cl}_{\approx} U$, then $U \cap \mu(x) \neq \varnothing$. Hence, on the basis of **4.1.10** (1) we can construct a filter $\mathfrak{F}$ such that $A \in \mathfrak{F} \leftrightarrow A \supset U \cap \mu(x)$. Obviously, this is the filter sought.

(3) → (4). Let us set $\Xi := \tau(x)$ and $\xi_1 \leq \xi_2 \leftrightarrow \xi_1 \supset \xi_2$. Let us define x_ξ as an arbitrary point of a $F \in \mathfrak{F}$ such that $F \subset \xi$. Obviously, $(x_\xi)_{\xi \in \Xi}$ is the sought net. Indeed, by construction, $x_\xi \in \mu(x)$ for $\xi \in {}^a\Xi$.

(4) → (1). Let V be a standard neighbourhood of x, and η be an arbitrary large index

of Ξ. Obviously, $x_\xi \in V$ for $\xi \geq \eta$, since $\mu(x) \subset V$ and $\xi \in^a \Xi$. Hence, $V \cap U \neq \varnothing$ (since, by hypothesis, $x_\xi \in U$).

4.2.7. Nonstandard criteria for continuity. *Let* (X,τ) *and* (Y,σ) *be standard topological spaces,* $f: X \to Y$ *be a standard mapping, and* x *be a standard point in* X. *The following statements are equivalent*:

(1) f *is continuous at the point* x;

(2) *the function* f *sends points infinitely close to* x *to points infinitely close to* $f(x)$, *i.e.*,

$$(\forall x')\, x' \in \mu_\tau(x) \to f(x') \in \mu_\sigma(f(x)).$$

◁ It suffices to refer to **4.1.8**. ▷

4.2.8. For a set A in X by the symbol $\mu(A)$ we shall denote the intersection of standard open sets containing A. The set $\mu(A)$ is termed the *monad* of A. It should be remarked that $\mu(\varnothing) = \varnothing$. If $A \neq \varnothing$, then $\mu(A)$ is the monad of the neighbourhood filter of the set A.

4.2.9. *Let* (X,τ) *be a standard topological space. Then*

(1) (X,τ) *is a separated* (= T_1) *space iff* ${}^\circ\mu(x) = \{x\}$ *for any point* $x \in {}^\circ X$;

(2) (X,τ) *is a Hausdorff* (= T_2) *space iff* $\mu(x_1) \cap \mu(x_2) = \varnothing$ *for* $x_1, x_2 \in {}^\circ X$;

(3) (X,τ) *is regular if it is* T_1*-spaced and satisfies axiom* T_3: *for every closed standard* $A \subset X$ *and a standard point* $x \notin A$ *the following relation holds* $\mu(x) \cap \mu(A) = \varnothing$;

(4) (X,τ) *is normal if it is separated and satisfies axiom* T_4: *for any two disjoint closed sets* A *and* B *in* X *we have* $\mu(A) \cap \mu(B) = \varnothing$.

4.2.10. *The following statements are valid*:

(1) *a standard set is well-saturated iff it is open*;

(2) *the monad of an arbitrary set is well-saturated;*

(3) *the monad of a standard filter $\mathfrak{F}$ is well-saturated iff $\mathfrak{F}$ has a base of open sets;*

(4) *the monad $\mu(A)$ of an arbitrary open A is the least well-saturated set containing A, in which case the presentation $\mu(A) = \cup\{\mu(a) : a \in A\}$ is valid.*

◁ (1) If A is standard and well-saturated, then it is saturated and, hence, A is open (see **4.2.3**). If we know beforehand that A is standard and open, then, by the definition of monad, for $a \in A$ we get $\mu(a) \subset A$, i.e., A is well-saturated.

(2) The monad of a set is, by definition, the intersection of standard open sets. Hence, with (1) taken into account, it is well-saturated.

(3) If $\mathfrak{F}$ has a base of open standard sets then the proof follows from (1). If $\mu(\mathfrak{F})$ is well-saturated and V is a standard $\mathfrak{F}$ element, then $V \supset \mu(\mathfrak{F}) \supset \cup\{U_a : a \in F\}$, where F is an infinitely distant $\mathfrak{F}$ element, and U_a is an infinitely small neighbourhood of the point a. Since $\cup\{U_a : a \in F\} \in \mathfrak{F}$, the required result follows from the transfer principle.

(4) By (2), $\mu(A)$ is well-saturated. Moreover, according to (3) well-saturated is $B := \cup\{\mu(a) : a \in A\}$. We have to check if $B = \mu(A)$. The inclusion $B \subset \mu(A)$ is obvious. Let us assume, contrary to what has been proved, that $B \neq \mu(A)$, i.e., there is an $x \in \mu(A)$ such that $x \notin B$. Therefore, for every $a \in A$ there is a standard neighbourhood U_a of the point a with the property $x \notin U_a$. In other words, $(\forall a \in A)(\exists^{st} U_a)\, U_a \in \tau(a)$. Employing the idealization principle, we see that there is a standard finite set $\{a_1, \dots, a_n\} \subset A$ such that $A \subset U_{a_1} \cup \dots \cup U_{a_n}$. Hence, $x \in \mu(A) \subset U_{a_1} \cup \dots \cup U_{a_n}$, which is a contradiction. ▷

4.2.11. *Let (X,τ) be a separated topological space. The mapping $f : (X,\tau) \to (Y,\sigma)$ is continuous at the point x iff $f(\mu_\tau(x) \setminus U) \subset \mu_\sigma(f(x))$ for an infinitely small neighbourhood U of the point x.*

◁ By axiom T_1, $\mu_\tau(x) - U = \mu(x) - U$, where $\mu(x)$ is the monad of the filter $\tau(x)$ of the deleted neighbourhoods of x, i.e., $V \in \tau(x) \leftrightarrow V \cup \{x\} \in \tau(x)$. Obviously, $\mu(x) = \mu_\tau(x) - \{x\}$, in which case $U - \{x\}$ is an infinitely small element of $\tau(x)$. Using the principle of a granted horizon **4.1.18**, we see that $f(\mu(x) - U) \subset \mu_\sigma(f(x)) \leftrightarrow$ $\leftrightarrow f(\mu(x)) \subset \mu_\sigma(f(x)) \leftrightarrow f(\mu_\tau(x)) \subset \mu_\sigma(f(x))$. ▷

4.2.12. *Let $(Y_\xi, \sigma_\xi)_{\xi \in \Xi}$ be a family of topological spaces. Let, then, $(f_\xi : X \to Y_\xi)_{\xi \in \Xi}$ be* *a family of mappings, and* $\tau := \sup_{\xi\in\Xi} f_\xi^{-1}(\sigma_\xi)$ *be the initial topology in* X*, i.e., the weakest topology in which the mappings* f_ξ *are continuous for all* $\xi\in\Xi$*. In this case, for every standard point* $x\in X$,

$$\mu_\tau(x) = \bigcap_{\xi\in{}^\circ\Xi} f_\xi^{-1}(\sigma_\xi(f_\xi(x))).$$

◁ The required result is obtained from **4.1.8**. ▷

4.2.13. *A point* x' *of a Tychonoff product is infinitely close to the given point* x *provided the standard coordinates of* x' *are close to the corresponding standard coordinates of* x.

◁ Let, formally speaking, $(X_\xi, \tau_\xi)_{\xi\in\Xi}$ be a standard family of standard topological spaces. Let, then, $(\mathfrak{X}, \tau)$ be the Tychonoff product of $(X_\xi, \tau_\xi)_{\xi\in\Xi}$, i.e.,

$$\mathfrak{X} := \prod_{\xi\in\Xi} X_\xi; \quad \tau := \sup_{\xi\in\Xi} \mathrm{Pr}_\xi^{-1}(\tau_\xi),$$

where Pr_ξ is the projection operator of $\mathfrak{X}$ on X_ξ. Making use of **4.2.11** and **4.1.6** (5), for $x\in{}^\circ\mathfrak{X}$, we deduce

$$\mu(x) = \bigcap_{\xi\in{}^\circ\Xi} \mu(\mathrm{Pr}_\xi^{-1}(\tau_\xi(x_\xi))) = \bigcap_{\xi\in{}^\circ\Xi} \mathrm{Pr}_\xi^{-1}(\mu(\tau_\xi(x_\xi))).$$

It should be remarked that for $\xi\in{}^\circ\Xi$ we have $x'\in\mathrm{Pr}_\xi^{-1}(\mu(\tau_\xi(x_\xi))) \leftrightarrow \mathrm{Pr}_\xi\, x'\in\mu(\tau_\xi(x_\xi))$, i.e.,

$$\mathrm{Pr}_\xi^{-1}(\mu(\tau_\xi(x_\xi)) = \mu_{\tau_\xi}(x_\xi)\times \prod_{\eta\neq\xi} X_\eta.$$

Therefore, for every standard $\xi\in\Xi$ (cf. **4.1.6** (5)), we have

$$\mathrm{Pr}_\xi(\mu(x)) = \mu(\tau_\xi(x_\xi)),$$

which completes the proof. ▷

4.3. Nearstandardness and Compactness

Proximity to a standard point arising in topological spaces makes it possible to give convenient criteria of compact spaces. Obtaining these criteria is the basic topic of the present section.

4.3.1. A point x of the topological space (X,τ) is called *nearstandard* or, in more detail, τ-nearstandard, if $x \in \mathrm{nst}(X)$, i.e., if for a certain standard $x' \in {}^\circ X$ we have $x \in \mu(x')$.

4.3.2. *A point $x \in X$ is nearstandard iff for every standard open cover $\mathcal{E}$ of the set X we have $x \in \mu(\mathcal{E})$. In other words,*

$$\mathrm{nst}(X) = \bigcap\{\mu(\mathcal{E}) : \mathcal{E} \text{ is an open cover of } X\}.$$

◁ Let first $x \in \mathrm{nst}(X)$ and $x' \in {}^\circ X$ be such that $x \in \mu(x')$. For an open cover $\mathcal{E}$ there is a standard element $E \in \mathcal{E}$ such that $x' \in \mathcal{E}$, i.e., $\mu(x') \subset E$ (see **4.2.3**). Therefore, $x \in \mu(x') \subset E \subset \mu(\mathcal{E})$. Let now $x \notin \mathrm{nst}(X)$. Then for any $x' \in {}^\circ X$ we have $x \notin \mu(x')$. Hence, there is a standard open neighbourhood $U_{x'}$ of the point x' for which $x \notin U_{x'}$. The standardization $\mathcal{E} := {}^*\{U_{x'} : x' \in {}^\circ X\}$ is an open cover of X for which $x \notin \mu(\mathcal{E})$. ▷

4.3.3. *Every nearstandard point is infinitely close to the only standard point iff the space considered is Hausdorff.*

◁ If τ is a Hausdorff topology, and $x', x'' \in {}^\circ X$, then $\mu(x') \cap \mu(x'') \neq \varnothing \rightarrow x' = x''$. Let, on the contrary, $x \in \mu(x') \cap \mu(x'')$ for $x', x'' \in {}^\circ X$. Since x is nearstandard, $x' = x''$ by hypothesis. Hence, $x' \neq x'' \rightarrow \mu(x') \cap \mu(x'') = \varnothing$. ▷

4.3.4. Let us determine the external correspondence $\mathrm{st}(x) := \{x' \in {}^\circ X : x \in \mu(x')\}$. In the Hausdorff case st is a mapping of $\mathrm{nst}(X)$ on ${}^\circ X$.

4.3.5. *For every internal U, the following presentation holds $\mathrm{cl}_{*} U = {}^*\mathrm{st}(U)$. In particular, a standard set U is closed iff $U = {}^*\mathrm{st}(U)$.*

◁ The proof is in **4.2.5**. ▷

4.3.6. Nonstandard criteria for compactness. *For a standard space X the following statements are equivalent:*

(1) *X is compact;*

(2) *every point of X is nearstandard;*

(3) *the autohalo of X is an internal set.*

◁ (1) → (2). Let $\mathscr{E}$ be an open cover of X. The monad $\mu(\mathscr{E})$ coincides with X by **4.1.15** (since X is compact). By virtue of **4.3.2** we come to the conclusion: $\mathrm{nst}(X) = \bigcap_{\mathscr{E}} \mu(\mathscr{E}) = X$.

(2) → (3). This is obvious.

(3) → (1). Let $\mathscr{E}$ be an open cover of X. Since $(\forall x \in \mathrm{nst}(X))\,(\exists^{\mathrm{st}} E \in \mathscr{E})\; x \in E$, according to the idealization principle, $(\exists^{\mathrm{stfin}} \mathscr{E}_0 \subset \mathscr{E}) \cup \mathscr{E}_0 \supset \mathrm{nst}(X) \supset {}^{\circ}X$. Hence, by the transfer principle, $\mathscr{E}_0$ is a cover of X. ▷

4.3.7. *Let C be a set in a topological space X. The following statements are equivalent:*

(1) *C is compact in the induced topology;*

(2) *C lies in the halo $h(C)$;*

(3) *the monad $\mu(C)$ coincides with the halo $h(C)$.*

◁ (1) → (2). As C is compact in the induced topology, we have $C \subset \mathrm{nst}(C) \subset h(C)$ (see **4.3.6**).

(2) → (3). It is obvious that we always have $h(G) = \cup\{\mu(x) : x \in {}^{\circ}G\} \subset \mu(G)$. By hypothesis, for every $x \in C$ there is a $y \in {}^{\circ}C$ which obeys the relation $x \in \mu(y)$. By **4.2.8** (2), $\mu(x) \subset \mu(y)$. Therefore, making use of **4.2.8** (4), we get $\mu(C) = \cup\{\mu(x) : x \in C\} \subset \cup\{\mu(y) : y \in {}^{\circ}C\} = h(C)$.

(3) → (1). Let $\mathscr{E}$ be a standard cover of C. By definition, $C \subset \mu(C) \subset h(C)$. Therefore (cf. **4.3.2**), $C \subset \mu(\mathscr{E})$. Hence, in line with **4.1.5**, there is a finite subcover of

C in $\mathscr{E}$. ▷

4.3.8. Nonstandard criterion for relative compactness. *For a regular space X and a set C in X the following statements are equivalent:*

(1) *C is relatively compact (i.e., $\mathrm{cl}\,C$ is compact);*

(2) *C lies in the nearstandard part of X.*

◁ (1) → (2). With no additional hypotheses, **4.3.7** obviously yields:

$$C \subset \mathrm{cl}\,C \subset h(\mathrm{cl}\,C) \subset h(X) = h(X) \cap X = \mathrm{nst}(X).$$

(2) → (1). Let us consider the closure $\mathrm{cl}\,C$, and let $\mathscr{E}$ be an open cover of $\mathrm{cl}\,C$. Hence, for every $c \in C$ there is an $E \in \mathscr{E}$ containing c. Let E_c be a closed neighbourhood of c contained in E. Obviously, the family $\mathscr{E}' := \{E_c : c \in C\}$ comprises a standard cover of $\mathrm{cl}\,C$. The family $\mathscr{E}' \cup \{X - \mathrm{cl}\,C\}$ forms a cover of X and, hence, from **4.3.1**, we deduce $C \subset \mathrm{nst}(X) \subset \mu(\mathscr{E}') \cup \{X - \mathrm{cl}C\}$. By virtue of **4.1.15**, there is a finite set $\mathscr{E}_0 \subset \mathscr{E}'$ cover C. Obviously, $\cup\mathscr{E}_0$ is closed, i.e., $\mathscr{E}_0$ is a cover of $\mathrm{cl}\,C$. Every element of $\mathscr{E}_0$ is, by construction, a subset of an element of $\mathscr{E}$. Therefore, it is possible to refine a finite subcover of $\mathrm{cl}\,C$ from the initial $\mathscr{E}$. ▷

4.3.9. Criterion **4.3.8** allows strengthening. Namely, the microclosure of an arbitrary internal subset of the nearstandard part of an arbitrary Hausdorff space proves to be compact.

4.3.10. *Let $\mathcal{X} := \prod_{\xi \in \Xi} X_\xi$ be a standard product of standard topological spaces. A point $x \in \mathcal{X}$ is nearstandard iff nearstandard are its standard coordinates $x_\xi \in \mathrm{nst}(X_\xi)$ for $\xi \in {}^\circ\Xi$.*

◁ If $x \in \mathrm{nst}(\mathcal{X})$, then, by **4.1.12**, for a certain $y \in {}^\circ\mathcal{X}$ and any $\xi \in {}^\circ\Xi$ we get $x_\xi \in \mu(y_\xi)$. Now we have to remark that, by the transfer principle, $y_\xi \in {}^\circ X_\xi$. Assume that we know beforehand that $x_\xi \in \mathrm{nst}(X_\xi)$ for $\xi \in {}^\circ\Xi$. Consider the external function $y : \xi \to \mathrm{st}(x_\xi)$ from ${}^\circ\Xi$ to $\bigcup_{\xi \in \Xi} {}^\circ X_\xi$. Obviously, by virtue of **4.1.12**, for the standardization ${}^*y$ we get ${}^*y \in {}^\circ\mathcal{X}$ and $x \in \mu({}^*y)$. ▷

4.3.11. Tychonoff theorem. *The Tychonoff product of compact sets is compact.*

◁ According to the transfer principle one can assume that a standard family of standard spaces is considered here. In this case, making use of **4.3.10**, we conclude that every point of the product is nearstandard. ▷

4.3.12. Further on we will, as a rule, consider Hausdorff compact spaces. In conventional terminology such spaces are referred to in brief as *compacta.*

4.4. Infinite Proximity in Uniform Spaces

In uniform spaces there arises a symmetric, reflexive and transitive relation between internal points, i.e., their infinite proximity. Let us now study the most important constructions that pertain to this notion.

4.4.1. Let $(X,\mathcal{U})$ be a uniform space. This implies that $U := \{\varnothing\}$ if $X = \varnothing$. When $X \neq \varnothing$, then $\mathcal{U}$ is a filter in X^2 with the following properties:

(1) $\mathcal{U} \subset \widetilde{\{I_X\}}$;

(2) $(\forall U \in \mathcal{U})\ U^{-1} \in \mathcal{U}$;

(3) $(\forall V \in \mathcal{U})\ (\exists U \in \mathcal{U})\, U \circ U \subset V$.

4.4.2. Luxemburg criterion. *The filter $\mathcal{U}$ in X^2 is a uniformity on a (nonempty) set X iff the monad $\mu(\mathcal{U})$ is an external equivalence.*

◁ We have

$$\mu(\mathcal{U}) = \cap{}^{\circ}\mathcal{U} = \bigcap_{U \in {}^{\circ}\mathcal{U}} U = \bigcap_{U \in {}^{\circ}\mathcal{U}} U^{-1} = \mu(\mathcal{U})^{-1};$$

$$\mu(\mathcal{U}) \supset I_X;$$

$$\mu(\mathcal{U}) = \cap\{U \circ U : U \in {}^{\circ}\mathcal{U}\} \supset \mu(\mathcal{U}) \circ \mu(\mathcal{U}) \supset \mu(\mathcal{U}) \circ I_X \supset \mu(\mathcal{U}).$$

Here we have made use of the fact that U^{-1} and $U \circ U$ are standard under the condition that U is standard. Besides, by the definition of monad, $U \supset \mu(\mathcal{U})$ for $U \in {}^{\circ}\mathcal{U}$.

By virtue of **4.1.4**, the filter $\mathcal{U}$ is the standardization of all the supersets of its monad, i.e.,

$$U \in {}^\circ\mathcal{U} \leftrightarrow U \supset \mu(\mathcal{U}).$$

This implies $\mathcal{U} \subset \widetilde{\{I_X\}}$ and $U \in \mathcal{U} \to U^{-1} \in \mathcal{U}$. Let us consider an infinitely small element W of the filter $\mathcal{U}$. By virtue of the above, $U := W^{-1} \cap W \in \mathcal{U}$. Moreover, $U \circ U \subset \mu(\mathcal{U}) \circ \mu(\mathcal{U}) = \mu(\mathcal{U})$. Hence, for every standard $V \in \mathcal{U}$, there is a $U \in \mathcal{U}$ such that $U \circ U \subset V$. By the transfer principle we conclude that $\mathcal{U}$ is a uniformity. $\rhd$.

4.4.3. When applying the Luxemburg criterion it is expedient to bear in mind that not every equivalence relation on X^2 is a monad (i.e., it produces a uniformity in X). For instance, if we assume that $x, y \in \mathbf{R}$ are equivalent for $x - y \in {}^\circ\mathbf{R}$, then points equivalent to zero will compose the set ${}^\circ\mathbf{R}$ which is not a monad of any filter. This, in particular, implies that such an equivalence is produced by no standard uniformity.

4.4.4. If x, y are points of the space X with uniformity $\mathcal{U}$, then we call x and y *infinitely close* (relative to $\mathcal{U}$) and write $x \approx_{\mathcal{U}} y$, or simply $x \approx y$, provided $(x,y) \in \mu(\mathcal{U})$. For an arbitrary set A in X (possibly, external) the set $\mu_{\mathcal{U}}(A)$ is called the *microhalo* of the set A in X and denoted by ${}^{\approx}A$. If the set A is standard, then, taking the liberty of being inconsistent, the symbol ${}^{\approx}A$ can be also used to denote the halo $h(A)$ of the set A, meaning the equality $h(A) = {}^{\approx\circ}A$. It goes without saying that in this case the halo is calculated relative to the uniform topology $\tau_{\mathcal{U}}$ generated by $\mathcal{U}$. It should be remarked that in such a topology the monad of a standard point x consists, as it might be expected, of all the points infinitely close to it, i.e., it is the microhalo ${}^{\approx}x := {}^{\approx}\{x\}$ of this point. Sometimes, when a terminology less adequate to the essence of the matter is used, the microhalo ${}^{\approx}x$ of an internal point x is called the monad of this point.

4.4.5. A function f acting from a uniform space X into a uniform space Y and transferring infinitely close points into infinitely close ones, is termed *microcontinuous* over X.

4.4.6. *The following statements are valid:*

(1) *a standard function is microcontinuous iff it is uniformly continuous;*

(2) *a standard set consists of microcontinuous functions iff this set is (uniformly)* *equicontinuous.*

◁ (1) The uniform continuity of $f: X \to Y$ implies that $f^{\times}(\mathcal{U}_X) \supset \mathcal{U}_Y$, where $\mathcal{U}_X, \mathcal{U}_Y$ are the uniformities of X and Y, respectively, and $f^{\times}(x,x') := (f(x), f(x'))$ for $x, x' \in X$. Taking into account **4.1.8**, we deduce

$$f^{\times}(\mathcal{U}_X) \supset \mathcal{U}_Y \leftrightarrow \mu(f^{\times}(\mathcal{U}_X)) \subset \mu(\mathcal{U}_Y).$$

(2) The set $\mathcal{E} \subset Y^X$ is, as is known, termed equicontinuous, if $(\forall V \in \mathcal{U}_Y)$ $f^{\times -1}(V) = f^{-1} \circ V \circ f \in \mathcal{U}_X$. Therefore, for such an $\mathcal{E}$, by the transfer principle, we have $(\forall^{st} V \in \mathcal{U}_Y) \times (\exists^{st} U \in \mathcal{U}_X)\, (\forall f \in \mathcal{E})\, (\forall x, x' \in U)\, (f(x), f(x')) \in V$. In particular, if $x \approx x'$, then for every $f \in \mathcal{E}$ for any $V \in {}^{\circ}\mathcal{U}_Y$ we get $(f(x), f(x')) \in V$, i.e., $f(x) \approx f(x')$. Therefore, an equicontinuous standard set has only microcontinuous elements.

In order to prove the reverse implication, let us, for the sake of diversity, make use of the Cauchy principle (**4.1.17**). Indeed, for $V \in {}^{\circ}\mathcal{U}_Y$ and an arbitrary remote element $U \in {}^{a}\mathcal{U}_X$ we have $(\forall f \in \mathcal{E})\ f^{\times}(U) \subset V$. Hence, the same internal property is also valid for a certain standard $U \in \mathcal{U}_X$. Now we are to apply the transfer principle. ▷

4.4.7. *Let $(X, \mathcal{U}_X), (Y, \mathcal{U}_Y)$ be standard uniform spaces, and let f be an internal function; $f: X \to Y$. Let, then, ${}^{E}\mathcal{U}_X, {}^{E}\mathcal{U}_Y$ be the filters of external supersets of ${}^{\circ}\mathcal{U}_X, {}^{\circ}\mathcal{U}_Y$, respectively. In this case the following statements are valid:*

(1) *f is microcontinuous;*

(2) *$f: (X, {}^{E}\mathcal{U}_X) \to (Y, {}^{E}\mathcal{U}_Y)$ is uniformly continuous;*

(3) $(\forall^{st} V \in \mathcal{U}_Y)\, (\exists^{st} U \in \mathcal{U}_X)\ \ f^{\times}(U) \subset V$.

◁ (1) → (3). Let $V \in {}^{\circ}\mathcal{U}_Y$. For any remote element $U \in {}^{\circ}\mathcal{U}_X$ we have $(x, x') \in U \to x \approx x' \to f(x) \approx f(x')$, i.e., $f^{\times}(U) \subset V$. By the Cauchy principle (**4.1.17**), there is a standard U with the same property.

(3) → (1). Let us set $x \approx x'$ and a standard element $V \in \mathcal{U}_Y$. By hypothesis, for a certain standard $U \in \mathcal{U}_X$, we obtain $f^{\times}(U) \subset V$. In particular, $(f(x), f(x')) \in V$. Hence, $f(x) \approx f(x')$.

(3) → (2). This is obvious. ▷

4.4.8. Examples

(1) Let X be a set, and d be a *semimetric* (= *deviation*) on X. In other words, there are (standard) objects X and $d: X^2 \to \mathbf{R}$ such that

$$d(x,x) = 0 \quad (x \in \mathbf{X});$$
$$d(x,y) = d(y,x) \quad (x,y \in X);$$
$$d(x,y) \leq d(x,z) + d(z,y) \quad (x,y,z \in X).$$

Let us consider the cylinders $\{d \leq \varepsilon\} := \{(x,y) \in X_2 : d(x,y) \leq \varepsilon\}$ and the family $\mathcal{U}_d := \{\{d \leq \varepsilon\} : \varepsilon \in \mathbf{R}, \varepsilon > 0\}$. Obviously, $\mathcal{U}_d$ provides X with the structure of a uniform space, i.e., the conventional uniformity of a *semimetric space* (X,d). *It should be remarked that the monad of this uniformity is defined by the following equivalence relation:*

$$x \approx_d y \leftrightarrow d(x,y) \approx 0 \leftrightarrow d(x,y) \in \mu(\mathbf{R}).$$

(2) Let $(X, \mathfrak{M})$ be *a multimetric space*, i.e., $\mathfrak{M}$ is a *multimetric* (= a nonempty set of semimetrics on X). The monad $\mu(\mathfrak{M})$ is defined as the intersection of the monads of (standard) uniform spaces (X,d), where $d \in {}^{\circ}\mathfrak{M}$. Namely,

$$x \approx_{\mathfrak{M}} y \leftrightarrow (\forall d \in {}^{\circ}\mathfrak{M})\, d(x,y) \approx 0.$$

The monad $\mu(\mathfrak{M})$ is, undoubtedly, the monad of the uniformity $\mathcal{U}_{\mathfrak{M}} := \sup\{\mathcal{U}_d : d \in \mathfrak{M}\}$ of the multimetric space $(X, \mathfrak{M})$ under consideration. It would be expedient to recall that every uniform space $(X, \mathcal{U})$ is multimetrizable, i.e., $\mathcal{U} = \mathcal{U}_{\mathfrak{M}}$ for a suitable multimetric $\mathfrak{M}$.

(3) Let $(X, \mathcal{U})$ be a uniform space. Let us endow the space $\mathcal{P}(X)$ with the *Vietoris uniformity*, whose neighbourhood filterbase is composed of the sets:

$$\{(A,B) \in \mathcal{P}(X)^2 : B \subset U(A), A \subset U(B)\},$$

where $U \in \mathcal{U}$. Obviously, the monad $\mu_v := \mu_v(\mathcal{U})$ of the Vietoris uniformity has the form:

$$\mu_v = \{(A,B) : A \subset {}^{\bullet}B,\ B \subset {}^{\bullet}A\}.$$

(4) Let (X,τ) be a compactum, i.e., a Hausdorff compact space. This space is (uniquely) uniformizable, i.e., a filter of $\mathcal{U}$ such that the uniform topology $\tau_{\mathcal{U}}$ coincides with τ is the neighbourhood filter of the diagonal in X^2. Therefore, $\mu(\mathcal{U}) = \mu_{\tau \times \tau}(I_X)$. In other words, $x \approx x' \leftrightarrow \mathrm{st}(x) = \mathrm{st}(x')$, since $\mu_{\tau \times \tau}(x,x) = \mu_\tau(x) \times \mu_\tau(x)$ for a standard point

x (see **4.2.1**) and every point of X^2 is nearstandard (see **4.3.6**).

(5) Let X,Y be nonempty sets, $\mathcal{U}_Y$ be a uniformity in Y, and $\mathcal{B}$ be a family of subsets of X filtered upwards by inclusion. Let us consider the uniformity $\mathcal{U}$ in Y^X, which is called *the uniformity of uniform convergence on the sets of* $\mathcal{B}$. The family of $\mathcal{U}$ is a union of the supersets of the following elements:

$$V_{B,U} := \{(f,g) \in Y^X \times Y^X : g \circ I_B \circ f^{-1} \subset U\}$$

where $B \in \mathcal{B}$ and $U \in \mathcal{U}_X$. It is obvious that

$$(f,g) \in \mu(\mathcal{U}) \leftrightarrow (\forall^{\mathrm{st}} B \in \mathcal{B})(\forall^{\mathrm{st}} U \in U_Y)(\forall x \in B)\,(f(x),g(x) \in U$$
$$\leftrightarrow (\forall^{\mathrm{st}} B \in \mathcal{B})(\forall x \in B)\; f(x) \approx g(x) \leftrightarrow (\forall x \in \mu(\mathcal{B}))\; f(x) \approx g(x),$$

where, as usual, $\mu(\mathcal{B}) := \cup\, {}^{\circ}\mathcal{B}$ is the monad of the family $\mathcal{B}$. If $\mathcal{B} = \{X\}$, then we speak about *the strong uniformity* $\mathcal{U}_s$ on X. The following relation is obvious:

$$(f,g) \in \mu(\mathcal{U}_s) \leftrightarrow (\forall x \in X)\; f(x) \approx g(x).$$

If $\mathcal{B} = \mathcal{P}_{\mathrm{fin}}(X)$, then $\mu(\mathcal{B}) = {}^{\circ}X$ and, hence, for the corresponding *weak convergence* $\mathcal{U}_\omega$ (or, which is the same by definition, for *the uniformity of pointwise convergence*), we get

$$(f,g) \in \mu(\mathcal{U}_\omega) \leftrightarrow (\forall^{\mathrm{st}} x \in X)\; f(x) \approx g(x).$$

4.4.9. A set A is called *infinitely small* (relative to the uniformity $\mathcal{U}$), provided $A^2 \subset \mu(\mathcal{U})$, i.e., if any two points of A are infinitely close.

4.4.10. *For a standard filter* $\mathfrak{F}$ *in* $(X,\mathcal{U})$ *the following statements are valid:*

(1) *the monad* $\mu(\mathfrak{F})$ *is infinitely small;*

(2) *the filter* $\mathfrak{F}$ *is a Cauchy filter;*

(3) *for any* $U \in {}^{\circ}\mathcal{U}$ *there is an* $x \in {}^{\circ}X$ *such that* $\mu(\mathfrak{F})^2 \subset U(x)$.

$\triangleleft$ (1) $\rightarrow$ (2). Let $\mu(\mathfrak{F})^2 \subset \mu(\mathcal{U})$. Obviously, $\mu(\mathfrak{F})^2 = \mu(\mathfrak{F}^{\times})$, where $\mathfrak{F}^{\times} := \{F^2 : F \in \mathfrak{F}\}$, as

$$(x,y) \in \mu(\mathfrak{F}^{\times}) \leftrightarrow (\forall^{\mathrm{st}} F \in \mathfrak{F})\, x \in F \wedge y \in F \leftrightarrow x \in \mu(\mathfrak{F}) \wedge y \in \mu(\mathfrak{F}).$$

Therefore, $\mu(\mathfrak{F}^{\times}) \subset \mu(\mathcal{U})$, i.e., $\mathfrak{F}^{\times} \supset \mathcal{U}$. The last result implies that $\mathfrak{F}$ is a Cauchy filter.

(2) $\rightarrow$ (3). For $U \in {}^{\circ}\mathcal{U}$ there is a standard element $F \in \mathfrak{F}$ for which $F \times F \subset U$. If $x \in {}^{\circ}F$, then $(\forall^{\mathrm{st}} y \in F)\, y \in U(x)$. Hence, $F \subset U(x)$, and, moreover, $\mu(\mathfrak{F}) \subset U(x)$.

(3) $\rightarrow$ (1). Applying idealization, we see that $(\exists x \in X)\, \mu(\mathfrak{F}) \subset {}^{\approx}x$. Hence, $\mu(\mathfrak{F})$ is infinitely small. $\triangleright$

4.4.11. *A Cauchy filter converges iff its monad contains a nearstandard point.*

$\triangleleft$ $\rightarrow$ If $\mathfrak{F}$ is the filter under consideration, then $\mu(\mathfrak{F}) \subset \mu(x)$ as soon as $\mathfrak{F} \rightarrow x$. Any point of $\mu(\mathfrak{F})$ is nearstandard.

$\leftarrow$ Let $\mu(\mathfrak{F}) \cap {}^{\approx}x \neq \varnothing$. For $y \in \mu(\mathfrak{F})$ and $z \in \mu(\mathfrak{F}) \cap {}^{\approx}x$ we get $y \approx z \approx x$, i.e., $y \approx x$. Hence, $\mu(\mathfrak{F}) \subset \mu(x)$ and now we have to appeal to **4.1.7.** $\triangleright$

4.5. Pre-Neartandardness, Completeness and Total Boundedness

As is known, in uniform spaces a convenient indication of compactness, the classical Hausdorff criterion, is ensured. In the present section we consider its nonstandard analogues and the criteria of prenearstandardness pertaining to them in spaces of continuous functions.

4.5.1. *For an (internal) point of a (standard) uniform space X the following statements are equivalent:*

(1) *the microhalo of x is a monad of a certain (standard) filter in X;*

(2) *the microhalo of X is a monad of a certain Cauchy filter in X;*

(3) *the microhalo of x coincides with a monad of a minimal (by inclusion) Cauchy filter;*

(4) *the microhalo of x contains a certain (infinitely small) monad;*

(5) *There is a standard generalized sequence* $(x_\xi)_{\xi \in \Xi}$ *of* X *elements microconverging to* x, *i.e. such that for all remote elements* $\xi \in {}^a\Xi$ *we have* $x_\xi \approx x$.

$\lhd$ (1) → (2). If ${}^{\approx}x = \mu(\mathfrak{F})$ for a certain standard filter $\mathfrak{F}$, then $\mu(\mathfrak{F})$ is infinitely small (since the microhalo ${}^{\approx}x$ is infinitely small).

(2) → (3). Let ${}^{\approx}x = \mu(\mathfrak{F})$, $\mathfrak{F}'$ is a Cauchy filter and $\mathfrak{F}' \subset \mathfrak{F}$. Then, by virtue of **4.1.17**, $\mu(\mathfrak{F}') \supset \mu(\mathfrak{F}) = {}^{\approx}x$. If $y \in \mu(\mathfrak{F}')$, then since $\mu(\mathfrak{F}')$ is infinitely small, we get $y \approx x$, i.e., $\mu(\mathfrak{F}') = \mu(x) = \mu(\mathfrak{F})$. Hence, $\mathfrak{F}' = \mathfrak{F}$ (see **4.1.4**).

(3) → (4). It is obvious.

(4) → (1). Let us assume that ${}^{\approx}x \supset \mu(\mathfrak{F})$ and the filter $\mathfrak{F}$ is a Cauchy filter. Let us set $\mathfrak{F}' := \{U(F) : U \in \mathcal{U}_X, F \in \mathfrak{F}\}$. For $\mathcal{U} := \mathcal{U}_X$ we have the following relations:

$$
\begin{aligned}
{}^{\approx}\mu(\mathfrak{F}) &= \mu(\mathcal{U})(\mu(\mathfrak{F})) = \mu(\mathcal{U})\Big(\bigcap_{F \in {}^\circ\mathfrak{F}} F\Big) = \bigcap_{F \in {}^\circ\mathfrak{F}} \mu(\mathcal{U})(F) \\
&= \bigcap_{F \in {}^\circ\mathfrak{F}} \bigcap_{U \in {}^\circ\mathcal{U}} U(F) = \bigcap \{F' : F' \in {}^\circ\mathfrak{F}'\} = \mu(\mathfrak{F}').
\end{aligned}
$$

Obviously, ${}^{\approx}\mu(\mathfrak{F}) \supset {}^{\approx}x$. Hence, $\mu(\mathfrak{F}) = {}^{\approx}x = \mu(\mathfrak{F}')$.

(4) → (5). If $\mathfrak{F}$ is a filter and $\mu(\mathfrak{F}) \subset {}^{\approx}x$, then, choosing in a conventional way one point from each standard $F \in {}^\circ\mathfrak{F}$ and applying standardization, we construct the sequence required. Conversely, if $(x_\xi)_{\xi \in \Xi}$ microconverges to x, then the monad of the tail filter of this sequence is contained in the microhalo ${}^{\approx}x$. $\rhd$

4.5.2. A point x obeying one (and, hence, all) of the equivalent conditions **4.5.1** (1) - (4), is called *prenearstandard* in X. The external set of all prenearstandard points in X is denoted by $\mathrm{pst}(X)$.

4.5.3. *Nearstandard points (relative to the uniform topology) are prenearstandard.*

$\lhd$ Let $x \in \mathrm{nst}(X)$ for the space under study $(X, \mathcal{U})$. Hence, for a certain $y \in {}^\circ X$ we have $x \in {}^{\approx}y$. Therefore, ${}^{\approx}x \supset {}^{\approx}y = \mu(\tau_{\mathcal{U}}(y))$. In line with **4.5.1**, $x \in \mathrm{pst}(X)$. $\rhd$

4.5.4. *The image of a prenearstandard point under a uniformly continuous mapping is prenearstandard.*

◁ Let $\mathfrak{F}$ be a Cauchy filter and $\mu(\mathfrak{F}) \subset {}^{\approx}x$. Obviously, $f(\mathfrak{F})$ is a Cauchy filter in the image of X under the mapping f. Hence, $\mu(f(\mathfrak{F})) \subset {}^{\approx}f(x)$, i.e., $f(x)$ is a prenearstandard point (see **4.5.2**). ▷

4.5.5. *A point of the Tychonoff product of uniform spaces is prenearstandard iff its standard coordinates are prenearstandard.*

◁ Let $\mathfrak{X} := \prod_{\xi \in \Xi} X_\xi$ and $\mathcal{U}_{\mathfrak{X}} := \sup_{\xi \in \Xi} \Pr_\xi^{x-1}(\mathcal{U}_\xi)$ be the Tychonoff product of standard spaces $(X_\xi, \mathcal{U}_\xi)_{\xi \in \Xi}$. Let us choose an $x \in \mathrm{pst}(X)$. According to **4.5.1**, in $(\mathfrak{X}, \mathcal{U}_{\mathfrak{X}})$ there is a Cauchy filter $\mathfrak{F}$ such that ${}^{\approx}x = \mu(\mathfrak{F})$. For any standard $\xi \in \Xi$ we get, since $\Pr_\xi$ is continuous and by virtue of **4.4.6**, $\Pr_\xi({}^{\approx}x) \subset {}^{\approx} x_\xi$, i.e., ${}^{\approx}x_\xi \supset \Pr_\xi(\mu(\mathfrak{F})) = \mu(\Pr_\xi(\mathfrak{F}))$. Therefore, x_ξ is a prenearstandard point in X_ξ for $\xi \in {}^\circ\Xi$.

If for any $\xi \in {}^\circ\Xi$ we have ${}^{\approx}x_\xi = \mu(\mathfrak{F}_\xi)$ for a suitable choice of the filter $\mathfrak{F}_\xi$, then we can consider a filter

$$\mathfrak{F} := \sup_{\xi \in \Xi} \Pr_\xi^{-1}(\mathfrak{F}_\xi).$$

Obviously, the filter $\mathfrak{F}$ is standard and

$$\mu(\mathfrak{F}) = \bigcap_{\xi \in {}^\circ\Xi} \mu(\Pr_\xi^{-1}(\mathfrak{F}_\xi)) = \bigcap_{\xi \in {}^\circ\Xi} \Pr_\xi^{-1}(\mu(\mathfrak{F}_\xi))$$
$$= \bigcap_{\xi \in {}^\circ\Xi} \Pr_\xi^{-1}({}^{\approx}x_\xi) = \{y \in \mathfrak{X} : (\forall_\xi \in {}^\circ\Xi)\ y_\xi \approx x_\xi\} = {}^{\approx}x. \ ▷$$

4.5.6. Nonstandard criterion for completeness. *A standard space is complete iff each of its prenearstandard points is nearstandard.*

◁ → Let X be a complete space, i.e., such that every Cauchy filter in X converges. Let us choose an $x \in \mathrm{pst}(X)$. By **4.5.2**, for a certain Cauchy filter $\mathfrak{F}$ we have $\mu(\mathfrak{F}) = {}^{\approx}x$. According to completeness, there is a $y \in {}^\circ X$ such that $\mu(y) \supset \mu(\mathfrak{F})$. Hence, ${}^{\approx}y = \mu(y) \supset \mu(\mathfrak{F}) \supset {}^{\approx}x$. Therefore, ${}^{\approx}y = {}^{\approx}x$, i.e., $x \in \mathrm{nst}(X)$.

← Let $\mathrm{nst}(X) = \mathrm{pst}(X)$ and $\mathfrak{F}$ be a Cauchy filter in X. Let us choose an $x \in \mu(\mathfrak{F})$. Then ${}^{\approx}x \supset \mu(\mathfrak{F})$ (as $\mu(\mathfrak{F})$ is an infinitely small set). By **4.5.2**, $x \in \mathrm{pst}(X)$. Hence,

$x \in \mathrm{nst}(X)$. Now we have to use **4.4.11.** ▷

4.5.7. *The Tychonoff product of complete uniform spaces is complete.*

◁ By virtue of the transfer principle, it suffices to consider the case of standard parameters. If standard factors are complete, then their every prenearstandard point is nearstandard (see **4.5.5**). We now have to recall that nearstandard points are those with nearstandard standard coordinates (see **4.3.10**), while prenearstandard points are those with prenearstandard standard coordinates (by **4.5.5**). Besides, we make use of the fact that the uniform topology of a product is the product of the uniform topologies of factors. ▷.

4.5.8. *The space of functions acting into a complete space becomes complete when endowed with the strong uniformity.*

◁ Let $(Y,\mathcal{U})$ be a complete standard uniform space, X be a standard set. Choose a prenearstandard point $f \in Y^X$. By virtue of **4.5.2** and **4.4.8**, this implies that there is a standard sequence $(f_\xi)_{\xi \in \Xi}$ of the elements of Y^X, for which

$$(\forall \xi \in {}^a\Xi)\,(\forall x \in X) \quad f_\xi(x) \approx f(x).$$

According to **4.5.7**, f is nearstandard in the weak uniformity, i.e., there is a standard element $g \in Y^X$ such that

$$(\forall \xi \in {}^a\Xi)\,(\forall^{\mathrm{st}} x \in X) \quad f_\xi(x) \approx g(x).$$

Hence, for every standard $x \in X$ the sequence $(f_\xi(x))_{\xi \in \Xi}$ converges to $g(x)$. By the transfer principle, $(\forall x \in X)\ f_\xi(x) \to g(x)$. Hence, $(\forall U \in {}^\circ\mathcal{U})\,(\forall x \in X)\ (f(x),g(x)) \in U$, which insures the fact that f is infinitely close to g in the strong uniformity. The proof is completed by referring to **4.5.6** and the transfer principle. ▷

4.5.9. *Let E be a set in a uniform space $(X,\mathcal{U})$. The following statements are equivalent:*

(1) *the set E is totally bounded, i.e., for every $U \in \mathcal{U}$ there is a finite set $E_0 \subset E$ such that $E \subset U(E_0)$ (for every $U \in \mathcal{U}$ there is a finite U-net);*

(2) *there is an internal finite cover of E by infinitely small internal sets;*

(3) *the set* E *has a finite skeleton, i.e., there is an internal finite set* E_0 *in* X *such that* E *lies in the microhalo* ${}^{\approx}E_0$;

(4) *the set* E *lies in the microhalo of an internal totally bounded set.*

◁ (1) ⇔ (2). Using the definition and principle of idealization, we deduce:

$$
\begin{aligned}
&(\forall^{\mathrm{st}} U \in \mathcal{U})(\exists E_0)(E_0 \subset E \wedge E_0 \in \mathcal{P}_{\mathrm{fin}}(X) \wedge E \subset U(E_0)) \\
&\leftrightarrow (\forall^{\mathrm{st\,fin}} \mathcal{U}_0 \subset \mathcal{U})(\exists E_0)(\forall U \in \mathcal{U}_0)(E_0 \subset E \wedge E_0 \in \mathcal{P}_{\mathrm{fin}}(X) \wedge E \\
&\subset U(E_0)) \leftrightarrow (\exists E_0)(\forall^{\mathrm{st}} U \subset \mathcal{U})(E_0 \subset E \wedge E_0 \in \mathcal{P}_{\mathrm{fin}}(X) \wedge E \subset U(E_0)) \\
&\leftrightarrow (\exists E_0 \subset E)(E_0 \in \mathcal{P}_{\mathrm{fin}}(X) \wedge E \subset {}^{\approx}E_0).
\end{aligned}
$$

(1) ↔ (3). Obviously, E is totally bounded iff for any standard $U \in \mathcal{U}$ there is a finite cover $\{E_1,\dots,E_n\}$ of the set E such that $E_k \times E_k \subset U$ (i.e., E_k is small of order U) for $k = 1,\dots,n$. Now the idealization principle is to be used.

(3) ↔ (4). This is selfevident.

(4) ↔ (1). Let U be a standard entourage. There is a symmetric element $V \in {}^{\circ}\mathcal{U}$ for which $V \circ V \subset U$. Obviously, for a finite E' in X we have $V(E') \supset E_0$, where E_0 is a given totally bounded set with the property ${}^{\approx}E_0 \supset E$. Hence, $U(E') \supset V \circ V(E') \supset V(E_0) \supset E$. ▷

4.5.10. *In every standard uniform space there is a universal finite skeleton, i.e., a common internal finite skeleton for all totally bounded standard sets of the initial space.*

◁ Recalling that the union of a finite number of totally bounded sets is totally bounded and making use of **4.5.9**, for a space X, a finite standard family $\mathcal{E}$ of totally bounded sets and a standard finite family $\mathcal{U}_0 \subset \mathcal{U}_X$ one can choose a common finite set in X which serves as a U-net of any $E \in \mathcal{E}$ for any $U \in \mathcal{U}_0$. Now employ idealization. ▷

4.5.11. Nonstandard criteria for total boundedness. *For a uniform space* X *the following statements are equivalent:*

(1) X *is totally bounded;*

(2) *every point of* X *is prenearstandard;*

(3) *the set* $\mathrm{pst}(X)$ *is internal;*

(4) *the set X has a finite skeleton.*

◁ (1) → (2). Let $x \in X$. For any standard $U \in \mathcal{U}$ there is a standard point $x' \in {}^{\circ}X$ for which $x \in U(x')$ is an element of the finite standard U-net for X. Let us set $\mathfrak{F} := {}^*\{U(x') : U \in {}^{\circ}\mathcal{U}\}$. Obviously, F is a Cauchy filter (see **4.4.10**). In this case, by construction, $x \in \mu(\mathfrak{F})$, i.e., $x \in \mathrm{pst}(X)$.

(2) → (3). This is obvious.

(3) → (1). Let us assume that for a certain standard $U \in \mathcal{U}$ and any finite standard $E \subset X$ the inclusion $\mathrm{pst}(X) \subset U(E)$ is not valid. According to the idealization principle this implies that there is an internal point $x \in \mathrm{pst}(X)$ with the property $x \notin U(y)$ for any $y \in {}^{\circ}X$. By definition **4.5.2**, we have ${}^{\approx}x = \mu(\mathfrak{F})$ for a suitable Cauchy filter $\mathfrak{F}$. Let us choose $F \in {}^{\circ}\mathfrak{F}$ in such a way that $F \times F \subset U$. Then for any $y \in {}^{\circ}F$ we get $x \in \mu(\mathfrak{F}) \subset U(y)$, which contradicts the assumption. Therefore, $(\forall^{\mathrm{st}} U \in \mathcal{U})(\exists^{\mathrm{st\,fin}} E \subset X)\ \ U(E) \supset \mathrm{pst}(X)$. Now we have to recall that $\mathrm{pst}(X) \supset {}^{\circ}X$.

(1) → (4). For the proof see **4.5.9**. ▷

4.5.12. Hausdorff criterion. *A uniform space is compact iff it is complete and totally bounded.*

◁ → If a space X is compact (and standard), then every point in it is nearstandard and, hence, prenearstandard (by **4.5.3**), X is totally bounded (by **4.5.11**), and X is complete (by **4.5.6**).

Since X is totally bounded; therefore, according to **4.5.11**, $X = \mathrm{pst}(X)$. Since X is complete, by **4.6.6**, $\mathrm{pst}(X) = \mathrm{nst}(X)$. And, finally, $X = \mathrm{nst}(X)$, i.e., X is compact (see **4.3.6**). ▷

4.5.13. *Let X be an arbitrary set, Y be a uniform space, and $f : X \to Y$ be a (standard) function. The following statements are equivalent:*

(1) *f is a totally bounded mapping, i.e.,* $\mathrm{im} f$ *is totally bounded in Y;*

(2) *there is an internal finite cover $\mathfrak{E}$ of the set X such that $f(E)$ is infinitely small for every $E \in \mathfrak{E}$, i.e., f is a nearstep function relative to $\mathfrak{E}$;*

(3) *there is an internal* $n \in \mathbf{N}$ *and a set* $\{X_1,\dots,X_n\}$ *of external mutually disjoint sets such that* $X_1 \cup \dots \cup X_n = X$ *and* $f(x) \approx x'$ *for all* $x, x' = X_k$ *and every* $k := 1,\dots,n$.

◁ (1) → (2). In line with **4.5.9**, there is an internal finite cover $\mathcal{E}$ of the set $\operatorname{im} f$ such that $E \in \mathcal{E} \to E_2 \subset \mu(\mathcal{U}_Y)$. Let us set $\mathcal{E}' := \{f^{-1}(E) : E \in \mathcal{E}\}$. Obviously, $\mathcal{E}'$ is the sought cover of X.

(2) → (3). This is obvious.

(3) → (1). Let us choose $y_k \in f(X_k)$ and set $E := \{y_k : k = 1,\dots,n\}$. Obviously, E is a finite internal set. By hypothesis, E is a skeleton of $f(X)$. Hence, according to **4.5.9**, $\operatorname{im} f$ is totally bounded. ▷.

4.5.14. *The space* $CB(X,Y)$ *of totally bounded mappings from* X *to* Y *is complete in the strong uniformity.*

◁ By virtue of **4.5.8**, it suffices to establish that $CB(X,Y)$ is closed. Hence, let a standard $f : X \to Y$ be such that for a certain totally bounded function g we have $(\forall x \in X)\ f(x) \approx g(x)$. Obviously, $\operatorname{im} f \subset {}^{\approx}\operatorname{im} g$. Making use of the fact that $\operatorname{im} g$ is completely limited, and taking into account **4.2.5** and **4.5.9**, we deduce: $f \in \operatorname{cl} CB(X,Y) \to f \in CB(X,Y)$. ▷

4.5.15. A finite cover $\mathcal{E}$ of the set X is called *tiny* if it is refined into every standard finite cover $\mathcal{E}_0$ of the standard set X, i.e., if every set of $\mathcal{E}$ is contained in a certain set of $\mathcal{E}_0$. The mapping which acts from X into a uniform space and is nearstep relative to every tiny cover of X, is called *microstep* on X.

4.5.16. Criterion for prenearstandardness in $CB(X,Y)$. *A function* $f : X \to Y$, *where* Y *is a complete uniform space, is prenearstandard in* $CB(X,Y)$ *(relative to the strong uniformity) iff* f *is microstep on* X *and the image of* f *is composed of nearstandard points of* Y.

◁ On the basis of **4.5.11** and **4.5.6** we conclude that f is nearstandard in the strong uniformity. Therefore, for a certain $g \in {}^{\circ}CB(X,Y)$ we get $f(x) \approx g(x)$ for all $x \in X$. Obviously, $\operatorname{im} f \subset {}^{\approx}\operatorname{im} g$. Besides, $\operatorname{im} g \subset \operatorname{pst}(Y)$ (see **4.5.13**). If now $\mathcal{E}$ is a tiny cover, then, making use of the definition of total boundedness, for every standard $V \in \mathcal{U}_Y$ one can find a standard finite cover $\mathcal{E}'$ in X such that $g(E)^2 \subset V$ for any $E \in \mathcal{E}'$. Therefore,

$(\forall E \in \mathscr{E})$ $g(E)^2 \subset V$, i.e., g is nearstep on $\mathscr{E}$. Hence, for $E \in \mathscr{E}$ and $x, x' \in E$ we have $g(x) \approx f(x) \approx f(x') \approx g(x')$, i.e., f is also nearstep relative to $\mathscr{E}$. Since $\mathscr{E}$ is arbitrary, the mapping f is microstep.

$\leftarrow$ Since $\operatorname{im} f \subset \operatorname{nst}(Y)$, we have $(\forall x \in X)\,(\exists^{\mathrm{st}} y \in Y)\,(\forall^{\mathrm{st}} W \in \tau(y))\; f(x) \in W$. Applying the rule for introducing standard functions, we get $(\forall^{\mathrm{st}} W(\cdot))\,(\forall x \in X)$ $(\exists^{\mathrm{st}} y \in Y)$ $f(x) \in\ \in W(y)$. According to the idealization principle, we conclude $(\forall^{\mathrm{st}} W(\cdot))\,(\exists^{\mathrm{st}}$ $\{y_1, \dots, y_n\})$ $(\forall x \in X)\,(\exists k)\; f(x) \in W(y_k)$. Let us now choose $V \in \mathscr{U}_Y$. By hypothesis, for every tiny cover $\mathscr{E}$ of the set X and for $E \in \mathscr{E}$ we have $f(E)^2 \subset V$. Applying the Cauchy principle **4.1.17** (taking it into account that tiny covers are remote elements of the directed set of finite covers), we see that there is a standard finite cover $\mathscr{E}_V$ such that $f(E)^2 \subset V$ for $E \in \mathscr{E}_V$.

Let us choose a corresponding standard cover $\mathscr{E}_V$ and a standard finite set Y_0 of Y elements, for which $\operatorname{im} f \subset V(Y_0)$.

Using $\mathscr{E}_V$ and Y_0, we can easily construct a standard step function f_V such that $(\forall x \in X)$ $(f_V(x), f(x)) \in V$. Obviously, for $U \in {}^{\circ}\mathscr{U}_Y$ obeying the conditions $U = U^{-1}$ and $U \circ U \subset V$ we get $(f_{V'}(x), f_{V''}(x)) \in V' \circ V''^{-1} \subset U \circ U \subset V$ for any $V', V'' \subset U$. Therefore, the standard net $(f_V)_{V \in \mathscr{U}_Y}$ (in more detail, ${}^*\{f_V \colon V \in {}^{\circ}\mathscr{U}_Y\}$) is fundamental. Let us denote by g its standard limit in $CB(X,Y)$. As above, we have $(\forall^{\mathrm{st}} V \in \mathscr{U}_Y)\,(\forall_x \in X)\,(g(x), f(x)) \in V$. And, finally, $g \approx f$ in the strong uniformity. Therefore, f is nearstandard, and, hence, also prenearstandard by virtue of completeness of $CB(X,Y)$ discussed in **4.5.14**. $\triangleright$

4.5.17. Nonstandard criteria for relative compactness. *In a complete separated space X the following statements are equivalent for a set E:*

(1) *E is relatively compact;*

(2) *E is precompact (i.e., the completion of E is compact);*

(3) *E is totally bounded;*

(4) $E \subset \operatorname{pst}(X)$;

(5) $E \subset \operatorname{nst}(X)$;

(6) *E lies in the microhalo of a finite set;*

(7) $\operatorname{cl} U$ *has a finite skeleton.*

◁ Since X is complete, by **4.5.6**, pst(X) = nst(X). Therefore, (5) → (1) → (4) (see **4.3.8**). Obviously, (7) → (6) → (3) → (1) → (2). If (2) is fulfilled, then cl E is complete and totally bounded by the Hausdorff criterion. Making use of **4.5.11**, we deduce (2) → (7). ▷

4.5.18. Criteria for prenearstandardness in $C(X,Y)$. *Let X be a compact set, Y be a complete uniform space, and $C(X,Y)$ be the space of continuous functions from X to Y endowed with strong uniformity. For an internal element $f \in C(X,Y)$ the following statements are equivalent:*

(**1**) *f is prenearstandard;*

(**2**) *f is nearstandard;*

(**3**) *f is microcontinuous and transforms standard points into nearstandard points.*

◁ (1) → (2). Obviously, f is prenearstandard in Y^X in the strong uniformity by virtue of **4.5.4**, while by **4.5.8** and **4.5.6** f is nearstandard in Y^X, i.e., there is a standard $g \in Y^X$, for which $f(x) \approx g(x)$ for all $x \in X$. Let $(f_\xi)_{\xi \in \Xi}$ be a standard sequence in $C(X,Y)$ microconverging to f. Let us choose an $x' \approx x$ and remark that $f_\xi(x') \approx f_\xi(x)$ for all standard $\xi \in \Xi$ (as f_ξ is continuous and X is compact). Then (cf. **3.3.17** (3)) for a certain $\eta \in {}^a\Xi$ we get $f_\eta(x') \approx f_\eta(x)$. Hence, we deduce $g(x') \approx f(x') \approx f_\eta(x') \approx f_\eta(x) \approx f(x) \approx g(x)$. Therefore, the standard function g is microcontinuous and, hence, by **4.4.6**, $g \in C(X,Y)$.

(2) → (3). By hypothesis, for a certain continuous function g it is fulfilled that $g(x) \approx f(x)$ for all $x \in X$. Therefore, $f({}^\circ X) \subset {}^{\approx}g({}^\circ X) \subset {}^{\approx}g(X) \subset \mathrm{nst}(Y)$. Moreover, by **4.5.6**, g is microcontinuous and, hence, for $x' \approx x$ we get $f(x) \approx g(x) \approx g(x')$.

(3) → (1). By **4.5.3**, we are to make sure that (3) → (2). Let us choose a microcontinuous f for which $f({}^\circ X) \subset \mathrm{nst}(X)$. According to the construction principle, there is a standard g such that $g(x) \in {}^\circ f(x)$. Let us check that g is uniformly continuous. To this end, choose a standard entourage $V \in \mathcal{U}_Y$, and a standard $W \in \mathcal{U}_Y$ from the condition $W \circ W \circ W \subset V$. Making use of **4.5.7**, find a standard U of the unique uniformity $\mathcal{U}_X$ (see **4.4.8** (4)), so that $f^\times(U) \subset W$. At $(x,x') \in U$ for standard $x, x' \in {}^\circ X$ we get $(f(x), f(x')) \in W$, $(f(x'), g(x')) \in W$, $(g(x), f(x)) \in W$. Therefore, $(g(x), g(x')) \in W \circ W \circ W \subset V$. And, finally,

$$(\forall^{\mathrm{st}} V \in \mathcal{U}_Y)\,(\exists^{\mathrm{st}} U \in \mathcal{U}_X)\,(\forall^{\mathrm{st}} x, x' \in U)\,(g(x), g(x')) \in V.$$

Therefore, by the transfer principle, we get $g \in C(X,Y)$. Now for an arbitrary $x \in X$ we deduce $f(x) \approx f(x') \approx g(x') \approx g(x)$, where x' is the only standard point infinitely close to x. Hence, f is infinitely close to g in the strong uniformity. ▷

4.5.19. Ascoli-Arzela theorem. *Let X be a compact set, Y be a complete separated uniform space, and $E \subset C(X,Y)$. The set E is relatively compact in the strong uniformity iff E is equicontinuous and uniformly (totally) bounded (i.e., for a certain totally bounded C in Y we have $f(X) \subset C$ for all $f \in F$).*

◁ The proof follows from **4.5.18**, **4.5.17** and **4.4.6** (2). ▷

CHAPTER 5

INFINITESIMALS AND SUBDIFFERENTIALS

The nonstandard methods of analysis have been applied to various fields of mathematics. In the present chapter we shall consider the use of infinitesimals in *subdifferential calculus*, one of the new branches of functional analysis which originated from evolution of the theory of extremal problems. When studying optimization problems, a significant attention is paid to the search for convenient convex approximations to rather arbitrary functions and sets. The point is that for convex problems a quite powerful and effective technique of theoretical analysis has been developed and the corresponding calculation algorithms have been constructed. The ways of local approximation to sets and functions being developed in subdifferential calculus are related to constructing quite complex and often cumbersome formulas. The arising notions such as hypertangents, Rockafeller limits, and Clarke derivatives, seem to be difficult to understand when first encountered, since it is too complicated to comprehend the sense of their formal definitions.

Nonstandard analysis offers effective simplifying procedures since the use of the external notions legalized by it *'kills quantifiers'* essentially simplifies the complexity of perception of the standard constructions described. Below we shall basically study the evolution and examples of these operations for classifying one-sided tangents to arbitrary functions and sets.

It should be emphasized that many constructions described in the present chapter have a wider range of applicability than subdifferential calculus in the context of which the presentation below will be given.

5.1. Topologies in Vector Spaces

Studies of local approximations in vector spaces are associated with the characteristic features of the monads that introduce topologies agreeable with the structure available. It is these topologies that we shall study in the present section.

5.1.1. *Let U be a star-like set in a vector space, i.e.,* $[0,1]U \subset U$. *The set U absorbs a set*

V iff for any (and then for all) positive infinitesimal α we have $\alpha V \subset U$.

$\lhd$ As U absorbs V, by definition, there is a $\beta > 0$ for which $\beta V \subset U$. By the transfer principle, making use of the fact that both U and V are standard, we can conclude that $(\exists^{st}\beta > 0)\ \beta V \subset U$. If now $\alpha > 0$ and $\alpha \approx 0$, then $\alpha V = \alpha/\beta(\beta V) \subset \alpha/\beta U \subset U$. The remaining part of the statement is obvious. $\rhd$

5.1.2. Let x be a standard element of the standard vector space X under consideration. An external set $\{\alpha x\colon \alpha > 0, \alpha \approx 0\}$ is called *the radius-monad of x*, or *the infinitely small pointer to x*, or, finally, *the direction of x*. The family of the radius-monads of all the standard elements of X is termed *the direction monad* of this space and is denoted by $\mathrm{md}(X)$.

5.1.3. *A standard star-like set is absorbing in X iff it contains the direction monad* $\mathrm{md}(X)$ *of the space X.*

5.1.4. Nonstandard criterion for a vector topology. *Let X be a standard vector space over the basic field* $\mathbf{F}$, *and $\mathcal{N}$ be a standard filter in X. There is a vector topology τ on X such that $\mathcal{N} = \tau(0)$ iff the monad $\mu(\mathcal{N})$ of the filter $\mathcal{N}$ contains the direction monad* $\mathrm{md}(X)$ *and, besides, constitutes an external ${}^{\approx}\mathbf{F}$-submodule of X.*

(Here, as usual, ${}^{\approx}\mathbf{F} := \{t \in F\colon (\exists^{st} n \in \mathbf{N})\ |t| \le n\}$ is the finite part of the field $\mathbf{F}$ endowed with the natural structure of an external ring.)

$\lhd$ $\rightarrow$ Since addition is continuous at zero, $\mu(\mathcal{N}) + \mu(\mathcal{N}) = \mu(\mathcal{N})$; i.e., $\mu(\mathcal{N})$ is an external subgroup of X. Let $\alpha \in {}^{\approx}\mathbf{F}$ and $\mathcal{B}$ be a base of $\mathcal{N}$ consisting of balanced sets. If $n \in {}^{\circ}\mathbf{N}$ is such that $|\alpha| \le n$, then for $G \in {}^{\circ}\mathcal{B}$ and $x \in \mu(\mathcal{N})$ we have $\alpha/n\, x \in G$. Therefore, $\alpha/n\, x \in \cap\{G\colon G \in {}^{\circ}\mathcal{B}\} = \mu(\mathcal{B}) = \mu(\mathcal{N})$; hence, $\alpha x \in n\mu(\mathcal{N}) = \mu(\mathcal{N})$; and, finally, $\alpha\mu(\mathcal{N}) = \mu(\mathcal{N})$ for $\alpha \in {}^{\approx}\mathbf{F}$. We should observe that $\mathcal{N}$ has a base of absorbing sets and recall **5.1.3** in order to conclude $\mu(\mathcal{N}) \supset \mathrm{md}(X)$.

$\leftarrow$ Let us choose a $U \in {}^{\circ}\mathcal{N}$. By **4.1.4**, this implies $U \supset \mu(\mathcal{N})$. If W is an infinitely small element of $\mathcal{N}$, then its balanced hull V is also infinitely small (as $V \subset \mu(\mathcal{N})$). Besides, $V + V \subset \mu(\mathcal{N}) + \mu(\mathcal{N}) \subset \mu(\mathcal{N}) \subset U$. Hence,

$$(\forall^{st} U \in \mathcal{N})(\exists V \in \mathcal{N})\ \ (V \text{ is balanced} \wedge V + V \subset U).$$

By the transfer principle, we conclude that $\mathcal{N} + \mathcal{N} = \mathcal{N}$ and, moreover, that $\mathcal{N}$ has a base of balanced sets. By virtue of **5.1.3**, we also obsreve that $\mathcal{N}$ is composed of balanced standard sets. Therefore, $\mathcal{N}$, indeed, determines a vector topology on X. ▷

5.1.5. *For every point x of the monad $\mu(X) := \mu(\tau(0))$ of a topological vector space there is an infinitely large natural number $N \in \mathbf{N} - {}^{\circ}\mathbf{N}$ such that $Nx \in \mu(X)$.*

◁ If V is a standard neighbourhood of zero and $n \in {}^{\circ}\mathbf{N}$, then (see **5.1.4**) the set $A(n,V) := \{m \in \mathbf{N}: m \geq n \wedge mx \in V\}$ is nonempty (since $\mu(X) \subset V$). According to the transfer principle, there is an element N for which $(\forall^{\mathrm{st}} n \in \mathbf{N})(\forall^{\mathrm{st}} U \in \tau(0))\ \ N \in A(n,V)$. It is obvious that the element N is the sought one. ▷

5.1.6. In applications it is sometimes convenient to consider *nearvector topologies.* Such a topology τ in the space X is characterized by the following properties: first, multiplication of the vectors of X by every scalar of the basic field is continuous and, second, addition is also continuous in both variables. The pair (X,τ), as well as X itself, is in this case called *a neartopological vector space.* This notion is natural, which can be easily understood from the following obvious statement.

5.1.7. Nonstandard criterion for a nearvector topology. *Let X be a vector space over $\mathbf{F}$. There is a nearvector topology τ on X such that $\tau(0)$ coincides with a fixed filter $\mathcal{N}$ iff the monad $\mu(\mathcal{N})$ is an external vector space over the external field of standard scalars ${}^{\circ}\mathbf{F}$.*

5.1.8. In relation to **5.1.7** we should remark that the monad of the neighbourhood filter of zero of a nearvector space is an external convex set. An internal convex set U contains, obviously, arbitrary convex combinations of its elements, i.e., for finite sets $\{\alpha_1,\dots,\alpha_N\}$ of positive scalars comprising unity as a sum, and a set $\{u_1,\dots \mu_N\}$ of the U elements we have $\Sigma_{k=1}^{N}\alpha_k u_k \in U$. Here N is an arbitrary (internal !) element of $\mathbf{N}$. The formulated property termed *hyperconvexity* is not valid for external convex sets (since in the world of external sets the induction on internal natural numbers completely fails). The examples confirming the above statement can be easily found by making use of the following expedient proposition.

5.1.9. Nonstandard criterion for a locally convex topology. *A vector topology is locally convex iff the monad of its filter of zero neighbourhoods is hyperconvex.*

◁ → Standard neighbourhoods of a locally convex topology contain standard convex and, hence, hyperconvex neighbourhoods. The intersection of hyperconvex external sets is also hyperconvex.

← Every standard neighbourhood of zero of the topology in question τ contains the convex hull of an infinitely small neighbourhood (this hull as a whole lies in the monad $\tau(0)$ due to its hyperconvexity). By the transfer principle, we conclude that any neighbourhood in $\tau(0)$ contains a convex neighbourhood of zero. ▷

5.1.10. By way of concluding this section, let us, deviating a bit from the basic direction of presenting the material, remark that the nonstandard analysis of topological vector spaces and operators in them is associated with studying the location of points of different types. In this case, alongside with nearstandard and prenearstandard points we have already encountered, an important role is played by specific notions of a 'bornological type'. Let us recall here only the simplest notions of the kind.

5.1.11. *Let (X,τ) be a locally convex space, and x be an internal point of X. The following statements are equivalent*:

(1) *for any infinitely small $\alpha \in \mathbf{F}$ the relation $\alpha x \approx_\tau 0$ is valid*;

(2) $x \in \bigcap_{V \in \tau(0)} \bigcup_{n \in {}^\circ\mathbf{N}} nV$;

(3) *for any standard continuous seminorm p (an element of the spectrum of τ) we have $p(x) \in {}^{\text{▪}}\mathbf{F}$.*

◁ (1) ↔ (2). Let us make use of the Nelson algorithm:

$$
\begin{aligned}
&(\forall \alpha \in F)\quad (\alpha \approx 0 \to \alpha x \approx 0)\\
&\leftrightarrow (\forall^{st} V \in \tau(0))\,(\forall \alpha)\,((\forall^{st} n \in \mathbf{N})\quad |\alpha| \le n^{-1} \to \alpha x \in V)\\
&\leftrightarrow (\forall^{st} V \in \tau(0))\,(\forall \alpha)\,(\exists^{st} n \in \mathbf{N})\quad (|\alpha| \le n^{-1} \to \alpha x \in V)\\
&\leftrightarrow (\forall^{st} V \in \tau(0))\,(\exists^{st} n \in \mathbf{N})\quad x \in nV.
\end{aligned}
$$

(1) ↔ (3). If p is a continuous seminorm, then for any $t \in {}^{\text{▪}}\mathbf{R}$ we have $|t|\,p(x) = p(|t|\,x) \approx 0$ (see **4.2.7**). Hence, $p(x) \in {}^{\text{▪}}\mathbf{R}$.

(3) ↔ (1). For every standard continuous seminorm p we have $p(\alpha x) = |\alpha| p(x) \approx 0$ as soon as $|\alpha| \approx 0$. Now we have to obsrve that the last fact implies that αx is infinitesimal in the topology τ. ▷

5.1.12. A point x satisfying the equivalent conditions **5.1.11** (1)-(3) is called *finite* (or *accessible*) *in* (X,τ). In this case we write $x \in \operatorname{fin}(X,\tau)$, or simply $x \in \operatorname{fin}(X)$ if there is no necessity to indicate the topology, and say that x belongs to *the finite* (or *accessible*) *part* of the space X.

5.1.13. Nonstandard criterion for boundedness. *Let X be a standard locally convex space. A standard set U in X is limited iff it is composed of finite points of X, i.e., if $U \subset \operatorname{fin}(X)$.*

$\lhd$ $\rightarrow$ If U is bounded, then there is a standard $t \in {}^{\circ}\mathbf{R}$ such that $p(U) \leq t$ for a chosen continuous seminorm $p \in \mathfrak{M}_{\tau}$. Therefore, for $a \approx 0$ and $x \in U$ we get $p(\alpha x) \leq t\alpha$, i.e., $\alpha x \approx 0$.

Let us now make use of the sequential criterion of boundedness (for diversity). Hence, let (α_n) be a standard sequence of scalars convergent to zero, and (u_n) be a standard sequence of the points of U. We have to show that $\alpha_n u_n \rightarrow 0$ Let N be an infinitely large number. Then $\alpha_N \approx 0$ and, hence, by condition **5.1.11** and (1), we get $\alpha_N u_N \approx 0$. $\rhd$

5.1.14. A point x of a space X is called *bounded* and we write $x \in \operatorname{bd}(X)$ provided there is a standard bounded set containing x.

5.1.15. Nonstandard criteria for seminormability. *Let X be a (separeted) locally convex space. The following statements are equivalent:*

(1) *X is normable*;

(2) $\operatorname{bd}(X) = \operatorname{fin}(X)$;

(3) $\mu(X) \subset \operatorname{bd}(X)$.

$\lhd$ (1) $\rightarrow$ (2). Obviously, $\operatorname{bd}(X) = \operatorname{fin}(X)$ with no hypotheses on X. If X is normable, then $\operatorname{fin}(X) = \{x \in X : \|x\| \in {}^{\bullet}\mathbf{R}\}$, where $\|\cdot\|$ is a suitable norm. Therefore, $\operatorname{fin}(X)$ lies, for instance, in the ball $B_X := \{x \in X : \|x\| \leq 1\}$.

(2) $\rightarrow$ (3). Since $\mu(X)$ always lies in $\operatorname{fin}(X)$, the required result is obvious.

(3) $\rightarrow$ (1). Let U be an infinitely small neighbourhood in X. By condition, for every

$x \in U$ there is a standard set V such that V is bounded and $x \in V$. Therefore, according to the idealization principle, U lies in a certain bounded set. All we have to do now is to recall the Kolmogorov classical criterion. ▷

5.1.16. The above statement demonstrates, in particular, that in a general (not necessarily normable) case in a space there are more finite points than those bounded, while in a normable space X we, obviously, have $\mathrm{fin}(X) = \mathrm{bd}(X)$.

5.2. Classical Approximating and Regularizing Cones

In nonsmooth analysis there has been intensive search of convenient ways for local one-sided approximation to arbitrary functions and sets. A principal starting point of this search was the definition of subdifferential for a Lipshitz function given by F.Clarke [121]. Tangent cones and the corresponding derivatives constructed and studied in this respect are often defined by cumbersome and bulky formulas. Here we shall apply the nonstandard analysis as a method of 'killing quantifiers', i.e., simplifying complex formulas. Under a conventional supposition of standard entourage (in case when the free variables are standard (see **4.1.9**)) the Bouligand, Clarke and Hadamard cones and the regularizing cones pertaining to them prove to be determined by explicit infinitesimal constructions which appeal directly to infinitely close points and directions.

5.2.1. Let X be a real vector space. In this space let us, alongside with a fixed nearvector topology $\sigma := \sigma_X$ with the neighbourhood filter of zero $\mathcal{N}_\sigma := \sigma(0)$, single out a nearvector topology τ with a filter $\mathcal{N}_\tau := \tau(0)$. Following common practice, we introduce a relation of an infinite proximity associated with the corresponding uniformity: $x_1 \approx_\sigma x_2 \leftrightarrow x_1 - x_2 \in \mu(\mathcal{N}_\sigma)$, an analogous rule acting for τ. Below, if not otherwise stated, τ is considered to be a vector topology. In this case the monad of the neighbourhood filter $\sigma(x)$ will be denoted by $\mu(\sigma(x))$, while the monad $\mu(\sigma(0))$ simply by $\mu(\sigma)$.

5.2.2. In subdifferential calculus for fixed sets F in X and a point $x' \in X$ the following *Hadamard, Clarke, and Bouligand cones* are, in particular, considered:

$$\mathrm{Ha}(F, x') := \bigcup_{\substack{U \in \sigma(x') \\ \alpha'}} \mathrm{int}_\tau \bigcap_{\substack{x \in F \cap U \\ 0 < \alpha \le \alpha'}} \frac{F - x}{\alpha};$$

$$\mathrm{Cl}(F,x'):=\bigcap_{V\in\mathcal{N}_\tau}\ \bigcup_{\substack{U\in\sigma(x')\\ \alpha'}}\ \bigcap_{\substack{x\in F\cap U\\ 0<\alpha\le\alpha'}}\left(\frac{F-x}{\alpha}+V\right);$$

$$\mathrm{Bo}(F,x'):=\bigcap_{\substack{U\in\sigma(x')\\ \alpha'}}\mathrm{cl}_\tau\bigcup_{\substack{x\in F\cap U\\ 0<\alpha\le\alpha'}}\frac{F-x}{\alpha},$$

where, as usual, $\sigma(x'):=x'+\mathcal{N}$. If $h\in\mathrm{Ha}(F,x')$ then we sometimes say that F *is epi-Lipshitz* in x' along h. Obviously,

$$\mathrm{Ha}(F,x')\subset\mathrm{Cl}(F,x')\subset\mathrm{Bo}(F,x').$$

5.2.3. We also distinguish the *cone of hypertangents, a cone of feasible directions and the contingency of F* at the point x' by the following relations:

$$H(F,x'):=\bigcup_{\substack{U\in\sigma(x')\\ \alpha'}}\ \bigcap_{\substack{x\in F\cap U\\ 0<\alpha\le\alpha'}}\frac{F-x}{\alpha};$$

$$\mathrm{Fd}(F,x'):=\bigcap_{\alpha'>0}\frac{F-x'}{\alpha'};$$

$$K(F,x'):=\bigcap_{\alpha'}\mathrm{cl}_\tau\bigcap_{0<\alpha\le\alpha'}\frac{F-x'}{\alpha}.$$

For the sake of economizing words it is expedient to assume $x'\in F$. For instance, one can obviously say that the cones $F(F,x')$ and $K(F,x')$ are the Hadamard and Bouligand cones, respectively, for the case in which τ or σ is the discrete topology. Therefore, below we always assume $x'\in F$ with the following abbreviations taken to save the space:

$$(\forall^{\cdot}x)\varphi:=(\forall x\approx_\sigma x')\ \varphi:=(\forall x)\ (x\in F\wedge x\approx_\sigma x')\to\varphi,$$
$$(\forall^{\cdot}h)\varphi:=(\forall h\approx_\tau h')\ \varphi:=(\forall h)\ (h\in X\wedge h\approx_\tau h')\to\varphi,$$
$$(\forall^{\cdot}\alpha)\varphi:=(\forall\alpha\approx 0)\ \varphi:=(\forall\alpha)\,(\alpha>0\wedge\alpha\approx 0)\to\varphi.$$

The quantifiers $\exists^{\cdot}x$, $\exists^{\cdot}h$, $\exists^{\cdot}\alpha$ are determined in the natural way by duality, i.e., we assume that

$$(\exists^{\cdot}x)\varphi:=(\exists x\approx_\sigma x')\ \varphi:=(\exists x)\,(x\in F\wedge x\approx_\sigma x')\wedge\varphi,$$
$$(\exists^{\cdot}h)\varphi:=(\exists h\approx_\tau h')\ \varphi:=(\exists h)\ (h\in X\wedge h\approx_\tau h')\wedge\varphi,$$
$$(\exists^{\cdot}\alpha)\varphi:=(\exists\alpha\approx 0)\ \varphi:=(\exists\alpha)\,(\alpha>0\wedge\alpha\approx 0)\wedge\varphi.$$

Let us establish that the cones under discussion are defined by simple infinitesimal constructions.

5.2.4. *The Bouligand cone is the standardization of the* ∃∃∃-*cone*; *i.e., for a standard element* h' *we have*:

$$h' \in \mathrm{Bo}(F,x')\ (\exists x)\ (\exists \alpha)\ (\exists h)\ x + \alpha h \in F.$$

◁ The following equivalences follow from the definition of the Bouligand cone:

$$\begin{aligned}
& h' \in \mathrm{Bo}(F,x') \\
& \leftrightarrow (\forall U \in \sigma(x'))\,(\forall \alpha' \in \mathbf{R})\,(\forall V \in \mathcal{N}_\tau)\,(\exists x \in F \cap U) \\
& (\exists 0 < \alpha \le \alpha')\,(\exists h \in h' + V)\ x + \alpha h \in F \\
& \leftrightarrow (\forall U)\,(\forall \alpha')\,(\forall V)\,(\exists x)\,(\exists \alpha)\,(\exists h) \\
& (x \in F \cap U \wedge h \in h' + V \wedge 0 < \alpha \le \alpha' \wedge x + \alpha h \in F).
\end{aligned}$$

By virtue of the transfer principle we deduce:

$$\begin{aligned}
& h' \in \mathrm{Bo}(F,x') \leftrightarrow (\forall^{\mathrm{st}} U)\,(\forall^{\mathrm{st}} \alpha')\,(\forall^{\mathrm{st}} V)\ (\exists^{\mathrm{st}} x)\,(\exists^{\mathrm{st}} \alpha)\,(\exists^{\mathrm{st}} h) \\
& (x \in F \cap U \wedge h \in h' + V \wedge 0 < \alpha \le \alpha' \wedge x + \alpha h \in F).
\end{aligned}$$

Let us now use the weak idealization principle and obtain:

$$\begin{aligned}
& h' \in \mathrm{Bo}(F,x') \to (\exists x)\ (\exists \alpha)\,(\exists h)(\forall^{\mathrm{st}} U)\ (\forall^{\mathrm{st}} \alpha')\ (\forall^{\mathrm{st}} V) \\
& (x \in F \cap U \wedge h \in h' + V \wedge 0 < \alpha \le \alpha' \wedge x + \alpha h \in F) \\
& \to (\exists x \approx_\sigma x')\ (\exists \alpha \approx 0)\,(\exists h \approx_\sigma h')\ x + \alpha h \in F \\
& \to (\exists^{\cdot} x)\ (\exists^{\cdot} \alpha)\ (\exists^{\cdot} h)\ x + \alpha h \in F.
\end{aligned}$$

Let, in turn, a standard element h' belong to the standardization of the '∃∃∃-cone' Since standard elements of a standard filter contain the monads of this filter, we get

$$\begin{aligned}
& (\forall^{\mathrm{st}} U \in \sigma(x'))\,(\forall^{\mathrm{st}} \alpha' \in \mathbf{R})\,(\forall^{\mathrm{st}} V \in \mathcal{N}_\tau) \\
& (\exists x \in F \cap U)\ (\exists 0 < \alpha < \alpha')\,(\exists h \in h' + V)\ x + \alpha h \in F.
\end{aligned}$$

By virtue of the transfer principle, we conclude $h' \in \mathrm{Bo}(F,x')$. ▷

5.2.5. The just proved statement can be rewritten as

$$\mathrm{Bo}(F,x') = {}^*\{h' \in X\colon (\exists x)\,(\exists \alpha)\,(\exists h)\ x + \alpha h \in F\},$$

where, as usual, * is the symbol of standardization. In this respect the expressive notation is used:

$$\exists\exists\exists(F,x') := \mathrm{Bo}(F,x').$$

Further we shall use such notation without additional specification.

5.2.6. *The Hadamard cone is the standardization of the* $\forall\forall\forall$*-cone*:

$$\mathrm{Ha}(F,x') = {}^*\forall\forall\forall(F,x').$$

In other words, for standard h', F *and* x', *we have*

$$h' \in \mathrm{Ha}(F,x') \leftrightarrow (x' + \mu(\sigma)) \cap F + \mu(\mathbf{R}_+)(h' + \mu(\tau)) \subset F,$$

where $\mu(\mathbf{R}_+)$ *is the external set of positive infinitesimals.*

◁ The proof is obtained from **5.2.4** by duality, provided (which is by all means legitimate) we forget that F is present in $\exists x$. ▷

5.2.7. From the statements deduced above we can derive the following relations:

$$h' \in H(F,x') \leftrightarrow (\forall^{\bullet} x)(\forall^{\bullet} \alpha)\, x + \alpha h' \in F,$$
$$h' \in K(F,x') \leftrightarrow (\exists^{\bullet} x)(\exists^{\bullet} \alpha)\, x' + \alpha h \in F.$$

5.2.8. *For standard* h', F *and* x' *(under the conditions of weak idealization) the following statements are equivalent*:

(1) $h' \in \mathrm{Cl}(F,x')$;

(2) *there are infinitely small* $U \in \sigma(x')$, $V \in \mathcal{N}_\tau$ *and* $\alpha' > 0$ *such that*

$$h' \in \bigcap_{\substack{0<\alpha\le\alpha' \\ x\in F\cap U}} \left(\frac{F - x}{\alpha} + V\right);$$

(3) $(\exists U \in \sigma(x'))(\exists \alpha')(\forall x \in F \cap U)(\forall 0 < \alpha \le \alpha')(\exists h \approx_\tau h')\, x + \alpha h \in F.$

◁ Using obvious abbreviations, we can write

$$
\begin{aligned}
&h' \in \mathrm{Cl}(F,x') \\
&\leftrightarrow (\exists V)(\exists U)(\exists \alpha')(\forall x \in F \cap U)(\forall 0 < \alpha \le \alpha')(\exists h \in h' + V) \\
&x + \alpha h \in F.
\end{aligned}
$$

Applying the transfer principle and weak idealization, we get

$$
\begin{aligned}
&h' \in \mathrm{Cl}(F,x') \to (\forall^{\mathrm{st}} V)(\exists^{\mathrm{st}} U)(\exists^{\mathrm{st}} \alpha')(\forall x \in F \cap V) \\
&(\forall 0 < \alpha \le \alpha')(\exists h \in h' + V)(x + \alpha h \in F \\
&\to (\forall^{\mathrm{st}} \{V_1, \dots, Vn\})(\exists^{\mathrm{st}} U)(\exists^{\mathrm{st}} \alpha')(\exists^{\mathrm{st}} V)(\forall k = 1, \dots, n) \\
&V_k \supset V \wedge (\forall x \in F \cap U)(\forall 0 < \alpha \le \alpha')(\exists h \in h' + V)\, x + \alpha h \in F \\
&\to (\exists U)(\exists \alpha')(\exists V)(\forall^{\mathrm{st}} V')\, V' \supset V \wedge (\forall x \in F \cap U) \\
&(\forall 0 < \alpha \le \alpha')(\exists h \in h' + V)\, x + \alpha h \in F.
\end{aligned}
$$

Hence, we can obviously deduce that for some $V \in \mathfrak{N}_\tau$, $V \subset \mu(\tau)$ and $U \in \sigma(x')$, $U \subset \mu(\sigma) + x'$, as well as for an infinitesimal α we have (2) and, moreover, (3).

If, in turn, (3) is fulfilled, then, taking into account the definition of the relation $\approx$, we get

$$
\begin{aligned}
&(\forall^{\mathrm{st}} V)(\exists U)(\exists \alpha')(\forall x \in F \cap U)(\forall 0 < \alpha \le \alpha')(\exists h \in h' + V) \\
&x + \alpha h \in F.
\end{aligned}
$$

Thus, by the transfer principle, $h' \in \mathrm{Cl}(F,x')$. ▷

5.2.9. *The Clarke cone is (under the conditions of strong idealization) the standardization of the* $\forall\forall\exists$*-cone*

$$
\mathrm{Cl}(F,x') = {}^{*}\forall\forall\exists(F,x').
$$

In other words,

$$
h' \in \mathrm{Cl}(F,x') \leftrightarrow (\forall^{\cdot} x)(\forall^{\cdot} \alpha)(\exists^{\cdot} h)\, x + \alpha h \in F.
$$

◁ Let first $h' \in \mathrm{Cl}(F,x')$. Choose arbitrarily $x \approx_\sigma x'$ and $\alpha > 0$, $\alpha \approx 0$. For any standard neighbourhood of V, which is an element of the filter $\mathfrak{N}_\tau$, there is, by virtue of the transfer principle, an element h for which $h \in h' + V$ and $x + \alpha h \in F$. Applying strong idealization, we obtain

$$
\begin{aligned}
&(\forall^{\mathrm{st}} V)(\exists h)\,(h \in h' + V \wedge x + \alpha h \in F) \to (\exists h)(\forall^{\mathrm{st}} V)\, h \in h' \\
&+ V \wedge x + \alpha h \in F \to (\exists^{\cdot} h)\, x + \alpha h \in F,
\end{aligned}
$$

i.e., $h' \in \forall\forall\exists(F,x')$.

Let now $h' \in \forall\forall\exists(F,x')$. Choose an arbitrary standard neighbourhood V of the filter $\mathfrak{N}_\tau$. Let us fix an infinitesimal neighbourhood U of the point x' and a positive infinitesimal α'. Then, by condition, for a certain $h \approx_\tau h'$, we obtain

$$(\exists x \in F \cap U)\,(\forall 0 < \alpha \le \alpha')\; x + \alpha h \in F.$$

In other words,

$$(\forall^{\mathrm{st}} V)\,(\exists U)\,(\exists \alpha')\,(\forall x \in F \cap U)\,(\forall 0 < \alpha \le \alpha')\,(\exists h \in h' + V)$$
$$x + \alpha h \in F.$$

Then we apply the transfer principle and find $h' \in \mathrm{Cl}(F,x')$. ▷

5.2.10. Let us give an example of applying the obtained nonstandard criterion for the elements of the Clarke cone in order to deduce one of its basic (and well-known) properties. A more general statement will be derived below.

5.2.11. *The Clarke cone of an arbitrary set in a topological vector space is convex and closed.*

◁ By virtue of the transfer principle, it suffices to consider the situation in which the parameters (space, topology, set, etc.) are standard. Thus, take $h_0 \in \mathrm{cl}_\tau \mathrm{Cl}(F,x')$. Choose a standard neighbourhood V of $\mathfrak{N}_\tau$ and standard elements $V_1, V_2 \in \mathfrak{N}_\tau$ such that $V_1 + V_2 \subset V$. There is a standard element $h' \in \mathrm{Cl}(F,x')$ such that $h' - h_0 \in V'$. Besides, for any $x \approx_\tau x'$ and $\alpha > 0, \alpha \approx 0$ and for a certain h we have $h \in h' + V_2$ and $x + \alpha h \in F$. Obviously, $h \in h' + V_2 \subset h_0 + V_1 + V_2 \subset h_0 + V$ and, hence, $h_0 \in \mathrm{Cl}(F,x')$.

In order to prove that the Clarke cone is convex it suffices to observe that $\mu(\tau) + \mu(\mathbf{R}_+)\mu(\tau) \subset \mu(\tau)$, since the mapping $(x,\alpha,h) \to x + \alpha h$ is continuous. ▷

5.2.12. *Let θ be a vector topology and $\theta \ge \tau$ Then*

$$\forall\forall\exists(\mathrm{cl}_\theta F,x') \subset \forall\forall\exists(F,x').$$

Moreover, if $\theta \ge \sigma$, then

$$\forall\forall\exists(\mathrm{cl}_\theta F,x') = \forall\forall\exists(F,x').$$

◁ Let $h' \in \forall\forall\exists(\mathrm{cl}_\theta F, x')$ be a certain standard element of the cone in question. Let us choose elements $x \in F$ and $\alpha > 0$ such that $x \approx_\sigma x'$ and $\alpha \approx 0$. Obviously, $x \in \mathrm{cl}_\theta F$. Hence, for a certain $h \in_\tau h'$ we get $x + \alpha h \in \mathrm{cl}_\theta F$. Let us choose an infinitely small neighbourhood W from $\mu(\theta)$. The neighbourhood αW is also an element of $\theta(0)$ and, hence, for a certain $x'' \in F$ we have $x'' - (x + \alpha h) \in \alpha W$. Let us set $h'' := (x'' - x)/\alpha$. Obviously, $x + \alpha h'' \in F$ and, moreover, $\alpha h'' \in \alpha h + \alpha W$. Therefore, $h'' \in h + W \subset h' + \mu(\tau) + W \subset \subset h' + \mu(\tau) + \mu(\theta) \subset h' + \mu(\tau) + \mu(\tau) \subset h' + \mu(\tau)$, i.e., $h'' \approx_\tau h'$. Hence, $h' \in \forall\forall\exists(F, x')$.

Let now $\theta \geq \sigma$ and $h' \in \forall\forall\exists(F, x')$. Choose an arbitrary infinitely small α and an element $x \in \mathrm{cl}_\theta F$ such that $x \approx_\sigma x'$. Find an $x'' \in F$ for which $x - x'' \in \alpha W$, where $W \subset \mu(\theta)$ is an infinitely small symmetric neighbourhood of zero in θ. Since $\theta \geq \sigma$, we have $\mu(\theta) \subset \mu(\sigma)$, i.e., $x - x'' \in \mu(\theta) \subset \mu(\sigma)$ or, in other words, $x \approx_\sigma x' \approx_\sigma x''$. By definition (the element h', as usual, is considered standard!), for a certain $h \approx_\sigma h'$ we have $x'' + \alpha h \in F$. Let us set $h'' := (x'' - x)/\alpha + h$. Obviously, in this case we have

$$
\begin{aligned}
& h'' \in h + W \subset h + \mu(\theta) \subset h' + \mu(\theta) + \mu(\tau) \\
& \subset h' + \mu(\tau) + \mu(\tau) \subset h' + \mu(\tau),
\end{aligned}
$$

i.e., $h'' \approx_\tau h'$. Besides,

$$
x + \alpha h'' = x + (x'' - x) + \alpha h = x'' + \alpha h \in \mathrm{cl}_\theta F.
$$

And, finally, $h' \in \forall\forall\exists(\mathrm{cl}_\theta F, x')$. ▷

5.2.13. From the presentation obtained we can, in particular, deduce:

$$
\mathrm{Ha}(F, x') \subset H(F, x') \subset \mathrm{Cl}(F, x') \subset K(F, x') \subset \mathrm{cl}_\tau \mathrm{Fd}(F, x').
$$

Under the condition $\sigma = \tau$ for a convex F we obtain

$$
\mathrm{Fd}(F, x') \subset \mathrm{Cl}(F, x') \subset \mathrm{cl}\,\mathrm{Fd}(F, x');
$$

i.e.,

$$
\mathrm{Cl}(F, x') = K(F, x') = \mathrm{cl}\,\mathrm{Fd}(F, x').
$$

5.2.14. The nonstandard criteria of the Bouligand, Hadamard and Clarke cones presented above show these cones to be chosen from the list of eight possible cones with the infinitesimal prefix $(Qx)(Q\alpha)(Qh)$ (here Q is either $\forall$ or $\exists$). For a complete description of all these cones it obviously suffices to characterize $\forall\exists\exists$-cones and $\forall\exists\forall$-cones.

5.2.15. *The following presentation is valid:*

$$\forall\exists\exists(F,x') = \bigcap_{\substack{\alpha' \\ V\in\mathcal{N}_\tau}} \bigcup_{U\in\sigma(x')} \bigcap_{x\in F\cap U} \left(V + \bigcup_{0<\alpha\le\alpha'} \frac{F-x}{\alpha} \right).$$

◁ In order to prove this statement one should first of all realize that the required equality is an abbreviated presentation of the following statement: for standard h', F, x', we have:

$$(\forall^{\cdot} x)(\exists^{\cdot}\alpha)(\exists^{\cdot} h)\ x+\alpha h\in F$$
$$\leftrightarrow (\forall V\in\mathcal{N}_\tau)(\forall\alpha')(\exists U\in\sigma(x'))(\forall x\in F\cap U)$$
$$(\exists 0<\alpha\le\alpha')(\exists h\in h'+V)\ x+\alpha h\in F.$$

Therefore, for $h'\in\forall\exists\exists(F,x')$, and standard $V\in\mathcal{N}_\tau$ and $\alpha>0$ we can choose an internal subset of the monad $\mu(\sigma(x'))$ as the required neighbourhood of U. The consecutive application of transfer and strong idealization affords

$$(\forall^{st}V)(\forall^{st}\alpha')(\forall x\approx_\sigma x')(\exists 0<\alpha\le\alpha')(\exists h\in h'+V)\ x+\alpha h\in F$$
$$\to (\forall x\approx_\sigma x')(\forall^{st}\{V_1,\dots,V_n\})(\forall^{st}\{\alpha_1',\dots,\alpha_n'\})$$
$$(\exists h)(\exists\alpha)(\forall k:=1,\dots,n)\ (0<\alpha\le\alpha'\wedge h\in h'+V_k\wedge x+\alpha h\in F)$$
$$\to (\forall x\approx_\sigma x')(\exists h)(\exists\alpha)(\forall^{st}V)\ (h\in h'+V)\wedge(\forall^{st}\alpha')(0<\alpha\le\alpha'\wedge x$$
$$+\alpha h\in F)\to(\forall^{\cdot} x)(\exists^{\cdot} h)(\exists\alpha\approx 0)\ x+\alpha h\in F$$
$$\to h'\in{}^*\{h' : (\forall^{\cdot} x)(\exists^{\cdot}\alpha)(\exists^{\cdot} h)\ \ x+\alpha h\in F\}\to h'\in\forall\exists\exists(F,x').$$

Therefore, the proof is complete. ▷

5.2.16. Alongside with the eight infinitesimal cones of the classical series discussed above, there are nine more pairs of cones containing the Hadamard cone and lying in the Bouligand one. Such cones are evidently generated by changing the order of quantifiers. Five out of these pairs are constructed in a complex way by the type of the $\forall\exists\forall$-cone, the remaining pairs generated by permutations and dualizations of the Clarke and $\forall\exists\exists$ cones. For instance, in natural notation we have

$$\forall\alpha\forall h\exists x(F,x') = \bigcap_{U\in\sigma(x')\,\alpha'} \bigcup \operatorname{int}_\tau \bigcap_{0<\alpha\le\alpha'} \bigcup_{x\in F\cap U} \frac{F-x}{\alpha},$$
$$\exists h\exists x\forall\alpha(F,x') = \bigcup_{\alpha'} \bigcap_{U\in\sigma(x')} \operatorname{cl}_\tau \bigcup_{x\in F\cap U} \bigcap_{0<\alpha\le\alpha'} \frac{F-x}{\alpha},$$
$$\exists h\forall x\forall\alpha(F,x') = \bigcap_{\substack{U\in\sigma(x') \\ \alpha'}} \operatorname{cl}_\tau \bigcup_{\substack{0<\alpha\le\alpha' \\ x\in F\cap U}} \frac{F-x}{\alpha}.$$

The last cone is narrower than the Clarke cone and is convex when $\mu(\sigma) + \mu(\mathbf{R}_+)\,\mu(\tau) \subset \mu(\sigma)$, in which case it is denoted by $\mathrm{Ha}^+(F,x')$. It should be observed that

$$\mathrm{Ha}(F,x') \subset \mathrm{Ha}^+(F,x') \subset \mathrm{Cl}(F,x').$$

Also convex is the $\forall\alpha\exists h\forall x$-cone denoted by the symbol $\mathrm{In}(F,x')$. Obviously,

$$\mathrm{Ha}^+(F,x') \subset \mathrm{In}(F,x') \subset \mathrm{Cl}(F,x').$$

5.2.17. When calculating tangents to the composition of correspondence, use is made of special *regularizing cones.*

Namely, if $F \subset X \times Y$, where the vector spaces X and Y have topologies σ_X, τ_X and σ_Y, τ_Y, respectively, and $a' := (x', y') \in F$, we set $\sigma := \sigma_X \times \sigma_Y$ and

$$R^1(F,a') := \bigcap_{V \in \mathfrak{N}_{\tau_Y}} \bigcup_{\substack{W \in \sigma(a') \\ \alpha'}} \bigcap_{\substack{a \in W \cap F \\ 0 < \alpha \le \alpha'}} \left(\frac{F - a}{\alpha} + \{0\} \times V \right),$$

$$Q^1(F,a') := \bigcap_{V \in \mathfrak{N}_{\tau_Y}} \bigcup_{\substack{W \in \sigma(a') \\ \alpha' \\ U \in \mathfrak{N}_\sigma}} \bigcap_{\substack{a \in W \subset F \\ 0 < \alpha \le \alpha' \\ x \in U}} \left(\frac{F - a}{\alpha} + \{x\} \times V \right),$$

$$QR^2(F,a') := \bigcup_{\substack{W \in \sigma(a') \\ \alpha' \\ U \in \mathfrak{N}_\sigma}} \bigcap_{\substack{a \in W \cap F \\ 0 < \alpha \le \alpha' \\ x \in U}} \left(\frac{F - a}{\alpha} + (x, 0) \right).$$

The cones $R^2(F,a')$, $Q^2(F,a')$ and $QR^1(F,a')$ are determined by duality. Moreover, analogous notation is used for the case of product of more than two spaces, bearing in mind that the upper index over the symbol of an approximating set denotes the number of the coordinate on which the condition of the corresponding type is imposed. It should be also remarked that in applications we usually consider pairwise coinciding topologies: $\sigma_X = \tau_X$ and $\sigma_Y = \tau_Y$. Let us give obvious nonstandard criteria for the regularizing cones described.

5.2.18. *For standard vectors $s' \in X$ and $t' \in Y$ we have:*

$$(s',t') \in R^1(F,a')$$
$$\leftrightarrow (\forall a \approx_\sigma a',\, a \in F)\,(\forall \alpha \in \mu(\mathbf{R}_+))\,(\exists t \approx_{\tau_Y} t')\; a + \alpha(s',t) \in F,$$
$$(s',t') \in Q^1(F,a')$$
$$(\forall a \approx_\sigma a',\, a \in F)\,(\forall \alpha \in \mu(\mathbf{R}_+))\,(\forall s \approx_{\tau_X} s')\,(\exists t \approx_{\tau_Y} t')\,(a + \alpha(s,t) \in F,$$
$$(s',t') \in QR^2(F,a')$$
$$\leftrightarrow (\forall a \approx_\sigma a',\, a \in F)\,(\forall \alpha \in \mu(\mathbf{R}_+))\,(\forall s \approx_{\tau_X} s')\; a + \alpha(s,t') \in F.$$

5.2.19. As seen from **5.2.18**, the cones of the type QR^j are variations of the Hadamard cone, while the cones R^j are particular cases of the Clarke cone. In this case the cones R^j are also obtained by specialization of cones of the type Q^j by appropriate choice of discrete topologies. Under conventional suppositions the cones under discussion are convex. Let us prove this statement only for the cone Q^j, which is quite sufficient by virtue of what has been said above.

5.2.20. *If the mapping* $(a,\alpha,b)\to a+\alpha b$ *is continuous as acting from* $(X\times Y,\sigma)\times(\mathbf{R},\tau_{\mathbf{R}})\times(X\times Y,\tau_X\times\tau_Y)$ *to* $(X\times Y,\sigma)$, *then the cones* $Q^j(F,a')$ *are convex for* $j:=1,2$.

$\triangleleft$ By transfer, the proof can be carried out in standard entourage, i.e., the parameters considered can be assumed to be standard, and use can be made of criterion **5.2.18**. So, let (s',t') and (s'',t'') lie in $Q^1(F,x')$. For $a\approx_\sigma a'$ and $a\in F$, for a positive $\alpha\approx 0$ and $s\approx_{\tau_X}(s'+s'')$, we get, by virtue of **5.2.18**, $a_1:=a+\alpha(s-s'',t_1)\in F$ for a certain $t_1\approx_{\tau_Y} t'$. By condition, $\mu(\sigma)+\alpha(\mu(\tau_X)\times\mu(\tau_Y))\subset\mu(\sigma)$. Therefore, $a_1\approx_\sigma a$ and $a_1\in F$. Applying **5.2.18** again, we find $t_2\approx_{\tau_Y} t''$, for which $a_1+\alpha(s'',t_2)\in F$. Obviously, for $t:=t_1+t_2$ we get $t\approx_{\tau_Y}(t'+t'')$ and $a+\alpha(s,t)=a+\alpha(s-s'',t_1)+\alpha(s'',t_2)=a_1+$ $+\alpha(s'',t_2)\in F$, which was required, since the homogeneity of $Q_1(F,a')$ is ensured by stability of the monads of nearvector topologies under multiplication by standard scalars (see **5.1.4**). $\triangleright$

5.2.21. The analysis conducted shows that it is worthwhile introducing into consideration the cones P^j and S^j that employ the following direct standardizations:

$$(s',t')\in P^2(F,a')$$
$$(\exists s\approx_{\tau_X}s')\,(\forall t\approx_{\tau_Y}t')\,(\forall a\approx_\sigma a',\,a\in F)\,(\forall\alpha\in\mu(\mathbf{R}_+))\ a+\alpha(s,t)\in F,$$
$$(s',t')\in S^2(F,a')\leftrightarrow(\forall t\approx_{\tau_Y}t')\,(\exists s\approx_{\tau_X}s')\,(\forall a\approx_\sigma a',\,a\in F)$$
$$(\forall\alpha\in\mu(\mathbf{R}_+))\ a+\alpha(s,t)\in F.$$

The explicit forms of the cones P^j and S^j can, in principle, be written out (this problem will be discussed in the section to follow). It is, however, little use of the arising formulas (especially of that for S^j) since they are enormously cumbersome. But, as we have already convinced ourselves, formulas of the type obscure analysis by hiding the transparent 'infinitesimal' essence of the constructions.

5.2.22. *For* $j := 1,2$ *we have*

$$\mathrm{Ha}(F,a') \subset P^j(F,a') \subset S^j(F,a') \subset Q^j(F,a') \subset R^j(F,a') \subset \mathrm{Cl}(F,a').$$

In this case the cones in question are convex as soon as $\mu(\sigma) + \alpha(\mu(\tau_X) \times \mu(\tau_Y)) \subset \mu(\sigma)$ *for all* $\alpha > 0, \alpha \approx 0$.

◁ The inclusions to be proved are obvious from the nonstandard definitions of the corresponding cones. We have already pointed out that the majority of these cones is convex. Let us, to make the picture complete, establish that $S^2(F,a')$ is convex.

The fact that $S^2(F,a')$ is stable under multiplication by positive standard scalars results from indivisibility of a monad. Let us check if $S^2(F,a')$ is a semigroup. Hence, for standard (s',t') and (s'',t'') of $S^2(F,a')$, let us choose $t \approx_{\tau_Y} (t' + t'')$. Then $t - t'' \approx_{\tau_Y} t'$ and there is an $s_1 \approx_{\tau_X} s'$ which serves $t - t''$ in accordance with the definition of $S^2(F,a')$. Let us choose an $s_2 \approx_{\tau_X} s''$ which serves t'' in the same obvious sense. It is clear that $(s_1 + s_2) \approx_{\tau_X} (s' + s'')$. In this case for any $a \in F$ and $\alpha > 0$ such that $a \approx_\sigma a'$ and $\alpha \approx 0$ we get $a_1 := a + \alpha(s_1, t - t'') \in F$. Since a_1 is seen to be infinitely close (in the sense of σ) to a', from the choice of s_2 we conclude that $a_1 + \alpha(s_2, t'') \in F$. Hence, we can directly deduce $a + \alpha(s_1 + s_2, t) \in F$, i.e., $(s' + s'', t' + t'') \in S^2(F,a')$.

An analogous straightforward consideration can prove that $P^j(F,a')$ is convex. ▷

5.2.23. From the proof of **5.2.22** one can deduce that it is possible to consider convex extensions of the cones P^j and S^j, i.e., cones P^{+j} and S^{+j} obtained by 'leapfrogging the quantifier $\forall\alpha$'. For instance, the cone $P^{+2}(F,a')$ is determined by the relation

$$(s',t') \in P^{+2}(F,a') \leftrightarrow \forall\alpha \in \mu(\mathbf{R}_+))(\exists s \approx_{\tau_X} s')(\forall t \approx_{\tau_Y} t')$$
$$(\forall a \approx_\sigma a',\ a \in F)\ a + \alpha(s,t) \in F.$$

Obviously, it is also expedient (see **5.2.19**) to use the regularizations obtained by the specialization of the cone Ha^+ when choosing discrete topologies, the corresponding explicit formulas omitted. The importance of regularizing cones is associated with their role in subdifferentiating composite mappings which will be discussed in Section **5.5**.

5.3. Kuratowski and Rockafeller Limits

In the preceding section we have seen many constructions of interest for us to be associated with the procedure of transposing quantifiers in infinitesimal constructions. Similar constructions arise in various problems and pertain to certain facts of principal importance. Now we are going to discuss those which are most often encountered when subdifferentiating. Let us start with general observations concerning the Nelson algorithm.

5.3.1. *Let $\varphi = \varphi(x,y) \in$(ZFC), i.e., φ is a certain formula of Zermelo-Fraenkel theory containing no free variables but x, y. Then*

$$(\forall x \in \mu(\mathfrak{F}))\, \varphi(x,y) \leftrightarrow (\exists^{\mathrm{st}} F \in \mathfrak{F})\, (\forall x \in F)\, \varphi(x,y),$$
$$(\exists x \in \mu(\mathfrak{F}))\, \varphi(x,y) \leftrightarrow (\forall^{\mathrm{st}} F \in \mathfrak{F})\, (\exists x \in F)\, \varphi(x,y)$$

(here, as usual, $\mu(\mathfrak{F})$ is the monad of a standard filter $\mathfrak{F}$).

◁ It suffices to prove the implication $\rightarrow$ in the first of the equivalences. By hypothesis, for any remote element F of the filter $\mathfrak{F}$ the internal property $\psi := (\forall x \in F)\, \varphi(x,y)$ is fulfilled. Hence, by the Cauchy principle, ψ is valid for a standard F. ▷

5.3.2. *Let $\varphi = \varphi(x,y,z) \in$(ZFC) and $\mathfrak{F}, \mathfrak{G}$ be certain standard filters (in some standard sets). In this case*

$$(\forall x \in \mu(\mathfrak{F}))\, (\exists y \in \mu(\mathfrak{G}))\, \varphi(x,y,z)$$
$$\leftrightarrow (\forall^{\mathrm{st}} G \in \mathfrak{G})\, (\exists^{\mathrm{st}} F \in \mathfrak{F})\, (\forall x \in F)\, (\exists y \in G)\, \varphi(x,y,z)$$
$$\leftrightarrow (\exists^{\mathrm{st}} F(\cdot))\, (\forall^{\mathrm{st}} G \in \mathfrak{G})\, (\forall x \in F(G))\, (\exists y \in G)\, \varphi(x,y,z),$$
$$(\exists x \in \mu(\mathfrak{F}))\, (\forall y \in \mu(\mathfrak{G}))\, \varphi(x,y,z)$$
$$\leftrightarrow (\exists^{\mathrm{st}} G \in \mathfrak{G})\, (\forall^{\mathrm{st}} F \in \mathfrak{F})\, (\exists x \in F)\, (\forall y \in G)\, \varphi(x,y,z)$$
$$\leftrightarrow (\forall^{\mathrm{st}} F(\cdot))\, (\exists^{\mathrm{st}} G \in \mathfrak{G})\, (\exists x \in F(G))\, (\forall y \in G)\, \varphi(x,y,z)$$

(here the symbol $F(\cdot)$ denotes a function from $\mathfrak{G}$ to $\mathfrak{F}$).

◁ The proof consists of appealing to the idealization and construction principles with use made of **5.3.1.** ▷

5.3.3. *Let $\varphi = \varphi(x,y,z,u) \in$(ZFC) and let $\mathfrak{F}, \mathfrak{G}, \mathfrak{H}$ be three standard filters. When the set u is standard, the following relations are fulfilled:*

$$(\forall x \in \mu(\mathfrak{F}))(\exists y \in \mu(\mathfrak{B}))(\forall z \in \mu(\mathfrak{G}))\, \varphi(x,y,z,u)$$
$$\leftrightarrow (\forall G(\cdot))(\exists F \in \mathfrak{F})(\exists^{\mathrm{Fin}} \mathfrak{G}_0 \subset \mathfrak{G})(\forall x \in \mathfrak{F})$$
$$(\exists H \in \mathfrak{G}_0)(\exists y \in G(H))(\forall z \in H)\, \varphi(x,y,z,u),$$
$$(\exists x \in \mu(\mathfrak{F}))(\forall y \in \mu(\mathfrak{B}))(\exists z \in \mu(\mathfrak{G}))\, \varphi(x,y,z,u)$$
$$\leftrightarrow (\exists G(\cdot))(\forall F \in \mathfrak{F})(\forall^{\mathrm{Fin}} \mathfrak{G}_0 \subset \mathfrak{G})(\exists x \in \mathfrak{F})$$
$$(\forall H \in \mathfrak{G}_0)(\forall y \in G(H))(\exists z \in H)\, \varphi(x,y,z,u),$$

where $G(\cdot)$ is a function from $\mathfrak{G}$ to $\mathfrak{B}$, and the superscript $^{\mathrm{Fin}}$ *labelling a quantifier denotes its restriction to the class of nonempty finite sets.*

◁ By the Nelson algorithm, we deduce:

$$(\forall x \in \mu(\mathfrak{F}))(\exists y \in \mu(\mathfrak{B}))(\forall z \in \mu(\mathfrak{G}))\, \varphi$$
$$\leftrightarrow (\forall x \in \mu(\mathfrak{F}))(\forall^{\mathrm{st}} G(\cdot))(\exists^{\mathrm{st}} H \in \mathfrak{G})(\exists y \in G(H))(\forall z \in H)\, \varphi$$
$$\leftrightarrow (\forall^{\mathrm{st}} G(\cdot))(\forall x)(\exists^{\mathrm{st}} F \in \mathfrak{F})(\exists^{\mathrm{st}} H \in \mathfrak{G})$$
$$(x \in F \to (\exists y \in G(H))(\forall z \in H)\, \varphi)$$
$$\leftrightarrow (\forall^{\mathrm{st}} G(\cdot))(\exists^{\mathrm{stFin}} \mathfrak{F}_0)(\exists^{\mathrm{stFin}} \mathfrak{G}_0)(\forall x)(\exists F \in \mathfrak{F}_0)(\exists H \in \mathfrak{G}_0)$$
$$(F \in \mathfrak{F} \wedge H \in \mathfrak{G} \wedge (x \in F \to (\exists y \in G(H))(\forall z \in H)\, \varphi))$$
$$\leftrightarrow (\forall^{\mathrm{st}} G(\cdot))(\exists^{\mathrm{stFin}} \mathfrak{F}_0 \subset \mathfrak{F})(\exists^{\mathrm{stFin}} \mathfrak{G}_0 \subset \mathfrak{G})(\forall x)(\exists F \in \mathfrak{F}_0)$$
$$(x \in F \to (\exists H \in \mathfrak{G}_0)(\exists y \in G(H))(\forall z \in H)\, \varphi)$$
$$\leftrightarrow (\forall G(\cdot))(\exists^{\mathrm{Fin}} \mathfrak{F}_0 \subset \mathfrak{F})(\exists^{\mathrm{Fin}} \mathfrak{G}_0 \subset \mathfrak{G})(\forall x)$$
$$((\forall F \in \mathfrak{F}_0)\, x \in F \to (\exists H \in \mathfrak{G}_0)(\exists y \in G(H))(\forall z \in H)\, \varphi)$$
$$\leftrightarrow (\forall G(\cdot))(\exists^{\mathrm{Fin}} \mathfrak{F}_0 \subset \mathfrak{F})(\exists^{\mathrm{Fin}} \mathfrak{G}_0 \subset \mathfrak{G})(\forall x \in \cap \mathfrak{F}_0)$$
$$(\exists H \in \mathfrak{G}_0)(\exists y \in G(H))(\forall z \in H)\, \varphi.$$

Now we have to observe that for a nonempty finite $\mathfrak{F}_0$ lying in $\mathfrak{F}$ the relation $\cap \mathfrak{F}_0 \in \mathfrak{F}$ is valid by necessity. ▷

5.3.4. The discussed statement makes it possible to characterize explicitly the $\forall\exists\forall$-cones and similar constructions. The arising standard descriptions are obviously cumbersome. Let us now discuss the constructions most important for applications and pertaining to the prefixes of the type $\forall\exists$, $\forall\forall$, $\exists\forall$ and $\exists\exists$. Let us start with certain means allowing one to use the conventional language of infinitesimals for analyzing such constructions.

5.3.5. Let Ξ be a *direction*, i.e., a nonempty directed set. In line with the idealization principle, in Ξ there are internal elements majorizing ${}^{\circ}\Xi$. Let us recall (**4.1.6** (3)) that they are called *remote* or *infinitely large* in Ξ. Let us consider a standard basis of the tail filter

$\mathscr{B} := \{\sigma(\xi): \xi \in \Xi\}$, where σ is the order in Ξ. The monad of the tail filter is obviously composed of the remote elements of the direction considered. The following presentations are used: ${}^a\Xi := \mu(\mathscr{B})$ and $\xi \approx +\infty \leftrightarrow \xi \in {}^a\Xi$.

5.3.6. *Let* Ξ, H *be two directed sets, and* $\xi := \xi(\cdot): \mathrm{H} \to \Xi$ *be a mapping. The following statements are equivalent:*

(1) $\xi({}^a\mathrm{H}) \subset {}^a\Xi$;

(2) $(\forall \xi \in \Xi)\,(\exists \eta \in \mathrm{H})\,(\forall \eta' \geq \eta)\ \xi(\eta') \geq \xi$.

$\lhd$ Indeed, (1) implies that the tail filter of Ξ is coarser than the image of the tail filter of H, i.e., that in each tail of the direction of Ξ lies an image of a tail of H. The last statement is the contents of (2). $\rhd$

5.3.7. When equivalent conditions **5.3.6** (1) and **5.3.6** (2) are fulfilled, H is said to be a *subdirection* of Ξ (relative to $\xi(\cdot)$).

5.3.8. Let X be a certain set, and $x := x(\cdot): \Xi \to X$ be a net of elements of X (we also write $(x_\xi)_{\xi \in \Xi}$ or simply (x_ξ)). Let, then, $(y_\eta)_{\eta \in \mathrm{H}}$ be another net of elements of X. We say that (y_η) is a *Moore subnet* of the net (x_ξ), or a *strict subnet* of (x_ξ), if H is a subdirection of Ξ relative to such a $\xi(\cdot)$ that $y_\eta = x_{\xi(\eta)}$ for all $\eta \in \mathrm{H}$, i.e., $y = x \circ \xi$. It should be emphasized that by virtue of **4.1.6** (5) we have $y({}^a\mathrm{H}) \subset x({}^a\Xi)$ fulfilled.

5.3.9. The last property of Moore subnets is a cornerstone of a more free definition of a subnet which is attractive by its direct relation with filters. Namely, the set $(y_\eta)_{\eta \in \mathrm{H}}$ of X elements is termed a *subnet* (or a *subnet in a broader sense*) of the set $(x_\xi)_{\xi \in \Xi}$ of X elements, provided

$$(\forall \xi \in \Xi)\,(\exists \eta \in \mathrm{H})\,(\forall \eta' \geq \eta)\,(\exists \xi' \geq \xi)\quad x(\xi') = y(\eta'),$$

i.e., in the case when every tail of the net x contains a certain tail of y. It goes without saying that in terms of monads the condition $y({}^a\mathrm{H}) \subset x({}^a\Xi)$ is fulfilled, or, in a more expressive form,

$$(\forall \eta \approx +\infty)\,(\exists \xi \approx +\infty)\quad y_\eta = x_\xi.$$

In this case, for the sake of expressiveness it is often written that $(x_\eta)_{\eta\in H}$ is a subset of the net $(x_\xi)_{\xi\in\Xi}$ (which can result in ambiguity). It would be expedient to emphasize that in a general case subnets are not obligatorily Moore subnets. It should also be stressed that two nets in one set are called *equivalent* if each of them is a subnet of the other, i.e., if their monads coincide.

5.3.10. If $\mathfrak{F}$ is a filter in X, and (x_ξ) is a net of elements in X, then we say that the considered net is *subordinate to* $\mathfrak{F}$ under the condition $\xi \approx +\infty \rightarrow x_\xi \in \mu(\mathfrak{F})$. In other words, the net (x_ξ) is subordinate to $\mathfrak{F}$ provided the tail filter is finer than $\mathfrak{F}$. In this case a certain abuse of language is allowed when we write $x_\xi \downarrow (\mathfrak{F})$ meaning an analogy with topological notations of convergence. It should also be remarked here that when $\mathfrak{F}$ is an ultrafilter, $\mathfrak{F}$ coincides with the tail filter of any net (x_ξ) subordinate to it, i.e. such a (x_ξ) net is itself *an ultranet.*

5.3.11. Theorem. *Let* $\varphi = \varphi(x,y,z)$ *be a formula of Zermelo-Fraenkel theory containing no free parameters but* x, y, z, *where* z *is a standard set. Let, then,* $\mathfrak{F}$ *be a filter in* X, *and* $\mathfrak{G}$ *be a filter in* Y. *The following statements are equivalent:*

(1) $(\forall G\in\mathfrak{G})(\exists F\in\mathfrak{F})(\forall x\in F)(\exists y\in G)\ \varphi(x,y,z)$;

(2) $(\forall x\in\mu(\mathfrak{F}))(\exists y\in\mu(\mathfrak{G}))\ \varphi(x,y,z)$;

(3) *for any set* $(x_\xi)_{\xi\in\Xi}$ *of elements of* X *subordinate to* $\mathfrak{F}$ *we can find a net* $(y_\eta)_{\eta\in H}$ *of elements of* Y *subordinate to* $\mathfrak{G}$, *and a strict subnet* $(x_{\xi(\eta)})_{\eta\in H}$ *of the net* $(x_\xi)_{\xi\in\Xi}$ *such that for all* $\eta\in H$, *we get* $\varphi(x_{\xi(\eta)}, y_\eta, z)$, *i.e., symbolically,*

$$(\forall x_\xi \downarrow \mathfrak{F})(\exists y_\eta \downarrow \mathfrak{G})\quad \varphi(x_{\xi(\eta)}, y_\eta, z);$$

(4) *for any net* $(x_\xi)_{\xi\in\Xi}$ *of elements of* X *subordinate to* $\mathfrak{F}$ *there is a net* $(y_\eta)_{\eta\in H}$ *of elements of* Y *subordinate to* $\mathfrak{G}$, *and a subnet* $(x_\eta)_{\eta\in H}$ *of the net* $(x_\xi)_{\xi\in\Xi}$ *such that for all* $\eta\in H$ $\varphi(x_\eta, y_\eta, z)$; *i.e., symbolically,*

$$(\forall x_\xi \downarrow \mathfrak{F})(\exists y_\eta \downarrow \mathfrak{G})\quad \varphi(x_\eta, y_\eta, z);$$

(5) *for any ultranet* $(x_\xi)_{\xi\in\Xi}$ *of elements of* X *subordinate to* $\mathfrak{F}$ *there is an ultranet* $(y_\eta)_{\eta\in H}$ *subordinate to* , *and an ultranet* $(x_\eta)_{\eta\in H}$ *equivalent to* $(x_\xi)_{\xi\in\Xi}$ *such that for all*

$\eta \in \mathrm{H}$ *we have* $\varphi(x_\eta, y_\eta, z)$.

◁ (1) → (2). Let $x \in \mu(\mathfrak{F})$. By the transfer principle, for every standard G there is a standard F such that $(\forall x \in F)(\exists y \in G)\, \varphi(x,y,z)$. Therefore, for $x \in \mu(\mathfrak{F})$ we get $(\forall G \in {}^\circ\mathscr{B})(\exists y \in G)\, \varphi(x,y,z)$. Applying the idealization principle, we deduce: $(\exists y)\,(\forall G \in {}^\circ\mathscr{B})\, y \in G \wedge \varphi(x,y,z)$. Hence, $y \in \mu(\mathscr{B})$ and $\varphi(x,y,z)$.

(2) → (3). Let $(x_\xi)_{\xi \in \Xi}$ be a standard net in X subordinate to $\mathfrak{F}$. For every standard G of $\mathscr{B}$ let us set

$$A_{(G,\xi)} := \{\xi' \geq \xi : (\forall \xi'' \geq \xi')(\exists y \in G)\, \varphi(x_{\xi''}, y, z)\}.$$

By **4.1.8**, ${}^a\Xi \subset A_{(G,\xi)}$. Since $A_{(G,\xi)}$ is an internal set, we use the Cauchy principle and conclude: ${}^\circ A_{(G,\xi)} \neq \varnothing$. Therefore, on the direction $\mathrm{H} := \mathscr{B} \times \Xi$ (with the natural order) there are standard mappings $\xi : \mathrm{H} \to \Xi$ and $y : \mathrm{H} \to Y$ such that $\xi(\eta) \in A_{(G,\xi)}$ and $y_\eta \in G$ for $G \in \mathscr{B}$ and $\xi \in \Xi$, such that $\eta \in (G, \xi)$. Obviously, for $\eta \approx +\infty$ we have $\xi(\eta) \approx +\infty$ and $y_\eta \in \mu(\mathscr{B})$.

(3) → (4). This is obvious.

(4) → (1). If (1) is not fulfilled, then, by hypothesis,

$$(\exists G \in \mathscr{B})(\forall F \in \mathfrak{F})(\exists x \in F)(\forall y \in G)\, \neg\varphi(x,y,z),$$

For $F \in \mathfrak{F}$ we choose $x_F \in F$ in such a way that we had $\neg\varphi(x,y,z)$ for all $y \in G$. It should be remarked that the obtained net $(x_F)_{F \in \mathfrak{F}}$ of the elements of X, as well as the set G, can be considered standard by virtue of the transfer principle. No doubts, $x_F \downarrow \mathfrak{F}$ and, hence, by virtue of (3), there is a direction H and a subnet $(x_\eta)_{\eta \in \mathrm{H}}$ of the net $(x_F)_{F \in \mathfrak{F}}$ such that for a certain net $(y_\eta)_{\eta \in \mathrm{H}}$ we get $\varphi(x_\eta, y_\eta, z)$ for any $\eta \in \mathrm{H}$. By definition **5.3.9**, for any infinitely large η, x_η coincides with x_F for a certain remote F, i.e., $x_\eta \in \mu(\mathfrak{F})$. By condition, $y_\eta \in \mu(\mathscr{B})$ and, moreover, $y_\eta \in G$. In this case it appears that $\varphi(x_\eta, y_\eta, x)$ and $\neg\varphi(x_\eta, y_\eta, x)$, which is impossible. The contradiction obtained proves that the assumption made is false. Therefore, (1) is fulfilled (as soon as (4) is valid).

(1) → (5). In order to prove the equivalence required, it suffices to remark that the equivalence becomes evident in the case when $\mathfrak{F}$ and $\mathscr{B}$ are ultrafilters. Now we have to remark that every monad is a union of the monads of ultrafilters. ▷

5.3.12. In applications it is often convenient to consider concretizations of **5.3.11** corresponding to the cases when one of the filters is discrete. Thus, using natural notation, we deduce:

$$(\exists x \in \mu(\mathfrak{F}))\, \varphi(x,y) \leftrightarrow (\exists x_\xi \downarrow \mathfrak{F})\, \varphi(x_\xi, y);$$
$$(\forall x \in \mu(\mathfrak{F}))\, \varphi(x,y) \leftrightarrow (\forall x_\xi \downarrow \mathfrak{F})(\exists x_\eta \downarrow \mathfrak{F})\, \varphi(x_\eta, y).$$

5.3.13. Let $F \subset X \times Y$ be an internal correspondence from a standard set X to a standard set Y. Let us assume that in X there is a standard filter $\mathfrak{N}$, while in Y there is a topology τ. Let us set

$$\forall\forall(F) := {}^*\{y' : (\forall x \in \mu(\mathfrak{N}) \cap \operatorname{dom}(F))(\forall y \approx y')\quad (x,y) \in F\},$$
$$\exists\forall(F) := {}^*\{y' : (\exists x \in \mu(\mathfrak{N}) \cap \operatorname{dom}(F))(\forall y \approx y')\quad (x,y) \in F\},$$
$$\forall\exists(F) := {}^*\{y' : (\forall x \in \mu(\mathfrak{N}) \cap \operatorname{dom}(F))(\exists y \approx y')\quad (x,y) \in F\},$$
$$\exists\exists(F) := {}^*\{y' : (\exists x \in \mu(\mathfrak{N}) \cap \operatorname{dom}(F))(\exists y \approx y')\quad (x,y) \in F\},$$

where, as usual, * is the symbol of standardization, while the expression $y \approx y'$ means that $y \in \mu(\tau(y'))$. The set $Q_1Q_2(F)$ is called a *Q_1Q_2-limit* of F (here Q is one of the quantifiers, $\forall$ or $\exists$).

5.3.14. In applications they usually confine themselves to the case in which F is a standard correspondence determined on a certain element of the filter $\mathfrak{N}$, and then the $\exists\exists$-limit and the $\forall\exists$-limit are studied. The former is termed *the limit superior* or *upper limit*, the latter is called *the limit interior* or *lower limit* of F along $\mathfrak{N}$.

If we consider a net $(x_\xi)_{\xi \in \Xi}$ in the domain of the definition of F then, bearing in mind the tail filter of the net, we assign

$$\operatorname{Li}_{\xi \in \Xi} F := \liminf_{\xi \in \Xi} F(x_\xi) := \forall\exists(F),$$
$$\operatorname{Ls}_{\xi \in \Xi} F := \liminf_{\xi \in \Xi} F(x_\xi) := \forall\exists(F).$$

In such cases we speak about *Kuratowsky limits*.

5.3.15. *For a standard correspondence F we have the following presentations:*

$$\exists\exists(F) = \bigcap_{U \in \mathfrak{N}} \operatorname{cl} \bigcup_{x \in U} F(x);$$
$$\forall\exists(F) = \bigcap_{U \in \ddot{\mathfrak{N}}} \operatorname{cl} \bigcup_{x \in U} F(x),$$

where $\ddot{\mathscr{N}}$ *is the grill of* $\mathscr{N}$, *i.e., the family composed of all the subsets of* X *meeting the monad* $\mu(\mathscr{N})$. *In other words,*

$$\ddot{\mathscr{N}} = {}^*\{U' \subset X : U' \cap \mu(\mathscr{N}) \neq \varnothing\} = \{U' \subset X : (\forall U \in \mathscr{N})\, U \cap U' \neq \varnothing\}.$$

In this respect the following relations must be recalled:

$$\forall\exists(F) = \bigcap_{U \in \ddot{\mathscr{N}}} \operatorname{int} \bigcup_{x \in U} F(x),$$

$$\forall\forall(F) = \bigcap_{U \in \mathscr{N}} \operatorname{int} \bigcup_{x \in U} F(x).$$

5.3.16. Theorem **5.3.11** immediately yields a description for limits in the language of nets.

5.3.17. *An element* y *lies in the* $\forall\exists$*-limit of* F *iff for every net* $(x_\xi)_{\xi\in\Xi}$ *of elements of the* $\operatorname{dom}(F)$ *subordinate to* $\mathscr{N}$ *there is a subnet* $(x_\eta)_{\eta\in\mathrm{H}}$ *of the net* $(x_\xi)_{\xi\in\Xi}$ *and a net* $(y_\eta)_{\eta\in\mathrm{H}}$ *convergent to* y *such that* $(x_\eta, y_\eta) \in F$ *for all* $\eta \in \mathrm{H}$.

5.3.18. *An element* y *lies in the* $\exists\exists$*-limit of* F *iff there is a net* $(x_\xi)_{\xi\in\Xi}$ *of elements of the* $\operatorname{dom}(F)$ *subordinate to* $\mathscr{N}$, *and a net* $(y_\xi)_{\xi\in\Xi}$ *convergent to* y, *for which* $(x_\xi, y_\xi) \in F$ *for any* $\xi \in \Xi$.

5.3.19. *For any internal correspondence* F *we have*

$$\forall\forall(F) \subset \exists\forall(F) \subset \forall\exists(F) \subset \exists\exists(F).$$

In this case $\exists\exists(F)$, $\forall\exists(F)$ *are closed, while* $\forall\forall(F)$, $\exists\forall(F)$ *are open sets.*

$\lhd$ The inclusions sought are obvious. Therefore, taking into account the considerations of duality, let us, for the sake of definiteness, establish that the $\forall\exists$-limit is closed.

If V is a standard open neighbourhood of a point y' of $\operatorname{cl}\forall\exists(F)$, then there is a $y \in \forall\exists(F)$ for which $y \in V$. For an $x \in \mu(\mathscr{N})$ let us find an y'' so that we had $y'' \in \mu(\tau(y))$ and $(x, y'') \in F$. Obviously, $y'' \in V$ since V is a neighbourhood of y. Therefore,

$$(\forall x \in \mu(\mathscr{N}))\,(\forall V \in {}^\circ\tau(y'))\,(\exists y'' \in V)\,(x, y'') \in F.$$

Using now the idealization principle, we deduce: $y' \in \forall\exists(F)$. ▷

5.3.20. The general statements given above make it possible to characterize the elements of many approximating or regularizing cones in terms of nets which is common practice (see [121], [115]). It should be, in particular, observed that the Clarke cone $\mathrm{Cl}(F,x')$ of F in X is obtained by means of the Kuratowski limit:

$$\mathrm{Cl}(F,x') = \mathrm{Li}_{\tau(x')\times\tau_{\mathbf{R}^+}(0)}\Gamma_F ,$$

where Γ_F is the *homothety* associated with F, i.e.,

$$(x,\alpha,h) \in \Gamma_F \leftrightarrow h \in \frac{F-x}{\alpha} \ (x,h \in X, \alpha > 0).$$

5.3.21. In convex analysis use if often made of special variations of Kuratowski limits pertaining to the epigraphs of functions which operate into the extended numerical straight line $\overline{\mathbf{R}}$. Let us, first of all, recall important characteristics of the upper and lower limits.

5.3.22. *Let $f: X \to \overline{\mathbf{R}}$ be a standard function defined on a standard X, and let $\mathfrak{F}$ be a standard filter in X. For every standard $t \in \mathbf{R}$, we have*

$$\sup_{F\in\mathfrak{F}} \inf f(F) \le t \leftrightarrow (\exists x \in \mu(\mathfrak{F}))\, {}^\circ\! f(x) \le t,$$
$$\inf_{F\in\mathfrak{F}} \sup f(F) \le t \leftrightarrow (\exists x \in \mu(\mathfrak{F}))\, {}^\circ\! f(x) \le t.$$

◁ Let us first check the first equivalence. Applying the transfer and idealization principles, we deduce:

$$\sup_{F\in\mathfrak{F}} \inf f(F) \le t \to (\forall F \in \mathfrak{F}) \inf f(F) \le t \to$$
$$(\forall F \in \mathfrak{F})\, (\forall \varepsilon > 0) \inf f(F) < t + \varepsilon \to (\forall \varepsilon)\, (\forall F)\, (\exists x \in F)\ f(x) < t +$$
$$+\,\varepsilon \to (\forall^{\mathrm{st}}\varepsilon)\, (\forall^{\mathrm{st}} F)\, (\exists x)\, (x \in F \wedge f(x) < t + \varepsilon)$$
$$\to (\exists x)\, (\forall^{\mathrm{st}}\varepsilon)\, (\forall^{\mathrm{st}} F)\, (x \in F \wedge f(x) < t + \varepsilon)$$
$$\to (\exists x \in \mu(\mathfrak{F}))\, (\forall^{\mathrm{st}}\varepsilon > 0)\ \ f(x) < t + \varepsilon \to (\exists x \in \mu(\mathfrak{F}))\, {}^\circ\! f(x) \le t$$

(here use was made of **2.2.18** (3)). Let us now remark that for any standard element F of the filter $\mathfrak{F}$ we have $x \in \mu(\mathfrak{F}) \subset F$. Hence, $\inf f(F) \le t$ (as $\inf f(F) \le f(x) < t + \varepsilon$ for every $\varepsilon > 0$). Therefore, by virtue of the transfer principle, for an internal F of $\mathfrak{F}$ we have $\inf f(F) \le t$, which was required.

Taking into account the above-proved statements and the fact that $-f$ and t are standard, we deduce

$$\sup_{F\in\mathfrak{F}} \inf f(F) \ge t \leftrightarrow - \inf_{F\in\mathfrak{F}} \sup f(F) \le -t \leftrightarrow \sup_{F\in\mathfrak{F}} \inf(-f)(F) \ge t$$
$$\leftrightarrow (\exists x \in \mu(\mathfrak{F}))\, {}^\circ(-f(x)) \le -t \leftrightarrow (\exists x \in \mu(\mathfrak{F}))\, {}^\circ f(x) \ge t.$$

Therefore, we get

$$\inf_{F\in\mathfrak{F}} \sup f(F) < t \leftrightarrow \neg \left(\inf_{F\in\mathfrak{F}} \sup f(F) \ge t \right)$$
$$\leftrightarrow \neg((\exists x \in \mu(\mathfrak{F}))\, {}^\circ f(x) \ge t) \leftrightarrow (\forall x \in \mu(\mathfrak{F}))\, {}^\circ f(x) \le t.$$

And, finally, from the above, we conclude

$$\inf_{F\in\mathfrak{F}} \sup f(F) \le t \leftrightarrow (\forall \varepsilon > 0) \inf_{F\in\mathfrak{F}} \sup f(F) < t + \varepsilon$$
$$\leftrightarrow (\forall^{\mathrm{st}} \varepsilon > 0)(\forall x \in \mu(\mathfrak{F}))\, {}^\circ f(x) < t + \varepsilon$$
$$\leftrightarrow (\forall x \in \mu(\mathfrak{F}))(\forall^{\mathrm{st}} \varepsilon > 0)\, {}^\circ f(x) < t + \varepsilon \leftrightarrow (\forall x \in \mu(\mathfrak{F}))\, {}^\circ f(x) \le t,$$

since the number ${}^\circ f(x)$ is standard. ▷

5.3.23. *Let X, Y be standard sets, $f: X \times Y \to \overline{\mathbb{R}}$ be a standard function, and $\mathfrak{F}, \mathcal{B}$ be standard filters in X and Y, respectively. For any standard real number t we have*

$$\sup_{G\in\mathcal{B}} \inf_{F\in\mathfrak{F}} \sup_{x\in F} \inf_{y\in G} f(x,y) \le t \leftrightarrow (\forall x \in \mu(\mathfrak{F}))(\exists y \in \mu(\mathcal{B}))\, {}^\circ f(x,y) \le t.$$

◁ Assign $f_G(x) := \inf\{f(x,y) : y \in G\}$. Observe that f_G is a standard function only if G is a standard set. Now successively make use of the transfer principle, proposition **5.3.22** and (strong) idealization, and deduce:

$$\sup_{G\in\mathcal{B}} \inf_{F\in\mathfrak{F}} \sup_{x\in F} \inf_{y\in G} f(x,y) \le t \leftrightarrow (\forall G \in \mathcal{B}) \inf_{F\in\mathfrak{F}} \sup_{x\in F} f_G(x) \le t$$
$$\leftrightarrow (\forall^{\mathrm{st}} G \in \mathcal{B}) \inf_{F\in\mathfrak{F}} \sup_{x\in F} f_G(x) \le t \leftrightarrow (\forall^{\mathrm{st}} G \in \mathcal{B})(\forall x \in \mu(\mathfrak{F}))\, {}^\circ f_G(x)$$
$$\le t \leftrightarrow (\forall x \in \mu(\mathfrak{F}))(\forall^{\mathrm{st}} G \in \mathcal{B})(\forall^{\mathrm{st}} \varepsilon > 0) \inf_{y\in G} f(x,y) < t + \varepsilon$$
$$\to (\forall x \in \mu(\mathfrak{F}))(\forall^{\mathrm{st}} \varepsilon > 0)(\forall^{\mathrm{st}} G \in \mathcal{B})(\exists y \in G)\, f(x,y) < t + \varepsilon$$
$$\to (\forall x \in \mu(\mathfrak{F}))\ (\exists y \in \mu(\mathcal{B}))(\forall^{\mathrm{st}} \varepsilon > 0)\, f(x,y) < t + \varepsilon$$
$$\to (\forall x \in \mu(\mathfrak{F}))\ (\exists y \in \mu(\mathcal{B}))\ {}^\circ f(x,y) \le t.$$

For an internal element $F \subset \mu(\mathfrak{F})$ of the filter $\mathfrak{F}$ and a standard element G of the filter $\mathcal{B}$ the last relation yields (by virtue of the transfer principle):

$$\sup_{x \in F} \inf_{y \in G} f(x,y) \leq t \rightarrow \inf_{F \in \mathfrak{F}} \sup_{x \in F} \inf_{y \in G} f(x,y) \leq t$$
$$\rightarrow (\forall^{\mathrm{st}} G \in \mathcal{B}) \inf_{F \in \mathfrak{F}} \sup_{x \in F} \inf_{y \in G} f(x,y) \leq t$$
$$\rightarrow (\forall G \in \mathcal{B}) \inf_{F \in \mathfrak{F}} \sup_{x \in F} \inf_{y \in G} f(x,y) \leq t. \rhd$$

5.3.24. In relation with **5.3.23** the quantity

$$\limsup\nolimits_{\mathfrak{F}} \inf\nolimits_{\mathcal{B}} f := \sup_{G \in \mathcal{B}} \inf_{F \in \mathfrak{F}} \sup_{x \in F} \inf_{y \in G} f(x,y)$$

is often referred to as the *Rockafeller limit of* f.

If $f := (f_\xi)_{\xi \in \Xi}$ is a family of functions operating from the topological space (X,σ) in $\overline{\mathbf{R}}$, and if $\mathfrak{N}$ is a filter in Ξ, then we determine the *limit inferior* or *lower limit* at the point x' of X of the family f, and its *limit superior* or *upper limit*, or the Rockafeller limit

$$\mathrm{li}_{\mathfrak{N}} f(x') := \sup_{V \in \sigma(x')} \sup_{U \in \mathfrak{N}} \inf_{\xi \in U} \inf_{x \in V} f_\xi(x),$$
$$\mathrm{ls}_{\mathfrak{N}} f(x') := \sup_{V \in \sigma(x')} \inf_{U \in \mathfrak{N}} \sup_{\xi \in U} \inf_{x \in V} f_\xi(x).$$

The last limits are often termed *epilimits*. The essence of this term is revealed by the following obvious statement.

5.3.25. *The limit inferior and the limit superior of a family of epigraphs are the epigraphs of the respective limits of the family of functions under consideration.*

5.4. Approximations determined by a set of infinitesimals

In this section we shall study the problem of analysis of classical approximating cones of Clarke type by detalizing the contribution of infinitely small numbers participating in their definition. Such an analysis enables one to single out both new analogues of tangent cones and new descriptions for the Clarke cone.

5.4.1. Let us again consider a real vector space X with a linear topology σ and a nearvector topology τ. Let, then, in X there is a point x' of F. In line with **5.2**, these objects are considered as standard sets.

Let us fix a certain infinitesimal, a real number α for which $\alpha > 0$ and $\alpha \approx 0$. Let us set

$$\begin{aligned}
&\mathrm{Ha}_\alpha(F,x') := {}^*\{h' \in X\colon (\forall x \approx_\sigma x', x \in F)(\forall h \approx_\tau h')\, x + \alpha h \in F\},\\
&\mathrm{In}_\alpha(F,x') := {}^*\{h' \in X\colon (\exists h \approx_\tau h')\, (\forall x \approx_\sigma x', x \in F)\, x + \alpha h \in F\},\\
&\mathrm{Cl}_\alpha(F,x') := {}^*\{h' \in X\colon (\forall x \approx_\sigma x', x \in F)(\exists h \approx_\tau h')\, x + \alpha h \in F\},
\end{aligned}$$

where, as usual, * is the symbol of taking the standardization of an external set.

Let us now consider a nonempty and, generally speaking, external set of infinitesimals Λ, assigning

$$\begin{aligned}
&\mathrm{Ha}_\Lambda(F,x') := {}^* \bigcap_{\alpha \in \Lambda} \mathrm{Ha}_\alpha(F,x'),\\
&\mathrm{In}_\Lambda(F,x') := {}^* \bigcap_{\alpha \in \Lambda} \mathrm{In}_\alpha(F,x'),\\
&\mathrm{Cl}_\Lambda(F,x') := {}^* \bigcap_{\alpha \in \Lambda} \mathrm{Cl}_\alpha(F,x').
\end{aligned}$$

Let us pursue the same policy as regards notation for other types of the approximations introduced. As an example, it is worth emphasizing that by virtue of the definitions for a standard h' of X we have

$$\begin{aligned}
&h' \in \mathrm{In}_\Lambda(F,x')\\
&\leftrightarrow (\forall \alpha \in \Lambda)(\exists h \approx_\tau h')(\forall x \approx_\sigma x', x \in F)\; x + \alpha h \in F.
\end{aligned}$$

It is expedient to remark that when Λ is the monad of the corresponding standard filter $\mathfrak{F}_\Lambda$, where $\mathfrak{F}_\Lambda := {}^*\{A \subset \mathbf{R}\colon A \supset \Lambda\}$, then, say, for $\mathrm{Cl}_\Lambda(F,x')$ we get:

$$\mathrm{Cl}_\Lambda(F,x') = \bigcap_{V \in \mathfrak{N}} \; \bigcup_{\substack{U \in \sigma(x') \\ A \in \mathfrak{F}_\Lambda}} \; \bigcap_{\substack{x \in F \cap U \\ \alpha \in A, \alpha > 0}} \left(\frac{F - x}{\alpha} + V \right).$$

If Λ is not a monad (for instance, a singleton), then the implicit form of $\mathrm{Cl}_\Lambda(F,x')$ is associated with the model of analysis which is in fact under investigation. It should be emphasized that the ultrafilter $\mathcal{U}(\alpha) := {}^*\{A \subset \mathbf{R}\colon \alpha \in A\}$ has the monad not converging to the initial infinitesimal α, i.e., the set $\mathrm{Cl}_\alpha(F,x')$ is, generally speaking, broader than $\mathrm{Cl}_{\mu(\mathcal{U}(\alpha))}(F,x')$. At the same time, the introduced approximations prove to possess many advantages inherent to Clarke cones. When detalizing and corroborating the last statement, let us, without further specification, use, as was the case in **5.2**, the supposition that the mapping $(x,\beta,h) \to x+\beta h$ of the space $(X\times\mathbf{R}\times X, \sigma\times\tau_{\mathbf{R}}\times\tau)$ in X is continuous at zero (which 'in standard entourage' is equivalent to the inclusion $\mu(\sigma)+\mu(\mathbf{R}_+)\mu(\tau)\subset\mu(\sigma)$).

5.4.2. Theorem. *For every set* Λ *of positive infinitesimals the following statements are valid:*

(1) $\mathrm{Ha}_\Lambda(F,x')$, $\mathrm{In}_\Lambda(F,x')$, $\mathrm{Cl}_\Lambda(F,x')$ *are semigroups and, moreover,*

$$\mathrm{Ha}(F,x')\subset\mathrm{Ha}_\Lambda(F,x')\subset\mathrm{In}_\Lambda(F,x')\subset\mathrm{Cl}_\Lambda(F,x')\subset K(F,x'),$$
$$\mathrm{Cl}(F,x')\subset\mathrm{Cl}_\Lambda(F,x');$$

(2) *if* Λ *is an internal set, then* $\mathrm{Ha}_\Lambda(F,x')$ *is* τ*-open;*

(3) $\mathrm{Cl}_\Lambda(F,x')$ *is a* τ*-closed set and, for* F *convex, we have* $K(F,x')=\mathrm{Cl}_\Lambda(F,x')$ *as soon as* $\sigma=\tau$;

(4) *if* $\sigma=\tau$, *then the following equality is valid:*

$$\mathrm{Cl}_\Lambda(F,x')=\mathrm{Cl}_\Lambda(\mathrm{cl}F,x');$$

(5) *the Rockafeller formula holds*

$$\mathrm{Ha}_\Lambda(F,x')+\mathrm{Cl}_\Lambda(F,x')=\mathrm{Ha}_\Lambda(F,x');$$

(6) *if* x' *is a* τ*-boundary point of* F, *then, for* $F':=(X-F)\cup\{x'\}$,

$$\mathrm{Ha}_\Lambda(F,x')=-\mathrm{Ha}_\Lambda(F',x').$$

$\triangleleft$ (1) Let us make sure that $\mathrm{In}_\Lambda(F,x')$ is a semigroup. If standard h' and h'' belong to $\mathrm{In}_\Lambda(F,x')$, then, for every $\alpha\in\Lambda$ and a certain $h_1\approx_\sigma h'$, we get $x'':=x+\alpha h_1\in F$ as soon as $x\in F$ and $x\approx_\sigma x'$. By hypothesis, there is an $h_2\approx_\tau h''$ for which $x''+\alpha h_2\in F$, as $x''\approx_\sigma x$. Finally, $h_1+h_2\approx_\tau h'+h''$ and h_1+h_2 'serves' the membership $h'+h''\in\mathrm{In}_\Lambda(F,x')$.

If $h'\in\mathrm{Cl}_\Lambda(F,x')$ and h' is standard, then $x'+\alpha h\in F$ for some $\alpha\in\Lambda$ and $h\approx_\tau h'$, which implies $h'\in K(F,x')$. The rest of the inclusions written in (1) are obvious.

(2) If h' is a standard element of $\mathrm{Ha}_\Lambda(F,x')$, then

$$(\forall x\approx_\sigma x',\ x\in F)\,(\forall h\approx_\tau h')\,(\forall\alpha\in\Lambda)\ x+\alpha h\in F.$$

Taking into account **5.3.2** and using the fact that Λ is an internal set, we deduce:

$$(\exists^{st} V \in \mathcal{N}_\tau)(\exists^{st} U \in \sigma(x'))(\forall x \in U \cap F)(\forall h \in h' + V)(\forall \alpha \in \Lambda)$$
$$x + \alpha h \in F .$$

Let us choose standard neighbourhoods $V_1, V_2 \in \mathcal{N}_\tau$ in such a way that we had $V_1 + V_2 \subset V$. In this case, for all standard $h'' \in h' + V_1$ it is fulfilled that

$$(\forall x \in U \cap F)(\forall h \in h'' + V_2)(\forall \alpha \in \Lambda)\, x + \alpha h \in F,$$

i.e., $h'' \in \mathrm{Ha}_\Lambda(F, x')$ for any $h'' \in h' + V_1$.

(3) Let now h' be a standard element of $\mathrm{cl}_\tau \mathrm{Cl}_\Lambda(F, x')$. Let us choose an arbitrary standard neighbourhood V of the point h' and again choose standard $V_1, V_2 \in \mathcal{N}_\tau$ by the condition $V_1 + V_2 \subset V$. By definition, there is an $h'' \in \mathrm{Cl}_\Lambda(F, x')$ such that $h'' \in h' + V_1$. By **5.4.1** and **5.3.2**, we have

$$(\forall \alpha \in \Lambda)(\exists^{st} U \in \sigma(x'))(\forall x \in F \cap U)(\exists h \in h'' + V_2)\, x + \alpha h \in F.$$

In this case $h \in h'' + V_2 \subset h' + V_1 + V_2 \subset h' + V$. In other words,

$$(\forall^{st} V \in \mathcal{N}_\tau)(\forall \alpha \in \Lambda)(\exists^{st} U \in \sigma(x'))(\forall x \in F \cap U)(\exists h \in h' + V)$$
$$x + \alpha h \in F .$$

Therefore, $h' \in \mathrm{Cl}_\alpha(F, x')$ for every $\alpha \in \Lambda$, i.e., $h' \in \mathrm{Cl}_\Lambda(F, x')$.

If now $h' \in \mathrm{Fd}_\Lambda(F, x')$ and h' is standard, then for a certain standard $\alpha' > 0$ we have, by the transfer principle, $x' + \alpha' h' \in F$. If $x \approx_\sigma x'$ and $x \in F$, then $(x - x')/\alpha' \approx_\sigma 0$. For $h := h' + (x - x')/\alpha'$ we get $h \approx_\tau h'$ and, moreover, $x + \alpha' h \in F$. Taking into account the fact that F is convex, we have $x + (0, \alpha'] h \in F$. In particular, $x + \Lambda h \subset F$. Hence, $(\forall x \approx_\sigma x', x \in F)(\forall \alpha \in \Lambda)(\exists h \approx_\tau h')\, x + \alpha h \in F$, i.e, $h' \in \mathrm{Cl}_\Lambda(F, x')$. Therefore,

$$\mathrm{Fd}(F, x') \subset \mathrm{Cl}_\Lambda(F, x') \subset K(F, x') \subset \mathrm{cl}\, \mathrm{Fd}(F, x').$$

Taking into account the fact that $\mathrm{Cl}_\Lambda(F, x')$ is τ-closed, we conclude: $K(F, x') = \mathrm{Cl}_\Lambda(F, x')$.

(4) The proof is carried out as in **5.2.11**.

(5) For standard $k' \in \mathrm{Ha}_\Lambda(F, x')$ and $h' \in \mathrm{Cl}_\Lambda(F, x')$, for every $\alpha \in \Lambda$ and any $x \in F$ such that $x \approx_\sigma x'$, we get, having chosen h that enjoys the conditions $h \approx_\tau h'$ and $x + \alpha h \in F$,

$$x + \alpha(h' + k' + \mu(\tau)) = x + \alpha h + \alpha(k' + (h - h') + \mu(\tau))$$
$$\subset (x + \mu(\sigma)) \cap F + \alpha(k' + \mu(\tau) + \mu(\tau))$$
$$\subset (x + \mu(\sigma)) \cap F + \alpha(k' + \mu(\tau)) \subset F,$$

which implies the membership of $h' + k'$ in $\mathrm{Ha}_\Lambda(F,x')$.

(6) Let $-h \notin \mathrm{Ha}_\Lambda(F',x')$. Then for a certain $\alpha \in \Lambda$ there is an $h \approx_\tau h'$, so that for an appropriate $x \approx_\sigma x'$, $x \in F$, we have $x - \alpha h \in F$. While if $h \in \mathrm{Ha}_\Lambda(F,x')$, in particular, $h \in \mathrm{Ha}_\alpha(F,x')$ and $x = (x - \alpha h) + \alpha h \in F$, since $x - \alpha h \approx_\sigma x$. Hence, $x \in F \cap F'$, i.e., $x = x'$. Besides, $(x' - \alpha h) + \alpha(h + \mu(\tau)) \subset F$, since $h + \mu(\tau) \subset \mu(\tau(h'))$. Therefore, x' is a τ-internal point of F, which contradicts the condition. Hence, $h \notin \mathrm{Ha}_\Lambda(F,x')$, which ensures the inclusion $-\mathrm{Ha}_\Lambda(F,x') \subset \mathrm{Ha}_\Lambda(F',x')$. Substituting in the above considerations $F = (F')'$ for F', we come to the sought conclusion. ▷

5.4.3. It is important to emphasize that in many cases the described analogues of the Hadamard and Clarke cones are convex. In fact, the following propositions are valid.

5.4.4. *Let τ be a vector topology and $t\Lambda \subset \Lambda$ for a certain standard $t \in (0,1)$. Then* $\mathrm{Cl}_\Lambda(F,x')$ *is a convex cone. If in this case Λ is also an internal set, then* $\mathrm{Ha}_\Lambda(F,x')$ *is a convex cone as well.*

◁ Let us assume that we consider $\mathrm{Ha}_\Lambda(F,x')$ and that $h \in \mathrm{Ha}_\Lambda(F,x')$ is a standard element of this set. By virtue of **5.4.2** (2), $\mathrm{Ha}_\Lambda(F,x')$ is open in the topology τ. Moreover, $th \in \mathrm{Ha}_\Lambda(F,x')$, where t is the standard positive number mentioned in the hypothesis. ▷

5.4.5. *Let $t\Lambda \subset \Lambda$ for every standard $t \in (0,1)$. Then the sets* $\mathrm{Cl}_\Lambda(F,x')$, $\mathrm{In}_\Lambda(F,x')$ *and* $\mathrm{Ha}_\Lambda(F,x')$ *are convex cones.*

◁ Let us assume, for definiteness, that we consider $\mathrm{Cl}_\Lambda(F,x')$. Let h' be a standard vector of the set under discussion, and $0 < t < 1$ is a standard number. Let $x \approx_\sigma x'$, $x \in F$ and $\alpha \in \Lambda$. For x and $t\alpha \in \Lambda$ let us choose an h, for which $h \approx_\tau h'$ and $x + \alpha th \in F$. As long as $th \approx_\tau th'$ (see **5.1.7**), we have $th' \in \mathrm{Cl}_\alpha(F,x')$. In other words, by the transfer principle, $(0,1)\mathrm{Cl}_\Lambda(F,x') \subset \mathrm{Cl}_\Lambda(F,x')$. Now we are to recall **5.4.2** (1). ▷

5.4.6. A set Λ is called *representative*, provided $\mathrm{Ha}_\Lambda(F,x')$ and $\mathrm{Cl}_\Lambda(F,x')$ are (convex) cones, the propositions **5.4.4** and **5.4.5** giving examples of a representative Λ.

5.4.7. Let $f: X \to \overline{\mathbf{R}}$ be a function acting into the extended real line. For an infinitesimal α, a point x' of $\mathrm{dom}(f)$ and a vector $h' \in X$, we set:

$$f(\mathrm{Ha}_\alpha)(x')(h') := \inf\{t \in \mathbf{R}: (h',t) \in \mathrm{Ha}_\alpha(\mathrm{epi}(f),(x',f(x')))\},$$
$$f(\mathrm{In}_\alpha)(x')(h') := \inf\{t \in \mathbf{R}: (h',t) \in \mathrm{In}_\alpha(\mathrm{epi}(f),(x',f(x')))\},$$
$$f(\mathrm{Cl}_\alpha)(x')(h') := \inf\{t \in \mathbf{R}: (h',t) \in \mathrm{Cl}_\alpha(\mathrm{epi}(f),(x',f(x')))\}.$$

The derivatives $f(\mathrm{Ha}_\alpha)$, $f(\mathrm{In}_\alpha)$ and $f(\mathrm{Cl}_\alpha)$ are introduced in a natural way. It should be remarked that the derivative $f(\mathrm{Cl}) := f(\mathrm{Cl}_{\mu(\mathbf{R}_+)})$ is called the *Rockafeller derivative* and is denoted by the symbol $f^{\uparrow}$. Therefore, we write:

$$f_\alpha^{\uparrow}(x') := f(\mathrm{Cl}_\alpha)(x'), \quad f_\Lambda^{\uparrow}(x') := f(\mathrm{Cl}_\Lambda)(x').$$

If τ is the discrete topology, then $\mathrm{Ha}_\Lambda(F,x') = \mathrm{In}_\Lambda(F,x') = \mathrm{Cl}_\Lambda(F,x')$. In this case the Rockafeller derivative is termed the *Clarke derivative* and the following notation is used:

$$f_\alpha^{\circ}(x') := f_\alpha^{\uparrow}(x'), \quad f_\Lambda^{\circ}(x') := f_\Lambda^{\uparrow}(x').$$

For $\Lambda = \mu(\mathbf{R}_+)$, the indications of Λ are omitted.

When considering epiderivatives, the space $X \times \mathbf{R}$ is assumed to be endowed with the conventional product topologies $\sigma \times \tau_{\mathbf{R}}$ and $\tau \times \tau_{\mathbf{R}}$, where $\tau_{\mathbf{R}}$ is the conventional topology in $\mathbf{R}$. It is sometimes convenient to furnish $X \times \mathbf{R}$ with the pair of the topologies $\sigma \times \tau_0$ and $\tau \times \tau_{\mathbf{R}}$, where τ_0 is the trivial topology in $\mathbf{R}$. When using such topologies, we speak about *the Clarke and Rockafeller derivatives along effective domain* $\mathrm{dom}(f)$ and add the index d in the notation: f_d°, $f_{\Lambda,d}^{\uparrow}$, etc..

5.4.8. *The following statements are valid:*

$$f_\alpha^{\uparrow}(x')(h') \le t'$$
$$\leftrightarrow (\forall x \approx_\sigma x', t \approx f(x'), t \ge f(x))\, (\exists h \approx_\tau h')\, {}^{\circ}((f(x+\alpha h) - t)/\alpha) \le t';$$
$$f_\alpha^{\circ}(x')(h') < t'$$
$$\leftrightarrow (\forall x \approx_\sigma x', t \approx f(x'), t \ge f(x))\, (\forall h \approx_\tau h')\, {}^{\circ}((f(x+\alpha h) - t)/\alpha) < t';$$
$$f_{\alpha,d}^{\uparrow}(x')(h') \le t'$$
$$\leftrightarrow (\forall x \approx_\sigma x', x \in \mathrm{dom}(f))\, (\exists h \approx_\tau h')\, {}^{\circ}((f(x+\alpha h) - t)/\alpha) \le t';$$

$$f^{\circ}_{\alpha,d}(x')(h') < t'$$
$$\leftrightarrow (\forall x \approx_\sigma x', x \in \mathrm{dom}(f))\ (\forall h \approx_\tau h')\ {}^\circ((f(x+\alpha h) - t)/\alpha) < t'.$$

◁ For the proof, appeal to **2.2.18** (3). ▷

5.4.9. *If f is a lower semicontinuous function, then*

$$f^{\uparrow}_{\alpha}(x')(h') \le t'$$
$$\leftrightarrow (\forall x \approx_\sigma x', f(x) \approx f(x'))\ (\exists h \approx_\tau h')\ {}^\circ\left(\frac{f(x+\alpha h) - f(x)}{\alpha}\right) \le t;$$
$$f^{\circ}_{\alpha}(x')(h') < t'$$
$$\leftrightarrow (\forall x \approx_\sigma x', f(x) \approx f(x'))\ (\forall h \approx_\tau h')\ {}^\circ\left(\frac{f(x+\alpha h) - f(x)}{\alpha}\right) < t.$$

◁ It is only the implications to the right that have to be checked. Since such checks are identical, let us carry out the first of them. As f is lower semicontinuous, we can deduce: $x' \approx_\sigma x \to {}^\circ f(x) \ge f(x')$. Therefore, for x,t such that $t \approx f(x')$ and $t \ge f(x)$, we have ${}^\circ t \ge {}^\circ f(x) \ge f(x') = {}^\circ t$. In other words, ${}^\circ f(x) = f(x')$ and $f(x) \approx f(x')$. Choosing a suitable h from the conditions, we come to the conclusion

$${}^\circ(\alpha^{-1}(f(x+\alpha h) - t)) \le {}^\circ(\alpha^{-1}(f(x+\alpha h) - f(x))) \le t',$$

which ensures the sought result. ▷

5.4.10. *For a continuous function f the following equalities are valid:*

$$f^{\uparrow}_{\Lambda,d}(x') = f^{\uparrow}_{\Lambda}(x'),\quad f^{\circ}_{\Lambda,d}(x') = f^{\circ}_{\Lambda}(x').$$

◁ It suffices to remark that the continuity of f at a standard point yields $(x \approx_\sigma x', x \in \mathrm{dom}(f)) \to f(x) \approx f(x')$ (see **4.2.7**). ▷.

5.4.11. Theorem. *Let Λ be a monad. Then we have the following presentations:*

(1) *if f is a lower semicontinuous from function, then*

$$f^{\uparrow}_{\Lambda}(x')(h') = \lim_{\substack{x \to_f x' \\ \alpha \in \mathfrak{F}_\Lambda}} \sup \inf_{h \to h'} \frac{f(x+\alpha h) - f(x)}{\alpha},$$

$$f_\Lambda^\circ(x')(h') = \limsup_{\substack{x \to_f x' \\ \alpha \in \mathfrak{F}_\Lambda}} \frac{f(x+\alpha h') - f(x)}{\alpha},$$

where $x \to_f x'$ implies $x \to_\sigma x'$ and $f(x) \to f(x')$;

(2) *for a continuous function f we have*

$$f_{\Lambda,d}^\uparrow(x')(h') = \limsup_{\substack{x \to x' \\ \alpha \in \mathfrak{F}_\Lambda}} \inf_{h \to h'} \frac{f(x+\alpha h) - f(x)}{\alpha},$$

$$f_{\Lambda,d}^\circ(x')(h') = \limsup_{\substack{x \to x' \\ \alpha \in \mathfrak{F}_\Lambda}} \frac{f(x+\alpha h') - f(x)}{\alpha}.$$

◁ For the proof we have to recall the criterion for the Rockafeller limit **5.2.23**, as well as **5.4.9** and **5.4.10**. ▷

5.4.12. Theorem. *Let Λ be a representative set of infinitesimals. The following statements are valid:*

(1) *if f is a mapping directionally Lipshitz at the point x', i.e. such that* $\mathrm{Ha}(\mathrm{epi}(f), (x', f(x'))) \neq \varnothing$, *then*

$$f_\Lambda^\uparrow(x') = f_\Lambda^\circ(x');$$

if, moreover, f is continuous at the point x', then

$$f_\Lambda^\uparrow(x') = f_{\Lambda,d}^\uparrow(x') = f_{\Lambda,d}^\circ(x') = f_\Lambda^\circ(x');$$

(2) *if f is an arbitrary mapping, and the Hadamard cone of the effective domain of f at the point x' is nonempty, i.e.,* $\mathrm{Ha}(\mathrm{dom}(f), x') \neq \varnothing$, *then*

$$f_{\Lambda,d}^\uparrow(x') = f_{\Lambda,d}^\circ(x').$$

◁ The proof of both statements sought is carried out in the same pattern as that of theorem **5.4.2**. Let us consider in detail the case when f is directionally Lipshitz.

Let us set $\mathfrak{A} := \mathrm{epi}(f)$, $a' := (x', f(x'))$. By hypothesis, both $\mathrm{Cl}_\Lambda(\mathfrak{A}, a')$ and $\mathrm{Ha}_\Lambda(\mathfrak{A}, a')$ are convex cones. In this case $\mathrm{Ha}_\Lambda(\mathfrak{A}, a') \supset \mathrm{Ha}(\mathfrak{A}, a')$ and, hence, $\mathrm{int}_{\tau \times \tau_{\mathbb{R}}} \mathrm{Ha}_\Lambda(\mathfrak{A}, a') \neq \varnothing$. On the basis of the Rockafeller formula we deduce:

$$\mathrm{cl}_{\tau \times \tau_{\mathbf{R}}} \mathrm{Ha}_\Lambda(\mathfrak{A}, a') = \mathrm{Cl}_\Lambda(\mathfrak{A}, a').$$

This equality implies the required statement. ▷

5.4.13. Theorem. *Let $f_1, f_2 \colon X \to \overline{\mathbf{R}}$ be arbitrary functions, and let $x' \in \mathrm{dom}(f_1) \cap \mathrm{dom}(f_2)$. In this case*

$$(f_1 + f_2)^{\uparrow}_{\Lambda,d}(x') \le (f_1)^{\uparrow}_{\Lambda,d}(x') + (f_2)^{\circ}_{\Lambda,d}(x').$$

If, moreover, f_1 and f_2 are continuous at the point x', then

$$(f_1 + f_2)^{\uparrow}_{\Lambda}(x') \le (f_1)^{\uparrow}_{\Lambda}(x') + (f_2)^{\circ}_{\Lambda}(x').$$

◁ Let a standard element h' be chosen in the following way:

$$h' \in \mathrm{dom}((f_2)^{\circ}_{\Lambda,d}) \cap \mathrm{dom}((f_1)^{\uparrow}_{\Lambda,d}).$$

If there is no such an h', the estimates sought are obvious.

Let us choose $t' \ge (f_1)^{\uparrow}_{\Lambda,d}(x')(h')$ and $s' > (f_2)^{\circ}_{\Lambda,d}(x')(h')$. Then, by virtue of **5.4.8**, for every $x \approx_\sigma x'$, $x \in \mathrm{dom}(f_1) \cap \mathrm{dom}(f_2)$ and any $\alpha \in \Lambda$ there is an h for which $h \approx_\tau h'$, and, moreover,

$$\delta_1 := {}^{\circ}((f_1(x + \alpha h) - f_1(x) / \alpha) \le t';$$
$$\delta_2 := {}^{\circ}((f_2(x + \alpha h) - f_2(x) / \alpha) < s'.$$

Hence, we deduce: $\delta_1 + \delta_2 < t' + s'$, which ensures (1). If f_1 and f_2 are continuous at the point x, then we apply **5.4.10**. ▷

5.4.14. By way of concluding the present stage of discussion, let us consider special presentations of the Clarke cone which arise in a finite-dimensional space and pertain to the following remarkable result.

5.4.15. Cornet theorem. *In a finite-dimensional space the Clarke cone is the Kuratowski limit of contingencies:*

$$\mathrm{Cl}(F, x') = \mathrm{Li}_{\substack{x \to x' \\ x \in F}} K(F, x).$$

5.4.16. Corollary. *Let Λ be an (external) set of strictly positive infinitesimals, containing an (internal) sequence convergent to zero. Then the following equality is valid:*

$$\mathrm{Cl}_{\Lambda}(F,x') = \mathrm{Cl}(F,x').$$

◁ By the Leibniz principle, we can work in standard entourage. Since the inclusion $\mathrm{Cl}_{\Lambda}(F,x') \supset \mathrm{Cl}(F,x')$ is obvious, let us choose a standard point h' from $\mathrm{Cl}_{\Lambda}(F,x')$ and establish that h' lies in the Clarke cone $\mathrm{Cl}(F,x')$.

Since use is made of **5.3.13**, the following presentation is valid:

$$\mathrm{Li}_{\substack{x\to x'\\ x\in F}} K(F,x) = {}^*\{h' : (\forall x \approx x', x \in F)(\exists h \approx h')\, h \in K(F,x)\};$$

let us make sure that for $x \approx x'$, $x \in F$ we have $h \in K(F,x)$ for a certain element h infinitely close to h'.

If (α_n) is a sequence of Λ elements convergent to zero then, by hypothesis,

$$(\forall n \in \mathbf{N})(\exists h_n)\ X + \alpha_n h_n \in F \wedge h_n \approx h'.$$

For any standard $\varepsilon > 0$ and the conventional norm $\|\cdot\|$, in $\mathbf{R}^n$ we have $\|\bar{h} - h'\| \le \varepsilon$. Therefore, taking into account finite dimensions, we can find sequences $(\bar{\alpha}_n)$ and $(\bar{h}_n)$ such that

$$\bar{\alpha}_n \to 0,\ \ \bar{h}_n \to \bar{h},\ \ \|\bar{h} - h'\| \le \varepsilon,\ \ x + \bar{\alpha}_n \bar{h}_n \in F \quad (n \in \mathbf{N}).$$

Applying the strong idealization principle, we come to the conclusion that there are sequences $(\bar{\alpha}_n)$ and $(\bar{h}_n)$ serving simultaneously all standard positive numbers ε. Obviously, the corresponding limiting vector h is infinitely close to h', and, at the same time, by the definition of contingency, $h \in K(F, x)$. ▷

5.4.17. In the theorem given above we can use as a set Λ the monad of any filter convergent to zero, for instance, of the tail filter of a fixed standard sequence (α_n) composed of strictly positive numbers and vanishing. Let us recall the characteristics of the Clarke cone pertaining to this case and supplementing those already considered. For the formulation let use the symbol $d_F(x)$ to denote the distance from the point x to the set F.

5.4.18. Theorem. *For a sequence (α_n) of strictly positive numbers convergent to zero* *the following statements are equivalent:*

(1) $h' \in \mathrm{Cl}(F, x')$,

(2) $\limsup\limits_{\substack{x \to x' \\ n \to \infty}} \dfrac{d_F(x + \alpha_n h') - d_F(x)}{\alpha_n} \le 0$,

(3) $\limsup\limits_{x \to x'} \limsup\limits_{n \to \infty} \alpha_n^{-1}(d_F(x + \alpha_n h') - d_F(x)) \le 0$,

(4) $\lim\limits_{\substack{x \to x' \\ x \in F}} \limsup\limits_{n \to \infty} \alpha_n^{-1} d_F(x + \alpha_n h') = 0$,

(5) $\limsup\limits_{x \to x'} \liminf\limits_{n \to \infty} \alpha_n^{-1}(d_F(x + \alpha_n h') - d_F(x)) \le 0$,

(6) $\lim\limits_{\substack{x \to x' \\ x \in F}} \liminf\limits_{n \to \infty} \dfrac{d_F(x + \alpha_n h')}{\alpha_n} = 0$.

◁ Let us first of all observe that for $\alpha > 0$ the following equivalence is valid:

$$^{\circ}(\alpha^{-1} d_F(x + \alpha h')) = 0 \leftrightarrow (\exists h \approx h')\, x + \alpha h \in F,$$

where $^{\circ}t$ is, as usual, the standard part of the number t.

Indeed, in order to establish the implication to the left, set $y := x + \alpha h'$. In this case

$$d_F(x + \alpha h') / \alpha = \|x + \alpha h' - y\| / \alpha \le \|h - h'\|.$$

When checking the reverse implication, we get, applying the strong idealization principle, the following result:

$$\begin{aligned}
&^{\circ}(\alpha^{-1} d_F(x + \alpha h')) = 0 \to (\forall^{\mathrm{st}} \varepsilon > 0)\, d_F(x + \alpha h') / \alpha < \varepsilon \\
&\to (\forall^{\mathrm{st}} \varepsilon > 0)(\exists y \in F)\ \|x + \alpha h' - y\| / \alpha < \varepsilon \\
&\to (\exists y \in F)(\forall^{\mathrm{st}} \varepsilon > 0)\ \|h' - (y - x) / \alpha\| < \varepsilon \\
&\to (\exists y \in F)\, \|h - (y - x) / \alpha\| \approx 0.
\end{aligned}$$

Setting now $h := (y - x) / \alpha$, we see that $h \approx h'$, in which case $x + \alpha h \in F$.

Let us now go over to the proof of the sought equivalences.

Since the implications (3) → (4) → (6) and (3) → (5) → (6) are obvious, we only establish that (1) → (2) → (3) and (6) → (1).

(1) → (2). Working in standard entourage, let us choose $x \approx x'$ and $N \approx +\infty$. Choose

an $x'' \in F$ in such a way that we had $\|x = x''\| < d_F(x') + \alpha_N^2$. Since the inequality

$$d_F(x + \alpha_N h') - d_F(x'' + \alpha_N h') \leq \|x - x''\|,$$

is valid, we can deduce the following estimates:

$$(d_F(x + \alpha_N h') - d_F(x)) / \alpha_N \leq (d_F(x'' + \alpha_N h') + \|x - x''\| -$$
$$- d_F(x)) / \alpha_N \leq d_F(x'' + \alpha_N h') / \alpha_N + \alpha_N.$$

As $h' \in \mathrm{Cl}(F, x')$, and by the choice of x'' and N, for a certain $h \approx h'$ we get $x'' + \alpha_N h \in F$. Therefore, from the above, we infer $^\circ(d_F(x'' + \alpha_N h) / \alpha_N) = 0$. Hence,

$$(\forall x \approx x')(\forall N \approx +\infty)\, ^\circ(\alpha_N^{-1}(d_F(x + \alpha_N h') - d_F(x))) \leq 0.$$

This is, by **5.3.22**, the nonstandard criterion for (2) to be valid.

(2) → (3). It suffices to remark that for $f\colon U \times V \to \overline{\mathbf{R}}$, as well as for the filters $\mathfrak{F}$ in U and $\mathscr{B}$ in V, we have

$$\limsup_{\mathfrak{F}} \limsup_{\mathscr{B}} f(x,y) \leq t$$
$$\leftrightarrow (\forall x \in \mu(\mathfrak{F}))\, ^\circ\limsup_{\mathscr{B}} f(x,y) \leq t$$
$$\leftrightarrow (\forall x \in \mu(\mathfrak{F}))(\forall^{\mathrm{st}} \varepsilon > 0) \inf_{G \in \mathscr{B}} \sup_{y \in G} f(x,y) < t + \varepsilon$$
$$\leftrightarrow (\forall x \in \mu(\mathfrak{F}))(\forall^{\mathrm{st}} \varepsilon > 0)(\exists G \in \mathscr{B}) \sup_{y \in G} f(x,y) < t + \varepsilon$$
$$\leftrightarrow (\forall x \in \mu(\mathfrak{F}))(\exists G \in \mathscr{B})(\forall^{\mathrm{st}} \varepsilon > 0) \sup_{y \in G} f(x,y)\, t + \varepsilon$$
$$\leftrightarrow (\forall x \in \mu(\mathfrak{F}))(\exists G \in \mathscr{B})(\forall^{\mathrm{st}} \varepsilon > 0) \sup_{y \in G} f(x,y) \leq t + \varepsilon$$
$$\leftrightarrow (\forall x \in \mu(\mathfrak{F}))(\exists G \in \mathscr{B})(\forall y \in G)\, ^\circ f(x,y) \leq t.$$

Here, as usual, $\mu(\mathfrak{F})$ is the monad of the filter $\mathfrak{F}$.

(6) → (1). Let us start with the remark that in the notation of the preceding fragment of the proof, we have

$$\limsup_{\mathfrak{F}} \limsup_{\mathscr{B}} f(x,y) \leq t$$
$$\leftrightarrow (\forall x \in \mu(\mathfrak{F})) \sup_{G \in \mathscr{B}} \inf_{y \in \mathscr{B}} f(x,y) \leq t$$
$$\leftrightarrow (\forall x \in \mu(\mathfrak{F}))(\forall^{\mathrm{st}} \varepsilon > 0)(\forall G \in \mathscr{B}) \inf_{y \in G} f(x,y) \leq t + \varepsilon$$
$$\leftrightarrow (\forall x \in \mu(\mathfrak{F}))(\forall G \in \mathscr{B})(\forall^{\mathrm{st}} \varepsilon > 0) \inf_{y \in G} f(x,y) < t + \varepsilon$$

$$\leftrightarrow (\forall x \in \mu(\mathfrak{F}))(\forall G \in \mathcal{B})(\forall^{st}\varepsilon > 0)(\exists y \in G)\ f(x,y) < t + \varepsilon$$
$$\leftrightarrow (\forall x \in \mu(\mathfrak{F}))(\forall G \in \mathcal{B})(\exists y \in G)\ {}^{\circ}f(x,y) \leq t.$$

Using the conditions, from the established characteristic we deduce:

$$(\forall x \approx x', x \in F)\ (\forall n)\ (\exists N \geq n\ {}^{\circ}(\alpha_N^{-1} d_F(x + \alpha_N h')) = 0.$$

In other words, for a certain h_N such that $h_N \approx h'$, we get $x + \alpha_N h_N \in F$. Taking into account the considerations presented, we can, as was the case in **5.4.16**, deduce that h' lies in the lower Kuratowski limit of the contingencies of the set F at the points close to x', i.e., in the Clarke cone $\mathrm{Cl}(F, x')$. ▷

5.5. Approximation to the Composition of Sets

Let us now go over to studying tangents of the Clarke type and compositions of correspondences. In this case we have to start with some topological considerations pertaining to open and nearly open operators.

5.5.1. Take, alongside with the vector space X with topologies σ_Y and τ_X under consideration, another vector space Y with topologies σ_Y and τ_Y. Let us consider a linear operator T from X to Y and study, first of all, the problem of the relation between the approximating sets F at the point x', where $F \subset X$, and the image $T(F)$ at the point Tx'.

5.5.2. *The following statements are valid:*

(1) *the inclusion*

$$T(\mu(\sigma_X(x')) \cap F) \supset \mu(\sigma_Y(Tx')) \cap T(F)$$

is equivalent to the relation

$$(\forall U \in \sigma_X(x'))\ (\exists V \in \sigma_Y(Tx'))\ T(U \cap F) \supset V \cap T(F),$$

which is the condition of (relative) preopenness, or condition (ρ_-) *(for the parameters* T, F *and* x'*);*

(2) *condition* (ρ_-) *combined with the requirement that* T *be continuous as a mapping from* (X,σ_X) *to* (Y,σ_Y) *is equivalent to the following condition* (ρ_-), *the condition of (relative) openness*:

$$T(\mu(\sigma_X(x')) \cap F) \supset \mu(\sigma_Y(Tx')) \cap T(F);$$

(3) *the operator* T *obeys the condition of (relative) near-openness, or condition* (ρ_-), *i.e.*,

$$(\forall U \in \sigma_X(x'))(\exists V \in \sigma_Y((Tx'))\quad \mathrm{cl}_{\tau_Y} T(U \cap F) \supset V \cap T(F)$$

iff

$$(\forall W \in \mathcal{N}_{\tau_Y})\ T(\mu(\sigma_X(x')) \cap F) + W \supset \mu(\sigma_Y(Tx')) \cap T(F).$$

◁ Statements (1) and (2) are obtained by specialization of **5.3.2.**. To prove (3), let us denote

$$\mathcal{A} := T(\sigma_X(x') \cap F),\quad \mathcal{B} := \sigma_Y(Tx') \cap T(F),$$
$$\mathcal{N} := \{N \subset Y^2 : (\exists W \in \mathcal{N}_{\tau_Y})\quad N \supset \{(y_1,y_2) : y_1 - y_2 \in W\}\},$$

i.e., $\mathcal{N}$ is the uniformity in Y corresponding to the topology in question. Using the introduced notation and applying **5.3.2**, as well as the principles of idealization and transfer, we get:

$$\begin{aligned}
&(\forall N \in \mathcal{N})\ N(\mu(\mathcal{A})) \supset \mu(\mathcal{B})\\
&\leftrightarrow (\forall N \in \mathcal{N})\ (\forall b \in \mu(\mathcal{B}))\ (\exists a \in \mu(\mathcal{A}))\ b \in N(a)\\
&\leftrightarrow (\forall N \in \mathcal{N})\ (\forall^{\mathrm{st}} A \in \mathcal{A})\ (\exists^{\mathrm{st}} B \in \mathcal{B})\ (\forall b \in B)(\exists a \in A)\ b \in N(a)\\
&\leftrightarrow\ (\forall^{\mathrm{st}} A \in \mathcal{A})(\forall N \in \mathcal{N})(\exists^{\mathrm{st}} B \in \mathcal{B})\ B \subset N(A)\\
&\leftrightarrow\ (\forall^{\mathrm{st}} A \in \mathcal{A})(\exists^{\mathrm{st}} B \in \mathcal{B})(\forall N \in \mathcal{N})\ B \subset N(A)\\
&\leftrightarrow\ (\forall^{\mathrm{st}} A \in \mathcal{A})(\exists^{\mathrm{st}} B \in \mathcal{B})\ B \subset \mathrm{cl} A\\
&\leftrightarrow\ (\forall A \in \mathcal{A})(\exists B \in \mathcal{B})\ B \subset \mathrm{cl} A,
\end{aligned}$$

where the closure is calculated in the corresponding uniform topology. ▷

5.5.3. Theorem. *The following statements are valid*:

(1) *if the operator* T *obeys condition* (ρ) *and is continuous as a mapping from* (X,τ_X) *to* (Y,τ_Y), *then*

$$T(\mathrm{Cl}_\Lambda(F,x')) \subset \mathrm{Cl}_\Lambda(T(F),Tx'),$$

$$T(\mathrm{In}_\Lambda(F,x')) \subset \mathrm{In}_\Lambda(T(F),Tx');$$

if, moreover, T is an open mapping of (X,τ_X) in (Y,τ_Y), then

$$T(\mathrm{Ha}_\Lambda(F,x')) \subset \mathrm{Ha}_\Lambda(T(F),Tx');$$

(2) *if τ_Y is a vector topology, the operator $T\colon (X,\tau_X) \to (Y,\tau_Y)$ is continuous and obeys condition $(\bar{\rho})$, then*

$$T(\mathrm{Cl}_\Lambda(F,x')) \subset \mathrm{Cl}_\Lambda(T(F),Tx').$$

◁ (1) Let us, for instance, check the second of the required inclusions. To this end, having fixed $h' \in \mathrm{In}_\Lambda(F,x')$, for $\alpha \in \Lambda$, we choose an $h \approx_{\tau_X} h'$ such that for all $x \approx_{\sigma_X} x'$, $x \in F$ we have $x + \alpha h \in F$. Obviously, $Th \approx_{\sigma_X} Th'$ and $Tx + \alpha Th \in T(F)$. Applying condition (ρ), we conclude: $Th' \in \mathrm{In}_\Lambda(T(F),Tx')$.

Let now be known that T obeys the additional condition of openness, i.e., on the basis of **5.5.2** (1), $T(\mu(\tau_X)) \supset \mu(\tau_Y)$. Combined with the continuity of T, this implies that the obtained monads coincide. If now $y \in T(F)$, $y \approx_{\sigma_Y} Tx'$, then by condition (ρ), we get $y = Tx$, where $x \in F$ and $x \approx_{\sigma_X} x'$. In this case for $z \approx_{\tau_Y} Th'$ we can find an $h \approx_{\tau_X} h'$, for which $z = Th$. Therefore, for all $\alpha \in \Lambda$ we have $x + \alpha h \in F$, i.e., $y + \alpha z = Tx + \alpha Th \in T(F)$ as soon as a standard h' is such that $h' \in \mathrm{Ha}_\Lambda(F,x')$.

(2) Let us choose an infinitesimal $\alpha \in \Lambda$ and any standard element $h' \in \mathrm{Cl}_\Lambda(F,x')$. Let W is a certain infinitely small zero neighbourhood of τ_Y. Then, by hypothesis, αW is also a zero neighbourhood. On the basis of $(\bar{\rho})$, having taken $y \approx_{\sigma_X} Tx'$, $y \in T(F)$, we find $x \in \mu(\sigma_X(x')) \cap F$ in such a way that $y = Tx + \alpha\omega$ and $\omega \approx_{\tau_Y} 0$. By the condition of the containment of h' in the Clarke cone, there is an element $h'' \approx_{\tau_Y} h'$ for which $x + \alpha h'' \in F$. Hence, $y + \alpha(Th'' - \omega) = y - \alpha\omega + \alpha Th'' = T(x + \alpha h'') \in T(F)$. Indeed, from here we deduce: $Th'' - \omega \in Th' + \mu(\tau_Y) - \omega \in Th' + \mu(\tau_Y) + \mu(\tau_Y) = Th' + \mu(\tau_Y)$. Therefore, we have established: $Th' \in \mathrm{Cl}_\alpha(T(F),Tx')$. ▷

5.5.4. Let us now consider vector spaces X,Y,Z furnished with topologies σ_X,τ_X; σ_Y,τ_Y, respectively. Let, then, $F \subset X \times Y$ and $G \subset X \times Z$ be two correspondences, and let the point $d' := (x',y',z') \in X \times Y \times Z$ be such that $a' := (x',y') \in F$ and $b' := (x',z') \in G$. Introduce the following notation: $H := X \times G \cap F \times Z$, $c' := (x',z')$. It should be remarked that $G \circ F = \mathrm{Pr}_{X\times Z} H$, where $\mathrm{Pr}_{X\times Z}$ is the operator of natural projection.

Let us introduce the following abbreviations:

$$\sigma_1 := \sigma_X \times \sigma_Y;\quad \sigma_2 := \sigma_Y \times \sigma_Z;\quad \sigma := \sigma_X \times \sigma_Z;\quad \overline{\sigma} := \sigma_X \times \sigma_Y \times \sigma_Z;$$
$$\tau_1 := \tau_X \times \tau_Y;\quad \tau_2 := \tau_Y \times \tau_Z;\quad \tau := \tau_X \times \tau_Z;\quad \overline{\tau} := \tau_X \times \tau_Y \times \tau_Z.$$

It would be expedient to recall that the operator $\mathrm{Pr}_{X\times Z}$ is continuous and open (when 'same-letter' topologies are used). Still fixed is a certain set Λ composed of infinitesimals. Let us also recall a property of monads we need.

5.5.5. *The monad of a composition is the composition of monads..*

◁ Let $\mathcal{A}$ is a filter in $X \times Y$, while $\mathcal{B}$ is a filter in $Y \times Z$. We have

$$\mathcal{B} \circ \mathcal{A} := \widetilde{\{B \circ A : A \in \mathcal{A}, B \in \mathcal{B}\}},$$

in which case the sets operating in the definition of $\mathcal{B} \circ \mathcal{A}$ can be considered to be nonempty. It is obvious that $B \circ A = \mathrm{Pr}_{X\times Z}(A \times Z \cap X \times B)$. Therefore, the filter under consideration, $\mathcal{B} \circ \mathcal{A}$, is the image $\mathrm{Pr}_{X\times Z}(\mathcal{C})$, where $\mathcal{C} := \mathcal{C}_1 \vee \mathcal{C}_2$ and $\mathcal{C}_1 := \mathcal{A} \times \{Z\}$, $\mathcal{C}_2 := \{X\} \times \mathcal{B}$. Since the monad of a product is the product of monads, and the monad of the least upper bound of filters is the intersection of their monads, then, making use of **4.1.6** (5), we come to the relation

$$\mu(\mathcal{B} \circ \mathcal{A}) = \mathrm{Pr}_{X\times Z}(\mu(\mathcal{A}) \times Z \cap X \times \mu(\mathcal{B})) = \mu(\mathcal{B}) \circ \mu(\mathcal{A}),$$

which was required. ▷

5.5.6. *The following statements are equivalent:*

(1) *for the operator* $\mathrm{Pr}_{X\times Z}$, *the correspondence* H *and a point* c', *condition* (ρ) *is fulfilled*;

(2) $G \circ F \cap \mu(\sigma(c')) = G \cap \mu(\sigma_2(b')) \circ F \cap \mu(\sigma_1(a'))$;

(3) $(\forall V \in \sigma_Y(y'))(\exists U \in \sigma_X(x'))(\exists W \in \sigma_Z(z'))$
$G \circ F \cap U \times W \subset G \circ I_V \circ F,$

where I_V *is, as usual, the identity relation on* V.

◁ Applying **5.3.2**, let us rewrite (3) in equivalent form:

$$(\forall V \in \sigma_Y(y'))(\exists O \in \sigma(c'))(\forall (x,z) \in O, (x,z) \in G \circ F)$$
$$(\exists y \in V)\,(x,y) \in F \wedge (y,z) \in G \leftrightarrow (\forall (x,z) \approx_\sigma c', (x,z) \in G \circ F)$$
$$(\exists y \approx_{\sigma_Y} y')\,(x,y) \in F \wedge (y,z) \in G \leftrightarrow \mu(\sigma(c')) \cap G \circ F$$
$$\subset \mu(\sigma_2(b')) \cap G \circ \mu(\sigma_1(a')) \cap F.$$

Now we have to observe that

$$\mathrm{Pr}_{X \times Z}(\mu(\bar{\sigma}(d')) \cap H)$$
$$= \{(x,z) \in G \circ F : x \approx_{\sigma_X} x' \wedge z \approx_{\sigma_Z} z' \wedge (\exists y \approx_{\sigma_Y} y')\,(x,y) \in F \wedge (y,z) \in G\}$$
$$= \mu(\sigma_2(b')) \cap G \circ \mu(\sigma_1(a')) \cap F. \ \triangleright$$

5.5.7. *The following statements are equivalent:*

(1) *for the operator* $\mathrm{Pr}_{X \times Z}$, *the correspondence* H *and a point* c', *condition* $(\bar{p})$ *is fulfilled*;

(2) $(\forall W \in \mathfrak{N}_\tau)\ \mu(\sigma_2(b')) \cap G \circ \mu(\sigma_1(a')) \cap F + W \supset \mu(\sigma(c')) \cap G \circ F;$

(3) $(\forall V \in \sigma_2(b'))(\forall U \in \sigma_1(a'))(\exists W \in \sigma(c'))$
$W \cap G \circ F \subset \mathrm{cl}_\tau(V \cap G \circ U \cap F);$

(4) $(\forall U \in \sigma_X(x'))(\forall V \in \sigma_Y(y'))(\forall W \in \sigma_Z(z'))(\exists V \in \sigma(c'))$
$O \cap G \circ F \subset \mathrm{cl}_\tau(G \circ I_V \circ F \cap U \times W);$

(5) *if* $\tau \geq \sigma$, *then*

$$(\forall V \in \sigma_Y(y'))(\exists U \in \sigma_X(x'))(\exists W \in \sigma_Z(z'))$$
$$G \circ F \cap U \times W \subset \mathrm{cl}_\tau(G \circ I_V \circ F),$$

i.e., condition $(\overline{\rho c})$ is said to be fulfilled for the point $d' := (x', y', z')$.

◁ From supposition **5.5.2** (3) and the proof of **5.5.2** (3) we directly conclude: (1) ↔ (2) ↔ (3).

In order to prove the equivalence (3) ↔ (4), it suffices to remark:

$$
\begin{aligned}
&(V \times W) \cap G \circ (U \times V) \cap F \\
&= \{(x,z) \in X \times Z \colon x \in U \wedge z \in W \wedge (\exists y \in V)\,(x,y) \in F \wedge (y,z) \in G\} \\
&= G \circ I_V \circ F \cap U \times W
\end{aligned}
$$

for any $U \subset X, V \subset Y, W \subset Z$.

Therefore, it now remains to be established that (4) ↔ (5), this implication, however, being obvious, since (5) is obtained by a specialization of (4) for $U := X$ and $W := Z$.

In order to check (5) ↔ (4) let us, having set $V \in \sigma_Y(y')$, choose an open neighbourhood $C \in \sigma(c')$, so that we had $G \circ F \cap C \subset \mathrm{cl}_\tau A$, where $A := G \circ U_V \circ F$. Having chosen open $U \in \sigma_X(x')$ and $W \in \sigma_Z(z')$, let us set $B := U \times W$ and $O := B \cap C$. Obviously, $G \circ F \cap O \subset (\mathrm{cl}_\tau A) \cap B$. Working in standard entourage, for an $a \in (\mathrm{cl}_\tau A) \cap B$ let us find a point $a \in A$ such that $a' \approx_\tau a$. Obviously, $a' \approx_\sigma a$, since $\mu(\tau) \subset \mu(\sigma)$ by condition. As B is σ-open, we get $a' \in B$, i.e., $a' \in A \cap B$ and $a \in \mathrm{cl}_\tau(A \cap B)$. Finally, $G \circ F \cap O \subset \mathrm{cl}_\tau(A \cap B)$, which was to be ensured. ▷

5.5.8. *The following inclusions are valid:*

(1) $\mathrm{Ha}_\Lambda(H,d') \supset X \times \mathrm{Ha}_\Lambda(G,b') \cap \mathrm{Ha}_\Lambda(F,a') \times Z$;

(2) $R^2_\Lambda(H,d') \supset X \times R^1_\Lambda(G,b') \cap R^2_\Lambda(F,a') \times Z$;

(3) $\mathrm{Cl}_\Lambda(H,d') \supset X \times Q^1_\Lambda(G,b') \cap \mathrm{Cl}_\Lambda(F,a') \times Z$;

(4) $\mathrm{Cl}_\Lambda(H,d') \supset X \times \mathrm{Cl}(G,b') \cap Q^2_\Lambda(F,a') \times Z$;

(5) $\mathrm{Cl}^2(H,d') \supset X \times P^2(G,b') \cap S^2(F,a') \times Z$, *where the cone* $\mathrm{Cl}^2(H,d')$ *is determined (in standard entourage) by the relation*

$$\mathrm{Cl}^2(H,d') := {}^*\{(s',t',r')\in X\times Y\times Z:\ (\forall d\approx_{\bar\sigma} d', d\in H)$$
$$(\forall\alpha\in\mu(\mathbf{R}_+))\ (\exists s\approx_{\tau_X} s')\,(\forall t\approx_{\tau_Y} t')\,(\exists r\approx_{\tau_Z} z')\ d+\alpha(s,t,r)\in H\}.$$

◁ Only (1) and (5) are to be checked, the remaining statements proved by the same scheme.

(1) Let the element (s',t',r') be standard and a member of the right-hand side of the relation under study. Let us choose a $d\approx_{\bar\sigma} d'$ and $\alpha\in\Lambda$, where $d:=(x,y,z)\in H$. Obviously, $a:=(x,y)\in F$ and $a\approx_{\sigma_1} a'$, while $b:=(y,z)\in G$, $b\approx_{\sigma_2} b'$. In this respect, for $\alpha\in\Lambda$ and $(s,t,r)\approx_{\bar\tau}(s',t',r')$ we get $a+\alpha(s,t)\in F$ and $b+\alpha(t,r)\in G$. Hence,

$$d+\alpha(s,t,r) = (a+\alpha(s,t),\ z+\alpha r)\in F\times Z,$$
$$d+\alpha(s,t,r) = (x+\alpha s,\ b+\alpha(t,r))\in X\times G,$$

i.e., $(s',t',r')\in \mathrm{Ha}_\Lambda(H,d')$.

(5) Let us take a standard element (s',t',r') from the right-hand side of (4). By definition, there is an element $s\approx_{\tau_X} s'$ such that for any $t\approx_{\tau_Y} t'$ for a certain $r\approx_{\tau_Z} r'$ and all $a\approx_{\sigma_1} a'$ and $b\approx_{\sigma_2} b'$, we have $a+\alpha(s,t)\in F$ and $b+\alpha(t,r)\in G$. Obviously, we get $d+\alpha(s,t,r)\in H$ as soon as $b\approx_{\bar\sigma} d'$ and $d\in H$. ▷

5.5.9. It should be emphasized that the mechanism of 'leapfrogging' demonstrated in **5.5.8**, can be modified in accord with the purposes of investigation. Such purposes include, as a rule, the estimates of the approximation to the composition of sets. In this case it would be most convenient to use the scheme based on the use of the method of general position [115, 121], as well as the results discussed above, both detalizing and generalizing this scheme. Let us formulate one of the possible results.

5.5.10. Theorem. *Let τ be a vector topology, $\tau\ge\sigma$, and correspondences $F\subset X\times Y$ and $G\subset Y\times Z$ be such that* $\mathrm{Ha}(F,a')\ne\varnothing$ *and the cones* $Q_2(F,a')\times Z$ *and* $X\times\mathrm{Cl}(G,b')$ *are in general position (relative to the topology $\bar\tau$), then*

$$\mathrm{Cl}(G\circ F,c')\supset \mathrm{Cl}(G,b')\circ\mathrm{Cl}(F,a'),$$

provided condition $(\overline{pc})$ *is fulfilled at the point* d'.

◁ The proof is carried out by the scheme of **5.3.13** in [121], and consists in verifying if the (primarily-set) conditions ensuring validity for the following statements:

$$\begin{aligned}
&\mathrm{Cl}(G \circ F, c') = \mathrm{Cl}(\mathrm{Pr}_{X \times Z} H, \mathrm{Pr}_{X \times Z} d') \supset \mathrm{cl}_{\tau}\, \mathrm{Pr}_{X \times Z} \mathrm{Cl}(H, d') \\
&\supset \mathrm{Pr}_{X \times Z}\, \mathrm{cl}_{\bar{\tau}}(X \times \mathrm{Cl}(G, b') \cap Q^2(F, a') \times Z) \\
&= \mathrm{Pr}_{X \times Z}(\mathrm{cl}_{\bar{\tau}}(X \times \mathrm{Cl}(G, b')) \cap \mathrm{cl}_{\bar{\tau}}(Q^2(F, a') \times Z)) \\
&= \mathrm{Pr}_{X \times Z}(X \times \mathrm{Cl}(G, b') \cap \mathrm{Cl}(F, a') \times Z) = \mathrm{Cl}(G, b') \circ \mathrm{Cl}(F, a'). \ \triangleright
\end{aligned}$$

5.6. Infinitesimal Subdifferentials

In the theory of extreme problems some attention is paid to the problem of taking into account the accuracy of observing optimality criteria in practical calculations. The adopted qualitative approach to the problem in question has been implemented in the so-called convex ε-programming which ensures an apparatus for estimating approximation to an optimum relative to a functional. The apparatus developed within this approach is quite specific and proves, in a certain sense, to be artificially complicated. At the same time, it is not quite well-correlated with the conventional techniques based on the search for the 'practical optimum' with the help of 'practically exact' satisfaction of complementary slackness, which correspond to the classical case $\varepsilon = 0$. As a result, we have to state a certain discrepancy, and even a gap between theoretical and practical viewpoints.

In this section we shall discuss an approach to overcome the present difficulties within the radical stance on nonstandard analysis. The approach is based on the introduction of the notion of an infinitesimally optimal solution, i.e., a feasible point, the value of the goal function in which is infinitely close to the ideal, i.e., to the value of the program not obligatory assumed. Therefore, infinitesimal optimum becomes a suitable challenger for the role of 'practical' optimum, since no feasible procedures are able to distinguish it from the conventional 'theoretical' optimum. The basic formulas of the calculus of infinitesimal subdifferentials corresponding to the concept of optimum presented above are given. The rules for external sets obtained coincide in form with their classical analogues of conventional convex analysis. In this case in the criteria of infinitesimal optimality there indeed arises approximately met complimentary slackness.

5.6.1. Let X be a vector space, $E^{\cdot}$ an ordered vector space with the greatest element $+\infty$ adjoined. Let us consider a convex operator $f: X \to E^{\cdot}$, and a point $\bar{x}$ of the effective domain $\mathrm{dom}(f) := \{x \in X: f(x) < +\infty\}$ of the operator f. For an element $\varepsilon \geq 0$ (of the cone of positive elements E^{+} of the space E) let us use the conventional definition of the ε-*subdifferential of f* at the point $\bar{x}$, i.e., the set

$$\partial_{\varepsilon} f(\bar{x}) := \{T \in L(X,E): (\forall x \in X)\ Tx - T\bar{x} \leq f(x) - f(\bar{x}) + \varepsilon\},$$

where $L(X,E)$ is the space of linear operators acting from X to E.

5.6.2. Let a family $\mathcal{E}$ of positive elements filtered upwards be chosen in E. Viewing E and $\mathcal{E}$ as standard sets, let us determine the monad $\mu(\mathcal{E})$ by the relation $\mu(\mathcal{E}) := \bigcap\{[0,\varepsilon]: \varepsilon \in {}^{\circ}\mathcal{E}\}$. The elements of $\mu(\mathcal{E})$ are termed (positive) *infinitely small* or *infinitesimal* (*relative to* $\mathcal{E}$). Henceforth it will be implied without further specifications that E is a K-space, the monad $\mu(\mathcal{E})$ is an external cone over ${}^{\circ}\mathbf{R}$ and, besides, $\mu(\mathcal{E}) \cap {}^{\circ}E = 0$. (In application, as a rule, $\mathcal{E}$ is the unit filter of E). Use also will be made of the relation of infinite proximity between the elements of E, i.e.,

$$e_1 \approx e_2 \leftrightarrow e_1 - e_2 \in \mu(\mathcal{E}) \wedge e_2 - e_1 \in \mu(\mathcal{E}).$$

5.6.3. *The following equality is valid*:

$$\bigcap_{\varepsilon \in {}^{\circ}\mathcal{E}} \partial_{\varepsilon} f(\bar{x}) = \bigcap_{\varepsilon \in \mu(\mathcal{E})} \partial_{\varepsilon} f(\bar{x}).$$

$\lhd$ For $T \in L(X,E)$ we deduce:

$$\begin{aligned}
&T \in \bigcap_{\varepsilon \in {}^{\circ}\mathcal{E}} \partial_{\varepsilon} f(\bar{x}) \leftrightarrow (\forall^{\mathrm{st}} \varepsilon \in \mathcal{E})\ (\forall x \in X)\ Tx - T\bar{x} \leq f(x) - f(\bar{x}) + \varepsilon \\
&\leftrightarrow (\forall^{\mathrm{st}} \varepsilon \in \mathcal{E})\ f^{*}(T) := \sup_{x \in \mathrm{dom}(f)} (Tx - f(x)) \leq T\bar{x} - f(\bar{x}) + \varepsilon \\
&\leftrightarrow (\forall^{\mathrm{st}} \varepsilon \in \mathcal{E})\ 0 \leq f^{*}(T) - (T\bar{x} - f(\bar{x})) \leq -\varepsilon \leftrightarrow f^{*}(T) - (T\bar{x} - f(\bar{x})) \approx 0 \\
&\leftrightarrow (\exists \varepsilon \in E^{+})\ \varepsilon \approx 0 \wedge f^{*}(T) = T\bar{x} - f(\bar{x}) + \varepsilon \\
&\leftrightarrow T \in \bigcup_{\varepsilon \in \mu(\mathcal{E})} \partial_{\varepsilon} f(\bar{x}). \ \rhd
\end{aligned}$$

5.6.4. The external set occurring in both parts of equality **5.6.3** is called the *infinitesimal subdifferential* of the function f at the point $\bar{x}$ and is denoted by $Df(\bar{x})$. The elements of

$Df(\bar{x})$ are called *infinitesimal subgradients* of f at the point $\bar{x}$. The possibility of ambiguity being in this case insignificant, no additional specifications of the set $\mathscr{E}$ are made.

5.6.5. Let the assumption of standard entourage be fulfilled, i.e., the parameters $X, f, \bar{x}$ are assumed to be standard sets. *The standardization of the infinitesimal subdifferential of f at the point $\bar{x}$ coincides with the (zero) subdifferential of f at $\bar{x}$, i.e.,*

$$^*Df(\bar{x}) = \partial f(\bar{x}).$$

$\triangleleft$ By virtue of the transfer principle, for a standard $T \in {}^\circ L(X,E)$ we have:

$$\begin{aligned}
&T \in {}^*Df(\bar{x}) \leftrightarrow T \in Df(\bar{x}) \\
&\leftrightarrow (\forall^{st} \varepsilon \in \mathscr{E})\,(\forall x \in X)\; Tx - T\bar{x} \le f(x) - f(\bar{x}) + \varepsilon \\
&\leftrightarrow (\forall \varepsilon \in \mathscr{E})\,(\forall x \in X)\; Tx - T\bar{x} \le f(x) - f(\bar{x}) + \varepsilon \\
&\leftrightarrow T \in \partial f(\bar{x});
\end{aligned}$$

for $\inf \mathscr{E} = 0$ by virtue of the relation $\mu(\mathscr{E}) \cap {}^\circ E = 0$. $\triangleright$

5.6.6. *Let F be a standard K-space, and $g: E \to F$ be an increasing convex operator. If the sets $X \times$ epi(g) and epi$(f) \times F$ are in general position, then*

$$D(g \circ f)(\bar{x}) = \bigcup_{T \in Dg(f(\bar{x}))} D(T \circ f)(\bar{x}).$$

If, moreover, the parameters (except, possibly, the point $\bar{x}$) are standard, then the following presentation is valid for standard cores:

$$^\circ D(g \circ f)(\bar{x}) = \bigcup_{T \in {}^\circ Dg(f(\bar{x}))} {}^\circ D(T \circ f)(\bar{x}).$$

$\triangleleft$ It should be remarked that by condition the monad $\mu(\mathscr{E})$ is a normal external subsemigroup in F, i.e.,

$$\begin{aligned}
&\varepsilon \in \mu(\mathscr{E}) \to [0, \varepsilon] \subset \mu(\mathscr{E}), \\
&\mu(\mathscr{E}) + \mu(\mathscr{E}) \subset \mu(\mathscr{E}).
\end{aligned}$$

Taking into account this peculiarity and **5.6.3**, as well as the rules of calculating ε-subdifferentials, we get:

$$D(g \circ f)(\bar{x}) = \bigcup_{\varepsilon \in \mu(\mathscr{E})} \partial_\varepsilon (g \circ f)(\bar{x})$$

$$= \bigcup_{\varepsilon \in \mu(\mathbb{E})} \bigcup_{\substack{\varepsilon_1+\varepsilon_2=\varepsilon \\ \varepsilon_1 \geq 0,\, \varepsilon_2 \geq 0}} \bigcup_{T \in \partial_{\varepsilon_1} g(f(\bar{x}))} \partial_{\varepsilon_2}(T \circ f)(\bar{x})$$

$$= \bigcup_{\substack{\varepsilon_1 \geq 0,\, \varepsilon_2 \geq 0 \\ \varepsilon_1 \approx 0,\, \varepsilon_2 \approx 0}} \bigcup_{T \in \partial_{\varepsilon_1} g(f(\bar{x}))} \partial_{\varepsilon_2}(T \circ f)(\bar{x})$$

$$= \bigcup_{\varepsilon_1 \geq 0,\, \varepsilon_1 \approx 0} \bigcup_{T \in \partial_{\varepsilon_1} g(f(\bar{x}))} \bigcup_{\varepsilon_2 \geq 0,\, \varepsilon_2 \approx 0} \partial_{\varepsilon_2}(T \circ f)(\bar{x})$$

$$= \bigcup_{\varepsilon_1 \geq 0,\, \varepsilon_1 \approx 0} \bigcup_{T \in \partial_{\varepsilon_1} g(f(\bar{x}))} D(T \circ f)(\bar{x}).$$

Let now the assumption of standard entourage be fulfilled, and let $S \in {}^{\circ}D(g \circ f)(\bar{x})$. Then for a certain infinitely small ε we have

$$(g \circ f)^*(S) = \sup_{x \in \operatorname{dom}(g \circ f)} (Sx - g \circ f(x)) \leq S\bar{x} - g(f(\bar{x})) + \varepsilon.$$

By the change-of-variable formula for the Young-Fenchel transform, and making use of the transfer principle, we infer that there is a standard operator $T \in {}^{\circ}L(E,F)$ such that T is positive; i.e., $T \in L^+(E,F)$ and, moreover,

$$(g \circ f)^*(S) = (T \circ f)^*(S) + g^*(T).$$

This givess

$$\varepsilon \geq \sup_{x \in \operatorname{dom}(f)} (Sx - T \circ f(x)) + \sup_{e \in \operatorname{dom}(g)} (Te - g(e)) - S\bar{x} + g(f(\bar{x}))$$

$$= \sup_{x \in \operatorname{dom}(f)} (Sx - S\bar{x} - (T \circ f(x) - T \circ f(\bar{x})))$$

$$+ \sup_{e \in \operatorname{dom}(g)} (Te - T \circ f(\bar{x}) - (g(e) - g(f(\bar{x})))).$$

Let us set:

$$\varepsilon_1 := \sup_{e \in \operatorname{dom}(g)} (Te - T \circ f(\bar{x}) - (g(e) - g(f(\bar{x})))),$$

$$\varepsilon_2 := \sup_{x \in \operatorname{dom}(f)} (Sx - S\bar{x} - (T \circ f(x) - T \circ f(\bar{x}))).$$

Obviously, $S \in \partial_{\varepsilon_2}(T \circ f)(\bar{x})$, i.e., $S \in {}^{\circ}D(T \circ f)(\bar{x})$, and $T \in \partial_{\varepsilon_1} g(f(\bar{x}))$, i.e., $T \in {}^{\circ}Dg(f(\bar{x}))$, since $\varepsilon_1 \approx 0$ and $\varepsilon_2 \approx 0$. ▷

5.6.7. *Let $f_1, \ldots, f_n : X \to E$ be convex operators, and n be a standard natural number. If $f_1, \ldots, f_n$ are in general position, then for a point $\bar{x} \in \operatorname{dom}(f_1) \cap \ldots \cap \operatorname{dom}(f_n)$ we have*

$$D(f_1 + \ldots + f_n)(\bar{x}) = Df_1(\bar{x}) + \ldots Df_n(\bar{x}).$$

◁ The proof consists in applying **5.6.3** and the rule of ε-subdifferention for sum, with use made of the fact that the sum of a standard number of infinitely small addends is again infinitely small. ▷.

5.6.8. *Let $f_1,\dots,f_n\colon X\to E$ be convex operators, with n a standard number. Assume that $f_1,\dots,f_n$ are in a general position, E is a vector lattice, and $\bar{x}\in\operatorname{dom}(f_1\vee\dots\vee f_n)$. If F is a standard K-space and $T\in L^+(E,F)$ is a positive linear operator, then an element $S\in L(X,F)$ is an infinitesimal subgradient of the operator $T\circ(f_1\vee\dots\vee f_n)$ at a point $\bar{x}$ iff the following system of conditions is compatible*:

$$T=\sum_{k=1}^{n}T_k\,;T_k\in L^+(E,F)\quad (k:=1,\dots,n);$$
$$\sum_{k=1}^{n}T_k\bar{x}\approx T(f_1(\bar{x})\vee\dots\vee f_n(\bar{x}));\quad S\in\sum_{k=1}^{n}D(T_k\circ f_k)(\bar{x}).$$

◁ Let us determine the following operators:

$$(f_1,\dots,f_n)\colon X\to(E^n),\quad (f_1,\dots,f_n)(x)\colon=(f_1(x),\dots,f_n(x))\ ;$$
$$\aleph\colon E^n\to E,\ \aleph(e_1,\dots,e_n)\colon=e_1\vee\dots\vee e_n.$$

In this case the following presentation is valid:

$$T\circ f_1\vee\dots\vee f_n=T\circ\aleph\circ(f_1,\dots,f_n).$$

From now on, allowing for **5.6.5** and recalling that $T\circ\aleph$ is a sublinear operator, we deduce the required result. ▷

5.6.9. Let X be a vector space, E be a certain K-space and $\mathfrak{T}$ be a pointwise order-bounded set in $L(X,E)$. Let us consider a convex operator $f:=\varepsilon_{\mathfrak{T}}\circ\langle\mathfrak{T}\rangle^e$, where, as usual, $\varepsilon_{\mathfrak{T}}$ is the *canonical sublinear operator*

$$\varepsilon_{\mathfrak{T}}\colon l_\infty(\mathfrak{T},E)\to E,\quad \varepsilon_{\mathfrak{T}}(f)\colon=\sup f(\mathfrak{T}),$$

and the affine operator $\langle\mathfrak{T}\rangle^e$ for $e\colon l_\infty(\mathfrak{T},E)$ acts by the rule $\langle\mathfrak{T}\rangle^e x:=\langle\mathfrak{T}\rangle x+e$, $\langle\mathfrak{T}\rangle x\colon T\in\mathfrak{T}\to Tx$.

5.6.10. *If $g\colon E \to F$ is an increasing convex operator acting into a standard K-space F, and the image $f(X)$ contains an algebraic interior point* $\operatorname{dom}(g)$, *while an element $\bar{x}$ of X is such that $f(\bar{x}) \in \operatorname{dom}(g)$, then the following presentation is valid:*

$$D(g \circ f)(\bar{x})$$
$$= \{T \circ \langle \mathfrak{U} \rangle \colon T \circ \Delta_{\mathfrak{U}} \in Dg(f(\bar{x})),\quad T \geq 0,\ T \circ \Delta_{\mathfrak{U}} f(\bar{x}) \approx T \circ \langle \mathfrak{U} \rangle^{e^-}\ x\}.$$

◁ If $S \in D(g \circ f)(\bar{x})$, then, by **5.6.3**, $S \in \partial_g(g \circ f)(\bar{x})$ for a certain $\varepsilon \approx 0$, and all we have to do is to apply the corresponding rule of ε-subdifferentiation. Whereas if $T \geq 0, T \circ \Delta_{\mathfrak{U}} \in Dg(f(\bar{x}))$ and $T \circ \Delta_{\mathfrak{U}} f(\bar{x}) \approx T \circ \langle \mathfrak{U} \rangle^e \bar{x}$, then for a certain $\varepsilon \approx 0$ we obviously get $T \circ \Delta_{\mathfrak{U}} \in \partial_\varepsilon g(f(\bar{x}))$. Let us, in addition, assign $\delta := T \circ \Delta_{\mathfrak{U}} f(\bar{x}) - T \circ \langle \mathfrak{U} \rangle^e \bar{x}$. Then, by hypothesis, $\delta \geq 0$ and $\delta \approx 0$. Therefore, $T \circ \langle \mathfrak{U} \rangle \in \partial_{\varepsilon+\delta}(g \circ f)(\bar{x})$. Now we have to remark that $\varepsilon + \delta \approx 0$. ▷

5.6.11. *Under condition **5.6.5** let the mapping g be a sublinear Maharam operator. Then*

$$D(g \circ f)(\bar{x}) = \bigcup_{T \in Dg(f(\bar{x}))} \ \bigcup_{\delta \geq 0, T\delta \approx 0} T(\partial_\delta f(\bar{x})).$$

◁ By virtue of **5.6.5**, we can assume $g := T$. If for any $x \in X$ we have $Cx - C\bar{x} \leq f(x) - f(\bar{x}) + \delta$ and $T\delta \approx 0$, then, obviously, $TC \in \partial_{T\delta}(T \circ f)(\bar{x}) \subset \subset D(T \circ f)(\bar{x})$. To complete the proof, let us take $S \in D(T \circ f)(\bar{x})$. By virtue of **5.6.3**, there is an infinitely small ε, such that $S \in \partial_\varepsilon(T \circ f)(\bar{x})$. Applying the corresponding rule of ε-subdifferentiation, we find $\delta \geq 0$ and $C \in \partial_\delta f(\bar{x})$ such that $T\delta \leq \varepsilon$ and $S = TC$, which was required. ▷

5.6.12. *Let Ξ be a set and $(f_\xi)_{\xi \in \Xi}$ be a uniformly regular family of convex operators. The following presentations are valid:*

$$D\Big(\sum_{\xi \in \Xi} f_\xi\Big)(\bar{x}) = \bigcup_{\substack{\delta \in l_1(\Xi E) \\ \delta \geq 0, \delta \approx 0}} \sum_{\xi \in \Xi} \partial_{\delta(\xi)} f_\xi(\bar{x});$$

$$D\Big(\sup_{\xi \in \Xi} f_\xi\Big)(\bar{x})$$
$$= \bigcup \Big\{ \sum_{\xi \in \Xi} \alpha_\xi \partial_{\delta(\xi)}(\bar{x}) \colon 0 \leq \alpha_\xi \leq 1_E,\ \sum_{\xi \in \Xi} \alpha_\xi = 1_E,$$
$$\sum_{\xi \in \Xi} \alpha_\xi f_\xi(\bar{x}) \approx \sup_{\xi \in \Xi} f_\xi(\bar{x}),\quad \sum_{\xi \in \Xi} \alpha_\xi \delta(\xi) \approx 0 \Big\}.$$

◁ The proof results immediately from **5.6.11** with the rule of disintegration taken into account (see [115]). ▷

5.6.13. It is expedient to remark that the formulas **5.6.7** - **5.6.12** allow one to introduce refinements analogous to the case of standard entourage in **5.6.6** (which, possibly, does not include the point $\bar{x}$). It should be also emphasized that the given patterns enable one to deduce the whole spectrum of all possible formulas of subdifferential calculus (convolutions, Lebesque sets, etc.).

5.6.14. Let, as above, $f: X \to E^{\cdot}$ be a convex operator acting into a standard K-space E, and let $\mathcal{X} := (\cdot)$ be *a generalized point in* $\mathrm{dom}(f)$, i.e., a net of elements of the $\mathrm{dom}(f)$. The operator $T \in L(X, E)$ is said to be *an infinitesimal subgradient of f at the generalized point $\mathcal{X}$, if for a certain infinitely small ε we have*

$$f^*(T) \leq \liminf (T\mathcal{X} - f(\mathcal{X})) + \varepsilon$$

(here, of course, the rule $T\mathcal{X} := T \circ \mathcal{X}$ is in action). Therefore, in standard entourage the infinitesimal subgradient is a common support operator at a generalized point (see [1, 115]). Let us denote by the symbol $Df(\mathcal{X})$ the union of all infinitesimal subgradients f at $\mathcal{X}$. This set is termed, by well-understandable reasons, the *infinitesimal subdifferential of f* at $\mathcal{X}$. Let us deduce the two basic rules of subdifferetiation at a generalized point, which are of interest since no exact formulas for ε-subdifferentials are known.

5.6.15. *Let $f_1, \dots, f_n$ be a standard set of convex operators in general position, and let a generalized point $\mathcal{X}$ lie in* $\mathrm{dom}(f_1) \cap \dots \cap \mathrm{dom}(f_n)$. *Then*

$$D(f_1 + \dots + f_n)(\mathcal{X}) = Df_1(\mathcal{X}) + \dots + Df_n(\mathcal{X}).$$

$\lhd$ Let $T_k \in Df_k(\mathcal{X})$ for $k := 1, \dots, n$, i.e.,

$$f_k^*(T_k) \leq \liminf (T_k\mathcal{X} - f_k(\mathcal{X})) + \varepsilon_k$$

for a suitable infinitely small $\varepsilon_1, \dots, \varepsilon_n$. In this case

$$(f_1 + \dots + f_n)^*(T_1 + \dots + T_n) \leq \sum_{k=1}^{n} f_k^*(T_k)$$
$$\leq \sum_{k=1}^{n} (\liminf (T_k\mathcal{X} - f_k(\mathcal{X})) + \varepsilon_k)$$
$$\leq \liminf \sum_{k=1}^{n} (T_k\mathcal{X} - f_k(\mathcal{X})) + \sum_{k=1}^{n} \varepsilon_k$$

by virtue of conventional properties of the Young-Fenchel transform and the lower limit. Now we have to remark that $\varepsilon_1+\ldots+\varepsilon_n \approx 0$ and conclude that the inclusion $\supset$ is valid for the sets considered in the equality we are interested in.

In order to check the reverse inclusion, let us, having reduced the case to $n = 2$, choose a $T \in D(f_1 + f_2)(\mathcal{X})$. Then, for some $\varepsilon \approx 0$ and T_1, T_2, such that $T_1 + T_2 = T$, we get

$$(f_1 + f_2)^*(T) = f_1^*(T_1) + f_2^*(T_2),$$
$$f_1^*(T_1) + f_2^*(T_2) - \liminf(T\mathcal{X} - (f_1 + f_2)(\mathcal{X})) \le \varepsilon.$$

Let us, by definition, assign

$$\delta_1 := f_1^*(T_1) - \liminf(T_1\mathcal{X} - f_1(\mathcal{X})),$$
$$\delta_2 := f_2^*(T_2) - \liminf(T_2\mathcal{X} - f_2(\mathcal{X})).$$

Obviously, for $k := 1,2$ we have

$$0 \le \sup_{x \in \operatorname{dom}(f_k)} (T_k x - f_k(x)) - \limsup(T_k\mathcal{X} - f_k(\mathcal{X})) \le \delta_k.$$

Therefore, we now have to make sure that δ_1 and δ_2 are infinitesimals. We consequitively derive

$$\begin{aligned}
&\delta_1 + \delta_2 \\
&\le \varepsilon + \liminf(T\mathcal{X} - (f_1 + f_2)(\mathcal{X})) - \sum_{k=1}^{2} \liminf(T_k\mathcal{X} - f_k(\mathcal{X})) \\
&\le (\varepsilon + \limsup(T_1\mathcal{X} - f_1(\mathcal{X})) - \liminf(T_1\mathcal{X} - f_1(\mathcal{X}))) \\
&\wedge (\varepsilon + \limsup(T_2\mathcal{X} - f_2(\mathcal{X})) - \liminf(T_2\mathcal{X} - f_2(\mathcal{X})) \\
&\le (\varepsilon + f_1^*(T_1) - \liminf(T_1\mathcal{X} - f_1(\mathcal{X}))) \\
&\wedge (\varepsilon + f_2^*(T_2) - \liminf(T_2\mathcal{X} - f_2(\mathcal{X}))) \\
&\le \varepsilon + \delta_1 \wedge \delta_2.
\end{aligned}$$

Hence, $0 \le \delta_1 \vee \delta_2 \le \varepsilon$, which completes the proof. ▷

5.6.16. *Let F be a standard K-space, and let $g: E \to F^{\cdot}$ be an increasing convex operator. If the sets $X \times \operatorname{epi}(g)$ and $\operatorname{epi}(f) \times F$ are in general position, then, for a generalized point $\mathcal{X}$ in $\operatorname{dom}(g \circ f)$, we have*

$$D(g \circ f)(\mathcal{X}) = \bigcup_{T \in Dg(f(\mathcal{X}))} D(T \circ f)(\mathcal{X}).$$

◁ If it is known that

$$(T \circ f)^*(S) \le \liminf (S\mathcal{X} - T \circ f(\mathcal{X})) + \varepsilon_1,$$
$$g^*(T) \le \liminf (T \circ f(\mathcal{X}) - g \circ f\mathcal{X})) + \varepsilon_2$$

for some infinitely small ε_1 and ε_2, then

$$(g \circ f)^*(S) \le (T \circ f)^*(S) + g^*(T)$$
$$\le \liminf (S\mathcal{X} - T \circ f(\mathcal{X})) + \varepsilon_1 + \liminf (T \circ f(\mathcal{X}) - g \circ f(\mathcal{X})) + \varepsilon_2$$
$$\le \liminf (S\mathcal{X} - g \circ f(\mathcal{X})) + \varepsilon_1 + \varepsilon_2.$$

Therefore, $S \in D(g \circ f)(\mathcal{X})$ and the right-hand side of the formula under study symbolize the set incorporated in its left-hand side.

To complete the proof, let us take $S \in D(g \circ f)(\mathcal{X})$. Then there is an infinitely small ε and an operator T, such that

$$(g \circ f)^*(S) = (T \circ f)^*(S) + g^*(T) \le \liminf (S\mathcal{X} - g \circ f(\mathcal{X})) + \varepsilon.$$

Let us set

$$\delta_1 := (T \circ f)^*(S) - \liminf (S\mathcal{X} - T \circ f(\mathcal{X})),$$
$$\delta_2 := g^*(T) - \liminf (T \circ f(\mathcal{X}) - g \circ f(\mathcal{X})).$$

By the properties of the upper and lower limits, we deduce, first,

$$\delta_1 \ge (T \circ f)^*(S) - \limsup (S\mathcal{X} - T \circ f(\mathcal{X})) \ge 0,$$
$$\delta_2 \ge g^*(T) - \limsup (T \circ f(\mathcal{X}) - g \circ f(\mathcal{X})) \ge 0,$$

and, second,

$$\delta_1 + \delta_2 \le$$
$$\le \liminf (S\mathcal{X} - g \circ f(\mathcal{X})) + \varepsilon - \liminf (S\mathcal{X} - T \circ f(\mathcal{X}))$$
$$- \liminf (T \circ f(\mathcal{X}) - g \circ f(\mathcal{X})(\le (\limsup (S\mathcal{X} - T \circ f(\mathcal{X}))$$
$$- \liminf (S\mathcal{X} - T \circ f(\mathcal{X})) + \varepsilon) \wedge (\limsup (T \circ f(\mathcal{X}) - g \circ f(\mathcal{X}))$$
$$- \liminf (T \circ f(\mathcal{X}) - gf(\mathcal{X})) + \varepsilon) \le \delta_1 \wedge \delta_2 + \varepsilon,$$

since the following obvious equalities are valid:

$$\limsup (T \circ f(\mathcal{X}) - g \circ f(\mathcal{X})) \le g^*(T),$$
$$\limsup (S\mathcal{X} - T \circ f(\mathcal{X})) \le (T \circ f)^*(S).$$

Therefore, $0 \le \delta_1 \vee \delta_2 \le \varepsilon$ and $\delta_1 \approx 0, \delta_2 \approx 0$, which implies $T \in Dg(f(\mathcal{X}))$ and $S \in D(T \circ f)(\mathcal{X})$. ▷

5.6.17. The point $\bar{x} \in \mathrm{dom}(f)$ is called an *infinitesimal solution* of the unconstrained program $f(x) \to \inf$, where $f: X \to E^{\cdot}$, provided $0 \in Df(\bar{x})$, i.e., if $\bar{x}$ is feasible and $f(\bar{x}) \approx \inf\{f(x): x \in X\}$. The infinitesimal solution of an arbitrary program is understood in the natural way.

5.6.18. *In a standard unconstrained program $f(x) \to \inf$ there is an infinitesimal solution iff, first, the image $f(X)$ is bounded from below, and second, there is a standard generalized solution $(x_\varepsilon)_{\varepsilon \in \mathcal{E}}$ of the program under consideration, i.e., $x_\varepsilon \in \mathrm{dom}(f)$ and $e \le f(x_\varepsilon) \le e + \varepsilon$ for all $\varepsilon \in \mathcal{E}$, where $e := \inf f(X)$ is the value of the program.*

◁ By virtue of the idealization and transfer principles, as well as of **5.6.3**, we deduce:

$$
\begin{aligned}
&(\exists \bar{x} \in X)\ 0 \in Df(\bar{x}) \leftrightarrow (\exists x \in X)(\forall^{st} \varepsilon \in \mathcal{E})\ 0 \in \partial_\varepsilon f(x) \\
&\leftrightarrow (\forall^{st\,fin} \mathcal{E}_0 \subset \mathcal{E})(\exists x \in X)(\forall \varepsilon \in \mathcal{E}_0)\ 0 \in \partial_\varepsilon f(x) \\
&\leftrightarrow (\forall^{st} \varepsilon \subset \mathcal{E})(\exists x_\varepsilon \in X)\ 0 \in \partial_\varepsilon f(x_\varepsilon) \\
&\leftrightarrow (\forall \varepsilon \subset \mathcal{E})(\exists x_\varepsilon \in X)(\forall x \in X)\ f(x) \ge f(x_\varepsilon) - \varepsilon.\ ▷
\end{aligned}
$$

5.6.19. Let us consider a *regular convex program*

$$g(x) \le 0, \quad f(x) \to \inf.$$

Therefore, $g, f: X \to E^{\cdot}$ (for simplicity, $\mathrm{dom}(f) = \mathrm{dom}(g) = X$), for every $x \in X$ we have either $g(x) \le 0$ or $g(x) \ge 0$ and, moreover, for a certain $x_0 \in X$ the element $-g(x_0)$ is a unit in E.

5.6.20. *In standard entourage a feasible interior point $\bar{x}$ is an infinitesimal solution of the regular program under consideration iff the following system of conditions is compatible:*

$$
\begin{aligned}
&\alpha, \beta \in {}^\circ[0, 1_E],\ \alpha + \beta = 1_E, \ker(\alpha) = 0; \\
&\beta \circ g(\bar{x}) \approx 0, 0 \in D(\alpha \circ f)(\bar{x}) + D(\beta \circ g)(\bar{x}).
\end{aligned}
$$

◁ ← When the system of conditions is compatible, for a feasible x for some infinitely small ε_1 and ε_2 we have

$$\alpha f(\bar{x}) \leq \alpha f(x) + \beta g(x) - \beta g(\bar{x}) + \varepsilon_1 + \varepsilon_2 \leq \alpha f(x) + \varepsilon$$

for every standard $\varepsilon \in {}^\circ\mathfrak{E}$. In particular, $\alpha(f(\bar{x}) - f(x)) \leq \alpha\varepsilon$ for $\varepsilon \in {}^\circ\mathfrak{E}$, since α is a standard mapping. By condition, $\ker(\alpha) = 0$ and the general properties of multipliers we see that $\bar{x}$ is an infinitesimal solution.

→ Let $e := \inf\{f(x) : x \in X, g(x) \leq 0\}$ be the value of the program under discussion. By hypothesis and by virtue of the transfer principle, e is a standard element. Therefore, again applying the transfer principle, by the vector minimax theorem we can find standard multipliers $\alpha, \beta \in {}^\circ[0, 1_E]$ such that

$$\alpha + \beta = 1_E, \quad 0 = \inf_{x \in X} (\alpha(f(x) - e) + \beta \circ g(x)) .$$

Using conventional considerations (see [1]), we can check if $\ker(\alpha) = 0$. Moreover, since $\bar{x}$ is an infinitesimal optimal solution, for a certain infinitely small ε we get $f(\bar{x}) - e = \varepsilon$. Hence, for any $x \in X$ the estimates $-\alpha\varepsilon \leq \alpha f(x) - \alpha f(\bar{x}) + \beta g(x)$ are valid. In particular, $0 \geq \beta g(\bar{x}) \geq -\alpha\varepsilon \geq -\varepsilon$, i.e., $\beta g(\bar{x}) \approx 0$ and

$$0 \in \partial_{\alpha\varepsilon + \beta g(\bar{x})}(\alpha \circ f + \beta \circ g)(\bar{x}) \subset D(\alpha \circ f + \beta \circ g)(\bar{x}),$$

as $\alpha\varepsilon + \beta g(\bar{x}) \approx 0$. ▷

5.6.21. Let us consider a *program regular in the Slater sense*

$$Ax = A\bar{x}, \quad g(x) \leq 0, \quad f(x) \to \inf ,$$

i.e., first, $A \in L(X, ƛ)$ is a linear operator with values in a certain vector space $ƛ$, the mappings $g : X \to E^{\cdot}$ and $f : X \to E^{\cdot}$ are convex operators (for convenience, $\mathrm{dom}(f) = \mathrm{dom}(g) = X$); second, F is an Archimedean ordered vector space, E is a standard K-space of bounded elements; and, finally, for a certain feasible point x_0 the element $-g(x_0)$ is a strong unit in F.

5.6.22. Criterion for infinitesimal optimality. *A feasible point $\bar{x}$ is the infinitesimal solution to a Slater-regular program iff the following system of conditions is compatible:*

$$\gamma \in L^+(F, E), \quad \mu \in L(ƛ, E), \quad \gamma g(\bar{x}) \approx 0,$$
$$0 \in Df(\bar{x}) + D(\gamma \circ g)(\bar{x}) + \mu \circ A .$$

◁ ← When the system of conditions is compatible, for a feasible x and some infinitely small ε_1 and ε_2 we have

$$f(\bar{x}) \le f(x) + \varepsilon_1 + \gamma g(x) - \gamma g(\bar{x}) + \varepsilon_2 - \mu(Ax) + \mu(A\bar{x})$$
$$\le f(x) + \varepsilon_1 + \varepsilon_2 - \gamma g(\bar{x}) \le f(x) + \varepsilon$$

for any standard $\varepsilon \in {}^{\circ}\mathcal{E}$.

→ If $\bar{x}$ is an infinitesimal solution, it is also an ε-solution for a suitable infinitely small ε. Now we have to apply the corresponding criterion of ε-optimum. ▷

5.6.23. A feasible point $\bar{x}$ is called *infinitesimal Pareto-optimal* in the program **5.6.21**, provided $\bar{x}$ is Pareto-ε-optimal for some infinitely small ε (relative to the (strong)order unit 1_E in the space E), i.e., if for a feasible x we have $f(x) - f(\bar{x}) \le \varepsilon 1_E$, then $f(x) - f(\bar{x}) = \varepsilon 1_E$ for $\varepsilon \in \mu(\mathbf{R}_+)$.

5.6.24. *Let a point $\bar{x}$ be infinitesimal Pareto-optimal in a Slater-regular program. Then for certain linear functionals α, β, γ on the spaces E, F and $\mathfrak{X}$, respectively, the following system of conditions is compatible*:

$$\alpha > 0, \quad \beta \ge 0, \quad \beta g(\bar{x}) \approx 0,$$
$$0 \in D(\alpha \circ f)(\bar{x}) + D(\beta \circ g)(\bar{x}) + \gamma \circ A.$$

If, in turn, the above relations are valid for a certain feasible point $\bar{x}$, and $\alpha(1_E) = 1$, $\ker(\alpha) \cap E^+ = 0$, *then $\bar{x}$ is an infinitesimal Pareto-optimal solution to the program under consideration.*

◁ The first part of the statement being proved results from the well-known characteristic of Pareto-ε-optimum with the properties of the infinitely small discussed earlier taken into account. If the hypothesis of the second part of the statement under discussion is fulfilled, then, making use of the definitions, for any feasible $x \in X$ we deduce:

$$0 \le \alpha(f(x) - f(\bar{x})) + \beta g(x) - \beta g(\bar{x}) + \varepsilon_1 + \varepsilon_2$$
$$\le \alpha(f(x) - f(\bar{x})) + \varepsilon_1 + \varepsilon_2 - \beta g(\bar{x})$$

for an appropriate infinitely small $\varepsilon_1, \varepsilon_2$. Let us set $\varepsilon := \varepsilon_1 + \varepsilon_2 - \beta g(\bar{x})$. Obviously, $\varepsilon \approx 0$ and, moreover, $\varepsilon \ge 0$. If now for a feasible x valid is $f(x) - f(\bar{x}) \le -\varepsilon 1_E$, then we get the equality $\alpha(f(x) - f(\bar{x})) = \varepsilon$. In other words, $\alpha(f(x) - f(\bar{x}) - \varepsilon 1_E) = 0$ and

$f(x) - f(\bar{x}) = \varepsilon 1_E$. This implies that $\bar{x}$ is a Pareto-ε-optimal solution. ▷

5.6.25. Using the pattern described above, one can obtain the properties of infinitesimal solutions for other basic forms of the problems of convex programming. One can, for instance, deduce nonstandard analogues of the theorem on characteristics of naturally defined infinitesimal optimal paths of finite-stage terminal dynamic problems.

PART 2

BOOLEAN-VALUED ANALYSIS

Boolean-valued analysis owes its origination to an outstanding achievement by P.J.Cohen who in the beginning of the sixties established that the addition of the negation of the continuum-hypothesis, CH, to the other axioms of Zermelo-Fraenkel set theory, ZFC, is consistent. Combined with an earlier result by K.Gödel on the CH-ZFC consistency, the fact proved by P.J.Cohen implies that CH is independent of the other ZFC axioms.

The discovery made by P.J.Cohen is associated with his overcoming a principal difficulty marked by G.Sheperdson and absent in the case considered by K.Gödel. The proof of $(\mathrm{ZFC}) + (\neg\,\mathrm{CH})$ consistency is impossible by means of standard models. To be more exact, having chosen a realization of the von Neumann universe, we cannot find a subclass in it which serves a $(ZFC) + (\neg CH)$ model, if we make use of THE available interpretation of the predicate of containment. P.J.Cohen managed to propose a new powerful method of constructing non-internal, i.e., nonstandard, ZHC models which he termed the forcing method. The techniques used by P.J.Cohen, such as the use of the axiom of the existence of a standard transitive model for ZFC and a forced transformation of the latter into a principally nonstandard model with the constraint method contradict the conventional mathematical intuition stemming, by the words of P.J.Cohen himself, "from our belief into a natural nearly physical model of the mathematical world" [28, p.202].

The difficulties hampering the understanding of the results by P.J.Cohen were, long before their origination, well expressed by N.N.Luzin in his famous report 'The present state of the theory of functions in a real variable' made by him at the All-Russian Congress of Mathematicians in 1927: "The first idea that occurs is that the determination of the cardinality of a continuum is a matter of a free axiom, as for parallel lines in geometry. At the same time, both when all other Eucledean axioms in geometry are presented, and we vary the axiom on parallel lines, the very sense of the uttered or written terms undergoes changes: 'a point', 'a straight line', etc., the sense of which words should change when we make the cardinality of the continuum moving along the alephic scale, proving all the time the consistency of this motion? The cardinality of the continuum, if only viewed as a set of points, is a unique reality, and it should have a specific location on the alephic scale, irrespective of the fact that the definition of the location is difficult or, as J.Hadamard would add, "even impossible for us, people" [163, pp. 11-12].

A typical viewpoint has been offered by P.S.Novikov: "...it is plausible (I share this opinion myself) that the Cohen result has purely negative value and exposes the end of the evolution of the 'naive' set theory in the Cantor sense" [200, p. 209].

The desire to eliminate the afore-mentioned difficulties in the understanding of the results and methods by P.J.Cohen led D.Scott and P.Solovay to the construction of Boolean-valued models for ZFC both possessing an appealing visualization from the standpoint of classical mathematicians and, at the same time, suitable for obtaining theorems on independence. Analogous models were also constructed by P.Vopenka at the same period, i.e., at the beginning of the sixties.

From all the above-said, the Boolean-valued models which achieve the same aims that those constructed by P.J.Cohen via forcing should be, in a sense, nonstandard, i.e., they should possess features not characteristic of conventional models.

In a qualitative sense, *the notion of Boolean-valued model includes a new concept of modelling*, which could be termed modelling by communication, or modelling by telephone. Let us clarify the essence of this concept by way of comparing it with traditional approaches. In a classical sense, when comparing two models of the same theory, we try to establish a one-to-one correspondence between the universes involved. Once such a bijection has been found by transferring the predicates and operators of one model into their analogs in the other one, we say that the models are isomorphic. Therefore, the presentation of isomorphism described implies a visual comparison of the models, i.e., the presentation of a bijection of the universes.

Let us imagine that we are deprived of the possibility of comparing models simultaneously physically element-by-element, but can exchange information with the owner of the counter-model by, for instance, telephone. In the process of communication we could easily establish that our counter-partner uses his model for studying objects which he calls by familiar terms when speaking about sets, their comparison and membership. As long as we are interested in ZFC, we ask him if the axioms of ZFC are true. Having worked with his model, he answers us in the affirmative. Having also checked that he is using the same deduction rules as adopted by us, we must admit that the model at his disposal is a model of the theory we are interested in. It would be expedient to emphasize that having come to this conclusion we have known nothing either about the objects comprising his model, not about the procedures he uses to distinguish true statements from false ones.

Therefore, *the new concept of modelling is associated with both the refusal to identify universes of discourse and the acceptance of new procedures for verifying statements.* In particular, when considering a Boolean-valued model $\mathbf{V}^{(B)}$ for ZFC, to each formula φ we assign an element $[\varphi]\in B$ lying in a given Boolean algebra B. The quantity $[\varphi]$ is called the truth-value of the formula φ. In this case a theorem of ZFC obtains the truth-value $\mathbf{1}_B$. In Boolean-valued models the construction starts with a fixed complete Boolean algebra B which serves as a base for constructing the Boolean-valued universe $\mathbf{V}^{(B)}$ and the domain of arrival to the truth-value by putting a certain element of the algebra B in correspondence to a formula of ZFC.

A detailed presentation of the afore-mentioned constructions can be found in Chapters **1** - **3** of this part of the book. The presented constructions and, first of all, the procedures of ascent and descent implementing the functorial relations between the von Neumann universe, $\mathbf{V}$, and the Boolean-valued universe, $\mathbf{V}^{(B)}$, comprise a technical foundation of applying Boolean-valued models to the problems of analysis. In the concluding chapters we we shall demonstrate the most important possibilities provided by Boolean-valued analysis, such as methods of transforming functional spaces into numerical sets, operators into functionals, vector-functions into conventional mappings, etc.. And again, as was the case in the first part of the present book, the choice of the circle of applications to the problems of functional analysis has been basically prompted by out personal scientific interests.

CHAPTER 1

UNIVERSES OF SETS

The credo of naive set theory includes, as is known, the dream about the 'Cantor paradise', i.e., about the universe, the world of sets, containing all possible formations that can be thought of as distinct. Realistic approximations to the unattainable ideal, i.e., adequate formal schemes making it possible to present a large spectrum of concrete sets while remaining within comfortable conditions of sufficient logical accuracy, are the subject of the modern set theory. In the first part of the book we have already encountered the ideas underlying a number of axiomatic set theories, such as Zermelo-Fraenkel set theory, the theories of external and internal sets.

The essence of these theories is the construction of universes ensuring 'approximations from below', to the world of naive sets, satisfactory for some specific purposes. It is within the frameworks of the corresponding axiomatics that we can exactly substantiate and realize in detail the qualitative phenomenological principles on which the standard and nonstandard mathematical models are based. In the present chapter we consider the formal apparatus of constructing universes of sets by transfinite processes of creating the so-called cumulative hierarchies. We are going to concentrate on a detailed description of the construction of the von Neumann universe, which often occurred in the first part of the present monograph. In this respect a thorough analysis will be carried out as regards the classes of sets within the formal system stemming from J.von Neumann, K.Gödel and P.Bernays and serving as a conservative extension of Zermelo-Fraenkel set theory.

1.1. Boolean Algebras

In this section we shall schematically present only those facts about Boolean algebras which are needed in the sequel. A more detailed presentation can be found in a number of monographs [74, 233, 265].

1.1.1. With the view of fixing terminology, let us recall some well-known notions partly used in the preceding sections.

An ordered set is a pair $(M,\le)$, where $\le$ is an order relation on M (see **I.3.1.10**). An *upper bound* of a subset X in the ordered set M is an element $a \in M$ such that $x \le a$ for all $x \in X$. The least element of the set of upper bounds of the subset X is called its *least upper bound* or its *supremum* and is denoted by $\sup X$. In other words, $a = \sup X$ iff a is an upper bound of X and $a \le b$ for any upper bound b of the set X. By reversal, i.e., by passing from $\le$ to $\le^{-1}$, we determine a *lower bound* or a *greatest lower bound*, $\inf X$, termed the *infimum* of the set X. If either the least upper or the greatest lower bound of a given set exists, it is unique and thus deserves the definite article Either of the bounds, $\inf X$ and $\sup X$, is referred to as *exact.*

A lattice is an ordered set L in which any two-element set $\{x, y\}$ has the supremum $x \vee y := \sup\{x,y\}$ and the infimum $x \wedge y := \inf\{x,y\}$. For lattices the following notation is used:

$$\vee X := \sup X, \quad \wedge X := \inf X,$$
$$\bigvee_{\alpha\in A} x_\alpha := \vee\{x_\alpha : \alpha \in A\}, \quad \bigwedge_{\alpha\in A} x_\alpha := \wedge\{x_\alpha : \alpha \in A\},$$
$$\bigvee_{k=1}^{n} x_k := x_1 \vee \ldots \vee x_n := \sup\{x_1,\ldots x_n\},$$
$$\bigwedge_{k=1}^{n} x_k := x_1 \wedge \ldots \wedge x_n := \inf\{x_1,\ldots x_n\}.$$

The binary operations $(x,y) \to x \vee y$ and $(x,y) \to x \wedge y$ arising in any lattice L satisfy the following laws:

(1) *commutativity:*

$$x \vee y = y \vee x, \quad x \wedge y = y \wedge x;$$

(2) *associativity:*

$$x \vee (y \vee z) = (x \vee y) \vee z, \quad x \wedge (y \wedge z) = (x \wedge y) \wedge z.$$

By induction, from (2) we deduce that *any nonempty finite set in a lattice has exact bounds.* If every subset of a lattice has such bounds, the lattice is called *complete.*

A lattice L is said to be *distributive*, provided the following relations hold in it:

(3) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$,

(4) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

If there exists the least (greatest) element of the lattice, it is called *zero* (*unity*, or *unit*), respectively. The zero and unity in the lattice L are denoted by the symbols $\mathbf{0}_L, \mathbf{1}_L$, respectively, or simply by $\mathbf{0}, \mathbf{1}$, provided it is clear what lattice L is discussed. It should be remarked that both $\mathbf{0}$ and $\mathbf{1}$ are neutral elements:

(5) $\mathbf{0} \vee x = x, \quad \mathbf{1} \wedge x = x$.

According to conventional definitions, $\vee \varnothing = \sup \varnothing := \mathbf{0}, \wedge \varnothing = \inf \varnothing := \mathbf{1}$. A *complement* x^* of the element x in the lattice L with zero and unity is determined as such an element of L that

(6) $x \wedge x^* = \mathbf{0}, \quad x \vee x^* = \mathbf{1}$.

Since the elements x and y in L are called *disjoint*, if $x \wedge y = \mathbf{0}$, we see that each element x is disjoint to each of its complements x^*. It should be, finally, remarked that if each element in L has at least one complement, then L is said to be a lattice with complements.

1.1.2. *A Boolean algebra* is a distributive lattice with complements. In particular, in a Boolean algebra B there is, by definition, zero, $\mathbf{0} := \mathbf{0}_B$, and unity, $\mathbf{1} := \mathbf{1}_B$.

At a first glance, the above definition might seem a bit strange, as it does not reveal why and what for a distributive lattice could be termed an algebra, since the term 'algebra' refers to conventional objects (cf.: a Lie algebra, a Banach algebra, a C^*-algebra, etc.). The arising ambiguity is easily eliminated, since in fact a Boolean algebra is an algebra over the two-element field. The principal importance of this peculiarity will be partially discussed in the section to follow. At the same time, it would be quite natural to view Boolean algebras in different contexts at different angles. Below we shall consider a Boolean algebra primarily as a distributive lattice with complements. It should be emphasized that concrete Boolean algebras important for functional analysis often arise as distributive lattices with complements.

It should be remarked that a formal example of a Boolean algebra is the one-element lattice, i.e., a singleton of the type $\{x\}$ with the obvious order $x \leq x$. This algebra is termed *degenerate*. A simplest nondegenerate Boolean algebra is the two-element lattice $\{\mathbf{0}, \mathbf{1}\}$ with the order $\mathbf{0} \leq \mathbf{1}, \mathbf{0} \leq \mathbf{0}, \mathbf{1} \leq \mathbf{1}$. A degenerate Boolean algebra is natural as an algebraic system but absurd in the context of Boolean-valued analysis we are interested in. At the same time, being the simplest algebraic system of little interest, the two-element lattice plays an important role in the chapters to follow. Therefore, let us make an agreement: speaking about a Boolean algebra B we shall always assume $\mathbf{0}_B \neq \mathbf{1}_B$, i.e., the degenerate algebra will be excluded from considerations.

In a Boolean algebra B each element $x \in B$ has the unique complement denoted by the symbol x^*. The arising mapping $x \to x^* (x \in B)$ is idempotent (i.e., $(\forall x \in B)\ (x^{**} := (x^*)^* = x)$) and it presents a *dual isomorphism* (=*antiisomorphism*) onto itself (i.e., it is an isomorphism of the ordered sets $(B,\le)$ and $(B,\le^{-1})$). In particular, the De Morgan formulas are valid:

$$\left(\bigvee_{\alpha\in A} x_\alpha\right)^* = \bigwedge_{\alpha\in A} x_\alpha^*, \quad \left(\bigwedge_{\alpha\in A} x_\alpha\right)^* = \bigvee_{\alpha\in A} x_\alpha^*,$$

where $x_\alpha \in B (\alpha \in A)$.

1.1.3. Thus, in an arbitrary algebra B the three operations $\vee, \wedge$ and $*$ are given; we call them *Boolean.* It is possible to give an equivalent definition to the Boolean algebra B, characterizing it as a universal algebra $(B, \vee, \wedge, *, \mathbf{0}, \mathbf{1})$ with two binary operations $\vee$ and $\wedge$, one unary operation $*$, and two chosen elements $\mathbf{0}$ and $\mathbf{1}$ obeying the conditions:

(1) the operations $\vee$ and $\wedge$ are commutative and associative (**1.1.1** (1,2));

(2) the operations $\vee$ and $\wedge$ are dually distributive relative to one another (**1.1.1** (3,4));

(3) the elements x and x^* are mutually complementary (**1.1.1** (6));

(4) $\mathbf{0}$ and $\mathbf{1}$ are neutral elements for the operations $\vee$ and $\wedge$, respectively (**1.1.1** (5)).

Having determined such a universal algebra B, we can introduce in it a relation of order, setting $x \le y$ if $x \wedge y = x$. In this case it appears that $(B, \le, \mathbf{0}, \mathbf{1})$ is a distributive lattice with complements, where $\vee$ and $\wedge$ coincide with the lattice operations, $*$ is complementation, and $\mathbf{0}$ and $\mathbf{1}$ are the least and the greatest elements, respectively. In literature one can find many equivalent systems of axioms which characterize Boolean algebras.

1.1.4. Using the basic operations $\vee$, $\wedge$ and $*$, other operations are introduced:

$$x - y := x \wedge y^*, \quad x \Rightarrow y := x^* \vee y,$$
$$x \Delta y := (x - y) \wedge (y - x) = (x \wedge y^*) \vee (y \wedge x^*),$$
$$x \Leftrightarrow y := (x \Rightarrow y) \wedge (y \Rightarrow x) = (x^* \vee y) \wedge (y^* \vee x).$$

Let us consider some easily checkable relations which will be repeatedly used below:

(1) $x \Rightarrow y = (x - y)^*$, $x \Leftrightarrow y = (x\Delta y)^*$;

(2) $x \Rightarrow (y \Rightarrow z) = (x \wedge y) \Rightarrow z = (x \wedge y) \Rightarrow (x \wedge z)$;

(3) $x \le y \Rightarrow z \leftrightarrow x \wedge y \le z \leftrightarrow y - z \le x^*$;

(4) $x \le y \leftrightarrow x \Rightarrow y = \mathbf{1} \leftrightarrow x - y = 0$;

(5) $x = y \leftrightarrow x \Leftrightarrow y = \mathbf{1} \leftrightarrow x\Delta y = \mathbf{0}$.

It should be emphasized that the operation Δ, termed the *symmetric difference*, has the properties of a metric:

(6) $x\Delta y = \mathbf{0} \leftrightarrow x = y$;

(7) $x\Delta y = y\Delta x$;

(8) $x\Delta y \le (x\Delta z) \vee (z\Delta y)$.

In this case, relative to such a metric the lattice operations become contractive, while complementation becomes an isometry:

$$(x \vee y)\Delta(u \vee v) \le (x\Delta u) \vee (y\Delta v),$$
$$(x \wedge y)\Delta(u \wedge v) \le (x\Delta u) \vee (y\Delta v),$$
$$x^* \Delta y^* = x\Delta y.$$

1.1.5. A Boolean algebra B is called *complete* (*σ-complete*), if any set (any countable set) in B has exact bounds. Instead of σ-complete algebras we often simply use the term *σ-algebras*. Associated with a Boolean algebra B are the mappings $\vee, \wedge: \mathcal{P}(B) \to B$, putting into correspondence to the set in B its supremum and infimum, respectively. These mappings are sometimes termed *infinite operations*. For these operations we have important relations such, for instance, as the distributive laws:

(1) $x \vee \bigwedge_{\alpha \in A} x_\alpha = \bigwedge_{\alpha \in A} x \vee x_\alpha$;

(2) $x \wedge \bigvee_{\alpha \in A} x_\alpha = \bigvee_{\alpha \in A} x \wedge x_\alpha$;.

From (1), (2) the following often used relations ensue:

(3) $\left(\bigvee_{\alpha\in A}\right) \Rightarrow x = \bigwedge_{\alpha\in A}(x_\alpha \Rightarrow x);$

(4) $\left(\bigwedge_{\alpha\in A}\right) \Rightarrow x = \bigvee_{\alpha\in A}(x_\alpha \Rightarrow x);$

(5) $x \Rightarrow \left(\bigvee_{\alpha\in A} x_\alpha\right) = \bigvee_{\alpha\in A}(x \Rightarrow x_\alpha);$

(6) $x \Rightarrow \left(\bigwedge_{\alpha\in A} x_\alpha\right) = \bigwedge_{\alpha\in A}(x \Rightarrow x_\alpha).$

Ensured are also the commutativity and associativity of the exact bounds, recalled earlier in some particular cases in **1.1.1** (1,2):

(7) $\bigvee_{\alpha\in A}\bigvee_{\beta\in B} x_{\alpha,\beta} = \bigvee_{\beta\in B}\bigvee_{\alpha\in A} x_{\alpha,\beta};$

(8) $\bigwedge_{\alpha\in A}\bigwedge_{\beta\in B} x_{\alpha,\beta} = \bigwedge_{\beta\in B}\bigwedge_{\alpha\in A} x_{\alpha,\beta};$

(9) $\vee\left(\bigcup_{\alpha\in A} X_\alpha\right) = \bigvee_{\alpha\in A} \vee X_\alpha;$

(10) $\wedge\left(\bigcup_{\alpha\in A} X_\alpha\right) = \bigwedge_{\alpha\in A} \wedge X_\alpha,$

where $X_\alpha \subset B (\alpha \in A)$. It should be emphasized that rules (1) - (6) are valid in an arbitrary Boolean algebra, while rules (7) - (10) hold in any ordered set under the obvious assumptions of existence of exact bounds.

1.1.6. Let us consider some ways of forming Boolean algebras.

(1) A nonempty subset B_0 of a Boolean algebra B is termed *a subalgebra* of B, if B_0 is closed relative to the Boolean operation $\vee, \wedge$ and $*$, i.e., $\{x \vee y, x \wedge y, x^*\} \subset B_0$, whatever $x, y \in B_0$. Concerning the order induced from B, the subalgebra B_0 is a Boolean algebra with the same zero and unit that B has. In particular, $B_0 := \{\mathbf{0}_B, \mathbf{1}_B\}$ is a subalgebra of B.

A subalgebra $B_0 \subset B$ is termed *regular* (*σ-regular*) iff for any set (any countable set) $A \subset B$ the exact bounds $\vee A$ and $\wedge A$ existing in B are in B_0. The intersection of an arbitrary family of subalgebras is a subalgebra as well. The same is also true for regular (σ-regular) subalgebras, which makes the definition to follow sound. The least subalgebra of the algebra

B containing a nonempty subset $M \subset B$ is called the *subalgebra generated by the set M*. *The regular (σ-regular) subalgebra generated by the set M* is introduced in an analogous way.

(2) *An ideal* of a Boolean algebra B is any nonempty set $J \subset B$ obeying the conditions:

$$x \in J \wedge y \in J \rightarrow x \vee y \in J,$$
$$x \in J \wedge y \le x \rightarrow y \in J.$$

Examples of such ideals are sets $B_a := \{x \in B : x \le a\}$, where $a \in B$. Those are called *principal*. If $\mathbf{0} \ne e \in B$, then the principal ideal B_e is an independent Boolean algebra relative to the order induced from B. In B_e the role of unit is played by the element e. The lattice operations are inherited from B, while the complementation of B_e has the form $x \rightarrow e - x \ (x \in B)$. The ideal J is termed *proper* provided $J \ne B$.

(3) Let us choose Boolean algebras B and B'. The mapping $h: B \rightarrow B'$ is called a *(Boolean) homomorphism*, if for any $x, y \in B$ the following equalities are fulfilled:

$$h(x \vee y) = h(x) \vee h(y),$$
$$h(x \wedge y) = h(x) \wedge h(y),$$
$$h(x^*) = h(x)^*.$$

The homomorphism h is *isotonic* $(x \le y \rightarrow h(x) \le h(y))$. If h is a homomorphism, then the image $h(B)$ of the algebra B is a subalgebra of B'. If h is bijective, then it is called an *isomorphism*, while the algebras B and B' themselves are called *isomorphic*. An injective homomorphism is usually called a *monomorphism*.

Let C be an arbitrary set, and let a bijection $h: B \rightarrow C$ be given. Then we can introduce an order in C by setting $h(x) \le h(y)$ iff $x \le y$. In this case C turns into a Boolean algebra, while h becomes an isomorphism of Boolean algebras.

(4) Let J be a proper ideal of a Boolean algebra B. Let us introduce an equivalence relation $\sim$ by using the rule

$$x \sim y \leftrightarrow x \Delta y \in J \quad (x, y \in B).$$

Let us denote by φ the canonical (factor-) mapping of the algebra B on the factor-set $B / J := B / \sim$. For the equivalence classes (cosets) $u, v \in B / J$ we denote $u \le v$ if there are elements $x \in U$ and $y \in v$ such that $x \le y$. An order relation in B / J has thus been determined. In this case B / J becomes a Boolean algebra which is termed the *factor-algebra of B by J*. The Boolean operations induced in B / J are such that φ becomes a homomorphism. If $h: B \to B'$ is a homomorphism, then $\ker h = \{x \in B: h(x) = \mathbf{0}\}$ is an ideal, and there is a unique monomorphism $g: B / \ker h \to B'$ for which $g \circ \varphi = h$, where $\varphi: B \to B' / \ker h$ is the factor-homomorphism. Therefore, any homomorphic image of a Boolean algebra is isomorphic to its factor-algebra by a suitable ideal.

(5) Let us choose a family of Boolean algebras $(B_\alpha)_{\alpha \in A}$. Let us supply the product $B := \Pi_{\alpha \in A} B_\alpha$ with coordinatewise order, setting $x \le y$ for $x, y \in B$ if $x(\alpha) \le y(\alpha)$ for all $\alpha \in A$. In this case B is a Boolean algebra. The Boolean operations in B coincide with the corresponding coordinatewise operations in the algebras B_α. The zero θ and unity e in B are determined by the equalities $\theta(\alpha) := \mathbf{0}_\alpha$ and $e(\alpha) := \mathbf{1}_\alpha$ $(\alpha \in A)$, where $\mathbf{0}_\alpha$ and $\mathbf{1}_\alpha$ are the zero and unity in B_α. The Boolean algebra B is termed the *Cartesian product* of the family of Boolean algebras $(B_\alpha)_{\alpha \in A}$.

(6) Let us again consider a family of Boolean algebras $(B_\alpha)_{\alpha \in A}$. There is a Boolean algebra B and a family of monomorphisms $\iota_\alpha: B_\alpha \to B$ $(\alpha \in A)$ obeying the following conditions:

(1) the family of subalgebras $(\iota_\alpha(B_\alpha))_{\alpha \in A}$ of the algebra B is independent, i.e., for any finite set of nonzero elements $x_k \in \iota_{\alpha_k}(B_{\alpha_k})$, where $\alpha_1, \dots, \alpha_n \in A$ and $\alpha_k \ne \alpha_l$ for $k \ne l$, fulfilled is $x_1 \wedge \dots \wedge x_n \ne \mathbf{0}$;

(2) the subalgebra in B generated by the union of all $\iota_\alpha(B_\alpha)$ coincides with B.

If a Boolean algebra B' and a family of monomorphisms $\iota'_\alpha: B_\alpha \to B'$ $(\alpha \in A)$ obey the same conditions, (1) and (2), then there is an isomorphism h of the algebra B on the algebra B' such that $\iota_\alpha \circ h = \iota'_\alpha$ $(\alpha \in A)$. The pair $(B, (\iota_\alpha)_{\alpha \in A})$ is called *the Boolean* (or *tensor*) *product of the family* $(B_\alpha)_{\alpha \in A}$ and is denoted by the symbol $\otimes_{\alpha \in A} B_\alpha$.

(7) *A completion* of a Boolean algebra B is a pair (ι, A) provided the following conditions are met:

(a) A is a complete Boolean algebra;

(b) ι is a monomorphism from B to A preserving the exact bounds of all sets;

(c) the proper algebra in A generated by the set $\iota(B)$ coincides with A.

The pairs (ι, A) and (ι', A') are said to *be isomorphic* if there is an isomorphism $h: A \to A'$ such that $h \circ \iota = \iota'$. For any Boolean algebra B there is a completion unique up to isomorphism (and thus referred to as the completion of B), which can be, for instance, obtained by the classical method of sections (stemming from Dedekind).

1.1.7. Examples

(1) For a nonempty set X, the set of subsets $\mathcal{P}(X)$ ordered by inclusion is a complete Boolean algebra, sometimes called *the Boolean* of X. In this case the Boolean operations coincide with the set-theoretic operations of union, intersection and complement.

(2) Let X be a topological space. The set of all clopen (i.e., open and closed simultaneously) subsets of the space X ordered by inclusion is a subalgebra of the Boolean $\mathcal{P}(X)$. Let us denote this subalgebra by the symbol $\mathbb{E}(X)$. The Boolean operations in $\mathbb{E}(X)$ are inherited from $\mathcal{P}(X)$ and, hence, coincide with the set-theoretic ones. However, $\mathbb{E}(X)$ is not a regular subalgebra, i.e., the infinite operations in $\mathcal{P}(X)$ and $\mathbb{E}(X)$ can differ essentially.

(3) A closed subset F of a topological space X is called *regular* if $F = \mathrm{cl}\,\mathrm{int}\,F$, i.e., if F coincides with the closure of the set of its interior points. Analogously, *a regular open set* G is defined by the relation $G = \mathrm{int}\,\mathrm{cl}\,G$. Let $\mathrm{RC}(X)$ and $\mathrm{RO}(X)$ be sets of regular closed subsets and regular open subsets, respectively, of the topological space X. The sets $\mathrm{RC}(X)$ and $\mathrm{RO}(X)$ ordered by inclusion are complete Boolean algebras. The mapping $F \to \mathrm{int}\,F$ ($F \in \mathrm{RC}(X)$) establishes an isomorphism between them. The algebras of $\mathrm{RC}(X)$ and $\mathrm{RO}(X)$ are contained in the Boolean $\mathcal{P}(X)$, not being, however, its subalgebras. Thus, for instance, in $\mathrm{RC}(X)$ the Boolean operations have the form

$$E \vee F = E \cup F, \quad E \wedge F = \mathrm{cl}\,\mathrm{int}\,E \cap F, \quad F^* = \mathrm{cl}(X - F).$$

(4) Let $\mathcal{B}(X)$ be the Borel σ-algebra of a topological space X (= σ-regular subalgebra of the Boolean $\mathcal{P}(X)$ generated by the topology). In $\mathcal{B}(X)$ let us consider an ideal $\mathcal{N}$ consisting of all meager sets (= first-category sets). Then the factor-algebra $\mathcal{B}(X)/\mathcal{N}$ is a complete Boolean algebra termed *the algebra of Borel sets modulo meager sets*. An isomorphic algebra is obtained if instead of $\mathcal{B}(X)$ we take an σ-algebra with the Baire property. (A set $M \subset X$ *has the Baire property* if for a certain open $G \subset X$ the symmetrical difference $M \Delta G$ is a meager set). If the space X is a Baire one, i.e., if in it there are no nonempty open meager sets, then the algebra in question is isomorphic to the algebra of regular closed sets $\mathrm{RC}(X)$.

(5) Let $\mathcal{B}$ be a σ-complete Boolean algebra, and let a positive countably additive function $\mu : \mathcal{B} \to \mathbf{R}$ be given. *Countable additivity*, as usual, means

$$\mu\left(\bigvee_{n=1}^{\infty} x_n\right) = \sum_{n=1}^{\infty} \mu(x_n)$$

for any sequence (x_n) of mutually disjoint elements of $\mathcal{B}$. Let us set $\mathcal{N} := \{x \in \mathcal{B} : \mu(x) = 0\}$, in which case $\mathcal{N}$ is a σ-complete ideal. On the factor-algebra $B := \mathcal{B} / \mathcal{N}$ there is a unique countably additive function $\bar{\mu}$, for which $\mu = \bar{\mu} \circ \varphi$, where $\varphi : \mathcal{B} \to B$ is a factor-homomorphism. The algebra B is complete, while the function $\bar{\mu}$ is strictly positive, i.e., $\bar{\mu}(x) = 0 \to x = \mathbf{0}$. If $\rho(x,y) := \bar{\mu}(x \Delta y)$, then ρ is a metric, and the metric space (B,ρ) is complete.

Let $(X, \mathcal{B}, \mu)$ be a space with finite measure, i.e., X is a nonempty set, $\mathcal{B}$ is a σ-complete subalgebra in $\mathcal{P}(X)$, and μ is the same as above. Then the algebra B is termed *an algebra of measurable sets modulo sets of measure zero.*

(6) Let $(X, \mathcal{B}, \mu)$ be the same as in (5), and let us denote by the symbol $M(\mu) := M(X, \mathcal{B}, \mu)$ the space of equivalence classes of μ-measurable functions almost everywhere finite on X. The measurable functions *are equivalent* provided they can assume different values only on the set of measure zero. In the space $M(\mu)$ an order is introduced by setting $\bar{f} \le \bar{g}$ iff $f(x) \le g(x)$ for almost all $x \in X$. Here $\bar{f}$ is the class of equivalence of the function f. Then $M(\mu)$ is a lattice. Let $\mathbf{1}$ the equivalence class of the function identically equal to unity on X. Let us set $B := \{e \in M(\mu) : e \wedge (\mathbf{1} - e) = \mathbf{0}\}$. In this case B is a complete Boolean algebra with respect to the order induced from $M(\mu)$

$$c \vee e = c + e - c \cdot e, \quad c \wedge e = c \cdot e, \quad e^* = \mathbf{1} - e \quad (c, e \in B),$$

where $+$, $\cdot$, $-$ are the signs of addition, multiplication and subtraction, respectively, in the ring $M(\mu)$.

(7) Let H be a complex Hilbert space, and $B(H)$ be the algebra of all bounded linear operators in H. The *commutant* $\mathcal{U}'$ of a set $\mathcal{U} \subset B(H)$ is introduced by the formula $\mathcal{U}' := \{T \in B(H) : (\forall S \in \mathcal{U})(TS = ST)\}$, while the *bicommutant* is introduced by the rule $\mathcal{U}'' := (\mathcal{U}')'$. *The von Neumann algebra* is any selfadjoint $(T \in \mathcal{U} \to T^* \in \mathcal{U})$ subalgebra $\mathcal{U} \subset B(H)$ which coincides with its bicommutant. Let us choose a commutative von Neumann algebra $\mathcal{U}$. The set of all orthoprojections contained in $\mathcal{U}$ will be denoted by the symbol $B_{\mathcal{U}}$. The order relation in $B_{\mathcal{U}}$ is introduced as follows:

$$\pi \le \rho \leftrightarrow \pi(H) \subset \rho(H) \quad (\pi, \rho \in B_{\mathcal{U}}).$$

In this case $B_{\mathcal{U}}$ is a complete Boolean algebra, the Boolean operations having the form:

$$\pi \vee \rho = \pi + \rho - \pi \circ \rho, \quad \pi \wedge \rho = \pi \circ \rho, \quad \pi^* = I_H - \pi.$$

1.1.8. Remarks

(1) The theory of Boolean algebras originated from the classical work by G.Boole 'An Investigation of the Laws of Thought, on Which Are Founded the Mathematical Theories of Logic and Probabilities' [17]. The purpose of this monograph was formulated by the author in the following way: "In the treatise offered to the attention of the reader we intend to investigate the fundamental laws of those operations which refine the reason in the process of consideration, in order to express them in a symbolic language of calculus and on this foundation to construct the science of logic and its method." Following this doctrine, G.Boole carries out, in fact, algebrization of the logical system that underlies classical mathematical considerations. As a result, he has become the author of the algebraic system termed a Boolean algebra.

(2) One of the most important examples considered in the afore-mentioned book is the propositional algebra. In modern terms, the *propositional algebra* is a Boolean algebra arising as a result of identifying the equivalent formulas in a set of all formulas of the propositional calculus.

The above-said can be in general general formalized as follows. Let $\mathfrak{T}$ be a first-order theory based on the classical (two-valued) logic. In the set of all formulas Φ in the theory $\mathfrak{T}$ let us introduce a relation of preorder, setting $\varphi \le \psi$ iff the formula $\varphi \to \psi$ is a theorem of the theory $\mathfrak{T}$. Let us consider the equivalence relation $\sim$ in Φ:

$$\varphi \sim \psi \leftrightarrow \varphi \le \psi \wedge \psi \le \varphi \quad (\varphi, \psi \in \Phi).$$

Let $\mathfrak{A} = \Phi/\sim$ be the corresponding factor-set supplied with the induced order. In more detail, if $|\varphi|$ is the equivalence class of the formula $\varphi \in \Phi$, then $|\varphi| \le |\psi|$ implies $\varphi \le \psi$. The arising ordered set $\mathfrak{A} := \mathfrak{A}(\mathfrak{T})$ is a Boolean algebra, which is sometimes termed a Lindenbaum-Tarski algebra of the theory $\mathfrak{T}$. The Boolean operations in the algebra $\mathfrak{A}(\mathfrak{T})$ have the form

$$|\varphi| \vee |\psi| = |\varphi \vee \psi|,$$
$$|\varphi| \wedge |\psi| = |\varphi \wedge \psi|,$$
$$|\varphi|^* = |\neg \varphi|.$$

The translation of the logical problems of formal theories into the language of the corresponding Boolean algebras, the Lindenbaum-Tarski algebras, is called the *Boolean method.*

(3) The classical ways of making conclusions (syllogisms, the excluded middle,

modus ponens, generalization, etc.) are abstractions originated as a result of idealization of those real operations carried out in the brain in the process of reasoning. Inevitably making the reality rougher, the two-valued logic gives, strictly speaking, only a hardly approximate, incomplete description of the laws of reasoning, which explains the interest to non-classical logical systems. One of such systems has been elaborated within the framework of the intuitionistic approach. Without going into details, let us briefly describe the corresponding propositional algebra.

The pseudo-Boolean algebra is a lattice L with zero and unity, where for any $x, y \in L$ there is a pseudo-complement $x \Rightarrow y$ of the element x relative to y. By definition, *the pseudo-complement* $x \Rightarrow y$ is the greatest of the elements $z \in L$ obeying the inequality $z \wedge x \leq y$. Hence, the following equivalence is valid (cf.: **1.1.4** (3)):

$$z \leq x \Rightarrow y \leftrightarrow x \wedge z \leq y \quad (x, y, z \in L),$$

which may be also considered as the definition of $x \Rightarrow y$. A pseudo-Boolean algebra is a distributive lattice. A complete lattice is a pseudo-Boolean algebra iff the following distributive law is fulfilled in it:

$$x \wedge \bigvee_{\alpha \in A} x_\alpha = \bigvee_{\alpha \in A} x \wedge x_\alpha \quad (x, x_\alpha \in L).$$

An example of a complete pseudo-Boolean algebra is the set of all open subsets of a topological space ordered by inclusion. Pseudo-Boolean algebras are termed *Brouwer lattices* or, most often, *Heyting algebras.*

We can demonstrate that the Lindenbaum algebra of the intuitionistic logic is a Heyting algebra. Therefore, Heyting algebras characterize the intuitionistic logic is the same way as Boolean algebras characterize the classical logic (for details, see [13, 214, 215]).

(**4**) Investigation of certain types of nonclassical logics results, as was in the case of the intuitionistic logic, to various classes of algebraic systems which are distributive lattices. The most well-known varieties are as follows: an implicative lattice (= the lattice with relative pseudo-complements), a topological Boolean algebra (= a Boolean algebra B with the operation $\mathbf{I}: B \rightarrow B$, obeying the internal axioms: $\mathbf{I}: (x \wedge y) = \mathbf{I}x \wedge \mathbf{I}y$; $x \leq y \rightarrow \mathbf{I}x \leq \mathbf{I}y$, $\mathbf{I}^2 = \mathbf{I}$, $\mathbf{I0} = \mathbf{0}$, $\mathbf{I1} = \mathbf{1}$), a Post algebra, etc. (see [13, 72, 214, 215]). A general theory of lattices is an individual branch with its numerous and in-depth relations with various sections of mathematics.

(**5**) The origination of the above-mentioned logics (=lattices) is associated with 'the investigation of the laws of thought' in the sense of the the Boole's program we discussed above. The analysis of the laws of microworld has given rise to a principally different type of logic. The logic of quantum mechanics differs significantly from the classical, intuitionistic and modal logics.

The ortholattice is a lattice L with zero, unity and a unary operation (of orthocomplementation) $(\cdot)^{\perp}: L \rightarrow L$ obeying the following conditions:

$$x \wedge x^{\perp} = \mathbf{0}, \quad x \vee x^{\perp} = \mathbf{1};$$
$$x^{\perp\perp} := (x^{\perp})^{\perp} = x;$$
$$(x \vee y)^{\perp} = x^{\perp} \wedge y^{\perp}, \quad (x \wedge y)^{\perp} = x^{\perp} \vee y^{\perp}.$$

The distributive ortholattice is a Boolean algebra. The elements x and y are called *orthogonal*, and we write $x \perp y$ if $x \leq y^{\perp}$ or, which is equivalent, $y \leq x^{\perp}$. The ortholattice L is termed an *orthomodular lattice* or a *(quantum) logic*, provided for any $x, y \in L, x \leq y$ there is an element $z \in L$ such that $x \perp z$ and $x \vee z = y$. The last peculiarity is equivalent to the fact that from $x \leq y$ we get $y = x \vee (y \wedge x^{\perp})$. An example of the quantum logic is a lattice of all closed subspaces of a Hilbert space with the operation of orthogonal complementation.

1.2. Representation of Boolean Algebras

A principally important possibility of representing a Boolean algebra as an algebra of clopen subsets of a compact space is guaranteed by the Stone theorem. The basic goal of this section is to prove the theorem and to describe some opportunities that are granted by it.

1.2.1. Let $\mathbf{2} := \mathbb{Z}_2 := \mathcal{P}(\{\varnothing\}) := \{\mathbf{0}, \mathbf{1}\}$ be a two-element set with the field structure determined by the relations:

$$\mathbf{0} + \mathbf{0} := \mathbf{0}, \quad \mathbf{0} + \mathbf{1} = \mathbf{1} + \mathbf{0} := \mathbf{1}, \quad \mathbf{1} + \mathbf{1} := \mathbf{0},$$
$$\mathbf{0} \cdot \mathbf{1} = \mathbf{1} \cdot \mathbf{0} := \mathbf{0}, \quad \mathbf{0} \cdot \mathbf{0} := \mathbf{0}, \quad \mathbf{1} \cdot \mathbf{1} := \mathbf{1}.$$

It should be remarked that all the elements of the field $\mathbf{2}$ are idempotents. Let us now consider an arbitrary set B with the structure of an associative ring whose elements are idempotents: $b \in B \rightarrow b^2 = b$. In this case B is called a *Boolean ring*. Such a ring is commutative and obeys the identity $b = -b$ for $b \in B$. The Boolean ring is obvious to be a vector space over the field $\mathbf{2}$ and, moreover, a commutative algebra over this field.

It should be recalled that the unity of an algebra is by definition different from zero. The field $\mathbf{2}$ can be naturally identified with the subring of a Boolean ring composed of the zero and unity of the latter. This fact is reflected in the notation: for the zero of any ring, use is made of the symbol $\mathbf{0}$, whereas for the unity, of the symbol $\mathbf{1}$. Such an agreement, naturally, results in a quite common collision of notation (in the field $\mathbf{2}$ the addition and multiplication can be redefined, in which case $\mathbf{0}$ starts playing the role of $\mathbf{1}$, and vice versa).

A Boolean ring B is always considered with the order relation determined by the rule:

$$b_1 \geq b_2 \leftrightarrow b_1 b_2 = b_1 \quad (b_1, b_2 \in B).$$

The ordered set $(B,\leq)$ can be directly proved to be a distributive lattice with the least element **0**, and the greatest element **1**. In this case the lattice operations are related with the ring ones in the following way:

$$x \vee y = x + y + xy, \quad x \wedge y = xy.$$

Moreover, each element $b \in B$ has a unique complement, i.e. such an element b^*, that

$$b^* \vee b = \mathbf{1}, \quad b^* \wedge b = \mathbf{0}.$$

Obviously, $b^* = \mathbf{1} + b$. Hence, any Boolean ring is a Boolean algebra provided the order in it is determined in the way just discussed.

In turn, we can introduce the structure of a ring in the Boolean algebra B, setting

$$x + y := x \Delta y, \quad xy := x \wedge y \quad (x, y \in B).$$

In this case $(B, +, \cdot, \mathbf{0}, \mathbf{1})$ becomes a Boolean ring with unity, for which the newly-arisen order relation coincides with the one available.

Therefore, a Boolean algebra can be viewed as an algebra with unity over the ring **2**, whose every element is idempotent.

1.2.2. Let B be an arbitrary Boolean algebra.

(1) *A character* of the algebra B is a (Boolean, or, which is the same, ring) homomorphism $\chi: B \to \mathbf{2}$. By the symbol $X(B)$ we shall denote a set of all the characters of B with the topology of pointwise convergence. In more detail, the topology in $X(B)$ is induced by the product topology of $\mathbf{2}^B$, in which case **2** has a unique compact Hausdorff (discrete) topology. The arising space $\mathbf{2}^B$, a Cantor discontinuum, is compact and totally disconnected. The last fact implies that any two different points in this space have disjoint clopen neighbourhoods. It is also evident that $X(B)$ is a closed subset of $\mathbf{2}^B$. Therefore, $X(B)$ is compact and totally disconnected. The set $X(B)$ is termed *the character space* of the Boolean algebra B.

(2) As is known, a nonempty set $\mathfrak{F} \subset B$ is called *a filter* provided

$$x \in \mathfrak{F} \wedge y \in \mathfrak{F} \to x \wedge y \in \mathfrak{F},$$
$$x \in \mathfrak{F} \wedge x \leq y \to y \in \mathfrak{F}.$$

A filter different from B is termed *proper*. Elements of the set of all proper filters that are maximal by inclusion are called *ultrafilters*. Let $U(B)$ be the set of all ultrafilters in B, and let $U(b)$ be the set of ultrafilters containing b. Let us supply $U(B)$ with a topology, having chosen the system of sets $\{U(b): b \in B\}$ to be the basis of the topology. Such a definition of a topology is sound since, as can be easily checked, $U(x \wedge y) = U(x) \cap U(y)$ $(x, y \in B)$. The topological space of $U(B)$ is often referred to as *the Stone space of the Boolean algebra B* and is denoted by $\mathfrak{S}(B)$.

(3) Let $M(B)$ be a set of all maximal (proper) ideals of the algebra B. An ideal here can be understood both in accordance with **1.1.6** (2) and in the standard sense of the theory of rings. A set $J \subset B$ is an ideal iff $J^* := \{x^* : x \in J\}$ is a filter in B. Moreover, $J \in M(B) \leftrightarrow J^* \in U(B)$. Therefore, the mapping $J \to J^*$ presents a bijection between $M(B)$ and $U(B)$. The set $M(B)$ is usually called *the space of maximal ideals* and is assumed to have the induced topology, the one which turns the mapping $J \to J^*$ into a homeomorphism.

1.2.3.

(1) *A Boolean ring B is a field iff it contains exactly two elements* **0** *and* **1**. *Hence,* **2** *is a Boolean field unique up to isomorphism.*

◁ Indeed, a nonzero element $x \in B$ is invertible, and, hence, the following implications are valid:

$$xx^{-1} = \mathbf{1} \to xxx^{-1} = \mathbf{1} \to xx^{-1} = x \to x = \mathbf{1}. \ \triangleright$$

For $\chi \in X(B)$ by the symbol χ^* we shall denote the mapping $x \to \chi(x)^*$ $(x \in B)$. As is seen, $\ker \chi := \{x \in B : \chi(x) = \mathbf{0}\}$ is an ideal, while $\ker \chi^*$ is a filter.

(2) *The mappings* $\chi \mapsto \ker(\chi)$ $(\chi \in X(B))$ *and* $\chi \to \ker \chi^*$ $(\chi \in X(B))$ *are homeomorphisms of* $X(B)$ *on* $M(B)$ *and* $U(B)$, *respectively.*

◁ The mapping $\chi \mapsto \ker(\chi)$ is injective. If $J \in M(B)$, then B / J is a field and, according to (1), it is isomorphic to **2**. Let us set $\chi := \lambda \circ \varphi$, where $\varphi : B \to B / J$ is a factor-homomorphism, while $\lambda : B / J \to \mathbf{2}$ is an isomorphism. Obviously, $\ker \chi = J$ and, hence, the mapping under discussion is bijective. The remaining statements are obvious. ▷

(3) *The element* $x \in B$ *is equal to zero iff* $\chi(b) = \mathbf{0}$ *for all* $\chi \in X(B)$.

◁ Let us assume $x \neq \mathbf{0}$. Then the principal ideal $\{y \in B : y \leq x^*\}$ is proper, and it can be extended to a maximal ideal $J \in M(B)$. This statement, the Krull theorem, is directly deduced from the Kuratowski-Zorn Lemma (see **I.2.3.9**). By virtue of **(2)**, $J = \ker\chi$ for a certain $\chi \in X(B)$. Since $x \notin J$, we must get $\chi(x) \neq \mathbf{0}$. ▷

1.2.4. Stone theorem. *Every Boolean algebra B is isomorphic to the Boolean algebra of clopen sets of a totally disconnected compact set unique up to homeomorphism, the Stone space of the algebra B.*

◁ Let $C(X(B),\mathbf{2})$ be the algebra of continuous $\mathbf{2}$-valued functions determined on a totally disconnected compact $X(B)$. The Gelfand transform $\mathscr{G}_B$ assigns to an element $x \in B$ the $\mathbf{2}$-valued function

$$\hat{x}: \chi \to \chi(x) \quad (\chi \in X(B)).$$

It is obvious that $\mathscr{G}_B: B \to C(X(B),\mathbf{2})$ is a homomorphism which is injective (by **1.2.3** (3)) Let us take an $f \in C(X(B),\mathbf{2})$ and set $V_f := \{\chi \in X(B): f(\chi) = \mathbf{1}\}$. The set V_f is clopen. By the definition of the topology, of $X(B)$ there are $b_1,\dots,b_k \in B$ and $c_1,\dots,c_l \in B$ such that

$$V_f := \{\chi \in X(B): \chi(b_n) = \mathbf{1}\ (n \leq k), \chi(c_m) = \mathbf{0}\ (m \leq l)\}.$$

Let us set $b_0 := b_1 \wedge \dots \wedge b_k$, $c_0 := c_1 \vee \dots \vee c_l$ and $b := b_0 \wedge c_0^*$. The set V_f can be presented as follows:

$$V_f = \{\chi \in X(B): \chi(b_0) = \mathbf{1} \wedge \chi(c_0) = \mathbf{0}\} =$$
$$= \{\chi \in X(B): \chi(b) = \mathbf{1}\} = \{\chi \in X(B): \hat{b}(\chi) = \mathbf{1}\}.$$

Therefore, $f = \hat{b}$, and, hence, $\mathscr{G}_B$ is an isomorphism.

Let us assume now that Q_1 and Q_2 are totally disconnected compacta, and that the mapping $h: C(Q_1,\mathbf{2}) \to C(Q_2,\mathbf{2})$ is an isomorphism of the algebras. If χ is a character of the algebra $C(Q_2,\mathbf{2})$, then $\chi \circ h$ is that of $C(Q_2,\mathbf{2})$. In this case the mapping $\chi \to \chi \circ h$ implements the homeomorphism of the character spaces. On the other hand, the character space of the algebra $C(Q_k,\mathbf{2})$ is homeomorphic to the compactum Q_k. Therefore, the compacta Q_1 and Q_2 are homeomorphic. Now we have to notice that the algebra $C(X(B),\mathbf{2})$ is isomorphic to the algebra of clopen sets of the space $X(B)$ and, hence, of the space $U(B)$ as well. ▷

1.2.5. Further on we, as a rule, shall consider complete Boolean algebras. The notion of a complete Boolean algebra is closely associated with *extremal compacta*, i.e., compacta which are totally disconnected spaces. It should be recalled that a topological space is termed *totally disconnected* or *extremally disconnected* or, to put it short, extremal, if the closure of its every open subset is open.

Ogasawara theorem. *A Boolean algebra is complete iff its Stone space is extremal.*

$\lhd$ Let B be a complete Boolean algebra, and h be an isomorphism of B on the algebra of clopen sets of the compactum Q: $U(B)$. Let us choose an open set $G \subset Q$. As Q is totally disconnected, then $G \cup \mathfrak{U}$, where $\mathfrak{U}$ is the union of the clopen sets contained in $\mathcal{C}$. Let $\mathfrak{U}' := \{h^{-1}(U): U \in \mathfrak{U}\}$ and $b := \vee \mathfrak{U}'$. It is the clopen set $h(b)$ that is the closure of G. Indeed, $\mathrm{cl}\, G \subset h(b)$ and $h(b) \setminus \mathrm{cl}\, G$ is open. If the latter set is nonempty, then $h(c) \subset h(b) \setminus \mathrm{cl}\, G$ for a certain $\mathbf{0} \neq c \in B$. This, however, implies that $h(c) \vee h(u) \leq h(b)$ for all $u \in \mathfrak{U}'$, which contradicts the equality $b = \vee \mathfrak{U}$. Therefore, $\mathrm{cl}\, G = h(b)$ is an open set.

Let us now assume that the compactum Q is extremal. Let $\mathcal{C}$ be the collection of clopen subsets of Q, and let $G := \cup \mathcal{C}$. The set G is open and its closure $\mathrm{cl}\, G$ must also be open, since Q is extremal. Obviously, $\mathrm{cl}\, G$ is the least upper bound of the set $\mathcal{C}$ in the Boolean algebra of clopen sets $\mathfrak{B}(Q)$. $\rhd$

1.2.6. Examples

(1) The Stone space of the algebra $\{\mathbf{0}, \mathbf{1}\}$ is a singleton. If a Boolean algebra is finite, it has $\mathbf{2}^n$ elements for a certain $n \in \mathbf{N}$, and its Stone space contains exactly n points.

(2) Let us choose a nonempty set X. The Stone space of the Boolean $\mathcal{P}(X)$ is a Stone-Cech compactification $\beta(X)$ of the set X which is viewed as a discrete topological space.

(3) If Q is a totally disconnected compact space, then the Stone space of the algebra $\mathfrak{B}(Q)$ is homeomorphic to Q.

(4) Let B, B' be Boolean algebras, and $h: B \to B'$ be a homomorphism. Let $\iota: B \to \mathfrak{B}(\mathfrak{S}(B))$ and $\iota': B' \to \mathfrak{B}(\mathfrak{S}(B'))$ be the Stone presentations of the algebras B and B'. There is a unique continuous mapping $\theta: \mathfrak{S}(B') \to \mathfrak{S}(B)$ such that

$$h(x) = \iota^{-1}\theta^{-1}(\iota(x)) \quad (x \in B).$$

The mapping $h \to \mathfrak{S}(h) := \theta$ is a bijection between the sets of homomorphisms from B to B' and the continuous mappings from $\mathfrak{S}(B')$ to $\mathfrak{S}(B)$. If B'' is another Boolean algebra, and $g: B' \to B''$ is a homomorphism, then $\mathfrak{S}(g \circ h) = \mathfrak{S}(h) \circ \mathfrak{S}(g)$. Moreover,

$\mathfrak{S}(I_B) = I_{\mathfrak{S}(B)}$. Let Boole be the category of Boolean algebras and homomorphisms, while Comp be a category of compacta and continuous mappings. Then the above statements can be formulated in the following way.

Theorem. *The mapping* $\mathfrak{S}$ *is a contravariant functor from the category* Boole *to the category* Comp.

Two important particular cases of the situation under consideration deserve special attention.

(5) A Boolean algebra B_0 is isomorphic to the subalgebra of a Boolean algebra B iff the compact set $\mathfrak{S}(B_0)$ is a continuous image of the compact set $\mathfrak{S}(B)$.

(6) A Boolean algebra B' is a homomorphic image of the algebra B (or isomorphic to the factor-algebra B) (see **I.1.1** (4)) iff the compact set $\mathfrak{S}(B')$ is homeomorphic to a closed subset of the compact set $\mathfrak{S}(B)$.

(7) Let $B := \Pi_{\alpha\in A} B_\alpha$, where $(B_\alpha)_{\alpha\in A}$ is a nonempty family of Boolean algebras. The Stone space $\mathfrak{S}(B)$ of the algebra B coincides with the Stone-Cech compactification of the topological sum $\cup_{\alpha\in A}\mathfrak{S}(B_\alpha)\times\{\alpha\}$ of spaces $\mathfrak{S}(B_\alpha)$.

(8) Let $B := \otimes_{\alpha\in A} B_\alpha$ be the Boolean product of a nonempty family of Boolean algebras (**I.1.6** (6)). Then the Stone space $\mathfrak{S}(B)$ of the algebra B is homeomorphic to the product $\Pi_{\alpha\in A}\mathfrak{S}(B_\alpha)$.

(9) An *absolute* of a compact set X is such a compact set $\dot{X}$ that meets the following conditions: **(a)** X is a continuous irreducible preimage of $\dot{X}$ (i.e., there is a continuous surjection of $\dot{X}$ on X, and X is not a continuous image of any proper closed subset of $\dot{X}$); **(b)** any compact continuous irreducible preimage of the compactum X is homeomorphic to $\dot{X}$. If $\dot{B}$ is the completion of a Boolean algebra B, then $\mathfrak{S}(\dot{B}) = \mathfrak{S}(B)$, i.e., an absolute of the Stone space of the algebra B is homeomorphic to the Stone space of its completion $\dot{B}$.

1.2.7. An *atom* of a Boolean algebra B is a nonzero element a of it such that $\{x\in B: \mathbf{0}\le x\le a\} = \{\mathbf{0}, a\}$, or, which is equivalent, $a\ne\mathbf{0}$ is an atom of the Boolean algebra B if for any $x\in B$ we have either $a\le x$ or $a\le x^*$. The algebra B is said to be *atomic* if for any nonzero element $x\in B$ there exists an atom $a\le x$. A Boolean algebra is termed *atomless* if it contains no atom.

Let us call the Boolean algebra B *fully distributive* if

$$\bigwedge_{m\in M}\bigvee_{n\in N} x_{m,n} = \bigvee_{f\in N^M}\bigwedge_{m\in M} x_{m,f(m)},$$

where $x_{m,n} \in B$ $(m \in M,\ n \in N)$, M and N are arbitrary sets, and $N^M := \{f: f: M \to N\}$.

Theorem. *Let B be a complete Boolean algebra. The following statements are equivalent:*

(1) *B is isomorphic to the Boolean $\mathcal{P}(A)$ for a nonempty A;*

(2) *B is fully distributive;*

(3) *B is atomic.*

⊲ (1) → (2) It suffices to remark that both the set-theoretic union (join) and intersection (meet) obey full distributivity.

(2) → (3) Let us consider a double family $\{x_{b,t} \in B: b \in B, t \in \mathbf{2}\}$, where $\mathbf{2} := \{\mathbf{0},\mathbf{1}\}$, $x_{b,0} := b^*$ and $x_{b,1} := b$. In this case

$$\mathbf{1} = \bigwedge_{b\in B} x_{b,0} \vee x_{b,1} = \bigwedge_{b\in B}\bigvee_{t\in \mathbf{2}} x_{b,t}.$$

Since the Boolean algebra B is fully distributive, we have

$$\mathbf{1} = \vee\{c(f): f: B \to \mathbf{2}\},$$

where $c(f) := \wedge\{x_{b,f(b)}: b \in B\}$. This implies that for $b \in B$ valid is $b = \vee\{b \wedge c(f): f \in \mathbf{2}^B\}$. Hence, for a non-zero $b \in B$ there is a $g \in \mathbf{2}^B$ such that $b \wedge c(g) \neq 0$. On the other hand, for arbitrary $b \in B$ and $f \in \mathbf{2}^B$ only the two following cases are possible:

(a) $f(b) = \mathbf{0} \to x_{b,f(b)} = b^* \to c(f) \leq b^* \leftrightarrow b \wedge c(f) = 0$,

(b) $f(b) = \mathbf{1} \to x_{b,f(b)} = b \to c(f) \leq b$.

Therefore, if $b \neq \mathbf{0}$, then either $b \wedge c(g) = 0$ or $c(f) \leq b$, i.e., $c(f)$ is an atom of B provided $c(f) \neq \mathbf{0}$. However, since there are sufficiently many nonzero $c(f)$, then B is atomic.

(3) → (1) Let A be a set of all the atoms of the Boolean algebra B. For $x \in B$ let us denote by the symbol $h(x)$ the set of all atoms $a \in B$ such that $a \leq x$. The mapping $h: B \to \mathcal{P}(A)$ can be easily checked to be an isomorphism of the Boolean algebras. ⊳

1.2.8. Remarks

(1) As is seen from theorem **1.2.4**, a Boolean algebra is completely determined by its Stone space. To be more exact, any property of the Boolean algebra B can be translated into the topological language, after which it becomes a property of the Stone space $\mathfrak{S}(B)$. This method of studying Boolean algebras is called *the realization method.*

(2) The basic idea underlying the Stone theorem **1.2.4** is also valid for the case of distributive lattices. For a distributive lattice L the role of the Stone space $\mathfrak{S}(L)$ is played by the set of all simple ideals (or filters) topologized in a special way. The proper ideal $J \subset L$ is called *simple* when

$$x \wedge y \in J \rightarrow x \in J \vee y \in J.$$

The Stone spaces of distributive lattices can be used either for constructing new lattices or for the topological description of lattice-theoretical properties (the realization method!) (see [13, 72, 214]).

1.3. Von Neumann-Gödel-Bernays Theory

As has been earlier remarked in **I.3.2.5**, the axiom schemata of replacement ZF_4^{φ} embraces an infinite number of axioms due to arbitrariness in the choice of a formula φ. One, however, can try to introduce new undefinable primitive objects which can be determined by formulas φ of ZF_4^{φ}. Then a great deal of the statements contained in the schemata ZF_4^{φ} could be formulated as a single axiom on such objects. In this case required are the axioms that could imply existence for an object corresponding to the formula. Since all the formulas are constructed by the same procedure within a finite number of sets, we cannot exclude the possibility of managing with a finite number of axioms. It is this basic idea stemming from von Neumann, that became a cornerstone of the axiomatics of set theory developed by Gödel and Bernays and designated by NBG.

The initial undefinable object (notion) of NBG is a class. A class which is an element of a class is called a *set*, the other classes termed *proper*. It is turning classes into objects that constitutes the basic difference between NBG and ZFC, the metalanguage of the latter treating 'a class' and 'a property' as synonyms.

When presenting the axiomatic theory NBG use, as a rule, is made of one of the two available modifications of the language of ZFC. The first modification consists in adding a new unary predicate symbol M to the language of ZFC, with $M(X)$ semantically implying that X is a set. The second modification uses different types of variables for sets and classes. It should be emphasized that the techniques mentioned are not obligatory for describing NBG, but are rather used for convenience.

1.3.1. The system NBG is a first-order theory with equality. Strictly speaking, the language of NBG does not differ from that of ZFC. However, capital Latin letters, $X, Y, Z, \ldots$ (with indices) are commonly used for variables, while lowercase Latin letters are left for the argo arising as a result of introducing abbreviated symbols which are not used in the language of NBG.

Let $M(X)$ be an abbreviation of the formula $(\exists Y)(X \in Y)$ (read as 'X is a set'). Let us introduce lowercase Latin letters $x, y, z \ldots$ (with indices) for the variables bounded by the sets. To be more exact, the formulas $(\forall x)\,\varphi(x)$ and $(\exists x)\,\varphi(x)$ are abbreviations of the formulas $(\forall X)\,(M(X) \rightarrow \varphi(X))$ and $(\exists X)\,(M(X) \wedge \varphi(X))$, respectively. Semantically these formulas imply: 'for any set φ is valid' and 'there is a set for which φ is valid', respectively. When using these abbreviated denotations the variable X must not enter the formula φ, as well as in the formulas comprising these abbreviations. The above rules of using uppercase and lowercase letters will, however, be observed only within the present section. After we have proved that the theory of classes is, in principle, formalizable, we will gradually return to the conventional and, hence, freer mathematical language.

Let us now get down to formulating special axioms of NBG.

1.3.2. The axiom of extensionality (for classes), NBG_1: *two classes coincide if (and only iff) they consist of the same elements*

$$(\forall X)\,(\forall Y)\,(X = Y \leftrightarrow (\forall Z)\,(Z \in X \leftrightarrow Z \in Y)).$$

1.3.3. The axioms for sets:

(1) the axiom of an (unordered) pair, NBG_2:

$$(\forall x)\,(\forall y)\,(\exists z)\,(\forall u)\,(u \in z \leftrightarrow u = x \vee u = y);$$

(2) the axiom of union, NBG_3:

$$(\forall x)\,(\exists y)\,(\forall z)\,(z \in y \leftrightarrow (\exists u)\,(u \in x \wedge z \in y));$$

(3) the axiom of powersets, NBG_4:

$$(\forall x)\ (\exists y)(\forall z)(z \in y \leftrightarrow z \subset x);$$

(4) the axiom of infinity, NBG_5:

$$(\exists x)\ (0 \in x \wedge (\forall y)\ (y \in x \leftrightarrow y \cup \{y\} \in x)).$$

These axioms are seen to coincide with their analogues of ZF formulated in **1.3.2.3**, **1.3.2.4**, **1.3.2.7** and **1.3.2.8**. One should, however, bear in mind that in verbal formulation the word 'set' can already mean here a class which is an element of the class, while in symbolic presentations of the axioms small Latin letters denote abbreviations (see **1.3.1**). Thus, for instance, a partially expanded axiom of powersets, NBG_4, has the form

$$(\forall X)(M(X) \rightarrow (\exists Y)\ (M(Y) \wedge (\forall Z)(M(Z) \rightarrow (Z \in Y \leftrightarrow Z \subset X)))).$$

In the presentation of the axiom of infinity use has been made of the following abbreviation:

$$0 \in x := (\exists y)\ (y \in x \wedge (\forall u)(u \notin y)).$$

The existence of an empty set is not assumed beforehand but results from the axiom of infinity. Nevertheless, this statement is sometimes included into the list of NBG as a separate axiom:

(5) $(\exists y)(\forall u)(u \notin y)$.

1.3.4. The axiom of replacement, NBG_6: *if the class* X *is single-valued, then for any set* y *the class of the second components of those pairs of* X*, whose first components belong to* y*, is a set:*

$$(\forall X)(\mathrm{Un}(X) \rightarrow (\forall y)(\exists z)(\forall u)(u \in z \leftrightarrow (\exists v)((v, z) \in X \wedge v \in y))).$$

As it has been assumed, the schema ZF_4^{φ} has turned into a single axiom. Let us here remark that the axiom schemata of replacement of ZF (see **I.3.2.5**) is also corresponded to by a single axiom, the axiom of replacement. This axiom states that for any set x and any class Y there is a set which consists of the elements common for both x and Y, i.e.,

$$(\forall x)(\forall Y)(\exists z)(\forall u)\ (u \in z \leftrightarrow u \in x \wedge u \in Y).$$

This axiom is weaker than the axiom of replacement (deduced from NBG_6 and theorem **1.3.14** to be proved below), but is in many cases more convenient for practical purposes.

The group of axioms to follow, NBG_6-NBG_{13}, is designated for the formation of classes. These axioms state that for certain properties expressed by formulas there exist classes of all sets possessing the required properties. In this case uniqueness results, as usual, from the axiom of extensionality, NBG_1.

1.3.5. The axiom of $\in$-relation, NBG_7: *there is a class which consists exactly of those ordered pairs of sets whose first component is an element of the second one*:

$$(\exists X)(\forall y)(\forall z)\,((y,z)\in X \leftrightarrow y\in z).$$

1.3.6. The axiom of intersection, NBG_8: *for any two classes there is their intersection*:

$$(\forall X)(\forall Y)(\exists Z)(\forall u)\,(u\in Z \leftrightarrow u\in X \wedge u\in Y).$$

1.3.7. The axiom of complementation, NBG_9: *for any class there exists a class complementing it*:

$$(\forall X)(\exists Y)(\forall u)\,(u\in Y \leftrightarrow u\notin X).$$

This implies the existence of the universal class $\mathbf{U} := \overline{\varnothing}$ which is the complement of the empty class $\varnothing$.

1.3.8. The axiom of domain, NBG_{10}: *for every class X of ordered pairs there is a class $Y := \operatorname{dom} X$, whose elements are exactly the first elements of the class X*:

$$(\forall X)(\exists Y)(\forall u)\,(u\in Y \leftrightarrow (\exists v)\,((u,v)\in X)).$$

1.3.9. The axiom of the Cartesian product, NBG_{11}: *for every class X there is a class $Y := X\times\mathbf{U}$ consisting of all possible ordered pairs whose first components are elements of the class X*:

$$(\forall X)(\exists Y)(\forall u)(\forall v)((u,v)\in Y \leftrightarrow u\in X).$$

1.3.10. The axioms of permutation, NBG_{12} and NBG_{13}. Let $\sigma := (i_1,i_2,i_3)$ be permutations of the set $\{1,2,3\}$. The class Y will be called the *σ-conjugation* of the class X provided $(x_1,x_2,x_3)\in Y$ iff $(x_{i_1},x_{i_2},x_{i_3})\in X$. *For any class X there is its* $(2,3,1)$- *and* $(1,3,2)$ *-conjugations*:

$$(\forall X)(\exists Y)(\forall u)(\forall v)(\forall \omega)((u,v,\omega)\in Y \leftrightarrow (v,\omega,u)\in X);$$
$$(\forall X)(\exists Y)(\forall u)(\forall v)(\forall \omega)((u,v,\omega)\in Y \leftrightarrow (u,\omega,v)\in X).$$

1.3.11. The axiom of foundation, NBG_{14}: *in every nonempty class there is an element having no common elements with it*:

$$(\forall X)(X\neq\varnothing \rightarrow (\exists y)(y\in X \wedge y\cap X=\varnothing)).$$

1.3.12. The axiom of choice, NBG_{15}: *for every class X there is a selecting function, i.e., a single-valued class assigning an elemeny of X to each nonempty set of X*:

$$(\forall X)(\exists Y)(\forall u)(u\neq\varnothing \wedge u\in X \rightarrow (\exists! v)(v\in u \wedge (u,v)\in Y)).$$

This is a very strong form of the axiom of choice. It is equivalent to the existence of a simultaneous choice of one element from every nonempty set.

The above axiom makes the list of the axioms of NBG complete. The system NBG, unlike the theory ZFC, is seen to have a finite number of axioms. Another convenient feature of NBG is that in fact it treats sets and properties of sets as formal objects, thus implementing objectivization inaccessible to the expressive means of the language of ZFC.

1.3.13. From the group of the axioms on the formation of classes let us deduce some statements to be used to prove general theorems on existence of classes.

(1) *For any class there is its* $(2,1)$*-conjugation*:

$$(\forall X)(\exists Z)(\forall u)(\forall v)((u,v)\in Z \leftrightarrow (v,u)\in X).$$

◁ The axiom of the Cartesian product guarantees existence for the class $X \times \mathbf{U}$. If we consecutively apply the axioms of the (2,3,1)-conjugation and (1,3,2) -conjugation to the class $X \times \mathbf{U}$, we get a class Y of all triplets (v,u,ω) such that $(v,u) \in X$. Now we make use of the axiom of domain to see that $Z := \operatorname{dom}(Y)$ is the sought class. ▷

(2) *For any two classes, there is their Cartesian product:*

$$(\forall X)(\forall Y)(\exists Z)(\forall \omega)\,(\omega \in Z \leftrightarrow (\exists u \in X)(\exists v \in Y)(\omega = (u,v))).$$

◁ To prove the above statement we shall consecutively use the axiom of the Cartesian product, statement **(1)**, the axiom of intersection, and set $Z := (\mathbf{U} \times Y) \cap (X \times \mathbf{U})$. ▷

By virtue of **1.3.13** (2) for $n \geq 2$ determined is the class $\mathbf{U}^n$ of all ordered tuples.

(3) *For any class X there is a class $Z := (\mathbf{U}^n \times \mathbf{U}^m) \cap (X \times \mathbf{U}^m)$:*

$$(\forall X)(\exists Z)(\forall x_1)\ldots(\forall x_n)(\forall y_1)\ldots(\forall y_m)$$
$$((x_1,\ldots,x_n,y_1,\ldots,y_m) \in Z \leftrightarrow (x_1,\ldots,x_n) \in X).$$

(4) *For any class X there is a class $Z := (\mathbf{U}^m \times \mathbf{U}^n) \cap (\mathbf{U}^m \times X)$:*

$$(\forall X)(\exists Z)(\forall x_1)\ldots(\forall x_n)(\forall y_1)\ldots(\forall y_m)$$
$$((y_1,\ldots,y_m,x_1,\ldots,x_n) \in Z \leftrightarrow (x_1,\ldots,x_n) \in X).$$

◁ In order to prove **(3)** and **(4)** we should apply the axiom of the Cartesian product and the axiom of intersection. ▷

(5) *For any class X there is a class Z such that*

$$(\forall x_1)\ldots(\forall x_n)(\forall y_1)\ldots(\forall y_m)$$
$$((x_1,\ldots,x_{n-1},y_1,\ldots,y_m,x_n) \in Z \leftrightarrow (x_1,\ldots,x_n) \in X).$$

◁ Use should be made of the axiom of permutation and of the axiom of the Cartesian product. ▷.

1.3.14. Theorem. *Let φ be a formula in the construction of which only variables of $X_1,\ldots,X_n$, $Y_1,\ldots,Y_m$ occur, and which is predicative, i.e., all bound variables of φ are restricted to sets. Then in* NBG *the following statement is provable:*

$$(\forall Y_1)\dots(\forall Y_m)(\exists Z)(\forall x_1)\dots(\forall x_n)$$
$$((x_1,\dots,x_n)\in Z \leftrightarrow \varphi(x_1,\dots,x_n,Y_1,\dots,Y_m)).$$

◁ Let the formula φ be written, with the adopted abbreviations taken into account, in such a way that the only bound variables of φ are those for sets. It suffices to consider only φ containing no subformulas of the type $Y\in W$ and $X\in X$, since the latter are replaced with equivalent ones, $(\exists x)(x=Y\wedge x\in W)$ and $(\exists u)(u=X\wedge u\in X)$. Moreover, the symbol of equality can be eliminated from φ by substituting, in line with the axiom of extensionality, the expression $(\forall u)(u\in X\leftrightarrow u\in Y)$ for $X=Y$. The proof is carried out by induction on length k of the formula φ, i.e., by the number k of logical connectives and quantifiers belonging to φ.

For $k=0$ the formula φ is atomic and has the form $x_i\in x_j$, or $x_j\in x_i$, or $x_i\in Y_l$ $(i<j\le n, l<m)$. If $\varphi:=x_i\in x_j$, then, according to the axiom of $\in$-relation, there is a class W_1 for which

$$(\forall x_i)(\forall x_j)((x_i,x_j)\in W_1 \leftrightarrow x_i\in x_j).$$

If $\varphi:=x_j\in x_i$, then we first, using the same axiom, find a class W_2 with the property

$$(\forall x_i)(\forall x_j)((x_j,x_i)\in W_2 \leftrightarrow x_j\in x_i),$$

and then apply **1.3.13** (1). As a result, we choose a class W_3 for which

$$(\forall x_i)(\forall x_j)((x_i,x_j)\in W_3 \leftrightarrow x_j\in x_i).$$

Hence, in any of these two cases there is a class W such that the following formula is valid:

$$\Phi: (\forall x_i)(\forall x_j)((x_i,x_j)\in W \leftrightarrow \varphi(x_1,\dots,x_n,Y_1,\dots,Y_m)).$$

By virtue of **1.3.13** (4), in the formula Φ we can replace the subformula $(x_i,x_j)\in W$ with $(x_1,\dots,x_{i-1},x_i)\in Z_1$ for a certain other class Z_1, and add the quantifiers $(\forall x_1)\dots(\forall x_{i-1})$ in the beginning of the formula. Let Ψ be the formula obtained in this case. According to **1.3.13** (5), in the formula Ψ for a certain other class Z_2 it is possible to write $(x_1,\dots,x_i,x_{i+1},\dots,x_j)\in Z_2$ instead of the subformula $(x_1,\dots,x_{i-1},x_i,x_j)\in Z_1$ and to add the quantifiers $(\forall x_{i+1})\dots(\forall x_{j-1})$ at the beginning of the formula Ψ. And, finally, applying to Z_2 **1.3.13** (3), we find a class Z for which the following formula is true:

$$(\forall x_1)\ldots(\forall x_n)((x_1,\ldots,x_n)\in Z \leftrightarrow \varphi(x_1,\ldots,x_n, Y_1,\ldots,Y_m)).$$

For the remaining case, $x_i \in Y_l$ the sought requirement follows from the existence of the Cartesian products $W := \mathbf{U}^{i-1} \times Y_l$ and $Z := W \times \mathbf{U}^{n-i}$. Therefore, the theorem has been proved for $k = 0$.

Let us assume that for all $k < p$ the theorem has been proved and the formula φ has p logical connectives and quantifiers. It suffices to consider the cases when φ is obtainable from some formulas using negation, implications and the universal quantifier.

(a) $\varphi := \neg\psi$. By the induction hypothesis, there is a class V such that

$$(\forall x_1)\ldots(\forall x_n)((x_1,\ldots,x_n)\in V \leftrightarrow \psi(x_1,\ldots,x_n, Y_1,\ldots,Y_m)).$$

In accordance with the axiom of complement, there is a class $Z := \mathbf{U} - V := \mathbf{U} \setminus V$ obeying the required conditions.

(b) $\varphi := \psi \to \theta$. Again, by the induction hypothesis, there are classes V and W, such that for V and ψ all the statements of **(a)** are valid and, moreover,

$$(\forall x_1)\ldots(\forall x_n)((x_1,\ldots,x_n)\in W \leftrightarrow \theta(x_1,\ldots,x_n, Y_1,\ldots,Y_m)).$$

The sought class, $Z := \mathbf{U} - (V \cap (\mathbf{U} - W))$ does exist by virtue of the axiom of intersection and that of complementation.

(c) $\varphi := (\forall x)\psi$. Let V and ψ be the same as in **(a)**. If we apply the axiom of the domain to the class $X := \mathbf{U} - V$, then we obtain the class Z_1 for which

$$(\forall x_1)\ldots(\forall x_n)((x_1,\ldots,x_n)\in Z \leftrightarrow (\exists x)\ \neg\psi(x_1,\ldots,x_n, Y_1,\ldots,Y_m)).$$

The class $Z := \mathbf{U} - Z_1$ that exists by virtue of the axiom of complementation is the sought one, since $(\forall x)\psi$ is equivalent to $\neg(\exists x)(\neg\psi)$. ▷

1.3.15. Each of the axioms for the formation of classes, NBG_7-NBG_{13}, is a corollary to theorem **1.3.14**, provided the formula φ is chosen in an appropriate way. On the other hand, the theorem itself is seen from the proof to be deduced from the axioms of the formation of classes. It is remarkable that instead of an infinite number of statements in **1.3.14** we can manage with a finite number of the axioms NBG_7-NBG_{13}.

Theorem **1.3.14** allows one to prove the existence of various classes. Thus, for any class Y there exists class of all its subsets $\mathcal{P}(Y)$, as well as the union of all elements of the class $\cup(Y)$ determined by conventional formulas

$$(\forall u)(u \in \mathcal{P}(Y) \leftrightarrow u \subset Y),$$
$$(\forall u)(u \in \cup(Y) \leftrightarrow (\exists v)(v \in Y \wedge u \in v)).$$

The above statements can be easily proved if we choose $\varphi(X,Y) := X \subset Y$ and $\varphi(X,Y) := (\exists u)(x \in u \wedge u \in Y)$. By analogous considerations sound are the definitions of Z^{-1}, $\operatorname{im} Z$, $Z \upharpoonright Y$, $Z''Y$, $X \cup Y$, etc., where X, Y and Z are some classes.

1.3.16. Theorem. *Every theorem of* ZFC *is a theorem of* NBG.

◁ All the axioms of ZFC are theorems of NBG. Let us prove the only not obvious part of this statement which concerns the axiom of replacement ZF_4^{φ}. Let the formula φ contain no free occurrences of the variable y, and let $\{x,t,z_1,\dots,z_m\}$ be complete set of variables used in constructing φ. Let us then assume that, for all $x,u,v,z_1,\dots,z_m$, the following relation holds:

$$\varphi(x,u,z_1,\dots,z_m) \wedge \varphi(x,v,z_1,\dots,z_m) \rightarrow u = v.$$

The formula φ is predicative as all the variables in it are restricted to sets. By theorem **1.3.14**, there is a class Z such that

$$(\forall x)(\forall u)((x,u) \in Z \leftrightarrow \varphi(x,u,z_1,\dots,z_m)).$$

This property of φ shows that the class Z is single-valued, i.e., that $\operatorname{Un}(Z)$ is provable within NBG. By virtue of the axiom of replacement, NBG_6, there is a set y for which

$$(\forall v)(v \in y \leftrightarrow (\exists u)((u,v) \in \mathbf{Z} \wedge u \in x)).$$

Obviously, for y the required relation

$$(\forall z_1)\dots(\forall z_m)(\forall v)(v \in y \leftrightarrow (\exists u \in x)\,\varphi(x,v,z_1,\dots,z_m))$$

is fulfilled. ▷

1.3.17. Theorem. *Every theorem of* NBG *dealing with sets is a theorem of* ZFC.

◁ The proof can be, for instance, found in [28]. It requires the use of some facts of model theory which go beyond the framework of the present book.

Statements **1.3.16** and **1.3.7** are often formulated in the following form.

1.3.18. Theorem. *Von Neumann-Gödel-Bernays set theory* NBG *is conservative over Zermelo-Fraenkel set theory* ZFC.

1.3.19. Among the other axiomatic set theories we should mention the Bernays-Morse theory that extends the theory NBG. The theory in question has special axioms, NBG_1-NBG_5, NBG_{14}, and the following schemata of the axiom of comprehension:

$$(\exists X)(\forall Y)\ (Y \in X \leftrightarrow M(Y) \wedge \varphi(Y, X_1, \dots, X_n)),$$

where φ is an arbitrary formula containing no occurrences of the variable X.

It is obvious from **1.3.14** that if in the formula φ all the domains of the definition of quantifiers is restricted to sets, then the axiom schema of comprehension is a theorem of NBG. The Bernays-Morse set theory allows quantification over arbitrary classes in the schemata of axioms of comprehension. This theory can be also modified by the axiom of choice, NBG_{15}.

1.4. Ordinals.

The concept of an ordinal is the key one in studying infinite sets. It is designated for transfinite iteration of various mathematical constructions, or considerations, as well as for measuring cardinality. The topic of the present section is to consider how to do it.

1.4.1. Let us consider classes X and Y. We say that$X$ is an *order relation*, or simply an *order* on Y provided X is an antisymmetric, reflexive and transitive relation on Y. The antiasymmetry, reflexivity and transitivity of a relation are written in the same way as in the language of ZFC (see **I.3.1.10**). The order of X on Y is called *linear* or *total* if $Y \times Y \subset X \cup X^{-1}$. The relation X is said *to well-order* Y or *to be well-ordering*, or Y is said *to be well-ordered* (sometimes an abusive term *totally-ordered* is used, too), provided that X is an order on Y and any nonempty subclass of the class Y has a least element (relative to X). Classes X_1 andX_2 ordered by the relations R_1 and R_2, respectively, are termed ***similar*** or ***equipotent*** if there is exists a bijection h from X_1 on X_2 such that $(x,y) \in R_1 \leftrightarrow (h(x),h(y)) \in R_2$ for all $x, y \in X_1$.

1.4.2. Let us introduce the relation E by the formula

$$(x,y) \in E \leftrightarrow (x \in y) \vee x = y.$$

The class E does exist by virtue of the axiom of $\in$-relation, NBG_7, and theorem **1.3.14**. As is seen, E is an order relation on the universal class $\mathbf{U}$.

The class X is called *transitive* (not to be mixed up with a transitive relation!) if each of its elements is also its subset:

$$\mathrm{Tr}(X) := (\forall y)(y \in X \rightarrow y \subset X).$$

An ordinal class is the name of any transitive class well-ordered by the relation E. The record $\mathrm{Ord}(X)$ implies that X is ordinal. The ordinal class that is a set is termed *an ordinal* (or *an ordinal number*, or *a transfinite number*). The class of all ordinals is denoted by On. It should be recalled that ordinals are, as a rule, denoted by lowercase Greek letters, the following abbreviations used in this case:

$$\alpha < \beta := \alpha \in \beta, \quad \alpha \leq \beta := (\alpha \in \beta) \vee (\alpha = \beta), \quad \alpha + 1 := \alpha \cup \{\alpha\}.$$

If $\alpha < \beta$, then α is said to proceed β, while β is said to succeed α. Using the axiom of foundation, NBG_{14}, we can easily prove the following statement.

1.4.3. *A class is ordinal iff it is transitive and well-ordered by the relation E.*

⊲ Let a transitive class X be totally ordered by the relation E. Choose a nonempty subclass $Y \subset X$ and show that Y has a least element. There is at least one element $y \in Y$. If $y = 0$, then y is the sought least element in Y. If $y \neq 0$, then, according to the axiom of foundation, we can find an element $x \in y$ such that $x \cap y = 0$. In this case x is the least element of the set y, since y is totally ordered. As the class Y is totally ordered by the relation E, the element x will be the least in the class Y as well. Hence, X is an ordinal class. The sufficiency of the given condition is thus proved, its necessity being obvious. ⊳

Therefore, both in NBG and ZFC use can be made of a simpler definition of ordinal:

$$\mathrm{Ord}(X) \leftrightarrow \mathrm{Tr}(X) \wedge (\forall u \in X)(\forall v \in X)\,(u \in v \vee u = v \vee v \in u).$$

It would be expedient to emphasize that the equivalence of the definitions of ordinal considered can also be established without the axiom of choice.

The majority of the properties of ordinals given below can be deduced without employing the axiom of foundation, making use only of the initial definition of ordinal. This peculiarity which is important to, for instance, substantiate consistency of the axiom of foundation with the remaining axioms of ZF, is insignificant for our further purposes.

1.4.4. Below we shall require some additional facts. Let X and Y be arbitrary classes.

(1) *If X is ordinal, Y is transitive, and $X \neq Y$, then the relations $Y \subset X$ and $Y \in X$ are equivalent.*

◁ For $Y \in X$ the class Y is a set, and $Y \subset X$ since X is transitive. Let us, in turn, assume that $Y \subset X$. Since $X \neq Y$, then $Z := X - Y \neq \varnothing$. The class Z has the least element, $x \in Z$ (in the sense of the order relation E). This implies that $x \cap Z = \varnothing$ or $x \subset Y$. Besides, $x \subset X$, since $x \in X$ and X is transitive. Let us choose an element $y \in Y$. Since X is linearly ordered, then $x \in y$ or $x = y$, or, finally, $y \in x$. With transitivity of Y taken into account, the first two relations yield $x \in Y$, which contradicts the membership $x \in Z$. Therefore, $y \in x$ and, hence, $Y \subset x$. Taking into consideration the inclusion $x \subset Y$ proved above, we get $x = Y$ and, finally, $x = Y \wedge x \in X \rightarrow Y \in X$. ▷.

(2) *The intersection of any two ordinal classes is an ordinal class.*

◁ This is obvious. ▷

(3) *If X and Y are ordinal classes, then*

$$X \in Y \vee X = Y \vee Y \in X.$$

◁ Let the intersection $X \cap Y = Z$ coincide with none of the classes X and Y. Then, according to **(1)** and **(2)**, $Z \in X$ and $Z \in Y$, i.e., $Z \in X \cap Y = Z$. For the set $Z \in X$, however, the relation $Z \in Z$ is impossible. Hence, either $Z = X$ and then $Y \subset X$, or $Z = Y$ and then $X \subset Y$. Now we have to refer to **(1)**. ▷

1.4.5. Theorem. *The following statements are valid:*

(1) *only ordinals can be elements of any ordinal class;*

(2) *the class* On *is the only ordinal class that is not an ordinal;*

(3) *for any ordinal* α, *the set* $\alpha+1$ *is an ordinal, and the least one of all the ordinals succeeding* α;

(4) *whatever the nonempty class of ordinals* $X \subset \mathrm{On}$ *might be, the union* $\cup(X)$ *is an upper bound of the set* X *in the ordered class* On .

$\triangleleft$ (1) Let us choose an ordinal class X and an element $x \in X$. Since X is transitive, we have $x \subset X$ and, hence, x is linearly ordered by the relation E. Let us prove $\mathrm{Tr}(x)$. If $z \in y \in x$, then $z \in X$ since X is transitive. Of three possible cases, $z = x$, $x \in z$ and $z \in x$, the first two result in closed cycles, $z \in y \in z$ and $z \in y \in x \in z$, respectively, which contradict the axiom of foundation. Therefore, $z \in x$ and, hence, $z \in y \rightarrow z \in x$, i.e., $y \subset x$, which proves $\mathrm{Tr}(x)$ and, at the same time, $\mathrm{Ord}(x)$.

(2) By **1.4.4** (3), the class On is linearly ordered, by **(1)**, it is transitive, and, hence, $\mathrm{Ord}(\mathrm{On})$. If On is a set, then On is an ordinal, which results in a contradiction, $\mathrm{On} \in \mathrm{On}$. Hence, On is an ordinal class but not an ordinal. For an arbitrary ordinal class X $X \notin \mathrm{On}$ yields $X = \mathrm{On}$. Indeed, statement **1.4.4** (3) opens only one possibility, $\mathrm{On} \in X$ which contradicts the fact that On is a proper class.

(3) If α is an ordinal, then, obviously, the set $\alpha + 1 := \alpha \cup \{\alpha\}$ is linearly ordered. For $x \in \alpha + 1$ we get either $x \in \alpha$ or $x = \alpha$, and in both cases $x \subset \alpha$. However, $\alpha \subset \alpha + 1$ and, hence, $x \subset \alpha + 1$, which proves that $\alpha + 1$ is transitive. And, finally, $\alpha + 1$ is an ordinal and $\alpha < \alpha + 1$. If $\alpha < \beta$ for a certain ordinal β, then $\alpha \in \beta$ and $\alpha \subset \beta$, i.e., $\alpha \cup \{\alpha\} \subset \beta$. By **1.4.4** (1), we have either $\alpha \cup \{\alpha\} \in \beta$ or $\alpha \cup \{\alpha\} = \beta$. Hence, $\alpha + 1 \leq \beta$.

(4) Let us assume $X \subset \mathrm{On}$ and $y \in Y := \cup X$, and choose such an element $x \in X$ that $y \in x$. Since x is an ordinal, then $y \subset x$ and, moreover, $y \subset Y$. As the class On is transitive (see **(2)**), $x \in X$ yields $x \subset \mathrm{On}$ and, hence $Y \subset \mathrm{On}$. Therefore, Y is a transitive subclass On and, hence, Y is an ordinal. If $\alpha \in X$, then $\alpha \subset Y$ and, in accord with **1.4.4** (1), $\alpha \leq Y$. While if β is an ordinal and $\beta \geq \alpha$ for all $\alpha \in X$, then $Y \subset \beta$ and, again by **1.4.4** (1), $Y \leq \beta$. Hence, $Y = \sup X$. $\triangleright$

1.4.6. The least upper bound of a set of ordinals x is usually denoted by $\lim(x)$. The ordinal α is called *limiting* if $\alpha \neq \varnothing$ and $\lim(\alpha) = \alpha$. The term *'limit ordinal'* is also employed. In equivalent words, α is a limiting ordinal if it is not presentable in the form $\alpha = \beta + 1$ with a $\beta \in \mathrm{On}$. The class of all limiting ordinals is designated by the symbol K_{II}. The ordinals not belonging to K_{II} form a class of nonlimiting ordinals $K_{\mathrm{I}} := \mathrm{On} - K_{\mathrm{II}} = \{\alpha \in \mathrm{On}\ (\exists \beta \in \mathrm{On})(\alpha = \beta + 1)\}$. Let us denote by the letter ω the least limiting ordinal (whose existence is ensured by theorem **1.4.5** and the axiom of infinity). We can show that ω coincides with the class of nonlimiting ordinals α such that each predecessor of α is also nonlimiting:

$$\omega = \{\alpha \in \mathrm{On}\colon \ \alpha \cup \{\alpha\} \in K_{\mathrm{I}}\}.$$

The ω elements are called *finite ordinals*, or *natural numbers*, or *positive integers*. The least ordinal, the zero set $0{:=}\varnothing$, is contained in ω. The successor $1{:=}\ 0+1 = 0 \cup \{0\} = \{\varnothing\}$ contains the only element 0. Then, $2{:} = 1 \cup \{1\} = \{0\} \cup \{1\} = \{0, 1\} = \{0,\{0\}\}$, $3{:} = 2 \cup \{2\} = \{0, \{0\}, \{\{0, \{0\}\}\}$, etc.. Therefore,

$$\omega{:} = \{0, \{0\}, \{0, \{0\}\}, \ldots\} = \{0,1,2,\ldots\}.$$

The following notation is also used:

$$\mathbf{N}{:} = \omega - \{0\} = \{1,2,\ldots\}.$$

The next statement enumerates the basic properties of a set of natural numbers ω known as a whole as *the system of Peano axioms.*

1.4.7. Theorem. *The following statements are valid*:

(1) *zero is a natural number, i.e.,* $0 \in \omega$ *;*

(2) *for every natural number* $\alpha \in \omega$ *the successor* $\alpha + 1$ *is also a natural number*;

(3) $0 \neq \alpha + 1$ *for any natural number* α ;

(4) *for natural numbers* α *and* β $\alpha + 1 = \beta + 1$ *implies* $\alpha = \beta$;

(5) *if the class* X *contains an empty set and is such that for every ordinal, its successor is also in* X*, then* $\omega \subset X$.

1.4.8. Theorem (the principle of transfinite induction). *Let* G *be a certain class with the following properties:* (1) $0 \in X$; (2) *if* α *is an ordinal and* $\alpha \in X$, *then* $\alpha + 1 \in X$; (3) *if* x *is a set of ordinals contained in* X, *then* $\lim (x) \in X$. *Then* $\mathrm{On} \subset X$.

$\lhd$ Let us assume $\mathrm{On} \not\subset X$. Then the nonempty subclass $\mathrm{On} - X$ of the well-ordered class On has the least element $\alpha \in \mathrm{On} - X$, in which case this implies that $\alpha \cap (\mathrm{On} - X) = 0$ or $\alpha \subset X$ and $\alpha \neq 0$ (see **(1)**). If $\alpha \in K_{\mathrm{I}}$, i.e., $\alpha = \beta + 1$ for a certain $\beta \in \mathrm{On}$, then

$\beta \in \alpha \subset X \to \beta \in X$ and, by condition (2), $\alpha = \beta + 1 \in X$. While if $\alpha \in K_{\text{II}}$, then, by condition (3), we deduce $\alpha = \lim(\alpha) \in X$. In both cases we have $\alpha \in X$, which contradicts the inclusion $\alpha \in \text{On} - X$. ▷

1.4.9. Theorem (the principle of transfinite recursion). *Let G be a certain class which is a function. Then there is a unique function F for which*

(1) $\operatorname{dom}(F) = \text{On}$;

(2) $F(\alpha) = (F{\restriction}\alpha)$ *for any* $\alpha \in \text{On}$, *where* $F{\restriction}\alpha := F \cap (\alpha \times \text{U})$ *is the restriction of F to α.*

◁ Let us determine the class Y by the relation

$$f \in Y \leftrightarrow \text{Func}(f) \wedge \operatorname{dom}(f) \in \text{On} \wedge (\forall \alpha \in \operatorname{dom}(f))(f(\alpha) = G(f{\restriction}\alpha)).$$

If $f, g \in Y$, then either $f \subset g$ or $g \subset f$. Indeed, if $\beta := \operatorname{dom}(f)$ and $\gamma := \operatorname{dom}(g)$, then either $\beta \le \gamma$ or $\gamma \le \beta$. Assuming, for instance, $\gamma < \beta$, let us set $z := \{\alpha \in \text{On}: \alpha < \gamma \wedge f(\alpha) \ne g(\alpha)\}$. If $z \ne 0$, then there is the least element $\delta \in z$. In this case for all $\alpha < \delta$ we get $f(\alpha) = g(\alpha)$, i.e., $f{\restriction}\delta = g{\restriction}\delta$. By the definition of the class Y, however, we also have $f(\delta) = G(f{\restriction}\delta)$ and $g(\delta) = G(g{\restriction}\delta)$, and, hence, $f(\delta) = g(\delta)$ and $\delta \notin z$. This contradicts the choice of δ and, thus, $z = 0$, i.e., $f(\alpha) = g(\alpha)$ for all $\alpha < \gamma$, which yields the required inclusion $g \subset f$. Let us set $F = \cup Y$. Obviously, F is a function, $\operatorname{dom}(F) \subset \text{On}$ and $F(\alpha) = G(F{\restriction}\alpha)$ for all $\alpha \in \operatorname{dom}(F)$. If $\alpha \in \operatorname{dom}(F)$, then $\alpha, G(F{\restriction}\alpha) \in f$ for a certain $f \in Y$. Then $\alpha \in \beta := \operatorname{dom}(f) \subset \operatorname{dom}(F)$ and, since β is transitive, we get $\alpha \subset \operatorname{dom} F$. Therefore, the class $\operatorname{dom} F$ is transitive and, according to **1.4.4** (1), we have either $\operatorname{dom} F = \text{On}$ or $\operatorname{dom} F \in \text{On}$, the latter being impossible. Indeed, it follows from $\delta := \operatorname{dom} F \in \text{On}$ that the function $f := F \cup \{(\delta, G(F))\}$ is included in Y, and, hence, $f \subset F$, which results in a contradiction, $f \subset F \to \operatorname{dom}(F) \subset \operatorname{dom}(F) \to \delta \in \operatorname{dom}(F) = \delta$. ▷

1.4.10. A binary relation R is termed *well-founded* if for any $x \in \text{V}$ the class $R^{-1}(x)$ is a set and for any nonempty $x \in \text{V}$ there is an element $y \in x$ such that $x \cap R^{-1}(y) = 0$. The last condition (assuming the axiom of choice) is equivalent to the fact that there is no infinite sequence (x_n) with the property $x_n \in R(x_{n+1})$ for all $n \in \omega$. An example of a well-founded relation is the relation $\in$. It is often more convenient to apply the principles of transfinite induction and recursion in the following form.

1.4.11. Theorem. *Let R be a well-founded relation. Then the following statements are valid*:

(1) (induction on R) *if the class X is such that for any $x \in \mathbf{V}$ the relation $R^{-1}(x) \subset X$ implies $x \in X$, then $X = \mathbf{V}$*;

(2) (recursion on R) *for any function $C: \mathbf{V} \to \mathbf{V}$ there is such a function F that* $\mathrm{dom}(F) = \mathbf{V}$ *and $F(x) = G(F|R^{-1}(x))$ for all $x \in \mathbf{V}$*.

1.4.12. Two sets are said to be *equipotent* (or *of the same cardinality*) if there is a one-to-one mapping of one onto the other. The ordinal which is equipotent to no preceding ordinal is termed a *cardinal*, any natural number being a cardinal. The cardinal that is not a natural number is called *infinite*. Therefore, ω is the least infinite cardinal. For any ordinal α by the symbol ω_α we shall denote the infinite cardinal for which the ordered set of all infinite cardinals less than ω_α is similar to α. If such a cardinal exists, then it is unique.

1.4.13. Theorem (the principle of measuring cardinality). *The following statements are valid*:

(1) *infinite cardinals form a well-ordered proper class*;

(2) *for any ordinal α there is a cardinal ω_α, in which case the mapping $\alpha \to \omega_\alpha$ is a similarity of the class of ordinals and that of infinite cardinals*;

(3) *there is a mapping $|\cdot|$ from the universal class $\mathbf{V}$ on the class of all cardinals such that the sets x and $|x|$ are equipotent for any $x \in \mathbf{V}$*.

⊲ The proof can, for instance, be found in [180]. ⊳

The cardinal $|x|$ is called *the cardinality* or *cardinal number of the set x*. Hence, any set is equipotent to a unique cardinal, namely, to its cardinal number. The set x is *countable* provided $|x| = \omega_0 := \omega$, and it is *at most countable* if $|x| \leq \omega_0$.

1.4.14. For an arbitrary ordinal α by the symbol $\mathbf{2}^{\omega_\alpha}$ we shall denote the cardinality of the set $\mathcal{P}(\omega_\alpha)$, i.e., $\mathbf{2}^{\omega_\alpha} = |\mathcal{P}(\omega_\alpha)|$. Such a denotation is justified by the fact that $\mathbf{2}^x$ and $\mathcal{P}(x)$ are equipotent for any x where $\mathbf{2}^x$ is the class of all mappings from x to $\mathbf{2}$. A theorem proved by Cantor states that $|x| < |\mathbf{2}^x|$ whatever the set x is. In particular, $\omega_\alpha < 2^{\omega_\alpha}$ for any ordinal α. In this case, by theorem **1.4.13**, we get $\omega_{\alpha+1} \leq 2^{\omega_\alpha}$. Whether there are intermediate powers between $\omega_{\alpha+1}$ and 2^{ω_α} or not, i.e., whether the equality $\omega_{\alpha+1} = 2^{\omega_\alpha}$ holds or not, that is the contents of the generalized problem of the continuum. For $\alpha = 0$ this is a classical *problem of the continuum.* The *continuum hypothesis,* CH (generalized continuum hypothesis, GCH) is the equality $\omega_1 = 2^\omega$ (or, respectively, the equality $\omega_{\alpha+1} = 2^{\omega_\alpha}$).

1.4.15. In the class $\mathrm{On} \times \mathrm{On}$ let us introduce an order which will be called *canonical.* Let us consider $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathrm{On}$. Let us also assume that for $(\alpha_1, \alpha_2) \leq (\beta_1, \beta_2)$ one of the following conditions is fulfilled:

(1) $\alpha_1 = \beta_1$ and $\alpha_2 = \beta_2$;

(2) $\sup\{\alpha_2, \alpha_2\} < \sup\{\beta_1, \beta_2\}$;

(3) $\sup\{\alpha_2, \alpha_2\} = \sup\{\beta_1, \beta_2\}$ and $\alpha_1 < \beta_1$;

(4) $\sup\{\alpha_2, \alpha_2\} = \sup\{\beta_1, \beta_2\}$ and $\alpha_1 = \beta_1$ and $\alpha_2 < \beta_2$.

Therefore, the pairs (α, β) are compared relative to $\sup\{\alpha, \beta\}$, while the set of ordered pairs (α, β) with the same $\sup\{\alpha, \beta\}$ has the lexicographic order. We can easily prove that the class $\mathrm{On} \times \mathrm{On}$ with the canonical order is a well-ordered class. In an analogous way it can be checked that the class $\mathrm{On} \times \mathrm{On} \times \mathrm{On}$ is canonically well-ordered, etc..

1.4.16. Remarks

(1) The idea of transfinite iteration is one of the most fundamental and original discoveries made by G.Cantor. Based on it, he created a powerful method of the qualitative analysis of the notion of infinity and penetrated into the essence of the infinite. The notion of infinity can be traced in religious and philosophical doctrines since the ancient times. The whole totality of the ideas on the infinite was, however, preferably a humanitarian subject before G.Cantor, who made the very notion of the infinite a subject for mathematical investigation.

(2) The problem of the continuum stems from G.Cantor and was the first to be formulated in the famous report by D.Hilbert. Remaining unsolved for decades, this problem gave rise to in-depth studies of the fundamentals of set theory. In 1939 K.Gödel established consistency of the generalized continuum hypothesis with ZFC [60]. In 1963 P.J.Cohen proved that the negation of the generalized continuum hypothesis is also consistent with ZFC. Both these results brought about new ideas, methods and problems.

(3) Following G.Cantor, an ordinal is the order-type of a certain well-ordered set x; i.e., the class of all ordered sets similar to x. All the order-types, however, except for that of the empty set, are proper classes. This peculiarity makes it impossible to develop the theory of order-types (within NBG) since it is impossible to consider the class of order-types. The definition of ordinal **1.4.2**, by J.von Neumann, singles a canonical representative out of every order-type.

(4) In this section we have given only the basic facts on ordinals, a more detailed information can be found in [112, 180].

1.5. Hierarchies of Sets

The recursive definitions based on theorem **1.4.9** or its modifications give, in particular, decreasingly (or increasingly) nested transfinite sequences of sets called cumulative hierarchies. Of a special interest for us are the hierarchies resulting in the models of set theory.

1.5.1. Let us consider a certain set x_0 and two single-valued classes Q and R. Starting with them, let us construct a new single-valued class G. Let us first set $G(0) := x_0$. Then, if x is a function and $\operatorname{dom}(x) = \alpha + 1$ for a certain $\alpha \in \mathrm{On}$, then $G(x) := Q(x(\alpha))$, while if $\operatorname{dom}(\alpha) = \alpha$ is a limiting ordinal, then in order to obtain $G(x)$ let us first accumulate the set of the values of $x(\beta)$ for $\beta < \alpha$, and then apply R to the obtained set, i.e., $G(x) := R(\bigcup \operatorname{im}(x))$. In all the remaining cases we assume $G(x) = 0$. By theorem **1.4.9** on transfinite recursion, there exists a single-valued class F satisfying the conditions:

$$F(0) = x_0,$$
$$F(\alpha + 1) = Q(F(\alpha)),$$
$$F(\alpha) = R\left(\bigcup_{\beta<\alpha} F(\beta)\right) \quad (\alpha \in K_{\mathrm{II}}).$$

Such a function F is often called a cumulative hierarchy. The union of the elements of the class $\operatorname{im}(F)$, i.e., the class

$$\bigcup_{\alpha \in \mathrm{On}} F(\alpha) := \bigcup \operatorname{im}(F),$$

is often termed the *limit of the cumulative hierarchy* $(F(\alpha))_{\alpha \in \mathrm{On}}$.

1.5.2. Further on we shall be interested only in a special case when x_0 is the empty set, R is the identity mapping of the universal class $\mathbf{U}$, and Q is a class-function, $\mathrm{dom}(Q) = \mathbf{U}$. In this case cumulative hierarchies are constructed inductively, starting with the empty set, by successively applying the operation Q. Varying Q, we get different cumulative hierarchies.

The least ordinal $\mathbf{a}$ for which $x \in F(\mathbf{a}+1)$ is called the *(ordinal) rank of the set x relative to the hierarchy* $(F(\alpha))_{\alpha \in \mathrm{On}}$ and is denoted by $\mathrm{rank}x$. This definition is obviously determined by theorem **1.3.14**, according to which there we can find the class 'rank' obeying the condition

$$(\forall x)(\forall y)((x,y) \in \mathrm{rank} \leftrightarrow \varphi(x,y,F,\mathrm{On})),$$

where φ is a predicative formula

$$(\exists \alpha \in \mathrm{On})(y = \alpha \wedge x \in F(\alpha+1) \wedge (\forall \beta \in \mathrm{On})(x \in F(\beta+1) \rightarrow \alpha \leq \beta)).$$

In this case $\mathrm{Un}(\mathrm{rank})$, $\mathrm{dom}(\mathrm{rank}) = \cup \mathrm{im} F$ and $\mathrm{im}(\mathrm{rank}) \subset \mathrm{On}$ hold, i.e., rank is a function from $\cup \mathrm{im}(F)$ to On. The notation of the rank will not include F since we know which hierarchy is considered.

1.5.3. As the simplest, let us consider the case when $(x_0 = 0, R = I_{\mathrm{U}}) Q := \mathcal{P}_{tr}$, where $\mathcal{P}_{tr}$ put into correspondence to any $x \in \mathbf{U}$ a class $\mathcal{P}_{tr}(x)$ of all transitive subsets of the set x. As long as a transitive subset of an ordinal is an ordinal, then $Q(\alpha) = \alpha \cup \{\alpha\} = \alpha + 1$ and $F(\alpha+1) = \alpha + 1$ for every ordinal α. If α is determined, then

$$F(\alpha) = \bigcup_{\beta<\alpha} F(\beta) = \bigcup_{\beta+1<\alpha} F(\beta+1) = \bigcup_{\beta+1<\alpha} \beta+1 = \alpha.$$

Therefore, the limit of the increasinly nested cumulative hierarchy is the class of ordinals On.

1.5.4. If the role of Q is played by the operation of taking powersets $\mathcal{P}$, then we obtain a familiar (see **1.3.1**) cumulative hierarchy (we put $x_0 = 0, R = I_{\mathrm{U}}$):

$$V_0 := 0,$$
$$V_{\alpha+1} := \mathcal{P}(V_\alpha) \; (\alpha \in \mathrm{On}),$$
$$V_\alpha := \bigcup_{\beta<\alpha} V_\beta \; (\alpha \in K_{\mathrm{II}}).$$

The class $\mathbf{V} := \bigcup_{\alpha \in \mathrm{On}} V_\alpha$ is the von Neumann universe. It should be recalled that its lower levels have the form $V_1 = \mathcal{P}(0) = \{0\} = 1$, $V_2 = \mathcal{P}(1) = \{0, \{0\}\} = 2$, $V_3 = \mathcal{P}(V_2) = = \{0, \{0\}, \{\{0\}\}, \{0, \{0\}\}\} \neq 3$, etc..

1.5.5. *The following statements are valid:*

(1) V_α *is a transitive set for every* $\alpha \in \mathrm{On}$;

(2) $V_\beta \in V_\alpha$ *and* $V_\beta \subset V_\alpha$ *for any* $\alpha, \beta \in \mathrm{On}, \beta < \alpha$;

(3) *if* $x \in y \in \mathbf{N}$, *then* $\mathrm{rank}(x) < \mathrm{rank}(y)$;

(4) *the class of ordinals* On *is contained in the universe* $\mathbf{V}$;

(5) $\mathrm{rank}(\alpha) = \alpha$ *for* $\alpha \in \mathrm{On}$;

(6) *if* x *is a set and* $x \subset \mathbf{V}$, *then* $x \in \mathbf{V}$.

$\lhd$ (1) Let us proceed by transfinite induction. For $\alpha = 0$ the class $V_0 = 0$ is a transitive set. Assume that the set V_α is transitive. As $V_{\alpha+1} = \mathcal{P}(V_\alpha)$, we set that $V_{\alpha+1}$ is a set and for any x and y it follows from $x \in y \in V_{\alpha+1}$ that $y \subset V_\alpha$ and $x \in V_\alpha$. By the induction hypothesis, either $x \subset V_\alpha$ or $x \in V_{\alpha+1}$, and, hence, $y \in V_{\alpha+1}$. If $\alpha \in K_{\mathrm{II}}$ and V_β is a transitive set for all $\beta < \alpha$, then for any $x \in V_\alpha$ we get

$$(\exists \beta < \alpha)(x \in V_\beta) \to (\exists \beta < \alpha)(x \subset V_\beta) \to x \subset V_\alpha .$$

Besides, V_α is a set as the union of a set of sets.

(2) We have established in (1) that V_α is transitive. Therefore, it suffices to demonstrate that $V_\beta \in V_\alpha$ $(\beta < \alpha)$. Let us carry out transitive induction on α. For $\alpha = 1$ we have nothing to prove. Let $\alpha > 1$ and $V_\beta \in V_\alpha$ for all $\beta < \alpha$. The inequality $\beta < \alpha + 1$ holds only when $\alpha = \beta$ or $\beta < \alpha$. If $\alpha = \beta$, then

$$V_\beta = V_\alpha \in \mathcal{P}(V_\alpha) = V_{\alpha+1}.$$

If $\beta < \alpha$, then, by the induction hypothesis we have $V_\beta \in V_\alpha$, while by (1) we get $V_\alpha \subset V_{\alpha+1}$ and, hence, $V_\beta \in V_{\alpha+1}$. Now we have to remark that for $\beta < \alpha$ with the limiting ordinal $\alpha \in K_{\mathrm{II}}$ we always have $V_\beta \in V_\alpha$ since

$$V_\beta \in V_{\beta+1} \subset \bigcup_{\gamma<\alpha} V_\gamma = V_\alpha .$$

(3) It is obvious that $\alpha = \mathrm{rank}(x)$ iff $x \in V_{\alpha+1}$ and $x \notin V_\alpha$. Hence, if $x \in y$, then $y \not\subset V_\alpha$ and, therefore, $y \not\subset V_{\alpha+1}$. By definition, $\mathrm{rank}(y) > \alpha$.

(4), (5) Let us again make use of transitive induction. For $\alpha = 0$ we have $0 \in V_0 \subset \mathbf{V}$ and $\mathrm{rank}(0) = 0$, as $0 \notin V_0$. Put $\alpha \in \mathbf{V}$ and $\mathrm{rank}(\alpha) = \alpha$. Then $\alpha + 1 = \mathbf{a} \cup \{\alpha\} \subset V_{\alpha+1}$, or $\alpha + 1 \in \mathcal{P}(V_{\alpha+1}) = V_{\alpha+2}$. On the other hand, if $\alpha + 1 \in V_{\alpha+1}$, then $\alpha \cup \{\alpha\} \subset V_\alpha$ and we come to a contradiction $\alpha \in V_\alpha$. Therefore, $\alpha + 1 \notin V_{\alpha+1}$ and, hence, $(\alpha + 1) = \alpha + 1$. Let us assume that $\alpha \in K_{\mathrm{II}}$, and for all $\beta < \alpha$ we have $\beta \in \mathbf{V}$ and $\mathrm{rank}(\beta) = \beta$. In this case

$$\alpha = \{\beta \in \mathrm{On} : \beta < \alpha\} \subset \bigcup_{\beta<\alpha} V_{\beta+1} \subset V_\alpha ;$$

whence we deduce $\alpha \in V_{\alpha+1}$. Besides, the relation $\alpha \in V_\alpha$ implies that $\alpha \in V_\beta$ for a certain $\beta < \alpha$. Applying (3), and the induction hypothesis, we immediately arrive at a contradiction:

$$\beta = \mathrm{rank}(\beta) < \mathrm{rank}(\alpha) < \beta .$$

(6) Let us put $\alpha = \sup\{\mathrm{rank}(y) : y \in x\}$. Obviously, $x \subset V_{\alpha+1}$ and $x \subset V_{\alpha+2} \subset \mathbf{V}$. ▷

1.5.6. Theorem. *The axiom of foundation,* NBG_{14}, *is equivalent to the statement* $\mathbf{U} = \mathbf{V}$, *i.e., to the coincidence of the universal class with the von Neumann universe.*

◁ Let $\mathbf{U} = \mathbf{V}$ and let us choose a nonempty class X. There is an element $x \in X$ with the least rank α, i.e., $\mathrm{rank}(x) = \alpha$ and $\mathrm{rank}(x) \le \mathrm{rank}(y)$ for all $y \in X$. If $u \in x \cap X$, then, by virtue of **1.5.5** (3), $\mathrm{rank}(u) < \alpha = \mathrm{rank}(x)$, which contradicts the definition of α. Hence, $x \cap X = 0$.

Let us now prove that $\mathbf{U} \ne \mathbf{V}$ contradicts the axiom of foundation. Indeed, applying the axiom to a nonempty class $\mathbf{U} - \mathbf{V}$ we find a set $y \in \mathbf{U} - \mathbf{V}$ for which $y \cap (\mathbf{U} - \mathbf{V}) = 0$. The last relation yields $y \subset \mathbf{V}$, while from **1.5.5** (6) we deduce $y \in \mathbf{V}$, which contradicts the choice of y. ▷

1.5.7. Theorem. *The following statements are valid*:

(1) ($\in$-induction): *if the class* X *is such that for any set* x *it follows from* $x \subset X$ *that* $x \in X$, *then* $X = \mathbf{V}$;

(2) ($\in$-recursion): *if* G *is a single-valued class, then there is a unique function* F

determined over $\mathbf{V}$, *for which* $F(x) = G(\mathrm{im}(F|x))$ *for* $x \in \mathbf{V}$;

(3) (induction on rank): *if for the class* X *and every set* x *it follows from* $\{y \in \mathbf{V}: \mathrm{rank}(y) < \mathrm{rank}(x)\} \subset X$ *that* $x \in \mathbf{V}$, *then* $\mathbf{V} = X$..

◁ As has been established in **1.5.6**, the universe $\mathbf{V}$ coincides with the class of all sets $\mathbf{U}$. Therefore, the required statements result directly from **1.1.11** under the condition that the relation $\in = \{(x,y) \in \mathbf{V}^2 : x \in y\}$ and $R := \{(x,y) \in \mathbf{V}^2 : \mathrm{rank}(x) < \mathrm{rank}(y)\}$ are well-founded. For $\in$, the necessary property results from the axiom of foundation (see **1.4.10**). Let us now choose such a sequence $(x_n)_{n \in \omega}$ of the sets $x_n \in \mathbf{V}$, that $x_{n+1} \in R(x_n)$ $(n \in \omega)$. Then the sequence of the ordinals $\alpha_n := \mathrm{rank}(x_n)$ obeys the condition $\alpha_{n+1} < \alpha_n$ $(n \in \omega)$ (see **1.5.5** (3)). This contradicts the fact that the class On is well-ordered and, hence R is well-founded. ▷

1.5.8. Let $\sim$ be an equivalence on a class W. The union of all W elements which are equivalent to a given $x \in W$ forms, generally speaking, a proper class, which hampers the formation of a factor-class. This difficulty can be overcome by using the ordinal rank.

Frege-Russel-Scott theorem. *There is a function* $F: W \to \mathbf{V}$ *such that for all* $x, y \in W$ *we have*

$$F(x) = F(y) \leftrightarrow x \sim y.$$

◁ By theorem **1.3.14**, there is a class F such that for all $x, y \in W$, we obtain

$$(x,y) \in F \leftrightarrow \varphi(x,y,W,\sim,\mathrm{rank}),$$

where the predicative formula φ has the form

$$(\forall z)(z \in y \leftrightarrow x \in W \wedge x \sim z \wedge (\forall u)(x \sim u \to \mathrm{rank}(z) \leq \mathrm{rank}(u))).$$

Therefore, F is a function, and $y = F(x)$ is the class of sets z equivalent to x and having the least ordinal rank among such sets. If $\alpha = \mathrm{rank}(x)$, then $F(x) \subset W \cap V_{\alpha+1}$ and, hence, $F(x)$ is a set. Besides, $\mathrm{dom}(F) = W$, and for any $x, y \in W$ we have $x \sim y \leftrightarrow F(x) = F(y)$. Indeed, if $F(x) = F(y)$, then there is a $\omega \in W$, for which $x \sim \omega$ and $y \sim \omega$, i.e., $x \sim y$. The reverse implication is obvious. ▷

If follows from the axiom of domain NBG_{10}, and **1.3.13** **(1)** that there is a class $\mathrm{im} F := \{F(x): x \in W\}$. Let us call this class the *factor-class of the class* W by the equivalence $\sim$, i.e., $W/\sim := \mathrm{im} F$. In this case we say that F *is the canonical factor-homomorphism* or *the canonical projection.*

1.5.9. Let B be a fixed set containing more than one element. Let us put $Q := \mathcal{P}^{(B)} : x \to B^x$ $(x \in \mathbf{V})$, where B^x is, as usual, the set of all mappings from x to B. The cumulative hierarchy arising in this case (see **1.5.1**, where $x_0 = 0, R = I_{\mathbf{V}}$) will be denoted by the symbol $(V_\alpha^{(B)})_{\alpha \in \mathrm{On}}$. It is obvious that the *B-valued universe*

$$\mathbf{V}^{(B)} := \bigcup_{\alpha \in \mathrm{On}} V_\alpha^{(B)}$$

is a subclass of the class $\mathbf{V}$ and consists of B-valued functions determined on the sets of B-valued functions. The standard interpretation of the symbol $\in$ in $\mathbf{V}^{(B)}$ yields nothing of interest, since for the B-valued functions u,v the relation $u \in v$ is valid only in trivial cases. The hierarchies (V_α) and $(V_\alpha^{(B)})$ are, however, essentially different and this peculiarity can give grounds for nonstandard interpretations of set theory in the universe $\mathbf{V}^{(B)}$, which will be discussed in more detail in Chapter **2** below.

1.5.10. For the sake of completeness let us consider one more cumulative hierarchy. The following operations with sets are called *Gödel operations* (they are eight all in all): the formation of an unordered pair, set-theoretic difference, Cartesian product; the (2,3,1)-, (3,2,1)- and (1,3,2)-conjugations (see **1.3.10**), as well as $X \to X^2 \cap \in$ and $X \to \mathrm{dom}(X)$. For any set (sets) X the closure $\mathrm{cl}_G(X)$ is the least set containing X and closed relative to the Gödel operations. Let us now set $Q(x) := \mathcal{P}(x) \cap \mathrm{cl}_G(x \cup \{x\})$. The hierarchy resulting in this case is termed the *constructible hierarchy* and denoted by $(L_\alpha)_{\alpha \in \mathrm{On}}$. The constructible universe is a class $\mathbf{L}$: $\bigcup_{\alpha \in \mathrm{On}} L_\alpha$; the elements of $\mathbf{L}$ being constructible sets (for details see [96, 184]).

1.5.11. Remarks

(1) The cumulative hierarchy $(V_\alpha)_{\alpha \in \mathrm{On}}$ was first considered by J.von Neumann. The relativization of the axiom of foundation to the class $\mathbf{V}$ is provable in the theory NBG-$\{\mathrm{NBG}_{14}\}$, which implies that NBG_{14} is consistent with the rest of the axioms of NBG. Other means can be employed to show that $\neg\,\mathrm{NBG}_{14}$ is also consistent with the other NBG axioms, i.e., that NBG_{14} is an independent axiom.

(2) If B is a complete Heyting lattice (see **1.1.8** (8)), then the universe $\mathbf{V}^{(B)}$ can be transformed into a model of intuitionistic set theory by using the structure of B and the hierarchy $(V_\alpha^{(B)})$. In particular, if B is a complete Boolean algebra, then there arises a

Boolean-valued model of set theory (for more details see **2.1.10** (3)).

(**3**) If $B := [0,1]$ is the interval of a real line from zero to unity, then the class $\mathbf{V}^{(B)}$ is naturally called the *universe of Zadeh-fuzzy sets* [280]. This universe can serve a model for a certain set theory with an appropriate many-valued logic, as well as constitute some basis for studying indistinct sets.

(**4**) The constructible universe $\mathbf{L}$ is the least transitive model of ZFC containing all ordinals. The class $\mathbf{L}$ satisfies the axiom of choice and the generalized continuum hypothesis. Therefore, both AC and GCH are consistent with ZF. The statement that all sets are constructible is termed the *axiom of constructibility* and is presented as $\mathbf{V} = \mathbf{L}$. The relativization of the formula $\mathbf{V} = \mathbf{L}$ to the class $\mathbf{L}$ is provable in ZF. Hence, $\mathbf{V} = \mathbf{L}$ is consistent with ZF. All these results, as well as the definition of constructible sets belong to K.Gödel [61] (see also [96, 184]). The corresponding statements on the consistency of the axiom of choice and GCH are also valid for NBG (see [96, 29, 180, 184]).

(**5**) It has been proved [255] that if B is a quantum logic (see **1.1.8** (5)), then the universe $\mathbf{V}^{(B)}$ serves as a model for a certain quantum set theory in the sense analogous to that discussed below in **2.4**. Traeting quantum theories as logic systems, constructing quantum set theory and developing a corresponding quantum mathematics, all these are interesting and actual problems, and a lot has been done in this direction. Adequate mathematical means and correct reference points can, possibly, be traced in the theory of von Neumann algebras and in various 'noncommutative' trends that has arisen from it (noncommutative probability theory, noncommutative integration, etc.).

CHAPTER 2

BOOLEAN-VALUED UNIVERSES

Various nonstandard methods of analysis are unified by studying special quite unconventional models of set theory. In particular, the apparatus of Boolean-valued analysis is based on the properties of a certain cumulative hierarchy $\mathbf{V}^{(B)}$, with its every succeeding layer composed of all possible functions departing from the preceding layers and arriving at a complete Boolean algebra B chosen beforehand. The principal topics of the present section is constructing such a hierarchy, i.e., the Boolean-valued universe $\mathbf{V}^{(B)}$, as well as studying the general properties of $\mathbf{V}^{(B)}$.

Special attention will be paid to introducing truth-values for formulas and exactly explaning the sense in which $\mathbf{V}^{(B)}$ can be viewed as a model of set theory. Presented in detail are the basic techniques that lay grounds for Boolean-valued analysis such as the principles of transfer, mixing and maximum. Considerations of logical rigour and independence made us pay special attention to constructing a separated universe and interpreting NBG in $\mathbf{V}^{(B)}$. The reader interested only in applications to analysis can, in the first reading, get acquainted with these more sophisticated fragments but quite briefly.

2.1 The Universe Over a Boolean Algebra

In this section a Boolean-valued universe is determined, Boolean truth-values are constructed for set-theoretical formulas and the simplest related facts are given.

2.1.1. Let us start with informal heuristic considerations which could facilitate acquaintance with some features of Boolean-valued universes and Boolean truth-values. Let $\mathbf{2} := \{\mathbf{0},\mathbf{1}\}$ be the two-element Boolean algebra (it is customary to identify all of them). Let us choose an arbitrary set $x \in \mathbf{V}$ and associate with it a certain (characteristic) function χ_x with the values in $\mathbf{2}$ determined (generally speaking, nonuniquely) by the conditions that $x \subset \operatorname{dom}(\chi_x)$ and $\chi_x(t) = \mathbf{1}$ whenever $t \in x$. Clearly, there are sound reasons to identify x with any such function χ_x. For these elements of the domain of definition $\operatorname{dom}(\chi_x)$ of the two-valued function χ_x to be interpretable as two-valued functions, we should, of course, have substituted all the elements on the level $V_\beta, \beta < \mathrm{rank}(x)$ where $\mathrm{dom}(\chi_x)$ is located with appropriate characteristic functions. If one wants to serve, in this sense, the whole world of sets, i.e., the universe $\mathbf{V}$, then one should start from the level zero $\varnothing$. By formalizing these observations, we come to the notion of the $\mathbf{2}$-valued universe

$$\mathbf{V}^{(2)} := \{x : (\exists \alpha \in \mathrm{On})(x \in V_\alpha^{(2)})\},$$

where $\mathbf{V}^{(2)} := \varnothing, V_1^{(2)} := \{\varnothing\}, V_2^{(2)} := \{\{\varnothing\}, (\{\varnothing\}, \mathbf{1})\}$, etc.. In more detail, by analogy with $\mathbf{V}$, by $\in$-recursion we determine the cumulative hierarchy

$$V_\alpha^{(2)} := \{x : \mathrm{Fnc}(x) \wedge \mathrm{im}(x) \subset \mathbf{2} \wedge (\exists \beta < \alpha)(\mathrm{dom}(x) \in V_\alpha^{(2)})\}.$$

Obviously, $\mathbf{V}^{(2)}$ consists of two-valued functions, in which case we associate with every element $x \in \mathbf{V}^{(2)}$ a unique set $\bar{x} := \{y \in \mathbf{V}^{(2)} : x(y)\} = \mathbf{1}$. However, different elements of $\mathbf{V}^{(2)}$ can correspond to the same set. Therefore, let us identify the functions x and $y \in \mathbf{V}^{(2)}$, for which $\bar{x} = \bar{y}$, paying no attention to formal difficulties and restrictions which are to be met on this way. Let us choose arbitrary $x, y \in \mathbf{V}^{(2)}$. By virtue of the above identification, the equality $x = y$ is valid iff $\bar{x} = \bar{y}$. At the same time, it is natural to assume that the formula $x \in y$ is true only if $x \in \bar{y}$. Let us put $[x = y] := \mathbf{1}$, $[x \in y] := \mathbf{1}$ in the case when the formulas $x = y$, $x \in y$ are valid, and let $[x = y] := \mathbf{0}$, $[x \in y] := \mathbf{0}$ in the opposite case. In this case the following presentations are valid:

$$[x \in y] = \bigvee_{t \in \mathrm{dom}(y)} y(t) \wedge [t \in x],$$
$$[x = y] = \bigwedge_{t \in \mathrm{dom}(x)} x(t) \Rightarrow [t \in y] \wedge \bigwedge_{t \in \mathrm{dom}(y)} y(t) \Rightarrow [t \in x].$$

It would be expedient to compare these relations with the tautologies of set theory

$$u \in v \leftrightarrow (\exists \omega)(\omega \in v \wedge \omega = u),$$
$$u = v \leftrightarrow (\forall \omega)(\omega \in u \rightarrow \omega \in v) \wedge (\omega \in v \rightarrow \omega \in u).$$

2.1.2. Let B be a fixed complete algebra which is an element of the von Neumann universe $\mathbf{V}$. The Boolean-valued universe $\mathbf{V}^{(B)}$ arises as the limit of the cumulative hierarchy (**1.5.1**) provided $x_0 := 0$, $R := I_{\mathbf{V}}$, while Q is determined by the formula

$$y \in Q(x) \leftrightarrow \mathrm{Fnc}(y) \wedge \mathrm{dom}(y) \subset x \wedge \mathrm{im}(y) \subset B.$$

Therefore, the hierarchy $(V_\alpha^{(B)})_{\alpha \in \mathrm{On}}$ has the form

$V_0^{(B)} := 0,$
$V_{\alpha+1}^{(B)} := \{y : \mathrm{Fnc}(y) \wedge \mathrm{dom}(y) \subset V_\alpha^{(B)} \wedge \mathrm{im}(y) \subset B\},$
$V_\alpha^{(B)} := \cup\{V_\alpha^{(B)} : \beta < \alpha\} \quad (\alpha \in K_{\mathrm{II}}).$

Therefore, by definition we assign

$$\mathbf{V}^{(B)} := \bigcup_{\alpha \in \mathrm{On}} V_\alpha^{(B)}.$$

Since the empty set is a function with the empty domain of definition, let us write down the first and the second layers of the Boolean-valued universe: $V_1^{(B)} = \{0\}$, $V_2^{(B)} = \{0\} \cup \{(0,b) : b \in B\}$. The ordinal rank of the element $x \in \mathbf{V}^{(B)}$ will be denoted by $\rho(x)$.

2.1.3. Since the relation $y \in \mathrm{dom}(x)$ is well-founded, the following induction principle results from **1.4.11** (1) for $\mathbf{V}^{(B)}$:

$$(\forall x \in \mathbf{V}^{(B)}) \quad ((\forall y \in \mathrm{dom}(x))\, \varphi(y) \to \varphi(x)) \to (\forall x \in \mathbf{V}^{(B)})\, \varphi(x),$$

where φ is an arbitrary formula of ZFC.

2.1.4. Our nearest task is to ascribe a truth-value to every formula of ZFC, where free variables are replaced with elements of $\mathbf{V}^{(B)}$. Such a value must be an element of B and have the property that the theorems of ZFC become 'true' in $\mathbf{V}^{(B)}$, i.e., that that they get the highest truth-value, unity.

Let us, before all, introduce the truth-value for atomic formulas $x \in y$ and $x = y$. This is performed with two class-functions, $[\cdot \in \cdot]$ and $[\cdot = \cdot]$, from $\mathbf{V}^{(B)} \times \mathbf{V}^{(B)}$ to B. For arbitrary $x, y \in \mathbf{V}^{(B)}$ we will set

$$(1)\ [x \in y] := \bigvee_{z \in \mathrm{dom}(y)} y(z) \wedge [z = x],$$

$$(2)\ [x = y] := \bigwedge_{z \in \mathrm{dom}(y)} x(z) \Rightarrow [z \in x] \wedge \bigwedge_{z \in \mathrm{dom}(x)} x(z) \Rightarrow [z \in y].$$

By recursion on $(\rho(x), \rho(y))$, the above formulas determine the functions $[\cdot \in \cdot]$ and $[\cdot = \cdot]$, provided $\mathrm{On} \times \mathrm{On}$ is canonically well-ordered (see **1.4.15**). Indeed, at the level zero when $(\rho(x), \rho(y)) = (0,0)$ we get (see **1.1.1**):

$$[0 \in 0] = \vee\varnothing = \mathbf{0}_B, [0 = 0] = \wedge\varnothing = \mathbf{1}_B.$$

Besides, for $z \in \mathrm{dom}(y)$ (or $z \in \mathrm{dom}(x)$) we have $(\rho(x),\rho(z)) < (\rho(x),\rho(y))$ (and, respectively, $(\rho(z),\rho(y)) < (\rho(x),\rho(y))$).

One can choose another way and make use of transfinite recursion **1.4.9**. Namely, if for all $u,v \in V_\alpha^{(B)}$ the values of $[u \in v]$ and $[u = v]$ are determined, then for $x,y \in V_{\alpha+1}^{(B)}$ we can calculate

$$[x = y] = \bigwedge_{u \in \mathrm{dom}(x)} \left(x(u) \Rightarrow \bigvee_{v \in \mathrm{dom}(y)} y(v) \wedge [u = v] \right)$$
$$\wedge \bigwedge_{v \in \mathrm{dom}(y)} \left(y(v) \Rightarrow \bigvee_{u \in \mathrm{dom}(x)} x(u) \wedge [u = v] \right),$$

since $\mathrm{dom}(x) \subset V_\alpha^{(B)}$ and $\mathrm{dom}(y) \subset V_\alpha^{(B)}$. Now the values of $[x = z]$ for all $z \in \mathrm{dom}(y)$. Therefore, we can calculate

$$[x \in y] = \bigvee_{z \in \mathrm{dom}(y)} y(z) \wedge [z = x].$$

The case of a limiting ordinal causes no problem.

2.1.5. Let us consider the substantiation of the discussed recursive definition **2.1.4** in more detail. For $k := 1,2,3,4$ let us set

$$\pi_x^k(u,v) := \vee \{ b \in B : (\exists c_1,c_2,c_3,c_4 \in B)((u,v,c_1,c_2,c_3,c_4) \in x \wedge c_k = b) \}.$$

Let π_1 and π_2 be functions putting into correspondence to every ordered hexad (u,v,c_1,c_2,c_3,c_4) the first and the second, respectively, components, u and v. Using this notation, let us describe a certain single-valued class Q. For an arbitrary $x \in \mathbf{V}$ the set $Q(x)$ consists of all possible hexads (u,v,c_1,c_2,c_3,c_4) obeying the following conditions:

$$\mathrm{Fnc}(u), \mathrm{Fnc}(v), \quad \mathrm{im}(u) \cup \mathrm{im}(v) \subset B,$$
$$\mathrm{dom}(u) \subset \pi_1 {}^{\backprime\backprime} x, \quad \mathrm{dom}(v) \subset \pi_2 {}^{\backprime\backprime} x;$$
$$b_1 = \bigvee_{z \in \mathrm{dom}(v)} v(z) \wedge \pi_x^3(u,z),$$
$$b_2 = \bigvee_{z \in \mathrm{dom}(u)} u(z) \wedge \pi_x^4(v,z),$$
$$b_3 = b_4 = \bigwedge_{z \in \mathrm{dom}(u)} u(z) \Rightarrow \pi_x^1(z,v) \wedge \bigwedge_{z \in \mathrm{dom}(v)} v(z) \Rightarrow \pi_x^2(u,z).$$

By **1.5.1**, there is a cumulative hierarchy $(F(\alpha))_{\alpha \in \mathrm{On}}$, for which

$$F(0) = (0,0,\mathbf{0}_B,\mathbf{0}_B,\mathbf{1}_B,\mathbf{1}_B),$$

$$F(\alpha+1) = Q(F(\alpha)) \quad (\alpha \in \mathrm{On}),$$
$$F(\alpha) = \bigcup_{\beta<\alpha} F(\beta) \quad (\alpha \in K_{\mathrm{II}}).$$

The class $X := \mathrm{im}(F)$ is obviously a function with $\mathrm{im}(X) \subset B^4$ and $\mathrm{dom}(X) \subset \mathbf{V}^{(B)} \times \mathbf{V}^{(B)}$. If $P_k: B^4 \to B$ is the k-th projection then, according to the definition, we put

$$[\cdot \in \cdot] := P_1 \circ X, [\cdot = \cdot] := P_3 \circ X.$$

2.1.6. Let us now describe a way of considering any formula of set theory as a statement about the elements of the Boolean-valued universe. In other words, we are going to define the interpretation of set theory in $\mathbf{V}^{(B)}$ by using the functions $[\cdot \in \cdot]$, $[\cdot = \cdot]$ discussed in **2.1.4**. Let us, first of all, determine the *interpretation class* I as a class of all mappings from the set of symbols of the variables in the language of set theory into the universe $\mathbf{V}^{(B)}$. By the *interpretation of a variable* x we mean the evaluation that assigns to each $v \in I$ the element $\bar{x}(v) := v(x)$. As interpretations of the formulas $x \in y$ and $x = y$ let us choose the following functions:

$$v \to [\bar{x}(v) \in \bar{y}(v)], \quad v \to [\bar{x}(v) = \bar{y}(v)] \quad (v \in I).$$

For every formula $\varphi(x_1,\dots,x_n)$ with n free variables let us now determine the interpretation $Y \to [\varphi(\bar{x}_1(v),\dots,\bar{x}_n(v))]$ by induction on the length of the formula φ, using the following rules:

$$[\varphi(x) \wedge \psi(y)]: v \to [\varphi(\bar{x}(v))] \wedge [\psi(\bar{y}(v))],$$
$$[\varphi(x) \vee \psi(y)]: v \to [\varphi(\bar{x}(v))] \vee [\psi(\bar{y}(v))],$$
$$[\neg \varphi(x)]: v \to [\varphi(\bar{x}(v))]^*,$$
$$[\varphi(x) \wedge \psi(y)]: v \to [\varphi(\bar{x}(v))] \Rightarrow [\psi(\bar{y}(v))],$$
$$[(\forall t)\, \varphi(t,x)]: v \to \wedge\{[\varphi(\bar{t},(v'),\bar{x}(v'))]: v' \in I_v(x)\},$$
$$[(\exists t)\, \varphi(t,x)]: v \to \vee\{[\varphi(\bar{t},(v'),\bar{x}(v'))]: v' \in I_v(x)\},$$

where $x := (x_1,\dots,x_n)$, $y := (y_1,\dots,y_m)$, $\bar{x}(v) := (\bar{x}_1(v),\dots,\bar{x}_n(v))$, $\bar{y}(v) := (\bar{y}_1(v),\dots,\bar{y}_m(v))$, $I_v(x) := \{v' \in I: v(x) = v'(x)\}$, and all free variables of the formulas φ and ψ are contained within $t, x_1,\dots,x_n$ and $t, y_1,\dots,y_m$, respectively. It should be remarked that $[\varphi(\bar{x}(v))]$ depends only on the values $\bar{x}_k(v) = v(x_k)$ $((k := 1,\dots,n)$; and, therefore, we shall write $[\varphi(u_1,\dots,u_n)]$ instead of $[\varphi(\bar{x}(v))] = [\varphi(\bar{x}_1(v),\dots,\varphi(\bar{x}_n(v))]$ provided $u_k := \bar{x}_k(v) \in \mathbf{V}^{(B)}$ $(k := 1,\dots,n)$. If $\varphi := \varphi(x_1,\dots,x_n)$ is a formula and $u_1,\dots,u_k \in \mathbf{V}^{(B)}$ then, by definition, we put

$$\mathbf{V}^{(B)} \models \varphi(u_1,\dots,u_n) \leftrightarrow [\varphi(u_1,\dots,u_n)] = \mathbf{1}_B.$$

In this case we say that φ *is true inside* $\mathbf{V}^{(B)}$ *for the given values* $u_1,\ldots,u_n$ *of the variables* $x_1,\ldots,x_n$, or simply that $\varphi(u_1,\ldots,u_n)$ *is valid in* $\mathbf{V}^{(B)}$.

2.1.7. The introduced notion of interpretation makes it possible to judge the elements in $\mathbf{V}^{(B)}$. More convenient for this purpose proves, however, to be a somewhat different language obtained by supplementing the alphabet of the language of set theory with one constant for every element in $\mathbf{V}^{(B)}$. In this case, as usual, the elements of $\mathbf{V}^{(B)}$ are identified with the corresponding symbols of the constants. The formulas and sentencess of the new language will be called *B-formulas* and *B-sentences*. Then every B-formula (B-expression) is obtained from a certain formula of set theory by inserting values of $\mathbf{V}^{(B)}$ in place of some (respectively, all) free variables. Let us now see in what way are the definitions of Boolean truth-values from **2.1.6** simplified. Namely, the Boolean estimate for any B-sentence can be obtained by letting

$$
\begin{aligned}
&[\sigma \wedge \tau] := [\sigma] \wedge [\tau],\\
&[\sigma \vee \tau] := [\sigma] \vee [\tau],\\
&[\neg \sigma] := [\sigma]^*,\\
&[\sigma \rightarrow \tau] := [\sigma] \Rightarrow [\tau],\\
&[(\forall x)\varphi(x)] := \wedge\{[\varphi(u)] : u \in \mathbf{V}^{(B)}\},\\
&[(\exists x)\varphi(x)] := \vee\{[\varphi(u)] : u \in \mathbf{V}^{(B)}\},
\end{aligned}
$$

where σ and τ are B-sentences, while φ is a B-formula with one free variable x. The B-sentence σ is said to be *true in (inside)* $\mathbf{V}^{(B)}$, and we write $\mathbf{V}^{(B)} \models \sigma$ if $[\sigma] = \mathbf{1}_B$. Herefrom, unless otherwise specified, we shall use both the linguistic means of **2.1.6** and **2.1.7**. We shall also use the same letters when denoting both variables and elements of the universe $\mathbf{V}^{(B)}$. If several Boolean algebras, $B, C, \ldots$ are considered simultaneously and there is a necessity to go into details, then, alongside with $[\varphi]$, we shall write $[\varphi]^B$, $[\varphi]^C$, etc..

2.1.8. Theorem. *If the formula* $\varphi(x_1,\ldots,x_n)$ *is provable in predicate calculus with equality, then* $\mathbf{V}^{(B)} \models \varphi(x_1,\ldots,x_n)$ *for any* $x_1,\ldots,x_n \in \mathbf{V}^{(B)}$. *In particular, the following relations are valid:*

(1) $[x = x] = \mathbf{1}$;

(2) $x(y) \leq [y \in x]$ *for all* $y \in \mathrm{dom}(x)$;

(3) $[x = y] = [y = x]$;

(4) $[x = y] \wedge [y = z] \leq [x = z]$;

(5) $[x \in y] \wedge [x = z] \leq [z \in y]$;

(6) $[y \in x] \wedge [x = z] \leq [y \in z]$;

(7) $[x = y] \wedge \varphi(x) \leq \varphi(y)$ *for any formula* φ.

◁ The axioms of predicate calculus are easily checked to be true inside $\mathbf{V}^{(B)}$, while the rules of inference preserve validity. To be more exact, if in predicate calculus the formula φ is deducible from the formulas $\varphi_1,\dots,\varphi_n$, then $[\varphi_1] \wedge \dots \wedge [\varphi_n] \leq [\varphi]$. Let us now prove validity for (1) -(7).

(1) This relation is established by induction on the well-founded relation $y \in \mathrm{dom}(x)$. Let us assume that $[y = y] = \mathbf{1}$ for all $y \in \mathrm{dom}(x)$. Then, by **2.1.4** (1), we get

$$[y \in x] = \bigvee_{t \in \mathrm{dom}(x)} x(t) \wedge [t = y] \geq x(y) \wedge [y = y] \geq x(y),$$

and, hence, according to **1.1.4** (4) we have

$$[x = x] = \bigwedge_{y \in \mathrm{dom}(x)} x(y) \Rightarrow [y \in x] = \mathbf{1}.$$

(2) Taking into account **2.1.4** (1) and what was proved in (1), for $y \in \mathrm{dom}(x)$ we get the following estimate:

$$[y \in x] \geq x(y) \wedge [y = y] = x(y).$$

(3) This relation results from the definition due to the symmetry of relation **2.1.4** (2) which gives the truth-value for equality.

Statements (4)-(6) are established by a simultaneous induction. Let $\rho(x,y,z) := (\alpha,\beta,\gamma) \in \mathrm{On}^3$ be such a permutation of the triple of ordinals $\rho(x)$, $\rho(y)$ and $\rho(z)$ that $\alpha \geq \beta \geq \gamma$. (The class On^3 is considered with the canonical well-ordering **1.4.15**.) Let us assume that $x, y, z \in \mathbf{V}^{(B)}$, and for all $u, v, \omega \in \mathbf{V}^{(B)}$ inequalities (4)-(6) hold if $\rho(u,v,\omega) < \rho(x,y,z)$. The induction step will be analysed in each case separately.

(4) Let $t \in \mathrm{dom}(x)$. Since $[x = y] \leq x(t) \Rightarrow [t \in y]$, by **1.1.4** (3), we have

$$x(t) \wedge [x = y] \leq [t \in y],$$
$$x(t) \wedge [x = y] \wedge [y = z] \leq [t \in y] \wedge [y = z].$$

Having remarked that $\rho(t,y,z) < \rho(x,y,z)$ and applying the induction hypothesis for (6), we get

$$[t \in y] \wedge [y = z] \le [t = z],$$
$$x(t) \wedge [y = x] \wedge [y = z] \le [t \in z].$$

Let us now again employ relation **1.1.4** (3). Then

$$[x = y] \wedge [y = z] \le x(t) \Rightarrow [t = z],$$

and, hence,

$$[x = y] \wedge [y = z] \le \bigwedge_{t \in \mathrm{dom}(x)} x(t) \Rightarrow [t \in z].$$

Analogously,

$$[x = y] \wedge [y = z] \le \bigwedge_{t \in \mathrm{dom}(z)} z(t) \Rightarrow [t \in x].$$

By virtue of **2.1.4** (2) we conclude: $[x = y] \wedge [y = z] \le [x = z]$.

(5) Let us consider $t \in \mathrm{dom}(y)$. Then $\rho(t,x,z) < \rho(x,y,z)$ and so, by the induction hypothesis for (6), we get

$$y(t) \wedge [t = x] \wedge [x = z] \le y(t) \wedge [t = z] \le [z \in y].$$

By **1.1.5** (2), this gives

$$[x = z] \wedge \bigvee_{t \in \mathrm{dom}(y)} y(t) \wedge [t = x] \le [|z \in y],$$

or $[x = z] \wedge [x \in y] \le [z \in y]$.

(6) Let again $t \in \mathrm{dom}(x)$. In this case

$$x(t) \wedge [x = z] \le [t \in z],$$
$$[t = y] \wedge x(t) \wedge [x = z] \le [t = y] \wedge [t \in z].$$

This time, once more, $\rho(t,y,z) < \rho(x,y,z)$ and, hence, by the induction hypothesis, for (5) and formula **1.1.5** (2), we derive

$$x(t) \wedge [x = z] \wedge [t = y] \le [y \in z],$$
$$[x = z] \wedge \bigvee_{t \in \mathrm{dom}(x)} x(t) \wedge [t = y] \le [y \in z].$$

Therefore, according to **2.1.4** (1), $[x = z] \wedge [y \in x] \le [y \in z]$.

(7) This relation is proved by induction on the length of the formula with the relations established above taken into account. ▷

As a corollary to theorem **2.1.8**, let us recall the following rules for calculating the Boolean truth-values of bounded formulas.

2.1.9. *For any B-formula* φ *with a single free variable* x *and for every* $u \in \mathbf{V}^{(B)}$ *the following relations are valid:*

$$[(\exists x \in u)\ \varphi(x)] = \bigvee_{v \in \mathrm{dom}(u)} u(v) \wedge [\varphi(v)],$$
$$[(\forall x \in u)\ \varphi(x)] = \bigwedge_{v \in \mathrm{dom}(u)} u(v) \Rightarrow [\varphi(v)].$$

◁ These formulas are mutually dual and, hence, it is sufficient to prove one of them, for instance the first. By **2.1.8** (2) the following inequality holds

$$[(\exists x \in u)\ \varphi(x)] = \bigvee_{v \in \mathrm{dom}(u)} u(v) \wedge [\varphi(v)].$$

On the other hand, applying **2.1.4** (1) and **2.1.8** (7), we get

$$[(\exists x \in u)\ \varphi(x)] = \bigvee_{t \in \mathbf{V}^{(B)}} \bigvee_{v \in \mathrm{dom}(u)} u(v) \wedge [t = v] \wedge [\varphi(t)]$$
$$\leq \bigvee_{v \in \mathrm{dom}(u)} u(v) \wedge [\varphi(v)]. \ ▷$$

Remarks

(1) For $u_1,\dots,u_n \in \mathbf{V}^{(B)}$ and $b \in B$ for every concrete formula φ of set theory, the expression $[u_1,\dots,u_n] = b$ is again a formula of set theory. In ZFC, however, the mapping $\varphi \to [\varphi]$ is not a definable class, allowing only a metalinguistic definition.

(2) The Boolean-valued universe $\mathbf{V}^{(B)}$ is used for proving relative consistency of set-theoretic propositions according to the following schema. Let $\mathfrak{T}$ and $\mathfrak{T}'$ be extensions of ZF such that consistency of ZF implies that of $\mathfrak{T}'$. Let us assume that B can be determined in such a way that $\mathfrak{T}' \vDash$ 'B is a complete Boolean algebra' and $\mathfrak{T}' \vDash [\varphi]^B = \mathbf{1}$ for every axiom φ of the theory $\mathfrak{T}$. In this case the consistency of ZF implies that of $\mathfrak{T}$ (see [10]).

(3) Let Ω be a complete Heyting lattice (see **1.1.8** (3)). The pseudo-complement b^* of an element $b \in \Omega$ is introduced by the formula $x^* := x \Rightarrow \mathbf{0}$, where $\Rightarrow$ is the operation of

relative pseudo-complementation. An insignificant modification of the formulas **2.1.4** determines the truth-values $[\cdot \in \cdot]^{\Omega}$ and $[\cdot = \cdot]^{\Omega}$ operating from $\mathbf{V}^{(\Omega)} \times \mathbf{V}^{(\Omega)}$ in Ω. The truth in $\mathbf{V}^{(\Omega)}$ is determined in the same way as in **2.1.6**. In this case in $\mathbf{V}^{(\Omega)}$ all the formulas of the intuitionistic predicate calculus prove to be true (see [56, 73, 246, 247].

2.2. Transformations of Boolean-valued universes

Any homomorphism of a Boolean algebra B induces a certain transformation of the universe $\mathbf{V}^{(B)}$. Studying such transformations and, in particular, elucidation of the problem how Boolean truth-values of formulas are in this case transformed is the topic to be discussed in this section.

2.2.1. Let π be a homomorphism of B in a complete Boolean algebra C. By recursion on a well-founded relation $y \in \mathrm{dom}(x)$ the mapping $\pi^*: \mathbf{V}^{(B)} \to \mathbf{V}^{(C)}$ is determined by the formulas

$$\mathrm{dom}(\pi * x): \{\pi * y: y \in \mathrm{dom}(x)\},$$
$$\pi * x: v \to \vee\{\pi(x(z)): z \in \mathrm{dom}(x) \wedge \pi * z = v\}.$$

If a homomorphism π is injective, then the mapping $\pi *$ is also injective. In this case

$$\pi * x: \pi * y \to \pi(x(y)) \quad (y \in \mathrm{dom}(x)).$$

◁ Indeed, it suffices to establish that for an arbitrary ordinal α, the restriction of $\pi *$ to $V_\alpha^{(B)}$ is injective. Let us assume that this statement holds for all $\beta < \alpha$. Let $x, y \in V_\alpha^{(B)}$ be such that $\pi * x: \pi * z \to \pi(x(z))$ $(z \in \mathrm{dom}(x))$ and $\pi * y: \pi * z \to \pi(y(z))$ $(z \in \mathrm{dom}(y))$. Therefore, we come to the inequality

$$\{(\pi * z, \pi(x(z))): z \in \mathrm{dom}(x)\} = \{(\pi * u, \pi(y(u))): u \in \mathrm{dom}(y)\}.$$

Since for a certain $\beta < \alpha$ the sets $\mathrm{dom}(x)$ and $\mathrm{dom}(y)$ are contained in $V_\beta^{(B)}$; therefore, $\pi *$ is injective on either of these sets. Since π is injective, we get

$$\{(z, x(z)): z \in \mathrm{dom}(x)\} = \{(u, (y(u)): u \in \mathrm{dom}(y)\},$$

or, which is the same, $x = y$. ▷

A homomorphism $\pi: B \to C$ is called *full* if $\pi(\vee M) = \vee \pi(M)$ for every set $M \subset B$. From now on π is a full homomorphism from B to a complete Boolean algebra C.

2.2.2. Theorem. *The following statements are valid:*

(1) *if ρ is a full homomorphism of an algebra C to a complete Boolean algebra D, then $(\rho \circ \pi)^* = \rho^* \circ \pi^*$;*

(2) *if a homomorphism π is injective (respectively, surjective), then the mapping π^* is injective (respectively, surjective);*

(3) *for all x and $y \in \mathbf{V}^{(B)}$ the following equalities hold:*

$$[\pi^* x = \pi^* y]^C = \pi([x = y]^B),$$
$$[\pi^* x \in \pi^* y]^C = \pi([x \in y]^B);$$

(4) *for any $x \in \mathbf{V}^{(B)}$ and $t \in \mathbf{V}^{(C)}$ the following equality holds:*

$$[t \in \pi^* x]^C = \bigvee_{u \in \mathbf{V}^{(B)}} \pi([u \in x]^B) \wedge [t = \pi^* u]^C.$$

◁ (1) Let us assume that $(\rho \circ \pi)^* y = (\rho^* \circ \pi^*) y$ for all $y \in \mathrm{dom}(x)$. Then for $u := (\rho \circ \pi)^* y$, where $y \in \mathrm{dom}(x)$, we deduce (see **1.1.5** (9)):

$$((\rho \circ \pi)^* x) u$$
$$= \vee\{(\rho \circ \pi)(x(z)) : z \in \mathrm{dom}(x) \wedge (\rho^* \circ \pi^*) z = (\rho^* \circ \pi^*) y\}$$
$$= \vee\{\rho(\vee\{\pi(x(z)) : z \in \mathrm{dom}(x) \wedge \pi^* z = v\}) : v \in \mathrm{dom}(\pi^* x) \wedge \rho^* v$$
$$= (\rho^* \circ \pi^*) y\} = \vee\{\rho((\pi^* x)(v)) : v \in \mathrm{dom}(\pi^* x) \wedge \rho^* v = \rho^*(\pi^* y)\}$$
$$= (\rho^*(\pi^* x))(\rho^*(\pi^* y)) = ((\rho^* \circ \pi^*) x) u.$$

Therefore, $(\rho \circ \pi)^* x = \rho^*(\pi^* x)$, and the required result follows from **2.1.3**.

(2) The case of an injective π has been discussed in **2.2.1**. Let us assume that π is a surjective mapping. In this case there is a principal ideal B_0 of a Boolean algebra B, and an isomorphism $\rho : C \xrightarrow{\text{on}} B_0$, for which ρ^{-1} coincides with the restriction π_0 of the homomorphism π to B_0. If $z \in \mathbf{V}^{(C)}$, then, according to (1), $x = I_C{}^* x = (\pi_0 \circ \rho)^* x = \pi_0{}^*(\rho^* x) \in \mathrm{im}(\pi_0{}^*)$. Hence, $\pi_0{}^*$ maps $\mathbf{V}^{(B_0)}$ on $\mathbf{V}^{(C)}$. Now we have to remark that $\mathbf{V}^{(B_0)} \subset \mathbf{V}^{(B)}$, and the restriction of π^* to $\mathbf{V}^{(B_0)}$ coincides with $\pi_0{}^*$.

(3) The proof is carried out by induction on $(\rho(x),\rho(y))$ when the class $\mathrm{On}\times\mathrm{On}$ is canonically well-ordered (see **1.4.15**). Let us assume that the required formulas are fulfilled for any $u,v\in\mathbf{V}^{(B)}$ for $(\rho(u),\rho(v))<(\rho(x),\rho(y))$. If $z\in\mathrm{dom}(x)$ or $z\in\mathrm{dom}(y)$, then, obviously, $\max\{(\rho(z),\rho(x)),(\rho(z),\rho(y))\}<(\rho(x),\rho(y))$. Hence, the following expressions are valid (see **1.1.5** (2,9)):

$$
\begin{aligned}
&[\pi * x \in \pi * y]\\
&= \bigvee_{t\in\mathrm{dom}(\pi^* y)} (\pi * y)(t)\wedge[t=\pi * x] = \bigvee_{z\in\mathrm{dom}(y)} (\pi * y)(\pi * z)\wedge[\pi * z=\pi * x]\\
&= \bigvee_{z\in\mathrm{dom}(y)} (\vee\{\pi(y(u)) : u\in\mathrm{dom}(y)\wedge \pi * u=\pi * z\})\wedge[\pi * z=\pi * x]\\
&= \bigvee_{z\in\mathrm{dom}(y)} \vee\{\pi(y(u))\wedge[\pi * z=\pi * x] : u\in\mathrm{dom}(y)\wedge\pi * u=\pi * z\}\\
&= \bigvee_{u\in\mathrm{dom}(y)} \pi(y(u))\wedge\pi([u=x]) = \pi\Big(\bigvee_{u\in\mathrm{dom}(y)} y(u)\wedge[u=x]\Big)\\
&= \pi([x\in y]).
\end{aligned}
$$

Analogous calculations are also used for the Boolean truth-values of equality (by successively applying **2.1.4** (2), **2.2.1**, **1.1.5** (10), and **2.1.4** (2):

$$
\begin{aligned}
&[\pi * x=\pi * y]\\
&= \bigvee_{t\in\mathrm{dom}(\pi^* y)} (\pi * y)(t)\Rightarrow[t=\pi * x]\wedge \bigvee_{z\in\mathrm{dom}(\pi^* x)} (\pi * x)(t)\Rightarrow[t\in\pi * y]\\
&= \bigwedge_{z\in\mathrm{dom}(y)} (\pi * y)(\pi * z)\Rightarrow[\pi * z\in\pi * x]\\
&\wedge \bigwedge_{z\in\mathrm{dom}(x)} (\pi * x)(\pi * z)\Rightarrow[\pi * z\in\pi * y]\\
&= \bigwedge_{z\in\mathrm{dom}(y)} \wedge\{\pi(y(u))\Rightarrow\pi([u\in x]) : u\in\mathrm{dom}(y)\wedge\pi * u=\pi * z\}\\
&\wedge \bigwedge_{z\in\mathrm{dom}(x)} \wedge\{\pi(x(u))\Rightarrow\pi([u\in y]) : u\in\mathrm{dom}(x)\wedge\pi * u=\pi * z\}\\
&= \bigwedge_{u\in\mathrm{dom}(x)} \pi(x(u)\Rightarrow[u\in y])\wedge \bigwedge_{u\in\mathrm{dom}(y)} \pi(y(u)\Rightarrow[u\in x]) = \pi([x=y]).
\end{aligned}
$$

(4) By virtue of (3) and **2.1.8** (4), the following estimates are fulfilled for $x\in\mathbf{V}^{(B)}$ and $t\in\mathbf{V}^{(C)}$:

$$
\begin{aligned}
&[t\in\pi * x]\\
&= \bigvee_{s\in\mathrm{dom}(\pi^* x)} (\pi * x)(s)\wedge[s=t] = \bigvee_{u\in\mathrm{dom}(x)} (\pi * x)(\pi * u)\wedge[\pi * u=t]\\
&\le \bigvee_{u\in\mathbf{V}^{(B)}} \pi([u=x])\wedge[\pi * u=t]\\
&= \bigvee_{u\in\mathbf{V}^{(B)}} [\pi * u=\pi * x]\wedge[\pi * u=t]\le[t\in\pi * x]. \ \rhd
\end{aligned}
$$

2.2.3. Theorem. *Let $\varphi(x_1,\dots,x_n)$ be a formula of* ZFC, $u_1,\dots,u_n \in \mathbf{V}^{(B)}$, *and π be a full homomorphism from B to C. Then the following statements are valid:*

(1) *if φ is a formula of class Σ_1 and the homomorphism π is arbitrary, then*

$$\pi\big([\![\varphi(u_1,\dots,u_n)]\!]^B\big) \le [\![\varphi(\pi^* u_1,\dots,\pi^* u_n)]\!]^C;$$

(2) *if φ is a bounded formula and π is arbitrary, or π is an epimorphism and φ is an arbitrary formula, then*

$$\pi\big([\![\varphi(u_1,\dots,u_n)]\!]^B\big) = [\![\varphi(\pi^* u_1,\dots,\pi^* u_n)]\!]^C.$$

$\lhd$ For atomic formulas this statement is ensured by **2.2.2**. The general case is established by induction on the complexity of the formula φ. In this case the nontrivial induction step arises only when φ has either the form $(\exists x)\varphi_0$ or $(\forall x)\varphi_0$. It is in this case than additional suppositions on φ and π are necessary.

(1) If in the induction step we have to do with a bounded universal quantifier, i.e., if φ has the form $(\forall x \in u)\varphi_0(x,u_1,\dots,u_n)$, then (see the definitions and **1.1.5** (3,10)), the following chain of equalities holds:

$$\begin{aligned}
&[\![\varphi(\pi^* u, \pi^* u_1,\dots,\pi^* u_2)]\!] \\
&= \bigwedge_{x\in \operatorname{dom}(\pi^* u)} (\pi^* u)(x) \Rightarrow [\![\varphi_0(x,\pi^* u_1,\dots,\pi^* u_n)]\!] \\
&= \bigwedge_{x\in \operatorname{dom}(u)} (\pi^* u)(\pi^* x) \Rightarrow [\![\varphi_0(\pi^* x,\pi^* u_1,\dots,\pi^* u_n)]\!] \\
&= \bigwedge_{x\in \operatorname{dom}(u)} \bigwedge \{\pi(u(z)) \Rightarrow [\![\varphi_0(\pi^* x, \pi^* u_1,\dots, \pi^* u_n)]\!] : z \in \operatorname{dom}(u) \wedge \pi^* z \\
&= \pi^* x\} = \bigwedge_{x \in \operatorname{dom}(u)} \pi(u(x) \Rightarrow [\![\varphi_0(x,u_1,\dots,u_n)]\!]) \\
&= \pi[\![(\forall x \in u)\varphi_0(x,u_1,\dots,u_n)]\!] = \pi[\![\varphi(u,u_1,\dots,u_n)]\!].
\end{aligned}$$

Then, for an unbounded existential quantifier we immediately deduce from the definition

$$\begin{aligned}
&[\![(\exists x)\varphi_0(x,\pi^* u_1,\dots,\pi^* u_n)]\!] \\
&\ge \bigvee\{[\![\varphi_0(x,\pi^* u_1,\dots,\pi^* u_n)]\!] : x \in \operatorname{im}(\pi^*)\} \\
&= \bigvee\{[\![\varphi_0(\pi^* u,\pi^* u_1,\dots, \pi^* u_n)]\!] : u\in \mathbf{V}^{(B)}\} \\
&= \bigvee\{\pi\big([\![\varphi_0(u,u_1,\dots,u_n)]\!]\big) : u \in \mathbf{V}^{(B)}\} = \pi\big([\![(\exists x)\varphi_0(x,u_1,\dots,u_n)]\!]\big).
\end{aligned}$$

(2) It should be, first of all, remarked that if π is a surjection, then π^* is also a surjection, i.e., $\operatorname{im}(\pi^*) = \mathbf{V}^C$ (see **2.2.2** (2)). Therefore, for the formula $\varphi := (\exists x)\varphi_0$ we get

$$
\begin{aligned}
&[\varphi(\pi * u_1,\dots,\pi * u_n)] \\
&= \vee\{[\varphi_0(x,\pi * u_1,\dots,\pi * u_n)]\colon x \in \mathbf{V}^{(C)} = \mathrm{im}(\pi^*)\} \\
&= \vee\{[\varphi_0(\pi * u,\pi * u_1,\dots,\pi * u_n)]\colon u \in \mathbf{V}^{(B)}\} \\
&= \vee\{\pi([\varphi_0(u,u_1,\dots,u_n)])\colon u \in \mathbf{V}^{(B)}\} = \pi([\varphi(u_1,\dots,u_n)]).
\end{aligned}
$$

The same considerations are also applicable to the formula φ of type $(\forall x)\varphi_0(x,u_1,\dots,u_n)$.

If the domain of the existential quantifier under consideration is bounded, i.e., if $\varphi(u_1,\dots,u_n)$ has the form $(\exists x \in u)\varphi_0(x,u_1,\dots,u_n)$ and $u,u_1,\dots,u_n \in \mathbf{V}^{(B)}$, then (see the definitions and **1.1.5** (2,9)) the following calculations hold:

$$
\begin{aligned}
&[\varphi(\pi * u, \pi * u_1,\dots,\pi * u_n)] \\
&= \bigvee_{x \in \mathrm{dom}(\pi^* u)} (\pi * u)(x) \wedge [\varphi_0(x, \pi * u_1,\dots,\pi * u_n)] \\
&= \bigvee_{x \in \mathrm{dom}(u)} (\pi * u)(\pi * x) \wedge [\varphi_0(\pi * x,\pi * u_1,\dots,\pi * u_n)] \\
&= \bigvee_{z \in \mathrm{dom}(u)} \pi(u(z) \wedge [\varphi_0(z,u_1,\dots,u_n)]) = \pi([\varphi(u,u_1,\dots,u_n)]).
\end{aligned}
$$

The case of a bounded universal quantifier has been considered earlier. ⊳

2.2.4. Corollary. *Let π,φ and $u_1,\dots,u_n$ be the same as in* **2.2.3**, *and let one of the following statements be fulfilled:*

(1) *$\varphi(x_1,\dots,x_n)$ is a formula of class Σ_1, π is arbitrary;*

(2) *π is an epimorphism and $\varphi(x_1,\dots,x_n)$ is an arbitrary formula.*

Then

$$
\mathbf{V}^{(B)} \models \varphi(u_1,\dots,u_n) \to \mathbf{V}^{(C)} \models \varphi(\pi * u_1,\dots,\pi * u_n).
$$

2.2.5. Corollary. *Let π,φ and $u_1,\dots,u_n$ be the same as in* **2.2.3**, *and, moreover, let one of the following statements be fulfilled:*

(1) *φ is bounded and π is a monomorphism;*

(2) *π is an isomorphism and φ is arbitrary. In this case we have*

$$\mathbf{V}^{(B)} \models \varphi(u_1,\dots,u_n) \leftrightarrow \mathbf{V}^{(C)} \models \varphi(\pi^* u_1,\dots,\pi^* u_n).$$

2.2.6. Let us now consider a special case of the situation under study. Let B_0 be a regular subalgebra of a complete Boolean algebra B. This implies that B_0 is a complete subalgebra, and the exact bounds of any set in B_0 are independent of the fact whether they are calculated in B_0 or B. Under such circumstances $\mathbf{V}^{(B_0)} \subset \mathbf{V}^{(B)}$, in which case if $\imath$ is the identical embedding of B_0 in B, then $\imath^*$ is an embedding of $\mathbf{V}^{(B_0)}$ in $\mathbf{V}^{(B)}$. It follows from **2.2.5** (1) that if $\varphi(x_1,\dots,x_n)$ is a bounded formula, and if $u_1,\dots,u_n \in \mathbf{V}^{(B_0)}$, , then

$$\mathbf{V}^{(B_0)} \models \varphi(u_1,\dots,u_n) \leftrightarrow \mathbf{V}^{(B)} \models \varphi(u_1,\dots,u_n).$$

Since the two-element algebra $\mathbf{2} := \{\mathbf{0},\mathbf{1}\}$ can be viewed as a regular subalgebra of the Boolean algebra B, then the above is also valid for the universe $\mathbf{V}^{(2)}$. Below we shall see that $\mathbf{V}^{(2)}$ is naturally isomorphic to the von Neumann universe $\mathbf{V}$.

2.2.7. For an arbitrary set $x \in \mathbf{V}$ let us determine an element $x^\wedge \in \mathbf{V}^{(2)} \subset \mathbf{V}^{(B)}$ by recursion on the well-founded relation $y \in x$. To this end let us put

$$\operatorname{dom}(x^\wedge) := \{y^\wedge : y \in x\}, \quad \operatorname{im}(x^\wedge) := \{\mathbf{1}_B\}.$$

From **2.2.2** (3) for any $x, y \in V$ it follows:

$$[x^\wedge \in y^\wedge]^B \in \mathbf{2}, \quad [x^\wedge = y^\wedge]^B \in \mathbf{2}.$$

The mapping $x \to x^\wedge$ $(x \in \mathbf{V})$ is called the *canonical embedding* of the class of all sets $\mathbf{V}$ in the Boolean-valued universe $\mathbf{V}^{(B)}$. The elements of $\mathbf{V}^{(B)}$ which have the form $x^\wedge$ for a certain $x \in \mathbf{V}$ are termed *standard.* Sometimes $x^\wedge$ is called the *standard name of the set x in* $\mathbf{V}^{(B)}$.

2.2.8. Theorem. *The following statements are valid*:

(1) *if $x \in \mathbf{V}$ and $y \in \mathbf{V}^{(B)}$, then*

$$[y \in x^\wedge] = \vee\{[y = u^\wedge] : u \in x\};$$

(2) *if $x, y \in \mathbf{V}$, then*

$$x \in y \leftrightarrow \mathbf{V}^{(B)} \models x^{\wedge} \in y^{\wedge}, \quad x = y \leftrightarrow \mathbf{V}^{(B)} \models x^{\wedge} = y^{\wedge};$$

(3) *the mapping* $x \to x^{\wedge}$ *is injective*;

(4) *for any* $y \in \mathbf{V}^{(2)}$ *there is a unique element* $x \in \mathbf{V}$ *such that* $\mathbf{V}^{(B)} \models x^{\wedge} = y$;

(5) *if* π *is a full homomorphism from* B *to* C, *then for every* $x \in \mathbf{V}$ *we have* $\pi * x^{\wedge} = \overset{\wedge}{x^{\wedge}}$, *where* $\overset{\wedge}{(\,)^{\wedge}}$ *is a canonical embedding of* $\mathbf{V}$ *to* $\mathbf{V}^{(C)}$.

$\lhd$ Direct calculations with account taken of definitions **2.1.4** and **2.2.7** afford

$$[y \in x^{\wedge}] = \bigvee_{t \in \mathrm{dom}(x^{\wedge})} x^{\wedge}(t) \wedge [t = y]$$
$$= \bigvee_{t \in x} x^{\wedge}(t^{\wedge}) \wedge [t^{\wedge} = y] = \bigvee_{t \in x} [t^{\wedge} = y].$$

(2) Let us assume that, for all $z \in \mathbf{V}$ such that $\mathrm{rank}(z) < \mathrm{rank}(y)$, the following relations hold:

$$(\forall x)\big(x \in z \leftrightarrow [x^{\wedge} \in z^{\wedge}] = \mathbf{1}\big),$$
$$(\forall x)\big(x = z \leftrightarrow [x^{\wedge} = z^{\wedge}] = \mathbf{1}\big),$$
$$(\forall x)\big(z \in x \leftrightarrow [z^{\wedge} \in x^{\wedge}] = \mathbf{1}\big).$$

According to (1), $[x^{\wedge} \in y^{\wedge}] = \vee\big\{[t^{\wedge} = x^{\wedge}] : t \in y\big\}$. Since $\mathrm{rank}(t) < \mathrm{rank}(y)$ for $t \in y$, by the inductive hypothesis we conclude that $[x^{\wedge} \in y^{\wedge}] = \mathbf{1}$ iff $[t^{\wedge} = x^{\wedge}] = \mathbf{1}$ or $t = x$ for some $t \in y$. Then, by definition,

$$[x^{\wedge} = y^{\wedge}] = \bigwedge_{t \in x} [t^{\wedge} \in y^{\wedge}] \wedge \bigwedge_{s \in y} [s^{\wedge} = x^{\wedge}]$$

and $\mathrm{rank}(s) < \mathrm{rank}(y)$ for $s \in y$. Therefore, taking into consideration the above and the inductive hypothesis, we deduce that the right-hand side of the last equality is equal to unity iff $t \in y$ for all $t \in x$, and $s \in x$ for all $s \in y$, i.e., if $x = y$. If we again use (1), we get

$$[y^{\wedge} \in x^{\wedge}] = \vee\{[y^{\wedge} = t^{\wedge}] : t \in x\}.$$

Hence, $[y^{\wedge} \in x^{\wedge}] = \mathbf{1}$ is valid only if $[y^{\wedge} \in t^{\wedge}] = \mathbf{1}$ for some $t \in x$. The last statement is, by virtue of the above, equivalent to the relation $(\exists t \in x)(t \in y)$, i.e., $y \in x$.

(3) This statement ensues from (2).

(4) Let us assume that $y \in \mathbf{V}^{(2)}$ and for any $t \in \mathrm{dom}(y)$ there are such $u \in \mathbf{V}$ that $[t = u^\wedge] = \mathbf{1}$. Let us determine $x \in \mathbf{V}$ by the identity

$$x := \{u \in \mathbf{V} : (\exists t \in \mathrm{dom}(y))(y(t) = \mathbf{1} \wedge [u^\wedge = t] = \mathbf{1})\}.$$

In this case for $u \in x$ we get

$$[u^\wedge = y] = \bigvee_{t \in \mathrm{dom}(y)} y(t) \wedge [t = u^\wedge] = 1.$$

Besides, using the inductive hypothesis, we deduce for $t \in \mathrm{dom}(y)$:

$$y(t) \le [t \in x^\wedge] = \bigvee_{u \in x} [t = u^\wedge].$$

Summing up the above, we can conclude

$$[x^\wedge = y] = \bigwedge_{t \in \mathrm{dom}(y)} y(t) \Rightarrow [t \in x^\wedge] \wedge \bigwedge_{u \in x} [u^\wedge \in y] = \mathbf{1}.$$

(5) Let us carry out induction on the well-founded relation $y \in x$. Let us assume $(\forall y \in x)\left(\pi * y^\wedge = \widehat{y^\wedge}\right)$. In this case

$$\mathrm{dom}(\pi * x^\wedge) = \{\widehat{y^\wedge} : y \in x\} = \mathrm{dom}(\widehat{x^\wedge}).$$

Therefore, for $y \in x$ we have

$$\begin{aligned}
(\pi * x^\wedge)\left(\widehat{y^\wedge}\right) &= (\pi * x^\wedge)(\pi * y^\wedge) = \\
&= \vee\{\pi(x^\wedge(y^\wedge)) : z \in \mathrm{dom}(x) \wedge \pi * z = \pi * y^\wedge\} \ge \\
&\ge \pi\left(x^\wedge(y^\wedge)\right) = \mathbf{1}_B = \widehat{x^\wedge}(\widehat{y^\wedge}).
\end{aligned}$$

Therefore, $\pi * x^\wedge = \widehat{x^\wedge}$, which ensures the inductive step. ▷

2.2.9. *Let* $u_1, \dots, u_n \in \mathbf{V}$, *and* $\varphi(x_1, \dots, x_n)$ *be a formula of* ZFC. *Then the following statements hold*:

(1) $\varphi(u_1,\dots,u_n)\in\mathbf{V}^{(2)}\models\varphi(u_1^{\wedge},\dots,u_n^{\wedge})$;

(2) *if* φ *is a bounded formula, then*

$$\varphi(u_1,\dots,u_n)\leftrightarrow\mathbf{V}^{(B)}\models\varphi(u_1^{\wedge},\dots,u_n^{\wedge});$$

(3) *if* φ *is a formula of class* Σ_1, *then*

$$\varphi(u_1,\dots,u_n)\to\mathbf{V}^{(B)}\models\varphi(u_1^{\wedge},\dots,u_n^{\wedge}).$$

$\lhd$ Let us remark that only statement (1) has to be proved, as both (2) and (3) result from (1), **2.2.4** (1) and **2.2.5** (1). For the atomic formulas (1) is ensured by **2.2.8** (2). Then, due to complexity of the formula φ, a nontrivial step arises in induction only when the existential quantifier appears. Let us assume that φ has the form $(\exists x)\psi(x,u_1,\dots,u_n)$ and $\varphi[u_1^{\wedge},\dots,u_n^{\wedge}]=\mathbf{1}$, and let for ψ the statement (1) be fulfilled. In this case

$$\mathbf{1}=\vee\{[\![\psi(u,u_1^{\wedge},\dots,u_n^{\wedge})]\!]:u\in\mathbf{V}^{(2)}\}.$$

Therefore, $[\psi(v,u_1^{\wedge},\dots,u_n^{\wedge})]=\mathbf{1}$ for a certain $v\in\mathbf{V}^{(2)}$. By **2.2.8** (4), there is such an $u_0\in\mathbf{V}$ that $[u_0^{\wedge}=u]=1$. Hence, according to **2.1.8** (7), we get

$$\mathbf{1}=[\psi(v,u_1^{\wedge},\dots,u_n^{\wedge})]\wedge[v=u_0^{\wedge}]<[\psi(u_0^{\wedge},\dots,u_n^{\wedge})].$$

By the induction hypothesis, we have $\psi(u_0,\dots,u_n)$. Hence, $\varphi(u_1,\dots,u_n)$ is also valid. Vice versa, if $\varphi(u_1,\dots,u_n)$, then for some $u_0\in\mathbf{V}$ we get $\psi(u_0,u_1,\dots,u_n)$. And again, by the induction hypothesis, $[\psi(u_0^{\wedge},u_1^{\wedge},\dots,u_n^{\wedge})]=1$. Since, however, $[(\exists x)\psi(x,u_1^{\wedge},\dots,u_n^{\wedge})]\geq$ $\geq[\psi(u_0^{\wedge},u_1^{\wedge},\dots,u_n^{\wedge})]$; therefore, $[\varphi(u_1^{\wedge},\dots,u_n^{\wedge})]=\mathbf{1}$. $\rhd$

2.2.10. Remarks

(1) Let $\mathcal{U}$ be an ultrafilter in a Boolean algebra B, while $\mathcal{U}'$ be the ideal dual to it, i.e., $\mathcal{U}':=\{b^*: b\in\mathcal{U}\}$. Then the factor-algebra $B/\mathcal{U}'$ has two elements and can be identified with the Boolean algebra $\mathbf{2}:=\{\mathbf{0},\mathbf{1}\}$. The factor-homomorphism $\pi:B\to\mathbf{2}$ is not, generally speaking, full, and hence **2.2.4** and **2.2.5** cannot be applied to establish a relationship between the truth-values in $\mathbf{V}^{(B)}$ and $\mathbf{V}^{(2)}$. If, however, π is full then, by virtue of **2.2.5**, it is evident that for any formula $\varphi(x_1,\dots,x_n)$ and any set $u_1,\dots,u_n\in\mathbf{V}^{(B)}$ we get

$$\mathbf{V}^{(2)}\models\varphi(\pi*u_1,\dots,\pi*u_n)\leftrightarrow[\varphi(u_1,\dots,u_n)]\in\mathcal{U},$$

since for $b \in B$ the relations $\pi(b) = \mathbf{1}$ and $b \in \mathcal{U}$ are equivalent.

(**2**) By way of factorizing, a model other than $\mathbf{V}^{(2)}$ can be constructed from the universe $\mathbf{V}^{(B)}$ and the ultrafilter $\mathcal{U}$.

Let us introduce in $\mathbf{V}^{(B)}$ the relation $\sim_{\mathcal{U}}$ by the formula

$$\sim_{\mathcal{U}} := \{(x,y) \in \mathbf{V}^{(B)} \times \mathbf{V}^{(B)} : [x = y] \in \mathcal{U}\}.$$

It is obvious that $\sim_{\mathcal{U}}$ is an equivalence on $\mathbf{V}^{(B)}$. By the symbol $\mathbf{V}^{(B)}/\mathcal{U}$ let us denote the factor-class (see **1.5.8**) of the universe $\mathbf{V}^{(B)}$ by $\sim_{\mathcal{U}}$, considered with the binary relation

$$\in_{\sim\mathcal{U}} := \{(\tilde{x}, \tilde{y}) : x,y \in \mathbf{V}^{(B)} \wedge [x \in y] \in \mathcal{U}\},$$

where $x \to \tilde{x}$ is the canonical factor-mapping from $\mathbf{V}^{(B)}$ to $\mathbf{V}^{(B)}/\mathcal{U}$. We can also show that

$$\mathbf{V}^{(B)}/\mathcal{U} \models \varphi(\tilde{x}_1,\ldots,\tilde{x}_n) \leftrightarrow [\varphi(x_1,\ldots,x_n)] \in \mathcal{U}$$

for $x_1,\ldots,x_n \in \mathbf{V}^{(B)}$ and the formula φ.

The reader familiar with the theory of ultraproducts will recognize in (2) a known Łoš theorem (see [26,46, 96]). One can easily make sure in the in-depth relations between these phenomena. In (3) and (4) ultraproducts are obtained by the factorization of an appropriate Boolean-valued universe.

(**3**) Let T be a nonempty set of (not necessarily all) principal ultrafilters on a Boolean algebra B, and $\mathbf{V}^T$ be, as usual, the class of all mappings from T to $\mathbf{V}$. By virtue of **2.2.8** (4), for every $x \in \mathbf{V}^{(2)}$ there is a unique element $x^{\wedge} \in \mathbf{V}$ such that $[(x^{\wedge})^{\wedge} = x] = \mathbf{1}$. Let us now determine a mapping $h: \mathbf{V}^{(B)} \to \mathbf{V}^T$, assuming

$$h(x) := \{(t, \pi_t * x) : t \in T\} \quad (x \in \mathbf{V}^{(B)}),$$

where π_t is the full homomorphism from B to $\mathbf{2}$ determined by the ultrafilter t, i.e., $\pi_t(b) = \mathbf{1}$ if $b \in t$, and $\pi_t(b) = \mathbf{0}$ if $b \in t'$. It is also possible to demonstrate that h is a surjective mapping. On the other hand, h is injective iff any element $b \in B$ belongs to an ultrafilter $t \in T$, i.e., $(\forall b \in B)(\exists t \in T)\ (b \in t)$ (which asserts that T defines a dense set of points in the Stone space of the algebra B, or that B is atomic, or that B is isomorphic to the Boolean $\mathscr{P}(T)$). It is the last statement that is the Los theorem mentioned above. In this case for any $u_1,\ldots,u_n \in \mathbf{V}^{(B)}$ and the formula $\varphi(x_1,\ldots,x_n)$ we have

$$[\varphi(u_1,\ldots,u_n)] \leq b \leftrightarrow (\forall t \in T) \quad \big([\varphi(\pi_t * u_1,\ldots,\pi_t * u_n)] = \mathbf{1} \to b \in t\big).$$

(4) Let T be a set and $\mathcal{U}$ be an ultrafilter in the Boolean $\mathcal{P}(T)$. Let $\mathbf{V}^T/\mathcal{U}$ be the usual ultrapower of the class $\mathbf{V}$ over $\mathcal{U}$ with the canonical factor-mapping $g: \mathbf{V}^T \to \mathbf{V}^T/\mathcal{U}$ (see **1.5.7**). Let us put $\lambda(\tilde{x}) := g \circ h(x)$, where h is determined in (3), while $x \to \tilde{x}$ is the same as in (3). Therefore, a bijection λ is determined between $\mathbf{V}^{(\mathcal{P}(T))}/\mathcal{U}$ and $\mathbf{V}^T/\mathcal{U}$. In this case for any formula $\varphi(x_1,\dots,x_n)$ and functions $u_1,\dots,u_n \in \mathbf{V}^T$ we get

$$\mathbf{V}^T/\mathcal{U} \models \varphi(\tilde{u}_1,\dots,\tilde{u}_n) \leftrightarrow \{t \in T: \varphi(u_1(t),\dots,u_n(t))\} \in \mathcal{U}.$$

(5) It is expedient to compare **2.2.4** and **2.2.5** with the following statement. If M is a transitive model of ZFC (i.e., if M is a transitive class which is a model of ZFC), $u_1,\dots,u_n \in M$, $\varphi(x_1,\dots,x_n)$ is a bounded formula and $\psi(x_1,\dots,x_n)$ is a formula of class Σ_1, then

$$M \models \varphi(u_1,\dots,u_n) \leftrightarrow \varphi(u_1,\dots,u_n),$$
$$M \models \psi(u_1,\dots,u_n) \to \psi(u_1,\dots,u_n).$$

2.3. Mixings and the Maximum Principle

Let us consider a family of functions $(f_\xi)_{\xi\in\Xi}$ with domain A. If $(A_\xi)_{\xi\in\Xi}$ is a family of pairwise disjoint subsets of A, then we can define on A the function f whose restriction to A_ξ coincides with the restriction of f_ξ to A_ξ for all $\xi \in \Xi$. This function can be naturally called a disjunctive mixing of the family $(f_\xi)_{\xi\in\Xi}$. The Boolean-valued universe is complete in the sense that it contains all disjoint mixings of families of its elements. This peculiarity allows one to construct various special elements inside $\mathbf{V}^{(B)}$. Now let us go over to a more exact presentation.

2.3.1. A set consisting of mutually disjoint elements of a Boolean algebra is called an *antichain*. To be more exact, the set $A \subset B$ is called an antichain provided $a_1 \wedge a_2 = \mathbf{0}$ for any distinct $a_1, a_2 \in A$. If an antichain has the form $A := \{a_\xi : \xi \in \Xi\}$, then it is always assumed that $a_\xi \wedge a_\eta = \mathbf{0}$ as soon as $\xi \neq \eta$. The antichain $A \subset B$ is termed a *partition of the element* $b \in B$ (or a *partition of unity* when b is the unity of the algebra B), provided $b = \vee A$.

Let us consider an antichain $(b_\xi)_{\xi\in\Xi}$ in the Boolean algebra B and a family $(x_\xi)_{\xi\in\Xi}$ of elements of the universe $\mathbf{V}^{(B)}$. The *disjoint mixing* or simply the *mixing of the family*

(x_ξ) *relative to the antichain* (b_ξ) (sometimes they say *with probabilities* (b_ξ) or simply *by* (b_ξ)) is an element $x \in \mathbf{V}^{(B)}$ meeting the conditions

$$\mathrm{dom}(x) := \cup\{\mathrm{dom}(x_\xi) : \xi \in \Xi\},$$
$$x(t) := \vee\{b_\xi \wedge x_\xi(t) : \xi \in \Xi\} \quad (t \in \mathrm{dom}(x)).$$

The last equality implies that $x_\xi(t) = \mathbf{0}$ for $t \in \mathrm{dom}(x) - \mathrm{dom}(x_\xi)$. As long as $\alpha := \sup_{\xi \in \Xi} \rho(x_\xi) \in \mathrm{On}$, we have $\mathrm{dom}(x) \in \mathbf{V}^{(B)}_{\alpha+1}$. Hence, the given relation indeed determines a certain element $x \in \mathbf{V}^{(B)}$. The following conventional symbolic notation is used: $\mathrm{mix}_{\xi \in \Xi}(b_\xi x_\xi) := \mathrm{mix}\{b_\xi x_\xi : \xi \in \Xi\} : x$. In order to study the basic properties of mixings let us prove an auxiliary fact.

2.3.2. *Take* $x \in \mathbf{V}^{(B)}$ *and* $b \in B$. *Determine the function* bx *by the following relations*:

$$\mathrm{dom}(bx) := \mathrm{dom}(x), \quad bx : t \to b \wedge x(t) \quad (t \in \mathrm{dom}(x)).$$

Then $bx \in \mathbf{V}^{(B)}$, *and for any* x *and* $y \in \mathbf{V}^{(B)}$, *the following equalities are valid*:

$$[x \in by] = b \wedge [x \in y], \quad [bx = by] = b \Rightarrow [x = y].$$

◁ The first relation can be checked by a direct calculation of Boolean truth-values using the infinite distributive law **1.1.5** (2). Indeed,

$$[x \in by] = \bigvee_{t \in \mathrm{dom}(by)} (by)(t) \wedge [t = x]$$
$$= b \wedge \bigvee_{t \in \mathrm{dom}(y)} y(t) \wedge [t = x] = b \wedge [x \in y].$$

Then, using the preceding equality and successively applying **1.1.4** (2), **1.1.5** (6), **1.1.4** (4), **1.1.4** (2) and (**1.1.5** (6), we deduce

$$[bx = by]$$
$$= \bigwedge_{t \in \mathrm{dom}(by)} (by)(t) \Rightarrow [t \in bx] \wedge \bigwedge_{t \in \mathrm{dom}(bx)} (bx)(t) \Rightarrow [t \in by]$$
$$= \bigwedge_{t \in \mathrm{dom}(y)} (b \wedge y(t)) \Rightarrow (b \wedge [t \in x]) \wedge \bigwedge_{t \in \mathrm{dom}(x)} (b \wedge x(t)) \Rightarrow (b \wedge [t \in y])$$
$$= \bigwedge_{t \in \mathrm{dom}(y)} ((b \wedge y(t)) \Rightarrow b) \wedge ((b \wedge y(t)) \Rightarrow [t \in x])$$
$$\wedge \bigwedge_{t \in \mathrm{dom}(x)} ((b \wedge x(t)) \Rightarrow b) \wedge ((b \wedge x(t)) \Rightarrow [t \in y])$$
$$= \bigwedge_{t \in \mathrm{dom}(y)} b \Rightarrow (y(t) \Rightarrow [t \in x]) \wedge \bigwedge_{t \in \mathrm{dom}(x)} b \Rightarrow (x(t) \Rightarrow [t \in y])$$
$$= b \Rightarrow [x = y]. \ \triangleright$$

2.3.3. Theorem (the mixing principle). *Let $(b_\xi)_{\xi\in\Xi}$ be an antichain in B and $(x_\xi)_{\xi\in\Xi}$ be a family of elements of $\mathbf{V}^{(B)}$. Put $x := \operatorname{mix}_{\xi\in\Xi}(b_\xi x_\xi)$. Then*

$$[x = x_\xi] \geq b_\xi \quad (\xi\in\Xi).$$

If, moreover, $(b_\xi)_{\xi\in\Xi}$ is a partition of unity and an element $y\in\mathbf{V}^{(B)}$ obeys the relation $[y = x_\xi] \geq b_\xi$ for all $\xi\in\Xi$, then $[x = y] = 1$.

◁ By the definition of mixing, for any $\xi\in\Xi$ we have $b_\xi x = b_\xi x_\xi$. Applying **2.3.2**, we deduce

$$1 = [b_\xi x = b_\xi x_\xi] = b_\xi \Rightarrow [x_\xi = x].$$

Therefore, $[x_\xi = x] \geq b_\xi$ ($\xi\in\Xi$) (according to **1.1.4** (4)).

Let us now assume that (b_ξ) is a partition of unity and $[y = x_\xi] \geq b_\xi$ ($\xi\in\Xi$). Then by **2.1.8** (4), we get

$$b_\xi \leq [x = x_\xi] \wedge [x_\xi = y] \leq [x = y] \, (\xi\in\Xi).$$

Hence,

$$1 = \vee\{b_\xi : \xi\in\Xi\} \leq [x = y] \leq 1,$$

which completes proof. ▷

2.3.4. *Let $x\in\mathbf{V}^{(B)}$, and let us determine $\bar{x}\in\mathbf{V}^{(B)}$ by the relations*

$$\operatorname{dom}(\bar{x}) := \operatorname{dom}(x), \quad \bar{x}(t) := [t\in x] \quad (t\in\operatorname{dom}(x)).$$

In this case

$$\mathbf{V}^{(B)} \models x = \bar{x}.$$

◁ The aim can be achieved by performing the following simple calculations, making use of the definition of **2.1.4**, as well as of **1.1.4** (4) and **2.1.8** (2):

$$\begin{aligned}
&[x = \bar{x}] \\
&= \bigwedge_{t\in\mathrm{dom}(x)} x(t) \Rightarrow [t \in \bar{x}] \wedge \bigwedge_{t\in\mathrm{dom}(\bar{x})} [t\in x] \Rightarrow [t\in x] \\
&= \bigwedge_{t\in\mathrm{dom}(x)} x(t) \Rightarrow \Big(\bigvee_{u\in\mathrm{dom}(\bar{x})} \bar{x}(u) \wedge [u = t] \Big) \\
&\geq \bigwedge_{t\in\mathrm{dom}(x)} x(t) \Rightarrow x(t) \Rightarrow [t \in x] = \mathbf{1}. \ \rhd
\end{aligned}$$

2.3.5. *Let us choose a partition of unity* $(b_\xi)_{\xi\in\Xi} \subset B$ *and a family* $(x_\xi)_{\xi\in\Xi} \subset \mathbf{V}^{(B)}$. *Let us set* $x := \mathrm{mix}_{\xi\in\Xi}(b_\xi x_\xi)$. *Then the following statements are valid:*

(1) *if* $(x'_\xi)_{\xi\in\Xi} \subset \mathbf{V}^{(B)}$ *and* $\mathbf{V}^{(B)} \models x_\xi = x'_\xi$ $(\xi \in \Xi)$, *then*

$$\mathbf{V}^{(B)} \models x = \mathop{\mathrm{mix}}_{\xi\in\Xi}(b_\xi x'_\xi);$$

(2) *if an element* $y \in \mathbf{V}^{(B)}$ *is such that* $\mathrm{dom}(y) = \mathrm{dom}(x)$ *and*

$$y(t) := \bigvee_{\xi\in\Xi} b_\xi \wedge [t \in x_\xi] \quad (t \in \mathrm{dom}(y)),$$

then $\mathbf{V}^{(B)} \models x = y$.

$\lhd$ Let $x' := \mathrm{mix}_{\xi\in\Xi}(b_\xi x'_\xi)$. From the conditions we deduce

$$b_\xi \leq [x_\xi = x'_\xi] \wedge [x_\xi = x'] \wedge [x'_\xi = x'] \leq [x = x'],$$

and, hence, $[x = x'] = \mathbf{1}$. The last claim (2) follows from claim (1) and **2.3.4.** $\rhd$.

2.3.6. *For any* $b \in B$ *and* $x \in \mathbf{V}^{(B)}$ *the following formulas are valid:*

$$[bx = x] = b \vee [x = \varnothing], \quad [bx = \varnothing] = b^* \vee [x = \varnothing].$$

In particular,

$$\mathbf{V}^{(B)} \models bx = \mathrm{mix}\{bx, b^*, \varnothing\}.$$

$\lhd$ It should be remarked that $[t \in bx \to t \in x] = \mathbf{1}$, since, by virtue of **2.3.2**, $[t \in bx] = b \wedge [t \in x] \leq [t \in x]$. Therefore, $[bx = x \leftrightarrow (\forall t)(t \in x \to t \in bx)] = \mathbf{1}$. With this equality taken into account, we calculate

$$
\begin{aligned}
[bx = x] &= \bigwedge_{t\in V^{(B)}} [t\in x] \Rightarrow [t\in bx] \\
&= \bigwedge_{t\in V^{(B)}} [t\in x]^* \vee (b \wedge [t\in x]) \\
&= \bigwedge_{t\in V^{(B)}} (b \vee [t\in x]^*) \wedge ([t\in x]^* \vee [t\in x]) \\
&= \bigwedge_{t\in V^{(B)}} b \vee [t\in x]^* = b \vee \bigwedge_{t\in V^{(B)}} b \vee [t\in x]^* \\
&= b \vee [(\forall t)(t \notin x)] = b \vee [x = \varnothing].
\end{aligned}
$$

On the other hand, appealing to **2.3.2** again, and making use of the fact that $b\varnothing = \varnothing$, we can write

$$b^* \vee [x = \varnothing] = b \Rightarrow [x = \varnothing] = [bx = b\varnothing] = [bx = \varnothing]. \ \triangleright$$

2.3.7. *Let us assume that* (b_ξ) *is a partition of unity in* B, *and let a family* $(x_\xi) \subset V^{(B)}$ *be such that* $V^{(B)} \models x_\xi \neq x_\eta$ *for any* $\xi \neq \eta$. *Then there is an element* $x \in V^{(B)}$, *for which* $[x = x_\xi] = b_\xi$ *for all* ξ.

$\triangleleft$ Let us put $x := \operatorname{mix}(b_\xi x_\xi)$ and $a_\xi := [x = x_\xi]$. By hypothesis, we have

$$a_\xi \wedge a_\eta = [x = x_\xi] \wedge [x_\eta = x] \leq [x_\xi \neq x_\eta]^* = \mathbf{0}$$

for $\xi \neq \eta$. Moreover, due to the properties of mixing, $b_\xi \leq a_\xi$ for all ξ. Therefore, (a_ξ) is also a partition of unity in B. On the other hand,

$$b_\xi^* = \bigvee_{\eta\neq\xi} b_\eta \leq \bigvee_{\eta\neq\xi} a_\eta = a_\xi^*,$$

and, hence, $b_\xi^* \leq a_\xi^* \to b_\xi \geq a_\xi$. Therefore, the partitions of unity (b_ξ) and (a_ξ) coincide. $\triangleright$

The following fact, whose proof is based on a two-point mixing, often makes it possible to reduce the amount of bulky calculations.

2.3.8. *Let us consider* B*-formulas* $\varphi(x)$ *and* $\psi(x)$. *Assume that for a certain* $u_0 \in V^{(B)}$ *we have* $[\varphi(u_0)] := \mathbf{1}$. *Then*

$$[(\forall x)(\varphi(x) \to \psi(x))] = \wedge\{[\psi(u)] : u \in V^{(B)} \wedge [\varphi(u)] = \mathbf{1}\},$$
$$[(\exists x)(\varphi(x) \to \psi(x))] = \vee\{[\psi(u)] : u \in V^{(B)} \wedge [\varphi(u)] = \mathbf{1}\}.$$

◁ Prove the first equality. To begin with, it is evident (see **2.1.7**) that

$$c := [(\forall x)(\varphi(x) \to \psi(x))] = \bigwedge_{t \in \mathbf{V}^{(B)}} [\varphi(t)] \Rightarrow [\psi(t)]$$
$$\leq \bigwedge_{t \in \mathbf{V}^{(B)}, [\varphi(t)] = 1} [\varphi(t)]^* \vee [\psi(t)] = \bigvee_{t \in \mathbf{V}^{(B)}, [\varphi(t)] = 1} [\psi(t)] =: d.$$

In order to substantiate the reverse inequality $d \leq c$, let us choose an arbitrary element $t \in \mathbf{V}^{(B)}$ and put $u := \operatorname{mix}\{bt, b^* u_0\}$, where $b := [\varphi(t)]$. Then, by virtue of **2.1.8** (7) and **2.3.3**, we can estimate

$$b \leq [\varphi(t)] \wedge [t = u] \leq [\varphi(u)],$$
$$b^* \leq [\varphi(u_0)] \wedge [u = u_0] \leq [\varphi(u)].$$

Therefore, $[\varphi(u)] = 1$. In this case, by the same considerations,

$$b \wedge [\psi(u)] \leq [u = t] \wedge [\psi(u)] \leq [\psi(t)].$$

Hence, the following estimates are valid:

$$[\psi(u)] \leq b^* \vee (b \wedge [\psi(u)]) \leq b^* [\psi(t)]$$
$$= b \Rightarrow [\psi(t)] = [\varphi(t)] \Rightarrow [\psi(t)].$$

As long as $d \leq [\psi(u)]$, we have $d \leq [\varphi(t)] \Rightarrow [\psi(t)] \, (t \in \mathbf{V}^{(B)})$. Now, passing to the infimum over t on the right-hand side of the last inequality, we get $d \leq c$.

The last equality is dual to the previous one, since it is deduced from it according to the De Morgan formulas (see **1.1.2**). ▷

2.3.9. Let us now establish the central result of the present section, the maximum principle, stating that in the formula

$$[(\exists x)\varphi(x)] = \bigvee\{[\varphi(u)] : u \in \mathbf{V}^{(B)}\}$$

the least exact upper bound is attained at a certain element $u_0 \in \mathbf{V}^{(B)}$.

Let us first recall a fundamental property of complete Boolean algebras. Let B be a complete Boolean algebra. A set $E \subset B$ is called *minorizing*, or *minorant*, or *coinitial* in a subset $B_0 \subset B$ if for any $0 < b \in B_0$ there is such an $x \in E$ that $0 < x \leq b$.

(1) Theorem (the exhaustion principle). *Let M be a nonempty set of elements of a complete Boolean algebra B, and let E be a set minorizing in the component $B_0 \subset B$*

generated by the set M. Then there is an antichain $E_0 \subset E$ such that $\vee E_0 = \vee M$, and for each $x \in E_0$ there is a $y \in M$ for which $x \le y$.

◁ Let us consider a set $\mathfrak{U}$ of all antichains A obeying the following conditions: (1) $A \subset E$; (2) for any $x \in A$ there is a $y \in M$ for which $x \le y$. If $\mathbf{0} \ne y \in M$, then, by the condition of minorance, $y \ge x$ for some $\mathbf{0} \ne x \in E$. Hence, $\{x\} \in \mathfrak{U}$ and $\mathfrak{U}$ is non-empty. The set $\mathfrak{U}$ ordered by inclusion is easily checked to obey the conditions of the Kuratowski-Zorn lemma. Therefore, there is a maximal element $E_0 \in \mathfrak{U}$. The task is to show that the elements $b_0 := \vee E_0$ and $b := \vee M$ coincide. It follows from the definition of $\mathfrak{U}$ that $b_0 \le b$. If $b_0 \ne b$, then there are such elements $\mathbf{0} \ne x_0 \in B$ and $x \in M$, that $x_0 \wedge b_0 = \mathbf{0}$ and $x_0 \le x$. By the condition of minorance, $\mathbf{0} < y \le x$ for some $y \in E$. The set $E_0 \cup \{y\}$ is incorporated in $\mathfrak{U}$ and is essentially wider than E_0. This contradicts the fact that E_0 is minimal, and, hence, $b_0 = b$. ▷

(2) Corollary. *For any non-empty set $M \subset B$ there is an antichain $A \subset B$ with the following properties: $\vee A = \vee B$, and for any $x \in A$ there is a $y \in M$ such that $x \le y$.*

◁ We should choose a minorant set $E := \bigcup_{y \in M} [\mathbf{0}, y]$ and make use of **(1)**. ▷

2.3.10. Theorem (the maximum principle). *Let $\varphi(x, x_1, \dots, x_n)$ be a certain formula, and $u_1, \dots, u_n$ be arbitrary elements of $\mathbf{V}^{(B)}$. Then there is a $u_0 \in \mathbf{V}^{(B)}$ such that*

$$[(\exists x)\varphi(x, u_1, \dots, u_n)] = [\varphi(u_0, u_1, \dots, u_n)].$$

In particular, if $\mathbf{V}^{(B)} \models (\exists x)\varphi(x, u_1, \dots, u_n)$, then $\mathbf{V}^{(B)} \models \varphi(u_0, u_1, \dots, u_n)$ for some $u_0 \in \mathbf{V}^{(B)}$.

◁ By definition, we have

$$\begin{aligned} b &:= [(\exists x)\varphi(x, u_1, \dots, u_n)] \\ &= \bigvee_{u \in \mathbf{V}^{(B)}} [\varphi(u, u_1, \dots, u_n)]. \end{aligned}$$

The class $A := \{[\varphi(u, u_1, \dots, u_n)] : u \in \mathbf{V}^{(B)}\}$ is a subset of the algebra B. According to **2.3.9** (2), there is a partition $(b_\xi)_{\xi \in \Xi}$ of the element b and a family $(u_\xi)_{\xi \in \Xi}$ of the elements of $\mathbf{V}^{(B)}$, for which the following relations hold:

$$\begin{aligned} &b_\xi \le [\varphi(u_\xi, u_1, \dots, u_n)] \quad (\xi \in \Xi), \\ &b = \vee\{[\varphi(u_\xi, u_1, \dots, u_n)] : \ (\xi \in \Xi)\}. \end{aligned}$$

Let us set u_0: $\operatorname{mix}_{\xi\in\Xi}(b_\xi u_\xi)$ and recall that by **2.3.3** we have $b_\xi \le [u_0 = u_\xi]$ $(\xi \in \Xi)$. Obviously,

$$[\varphi(u_0,u_1,\dots,u_n)] \le b.$$

On the other hand, applying **2.1.8** (7), we get

$$b_\xi \le [u_0 = u_\xi] \wedge [\varphi(u_\xi, u_1,\dots,u_n)] \le [\varphi(u_0,\dots,u_n)].$$

Therefore,

$$[\varphi(u_0,\dots,u_n)] \ge \bigvee_{\xi\in\Xi} b_\xi = b.$$

The second part of the theorem is a direct corollary to the first one. ▷

2.4. The Transfer Principle

In this section we shall check if the universe $\mathbf{V}^{(B)}$ constructed over a complete Boolean algebra B can serve, together with the Boolean truth-values $[\cdot \in \cdot]$ and $[\cdot = \cdot]$, a Boolean model of ZFC. Or, more exactly, if the following fact is valid.

2.4.1. Theorem (the transfer principle). *Any theorem of* ZFC *is valid in* $\mathbf{V}^{(B)}$, *or, symbolically,* $\mathbf{V}^{(B)} \models ZFC$.

The proof of this theorem consists in checking the relations $\mathbf{V}^{(B)} \models ZF_k$ for $k = 1,2,\dots,6$ in $\mathbf{V}^{(B)} \models AC$. In this case the greater part of the effort is to be spent on routine calculations given below for the completeness of presentation.

2.4.2. *The axiom of extensionality* ZF_1 *is true in* $\mathbf{V}^{(B)}$:

$$\mathbf{V}^{(B)} \models (\forall x)(\forall y)\,(x = y \leftrightarrow (\forall z)(z \in x \leftrightarrow z \in y)).$$

◁ The proof results immediately from the definition of the Boolean truth-value of equality **2.1.4** (2) and from **2.1.9**. Indeed, for any x and $y \in \mathbf{V}^{(B)}$ we put

$$c := c(x,y) := [\![(\forall z \in x)(z \in y)]\!] = \bigwedge_{z \in \mathrm{dom}(x)} x(z) \Rightarrow [\![z \in y]\!].$$

Obviously, $c(x,y) \wedge c(y,x) = [\![x = y]\!]$ but, on the other hand,

$$c(x,y) \wedge c(y,x) = [\![(\forall z)(z \in x \leftrightarrow z \in y)]\!].$$

Hence, by **1.1.4** (5), we conclude

$$[\![x = y \leftrightarrow (\forall z)(z \in x \leftrightarrow z \in y)]\!] = \mathbf{1} \quad (x, y \in \mathbf{V}^{(B)}).$$

Now, taking infima over x and y, we complete the proof. ⊳

2.4.3. *The axiom of union* ZF_2 *is true in* $\mathbf{V}^{(B)}$:

$$\mathbf{V}^{(B)} \models (\forall x)(\exists y)(z \in y \leftrightarrow (\exists u \in x)(z \in u)).$$

⊲ Let us choose an arbitrary element $x \in \mathbf{V}^{(B)}$ and determine $y \in \mathbf{V}^{(B)}$ by the relations

$$\mathrm{dom}(y) := \bigcup \{\mathrm{dom}(u) : u \in \mathrm{dom}(x)\},$$
$$y(t) := [\![(\exists u \in x)(t \in u)]\!] \quad (t \in \mathrm{dom}(y)).$$

It suffices to show that $[\![y = \bigcup x]\!] = \mathbf{1}$.

In line with **2.1.9**, it holds that

$$[\![y \subset \bigcup x]\!] = [\![(\forall t \in y)(\exists u \in x)(t \in u)]\!]$$
$$= \bigwedge_{t \in \mathrm{dom}(y)} [\![(\exists u \in x)(t \in u)]\!] \Rightarrow [\![(\exists u \ni x)(t \in u)]\!] = \mathbf{1}.$$

Let us, then, remark that for $u \in \mathrm{dom}(x)$ and $z \in \mathrm{dom}(u)$ we have (see **2.1.8** (2) and **2.1.9**):

$$x(u) \wedge u(z) \le x(u) \wedge [\![z \in u]\!] \le \bigvee_{u \in \mathrm{dom}(x)} x(u) \wedge [\![z \in u]\!]$$
$$= [\![(\exists u \in x)(z \in u)]\!] = y(z) \le [\![(z \in y)]\!].$$

The above relation yields $x(u) \Rightarrow (u(z) \Rightarrow [\![z \in y]\!]) = \mathbf{1}$ (see **1.1.4** (2-4)). Taking into account this equality as well as $x(u) \Rightarrow (u(z) \Rightarrow [\![z \in y]\!]) = \mathbf{1}$ **2.1.9** and **1.1.5** (6), we can calculate

$$[\![\bigcup x \subset y]\!] = [\![(\forall u \in x)(\forall z \in u)(z \in y)]\!]$$
$$= \bigwedge_{u \in \mathrm{dom}(x)} x(u) \Rightarrow \left(\bigwedge_{z \in \mathrm{dom}(u)} u(z) \Rightarrow [\![z \in y]\!] \right)$$

$$= \bigwedge_{u \in \mathrm{dom}(x)} \bigwedge_{z \in \mathrm{dom}(u)} x(u) \Rightarrow (u(z) \Rightarrow [z \in y]) = \mathbf{1}.$$

Therefore, $[y = \cup x] = \mathbf{1}$, and, hence,

$$[(\exists u)(u = \cup x)] = \bigvee_{u \in \mathbf{V}^{(B)}} [u = \cup x] \geq [y = \cup x] = \mathbf{1}.$$

If we take the infimum over $x \in \mathbf{V}^{(B)}$, we get the required result

$$[(\forall x)(\exists y)(y = \cup x)] = \bigwedge_{x \in \mathbf{V}^{(B)}} [(\exists y)(y = \cup x)] = \mathbf{1}. \ \triangleright$$

2.4.4. *The axiom of powersets* ZF_3 *is true in* $\mathbf{V}^{(B)}$:

$$\mathbf{V}^{(B)} \models (\forall x)(\exists y)(\forall z)(z \in y \leftrightarrow z \subset x).$$

$\triangleleft$ Let us consider an arbitrary element $x \in \mathbf{V}^{(B)}$ and determine $y \in \mathbf{V}^{(B)}$ in such a way that

$$\mathrm{dom}(y) := B^{\mathrm{dom}(x)},$$
$$y(z) := [z \subset x] \quad (z \in \mathrm{dom}(y)).$$

It suffices to demonstrate that $[z \in y \leftrightarrow z \subset x] = \mathbf{1}$ for every $z \in \mathbf{V}^{(B)}$. It is obvious that

$$[z \in y] = \bigvee_{t \in \mathrm{dom}(y)} y(t) \wedge [t = z]$$
$$= \bigvee_{t \in \mathrm{dom}(y)} [t \subset x] \wedge [t = z] \leq [z \subset x].$$

Hence, $[z \in y \rightarrow z \subset x] = \mathbf{1}$ (see **1.1.4** (4)). Now we have to substantiate the equality $[z \subset x \rightarrow z \in y] = \mathbf{1}$. To this end, let us somewhat modify z, i.e., let us consider an element $z' \in \mathrm{dom}(y)$ such that $\mathrm{dom}(z') := \mathrm{dom}(x)$ and $z'(t) := [t \in z] \quad (t \in \mathrm{dom}(z'))$. Then for every $t \in \mathbf{V}^{(B)}$ we obtain

$$[t \in z'] = \bigvee_{u \in \mathrm{dom}(z')} z'(u) \wedge [t = u]$$
$$= \bigvee_{u \in \mathrm{dom}(z')} [u \in z] \wedge [u = t] \leq [t \in z],$$

and, hence, $[z' \subset z] = \mathbf{1}$. On the other hand, by virtue of **2.1.8** (5) and **2.1.9**, we obtain

$$[t \in z \cap x] = \bigvee_{u \in \mathrm{dom}(x)} x(u) \wedge [t = u] \wedge [t \in z]$$
$$\leq \bigvee_{u \in \mathrm{dom}(x)} z'(u) \wedge [t = u] \wedge [t \in z'],$$

and, thus, $[z \cap x \subset z'] = \mathbf{1}$ (and again **1.1.4** (4)!). Moreover,

$$[z \subset x] = \bigwedge_{t \in \mathbf{V}^{(B)}} [t \in z] \Rightarrow [t \in x] \leq \bigwedge_{t \in \mathrm{dom}(z')} z'(t) \Rightarrow [t \in x]$$
$$= [(\forall t \in z')(t \in x)] = [z' \subset x] = y(z') \leq [z' \in y].$$

Summing up all that has been said about z and z', we derive

$$[z \subset x] \leq [x \cap z \subset z'] \wedge [z' \subset z] \wedge [z \subset x] \leq [z = z'],$$
$$[z \subset x] \leq [z' \in y].$$

The last two relations immediately yield

$$[z \subset x] = [z \subset x] \wedge [z = z'] \leq [z' \in y] \wedge [z = z'] \leq [z \in y],$$

i.e., $[z \subset x] \leq [z \in y]$, which is equivalent to the sought result by virtue of **1.1.4** (4).

2.4.5. *The axiom of replacement* ZF_4^{φ} *is true in* $\mathbf{V}^{(B)}$*:*

$$\mathbf{V}^{(B)} \models (\forall u)(\forall v_1)(\forall v_2)(\varphi(u, v_1) \wedge \varphi(u, v_2) \rightarrow v_1 = v_2)$$
$$\rightarrow ((\forall x)(\exists y)(\exists s \in x)(\exists t)(\varphi(s, t) \rightarrow t \in y)).$$

⊲ In predicate calculus with equality the axiom of replacement can be deduced from that of choice (see **1.2.5**) and the formula

$$\Phi := (\forall x)((\forall t \in x)(\exists u)\varphi(t, u) \rightarrow (\exists y)(\forall t \in x)(\exists u \in y)\varphi(t, u))$$

(y does not occur freely in φ), i.e., $\Phi \wedge \Psi \rightarrow ZF_4^{\varphi}$, where Ψ is the axiom of choice. Therefore, it suffices to show that $\mathbf{V}^{(B)} \models \Phi$ and $\mathbf{V}^{(B)} \models \Psi$.

(a) $\mathbf{V}^{(B)} \models \Psi := (\forall x)(\exists y)(\forall t)\,(t \in y \leftrightarrow t \in x \wedge \psi(t))$.

Let us choose an arbitrary element $x \in \mathbf{V}^{(B)}$ and consider the function $y \in \mathbf{V}^{(B)}$ determined by the formulas

$$\mathrm{dom}(y) := \mathrm{dom}(x),$$
$$y(t) := x(t) \wedge [\psi(t)] \quad (t \in \mathrm{dom}(y)).$$

Then $[(\forall t)(t \in y \leftrightarrow t \in x \wedge \psi(t))] = a \wedge b$, where

$$a := [(\forall t \in y)(t \in x \wedge \psi(t))],\ b := [(\forall t \in x)(\psi(t) \rightarrow t \in y)].$$

From **2.1.8** (2) and **2.1.9**, however, it is easily deduced that $a = b = 1$. Indeed,

$$a = \bigwedge_{t \in \mathrm{dom}(y)} y(t) \Rightarrow [t \in x \wedge \psi(t)]$$
$$= \bigwedge_{t \in \mathrm{dom}(y)} x(t) \wedge [\psi(t)] \Rightarrow [t \in x] \wedge [\psi(t)] = 1.$$

Analogously,

$$b = \bigwedge_{t \in \mathrm{dom}(x)} x(t) \Rightarrow ([\psi(t)] \Rightarrow [t \in y])$$
$$= \bigwedge_{t \in \mathrm{dom}(x)} x(t) \wedge [\psi(t)] \Rightarrow [t \in x] \wedge [\psi(t)] = 1.$$

(**b**) $\mathbf{V}^{(B)} \models \Phi$. Let x be an arbitrary element of $\mathbf{V}^{(B)}$. Since B is a set, for every fixed $t \in \mathrm{dom}(x)$ the class

$$K := \{[\varphi(t,u)] : u \in \mathbf{V}^{(B)}\} \subset B$$

is a set. The axiom of replacement for sets (i.e., in $\mathbf{V}$) states that there is such an ordinal $\alpha(t)$ that

$$\{[\varphi(t,u)] : u \in \mathbf{V}^{(B)}_{\alpha(t)}\} = K.$$

Let us put $\alpha := \sup\{\alpha(t) : t \in \mathrm{dom}(x)\}$ and determine $y \in \mathbf{V}^{(B)}$ by the formulas

$$\mathrm{dom}(y) := \mathbf{V}^{(B)}_{\alpha},\ \mathrm{im}(y) = \{1\}.$$

In this case y is the sought element, as is shown by the following calculations:

$$[(\forall t \in x)(\exists u)\varphi(t,u)] = \bigwedge_{t \in \mathrm{dom}(x)} x(t) \Rightarrow \left(\bigvee_{u \in \mathbf{V}^{(B)}} [\varphi(t,u)]\right)$$
$$= \bigwedge_{t \in \mathrm{dom}(x)} x(t) \Rightarrow \left(\bigvee_{u \in \mathbf{V}^{(B)}_{\alpha(t)}} [\varphi(t,u)]\right)$$
$$= \bigwedge_{t \in \mathrm{dom}(x)} x(t) \Rightarrow \left(\bigvee_{u \in \mathbf{V}^{(B)}_{\alpha}} [\varphi(t,u)]\right)$$
$$= \bigwedge_{t \in \mathrm{dom}(x)} x(t) \Rightarrow [(\exists u \in y)\varphi(t,u)] = [(\forall t \in x)(\exists u \in y)\varphi(t,u)]. \ \triangleright$$

2.4.6. *The axiom of infinity* ZF_5 *is true in* $\mathbf{V}^{(B)}$:

$$\mathbf{V}^{(B)} \models (\exists x)(0 \in x \wedge (\forall t)(t \in x \to t \cup \{t\} \in x)).$$

$\lhd$ This axiom is fulfilled if we put $x := \omega^{\wedge}$ (see **2.2.7**). It is, first of all obvious that $[0^{\wedge} \in \omega^{\wedge}] = \mathbf{1}$, since $0^{\wedge} \in \mathrm{dom}(\omega^{\wedge})$. It should be remarked that for $t \in \mathbf{V}$ and $u := t \cup \{t\}$ we have $[u^{\wedge} = t^{\wedge} \cup \{t^{\wedge}\}] = \mathbf{1}$. Indeed, in line with **2.2.8** (1) we have

$$[v \in u^{\wedge}] = \bigvee_{s \in u} [s^{\wedge} = v] = [t^{\wedge} = v] \vee \bigvee_{s \in t} [s^{\wedge} = v]$$
$$[t^{\wedge} = v] \vee [v \in t^{\wedge}] = [t^{\wedge} = v \vee v \in t^{\wedge}] = [v \in t^{\wedge} \cup \{t^{\wedge}\}].$$

Taking into account this relation as well as **2.1.9** and **2.2.8** (2), we can easily calculate

$$[(\forall t \in \omega^{\wedge})(t \cup \{t\}) \in \omega^{\wedge}] = \bigwedge_{t \in \omega} [t^{\wedge} \cup \{t^{\wedge}\} \in \omega^{\wedge}]$$
$$= \bigwedge_{t \in \omega} [(t \cup \{t^{\wedge}\})^{\wedge} \in \omega^{\wedge}] = \mathbf{1}. \rhd$$

2.4.7. *The axiom of foundation* ZF_6 *is true in* $\mathbf{V}^{(B)}$*:*

$$\mathbf{V}^{(B)} \models (\forall x)(\exists y)(x = 0 \vee (y \cap x = 0)).$$

$\lhd$ Let us choose an arbitrary element $x \in \mathbf{V}^{(B)}$. Let us show that

$$b := [x \neq 0 \wedge (\forall y \in x)(y \cap x \neq 0)] = \mathbf{0}_B.$$

Let us assume $b \neq \mathbf{0}_B$. Since $b \leq [(\exists u)(u \in x)]$, there is an element $y_0 \in \mathbf{V}^{(B)}$, for which $[y_0 \in x] \wedge b \neq \mathbf{0}$ and $\rho(y_0) \leq \rho(y)$ for $[y \in x] \wedge b \neq \mathbf{0} (y \in \mathbf{V}^{(B)})$. Since, besides, for any $y \in \mathbf{V}^{(B)}$ the estimate

$$[y \in x] \wedge b \leq [y \cap x \neq 0] = \bigvee_{z \in \mathrm{dom}(y)} y(z) \wedge [z \in x]$$

is valid, $[z \in x] \wedge [y_0 \in x] \wedge b \neq \mathbf{0}$ for some $z \in \mathrm{dom}(y_0)$. However, $\rho(z) < \rho(y_0)$, which contradicts the choice of y_0. Therefore, $b = \mathbf{0}_B$ and, hence,

$$\mathbf{1}_B = b^* = [\neg(x \neq 0 \wedge (\forall y \in x)(y \cap x \neq 0))]$$
$$= [(\exists y)(x = 0 \vee (y \in x \wedge y \cap x = 0))].$$

The proof is completed by passing to the infimum over $x \in \mathbf{V}^{(B)}$. ▷

2.4.8 Now we have to check validity for the axiom of choice inside $\mathbf{V}^{(B)}$. To this end, some additional constructions will be necessary. Let us consider arbitrary elements $x, y \in \mathbf{V}^{(B)}$. Let us determine a singleton $\{x\}^B$, an unordered pair $\{x, y\}^B$ and an ordered pair $(x,y)^B$ inside $\mathbf{V}^{(B)}$ with the relations

$$\begin{aligned}&\operatorname{dom}(\{x\}^B) := \{x\}, \quad \operatorname{im}(\{x\}^B) := \{\mathbf{1}\};\\&\operatorname{dom}(\{x, y\}^B) := \{x, y\}, \quad \operatorname{im}(\{x, y\}^B) := \{\mathbf{1}\};\\&(x,y)^B := \{\{x\}^B, \ \{x, y\}^B\}^B.\end{aligned}$$

The elements $\{x\}^B$, $\{x, y\}^B$ and $(x,y)^B \in \mathbf{V}^{(B)}$ correspond to their names.

The following statements are valid:

$$\begin{aligned}&\mathbf{V}^{(B)} \models (\forall t)(t \in \{x\}^B \leftrightarrow t = x),\\&\mathbf{V}^{(B)} \models (\forall t)(t \in \{x, y\}^B \leftrightarrow t = x \vee t = y),\\&\mathbf{V}^{(B)} \models \text{‘}(x,y)^B \textit{ is the ordered pair of elements } x \textit{ and } y\text{’},\end{aligned}$$

or, in abbreviated form,

$$[\![\{x\}^B = \{x\}]\!] = [\![\{x,y\}^B = \{x, y\}]\!] = [\![(x, y)^B = (x, y)]\!] = \mathbf{1}.$$

◁ Let us check, for instance, the statement on an unordered pair. For any $t \in \mathbf{V}^{(B)}$ we have

$$\begin{aligned}[\![t \in \{x, y\}^B]\!] &= \vee\{[\![t = s]\!] : s \in \operatorname{dom}(\{x, y\}^B)\}\\&= [\![t = x]\!] \vee [\![t = y]\!] = [\![t = x \vee t = y]\!].\end{aligned}$$

Hence,

$$[\![(\forall t)(t \in \{x, y\}^B \leftrightarrow t = x \vee t = y)]\!] = \mathbf{1}. \ ▷$$

2.4.9. The notions introduced in the preceding item can be easily generalized for the case of an arbitrary $n > 2$. Let $x : n \to \mathbf{V}^{(B)}$. Then, by definition, $s := (x(0), \ldots x(n-1))^B \in \mathbf{V}^{(B)}$, provided there is a mapping $y : n \to \mathbf{V}^{(B)}$ such that

$$\begin{aligned}&y(0) = x(0), \quad y(n-1) = s,\\&y(k) = (y(k-1), x(k))^B \quad (0 < k \le n-1).\end{aligned}$$

Obviously, the function from $(\mathbf{V}^{(B)})^n$ to $\mathbf{V}^{(B)}$ is thus defined:

$$(x_0,\dots,x_{n-1}) \to (x_0,\dots,x_{n-1})^B \quad (x_0,\dots,x_{n-1} \in \mathbf{V}^{(B)}).$$

Let us note an important property of this function, confining ourselves for simplicity with the case in which $n = 2$. It should be recalled that for any $x, y, x', y' \in \mathbf{V}$ the equivalence

$$(x,y) = (x',y') \leftrightarrow x = x' \wedge y = y'$$

is valid. This statement is a theorem of ZF and, hence, it is also valid in the model $\mathbf{V}^{(B)}$ (by **2.4.2** - **2.4.7**). Therefore, for any $x, y, x', y' \in \mathbf{V}$ we have

$$[(x,y) = (x',y')] = [x = x'] \wedge [y = y'].$$

As long as $(x,y)^B$ is an ordered pair inside $\mathbf{V}^{(B)}$, we must have

$$[(x,y)^B = (x',y')^B] = [x = x'] \wedge [y = y'].$$

In particular,

$$\mathbf{V}^{(B)} \models (x,y)^B = (x',y')^B \leftrightarrow \mathbf{V}^{(B)} \models x = x' \wedge y = y',$$

i.e., the function $(\,;\,)^B$ is 'injective in the internal sense'. It goes without saying that it is also injective in the sense of $\mathbf{V}$, i.e., if $(x,y)^B$ and $(x',y')^B$ coincide as elements of $\mathbf{V}$, then $x = x'$ and $y = y'$. But still these are two different properties.

2.4.10. Let us recall that by theorem **1.4.3** an ordinal can be defined as a transitive set linearly ordered by the relation of membership E. In a symbolic presentation it is as follows:

$$\mathrm{Ord}(x) \leftrightarrow ((\forall u \in x)(\forall v \in u)(v \in x) \wedge$$
$$\wedge (\forall u \in x)(\forall v \in x)(u \in v \vee u = v \vee v \in u)).$$

We can see here that $\mathrm{Ord}(x)$ is a bounded formula, and, hence, according to **2.2.9** (2) valid is

$$\alpha \in \mathrm{On} \leftrightarrow \mathbf{V}^{(B)} \models \mathrm{Ord}(\alpha^{\wedge}).$$

Besides, in **2.2.8** **(2)** we have established that

$$[\alpha^{\wedge} = \beta^{\wedge}] = \mathbf{1} \leftrightarrow \alpha = \beta \; (\alpha, \beta \in \mathrm{On}).$$

2.4.11. *The axiom of choice* AC *is true in* $\mathbf{V}^{(B)}$:

$$\mathbf{V}^{(B)} \models (\forall x)(\exists y)\ (y \text{ is a function of choice on } x).$$

◁ In the theory of ZF it is provable that on the set x there is a function of choice provided there is an ordinal α and a function f such that $\alpha = \mathrm{dom}(f)$ and $\mathrm{im}(f) \supset u := \cup x$. Indeed, y can be determined by the formula

$$(t,s) \in y \leftrightarrow s \in t \wedge t \in x \wedge (\exists \alpha_0 \in \alpha)(f(\alpha_0) = s)$$
$$\wedge (\forall \beta \in \alpha)(f(\beta) \in t \rightarrow \alpha_0 \le \beta).$$

By virtue of **2.4.2** - **2.4.7**, this statement is also true inside $\mathbf{V}^{(B)}$, so it suffices to show that

$$\mathbf{V}^{(B)} \models (\forall u)(\exists \alpha)(\exists f)(\mathrm{Ord}(\alpha) \wedge \mathrm{Fnc}(f) \wedge \mathrm{dom}(f) = \alpha \wedge \mathrm{im}(f) \supset u).$$

Let us choose an arbitrary element $u \in \mathbf{V}^{(B)}$ and, using the axiom of choice for sets, find an ordinal α and a function g in such a way that $\mathrm{dom}(g) = \alpha$ and $\mathrm{dom}(u) \subset \mathrm{im}(g) \subset \mathbf{V}^{(B)}$. Let us determine $f \in \mathbf{V}^{(B)}$ via the relation

$$f := \left\{(\beta^\wedge, g(\beta))^B : \beta < \alpha\right\} \times \{\mathbf{1}_B\}.$$

Let us show that f obeys all the required conditions.

(a) $\mathbf{V}^{(B)} \models$ 'f is a binary relation'. Indeed, for an arbitrary $f \in \mathbf{V}^{(B)}$ we have

$$[t \in f] = \bigvee_{\beta < \alpha} [t = (\beta^\wedge, g(\beta))^B]$$
$$\le \vee\{[t = (x,y)^B] : x, y \in \mathbf{V}^{(B)}\} = [(\exists x)(\exists y)(t = (x,y))].$$

(b) $\mathbf{V}^{(B)} \models \mathrm{Fnc}(f)$. Taking into account **(a)**, we have only to show that f is unique inside $\mathbf{V}^{(B)}$. Let us choose arbitrary $t, s_1, s_2 \in \mathbf{V}^{(B)}$ and calculate, applying in succession **2.1.4** (1), **2.4.9**, **2.1.8** (4) and **2.2.8** (2):

$$[(t,s_1) \in f \wedge (t,s_2) \in f] = [(t,s_1)^B \in f] \wedge [(t,s_2)^B \in f]$$
$$= \bigvee_{\beta<\alpha} \bigvee_{\gamma<\alpha} [(t,s_1)^B = (\beta^\wedge, g(\beta))^B] \wedge [(t,s_2)^B = (\gamma^\wedge, g(\gamma))^B]$$
$$= \bigvee_{\beta<\alpha} \bigvee_{\gamma<\alpha} [t = \beta^\wedge] \wedge [t = \gamma^\wedge] \wedge [s_1 = g(\beta)] \wedge [s_2 = g(\gamma)]$$
$$\le \bigvee_{\beta<\alpha} \bigvee_{\gamma<\alpha} [\beta^\wedge = \gamma^\wedge] \wedge [s_1 = g(\beta)] \wedge [s_2 = g(\gamma)]$$

$$= \bigwedge_{\beta<\beta} [s_1 = g(\beta)] \wedge [s_2 = g(\beta)] \le [s_1 = s_2].$$

(c) $\mathbf{V}^{(B)} \models \mathrm{Ord}(\alpha^\wedge) \wedge \mathrm{dom}(f) = \alpha^\wedge$. The relation $\mathbf{V}^{(B)} \models \mathrm{Ord}(\alpha^\wedge)$ has been discussed in **2.4.10**. Then for $t \in \mathbf{V}^{(B)}$ we have

$$[t \in \mathrm{dom}(f)] = [(\exists s)(t,s) \in f] = \bigvee_{s \in \mathbf{V}^{(B)}} [(t,s) \in f]$$
$$= \bigvee_{s \in \mathbf{V}^{(B)}} \bigvee_{\beta < \alpha} [(t,s) = (\beta^\wedge, g(\beta))]$$
$$= \bigvee_{\beta < \alpha} \bigwedge_{s \in \mathbf{V}^{(B)}} [t = \beta^\wedge] \wedge [s = g(\beta)]$$
$$= \bigvee_{\beta < \alpha} [t = \beta^\wedge] = \bigvee_{\beta \in \mathrm{dom}(\alpha^\wedge)} [t = \beta] = [t \in \alpha^\wedge].$$

(d) $\mathbf{V}^{(B)} \models \mathrm{im}(f) \supset u$. Let us choose an $s \in \mathbf{V}^{(B)}$ and carry out the following calculations:

$$[s \in u] = \bigvee_{v \in \mathrm{dom}(u)} u(v) \wedge [s = v] \le \bigvee_{\beta < \alpha} [s = g(\beta)]$$
$$= \bigvee_{\beta < \alpha} \left([s = g(\beta)] \wedge \bigvee_{t \in \mathbf{V}^{(B)}} [\beta^\wedge = t] \right)$$
$$= \bigvee_{\beta < \alpha} \bigvee_{t \in \mathbf{V}^{(B)}} [(t,s) = (\beta^\wedge, g(\beta))]$$
$$= \bigvee_{t \in \mathbf{V}^{(B)}} [(t,s) \in f] = [(\exists t)(t,s) \in f] = [s \in \mathrm{im}(f)]. \ \rhd$$

The proof of theorem **2.4.1** is complete.

2.4.12. Remarks

(1) Substituting for the logical part of the language of ZF the laws of the intuitionistic logic (see **2.1.10** (3)), we come to the intuitionistic set theory ZF_1. The ZF_1 models can be constructed using the scheme presented. Namely, if Ω is a complete Heyting lattice, then the universe $\mathbf{V}^{(\Omega)}$ becomes a Heyting-valued model of the theory ZF_1 provided the corresponding truth-values $[\cdot \in \cdot]$ and $[\cdot = \cdot]$ acting from $\mathbf{V}^{(\Omega)} \times \mathbf{V}^{(\Omega)}$ to $\mathbf{V}^{(\Omega)}$ are determined. For details see [56, 73, 247].

(2) Let B be a (quantum) logic (see **1.5.11** (5)). If the truth-values $[\cdot \in \cdot]$ and $[\cdot = \cdot]$ are determined by **2.1.4** and the formulas are evaluated as in **2.1.7**, then in the universe $\mathbf{V}^{(B)}$ the axioms ZF_2-ZF_6 and AC are valid. Therefore, set theory can be developed in $\mathbf{V}^{(B)}$. In particular, the real numbers inside $\mathbf{V}^{(B)}$ will correspond to those observed in the mathematical model of a quantum-mechanical system (see [255]).

2.5.5. Separated Boolean-Valued Universe

In this section a separated Boolean-valued universe is built and the interpretation of NBG in it is given.

2.5.1. For elements x and y of the universe $\mathbf{V}^{(B)}$ the relation $\mathbf{V}^{(B)} \models x = y$ does not at all imply that x and y coincide as sets, i.e., as elements of $\mathbf{V}$. Indeed, if for every ordinal α we determine $x_\alpha \in \mathbf{V}^{(B)}$ by the formulas $\operatorname{dom}(x_\alpha) \in V_\alpha^{(B)}$, $\operatorname{im}(x_\alpha) := \{0\}$, then, as can be easily checked, $[x_\alpha = 0] = \mathbf{1}$ for all α. Therefore, each element of the class $\{x_\alpha : \alpha \in \mathrm{On}\}$ is presented as an empty set inside $\mathbf{V}^{(B)}$. We can make sure that for any $x \in \mathbf{V}^{(B)}$ there is a proper class $y \in \mathbf{V}^{(B)}$ such that $[x = y] = \mathbf{1}$. This peculiarity results in certain technical difficulties and, in particular, hampers translations from the language of $\mathbf{V}^{(B)}$ into that of $\mathbf{V}$. This defect of the $\mathbf{V}^{(B)}$ model is eliminated by a proper factorization (see **1.5.8**).

2.5.2. In the universe $\mathbf{V}^{(B)}$ let us introduce the equivalence $\sim$:

$$\sim := \{(x, y) \in \mathbf{V}^{(B)} \times \mathbf{V}^{(B)} : [x = y] = \mathbf{1}_B\}.$$

Let us consider a factor-class $\tilde{\mathbf{V}}^{(B)} := \mathbf{V}^{(B)} / \sim$, and let $\mathbf{V}^{(B)} \to \tilde{\mathbf{V}}^{(B)}$ be the canonical mapping. The class $\tilde{\mathbf{V}}^{(B)}$ is called a *separated Boolean-valued universe.* Let us introduce the Boolean truth-values for equality $[\cdot = \cdot]_s$, and of membership $[\cdot \in \cdot]_s$ for the class $\tilde{\mathbf{V}}^{(B)}$ by way of descending the corresponding functions $[\cdot = \cdot]_s$ and $[\cdot \in \cdot]_s$ on a factor-class:

$$[\cdot = \cdot]_s := [\cdot = \cdot] \circ (\pi^{-1} \times \pi^{-1}),$$
$$[\cdot \in \cdot]_s := [\cdot \in \cdot] \circ (\pi^{-1} \times \pi^{-1}).$$

Now for any formula $\varphi(u_1, \ldots, u_n)$ and for arbitrary $\tilde{x}_1, \ldots, \tilde{x}_n \in \tilde{\mathbf{V}}^{(B)}$ let us determine $[\varphi(\tilde{x}_1, \ldots, \tilde{x}_n)] \in B$ in the same way as in **2.1.7**. Then we must get

$$[\varphi(x_1, \ldots, x_n)] = [\varphi(\pi x_1, \ldots, \pi x_n)]_s \quad (x_1, \ldots, x_n \in \mathbf{V}^{(B)}).$$

The validity of formulas in $\tilde{\mathbf{V}}^{(B)}$ is set in the same way as in **2.1.6**:

$$\tilde{\mathbf{V}}^{(B)} \models \varphi(\tilde{x}_1, \ldots, \tilde{x}_n) \leftrightarrow [\varphi(\tilde{x}_1, \ldots, \tilde{x}_n)]_s = \mathbf{1}_B.$$

The soundness of the given definitions is obvious since by virtue of **2.1.8** (7) for any formula φ of ZFC we have

$$\mathbf{1} = [x = y] \to [\varphi(x)] = [\varphi(y)] \quad (x, y \in \mathbf{V}^{(B)}).$$

Therefore, when calculating Boolean values in a separated universe use can be made of arbitrary representatives of the required residue classes. This remark, in particular, yields that theorem **2.1.8** remains valid when $\tilde{\mathbf{V}}^{(B)}$ is substituted for $\mathbf{V}^{(B)}$ and when Boolean truth-values are supplied with the index s.

As a somewhat unexpected example, let us consider the following definition: for $\tilde{x} \in \tilde{\mathbf{V}}^{(B)}$ the symbol $\vee\tilde{x}$ denotes the *level of* $\tilde{x}$, i.e., the element

$$\vee\tilde{x} := \bigvee_{t \in \mathrm{dom}(x)} x(t),$$

where $\tilde{x}$ is the equivalence class of the element $x \in \mathbf{V}^{(B)}$. At first sight these definitions seem to be not quite legitimate, since the domain of the definition of the elements equal inside must not obligatorily coincide. At the same time,

$$[(\exists y \in \tilde{x})]_s = [(\exists y \in \tilde{x}) y = y]_s$$
$$= \bigvee_{t \in \mathrm{dom}(x)} x(t) \wedge [t = t] = \bigvee_{t \in \mathrm{dom}(x)} x(t) = \vee\tilde{x}.$$

Obviously, $\vee\tilde{x} = [x \neq \varnothing]_s$, and thus the notion of level is correct. In an analogous way, for any element $\tilde{x}$ of $\tilde{\mathbf{V}}^{(B)}$ and an element b of the Boolean algebra B, the element $\tilde{bx}: t \to b \wedge x(t)$ $((t \in \mathrm{dom}(x))$ is determined. Indeed, if $[x_1 = x_2] = \mathbf{1}$, then, by virtue of the earlier established **2.3.2**, $[bx_1 = bx_2] = b \Rightarrow [x_1 = x_2] = \mathbf{1}$. In this respect often is used the designation $0 = \varnothing$, which, in particular, implies that $0\varnothing = \varnothing = 0\tilde{x}$ for any $x \in \tilde{\mathbf{V}}^{(B)}$.

2.5.3. It should be remarked that the facts presented in **2.2** - **2.4** are, with some obvious specifications and refinements, also true in $\tilde{\mathbf{V}}^{(B)}$. Thus, in the sense of **2.4** $\tilde{\mathbf{V}}^{(B)}$ is a model of the theory of ZFC. Analogously, if ρ is a complete homomorphism of Boolean algebras, then ρ^* has any equivalence class invariant, and, hence ρ^* induces the only mapping of the corresponding separated universes which is also denoted by ρ^*, i.e., it is analogous with **2.2.2**, etc.. If $(x_\xi) \subset \mathbf{V}^{(B)}$, (b_ξ) is a disjoint family in B and $x = \mathrm{mix}(b_\xi x_\xi)$, then the element $\tilde{x} := \pi x$ will preserve the name 'a mixing' and the notation $\tilde{x} = \mathrm{mix}(b_\xi \tilde{x}_\xi)$ $(\tilde{x}_\xi = \pi x_\xi)$. Such a definition of mixing in $\tilde{\mathbf{V}}^{(B)}$ is correct (see **2.3.5** (1)). Therefore, if $\tilde{x} \in \tilde{\mathbf{V}}^{(B)}$ and $(\tilde{x}) \subset \tilde{\mathbf{V}}^{(B)}$, then the presentation $\tilde{x} = \mathrm{mix}(b_\xi \tilde{x}_\xi)$ implies that

$$b_\xi \leq [\tilde{x} = \tilde{x}_\xi]_s \ (\xi \in \Xi).$$

It should be remarked that if (b_ξ) is a partition of unity, then the mixing $\mathrm{mix}(b_\xi x_\xi)$ is unique (due to separation!) (see **2.3.3**).

The equality (see **2.4.9**)

$$[(x,y)^B = (x',y')^B] = [x = x'] \wedge [y = y']$$

shows the mapping $(\cdot,\cdot)^B$ to be stable relative to the equivalence relation in **2.5.2**. Hence, there is an injective embedding $\tilde{\mathbf{V}}^{(B)} \times \tilde{\mathbf{V}}^{(B)} \to \tilde{\mathbf{V}}^{(B)}$ denoted by the same symbol $(\cdot,\cdot)^B$, for which $(\pi x, \pi y)^B = \pi((x,y)^B)$. In this case

$$[(\tilde{x},\tilde{y})^B = (\tilde{x},\tilde{y})]_s = \mathbf{1} \ (\tilde{x},\tilde{y} \in \mathbf{V}^{(B)}).$$

The maximum principle is also preserved and has the following refinement.

2.5.4. *Let $\varphi(u,u_1,\dots,u_n)$ be a formula, $\tilde{x}_1,\dots,\tilde{x}_n \in \tilde{\mathbf{V}}^{(B)}$ and $\tilde{\mathbf{V}}^{(B)} \models (\exists! u)\varphi(u,\tilde{x}_1,\dots,\tilde{x}_n)$. Then there is a unique element $\tilde{x}_0 \in \tilde{\mathbf{V}}^{(B)}$ such that $\tilde{\mathbf{V}}^{(B)} \models \varphi(\tilde{x}_0,\tilde{x}_1,\dots,\tilde{x}_n)$.*

◁ Let $\tilde{x}_k := \pi(x_k)$, where $x_k \in \mathbf{V}^{(B)}$ $(k := 1,\dots,n)$. In this case $\mathbf{V}^{(B)} \models (\exists! u)\varphi(u,x_1,\dots,x_n)$. By transfer, there is an element $x_0 \in \mathbf{V}^{(B)}$, for which $\mathbf{V}^{(B)} \models \varphi(x_0,x_1,\dots,x_n)$. Let us put $\tilde{x}_0 := \pi(x_0)$. Obviously, $\tilde{\mathbf{V}}^{(B)} \models \varphi(\tilde{x}_0,\tilde{x}_1,\dots,\tilde{x}_n)$. If for an element $z \in \tilde{\mathbf{V}}^{(B)}$ we have $\tilde{\mathbf{V}}^{(B)} \models \varphi(z,\tilde{x}_1,\dots,\tilde{x}_n)$, then we get $\tilde{\mathbf{V}}^{(B)} \models \varphi(\tilde{x}_0,\dots,\tilde{x}_n) \wedge \varphi(z,\tilde{x}_1,\dots,\tilde{x}_n)$. By hypothesis, $\tilde{\mathbf{V}}^{(B)} \models z = \tilde{x}_0$, which implies, since $\tilde{\mathbf{V}}^{(B)}$ is separated, that $z = \tilde{x}_0$. ▷

2.5.5. For arbitrary b and $c \in B$ let us put (see **1.1.4**)

$$[b = c] := b \Leftrightarrow c = (b \Delta c)^* = (b \wedge c) \vee (b^* \wedge c^*).$$

It should be remarked that in line with **1.1.4** (3) $a \leq [b = c]$ iff $a \wedge b = a \wedge c$. Let us consider a function $f\colon \mathrm{dom}(f) \to B$, whose domain of definition $\mathrm{dom}(f)$ is contained in $\tilde{\mathbf{V}}^{(B)}$. They say that f is *extensional* if

$$[x = y]_s \leq [f(x) = f(y)] \ (x,y) \in \mathrm{dom}(f)).$$

The existence of f can be easily seen to be equivalent to the relation

$$f(x) \wedge [x = y]_s \leq f(y) \ (x,y) \in \mathrm{dom}(f)).$$

If $u\colon \mathrm{dom}(u) \to B$ is an arbitrary function and $\mathrm{dom}(u) \subset \tilde{\mathbf{V}}^{(B)}$, then u can be related with the extensional function $\bar{u}\colon \tilde{\mathbf{V}}^{(B)} \to B$ using the formula

$$\bar{u}\colon x \to \bigvee_{t \in \mathrm{dom}(u)} u(t) \wedge [t = x]_s \quad (x \in \tilde{\mathbf{V}}^{(B)}).$$

Another class of extensional functions arises in the following way. Let φ be a B formula. Then the following function is extensional

$$\bar{\varphi}\colon x \to [\varphi(x)]_s \quad (x \in \tilde{\mathbf{V}}^{(B)}).$$

2.5.6. Theorem. *If $u\colon \mathrm{dom}(u) \to B$ is a function, in which case $\mathrm{dom}(u) \subset \tilde{\mathbf{V}}^{(B)}$ and $\mathrm{dom}(u) \in \mathbf{V}$, then there exists a unique element $x \in \tilde{\mathbf{V}}^{(B)}$ such that $\bar{u}(t) = [t \in x]_s$ at all $t \in \tilde{\mathbf{V}}^{(B)}$. Vice versa, if $x \in \tilde{\mathbf{V}}^{(B)}$, then there exists a function $u\colon \mathrm{dom}(u) \to B$, for which $\mathrm{dom}(u) \subset \tilde{\mathbf{V}}^{(B)}$, $\mathrm{dom}(u) \in \mathbf{V}$ and $\bar{u}(t) = [t \in x]_s$ $(t \in \tilde{\mathbf{V}}^{(B)})$.*

◁ Let D be a subset of an unseparated universe whose image under the canonical factor-mapping π coincides with $\mathrm{dom}(u)$. Let us determine an element $x' \in \mathbf{V}^{(B)}$ by the formula

$$\mathrm{dom}(x') := D, \quad x'(t) := u(\pi t) \ (t \in D).$$

Let us, finally, put $x := \pi(x')$. Then for $t \in \tilde{\mathbf{V}}^{(B)}$ we get

$$[t \in x]_s = \bigvee_{y \in D} x'(y) \wedge [t \in \pi y]_s = \bigvee_{y \in \mathrm{dom}(u)} x(y) \wedge [y = t] = \bar{u}(t).$$

If some other element $z \in \tilde{\mathbf{V}}^{(B)}$ has this property, then $[t \in x]_s = [t \in z]_s$ for all $t \in \tilde{\mathbf{V}}^{(B)}$. Hence,

$$\tilde{\mathbf{V}}^{(B)} \models (\forall t)\,(t \in x \leftrightarrow t \in z).$$

By virtue of the axiom of extensionality, inside $\tilde{\mathbf{V}}^{(B)}$ we get $[x = z]_s = \mathbf{1}$. As $\tilde{\mathbf{V}}^{(B)}$ is separated, then $x = z$.

Now, conversely, let $x \in \tilde{\mathbf{V}}^{(B)}$, and let x' be such an element of the separated universe that $x = \pi(x')$. Let us put $\mathrm{dom}(u) := \pi``(\mathrm{dom}(x'))$ and determine $u\colon \mathrm{dom}(u) \to B$ in such way that $u(\pi t) = x'(t)$ $(t \in \mathrm{dom}(x'))$. In this case for any $t \in \tilde{\mathbf{V}}^{(B)}$ we have

$$[t \in x]_s = \bigvee_{y \in \mathrm{dom}(x')} x'(y) \wedge [t = \pi y]_s$$
$$= \bigvee_{y \in \mathrm{dom}(u)} u(y) \wedge [y = t]_s = \bar{u}(t). \triangleright$$

2.5.7. Below we shall basically operate with a separated Boolean-valued universe $\tilde{\mathbf{V}}^{(B)}$. In this case when calculating Boolean truth-values we often, without further specifications, replace elements of $\tilde{\mathbf{V}}^{(B)}$ with their representatives in $\mathbf{V}^{(B)}$ (it is, for instance, common practice when working with spaces of the equivalence classes of measurable functions). Besides, starting from the sentence to follow, we shall omit the sign $\sim$ and index s and write simply $\mathbf{V}^{(B)}$ $[\cdot = \cdot]$ and $[\cdot \in \cdot]$ instead of $\tilde{\mathbf{V}}^{(B)}$, $[\cdot = \cdot]_s$ and $[\cdot \in \cdot]_s$.

As is seen from **2.5.6**, any element of $\mathbf{V}^{(B)}$ determines a certain extensional mapping on $\mathbf{V}^{(B)}$ with values in B, but only some special-type extensional mappings from $\mathbf{V}^{(B)}$ in B are determined by elements from $\mathbf{V}^{(B)}$. This peculiarity serves as a motivation for the following definition.

2.5.8. *A class inside* $\mathbf{V}^{(B)}$ or the $\mathbf{V}^{(B)}$*-class* is any extensional mapping $X: \mathbf{V}^{(B)} \to B$, which is a class in the conventional sense, i.e., in the sense of $\mathbf{V}$. Therefore, we assign to each element $x \in \mathbf{V}^{(B)}$ the $\mathbf{V}^{(B)}$-class

$$< x > := [\cdot \in x] : t \to [t \in x] \quad (t \in \mathbf{V}^{(B)}).$$

Such a correspondence is obviously injective. Let us now introduce Boolean truth-values, setting for $\mathbf{V}^{(B)}$-classes X and Y and an element $z \in \mathbf{V}^{(B)}$:

$$[< z > \in X] := X(z),$$
$$[X = Y] := \bigwedge_{u \in \mathbf{V}^{(B)}} [< u > \in X] \Leftrightarrow [< u > \in Y],$$
$$[X \in Y] := \bigvee_{u \in \mathbf{V}^{(B)}} [< u > = X] \wedge [< u > \in Y].$$

The first and third formulas are consistent, since, as X is extensional, we have

$$[< z > \in X] = \bigvee_{u \in \mathbf{V}^{(B)}} X(u) \wedge [u = z]$$

and, moreover, $[< z > = < u >] = [z = u]$ at all $z \in \mathbf{V}^{(B)}$. It follows from the definitions that $[X = Y] = \mathbf{1}$ implies $X = Y$. The function $\mathbf{U}_B : x \to \mathbf{1}_B$ $(x \in \mathbf{V}_B)$ is a universal class inside $\mathbf{V}_B$. An empty $\mathbf{V}_B$-class is the function identical to zero on $\mathbf{V}_B$.

2.5.9. It should be recalled that a formula is termed *predicative* if only the variables for sets appear to be bounded in it (see **1.3.14**).

(1) Let us define the Boolean truth-value for a predicative formula. For propositional connectives it is carried out in the same way as in **2.1.7**, and we thus have to detalize only the case for the quantifiers, whose action is restricted to the class of sets. In this case we can consider only formulas containing no subformulas of the type $X_1 \in X_2$, since the latter is equivalent to the formula $(\exists x)(x = X_1 \wedge x \in X_2)$.

Thus, let φ be a predicative formula with free variables $X, X_1, \dots, X_n$, while $Y_1, \dots, Y_n$ be some $\mathbf{V}_B$-classes. Let us, by definition, put

$$[(\forall x)\varphi(x, Y_1, \dots, Y_n)] = \bigwedge_{y \in \mathbf{V}^{(B)}} [\varphi(y, Y_1, \dots, Y_n)],$$
$$[(\exists x)\varphi(x, Y_1, \dots, Y_n)] = \bigwedge_{y \in \mathbf{V}^{(B)}} [\varphi(y, Y_1, \dots, Y_n)].$$

Let us say that the predicative formula $\varphi(X_1, \dots, X_n)$ *is valid inside* $\mathbf{V}^{(B)}$ *for the given values* $Y_1, \dots, Y_n$ *of the variables* $X_1, \dots, X_n$ provided $[\varphi(Y_1, \dots, Y_n)] = \mathbf{1}$. As in **2.1.6**, we shall write

$$\mathbf{V}^{(B)} \models \varphi(Y_1, \dots, Y_n) \leftrightarrow [\varphi(Y_1, \dots, Y_n)] = \mathbf{1}.$$

(2) The notion of validity in the model $\mathbf{V}^{(B)}$ is spread on nonpredicative formulas in the following way. If $\varphi(X, X_1, \dots, X_n)$ is a predicative formula, then we set

$$\mathbf{V}^{(B)} \models (\forall X)\varphi(X, Y_1, \dots, Y_n) \quad (\mathbf{V}^{(B)} \models (\exists X)\varphi(X, Y_1, \dots, Y_n))$$

iff $[\varphi(Y, Y_1, \dots, Y_n)] = \mathbf{1}$ for any $\mathbf{V}^{(B)}$-class Y (respectively, there is such a $\mathbf{V}^{(B)}$-class Y that $[\varphi(Y, Y_1, \dots, Y_n)] = \mathbf{1}$).

The $\mathbf{V}^{(B)}$-class Y is called a $\mathbf{V}^{(B)}$*-set*, if $\mathbf{V}^{(B)} \models M(Y)$, where $M(X) := (\exists Z)(X \in Z)$ (see **1.3.1**). It would be simpler to say 'a B-set' instead of 'a $\mathbf{V}^{(B)}$-set'; this term, however, will be preserved to be used somewhere else (see **3.4**).

2.5.10. *For every* $x \in \mathbf{V}^{(B)}$ *a* $\mathbf{V}^{(B)}$*-class* $< x >$ *is a* $\mathbf{V}^{(B)}$*-set. Conversely, if a* $\mathbf{V}^{(B)}$*-class* X *is a* $\mathbf{V}^{(B)}$*-set, then* $x = X$ *for a certain* $x \in \mathbf{V}^{(B)}$.

$\lhd$ For an arbitrary element $x \in \mathbf{V}^{(B)}$ we have

$$[< x > \in < \{x\}^B >] = [< x > \in \{x\}^B] = \mathbf{1},$$

and, hence, $\mathbf{V}^{(B)} \models M(<x>)$. Let us assume that for a $\mathbf{V}^{(B)}$-class X $\mathbf{V}^{(B)} \models M(<X>)$ holds. Then, by definition (see **2.5.9** (2)), there is a $\mathbf{V}^{(B)}$-class Z, for which

$$\bigvee_{t \in \mathbf{V}^{(B)}} Z(t) \wedge [<t> = X] = \mathbf{1}.$$

Hence, by virtue of the exhaustion principle, we can choose such a partition of the unity, $(b_\xi)_{\xi \in \Xi}$ and such a family $(x_\xi)_{\xi \in \Xi} \subset \mathbf{V}^{(B)}$ that

$$[<x_\xi> = X] \geq b_\xi.$$

If $x := \operatorname{mix}(b_\xi x_\xi)$, then

$$[<x> = X] \geq [<x> = <x_\xi>] \wedge [<x_\xi> = X] \geq b_\xi,$$

and, hence, $[<x> = X] = \mathbf{1}$ or $<x> = X$. $\triangleright$

On the basis of the fact just established we shall henceforth identify an element $x \in \mathbf{V}^{(B)}$ and the corresponding $\mathbf{V}^{(B)}$-set $<x>$.

2.5.11. Let C be a complete Boolean algebra and $\pi : B \to C$ be a full homomorphism. Let us consider a $\mathbf{V}^{(B)}$-class X, and put by definition,

$$(x,b) \in \pi^* X \leftrightarrow b = \bigvee_{t \in \mathbf{V}^{(B)}} (\pi \circ X)(t) \wedge [x = \pi^* t]^C.$$

In this case $\pi * X$ is a class inside $\mathbf{V}^{(B)}$. Indeed, $\pi * X$ is a $\mathbf{V}$ subclass by virtue of theorem **1.3.14**, since

$$\pi * X = \{(x,b) : \varphi(x,b,B,C,X,\pi^*,[\cdot = \cdot],\mathbf{V}^{(B)})\}$$

for the predicative formula

$$\varphi(Y,Z,B,\ldots) := Z = \bigvee_{t \in \mathbf{V}^{(B)}} (\pi \circ X)(t) \wedge [Y = \pi * t].$$

Besides, $\pi * X$ is an extensional function:

$$(\pi * X)(x) \wedge [x = y] = \bigvee_{t \in \mathbf{V}^{(B)}} (\pi \circ X)(t) \wedge [x = \pi * t]$$
$$\wedge [x = y] \leq \bigvee_{t \in \mathbf{V}^{(B)}} (\pi \circ X)(t) \wedge [y = \pi * t] = (\pi * X)(y).$$

One can easily notice that statement **2.2.2** (1) remains valid for classes, i.e., if ρ is a full homomorphism, then

$$(\rho \circ \pi) * X = (\rho * \circ \pi^*) X.$$

Then, if $\mathbf{V}^{(B)} \models M(X)$, then $\mathbf{V}^{(C)} \models M(\pi * X)$. Indeed, if $X = < x >, x \in \mathbf{V}^{(B)}$, then, by **2.2.4** (4), we have

$$\begin{aligned}(\pi * x)(t) &= \bigvee_{u \in \mathbf{V}^{(B)}} \pi([u = x]) \wedge [t = \pi * u] \\ &= \bigvee_{u \in \mathbf{V}^{(B)}} (\pi \circ < x >)(u) \wedge [t = \pi * u] = (\pi^* < x >)(t).\end{aligned}$$

Therefore, $< \pi * x > = \pi^* < x > = \pi * X$. The converse statement is also valid if π is injective.

It should be also remarked that due to **2.2.2** (4) the above statement agrees with **2.2.1**.

2.5.12. For every $\mathbf{V}^{(B)}$-class X and for any predicative B-formula φ with a single free variable the following presentations are valid:

$$[(\forall x \in \pi * X)\ \varphi(x)]^C = \bigwedge_{t \in \mathbf{V}^{(B)}} \pi \circ X(t) \Rightarrow [\varphi(\pi * t)]^C,$$

$$[(\exists x \in \pi * X)\ \varphi(x)]^C = \bigwedge_{t \in \mathbf{V}^{(B)}} \pi \circ X(t) \wedge [\varphi(\pi * t)]^C.$$

$\triangleleft$ It suffices to substantiate one of these relations, for instance, the first one. Here are the corresponding calculations (where use has been made of **1.1.5** (3), **2.1.8** (7) and $(a \wedge b) \Rightarrow (c \wedge b) = (a \wedge b) \Rightarrow c$):

$$\begin{aligned}[(\forall x \in \pi * X)\ \varphi(x)] &= \bigwedge_{x \in \mathbf{V}^{(C)}} [x \in \pi * X)] \Rightarrow [\varphi(x)] \\ &= \bigwedge_{x \in \mathbf{V}^{(C)}} \left(\bigvee_{t \in \mathbf{V}^{(B)}} \pi \circ X(t) \wedge [x = \pi * t] \right) \Rightarrow [\varphi(x)] \\ &= \bigwedge_{t \in \mathbf{V}^{(B)}} \bigwedge_{x \in \mathbf{V}^{(C)}} (\pi \circ X(t) \wedge [x = \pi * t]) \Rightarrow [\varphi(x)] \\ &\le \bigwedge_{t \in \mathbf{V}^{(B)}} \pi \circ X(t) \Rightarrow [\varphi(\pi * t)] \\ &= \bigwedge_{t \in \mathbf{V}^{(B)}} \bigwedge_{x \in \mathbf{V}^{(C)}} (\pi \circ X(t) \wedge [x = \pi * t]) \Rightarrow ([\!|\varphi(\pi * t)] \wedge [x = \pi * t]) \\ &\le \bigwedge_{t \in \mathbf{V}^{(B)}} \bigwedge_{x \in \mathbf{V}^{(C)}} (\pi \circ X(t) \wedge [x = \pi * t]) \Rightarrow [\varphi(x)] \\ &= \bigwedge_{t \in \mathbf{V}^{(C)}} \left(\bigvee_{t \in \mathbf{V}^{(B)}} \pi \circ X(t) \wedge [x = \pi * t] \right) \Rightarrow [\varphi(x)]\end{aligned}$$

$$= \bigwedge_{x \in \mathbf{V}^{(C)}} [x \in \pi * X] \Rightarrow [\varphi(x)] = [(\forall x \in \pi * X)\, \varphi(x)]. \triangleright$$

2.1.13. For any $\mathbf{V}^{(B)}$-classes X *and* Y *the following relations hold*:

$$[\pi * X = \pi * Y] = \pi[X = Y],\ [\pi * X \in \pi * Y] = \pi[X \in Y].$$

$\triangleleft$ Let us first of all remark that $\pi \circ Y(t) = (\pi * Y)(\pi * t)$ or $\pi[t \in Y] = [\pi * t \in \pi * Y]$ for $t \in \mathbf{V}^{(B)}$ (this result follows from **2.5.8** and **2.5.11** by virtue of **2.2.2** (3)). Then, using the first formula of **2.5.12**, we easily deduce

$$[\pi * X \subset \pi * Y]^C = [(\forall x \in \pi * X)\,(x \in \pi * Y)]^C$$
$$= \bigwedge_{t \in \mathbf{V}^{(B)}} \pi \circ X(t) \Rightarrow [\pi * t \in \pi * Y]^C$$
$$= \bigwedge_{t \in \mathbf{V}^{(B)}} \pi\big([t \in X]^B \Rightarrow [t \in Y]^B\big) = \pi[X \subset Y].$$

These relations give

$$[\pi * X = \pi * Y] = [\pi * X \subset \pi * Y] \wedge [\pi * Y \subset \pi * X] = \pi[X = Y].$$

Finally, taking into account the above, according to the second formula of **2.5.12**, we get

$$[\pi * X \in \pi * Y]^C = [(\exists t \in \pi * Y)\,(t = \pi * X)]^C$$
$$= \bigvee_{t \in \mathbf{V}^{(B)}} \pi \circ Y(t) \wedge [\pi * t \in \pi * X]^C$$
$$= \bigvee_{t \in \mathbf{V}^{(B)}} \pi\big(Y(t) \wedge [t = X]^B\big) = \pi[X \in Y]^B. \triangleright$$

2.5.14. As in **2.2**, we can derive various corollaries from the results obtained above. Let us note only the following.

(1) *If* $\varphi(Y_1,\dots,Y_n)$ *is a bounded predicative formula, then for any* $\mathbf{V}^{(B)}$*-classes* $X_1,\dots,X_n$ *we have*

$$\pi[\varphi(X_1,\dots,X_n)] = [\varphi(\pi * X_1,\dots,\pi * X_n)].$$

This, in particular, implies that π *is a monomorphism, then*

$$\mathbf{V}^{(B)} \models \varphi(X_1,\dots,X_n) \leftrightarrow \mathbf{V}^{(C)} \models \varphi(\pi * X_1,\dots,\pi * X_n).$$

(2) *If φ is a predicative formula of class Σ_1, then for the same $X_1,\dots,X_n$ we get*

$$\pi[\varphi(X_1,\dots,X_n)] \le [\varphi(\pi * X_1,\dots,\pi * X_n)].$$

In particular, the following implication is valid:

$$\mathbf{V}^{(B)} \models \varphi(X_1,\dots,X_n) \to \mathbf{V}^{(C)} \models \varphi(\pi * X_1,\dots,\pi * X_n).$$

◁ The proof is carried out by the scheme of **2.2.3**. As an example, let us take the case of a bounded universal quantifier: $\varphi := (\forall x \in Y)\psi$. By **2.5.12** and **2.5.13**, for the $\mathbf{V}^{(B)}$-classes $X_1,\dots,X_n$ we have

$$\begin{aligned}
&[\varphi(\pi * Y, \pi * X_1,\dots,\pi * X_n)] \\
&= \bigwedge_{x \in \mathbf{V}^{(B)}} [\pi * x \in \pi * Y] \Rightarrow [\psi(\pi * x, \pi * X_1,\dots,\pi * X_n)] \\
&= \bigwedge_{x \in \mathbf{V}^{(B)}} \pi[X \in Y] \Rightarrow \pi[\psi(x, X_1,\dots,X_n)] \\
&= \pi\Big(\bigwedge_{x \in \mathbf{V}^{(B)}} [X \in Y] \Rightarrow [\psi(x, X_1,\dots,X_n)]\Big) \\
&= \pi[(\forall x \in Y\ \psi(x, X_1,\dots,X_n)] = \pi[\varphi(Y, X_1,\dots,X_n)].\ ▷
\end{aligned}$$

2.5.15. Making use of the canonical embedding $(\cdot)^\wedge : \mathbf{V} \to \mathbf{V}^{(B)}$, to each class $x \subset \mathbf{V}$ we assign a $\mathbf{V}^2$-class X' by the formula:

$$X'(t) := \mathbf{1}_2 \text{ if } (\exists x \in X)(t = x^\wedge),$$
$$X'(t) := \mathbf{0}_2 \text{ in the opposite case.}$$

It trivially follows from **2.1.8** (4) that X' is extensional. Then, let us put $X^\wedge := \iota * " X'$, where ι is the identical embedding of $\mathbf{2}$ in B. Hence, $X^\wedge$ is a $\mathbf{V}^{(B)}$-class, for which

$$X^\wedge(t) = \vee\{[t = x^\wedge] : x \in X\} \quad (t \in \mathbf{V}^{(B)}).$$

It should be remarked that since $\mathrm{Ord}(X)$ is a bounded predicative function, by **2.2.8** (4), **2.2.9** (1) and **2.5.14**, $\mathrm{On}^\wedge$ is an ordinal class inside $\mathbf{V}^{(B)}$, i.e., $\mathbf{V}^{(B)} \models \mathrm{Ord}(\mathrm{On}^\wedge)$.

Formulas **2.5.12** can obviously be specialized:

$$[(\forall x \in Y^\wedge)\ \varphi(x)] = \wedge\{[\varphi(x^\wedge)] : x \in Y\},$$
$$[(\exists x \in Y^\wedge)\ \varphi(x)] = \vee\{[\varphi(x^\wedge)] : x \in Y\}.$$

2.5.16. Let φ and ψ be predicative functions with free variables $X, X_1, \dots, X_n$, and let $Y_1, \dots, Y_n$ be some $\mathbf{V}^{(B)}$-classes. Then, if $[\varphi(x_0, Y_1, \dots, Y_n)] = \mathbf{1}$ for a certain $x_0 \in \mathbf{V}^{(B)}$, we have

$$
\begin{aligned}
&[(\exists x)(\varphi(x, Y_1, \dots, Y_n) \to \psi(x, Y_1, \dots, Y_n))] \\
&= \vee\{[\psi(x, Y_1, \dots, Y_n)] : x \in \mathbf{V}^{(B)} \wedge [\varphi(x, Y_1, \dots, Y_n)] = \mathbf{1}\}, \\
&[(\forall x)(\varphi(x, Y_1, \dots, Y_n) \to \psi(x, Y_1, \dots, Y_n))] \\
&= \wedge\{[\psi(x, Y_1, \dots, Y_n)] : x \in \mathbf{V}^{(B)} \wedge [\varphi(x, Y_1, \dots, Y_n)] = \mathbf{1}\}.
\end{aligned}
$$

◁ The proof is carried out by the same scheme as in **2.3.8.** ▷

2.5.17. Theorem (the maximum principle). *Let $\varphi(x)$ be a predicative B-formula with one free variable (which implies that φ can contain constants which are $\mathbf{V}^{(B)}$-classes or $\mathbf{V}^{(B)}$-sets). Then the following statements are valid:*

(1) *there is an element $x_0 \in \mathbf{V}^{(B)}$ for which*

$$[(\exists x)\, \varphi(x)] = [\varphi(x_0)];$$

(2) *if $\mathbf{V}^{(B)} \models (\exists x)\, \varphi(x)$, then there is an element $x_0 \in \mathbf{V}^{(B)}$ for which $\mathbf{V}^{(B)} \models \varphi(x_0)$;*

(3) *if $\mathbf{V}^{(B)} \models (\exists! x)\, \varphi(x)$, then there is a unique element $x_0 \in \mathbf{V}^{(B)}$ for which $\mathbf{V}^{(B)} \models \varphi(x_0)$.*

◁ The proof based on the mixing principle (see **2.5.3**) does not differ from the considerations given in **2.3.10** and **2.5.4.** ▷

2.5.18. Theorem (the transfer principle). *All the theorems of* NBG *are true within* $\mathbf{V}^{(B)}$.

◁ It suffices to make sure that the axioms of NBG are true inside $\mathbf{V}^{(B)}$.

(**a**) The axiom of extensionality for classes inside $\mathbf{V}^{(B)}$ is true, which follows directly from definitions **2.5.8** and **2.5.9**. The statement $\mathbf{V}^{(B)} \models \mathrm{NBG}_2$-$\mathrm{NBG}_5$ was proved in **2.4**.

(**b**) $\mathbf{V}^{(B)} \models \mathrm{NBG}_6$. The proof is carried out as in **2.4.5**, with the expressions $\varphi(t,u)$ substituted for by $(t,u) \in X$ throughout (see **2.4.5** and **1.3.4**).

(c) $\mathbf{V}^{(B)} \models \wedge_{k=7}^{13} \mathrm{NBG}_k$. It suffices to establish that inside $\mathbf{V}^{(B)}$ true is the statement **1.3.14**, the axioms NBG_7-NBG_{13} being its partial cases. Let the formula $\varphi(X_1,\dots,X_n,Y_1,\dots,Y_m)$ obey all the conditions of **1.3.14**. Let us consider arbitrary $\mathbf{V}^{(B)}$-classes $Y_1,\dots,Y_m$ and determine a $\mathbf{V}^{(B)}$-class Z by the formula

$$Z(t) := [(\exists x_1,\dots,x_n)(t = (x_1,\dots,x_n) \\ \wedge \varphi(x_1,\dots,x_n, Y_1,\dots,Y_m))].$$

It is easily checked that in this case

$$\mathbf{V}^{(B)} \models (\forall x_1,\dots,x_n)(\exists t)((t = (x_1,\dots,x_n) \\ \wedge t \in Z \leftrightarrow \varphi(x_1,\dots,x_n,Y_1,\dots,Y_N))).$$

(d) $\mathbf{V}^{(B)} \models \mathrm{NBG}_{14}$. We get the required result if we substitute the lowercase Latin letter x with the capital one X in **2.4.7**.

(e) $\mathbf{V}^{(B)} \models \mathrm{NBG}_{15}$. Let G be a function from On to $\mathbf{V}^{(B)}$. Let us put

$$F(t) := \vee\{[t = (\alpha^{\wedge}, G(\alpha))^{B}] : \alpha \in \mathrm{On}\}.$$

Then F is a $\mathbf{V}^{(B)}$-class and by analogy with **2.4.10**, we can successively calculate: $[\mathrm{Fnc}(F)] = \mathbf{1}$, $[\mathrm{Ord}(\mathrm{On}^{\wedge}) \wedge \mathrm{dom}(F) = \mathrm{On}^{\wedge}] = \mathbf{1}$ and $[\mathrm{im}(F) \supset \mathbf{U}_B] = \mathbf{1}$. Therefore, inside $\mathbf{V}^{(B)}$ the universal class $\mathbf{U}_B$ can be well-ordered. Hence, $\mathbf{V}^{(B)} \models$ 'there exists a selecting function for the class $\mathbf{U}_B$'. ▷

2.5.19. On the basis of theorem **2.1.18** there arises a possibility of operating with classes inside $\mathbf{V}^{(B)}$. As an example, let us consider the following definition of category in a Boolean-valued model.

A category $\mathfrak{K}$ inside $\mathbf{V}^{(B)}$ consists of classes $\mathrm{Ob}\,\mathfrak{K}$, $\mathrm{Mor}\,\mathfrak{K}$ and Com inside $\mathbf{V}^{(B)}$, which are called *the class of objects, the class of morphisms and the composition of the category* $\mathfrak{K}$,respectively, such that $\mathbf{V}^{(B)} \models (\mathfrak{K}\,1) - (\mathfrak{K}\,3)$:

($\mathfrak{K}$ 1) there are mappings D and R from $\mathrm{Mor}\,\mathfrak{K}$ to $\mathrm{Ob}\,\mathfrak{K}$ such that for any objects a and b the class $\mathfrak{K}(a,b) := H_{\mathfrak{K}}(a,b) := \{\alpha \in \mathrm{Mor}\,\mathfrak{K} : D(\alpha) = a, R(\alpha) = b\}$ is a set (termed the *set of morphisms from* a *to* b);

($\mathfrak{K}$ 2) Com is an associative partial binary operation on $\mathrm{Mor}\,\mathfrak{K}$ and

$$\mathrm{dom}(\mathrm{Com}) := \{(\alpha,\beta) \in (\mathrm{Mor}\,\mathfrak{K})^2 : D(\beta) = R(\alpha)\};$$

($\mathfrak{K}$ 3) for every object $a \in \mathrm{Ob}\,\mathfrak{K}$ there is a morphism 1_a which is called *the identity morphism of the object a* such that $D(1_a) = R(1_a) = a$, $\mathrm{Com}(1_a,\alpha) = \alpha$ for $R(\alpha) = a$ and $\mathrm{Com}(\beta,1_a) = \beta$ for $D(\beta) = a$.

We usually write $\beta\alpha$ or $\beta \circ \alpha$ instead of $\mathrm{Com}(\alpha,\beta)$.

2.5.20. Remarks

(1) The model $\mathbf{V}^{(B)}$ can be characterized axiomatically. Namely, there is a class $\mathbf{V}^{(B)}$ unique up to a bijection preserving all the Boolean truth-values and obeying the following conditions:

(a) there are two mappings, $[\cdot \in \cdot], [\cdot = \cdot]: \mathbf{V}^{(B)} \times \mathbf{V}^{(B)} \to B$ such that the conventional axioms of equality hold inside $\mathbf{V}^{(B)}$ (see **2.1.7**, **2.1.8**);

(b) $\mathbf{V}^{(B)}$ is separated, i.e., $[x = y] = 1_B$ implies that $x = y$ for $x, y \in \mathbf{V}^{(B)}$;

(c) the axioms of extensionality and foundation are true inside $\mathbf{V}^{(B)}$;

(d) for $\mathbf{V}^{(B)}$, statement **2.5.6** is valid.

(2) Let π be a full homomorphism from B to a complete Boolean algebra C. Then π^* is a unique mapping from $\mathbf{V}^{(B)}$ to $\mathbf{V}^{(C)}$, for which, first,

$$[\pi^* x = \pi^* y]^C = \pi[x = y]^B \quad (x,y \in \mathbf{V}^{(B)}),$$

and, second, for $y \in \mathbf{V}^{(B)}$ and $z \in \mathbf{V}^{(C)}$,

$$[z \in \pi^* y]^C \le \bigvee_{x \in \mathbf{V}^{(B)}} [z = \pi^* x].$$

CHAPTER 3

FUNCTORS OF BOOLEAN-VALUED ANALYSIS

Owing to the transfer and maximum principles, various constructions of conventional mathematical practice are possible within a Boolean-valued universe. In particular, in such a model there are fields of real numbers, Banach spaces, differential operators, etc.. The objects presenting them serve, in a certain sense, as a nonstandard realizations of the initial mathematical constructions. Therefore, assuming that the model $\mathbf{V}^{(B)}$ is a nonstandard presentation of the mathematical world and taking into account the fact that $\mathbf{V}^{(B)}$ is constructed within the von Neumann universe, we are in a position as if we look inside the Boolean-valued world and see a standard presentation of nonstandard objects. While examining algebras B item-by-item, an observer sees a great number of modifications of the same idea encoded in a set-theoretic formula. It is comparing them with one another that comprises the method of studying the mathematical idea they are based on. Besides, one discovers that essentially different analytical objects are nothing but presentations of the same concept. This fact enables one to clarify the internal reasons accounting for many analogies, as well as facilitates the appearance of new opportunities to study old objects.

(The situation presented above resembles the famous image of the Platon cave. If someone from this cave has managed to flee into the sunlight, then he, wishing to tell the others what he has seen, could light several fires outside this cave. In that event the things would throw not one but many distinctive shadows on the cave walls and the prisoners of the cave could penetrate into the essence of the things in-depth by way of comparing their shadows.)

The comparative analysis with the help of Boolean-valued models is carried out in two stages which can be conditionally called syntactic and semantic. At the syntactic stage the mathematical statement under investigation (a definition, a construction, a property, etc.) is transformed into a formal text of the symbolic language of set theory, or, to be more precise, into a text in a suitable jargon. In this stage we often have to investigate the complexity of the text under study and, in particular, to see if the text or some of its fragments is bounded formulas (Σ_1-formulas). The semantic stage is interpretation of a formal text in a Boolean-valued universe. In this stage we use the terms of the conventional set theory, i.e., the universe $\mathbf{V}$, to interpret (decode, translate) meaningful texts for objects of the Boolean-valued universe $\mathbf{V}^{(B)}$. This is carried out with elements and subsets of the Boolean-valued universe and the von Neumann universe by using exactly defined operations. In the present chapter we shall consider the basic operations of Boolean-valued analysis, i.e., the canonical embedding, descent, ascent and immersion. The most important properties of these operations can be conveniently explored by using the notions of category and functor, the latter being not obligatory. Therefore, assuming that the reader is not a specialist in the theory of categories, we shall make use of the initial notions of the aforementioned theory.

3.1. Canonical Embedding

This section is devoted to a detailed investigation of the method of embedding the class of all sets into a Boolean-valued universe.

3.1.1. Theorem. *The following statements are valid:*

(1) *if a class* $X \subset \mathbf{V}$ *and an element* $z \in \mathbf{V}^{(B)}$ *are such that* $\mathbf{V}^{(B)} \models z \in X^{\wedge}$, *then* $z = \operatorname{mix}_{x \in X}(b_x x^{\wedge})$ *for a certain partition of unity* $(b_x)_{x \in X}$ *in* B;

(2) *for a* $\mathbf{V}^{(2)}$*-class* Y *there is a unique class* $X \subset \mathbf{V}$ *such that* $\mathbf{V}^{(2)} \models X^{\wedge} = Y$;

(3) *for* $X \subset \mathbf{V}$ *and* $Y \subset \mathbf{V}$ *we have:*

$$X \in Y \leftrightarrow \mathbf{V}^{(B)} \models X^{\wedge} \in Y^{\wedge},\ X = Y \leftrightarrow \mathbf{V}^{(B)} \models X^{\wedge} = Y^{\wedge};$$

(4) *if* $\pi : B \to C$ *is a full homomorphism, then* $\pi^* : X^{\wedge} = \overset{\wedge}{X^{\wedge}}$ *for every class* $X \subset \mathbf{V}$, *where* $\overset{\wedge}{X^{\wedge}}$ *is the canonical embedding of* X *in* $\mathbf{V}^{(C)}$.

◁ (1) For $x \in X$ let us put $b_x := [x^{\wedge} = z]$. Then, by **2.2.8** (2), for $x, y \in X, x \neq y$,

$$b_x \wedge b_y \leq [x^{\wedge} = y^{\wedge}] = \mathbf{0}.$$

On the other hand,

$$\vee\{b_x : x \in X\} = X^{\wedge}(z) = [z \in X^{\wedge}] = \mathbf{1},$$

so that $(b_x)_{x \in X}$ is a partition of unity, and $z = \operatorname{mix}_{x \in X}(b_x x^{\wedge})$.

(2) The proof follows from **2.2.8**. Indeed, if $X' := \{y \in \mathbf{V}^{(2)} : [y \in Y] = \mathbf{1}_2\}$ and $X := \{x \in \mathbf{V} : x^{\wedge} \in X'\}$, then, according to **2.2.8** (3,4), for $t \in \mathbf{V}^{(2)}$ we get

$$X \wedge (t) = \vee\{[t = x^{\wedge}]^2 : x \in X\} = \vee\{[t = x^{\wedge}]^2 : Y(x) = \mathbf{1}_2\}$$

$$= \vee\{Y(x) \wedge [t = x^\wedge]^2 : x \in \mathbf{V}^{(2)}\} = Y(t).$$

The uniqueness follows from **2.2.8** (4) and **2.5.15**.

(3) We have to compare **2.5.15** and (2).

(4) If ι_1 and ι_2 are embeddings of the algebra **2** in B or C, respectively, then $\pi \circ \iota_1 = \iota_2$ and, in line with **2.5.11**,

$$\pi * X^\wedge = \pi * \circ \iota *_1 (X^\wedge) = \iota *_2 X^\wedge = X^{\overset{\wedge}{\wedge}}. \ \triangleright$$

3.1.2. *If x and y are sets then*

$$\{x\}^\wedge = \{x^\wedge\}^B, \quad \{x, y\}^\wedge = \{x^\wedge, y^\wedge\}^B, \quad (x,y)^\wedge = (x^\wedge, y^\wedge)^B.$$

$\triangleleft$ All these three formulas are bounded and, hence, from **2.2.9** we deduce

$$\mathbf{V}^{(B)} \models \{x\}^\wedge = \{x^\wedge\} \wedge \{x,y\}^\wedge = \{x^\wedge, y^\wedge\} \wedge (x, y)^\wedge = (x^\wedge, y^\wedge).$$

All we have to do now is to use the necessary relations of **2.4.8.** $\triangleright$

3.1.3. *Let a formula φ of class Σ_1 obey all the conditions of theorem* **1.3.14.** *Let us choose classes $Z_1,\dots,Z_n$, $Y_1,\dots,Y_m$, and let the class Y be determined by the formula*

$$Y := \{(x_1,\dots,x_n) : x_1 \in Z_1 \wedge \dots \wedge x_n \in Z_n$$
$$\wedge\, \varphi(x_1,\dots,x_n, Y_1,\dots,Y_m)\}.$$

In this case inside $\mathbf{V}^{(B)}$ the following relation holds:

$$Y^\wedge = \{(x_1,\dots,x_n) : x_1 \in Z_1^\wedge \wedge \dots \wedge x_n \in Z_n^\wedge \wedge \varphi(x_1,\dots,x_n, Y_1^\wedge,\dots,Y_m^\wedge)\}.$$

$\triangleleft$ According to theorem **1.3.14**, Y is the only class obeying the conditions $\Phi(Z_1,\dots,Z_n, Y_1,\dots,Y_m)$ and $\Psi(Z_1,\dots,Z_n, Y_1,\dots,Y_m)$, where Φ and Ψ have the form

$$\Phi := (\forall u \in Y)(\exists x_1 \in Z_1)\dots(\exists x_n \in Z_n)(u = (x_1,\dots,x_n) \wedge \varphi(x_1,\dots,Y_m)),$$
$$\Psi := (\forall x_1 \in Z_1)\dots(\forall x_n \in Z_n)(\exists u)$$
$$(u = (x_1,\dots,x_n) \wedge \varphi(x_1,\dots,Y_m) \to u \in Y).$$

Obviously, Φ and Ψ are formulas of class Σ_1 and, hence, by **2.5.14**, we have

$$\mathbf{V}^{(B)} \models \Phi(Z_1^{\wedge},\dots,Y_m^{\wedge}) \wedge \Psi(Z_1^{\wedge},\dots,Y_m^{\wedge}).$$

The relation obtained is equivalent to the sought one. ▷

3.1.4. *For any classes* $X \subset \mathbf{V}$ *and* $Y \subset \mathbf{V}$ *the following statements are valid:*

(1) $\mathbf{V}^{(B)} \models (X \cup Y)^{\wedge} = X^{\wedge} \cup Y^{\wedge}$;

(2) $\mathbf{V}^{(B)} \models (X \times Y)^{\wedge} = X^{\wedge} \times Y^{\wedge}$;

(3) $\mathbf{V}^{(B)} \models (\cup X)^{\wedge} = \cup(X^{\wedge})$;

(4) $\mathrm{Rel}(X) \to \mathbf{V}^{(B)} \models \mathrm{Rel}(X^{\wedge})$;

(5) $(F: X \to Y) \to \mathbf{V}^{(B)} \models F^{\wedge}: X^{\wedge} \to Y^{\wedge}$;

(6) $\mathrm{Rel}(X) \to \mathbf{V}^{(B)} \models (X``Y)^{\wedge} = (X^{\wedge})``(Y^{\wedge})$;

(7) $\mathrm{Rel}(X) \to \mathbf{V}^{(B)} \models \mathrm{dom}(X^{\wedge}) = \mathrm{dom}(X)^{\wedge} \wedge \mathrm{im}(X^{\wedge}) = \mathrm{im}(X)^{\wedge}$.

◁ Formulas (1) - (5) result from **3.1.3** (see **I.3.1.11**, **I.3.1.12**). Statements (6) and (7) are beyond the scope of applicability of **3.1.3** and they are therefore deduced by direct calculations (with use made of **2.4.9**, **3.1.1** and **3.1.2**):

(6) $$\begin{aligned}[t \in (X^{\wedge})``(Y^{\wedge})] &= [(\exists u \in X^{\wedge})(\exists v \in Y)(u = (v,t))] \\ &= \bigvee_{u \in X} \bigvee_{v \in Y} [u^{\wedge} = (v^{\wedge},t)] = \bigvee_{v \in Y} \bigvee_{(z,\omega) \in X} [z^{\wedge} = v^{\wedge}] \wedge [\omega^{\wedge} = t] \\ &= \wedge\{[\omega^{\wedge} = t] : v \in Y \wedge (v,\omega) \in X\} \\ &= [(\exists \omega \in (X^{\wedge})``(Y^{\wedge}))(t = \omega)] = [t \in (X``Y)^{\wedge}].\end{aligned}$$

(7) $$\begin{aligned}[t \in \mathrm{dom}(X^{\wedge})] &= [(\exists u \in X^{\wedge})(\exists v)(u = (t,v))] \\ &= \bigvee_{(z,\omega) \in X} \bigvee_{v \in \mathbf{V}^{(B)}} [z^{\wedge} = t] \wedge [\omega^{\wedge} = v] \\ &= \vee\{[z^{\wedge} = t] : z \in \mathrm{dom}(X)\} = [t \in \mathrm{dom}(X)^{\wedge}]. \ \triangleright\end{aligned}$$

3.1.5. Theorem. *Let X and Y be nonempty sets, $F \subset X \times Y$ and consider a correspondence $\Phi := (F,X,Y)$. Then an element $\Phi^\wedge \in \mathbf{V}^{(B)}$ obeys the following conditions:*

(1) $\mathbf{V}^{(B)} \models \Phi^\wedge$ *is a correspondence from* $X^\wedge$ *to* $Y^\wedge$, *and* $\mathrm{Gr}\Phi^\wedge = F^\wedge$;

(2) $\mathbf{V}^{(B)} \models \Phi^\wedge(A^\wedge) = \Phi(A)^\wedge$ *at all* $A \in \mathcal{P}(X)$;

(3) $\mathbf{V}^{(B)} \models (\Psi \circ \Phi)^\wedge = \Psi^\wedge \circ \Phi^\wedge$ *for any correspondence* Ψ;

(4) $\mathbf{V}^{(B)} \models (I_X)^\wedge = I_{X^\wedge}$.

◁ (1) If the formula $\varphi(X,Y,F,\Phi)$ states that Φ is a correspondence from $\varphi(X,Y,F,X)$ to Y and $F = \mathrm{Gr}\Phi$, then φ is a bounded formula and the required result follows from **2.2.9**.

(2) The result follows from **3.1.4** (6).

(3), (4). Here we again have to do with bounded formulas, and, hence, it suffices to refer to **2.2.9**. ▷

3.1.6. Corollary. *For any mapping $f: X \to Y$ the element $f^\wedge$ obeys the conditions:*

(1) $\mathbf{V}^{(B)} \models f^\wedge : X^\wedge) \to Y^\wedge$;

(2) $\mathbf{V}^{(B)} \models f^\wedge(x^\wedge) = (\mathrm{x})^\wedge$ *for all* $x \in X$;

(3) $\mathbf{V}^{(B)} \models (g \circ f)^\wedge = g^\wedge \circ f^\wedge$ *for any* $g: Y \to Z$.

3.1.7. Let us introduce the categories $\mathcal{V}_*$ and $\mathcal{V}_*^{(B)}$ that are associated with the universes $\mathbf{V}$ and $\mathbf{V}^{(B)}$. Without specifying it every time, let us assume that the classes of objects and morphisms of a category do not intersect (this can be achieved by using different indices). Let $\mathcal{V}_*$ be the category of nonempty sets and correspondences, so that $\mathrm{Ob}\,\mathcal{V}_* = \mathbf{V} \setminus \{\varnothing\}$ and $\mathcal{V}_*(x,y)$ is a set of all nonempty correspondences from x to y, the composition being the conventional superposition of correspondences.

The class of objects of the category $\mathcal{V}_*^{(B)}$ is formed by nonempty $\mathbf{V}^{(B)}$-sets:

$$\mathrm{Ob}\,\mathcal{V}_*^{(B)} := \{x \in \mathbf{V}^{(B)} : [x \neq \varnothing] = 1\}.$$

The set of morphisms from an object $x \in \mathrm{Ob}\, \mathcal{V}_*^{(B)}$ into an object $y \in \mathrm{Ob}\, \mathcal{V}_*^{(B)}$ is determined by the formula

$$\mathcal{V}_*^{(B)}(x,y) := \{\alpha \in \mathbf{V}^{(B)} : [\alpha \text{ is a correspondence from } x \text{ to } y \wedge \mathrm{Gr}\alpha \neq \varnothing] = \mathbb{1}\}.$$

If α and β are morphisms of the category $\mathcal{V}_*^{(B)}$, in which case $[D(\beta) = R(\alpha)] = \mathbb{1}$, then there is a unique element $\gamma \in \mathbf{V}^{(B)}$ (the principle of maximum!) such that $[\gamma = \beta \circ \alpha] = \mathbb{1}$. It is this element γ that is assumed to be the composition of the morphisms α and β in the category $\mathbf{V}^{(B)}$.

The subcategories of $\mathcal{V}_*$ and $\mathcal{V}_*^{(B)}$ consisting of the same objects and of the mappings as morphisms will be denoted by $\mathcal{V}$ and $\mathcal{V}^{(B)}$, respectively. By the symbol $\mathfrak{F}^\wedge$ we shall denote the mapping from $\mathcal{V}_*$ to $\mathcal{V}_*^{(B)}$, putting into correspondence to the set $x \in \mathbf{V} \setminus \{0\}$ and to the correspondence α the elements $x^\wedge \in \mathbf{V}^{(B)}$ and $\alpha^\wedge \in \mathbf{V}^{(B)}$. The following theorem results from **3.1.5** and **3.1.6**.

3.1.8. Theorem. *The mapping $\mathfrak{F}^\wedge$ is a covariant functor from the category $\mathcal{V}_*$ to the category $\mathcal{V}_*^{(B)}$ (as well as from the category $\mathcal{V}$ to the category $\mathcal{V}^{(B)}$).*

The functor $\mathfrak{F}^\wedge$ (as well as its restriction on the subcategory $\mathcal{V}$) is termed *the canonical embedding functor* or *the standard name functor.*

3.1.9. Let us dwell on the properties of ordinals inside $\mathbf{V}^{(B)}$.

(1) It is known (see **2.4.10**) that $\mathrm{Ord}(X)$ is a bounded formula. Since $\lim(\alpha) \leq \alpha$ for any ordinal α, then the formula $\mathrm{Ord}(x) \wedge x = \lim(x)$ can be written as

$$\mathrm{Ord}(x) \wedge (\forall t \in x)(\exists s \in x)(t \in s),$$

and, hence, it is also bounded. Finally, the presentation

$$\mathrm{Ord}(x) \wedge x = \lim(x) \wedge (\forall t \in x)(t = \lim(t) \rightarrow t = 0)$$

states that 'the least limiting ordinal' is also a bounded formula. Therefore, by **2.2.9**, α is the (least) limiting ordinal iff $\mathbf{V}^{(B)} \models$ '$\alpha^\wedge$ is the (least) limiting ordinal'. Since ω is the least limiting ordinal (see **1.4.6**); therefore, $\mathbf{V}^{(B)} \models$ '$\omega^\wedge$ is the least limiting ordinal'.

(2) It follows from **1.4.5** (2), **2.5.15** and **2.5.16** that $\mathbf{V}^{(B)} \models$ '$\mathrm{On}^\wedge$ is the only ordinal class which is not an ordinal'. Therefore, for any $x \in \mathbf{V}^{(B)}$ the following relation is valid:

$$[\![\mathrm{Ord}(x)]\!] = \vee\{[\![x = \alpha^\wedge]\!] : \alpha \in \mathrm{On}\}.$$

(3) *For an arbitrary* $x \in \mathbf{V}^{(B)}$, *the equality* $\mathbf{V}^{(B)} \models \mathrm{Ord}(x)$ *holds iff there is an ordinal* $\beta < \mathrm{On}$ *and a partition of unity* $(b_\alpha)_{\alpha\in\beta} \subset B$ *such that* $x = mix_{\alpha\in\beta}(b_\alpha \alpha^\wedge)$. *In other words, any ordinal inside* $\mathbf{V}^{(B)}$ *is a mixing of a certain set of standard ordinals.*

⊲ This result follows from (2) and **3.1.1** (1). ⊳

(4) From **2.5.15** we get the quantification formulas over ordinals:

$$[\![(\forall x)(\mathrm{Ord}(x) \to \psi(x)]\!] = \bigwedge_{\alpha\in\mathrm{On}} [\![\psi(\alpha^\wedge)]\!],$$
$$[\![(\exists x)(\mathrm{Ord}(x) \wedge \psi(x)]\!] = \bigwedge_{\alpha\in\mathrm{On}} [\![\psi(\alpha^\wedge)]\!].$$

3.1.10. A class X is called *finite* if it coincides with the image of a function determined on a finite ordinal. Symbolically this is expressed as $\mathrm{fin}(X)$, so that

$$\mathrm{fin}(X) := (\exists n)(\exists f)(n \in \omega \wedge \mathrm{Fnc}(f) \wedge \mathrm{dom}(f) = n \wedge \mathrm{im}(f) = X).$$

The above formula is obviously not bounded. By virtue of the axiom of replacement NBG_6, it is clear that $\mathrm{fin}(X) \to M(X)$, and, hence, we shall speak about finite sets instead of finite classes. The class of all finite subsets of the class X is denoted by $\mathcal{P}_{\mathrm{fin}}(X)$:

$$\mathcal{P}_{\mathrm{fin}}(X) := \{Y \in \mathcal{P}(X) : \mathrm{fin}(Y)\}.$$

Let us clarify what happens with finite sets under the canonical embedding of $\mathbf{V}$ in $\mathbf{V}^{(B)}$, i.e., what the class $\mathcal{P}_{\mathrm{fin}}(X)^\wedge$ is. Let us first show that

$$\mathbf{V}^{(B)} \models \mathcal{P}_{\mathrm{fin}}(X)^\wedge \subset \mathcal{P}_{\mathrm{fin}}(X^\wedge).$$

⊲ It should be remarked that if f is a mapping of a certain $n \in \omega$ in X, then $[\![(f^\wedge) \in \mathcal{P}_{\mathrm{fin}}(X^\wedge)]\!] = \mathbb{1}$. Indeed, according to **3.1.6**, $[\![f^\wedge : n^\wedge \to X^\wedge]\!] = [\![n^\wedge \in \omega^\wedge]\!] = \mathbb{1}$, and, hence,

$$[\![\mathrm{im}(f^\wedge) \in \mathcal{P}(X^\wedge) \wedge \mathrm{fin}(\mathrm{im}(f^\wedge))]\!] = \mathbb{1}.$$

For an arbitrary $t \in \mathbf{V}^{(B)}$ we can now easily calculate (see **2.2.8** (1), **3.1.4** (7), **3.1.6**):

$$
\begin{aligned}
&[t \in \mathcal{P}_{\text{fin}}(X)^\wedge] \\
&= \bigvee_{u \in \mathcal{P}_{\text{fin}}(X)} [t = u^\wedge] = \bigvee_{n \in \omega} \bigvee_{f:n \to X} [t = \operatorname{im}(f)^\wedge] \\
&= \bigvee_{n \in \omega} \bigvee_{f:n \to X} [t = \operatorname{im}(f^\wedge)] \wedge [n^\wedge \in \omega^\wedge] \wedge [f^\wedge : n^\wedge \to X^\wedge] \\
&\leq [t \in \mathcal{P}_{\text{fin}}(X^\wedge)].
\end{aligned}
$$

3.1.11. *For any class X we have*

$$\mathbf{V}^{(B)} \models \mathcal{P}_{\text{fin}}(X)^\wedge = \mathcal{P}_{\text{fin}}(X^\wedge) .$$

$\triangleleft$ Let us assume that for a $t \in \mathbf{V}^{(B)}$ the following relation holds:

$$[t \in \mathcal{P}_{\text{fin}}(X^\wedge)] = [(\exists n \in \omega^\wedge)(\exists f)(f:n \leftrightarrow X^\wedge \wedge t = \operatorname{im}(f)] = \mathbf{1}.$$

Then there is such a countable partition of unity $(b^{(n)})_{n \in \omega} \subset B$ that

$$[(\exists f)(f:n \to X^\wedge \wedge t = \operatorname{im}(f)] \geq b^{(n)} \quad (n \in \omega).$$

For every $n \in \omega$ we can, by virtue of the maximum principle, find such an $f_n' \in \mathbf{V}^{(B)}$ which obeys the inequality

$$[f_n' : n^\wedge \to X^\wedge] \wedge [t = \operatorname{im}(f_n')] \geq b^{(n)}.$$

Let us make use of statement **3.1.6**, and choose an $f_n'' \in \mathbf{V}^{(B)}$ in such a way that $[f_n'' : n^\wedge \to X^\wedge] \geq (b^{(n)})^*$, and let us set $f_n : \operatorname{mix}\{b^{(n)} f_n', (b^{(n)})^* f_n''\}$. Then $[f_n : n^\wedge \to X^\wedge] = \mathbf{1}$ and $[t = \operatorname{im}(f_n)] \geq b^{(n)}$. Then, for every $k \in n$ we have $[f_n(k^\wedge) \in X^\wedge] = \mathbf{1}$. Hence, $f_n(k) = \operatorname{mix}(b_x^{(k)} x^\wedge)$ for a certain partition of unity $(b_x^{(k)})_{x \in X}$ (see **3.1.1** (1)). Therefore,

$$[f_n(k^\wedge) = x^\wedge] \geq b_x^{(k)} \quad (x \in X, k \in n).$$

Let X^n be, as usual, a class of all mappings from n to X. It should be remarked that for $g \in X^n$ and $k \in n$ we get

$$[f_n(k^\wedge) = g^\wedge(k^\wedge)] = [f_n(k^\wedge) = g^\wedge(k)^\wedge] \geq b_{g(k)}^{(k)},$$

and, hence, $[f_n = g^\wedge] \geq b_{g,n}$, where $b_{g,n} := \wedge\{b_{g(k)}^{(k)} : k \in n\}$. In this case, however, the following relation also holds:

$$[\mathrm{im}(f) = \mathrm{im}(g^\wedge)] \geq b_{g,n} \quad (g \in X^n).$$

By definition, $\mathrm{im}(g) \in \mathcal{P}_{\mathrm{fin}}(X)$, while by virtue of **3.1.4** (7) we have

$$[\mathrm{im}(g^\wedge) \in \mathcal{P}_{\mathrm{fin}}(X)^\wedge] = \mathbf{1}.$$

Hence, we get

$$[t \in \mathcal{P}_{\mathrm{fin}}(X)^\wedge] \geq [t = \mathrm{im}(f_n)]$$
$$\wedge [\mathrm{im}(f_n) = \mathrm{im}(g^\wedge)] \wedge [\mathrm{im}(g^\wedge) \in \mathcal{P}_{\mathrm{fin}}(X)^\wedge] \geq b^{(n)} \wedge b_{g,n}.$$

Making use of the definition of the element $b_{g,n}$ and distributive laws **1.1.5** (1,2), we can calculate

$$\vee \{b^{(n)} \wedge b_{g,n} : n \in \omega \wedge g \in X^n\} = \bigvee_{n\in\omega} b^{(n)} \wedge \left(\bigvee_{g\in X^n} \bigwedge_{k\in n} b^{(k)}_{g(k)} \right)$$
$$= \bigvee_{n\in\omega} b^{(n)} \wedge \left(\bigwedge_{k\in n} \bigvee_{g\in X^n} b^{(k)}_{g(k)} \right) = \bigvee_{n\in\omega} b^{(n)} \wedge \left(\bigwedge_{k\in n} \bigvee_{g\in X} b^{(k)}_{x} \right) = \bigvee_{n\in\omega} b^{(n)} = \mathbf{1}.$$

As is seen, $[t \in \mathcal{P}_{\mathrm{fin}}(X)^\wedge] = \mathbf{1}$, and, applying **2.5.16**, we deduce $[\mathcal{P}_{\mathrm{fin}}(X^\wedge) \subset \mathcal{P}_{\mathrm{fin}}(X)^\wedge] = \mathbf{1}$. The reverse inclusion is established in **3.1.10**. ▷

3.1.12. *For any class X and every $n \in \omega$ the following relations are valid:*

(1) $\mathbf{V}^{(B)} \models (X^n)^\wedge = (X^\wedge)^{n\wedge}$;

(2) $\mathbf{V}^{(B)} \models \mathcal{P}(X)^\wedge \subset \mathcal{P}(X^\wedge)$.

◁ (1) Taking into account **3.1.6**, we can write for an arbitrary $t \in \mathbf{V}^{(B)}$:

$$[t \in (X^n)^\wedge] = \vee \{[t = u^\wedge] : u \in X^n\}$$
$$= \vee \{[t = u^\wedge] \wedge [u^\wedge : n^\wedge \to X^\wedge] : u \in X^n\}$$
$$\leq \vee \{[t = u] \wedge [u : n^\wedge \to X^n] : u \in \mathbf{V}^{(B)}\}$$
$$= [(\exists u)(u : n^\wedge \to X^\wedge \wedge t = u)] = [t \in (X^\wedge)^{n^\wedge}].$$

Therefore, we have established

$$[(X^n)^\wedge \subset (X^\wedge)^{n^\wedge}] = \mathbf{1}.$$

To prove the reverse inclusion, let us consider such an element $u \in \mathbf{V}^{(B)}$ that $[u{:}n^\wedge \to (X^\wedge] = \mathbf{1}$. In this case $[u(k^\wedge) \in X^\wedge] = \mathbf{1}$ $(k \in n)$ and, hence, $[u(k^\wedge) = \mathrm{mix}(b_x^{(k)} x^\wedge)] = \mathbf{1}$ for a certain partition of unity $(b_x^{(k)})_{x \in X}$. Going over to refined partitions of unity, we can, if necessary, choose such a partition of unity (b_ξ) and such families $(x_{k,\xi}) \subset X$ $(k \in n)$ that $[u(k^\wedge) = \mathrm{mix}(b_\xi x_{k,\xi}^\wedge)] = \mathbf{1}$ for all $k \in n$. Let us determine the functions $u_\xi : n \to X$ by the relations $u_\xi(k) := x_{k,\xi}$. Then $[u = u_\xi^\wedge] \geq b_\xi$ and $[u_\xi^\wedge \in (X^n)^\wedge] = \mathbf{1}$ and, hence, $[u \in (X^n)^\wedge] = \mathbf{1}$. By virtue of **2.5.16**, we get $[(X^\wedge)^{n\wedge} \subset (X^n)^\wedge] = \mathbf{1}$.

(2) This result is obtained by direct calculations. ▷

3.1.13. Remarks

(1) Cardinals inside the model $\mathbf{V}^{(B)}$ are a greater problem than ordinals (see **3.1.9**). One can easily notice that $\neg\mathrm{Card}(x)$ is a Σ_1-formula and, hence, $[\mathrm{Card}(\alpha^\wedge)] = \mathbf{1}$ $\to \mathrm{Card}(\alpha)$. The formula $\neg\mathrm{Card}(x)$ is not, however, of class Σ_1 and, hence, the reverse implication might be violated and an ordinal might lose the property to be a cardinal under the canonical embedding in $\mathbf{V}^{(B)}$. In fact for infinite cardinals $\lambda < \aleph$ one can choose such a complete Boolean algebra B that $\mathbf{V}^{(B)} \models |\lambda^\wedge| = |\aleph^\wedge|$, this peculiarity called the *displacement of cardinal numbers.* One can also choose such a B that $\mathbf{V}^{(B)} \models 2^{\omega_\alpha} = \omega_{\beta+1}$ for some $\alpha < \beta$. In this way one can establish the consistency of $\neg$ GCH with ZFC [10, 96, 248].

(2) In spite of what has been said in (1), cardinals in $\mathbf{V}^{(B)}$ behave themselves provided B is made to obey the countability of antichains, i.e., if any antichain in B is no more than countable (B is also said to be of *countable type*). For such a B we have

$$\mathbf{V}^{(B)} \models \mathrm{Card}(\alpha^\wedge) \leftrightarrow \mathrm{Card}(\alpha),$$
$$\mathbf{V}^{(B)} \models (\omega_\alpha)^\wedge = \omega_\alpha{}^\wedge.$$

(3) The properties of constructible sets (see **1.5.10**) inside $\mathbf{V}^{(B)}$ are like those of ordinals. Namely, if $L(x)$ is a formula stating that x is a constructible set, then

$$[L(u)] = \vee\{[u = v]: v \in \mathbf{L}\} \quad (u \in \mathbf{V}^{(B)})$$

and statements **3.1.9** (2) - **3.1.9** (4) are preserved, provided Ord is replaced with L in them (see [10, 96, 248]).

(4) By virtue of **3.1.11**, it might seem that in **3.1.12** (2) there is equality, i.e., $[\mathcal{P}(X^{\wedge}) = \mathcal{P}(X)^{\wedge}] = \mathbf{1}$. This, however, is not the case since if B is an algebra of regular closed subsets of the Cantorian ω-discontinuum, then $[\mathcal{P}(\omega^{\wedge}) \neq \mathcal{P}(\omega)^{\wedge}] = \mathbf{1}$.

3.2. The Descent Functor

Presented here are the basic techniques of translating propositions about elements of the universe $\mathbf{V}^{(B)}$ in statements on sets, the role of the translator performed by the operation of descent. The word 'descent' is used for denoting both the result and the method of presenting elements of $\mathbf{V}^{(B)}$ in the universe $\mathbf{V}$. Therefore, speaking not formally, the descent is acting from $\mathbf{V}^{(B)}$ to $\mathbf{V}$.

3.2.1. Let us consider an arbitrary $\mathbf{V}^{(B)}$-class $X: \mathbf{V}^{(B)} \to B$ and put

$$X{\downarrow} := \{x \in \mathbf{V}^{(B)} : [x \in X] = \mathbf{1}_B\}.$$

This equality defines a certain subclass $X{\downarrow}$ of the universal class $\mathbf{V}$ which is called the *descent* of the $\mathbf{V}^{(B)}$-class X. Let $X_\varphi := \overline{\varphi}$ be a class inside $\mathbf{V}^{(B)}$ which is definable by the B-formula φ (see **2.5.5**). Then the descent of the class X_φ has the form

$$X_\varphi{\downarrow} = \{x \in \mathbf{V}^{(B)} : [\varphi(x)] = \mathbf{1}\}.$$

In this case the formula $x \in X_\varphi{\downarrow}$ is expressed by the words 'x satisfies φ inside $\mathbf{V}^{(B)}$'. Thus, for instance, if $f \in \mathbf{V}^{(B)}$ and $[\mathrm{Fnc}(f)] = \mathbf{1}$, then f is said to be a *function inside* $\mathbf{V}^{(B)}$ or *in the model* $\mathbf{V}^{(B)}$. It is obvious that the descent of the universal $\mathbf{V}^{(B)}$-class $\mathbf{U}_B$ coincides with $\mathbf{V}^{(B)}$. Let us immediately remark two expedient formulas resulting immediately from **2.5**:

$$[X_\varphi \subset X_\psi] = \wedge\{[\psi(x)] : x \in X_\varphi{\downarrow}\},$$
$$[X_\varphi \cap X_\psi \neq \varnothing] = \vee\{[\psi(x)] : x \in X_\varphi{\downarrow}\},$$

where φ and ψ are arbitrary B-formulas.

Below we shall systematically use the following technique of abbreviation. Let the symbol f be a (conventional) notation for a certain n-ary function, for instance, $\{\cdot,\cdot\}$, $(\cdot,\cdot)$,

$\Phi(\cdot)$, $\pi_\Phi(\cdot)$, etc.. Then for any $x_1,\dots,x_n \in \mathbf{V}^{(B)}$ there exists a unique element $x_f \in \mathbf{V}^{(B)}$ such that

$$[x_f = f(x_1,\dots,x_n)] = [(\exists x)\,(x_1,\dots,x_n,x) \in f].$$

Under these circumstances instead of $x_f\downarrow$ we simply write $f(x_1,\dots,x_n)\downarrow$. For instance, $\Phi(A)\downarrow$ is the class determined by the relation

$$y \in \Phi(A) \leftrightarrow ([(\exists x \in A)(y \in \Phi(x))] = \mathbf{1}).$$

3.2.2. Let X be a subclass of the class $\mathbf{V}^{(B)}$ (i.e., $X \in \mathbf{V}^{(B)}$ in the sense of $\mathbf{V}$). They say that X is *cyclic* (or *extended*, or *disjointly complete*) and write Cyc$X$ provided X is closed relative to mixings of its any subfamilies over arbitrary partitions of unity. In other words, our class X is cyclic when for any partition of unity, $(b_\xi)_{\xi\in\Xi} \subset B$, and any family $(x_\xi)_{\xi\in\Xi} \subset X$ we have $\operatorname{mix}_{\xi\in\Xi}(b_\xi x_\xi) \in X$. The intersection of any set of cyclic sets is obviously a cyclic set, too.

The least cyclic set containing the given set $M \in \mathbf{V}^{(B)}$ is called the *cyclic envelope*, the *cyclic hull*, or the *cyclic extension* of M and is denoted by cyc(M). Obviously, the set $M \in \mathbf{V}^{(B)}$ is cyclic iff $M = \text{cyc}(M)$.

3.2.3. *Let X and Y are classes inside $\mathbf{V}^{(B)}$. The following statements are valid:*

(1) $[X \neq \varnothing] = 1 \to X\downarrow \neq \varnothing \wedge \text{Cyc}(X\downarrow)$;

(2) $X \in \mathbf{V}^{(B)} \to X\downarrow \in \mathbf{V}$;

(3) $X = Y \leftrightarrow X\downarrow = Y\downarrow$.

◁ (1) By the maximum principle, the class $X\downarrow$ is nonempty. If $(x_\xi)_{\xi\in\Xi} \subset X\downarrow$ and $(b_\xi)_{\xi\in\Xi}$ is a partition of unity, then for $x := \operatorname{mix}_{\xi\in\Xi}(b_\xi x_\xi)$ we have

$$[x \in X] \geq [x = x_\xi] \wedge [x_\xi \in X] \geq b_\xi \quad (\xi \in \Xi).$$

Therefore, $[x \in X] \geq \vee_{\xi\in\Xi} b_\xi = \mathbf{1}$ and $x \in X\downarrow$.

(2) Let us assume that $X \in \mathbf{V}^{(B)}$ and $x \in X\downarrow$. Let $u\colon \text{dom}(u) \to B$ be such a function that $\text{dom}(u) \subset \mathbf{V}^{(B)}$, $\text{dom}(u) \in \mathbf{V}$ and $\bar{u}(\cdot) = [\cdot \in X]$ (see **2.5.6**). Then

$$\vee\{u(t)\wedge[t=x]: t\in\mathrm{dom}(u)\}=\mathbf{1}.$$

Using the exhaustion principle **2.3.9**, we find a partition of unity $(b_\xi)\subset B$ and a family $(t_\xi)\subset\mathrm{dom}(u)$ such that $u(t_\xi)\wedge[x=t_\xi]\geq b_\xi$, which implies the equality $x=\mathrm{mix}(b_\xi t_\xi)$. By $\mathrm{Part}(B)$ we denote a set of all partitions of unity in B and put

$$Y:=\cup\{(\mathrm{dom}(u))^\theta:\theta\in\mathrm{Part}(B)\}.$$

Let us consider a function F which assigns to each x the set of ordered pairs (θ,v) for which $\theta\in\mathrm{Part}(B)$, $v:\theta\to\mathrm{dom}(u)$ and if $\theta:=(b_\xi)$. Then $x=\mathrm{mix}(b_\xi x_\xi)$, where $x_\xi:=v(b_\xi)$. Obviously, $\mathrm{dom}(F)\supset X\downarrow$, $\mathrm{im}(F)\subset\mathcal{P}(\mathrm{Part}(B)\times Y)$ and $F(x)\cap F(y)=\varnothing$ for $x\neq y$. Therefore, $|X\downarrow|\leq|\mathcal{P}(\mathrm{Part}(B)\times Y)|$ and $X\downarrow\in\mathbf{V}$.

(3) If $X\downarrow=Y\downarrow$, then, by **2.5.16**,

$$[X\subset Y]=\bigwedge_{t\in X\downarrow}[t\in Y]=\bigwedge_{t\in Y\downarrow}[t\in Y]=\mathbf{1}.$$

Analogously, $[Y\subset Y]=\mathbf{1}$ and, hence, $[X=Y]=\mathbf{1}$. ▷

3.2.4. Let X and Y be two $\mathbf{V}^{(B)}$-classes, while $X\times_B Y$ be their Cartesian product, inside $\mathbf{V}^{(B)}$, which exists by virtue of **1.3.13** (2) and **2.5.18**. The mapping

$$(\cdot,\cdot)^B:(x,y)\to(x,y)^B\quad(x\in X\downarrow, y\in Y\downarrow)$$

implements a bijection from the class $X\downarrow\times Y\downarrow$ onto the class $(X\times_B Y)\downarrow$. In this case

$$[\mathrm{Pr}_{X\downarrow}(x,y)=\mathrm{Pr}_X(x,y)]=[\mathrm{Pr}_{Y\downarrow}(x,y)=\mathrm{Pr}_Y(x,y)]=\mathbf{1}$$
$$(x\in X\downarrow, y\in Y\downarrow),$$

where $\mathrm{Pr}_{X\downarrow}, \mathrm{Pr}_{Y\downarrow}$ are the canonical projections on the factors $X\downarrow$ and $Y\downarrow$, respectively, while $\mathrm{Pr}_X, \mathrm{Pr}_Y$ are canonical projections inside $\mathbf{V}^{(B)}$ on X and Y, respectively. (One should bear in mind that Pr_X and Pr_Y are classes inside $\mathbf{V}^{(B)}$, while $\mathrm{Pr}_{X\downarrow}$ and $\mathrm{Pr}_{Y\downarrow}$ are classes in the sense of $\mathbf{V}$.)

◁ As has been noted earlier (see **2.4.9** and **2.5.3**), the function $(\cdot,\cdot)^B$ is an injective embedding of the class $\mathbf{V}^{(B)}\times\mathbf{V}^{(B)}$ in the class $\mathbf{V}^{(B)}$. It is hence sufficient to establish that $(\cdot,\cdot)^B$ maps $X\downarrow\times Y\downarrow\subset\mathbf{V}^{(B)}\times\mathbf{V}^{(B)}$ on $(X\times_B Y)\downarrow$. For any $x\in X\downarrow$ and $y\in Y\downarrow$ we have

$$
\begin{aligned}
&[(x,y)^B \in X \times Y] \\
&= [(\exists u)(\exists v)(u \in X \wedge v \in Y \wedge (u,v) = (x,y)^B)] \\
&= \bigvee_{u \in \mathbf{V}^{(B)}} \bigvee_{v \in \mathbf{V}^{(B)}} [u \in X] \wedge [v \in Y] \wedge [(u,v) = (x,y)^B] \\
&\geq [x \in X] \wedge [y \in Y] \wedge [(x,y) = (x,y)^B] = \mathbf{1}.
\end{aligned}
$$

Therefore, $(x,y)^B \in (X \times_B Y)\downarrow$. Let us now consider an arbitrary element $z \in (X \times_B Y)\downarrow$ and remark that according to the maximum principle we find elements x and $y \in \mathbf{V}^{(B)}$ such that

$$
\begin{aligned}
\mathbf{1} &= [z \in X \times Y] = [(\exists u \in X)(\exists v \in Y)(z = (u,v))] \\
&= [x \in X] \wedge [y \in Y] \wedge [z = (x,y)].
\end{aligned}
$$

Hence, $x \in X\downarrow$, $y \in Y\downarrow$ and $z = (x,y)^B$. Finally, for $x \in X\downarrow$, $y \in Y\downarrow$ and $z \in \mathbf{V}^{(B)}$ we get

$$
[z = \mathrm{Pr}_X(x,y)] = [((x,y),z) \in \mathrm{Pr}_X] = [z = x] = [z = \mathrm{Pr}_{X\downarrow}(x,y)],
$$

which ensures validity for the required relation on the canonical projection on X. The situation is analogous for the projection on the second factor. ▷

3.2.5. Let us consider a binary relation X inside $\mathbf{V}^{(B)}$. This implies that the class X is inside $\mathbf{V}^{(B)}$ and $[X \text{ is a binary relation }] = \mathbf{1}$. In accordance with **3.2.4** and the axiom of domain NBG, there is a class Y such that

$$
(x,y) \in Y \leftrightarrow (x,y)^B \in X\downarrow.
$$

Indeed, we must put

$$
Y := \mathrm{dom}((\cdot, \cdot)^B \cap (\mathbf{V}^{(B)} \times \mathbf{V}^{(B)} \times X\downarrow)).
$$

It is obvious that Y is a binary relation and that $(\cdot, \cdot)^B$ carries out a bijection between Y and $X\downarrow$. The class Y is called the *descent of the binary relation* X, and we shall preserve the symbol $X\downarrow$ for its notation. In an absolutely analogous way we determine the descent of an $n^\wedge$-ary relation X, namely:

$$
X\downarrow := \{(x_1,\dots,x_n) \in (\mathbf{V}^{(B)})^n : (x_1,\dots,x_n)^B \in X\downarrow\}.
$$

Therefore, the descent of the class X and that of the binary relation X are not the same, while the common notation $X\downarrow$ is a conveniently free choice, which should be taken into consideration to avoid ambiguity. For instance, the equality $(X \times_B Y)\downarrow = X\downarrow \times Y\downarrow$ is to be viewed only as another presentation of the first part of **3.2.4**. The same remarks are also true for the descents of correspondences, categories, etc., to be defined below.

3.2.6. Theorem. *For classes X and Y, the following formulas are valid:*

(1) $\mathrm{dom}(X)\downarrow = \mathrm{dom}(X\downarrow)$, $\mathrm{im}(X)\downarrow = \mathrm{im}(X\downarrow)$;

(2) $(X \cap Y)\downarrow = X\downarrow \cap Y\downarrow$;

(3) $(X|Y)\downarrow = (X\downarrow)|(Y\downarrow)$;

(4) $(X^{-1})\downarrow = (X\downarrow)^{-1}$;

(5) $(X \circ Y)\downarrow = (X\downarrow)\circ(Y\downarrow)$;

(6) $(X``Y)\downarrow = (X\downarrow)``(Y\downarrow)$;

(7) $(\mathbf{V}^{(B)} \models \mathrm{Fnc}(X)) \leftrightarrow \mathrm{Fnc}(X\downarrow)$;

(8) $(\mathbf{V}^{(B)} \models X \subset Y) \leftrightarrow X\downarrow \subset Y\downarrow$;

(9) $[x = y] \leq [X(x) = X(y)]$ $(x, y \in \mathbf{V}^{(B)})$;

(10) $(X\downarrow)^n = (X^{n^\wedge})\downarrow$ $(n \in \omega)$.

(The arrow $\downarrow$ denotes the descents of the classes in (2), (8) that are on the left-hand sides of equalities (1), (6), (10), as well as those of the relations in the remaining places.)

$\lhd$ (1) By virtue of the maximum principle, for any $x \in \mathbf{V}^{(B)}$ there is such a $y \in \mathbf{V}^{(B)}$ that

$$[x \in \mathrm{dom}(X)] = [(\exists u)((x,u) \in X)] = [(x,y)^B \in X].$$

It follows from the above relation that $x \in \mathrm{dom}(X)\downarrow$ results in $x \in \mathrm{dom}(X\downarrow)$. Conversely, if $x \in \mathrm{dom}(X\downarrow)$ then $[(x,y) \in X] = \mathbf{1}$ for some $y \in \mathbf{V}^{(B)}$. Hence,

$$[x \in \mathrm{dom}(X)] = \vee\{[(x,u) \in X]: u \in \mathbf{V}^{(B)}\} \geq [(x,y) \in X],$$

which affords $x \in \mathrm{dom}(X)\downarrow$. The second relation is proved analogously.

(2) By definition, for an arbitrary $x \in \mathbf{V}^{(B)}$ we have

$$[x \in X \cap Y] = [x \in X \wedge x \in Y] = [x \in X] \wedge [x \in Y].$$

Therefore, $x \in (X \cap Y)\downarrow$ iff we have $x \in X\downarrow$ and $x \in Y\downarrow$ simultaneously.

(3) Applying (2) and the definition of the restriction $X|Y$, we deduce

$$(X|Y)\downarrow = (X \cap (Y \times U_B))\downarrow = X\downarrow \cap (Y\downarrow \times \mathbf{V}^{(B)}) = (X\downarrow)|(Y\downarrow).$$

(4) The proof results from the definition of X^{-1}.

(5) For any class Z lets us denote by σX the class obtained from Z by a σ-conjugation, where $\sigma := (\iota_1, \iota_2, \iota_3)$ is a permutation of the set $\{1,2,3\}$ (see **1.3.10**). We then can easily make sure that $(\sigma Z)\downarrow = \sigma(Z\downarrow)$. If $Z \in \mathbf{V}^{(B)}$ is such that $\mathbf{V}^{(B)} \models Z = (Y \times \mathbf{U}_B) \cap (\mathbf{U}_B \times X)$ and $\sigma := \{1,2,3\}$, then

$$\mathbf{V}^{(B)} \models X \circ Y = \mathrm{dom}(\sigma Z).$$

Now, taking into account (1), (2) and **3.2.4**, we can write the following chain of equalities

$$(X \circ Y)\downarrow = \mathrm{dom}(\sigma Z)\downarrow = \mathrm{dom}(\sigma(Z\downarrow))$$
$$= \mathrm{dom}(\sigma((Y\downarrow \times \mathbf{V}^{(B)}) \cap (\mathbf{V}^{(B)} \times X\downarrow)) = (X\downarrow) \circ (Y\downarrow).$$

(6) If we successively apply (1) and (3), we get

$$(X``Y)\downarrow = (\mathrm{im}(X|Y))\downarrow = \mathrm{im}((X|Y)\downarrow)$$
$$= \mathrm{im}((X\downarrow)|(Y\downarrow)) = (X\downarrow)``(Y\downarrow).$$

(7) Let us assume that $[\mathrm{Fnc}(X)] = \mathbf{1}$. Then $X\downarrow$ is a binary relation and, besides,

$$[(x,y) \in X] \wedge [(x,z) \in X] \leq [y = z]$$

for any $x, y, z \in \mathbf{V}^{(B)}$. Hence, for $(x,y) \in X\downarrow$ and $(x,z) \in X\downarrow$ we get $[\![y = z]\!] = \mathbf{1}$, i.e.,

$y = z$. In other words, $\mathrm{Fnc}(X)\downarrow$ is fulfilled. In turn, if $X\downarrow$ is a single-valued binary relation, then by **2.5.16**, we deduce

$$[\mathrm{Fnc}(X)] = \bigwedge_{x\in \mathbf{V}^{(B)}} \wedge\{[y = z]: (x,y)\in X\downarrow \cap Y\downarrow, (x,z)\in X\downarrow\} = \mathbf{1}.$$

(8) Making use of (2) and **3.2.3** (3), we can write

$$\mathbf{1} = [X\subset Y] \leftrightarrow \mathbf{1} = [X\cap Y = X] \leftrightarrow X\downarrow \cap Y\downarrow = X\downarrow \leftrightarrow X\downarrow \subset Y\downarrow.$$

(9) The formula $(\forall x)(\forall y)(x = y \to X``\{x\} = X``\{y\})$ is a theorem of ZF, and has, therefore, the truth-value unity. Writing the truth values first for quantifiers and then for implication, we will get the required result.

(10) If $[t{:}n^\wedge \to X] = \mathbf{1}$, then for every $k\in n$ there is a unique element $x\in X\downarrow$, for which $[t(k^\wedge) = x] = \mathbf{1}$. Setting $s(k) := x$ for $k\in n$, we will obtain a mapping $s{:}n\to X\downarrow$, which will be denoted by $t\downarrow := s$. Hence,

$$[t\downarrow(k) = t(k^\wedge)] = \mathbf{1} \quad (k\in n).$$

Conversely, if $s{:}\, n\to X\downarrow$, then $t\in \mathbf{V}^{(B)}$ will be determined by the relation

$$t := \{(k^\wedge, s(k))^B : k\in n\}\times \mathbf{1}_B.$$

In this case $[t{:}n^\wedge \to X] = 1$, $[t(k^\wedge) = s(k)] = \mathbf{1}$ for $k\in n$, and $t\downarrow = s$. From all what was said above we can conclude that the mapping $t\to t\downarrow$ is a bijection between $\{x\in \mathbf{V}^{(B)}: [x\in X^{n^\wedge}] = \mathbf{1}\}$ and $(X\downarrow)^n$.

Let us now recall the definition $s := (x(0),\dots,x(n-1))^B$ (see **2.4.9**). Let $x{:}n\to X\downarrow$ and $y{:}\,n\to X\downarrow$ be such that $y(0) = x(0)$, $y(k) = (y(k-1), x(k))^B$ for $0\neq k\in n$ and $y(n-1) = s$. According to what has been earlier proved, there are such $p,q\in \mathbf{V}^{(B)}$ that $[p,q : n^\wedge \to X] = \mathbf{1}$, in which case $p\downarrow = x$ and $q\downarrow = y$. We then can easily check that

$$[p(0) = q(0) \wedge (\forall k\in n^\wedge)(k\neq 0 \to q(k) = (q(k-1), p(k)))] = \mathbf{1}.$$

Therefore, $[q(n^\wedge - 1) = (p(0^\wedge),\dots,p(n^\wedge - 1))\in X^{n^\wedge}] = \mathbf{1}$. On the other hand, $[s = q(n^\wedge - 1)] = \mathbf{1}$, and, hence, $s\in (X^{n^\wedge})\downarrow$. Thus, the mapping

$$(x(0),\dots,x(n-1)) \to (x(0),\dots,x(n-1))^B$$

is an injection of $(X\downarrow)^n$ in $(X^{n^\wedge})\downarrow$. Analogous considerations show that in this case the image of $(X\downarrow)^n$ is all $(X^{n^\wedge})\downarrow$. ▷

3.2.7. Somewhat different from **3.2.6** is the matter with the descents of the complement of a class and the families of classes. Let us consider an arbitrary class $Y \subset \mathbf{V}^{(B)}$. Since the formula $x \in \mathbf{V}^{(B)} \wedge (\forall y \in Y)\,([x = y] = 0)$ is predicative, there is a class Y^c determined by the relation

$$x \in Y^c \leftrightarrow x \in \mathbf{V}^{(B)} \wedge (\forall y \in Y)\,([x = y] = \mathbf{0}).$$

Let now X be a class inside $\mathbf{V}^{(B)}$. By the symbol X^c let us denote the $\mathbf{V}^{(B)}$-class that is the complement of the class X inside $\mathbf{V}^{(B)}$, i.e.,

$$\mathbf{V}^{(B)} \models (\forall x)(x \in X^c \leftrightarrow x \notin X).$$

The existence of a $\mathbf{V}^{(B)}$-class X^c results from **2.5.18**. Let us consider a formula

$$\varphi\big(y, B, Y, \mathbf{V}^{(B)}, [\cdot = \cdot]\big) := (\forall a)(\forall b)(\forall x)(b : a \to Y$$
$$\text{`}b \text{ is a partition of unity'} \wedge x\colon a \to Y \wedge y$$
$$= \operatorname*{mix}_{\alpha \in a}(b(\alpha) \cdot x(\alpha))),$$

stating that y is a mixing of a certain family of elements of the class Y. We can make sure that the formula is predicative and, hence, there is a class $\operatorname{mix}(Y)$ such that

$$(\forall y)(y \in \operatorname{mix}(Y) \leftrightarrow \varphi\big(y, B, Y, \mathbf{V}^{(B)}, [\cdot = \cdot]\big).$$

As an example, let us indicate the fact that for an arbitrary class $X \subset \mathbf{V}$ we have $X^\wedge\downarrow = \operatorname{mix}(X_1)$, where $X_1 := \{x^\wedge : x \in X\}$, while the injection of X to $\operatorname{mix}(X_1)$ is realized by the canonical embedding (see **3.1.1** (1)).

3.2.8. *If the class Y is a set then*

$$\operatorname{mix}(Y) = \operatorname{cyc}(Y).$$

◁ We only have to demonstrate that the set $\operatorname{mix}(Y)$ of all possible mixings $\operatorname{mix}_{y \in Y}(b_y y)$ of the elements of the set Y is cyclic. Let us consider a partition of unity $(b_\xi)_{\xi \in \Xi}$ and a family

$$y_\xi := \underset{y\in Y}{\mathrm{mix}}(b_{\xi,y}y) \quad (\xi \in \Xi).$$

Let us put $y_0 := \mathrm{mix}_{\xi\in\Xi}(b_\xi y_\xi)$ and $b_{(\xi,y)} := b_\xi \wedge b_{\xi,y}$ $(\xi \in \Xi, y \in Y)$. If $(\xi, y) \neq (\eta, z)$, then

$$b_{(\xi,y)} \wedge b_{(\eta,z)} = b_\xi \wedge b_\eta \wedge b_{\xi,y} \wedge b_{\eta,z} = \mathbf{0}.$$

Besides, we can easily calculate (see **1.1.5** (2)):

$$\bigvee_{(\xi,y)\in\Xi\times Y} b_{(\xi,y)} = \bigvee_{\xi\in\Xi}\left(b_\xi \wedge \bigvee_{y\in Y} b_{\xi,y}\right) = \mathbf{1}.$$

Therefore, $\left(b_{(\xi,y)}\right)$ is a partition of unity. Then, for any $y \in Y$ we get

$$[\![y_0 = y]\!] \geq [\![y_0 = y_\xi]\!] \wedge [\![y_\xi = y]\!] \geq b_\xi \wedge b_{\xi,y}.$$

We see here that $y_0 = \mathrm{mix}(Y)$, i.e., $\mathrm{mix}(Y)$ is a cyclic set. ▷

3.2.9. *For any nonempty classes X and Y inside $\mathbf{V}^{(B)}$ we have*

(1) $X^c{\downarrow} = X{\downarrow}^c$;
(2) $(X \cup Y){\downarrow} = \mathrm{mix}(X{\downarrow} \cup Y{\downarrow})$.

◁ (1) By virtue of definitions and proposition **2.5.16**, the following equivalences hold:

$$\begin{aligned}
&x \in X^c{\downarrow} \leftrightarrow [\![x \in X^c]\!] = \mathbf{1} \leftrightarrow [\![x \notin X]\!] = \mathbf{1}\\
&\leftrightarrow [\![x \in X]\!] = \mathbf{0} \leftrightarrow \vee\{[\![x = s]\!] : s \in X{\downarrow}\} = \mathbf{0}\\
&\leftrightarrow (\forall s \in X{\downarrow})([\![s = x]\!] = \mathbf{0}) \leftrightarrow x \in (X{\downarrow})^c.
\end{aligned}$$

(2) It is seen from proposition **3.2.6** (8) that $X{\downarrow} \cup Y{\downarrow} \subset (X \cup Y){\downarrow}$. Conversely, if $z \in (X \cup Y){\downarrow}$, then

$$(\exists x \in X)(\exists y \in Y)(x = z \vee y = z).$$

Using the maximum principle, we choose $x_0, y_0 \in \mathbf{V}^{(B)}$ in such a way that $b \vee c = 1$, where $b := [\![x_0 \in X]\!] \wedge [\![x_0 \in z]\!]$ and $c := [\![y_0 \in Y]\!] \wedge [\![y_0 \in z]\!]$. Pick arbitrary $x_1 \in X{\downarrow}$ and $y_1 \in Y{\downarrow}$, and put $x = \mathrm{mix}\{bx_0, b^* x_1\}$, $y = \mathrm{mix}\{cy_0, c^* y_1\}$. Then $x \in X{\downarrow}$, since

$$b \leq [x = x_0] \wedge [x_0 \in X] \leq [x \in X],$$
$$b^* \leq [x_1 = x] \wedge [x_1 \in X] \leq [x \in X].$$

By an analogous reason, $y \in Y\downarrow$. Besides,

$$b \leq [x = x_0] \wedge [x_0 = z] \leq [x = z],$$
$$b^* \leq c \leq [y = y_0] \wedge [y_0 = z] \leq [y = z],$$

i.e., $z = \mathrm{mix}\{bx, b^*y\}$ and $z \in \mathrm{mix}(X\downarrow \cup Y\downarrow)$. ▷

Here we should make an additional remark:

$$(3)\ (X \cup Y)\downarrow = \bigcup_{b \in B} bX\downarrow \oplus b^* Y\downarrow,$$

where $bX\downarrow \oplus b^*Y\downarrow$ is a set of elements of the type $\mathrm{mix}\{bx, b^*y\}$ $(x \in X\downarrow, y \in Y\downarrow)$.

3.2.10. Descending has to be sometimes used repeatedly. Let us see how this happens. Let X be a class. Organize a class-function Y by the formula

$$Y := \{(x,y) : x \in \mathbf{V}^{(B)} \wedge y = x\downarrow\}.$$

The double or repeated descent of the class X is the class $\cup \mathrm{im}(Y|(X\downarrow))$ denoted by $X\downarrow\downarrow$. Therefore,

$$X\downarrow\downarrow = \cup\{x\downarrow : x \in X\downarrow\}.$$

It goes without saying that if $X \in \mathbf{V}^{(B)}$, then $X\downarrow\downarrow \in \mathbf{V}$ (see **3.2.3** (2)).

3.2.11. *For any nonempty* $\mathbf{V}^{(B)}$*-class* X *the following relations are valid:*

(1) $(\cup X)\downarrow = \cup(X\downarrow\downarrow)$;

(2) $(\cap Y)\downarrow = \cap(X\downarrow\downarrow)$;

(3) $\mathcal{P}(X)\downarrow\downarrow \subset \mathcal{P}(X\downarrow)$.

◁ The proof rests on **2.5.16**. Here are the necessary calculations:

(1) $u \in \cup(X\downarrow\downarrow) \leftrightarrow (\exists v \in X\downarrow\downarrow)(u \in v)$
$\leftrightarrow (\exists z \in X\downarrow)(u \in z\downarrow) \leftrightarrow (\exists z \in X\downarrow)([u \in z] = \mathbf{1})$
$\leftrightarrow [(\exists z \in X)(u \in z)] = \mathbf{1} \leftrightarrow [u \in \cup X] = \mathbf{1} \leftrightarrow u \in (\cup X)\downarrow.$

(2) $u \in \cap(X\downarrow\downarrow) \leftrightarrow (\forall v \in X\downarrow\downarrow)(u \in v)$
$\leftrightarrow (\forall z \in X\downarrow)(u \in z\downarrow) \leftrightarrow (\forall z \in X\downarrow)([u \in z] = \mathbf{1})$
$\leftrightarrow [(\forall z \in X)(u \in z)] = \mathbf{1} \leftrightarrow [u \in \cap X] = \mathbf{1} \leftrightarrow u \in (\cap X)\downarrow.$

(3) $u \in \mathcal{P}(X)\downarrow\downarrow \leftrightarrow (\exists z \in \mathcal{P}(X)\downarrow)(u \in z\downarrow)$
$\leftrightarrow (\exists z)([z \subset X] = \mathbf{1} \wedge u = z\downarrow) \leftrightarrow (\exists z)(z\downarrow \subset X\downarrow \wedge u = z\downarrow)$
$\rightarrow u \subset X\downarrow \leftrightarrow u \in \mathcal{P}(X\downarrow)$. ▷

3.2.12. Theorem. *Let* $X, Y, f \in \mathbf{V}^{(B)}$ *be such that* $[X \neq \varnothing] = [Y \neq \varnothing] = [f: X \rightarrow Y] = \mathbf{1}$. *Then there is a unique mapping* $f\downarrow: X\downarrow \rightarrow Y\downarrow$, *the descent of* f *such that*

$$[f(x) = f\downarrow(x)] = \mathbf{1} \quad (x \in X\downarrow).$$

The descent of a mapping has the following properties:

(**1**) f *is extensional, i.e.,*

$$[x = x'] \leq [f\downarrow(x) = f\downarrow(x')] \quad (x \in X\downarrow);$$

(**2**) *if* $Z, g \in \mathbf{V}^{(B)}$ *are such that* $[\![Z \neq \varnothing]\!] = [\![g: Y \rightarrow Z]\!] = \mathbf{1}$, *then*

$$(g \circ f)\downarrow = g\downarrow \circ f\downarrow;$$

(**3**) $f\downarrow$ *is surjective (injective* or *bijective) iff* $[f$ *is surjective (injective* or *bijective)* $] = \mathbf{1}$.

◁ Let h be the descent of the relation f in the sense of **3.2.5**. It follows from **3.2.6** (1,7) that $h: X\downarrow \rightarrow Y\downarrow$. Then, since $(x, h(x))^B \in f\downarrow$ for any $x \in X\downarrow$, then

$$[h(x) = f(x)] = [(x, h(x)) \in f] = [(x, h(x))^B \in f] = \mathbf{1}.$$

The mapping h is uniquely determined by this property, since if $g: X\downarrow \rightarrow Y\downarrow$ has the same property, then

$$[h(x) = g(x)] \geq [g(x) = f(x)] \wedge [h(x) = f(x)] = \mathbf{1}$$

and $h(x) = g(x)$ for every $x \in X\downarrow$ (since $\mathbf{V}^{(B)}$ is separated). Let us use the determining property of the mapping h and **3.2.6** (9) and estimate

$$[x = x'] \leq [f(x) = f(x')] \wedge [f(x) = h(x)] \\ \wedge [f(x') = h(x')] \leq [h(x) = h(x')].$$

Therefore, we have established (1), while (2) follows from **3.2.6** (5). Now we have to establish (3). The statement on the surjectivity is easily deduced from **3.2.6** (6), while the bijectivity is the conjunction of surjectivity and bijectivity. The injectivity of f inside $\mathbf{V}^{(B)}$ is equivalent to the relation

$$[x = x'] = [f(x) = f(x')] = [h(x) = h(x')] \quad (x, x' \in X\downarrow).$$

Hence, $x = x'$ iff $h(x) = h(x')$, which fact implies that the mapping h is injective. ▷

3.2.13. Theorem. *Let $X, Y, F \in \mathbf{V}^{(B)}$ are such that $[X \neq \varnothing] = [Y \neq \varnothing] = [\varnothing \neq F \subset X \times Y] = 1$. Let $\Phi \in \mathbf{V}^{(B)}$ be a correspondence from X to Y with the graph F inside $\mathbf{V}^{(B)}$, i.e., $\mathbf{V}^{(B)} \models \Phi = (F, X, Y)$. Then the triplet $\Phi\downarrow := (F\downarrow, X\downarrow, Y\downarrow)$ which is the descent of Φ, is the unique correspondence obeying the equality*

$$\Phi\downarrow(x) = \Phi(x)\downarrow \quad (x \in X\downarrow).$$

The descent of a correspondence has the following properties:

(1) $\Phi(A)\downarrow \in \Phi\downarrow(A)\downarrow$ *for any* $A \in \mathbf{V}^{(B)}$ *such that* $[A \subset X] = 1$;

(2) $\pi_\Phi(A)\downarrow = \pi_{\Phi\downarrow}(A\downarrow)$ *for all* $A \in \mathbf{V}^{(B)}$ *for which* $[A \subset X] = 1$;

(3) $(\Psi \circ \Phi)\downarrow = \Psi\downarrow \circ \Phi\downarrow$;

(4) $(I_X)\downarrow = I_{X\downarrow}$.

◁ All these statements but (3) are elementarily deduced from **3.2.6**. It should be only remarked that the determining equality $\Phi\downarrow(x) = \Phi(x)\downarrow \quad (x \in X\downarrow)$ must be understood with use made of the remark on **3.2.1**. It is by virtue of the maximum principle that there exists a $\overline{\Phi} \in \mathbf{V}^{(B)}$ such that $[\overline{\Phi}: X \to \mathcal{P}(Y)] = 1$ and $[\Phi(x) = \overline{\Phi}(x)] = 1$ for all $x \in X\downarrow$. According to **3.2.12**, $\overline{\Phi}\downarrow: X\downarrow \to \mathcal{P}(Y)\downarrow$ and $[\overline{\Phi}(x) = \overline{\Phi}\downarrow(x)] = 1$ for $x \in X\downarrow$. In this case, however, $\overline{\Phi}\downarrow$ is determined by the relation

$$\Phi\downarrow(x) = (\overline{\Phi}\downarrow(x))\downarrow = \overline{\Phi}(x)\downarrow\downarrow \quad (x\in X\downarrow).$$

This, in particular, affords $\Phi\downarrow(A\downarrow) = \overline{\Phi}(A)\downarrow\downarrow$. Taking these remarks into account, let us now prove (2). Observe that we have

$$[\pi_\Phi(A) = \cap\overline{\Phi}(A)] = \mathbf{1};$$

i.e., $\pi_\Phi(A) = \cap\{\overline{\Phi}(a): a\in A\}$ is fulfilled inside $\mathbf{V}^{(B)}$. From this relation, making use of the rule **3.2.11** (2), we deduce:

$$\pi_\Phi(A)\downarrow = (\cap\overline{\Phi}(A))\downarrow = \cap\overline{\Phi}(A)\downarrow\downarrow)$$
$$= \cap\{\Phi\downarrow(a): a\in A\downarrow\} = \pi_{\Phi\downarrow}(A\downarrow). \rhd$$

3.2.14. (1) *Let X and Y be nonempty sets inside $\mathbf{V}^{(B)}$, while a family $(f_\xi)_{\xi\in\Xi}\subset\mathbf{V}^{(B)}$ be such that*

$$[f_\xi: X\to Y] = \mathbf{1} \quad (\xi\in\Xi).$$

Then for a partition of unity $(b_\xi)_{\xi\in\Xi}\subset B$, the mixing $\operatorname{mix}_{\xi\in\Xi}(b_\xi f_\xi)$ is a function from X to Y inside $\mathbf{V}^{(B)}$ and

$$\operatorname*{mix}_{\xi\in\Xi}(b_\xi f_\xi)\downarrow(x) = \operatorname*{mix}_{\xi\in\Xi}(b_\xi f_\xi\downarrow(x)) \quad (x\in X\downarrow).$$

$\lhd$ Let us set $g := \operatorname{mix}_{\xi\in\Xi}(b_\xi f_\xi)$. Since

$$b_\xi \le [g = f_\xi]\wedge[f_\xi: X\to Y] < [g: X\to Y],$$

then $[g: X\to Y] = \mathbf{1}$, i.e., g is a function from X to Y. Besides, by virtue of **3.2.12**, for every $x\in X\downarrow$ we have

$$b_\xi \le [g\downarrow(x) = g(x)]\wedge[g(x) = f_\xi(x)]$$
$$\wedge[f_\xi\downarrow(x) = f_\xi(x)] \le [g\downarrow(x) = f_\xi\downarrow(x)].$$

Hence, we get $g\downarrow(x) = \operatorname{mix}_{\xi\in\Xi}(b_\xi f_\xi\downarrow(x))$. $\rhd$

(2) *Let X, Y and (b_ξ) be the same as above, while $(\Phi_\xi)_{\xi\in\Xi}$ be a family of elements*

of $\mathbf{V}^{(B)}$ *which are correspondences from* X *to* Y *inside* $\mathbf{V}^{(B)}$. *Then the mixing* $\text{mix}_{\xi\in\Xi}(b_\xi\Phi_\xi)$ *is a correspondence from* X *to* Y, *in which case*

$$\text{mix}_{\xi\in\Xi}(b_\xi\Phi_\xi)\downarrow(x) = \text{mix}_{\xi\in\Xi}(b_\xi\Phi_\xi\downarrow(x)) \quad (x\in X\downarrow).$$

◁ The proof is analogous to **3.2.14** (1). ▷

3.2.15. By the symbol $\mathfrak{F}^{\downarrow}$ let us denote the mapping that assigns to each nonempty $\mathbf{V}^{(B)}$-set X its descent $X\downarrow$, and to every correspondence Φ inside $\mathbf{V}^{(B)}$, the correspondence $\Phi\downarrow$.

Theorem. *The mapping* $\mathfrak{F}^{\downarrow}$ *is a covariant functor from the category* $\mathfrak{V}_*^{(B)}$ *to the category* $\mathfrak{V}_*$ *(from the category* $\mathfrak{V}^{(B)}$ *to the category* $\mathfrak{V}$, *respectively).*

3.2.16. Theorem. *Let* $\mathfrak{K}$ *be a category inside* $\mathbf{V}^{(B)}$. *Then there is a unique category* $\mathfrak{K}'$ *(in the sense of* $\mathbf{V}$*) such that* $\text{Ob}\,\mathfrak{K}' = (\text{Ob}\,\mathfrak{K})\downarrow$, $\text{Mor}\,\mathfrak{K}' = (\text{Mor}\,\mathfrak{K})\downarrow$ *and* $\text{Com}' = \text{Com}\downarrow$, *where* Com' *is a composition of the category* $\mathfrak{K}'$ *and* $\mathbf{V}^{(B)}\models \text{Com}$ *is the composition of the category* $\mathfrak{K}$.

◁ It follows from **3.2.6** (7) that Com' is a partial binary operation on the class $(\text{Mor}\,\mathfrak{K})\downarrow$. As long as $[\text{Com}(\alpha,\beta) = \text{Com}'(\alpha,\beta)] = \mathbf{1}$ for any $(\alpha,\beta)\in\text{Mor}\,\mathfrak{K}'$ and since Com is associative inside $\mathbf{V}^{(B)}$, then Com' is also associative. Let D and R be $\mathbf{V}^{(B)}$-classes taking part in the definition of the category $\mathfrak{K}$ (see **2.5.19**). Let us set $D' := D\downarrow$ and $R' := R\downarrow$. By virtue of **3.2.6** (1) and **3.2.6** (7), D' and R' are mappings from $\text{Mor}\,\mathfrak{K}'$ to $\text{Ob}\,\mathfrak{K}'$. Employing **3.2.6** (1) again, we conclude that for $(\alpha,\beta)\in\text{Mor}\,\mathfrak{K}'$ the relations $(\alpha,\beta)\in\text{dom}(\text{Com}')$ and $[(\alpha,\beta)\in\text{dom}(\text{Com})] = \mathbf{1}$ are equivalent. On the other hand, the equality $R'(\alpha) = D'(\beta)$ is fulfilled only if $[R(\alpha) = D(\beta)] = \mathbf{1}$. The existence of identity morphisms in $\mathfrak{K}'$ is obvious. Hence, $\mathfrak{K}$ satisfies all the conditions of definition **2.5.19**. ▷

3.2.17. The category $\mathfrak{K}'$ of **3.2.16** is called the *descent of the category* $\mathfrak{K}$ and is denoted by $\mathfrak{K}\downarrow$. Let Set_*^B be the category of nonempty sets and correspondences inside $\mathbf{V}^{(B)}$. In more detail, $\text{MorSet}_*^B, \text{ObSet}_*^B, \text{Com} : \mathbf{V}^{(B)} \to B$ have the form:

$$\text{ObSet}_*^B : x \to [x \neq \varnothing],$$
$$\text{MorSet}_*^B : \alpha \to [(\exists x)(\exists y)(\exists f)(x \neq \varnothing \wedge y \neq \varnothing \wedge f \neq \varnothing \wedge f \subset x \times y \wedge \alpha = (f,x,y)],$$

$$\mathrm{Com}: u \to [(\exists\alpha)(\exists\beta)(\exists\gamma)(\alpha,\beta,\gamma \text{ are correspondences}) \wedge \gamma = \alpha \circ \beta \wedge u = (\alpha,\beta,\gamma)].$$

The descent of the category Set_*^B is easily seen to coincide with the category $\mathcal{V}_*^{(B)}$ introduced in **3.1.7**. The category of nonempty sets and mappings inside $\mathbf{V}^{(B)}$ is determined in an analogous way, and it is also obvious that $\mathcal{V}^{(B)} = \mathrm{Set}^B\downarrow$.

3.2.18. Remarks

(1) As has been noted in **3.2.5**, the general symbol $\downarrow$ is used for denoting different operations of the same nature, so that the designation $X\downarrow$ is uniquely understood only if some additional information is available on which object X is descending. This analogous to using the sign '+' for denoting different group operations: the addition of numbers, vectors, linear operators, etc.. The exact interpretation is easily restored contextually.

(2) The double descent **3.2.10** also arises with respect to other set-theoretic operations. Thus, for instance, if $\prod X$ is a class of all mappings f from X to $\cup X$ such that $f(x) \in X$ for any $x \in X$, while $\sum X := \cup\{x \times \{x\} : x \in X\}$, then for every $X \in \mathbf{V}^{(B)}$ there are natural bijections

$$(\prod X)\downarrow\downarrow = \prod(X\downarrow\downarrow), \quad (\sum X)\downarrow = \sum(X\downarrow\downarrow).$$

(In the expression $(\prod X)\downarrow\downarrow$ the repeated descent is that of mappings!).

(3) It is obvious that in **3.2.11** (3) the inclusion is strict (provided $B \neq 2$). It should be also remarked that $\mathcal{P}(X)\downarrow$ is an algebraic system of signature $(\vee, \wedge, {}^*, 0, 1)$. We can show that this is a complete Boolean algebra which is a completion of the set $\mathcal{P}(X)\downarrow\downarrow$ ordered by inclusion in the following sense. There is an order-preserving injection $\iota : \mathcal{P}(X)\downarrow\downarrow \to \mathcal{P}(X)\downarrow$, in which for $a \in \mathcal{P}(X)\downarrow$, $a < 1$ there is a $b \in \mathcal{P}(X)\downarrow\downarrow$, so that we have $a \leq \iota b < 1$, which is quite analogous to the notion of the completion of a Boolean algebra (see [96, 233]).

(4) When proving **3.2.6** (10) we have established that, in particular, for any $X \in \mathbf{V}^{(B)}$ the mapping $\downarrow$ implements a bijection between the sets $\mathcal{V}(n, X\downarrow)$ and $\mathcal{V}^{(B)}(n^\wedge, X)$. In reality this fact is of quite general character and reflects an in-depth interrelation between the functors $\mathfrak{F}^\wedge$ and $\mathfrak{F}^\downarrow$ (for details see **3.5**).

3.3. The Ascent Functor

In this section ascent is introduced as procedure reverse in relation to descent, the corresponding functor is determined and its properties are studied.

3.3.1. Let us consider an arbitrary subclass X of the class $\mathbf{V}^{(B)}$.

(1) *There is a* $\mathbf{V}^{(B)}$*-class* Y *given by the formula*

$$Y(t) := \vee\{[\![t = x]\!] : x \in X\} \quad (t \in \mathbf{V}^{(B)}).$$

$\lhd$ Indeed, by theorem **1.3.14**, there is a class Y (in the sense of $\mathbf{V}$) such that

$$(y,b) \in Y \leftrightarrow y \in \mathbf{V}^{(B)} \wedge b \in B \wedge \left(b = \bigvee_{x \in X} [\![x = y]\!]\right).$$

As is seen, the class Y is single-valued and $Y = \mathbf{V}^{(B)}$, i.e., Y is a mapping from $\mathbf{V}^{(B)}$ to B. Besides, the mapping is extensional since, by virtue of **2.1.8** (4),

$$\begin{aligned} Y(t) \wedge [\![t = s]\!] &= \vee\{[\![t = x]\!] \wedge [\![t = s]\!] : x \in X\} \\ &\leq \vee\{[\![s = x]\!] : x \in X\} = Y(s). \end{aligned}$$

Hence, Y is a class inside $\mathbf{V}^{(B)}$. $\rhd$

Thus, we assign to every class $X \subset \mathbf{V}^{(B)}$ a class Y inside $\mathbf{V}^{(B)}$, which is called the *ascent of the class* X and is denoted by $X\uparrow$. In the case when X is a set, there is a unique element $y \in \mathbf{V}^{(B)}$ such that $X\uparrow(t) = [\![t \in y]\!]$ for all $t \in \mathbf{V}^{(B)}$ (see **2.5.15**). It is this y that is hereafter assumed to be the ascent of the set X, according to the remarks made in **2.5.10**. As an example, let us note that for a class $X \subset \mathbf{V}$ the class $X^{\wedge}$ is the ascent of the class $X_1 := \{x^{\wedge} : x \in X\}$ (see **2.5.15**).

(2) Let us now assume that X is a binary relation such that $X \subset \mathbf{V}^{(B)} \times \mathbf{V}^{(B)}$. In order to ascend the relation X, it should first be embedded into $\mathbf{V}^{(B)}$, and then the above procedure should be used. To achieve our purpose, let us make use of the function $(x,y) \to (x,y)^B$ (see **3.2.4**). Therefore, we give the following definition of the ascent of a binary relation:

$$X\uparrow : t \to \vee\{[\![t = (x,y)^B]\!] : (x,y) \in X\}.$$

In particular, if X is the product of classes $Y \subset \mathbf{V}^{(B)}$ and $Z \subset \mathbf{V}^{(B)}$, then we can ascend the product

$$(Y \times Z)\uparrow : t \to \vee\{[t = (x,y)^B]: y \in Y \wedge z \in Z\}.$$

3.3.2. *Let $X \subset \mathbf{V}^{(B)}$ be a nonempty class and φ be a B-formula. In this case*

$$[(\forall u \in X\uparrow)\, \varphi(u)] = \wedge\{[\varphi(u)]: u{:}\, X\},$$
$$[(\exists u \in X\uparrow)\, \varphi(u)] = \vee\{[\varphi(u)]: u{:}\, X\}.$$

◁ Let us present the deduction of the latter formula (see **1.1.5** (2,7)):

$$[(\exists u \in X\uparrow)\, \varphi(u)] = [(\exists u)(u \in X\uparrow \wedge\, \varphi(u))]$$
$$= \bigvee_{v \in \mathbf{V}^{(B)}} \bigvee_{u \in X} [u = v] \wedge [\varphi(v)]$$
$$= \bigvee_{u \in X} \left(\bigvee_{v \in \mathbf{V}^{(B)}} [v = u] \wedge [\varphi(v)] \right) = \vee\{[\varphi(u)]: u \in X\}.$$

The case of a universal quantifier is considered in an analogous way. ▷

3.3.3. *Whatever a class $X \subset \mathbf{V}^{(B)}$ and a nonempty $\mathbf{V}^{(B)}$-class $Y\colon \mathbf{V}^{(B)} \to B$ might be, the following rules for cancelling arrows are valid:*

(1) $X\uparrow\downarrow = \operatorname{mix}(X)$;

(2) $Y\uparrow\downarrow = Y$.

◁ (1) The case of an empty X is trivial. If $x \in X$, then $[x \in X\uparrow] = 1$ and, hence, $x \in X\uparrow\downarrow$. It follows from this fact and from **3.2.3** that $\operatorname{mix}(X) \subset X\uparrow\downarrow$. The reverse inclusion is deduced from **3.3.2** and from the mixing principle.

(2) By virtue of **2.5.16**, for an arbitrary $y \in \mathbf{V}^{(B)}$ we have

$$[y \in Y\downarrow\uparrow] = \vee\{[y = t]: t \in Y\downarrow\} = [(\exists t \in Y)(t = y)] = [t \in Y]. \ ▷$$

(3) Further on, when speaking about the mixing of a family of ordered pairs, it would be expedient to make use of the following proposition.

Let $(b_\xi)_{\xi \in \Xi}$ be a partition of unity, while $(x_\xi)_{\xi \in \Xi}$ and $(y_\xi)_{\xi \in \Xi}$ be families of elements of $\mathbf{V}^{(B)}$. Then, for mixings, we have

$$\operatorname*{mix}_{\xi\in\Xi} b_\xi (x_\xi, y_\xi)^B = \left(\operatorname*{mix}_{\xi\in\Xi} b_\xi x_\xi, \operatorname*{mix}_{\xi\in\Xi} b_\xi y_\xi\right)^B.$$

◁ Let us first show that $b(x,y)^B = b(bx,by)^B$ for any $x,y \in \mathbf{V}^{(B)}$ and $b \in B$. Indeed, we only have to successively apply **2.3.2**, **2.4.9** and **2.3.6**:

$$[b(x,y)^B = b(bx,by)^B] = b \Rightarrow [(x,y)^B = (bx,by)^B]$$
$$= b \Rightarrow ([x = bx] \wedge [y = by]) = b \Rightarrow ((b^* \Rightarrow [x = \varnothing])$$
$$\wedge (b^* \Rightarrow [y = \varnothing])) = b^* \vee ((b \vee [x = \varnothing]) \wedge (b \vee [y = \varnothing]))$$
$$= (b^* \vee b \vee [x = \varnothing]) \wedge (b^* \vee b \vee [y = \varnothing]) = 1.$$

Let us now put

$$x := \operatorname*{mix}_{\xi\in\Xi} b_\xi x_\xi, \ y := \operatorname*{mix}_{\xi\in\Xi} b_\xi y_\xi.$$

In this case, taking account of what has been proved above, we get

$$b_\xi(x_\xi, y_\xi)^B = b_\xi(b_\xi x_\xi, b_\xi y_\xi)^B = b_\xi(b_\xi x, b_\xi y)^B = b_\xi(x,y)^B.$$

Now we have to refer to the mixing principle. ▷

The fact just established allows one to consider mixings in the class $\mathbf{V}^{(B)} \times \mathbf{V}^{(B)}$. Namely, we will assume that, by definition,

$$\operatorname*{mix}_{\xi\in\Xi} b_\xi (x_\xi y_\xi) := \left(\operatorname*{mix}_{\xi\in\Xi} b_\xi x_\xi, \operatorname*{mix}_{\xi\in\Xi} b_\xi y_\xi\right).$$

Then the mapping $(x,y) \to (x,y)^B$ preserves mixings.

3.3.4. Theorem. *For any classes $X \subset \mathbf{V}^{(B)}$ and $Y \subset \mathbf{V}^{(B)}$ the following statements are valid:*

(1) $\mathbf{V}^{(B)} \models X\uparrow \subset Y\uparrow$ *if* $X \subset Y$;

(2) $\mathbf{V}^{(B)} \models (X \cup Y)\uparrow = X\uparrow \cup Y\uparrow$;

(3) $\mathbf{V}^{(B)} \models (\operatorname{mix}(X) \cap \operatorname{mix}(Y))\uparrow = X\uparrow \cap Y\uparrow$;

(4) $\mathbf{V}^{(B)} \models (X \times Y)\uparrow = X\uparrow \times Y\uparrow$.

If, however, X and Y are relations and Z is a class, then the following relations are also fulfilled:

(5) $\mathbf{V}^{(B)} \models \mathrm{dom}(X)\uparrow = \mathrm{dom}(X\uparrow) \wedge \mathrm{im}(X)\uparrow = \mathrm{im}(X\uparrow)$;

(6) $\mathbf{V}^{(B)} \models (X^{-1})\uparrow = (X\uparrow)^{-1}$;

(7) $\mathbf{V}^{(B)} \models (\mathrm{mix}(X)``\mathrm{mix}(Z))\uparrow = (X\uparrow)``(Z\uparrow)$;

(8) $\mathbf{V}^{(B)} \models (\mathrm{mix}(X) \circ \mathrm{mix}(Y))\uparrow = (X\uparrow) \circ (Y\uparrow)$;

(9) $\mathbf{V}^{(B)} \models (Z^n)\uparrow = (Z\uparrow)^{n^\wedge}$.

⊲ (1) The result follows from the definition of descent.

(2) This fact is substantiated by the following obvious relations:

$$[t \in (X \cup Y)\uparrow] = \vee\{[t = u] : u \in X \cup Y\}$$
$$= \bigvee_{u \in X} [t = u] \vee \bigvee_{u \in Y} [t = u] = [t \in X\uparrow \vee t \in Y\uparrow].$$

(3) Let us assume that inside $\mathbf{V}^{(B)}$ the ascent of the intersection of the classes X and Y coincides with the intersection of their ascents. Then, according to **3.2.6** (2) and **3.3.3**, we get

$$\mathrm{mix}(X \cap Y) = (X \cap Y)\uparrow\downarrow = (X\uparrow \cap Y\uparrow)\downarrow$$
$$= X\uparrow\downarrow \cap Y\uparrow\downarrow = \mathrm{mix}(X) \cap \mathrm{mix}(Y).$$

Conversely, let the cyclic hull of the intersection of the classes X and Y be equal to the intersection of their cyclic hulls. Then, applying **3.2.6** (2) and **3.3.3** again, we get

$$(X \cap Y)\uparrow\downarrow = X\uparrow\downarrow \cap Y\uparrow\downarrow = (X\uparrow \cap Y\uparrow)\downarrow,$$

and, hence, $[(X \cap Y)\uparrow = X\uparrow \cap Y\uparrow] = \mathbf{1}$ (according to **3.2.3** (3)). To complete the proof, we have to apply the above-established facts to the classes $\mathrm{mix}(X)$ and $\mathrm{mix}(Y)$ and to use the rules for cancelling arrows **3.3.3**.

(4) In conformity with rules **3.3.2**, we calculate

$$[z \in X\uparrow \times Y\uparrow] = [(\exists u \in X\uparrow)(\exists v \in Y\uparrow) z = (u,v)]$$
$$= \bigvee_{u \in X} \bigvee_{v \in Y} [z = (u,v)] = \bigvee_{(u,v) \in X \times Y} [z = (u,v)^B] = [z \in (X \times Y)\uparrow].$$

(5) Assuming X to be a binary relation, we can easily check the validity of the following chain of equalities (see **1.1.5** (2,7)):

$$[x \in \mathrm{dom}(X\uparrow)] = [(\exists y)((x,y \in X\uparrow)] = \bigvee_{y \in \mathbf{V}^{(B)}} \bigvee_{(s,t) \in X} [(x,y)^B = (s,t)^B]$$
$$= \bigvee_{(s,t) \in X} \bigvee_{y \in \mathbf{V}^{(B)}} [x = s] \wedge [y = t] = \bigvee_{s \in \mathrm{dom} X} [x = s] = [x \in \mathrm{dom}(X)\uparrow].$$

The statement on $\mathrm{im}(X)$ is established analogously.

(6) $[(x,y) \in (X\uparrow)^{-1}] = [(y,x) \in X\uparrow]$
$$= \bigvee_{(s,t) \in X} [(s,t) = (y,x)] = \bigvee_{(t,s) \in X^{-1}} [(t,s) = (x,y)]$$
$$= [(x,y) \in (X^{-1})\uparrow] = \bigvee_{(t,s) \in X^{-1}} [(t,s) = (x,y)].$$

(7), (8) It is obvious that

$$\mathrm{mix}(X) \cap (\mathrm{mix}(Z) \times \mathbf{V}^{(B)}) = \mathrm{mix}(X) \cap \mathrm{mix}(Z \times \mathbf{V}^{(B)});$$
$$(\mathrm{mix}(Y) \times \mathbf{V}^{(B)}) \cap (\mathbf{V}^{(B)} \times \mathrm{mix}(X))$$
$$= \mathrm{mix}(Y \times \mathbf{V}^{(B)}) \cap \mathrm{mix}(\mathbf{V}^{(B)} \times X).$$

Then we have to follow the scheme of **3.2.6** (5,6), making use of (3), (4) and the fact that $[\mathbf{V}^{(B)}\uparrow = \mathbf{U}_B] = \mathbf{1}$.

(9) It should be observed that if remarks **3.3.3** (3) are taken into account, then we conclude that $\mathrm{mix}(Z^n) = \mathrm{mix}(Z)^n$. Thus, by virtue of **3.2.6** (10) and **3.3.3** (1), we infer

$$((Z\uparrow)^{n^\wedge})\downarrow = (Z\uparrow\downarrow)^n = (Z^n)\uparrow\downarrow,$$

which yields, according to **3.2.3** (3), the sought equality. ▷

3.3.5. Let us consider a class X whose elements are subsets of $\mathbf{V}^{(B)}$, i.e., $X \subset \mathcal{P}(\mathbf{V}^{(B)})$. The *double* or *repeated ascent* of the class X, denoted by $X\uparrow\uparrow$, is the ascent of the class $\{x\uparrow : x \in X\}$. Hence,

$$[t \in X\uparrow\uparrow] = \vee\{[t = x\uparrow] : x \in X\} \quad (t \in \mathbf{V}^{(B)}).$$

Let us introduce one more notation:

$$\text{mix}''X: = \{\text{mix}(u): u \in X\}.$$

It is obvious that $[X \uparrow \uparrow = (\text{mix}''X) \uparrow \uparrow] = \mathbf{1}$. By $\mathcal{P}_0(X)$ we shall denote a class of nonempty elements $\mathcal{P}(X)$, i.e.,

$$\mathcal{P}_0(X): = \{z: z \subset X \wedge z \neq \varnothing\}.$$

3.3.6. *Let X be a nonempty $\mathbf{V}^{(B)}$-class and $Y \subset \mathcal{P}(\mathbf{V}^{(B)})$. Then*

(1) $\mathbf{V}^{(B)} \models \cup(Y \uparrow \uparrow) = (\cup Y) \uparrow$;

(2) $\mathbf{V}^{(B)} \models \cap(Y \uparrow \uparrow) = \cap(\text{mix}''(Y \uparrow))$;

(3) $\mathbf{V}^{(B)} \models \cup X = (\cup(X \downarrow \downarrow)) \uparrow$;

(4) $\mathbf{V}^{(B)} \models \mathcal{P}_0(X \downarrow) \uparrow \uparrow = \mathcal{P}_0(X)$.

◁ The proof is suggested as an exercise. ▷

3.3.7. Let us now return to theorem **3.3.4** and remark that according to items (1) and (4) of this theorem, the ascent of a relation is again a relation. From the functional viewpoint, however, it is important that 'the images of points and sets', $X(t)$ and $X''A$, are also preserved in ascending, which is not always the case (see **3.3.4** (7)). Moreover, in ascending, the function can lose its property of being single-valued. This can be easily understood if we take it into account that the procedure 'ascent - descent' results in taking the cyclic hull (**3.3.3** (1)), while the functions obtained by way of descending extensional (**3.2.6** (9)). Here is an example. Let $X \subset \mathbf{V}^{(B)}$ be a cyclic set and $f: X \to \{0^\wedge, 1^\wedge\}$ be a two-valued function. Let us assume that $f(x) = 0^\wedge$ and $f(y) = 1^\wedge$ for some $x, y \in X, x \neq y$, while an element $b \in B$ is other than $\mathbf{0}$ and $\mathbf{1}$. In on an element $z := \text{mix}\{bx, b^* y\} \in X$ the function f assumes the value $0^\wedge$, then $0 < b^* \leq [z = y] \nleq [f(z) = f(y)] = \mathbf{0}$. Analogously, for $f(z) = 1^\wedge$ we get $0 < b \leq [z = y] \nleq [f(z) = f(y)] = \mathbf{0}$.

On the other hand, $[z = y] \leq [f \uparrow (z) = f \uparrow (y)]$ (see **3.2.6** (9)). Therefore, either $[f \uparrow (y) = f(y)] \neq 1$ or $[f \uparrow (x) = f(x)] \neq \mathbf{1}$, i.e., $[f \uparrow (x) = f(x)] = \mathbf{1}$ is fulfilled not for any $x \in X$. Hence, preservance for the functional dependence in ascending should be specially considered.

3.3.8. *For an arbitrary relation $X \subset \mathbf{V}^{(B)} \times \mathbf{V}^{(B)}$ the following conditions are equivalent:*

(1) *if $b \le [x_1 = x_2]$ for $x_1, x_2 \in \mathrm{dom}(X), b \in B$, then for any $u \in \mathbf{V}^{(B)}$ we get*

$$\vee\{b \wedge [y_1 = u]: y_1 \in X(x_1)\} = \vee\{b \wedge [y_2 = u]: y_2 \in X(x_2)\};$$

(2) *if $x_1, x_2 \in \mathrm{dom} X$ and $y_1 \in X(x_1)$, then*

$$[x_1 = x_2] \le \vee\{[y_1 = y_2]: y_2 \in X(x_2)\};$$

(3) $\mathrm{mix}(X(x)) = (\mathrm{mix} X)(x) \quad (x \in \mathrm{dom}(X))$;

(4) $[X\uparrow(x) = X(x)\uparrow] = \mathbf{1} \quad (x \in \mathrm{dom}(X))$;

(5) $[x_1 = x_2] \le [X(x_1)\uparrow = X(x_2)\uparrow] = \mathbf{1} \quad (x_1, x_2 \in \mathrm{dom}(X))$.

◁ (1) → (2). Let us set in (1) $b := \lfloor |x_1 = x_2| \rfloor$ and $u := y_1$.

(2) → (3). The inclusion $\subset$ is obvious. In order to prove the reverse inclusion, let us choose a partition of unity $(b_\xi) \subset B$ and a family $((x_\xi, y_\xi)) \subset X$ and denote $(x,y) = \mathrm{mix}(b_\xi(x_\xi y_\xi))$. The task is to establish that $y \in \mathrm{mix}(X(x))$. It follows from (2) that

$$b_\xi \le [x = x_\xi] \le \vee\{[y' = y_\xi]: y' \in X(x)\} = [y_\xi \in X(x)\uparrow].$$

Therefore, $b_\xi \le [y = y_\xi] \wedge [y_\xi \in X(x)\uparrow] \le [y \in X(x)\uparrow]$, so that $[y \in X(x)\uparrow] = \mathbf{1}$. But in this case $y \in X(x)\uparrow\downarrow = \mathrm{mix}(X(x))$, which completes the proof.

(3) → (4). In view of **3.3.3** (1) and **3.2.6** (6), we have

$$X(x)\uparrow\downarrow = \mathrm{mix}(X(x)) = (\mathrm{mix} X)(x) = (X\uparrow\downarrow)(x) = (X\uparrow(x))\downarrow.$$

Using **3.3.3** (2) again, we come the the required relation.

(4) → (5). It suffices to apply **3.2.6** (9).

(5) → (1). By **2.3.2**, if $b \le [x_1 = x_2]$ and $x_1, x_2 \in \mathrm{dom}(X)$, then $b(X(x_1)\uparrow) = b(X(x_2)\uparrow)$. On the other hand, according to the definition of descent,

$$[u \in b(X(x_k)\uparrow)] = \vee\{\wedge [u = y]: y \in X(x_k)\},$$

which leads us to the required result. ⊳

3.3.9. Let us now return to the notion of extensionality which we had to do with in **3.2.6** (9) and **3.2.12** (1) and which is now considered under more general conditions. The binary relation $R \subset \mathbf{V}^{(B)} \times \mathbf{V}^{(B)}$ is called *extensional in second coordinate*, provided it obeys one (and, hence, all) of the equivalent conditions **3.3.8** (1) - (5). Let us remark that if R is a function, then each of conditions (2) and (5) of **3.3.8** turns into the following relation (cf. **2.5.5**)

$$[x_1 = x_2] \le [R(x_1) = R(x_2)] \quad (x_1, x_2 \in \mathrm{dom}(R)).$$

Let $X \subset \mathbf{V}^{(B)}$ and $Y \subset \mathbf{V}^{(B)}$ be sets. A correspondence $\Phi := (F, X, Y)$ is called extensional if its graph F is a relation extensional in second coordinate. If, moreover, $\mathrm{dom}\,\Phi = \mathrm{mix}\,\mathrm{dom}\Phi$ and $\Phi(x) = \mathrm{mix}\,\Phi(x)$ for every $x \in \mathrm{dom}\Phi$, then Φ is said to be *fully extensional.* It is evident that the fact that Φ is fully extensional implies $F = (X \times Y) \cap \mathrm{mix}(F)$.

The sets A and $C \subset \mathbf{V}^{(B)}$ are said to *be in general position* provided

$$[a = c] \le \vee\{[a = b] \wedge [b = c] : b \in A \cap C\}$$

for any $a \in A$ and $c \in C$. When this condition is fulfilled, the last relation is in fact an equality , since $[a = b] \wedge [b = c] \le [a = c]$.

The following statements are equivalent:

(1) $\mathbf{V}^{(B)} \models (A \cap C)\uparrow = A\uparrow \cap C\uparrow$;

(2) $\mathrm{mix}(A \cap C) = \mathrm{mix}(A) \cap \mathrm{mix}(C)$;

(3) *A and C are in general position.*

⊲ The equivalence of (1) and (2) results from **3.2.6** (1), **3.3.3** (1) and **3.3.4** (3). Let us prove (1) → (3). It should be remarked that the inclusion $A\uparrow \cap C\uparrow \subset (A \cap C)\uparrow$ is equivalent to the formula

$$(\forall a \in A\uparrow)(\forall c \in C\uparrow)(a = c \to (\exists b \in A \cap C) \quad (a = b \wedge b = c)).$$

The Boolean truth-value of this formula is as follows:

$$\bigwedge_{a \in A, c \in C} [a = c] \Rightarrow \bigvee_{b \in A \cap C} [a = b] \wedge [b = c].$$

This implies that (3) is equivalent to the inclusion $A\uparrow \cap C\uparrow \subset (A\cap C)\uparrow$ inside $\mathbf{V}^{(B)}$. The reverse inclusion is always valid. ▷

Therefore, if $A\subset C$, then A and C are in general position by a trivial reason. In a general position there are any two sets of the type $A := \{a^{\wedge} : a \in A'\}$, where $A' \in \mathbf{V}$.

The *ascent of the correspondence* $\Phi := (F,X,Y)$ is by definition the element $\Phi\uparrow := (F\uparrow, X\uparrow, Y\uparrow)^B \in \mathbf{V}^{(B)}$, where $F\uparrow$ is the ascent of the relation F (see **3.3.1** (2)).

3.3.10. Theorem. *Let X and Y be subsets of the class $\mathbf{V}^{(B)}$ and Φ is an extensional correspondence from X to Y. The ascent $\Phi\uparrow$ is a unique correspondence from $X\uparrow$ to $Y\uparrow$ inside $\mathbf{V}^{(B)}$ such that*

$$[\![\mathrm{dom}(\Phi\uparrow) = (\mathrm{dom}\Phi)\uparrow]\!] = \mathbf{1},$$
$$[\![\Phi\uparrow(x) = \Phi(x)\uparrow]\!] = \mathbf{1} \quad (x \in \mathrm{dom}\Phi).$$

The ascent of a correspondence has the following properties:

(1) *if* $\mathrm{dom}\,\Phi$ *and a set* $A\subset X$ *are in general position, then*

$$\mathbf{V}^{(B)} \models \Phi(A)\uparrow = \Phi\uparrow(A\uparrow);$$

(2) *the composition* $\Psi\circ\Phi$ *of extensional correspondences* Φ *and* Ψ *is an extensional correspondence, and if, besides,* $\mathrm{dom}\,\Psi\circ\Phi = \mathrm{dom}\Phi$ *and the sets* $\mathrm{dom}\,\Psi$ *and* $\Phi(x)$ *are in general position for all* $x\in\mathrm{dom}\,\Phi$, *then*

$$\mathbf{V}^{(B)} \models (\Psi\circ\Phi)\uparrow = \Psi\uparrow\circ\Phi\uparrow;$$

(3) $\mathbf{V}^{(B)} \models (I_X)\uparrow = I_{X\uparrow}$.

◁ By virtue of **3.3.4** and **3.3.8** it suffices to verify uniqueness for $\Phi\uparrow$ and properties (1) - (3). The case of the empty correspondence is omitted as obvious. Let Ψ be a correspondence inside $\mathbf{V}^{(B)}$ obeying the same relations that $\Phi\uparrow$, i.e., $[\![\mathrm{dom}\Psi = (\mathrm{dom}\Phi)\uparrow]\!] = \mathbf{1}$ and $[\![\Psi(x) = \Phi(x)\uparrow]\!] = \mathbf{1}$ $(x\in\mathrm{dom}\Phi)$. In this case $\mathbf{V}^{(B)} \models \mathrm{dom}\Psi = \mathrm{dom}(\Phi\uparrow)$ and

$$[\![(\forall x \in \mathrm{dom}\Psi)\ \Psi(x) = \Phi\uparrow(x)]\!]$$
$$= \bigwedge_{x\in\mathrm{dom}\Phi} [\![\Psi(x) = \Phi\uparrow(x)]\!] = \bigwedge_{x\in\mathrm{dom}\Phi} [\![\Psi(x) = \Phi(x)\uparrow]\!] = \mathbf{1}.$$

(1) Making use of **3.3.9** (1) and the properties of $\Phi\uparrow$ established above, the following equivalences can be written for an arbitrary $y \in \mathbf{V}^{(B)}$:

$$y \in \Phi\uparrow(A\uparrow) \leftrightarrow (\exists x)(x \in (\mathrm{dom}\Phi))\uparrow \wedge x \in A\uparrow \wedge y \in \Phi\uparrow(x) \leftrightarrow$$
$$\leftrightarrow (\exists x)(x \in (A \cap \mathrm{dom}\,\Phi)\uparrow \wedge y \in \Phi\uparrow(x)) \leftrightarrow$$
$$\leftrightarrow (\exists x \in (A \cap \mathrm{dom}\Phi)\uparrow) y \in \Phi(x).$$

Hence, the next equalities hold:

$$[y \in \Phi\uparrow(A\uparrow)] = \bigvee_{x \in A \cap \mathrm{dom}\Phi} [y \in \Phi(x)\uparrow]$$
$$= \bigvee_{x \in A \cap \mathrm{dom}\Phi} \ \bigvee_{v \in \Phi(x)} [y = v] = \bigvee_{v \in \Phi(A)} [y = v] = [y \in \Phi(A)\uparrow].$$

(2) Let us show that the correspondence $\Theta := \Psi \circ \Phi$ is extensional. Take $x_1, x_2 \in \mathrm{dom}\Theta$, $y_1 \in \Phi(x_1)$ and $z_1 \in \Psi(y_1)$. According to **3.3.8** (2), the following estimates are valid:

$$\bigvee_{z_2 \in \Theta(x_2)} [z_1 = z_2] = \bigvee_{y_2 \in \Phi(x_2)} \left(\bigvee_{z_2 \in \Psi(y_2)} [z_1 = z_2] \right)$$
$$\geq \bigvee_{y_2 \in \Phi(x_2)} \lfloor |y_1 = y_2| \rfloor \geq \lfloor |x_1 = x_2| \rfloor.$$

Using **3.3.8** (2) again, we remark that Θ is extensional. Therefore, by what has already been proved, for Θ we infer:

$$[\Theta\uparrow(x) = \Theta(x)\uparrow] = \mathbf{1} \quad (x \in \mathrm{dom}(\Theta)).$$

Taking now into account the facts established in (1), we can write inside $\mathbf{V}^{(B)}$ the following:

$$\Theta\uparrow(x) = \Theta(x)\uparrow = \Psi(\Phi(x))\uparrow = \Psi\uparrow(\Phi(x)\uparrow) =$$
$$= \Psi\uparrow(\Phi\uparrow(x)) = (\Psi\uparrow \circ \Phi\uparrow)(x) \quad (x \in \mathrm{dom}(\Theta)).$$

Hence, **3.3.2** yields the next relation

$$\mathbf{V}^{(B)} \models (\forall x \in \mathrm{dom}(\Theta\uparrow) = \mathrm{dom}(\Phi\uparrow))(\Theta\uparrow(x) = (\Psi\uparrow \circ \Phi\uparrow)(x)),$$

which is equivalent to the sought result, since $(\Psi\uparrow \circ \Phi\uparrow) = \mathrm{dom}(\Theta\uparrow)$.

(3) This is obvious. ▷

3.3.11. Theorem. *Let X and Y be subsets of the class $\mathbf{V}^{(B)}$, while f be an extensional mapping from X to Y. Then $f\uparrow$ is a unique element of $\mathbf{V}^{(B)}$ for which*

$$[f\uparrow : X\uparrow \to Y\uparrow] = [f\uparrow(x) = f(x)] = \mathbf{1} \ (x \in X).$$

The ascent of a mapping has the following properties:

(1) *if Z is a subset of $\mathbf{V}^{(B)}$ and $g: Y \to Z$ is an extensional mapping, then the mapping $g \circ f$ is also extensional and*

$$\mathbf{V}^{(B)} \models (g \circ f)\uparrow = g\uparrow \circ f\uparrow;$$

(2) $\mathbf{V}^{(B)} \models f(A)\uparrow = f\uparrow(A\uparrow) \quad (A \subset X)$;

(3) $\mathbf{V}^{(B)} \models$ *'the mapping $f\uparrow$ is injective' iff f is injective;*

(4) $\mathbf{V}^{(B)} \models$ *'the mapping $f\uparrow$ is surjective' iff* $\mathrm{mix}(\mathrm{im} f) = \mathrm{mix}(Y)$.

3.3.12. Proposition **3.3.3** directly yields rules for cancelling arrows for correspondences and mappings.

Let Φ and f be extensional correspondences from X to Y, in which case let Ψ be a correspondence inside $\mathbf{V}^{(B)}$. Then the following equalities are valid:

(1) $\Phi\uparrow\downarrow(x) = \mathrm{mix}(\Phi(x)) \quad (x \in \mathrm{dom}(\Phi))$;

(2) $f\uparrow\downarrow(x) = f(x) \quad (x \in \mathrm{dom}(f))$;

(3) $\Psi\uparrow\downarrow = \Psi$;

(4) $\pi_{\Phi\uparrow\downarrow}(A) = \pi_{\Phi\uparrow}(A\uparrow)\downarrow \quad (A \subset X)$;

(5) $\pi_{\Phi\uparrow\downarrow}(A)\uparrow = \pi_{\Phi\uparrow}(A\uparrow) \quad (A \subset X)$.

If, moreover, Φ is fully extensional and $A \subset \mathrm{dom}\,\Phi$ then

(6) $\pi_{\Phi}(A)\uparrow = \pi_{\Phi\uparrow}(A\uparrow)$.

$\triangleleft$ (1) From **3.2.13**, **3.3.10** and **3.3.3** (1) we directly deduce $(x \in \mathrm{dom}(\Phi))$:

$$\Phi\uparrow\downarrow(x) = \Phi\uparrow(x)\downarrow = \Phi(x)\uparrow\downarrow = \mathrm{mix}(\Phi(x)).$$

(2), (3) These are obvious.

(4) For an arbitrary $A \subset X$ we get

$$z \in \pi_{\Phi\uparrow}(A\uparrow)\downarrow \leftrightarrow [(\forall a \in A\uparrow)\, z \in \Phi\uparrow(a)] = \mathbf{1}$$
$$\leftrightarrow \bigwedge_{a\in A} [z \in \Phi\uparrow(a)] = \mathbf{1} \leftrightarrow (\forall a \in A)(z \in \Phi\uparrow(a)\downarrow)$$
$$\leftrightarrow (\forall a \in A)\, z \in \Phi\uparrow\downarrow(a) \leftrightarrow z \in \pi_{\Phi\uparrow\downarrow}(A).$$

(5) The sought equality results from the above-proved by virtue of **3.3.3** (2).

(6) According to (1), for a fully extensional Φ we have

$$\pi_{\Phi\uparrow\downarrow}(A) = \bigcap_{a\in A} \Phi\uparrow\downarrow(a) = \bigcap_{a\in A} \Phi(a) = \pi_{\Phi}(A).$$

The required result now ensues from (5). ▷

3.3.13. Let us now consider the category $\mathcal{PV}_*^{(B)}$ that consists of nonempty subsets of the class $\mathbf{V}^{(B)}$ and extensional correspondences having a nonempty graph with the conventional superposition as the composition:

$$\mathrm{Ob}\,\mathcal{PV}_*^{(B)} := \mathcal{P}(\mathbf{V}^{(B)}) - \{\varnothing\};$$

$$\mathcal{PV}_*^{(B)}(X,Y) := \{\Phi : \Phi \text{ is an extensional correspondence from } X \text{ to } Y \text{ and } \mathrm{Gr}\Phi \neq \varnothing\},$$

$$\mathrm{Com}(\Phi,\Psi) := \Psi \circ \Phi \;(\Phi,\Psi \in \mathrm{Mor}\,\mathcal{PV}_*^{(B)}).$$

The subcategory of the category $\mathcal{PV}_*^{(B)}$ which consists of cyclic sets and fully extensional correspondences will be denoted by $\mathcal{CPV}_*^{(B)}$. Let $\mathcal{PV}^{(B)}$ and $\mathcal{CPV}^{(B)}$ be subcategories of the categories $\mathcal{PV}_*^{(B)}$ and $\mathcal{CPV}_*^{(B)}$, respectively, with with the same classes of objects but with classes of extensional mappings as morphisms. The soundness of definition is ensured by **3.3.10** and **3.3.11**. Let us now consider a mapping $\mathfrak{F}\uparrow$ assigning to every object X and every morphism Φ of the category $\mathcal{PV}_*^{(B)}$ their ascents $X\uparrow$ and $\Phi\uparrow$, respectively. By virtue of theorem **3.3.10**, $\mathfrak{F}\uparrow$ acts into the category $\mathcal{V}_*^{(B)}$ (see **3.1.7**).

3.3.14. Theorem. *The mapping $\mathfrak{F}\uparrow$ is a covariant functor from the category $\mathcal{PV}^{(B)}$ to the category $\mathcal{V}^{(B)}$.*

3.3.15. Remarks

(1) The use of the symbol $\uparrow$ for denoting various types of ascents is analogous to the situation of the notation of descents. Therefore, the warnings and agreements made in **3.2.5** and **3.2.18** (1) should be taken into account.

(2) The functors $\mathfrak{F}^{\wedge}$ and $\mathfrak{F}^{\uparrow}$ operate in the same category and in many respects resemble one another (compare, for instance, definitions **2.5.15** and **3.3.1** (1), formulas **3.3.2** with analogous formulas of **2.5.15**, **3.3.3** and **3.1.1** (1), **3.3.4** and **3.1.4**, **3.1.10** and **3.1.5**). A more detailed analogy will be discussed in the section to follow.

(3) Formulas **3.3.2** and their counterparts of **2.5.15** are the particular cases of the following rules. If φ and ψ are predicative formulas in $n+1$ and $m+1$ free variables, respectively, while $X_1,\dots,X_n$ and $Y_1,\dots,Y_m$ are certain $\mathbf{V}^{(B)}$-classes, then

$$[(\forall u)(\varphi(u,\bar{X})\to\psi(u,\bar{Y})]=\wedge\{[\psi(u,\bar{X})]: x\in A\},$$
$$[(\exists u)(\varphi(u,\bar{X})\to\psi(u,\bar{Y})]=\vee\{[\psi(u,\bar{X})]: x\in A\},$$

where A is any subclass of the class $\mathbf{V}^{(B)}$ obeying the condition

$$\operatorname{mix}(A)=\{x\in\mathbf{V}^{(B)}:[\varphi(x,\bar{X})]=1\}\quad(\bar{X}=(X_1,\dots,X_n)).$$

(4) Ascending has already been implicitly used in **2.4**. Let us dwell on this point. Let x be a subset of an unseparated universe, while $x'\subset\mathbf{V}^{(B)}$ be its image under factorization (see **2.5.2**, **2.5.7**): $x':=\pi``x:=\{\pi t: t\in x\}$. Let us determine an element y of the unseparated universe by the formulas: $\operatorname{dom}(y):=x$, $\operatorname{im}(y):=\{1\}$. Then $[\pi y=x'\uparrow]=1$. Indeed,

$$[\pi t\in x'\uparrow]=\bigvee_{u\in x'}[\pi t=u]=\bigvee_{u\in x}[\pi t=\pi u]$$
$$=\bigvee_{u\in\operatorname{dom}(y)}y(u)\wedge[t=u]=[t\in y]=[\pi t=\pi u].$$

Therefore, the element y of **2.4.5** (b), $\{x\}^B$ and $\{x,y\}^B$ of **2.4.8**, f of **2.4.11** (1-3), are all ascents in the unseparated universe. Besides, $X^{\wedge}$ is the ascent of the class $\{x^{\wedge}: x\in X\}$ (see **3.3.1** (1)).

(5) In statements **3.3.10** (1,2) the condition of general position cannot be omitted.

The corresponding counter-examples can be easily constructed on using the following considerations. Let us assume that $A \subset X$ and Φ is a correspondence from X to X with the graph $\{(x,x): x \in M\}$. If $A \subset X$, in which case $A \cap M = \varnothing$ but $A \cap \operatorname{mix} M \neq \varnothing$, then $\Phi(A) = \varnothing$ and $[\Phi(A){\uparrow} = \varnothing] = \mathbf{1}$. On the other hand, $[\Phi{\uparrow}(A{\uparrow}) \neq \varnothing] = \mathbf{1}$, since for $z \in A \cap \operatorname{mix} M$ we have $[z \in \Phi{\uparrow}(A{\uparrow})] = \mathbf{1}$.

It should be also remarked that in similar considerations ([114, 120, 121, 134]) the condition of general position is absent and, hence, it is always implicitly assumed that $A \subset \operatorname{dom}\Phi$ or $\operatorname{im}\Phi \subset \operatorname{dom}\Psi$. This might result in ambiguity when working with general correspondences. This is, however, not dangerous for correspondences defined everywhere and, in particular, for mappings. The remarks made refer to the rules for calculating polars as well (see **3.3.12** (6)).

3.4. The Immersion Functor

In applications of Boolean-valued models to analysis the following technique proves to be greatly expedient. The analytical object under investigation is immersed in a Boolean-valued universe in such a way that inside the model it becomes a simpler and (or) well-studied object. This procedure proves functional, i.e., it enables one to study not only the internal structure of certain objects, but also their interrelations.

3.4.1. The sets appearing as descents have a supplementary algebraic structure and, therefore, only objects associated in a necessary way with a complete Boolean algebra can be immersed in $\mathbf{V}^{(B)}$. Let us introduce the necessary terminology. Consider an arbitrary set X. The mapping $d: X \times X \to B$ is called a *B-semimetric* provided that for any $x, y, z \in X$ the following conditions are fulfilled:

(1) $d(x,y) = 0$;

(2) $d(x,y) = d(x,y)$;

(3) $d(x,y) \leq d(x,z) \vee d(z,y)$.

If, moreover, $d(x,y) = 0$ yields $x = y$, then d is called a *B-metric* or a *Boolean metric* on X. The pair (X,d) is termed a *B-set* or a *Boolean set*, provided X is a set, while d is a Boolean metric on X.

When the set X is contained in the class $\mathbf{V}^{(B)}$, then X is furnished with the canonical B-metric:

$$d(x,y) := [\![x \neq y]\!] = [\![x = y]\!]^* \quad (x,y \in X).$$

The fact that d is a B-metric follows from **2.1.8** (1,3,4) and the separation of $\mathbf{V}^{(B)}$. While considering the subsets of the class $\mathbf{V}^{(B)}$ as B-sets, we will always mean the Boolean metric mentioned above. Many notions of Chapter 2 are naturally transferred to B-sets by way of dualizing with the help of complementation in the algebra B. Thus, we sometimes omit some small details in introducing new notions.

3.4.2. Let (b_ξ) be a partition of unity in B, and let (x_ξ) be a family of elements of a B-set X. *The mixing of the family* (x_ξ) *by* (b_ξ) is an element $x \in X$ such that $b_\xi \wedge d(x,x_\xi) = 0$ for all ξ. As before, the mixing will be denoted by the symbol $x = \operatorname{mix} b_\xi x_\xi$. The mixing (if it exists) is unique. Indeed, if $y \in X$ and $(\forall \xi)(b_\xi \wedge d(y,x_\xi) = 0)$, then

$$b_\xi \wedge d(x,y) \leq b_\xi \wedge (d(x,x_\xi) \vee d(x_\xi,y)) = 0.$$

The infinite distributive law **1.1.5** (2) in B implies

$$d(x,y) = \vee\{b_\xi \wedge d(x,y)\} = 0,$$

and, hence, $x = y$.

It should be emphasized that unlike in the universe $\mathbf{V}^{(B)}$ (see **2.3**), not all mixings in a B-set exist.

3.4.3. Let us consider a B-set (X,d). For a subset $A \subset X$ by the symbol $\operatorname{mix} A$ we shall denote a set of all mixings of elements of A. If $\operatorname{mix} A = A$, then A is said to be a *cyclic subset* in X. The intersection of all cyclic sets containing A is denoted by $\operatorname{cyc}(A)$. A Boolean set X is called *extended* (or *fully cyclic*) if it contains mixings $\operatorname{mix}(b_\xi x_\xi)$ of any families $(x_\xi) \subset X$ relative to any partitions of unity $(b_\xi) \subset B$. In the case when such mixings exist only for finite sets of elements, the X itself is called *decomposable*. In the same way as in **3.2.8** it is shown that if X is an extended B-set, then $\operatorname{mix}(A) = \operatorname{cyc}(A)$ for any $A \subset X$. The cyclic subset of a B-set is not always an extended B-set. Each cyclic subset of $\mathbf{V}^{(B)}$ with its canonical B-metric is an extended B-set.

3.4.4. *Let A be a set, and let for any $\alpha \in A$ be given a B-set (X_α, d_α). Put $X := \prod_{\alpha \in A} X_\alpha$ and define the mapping $d: X \times X \to B$ in the following way:*

$$d(x,y) := \vee\{d_\alpha(x(\alpha), y(\alpha)) : \alpha \in A\}.$$

Then d is a Boolean metric on X, and, moreover, (X,d) is extended iff X_α is extended for any $\alpha \in A$.

$\triangleleft$ We can easily prove that the above mapping is a B-metric. Besides, if (b_ξ) is a partition of unity, while (x_ξ) is a family of elements of the product X, then $x = \text{mix}(b_\xi x_\xi(\alpha))$ iff $x(\alpha) = \text{mix}(b_\xi x_\xi(\alpha))$ for all $\alpha \in A$. Therefore, X can be claimed to be extended. $\triangleright$

Below the product of B-sets will be always viewed as a B-set with the Boolean metric given in **3.4.4.**

3.4.5. *Let A be a subset of an extended B-set (X,d). Then for any $x \in X$ the distance*

$$\text{dist}(x, A) := \wedge\{d(x,a) : a \in A\}$$

is attained for a certain $a \in \text{mix}\,A$. In other words, for every $x \in X$ there is an $a \in \text{mix}\,A$ such that $\text{dist}(x, A) = d(x,a)$.

$\triangleleft$ If $b_0 := \text{dist}(x, A)$, then there is a partition (b_ξ) of an element b_0^* and a family $(a_\xi) \subset A$ such that $b_\xi \wedge d(x, a_\xi) = 0$ for all ξ. Let us put $a := \text{mix}\{b_0 a_0, b_\xi a_\xi\}$, where a_0 is an arbitrary element of A. Since $(b_\xi) \cup \{b_0\}$, is a partition of unity then $a \in \text{mix}(A)$. Besides, for any ξ we get

$$b_\xi \wedge d(x,a) \le (b_\xi \wedge d(x, a_\xi)) \vee (b_\xi \wedge d(a_\xi, a)) = 0.$$

Hence, $b_0^* \wedge d(x,a) = \vee\{b_\xi \wedge d(x,a)\} = 0$ or $d(x,a) \le b_0$. The converse statement is immediate. $\triangleright$

3.4.6. Let us note three expedient corollaries to **3.4.5.**

(1) *The distance from a point $x \in X$ to the subset A of an extended B-set X is equal to zero iff $x \in \text{mix}(A)$.*

(2) The distance between two sets $A_1 \subset X$ and $A_2 \subset X$ is defined by the formula

$$\bar{d}(A_1, A_2) := \bigvee_{a \in A_1} \text{dist}(a, A_2) \vee \bigvee_{a \in A_2} \text{dist}(A_1, a).$$

We can easily check that $\bar{d}$ is a Boolean semimetric on $\mathcal{P}(X)$ but is not, generally speaking, a metric. It would be natural to call $\bar{d}$ a *Hausdorff B-semimetric associated with* $\bar{d}$. If X is extended, then $\bar{d}(A_1, A_2) = 0$ iff $\operatorname{mix} A_1 = \operatorname{mix} A_2$.

(3) *Let $\mathcal{P}_{\mathrm{cyc}}(X)$ be the set of all cyclic subsets of a B-set (X,d). Then (X,d) is extended iff $(\mathcal{P}_{\mathrm{cyc}}(X), \bar{d})$ is an extended B-set.*

◁ Indeed, let X be extended. Then, by virtue of (2), $\bar{d}$ is a metric on $\mathcal{P}_{\mathrm{cyc}}(X)$ and we have only to prove that $(\mathcal{P}_{\mathrm{cyc}}(X), \bar{d})$ is extended. To this end, let us consider a partition of unity (b_ξ) and a family (A_ξ) in $\mathcal{P}_{\mathrm{cyc}}(X)$. Define $A \subset X$ as the union of all mixings from $\operatorname{mix}(b_\xi x_\xi)$, where $x_\xi \in A_\xi$ for all ξ. In this case by virtue of the distributive laws **1.1.5** (1,2) for any $x \in A$ and $x' \in A_\xi$, we get

$$b_\xi \wedge \operatorname{dist}(x', A) = \wedge\{b_\xi \wedge d(x', a) \colon a \in A\} = 0,$$
$$b_\xi \wedge \operatorname{dist}(x, A_\xi) = \wedge\{b_\xi \wedge d(x, a) \colon a \in A_\xi\} = 0$$

and, finally, $b_\xi \wedge \bar{d}(A, A_\xi) = 0$. The last equality is valid for all ξ and, hence, $A = \operatorname{mix}(b_\xi A_\xi)$. Using the scheme of **3.2.8**, we can prove that A is cyclic. The converse statement results from the fact that the mapping $x \to \{x\}$ is an injection of X to $\mathcal{P}_{\mathrm{cyc}}(X)$, in which case $\bar{d}(\{x\},\{y\}) = d(x,y)$ for any $x, y \in X$. ▷

3.4.7. Let us consider B-sets (X, d_X) and (Y, d_Y). The correspondence Φ from X to Y is called *contractive*, a *contraction correspondence*, or simply a *contraction* provided

$$\bar{d}_Y(\Phi(x), \Phi(y)) \le d_X(x,y) \quad (x, y \in \operatorname{dom}\Phi),$$

where $\bar{d}_Y$ is a Hausdorff B-semimetric associated with d_Y.

(1) *The property of being a contraction for a correspondence Φ is equivalent to each of the conditions* (cf. **3.3.8** (1,2)):

(a) *if $d_X(x_1, x_2) \le b$ $(x_1, x_2 \in \operatorname{dom}\Phi)$, then for every $y \in Y$ we have*

$$b \vee \operatorname{dist}(y, \Phi(x_1)) = b \vee \operatorname{dist}(y, \Phi(x_2));$$

(b) $\operatorname{dist}(y_1, \Phi(x_2)) \le d_X(x_1, x_2)$ *for all $x_1, x_2 \in \operatorname{dom}\Phi$ and $y_1 \in \Phi(x_1)$.*

If X and Y are subsets in $\mathbf{V}^{(B)}$, then to denote the same property of a correspondence after the introduced definition we have to use two (contrasting in the conventional sense) terms, i.e., contraction and extensionality. To avoid ambiguity one has to remember that extensionality is interpreted with the Boolean truth-value of equality $[\cdot = \cdot]$, while contraction pertains to the B-*metric* under study.

A correspondence Φ will be termed *fully contractive* if it is contractive and

$$\Phi(x) = \operatorname{mix}(\Phi(x)) \quad (x \in \operatorname{dom}\Phi).$$

(2) *The descent of any correspondence is a fully contractive (or, which is the same, fully extensional) correspondence.*

◁ The result sought implies that if Ψ is a correspondence inside $\mathbf{V}^{(B)}$ and $\Phi := \Psi\downarrow$, then Φ is an extensional correspondence and $\Phi(x)$ is a cyclic set for every $x \in \operatorname{dom}\Phi$. By virtue of **3.2.6** (9), **3.2.13** and **3.3.8** (5), Φ is extensional, while by **3.2.3** (1) and **3.2.13** (1) $\Phi(x)$ is cyclic. ▷

The mapping $f: X \to Y$ is contractive (a contraction mapping) if

$$d_Y(f(x), f(x')) \leq d_X(x, x') \quad (x, x' \in X).$$

If in the last relation the equality is fulfilled, then f is said to be a *B-isometry*. A bijective B-isometry is termed the *isomorphism* of B-sets.

3.4.8. Any set $X \in \mathbf{V}$ can be turned into a B-set if we define on it the discrete B-metric:

$$d(x,y) := \begin{cases} \mathbf{1}_B, & \text{if } x \neq y, \\ \mathbf{0}_B, & \text{if } x \neq y. \end{cases}$$

In this case the pair (X,d) is called a *discrete B-set*. In a discrete B-set there is no mixing $\operatorname{mix}(b_\xi x_\xi)$ only if the set of elements (x_ξ) contains more than one element, while the partition of unity (b_ξ) is different from the trivial partition $\{\mathbf{0}_B, \mathbf{1}_B\}$. Any correspondence given on a discrete B-set is contractive.

Discrete and extended B-sets are two extreme examples of 'B-qualification' presented by the elements of the universes $\mathbf{V}$ and $\mathbf{V}^{(B)}$ (see **3.2.3**), compromise variations presented by the class $\mathcal{P}(\mathbf{V}^{(B)})$. In analysis, B-sets of other origin are often encountered.

3.4.9. *Let π be a full monomorphism of B to a Boolean algebra C. Let us put*

$$d_\pi(x,y) := \wedge\{b^* : \pi(b) \wedge x = \pi(b) \wedge y\} \quad (x,y \in C).$$

Then d_π is a B metric on C and Boolean operations on C are contractive mappings.

$\lhd$ If $\pi = I_B$, then $d_\pi(b,b') = (b \Leftrightarrow b')^* = b\Delta b'$. Let us consider one more complete Boolean algebra C' and a full monomorphism $\pi' : B \to C'$. Then the homomorphism $h: C \to C'$ will be a contractive mapping from B-sets (C, d_π) to $(C', d_{\pi'})$ iff $h \circ \pi = \pi'$. Indeed, the fact that h is contractive in the metrics d_π and $d_{\pi'}$ implies that $\pi(b) \wedge x = \pi(b) \wedge y$ implies $\pi'(b) \wedge h(x) = \pi'(b) \wedge h(y)$ for any $x, y \in C$ and $b \in B$. If $\pi' = h \circ \pi$, then, applying h to the equality $\pi(b) \wedge x = \pi(b) \wedge y$, we obtain $\pi'(b) \wedge h(x) = \pi'(b) \wedge h(y)$. Conversely, if in the last equality we choose $x = \mathbf{1}_C$ and $y := \pi(b)$, then we get either $\pi'(b) = \pi'(b) \wedge h\pi(b)$ or $\pi'(b) \le h \circ \pi(b)$. Since $b \in B$, we deduce $\pi' = h \circ \pi$. $\rhd$

3.4.10. Let us consider another construction with B-sets analogous to **2.2.10**. Let ψ be an ultrafilter on a Boolean algebra D. Let us consider a Boolean set (X, d_X) with a D-valued B-metric d_X. Introduce a binary relation $\sim_\psi$ in X by the formula

$$(x,y) \in \sim_\psi \leftrightarrow d_X(x,y)^* \in \psi.$$

The definition of a Boolean metric implies that $\sim_\psi$ is an equivalence. Let $X/\sim_\psi$ be the factor-set of the set X by the relation $\sim_\psi$, while $\pi_X : X \to X/\sim_\psi$ be the canonical mapping. If the same is carried out with the Boolean set (D, Δ), then as a $D/\sim_\psi$ we have the two-element Boolean algebra, so that $D/\sim_\psi \cong \{\mathbf{0}_D, \mathbf{1}_D\}$. As is seen, there is a unique mapping $\tilde{d} : X/\sim_\psi \to D/\sim_\psi$ such that $\tilde{d}(\pi_X x, \pi_X y) = \pi_D(d(x,y))$ $(x, y \in X)$. Besides, $\tilde{d}$ is a discrete Boolean metric on $X/\sim_\psi$. If d_X is a discrete metric, then $\sim_\psi = I_X$ and $X/\sim_\psi = X$. Some set-theoretical operations in X and $X/\sim_\psi$ are interrelated by simple relations. If (X_α) is a family of subsets of the set X, then $(\cup X_\alpha)/\sim_\psi = \cup(X_\alpha/\sim_\psi)$. Between $X^n/\sim_\psi$ and $(X/\sim_\psi)^n$ there is a natural bijection given by the formula

$$\pi_{X^n}(x_1, \dots, x_n) \to (\pi_X x_1, \dots, \pi_X x_n) \quad (x_1, \dots, x_n \in X).$$

It should be also remarked that if $A \subset X$, then $A/\sim_\psi = \pi_X(A)$ and $\pi_A = \pi_X | A$.

Let us choose one more B-set (Y, d_Y), and let $F \subset X \times Y$. It can be easily checked that in this case

$$\mathrm{dom}(F/\sim_\psi) = \mathrm{dom}(F)/\sim_\psi, \ \mathrm{im}(F/\sim_\psi) = \mathrm{im}(F)/\sim_\psi.$$

3.4.11. *Let ρ be an arbitrary automorphism (=homomorphism into itself) of a Boolean algebra B, and ψ_ρ be an element of $\mathbf{V}^{(B)}$ determined by the function $\{(b^\wedge, \rho(b)) : b \in B\}$ in line with* **2.5.6.** *Then the following statements are valid:*

(**1**) *$\rho(b) = [b^\wedge \in \psi_\rho]$ for any $b \in B$;*

(**2**) *for a set $A \subset B$ we have $[A^\wedge \subset \psi_\rho \rightarrow (\wedge A)^\wedge \in \psi_\rho] = \mathbf{1}$ iff $\rho(\wedge A) = \wedge \rho(A)$;*

(**3**) *$[\psi_\rho$ is an ultrafilter on $B^\wedge] = \mathbf{1}$.*

$\lhd$ (1) This result is checked by applying **2.2.8** (1,2).

(2) Using (1), we deduce for $A \subset B$:

$$[A^\wedge \subset \psi_\rho] = \bigwedge_{a \in A} [a \in \psi_\rho] = \bigwedge_{a \in A} \rho(a) = \wedge \rho(A).$$

Since $\rho(\wedge A) \le \wedge \rho(A)$ because ρ is isotonic, the inequality $[A^\wedge \subset \psi_\rho] \le [(\wedge A)^\wedge \in \psi_\rho]$ is equivalent to the equality $\rho(\wedge A) = \wedge \rho(A)$.

(3) Let us first of all observe that $V^{(B)} \models \psi_\rho \subset B^\wedge$. Indeed, for every $t \in \mathbf{V}^{(B)}$ we have

$$[t \in \psi_\rho] = \bigvee_{b \in B} \rho(b) \wedge [t = b^\wedge] \le \bigvee_{b \in B} [t = b^\wedge] = [t \in B^\wedge].$$

Then, it follows from (1) that $[0^\wedge \notin \psi_\rho] = \mathbf{1}$, while (2) yields that $[\psi_\rho$ is a filter base $] = \mathbf{1}$. Besides, if $b \in B$, then

$$[(\exists a \in \psi_\rho)(a \le b^\wedge)] = \bigvee_{a \in B} \rho(a) \wedge [a^\wedge \le b^\wedge] = \bigvee_{a \le b} \rho(a)$$
$$= \rho(b) = [b^\wedge \in \psi_\rho],$$

so that

$$[(\forall b \in B^\wedge)((\exists a \in \psi_\rho)\, a \le b) \rightarrow b \in \psi_\rho] = \mathbf{1}.$$

Therefore, ψ_ρ is a filter in $B^\wedge$ inside $\mathbf{V}^{(B)}$ and we have to show that $\mathbf{V}^{(B)} \models$ 'for any $b \in B^\wedge$, or $b \in \psi_\rho$, or $b^* \in \psi_\rho$'. This statement is validated by the following formulas:

$$[\![(\forall b \in B^\wedge)(b \in \psi_\rho \vee b^* \in \psi_\rho)]\!]$$
$$= \bigwedge_{b\in B} [\![b^\wedge \in \psi_\rho]\!] \vee [\![(b^*)^\wedge \in \psi_\rho]\!] = \bigwedge_{b\in B} \rho(b) \vee \rho(b^*)$$
$$= \wedge\{\rho(b \vee b^*) : b \in B\} = \rho(1) = 1. \ \triangleright$$

3.4.12. Let $\psi := \psi_\iota$, where ι is the identity homomorphism on B. According to **3.4.11**, $\mathbf{V}^{(B)} \models$ 'ψ is an ultrafilter on $B^\wedge$, and $A^\wedge \subset \psi$ implies $\wedge(A)^\wedge \in \psi$', whatever a set $A \subset B$.

Let us choose an arbitrary B-set (X,d). It is obvious from **3.1.16** that $(X^\wedge, d^\wedge)$ is a B-set inside $\mathbf{V}^{(B)}$. By virtue of **3.4.10, 3.4.11** and by the maximum principle, there are such $\tilde{X}, \sim := \sim_\psi$ and $\pi_X \in \mathbf{V}^{(B)}$ that

(1) $\mathbf{V}^{(B)} \models$ "$\sim$ is an equivalence relation on $X^\wedge$";

(2) $\mathbf{V}^{(B)} \models \tilde{X} := X^\wedge/\!\sim$;

(3) $\mathbf{V}^{(B)} \models$ "$\pi_X : X \to \tilde{X}$ is the factor-mapping';

(4) $[\![(x^\wedge, y^\wedge)^B \in \sim]\!] = d(x,y)^*$ $\quad (x, y \in X)$.

If we apply the described procedure to a B-set (B,Δ) (see **3.4.9**), then as $\tilde{B}$ we get the two-element Boolean algebra, so that $\mathbf{V}^{(B)} \models \tilde{B}\{\mathbf{0}_B^\wedge, \mathbf{1}_B^\wedge\}^B$. Therefore, inside $\mathbf{V}^{(B)}$ there is a unique $\{\mathbf{0}_B^\wedge, \mathbf{1}_B^\wedge\}$-valued Boolean metric $\tilde{d}$ on $\tilde{B}$, for which

$$\mathbf{V}^{(B)} \models (\forall x, y \in X^\wedge)\ d(\pi_X(x), \pi_X(y)) = \pi_B(d^\wedge(x,y)).$$

As is seen from **3.4.10**, for the discrete B-set (X,d) we get $\sim = I_{X^\wedge}$ and $X^\sim = X^\wedge$.

We shall say that subsets A and C of a certain B-set (X,d) *are in general position* provided

$$d(a,c) \geq \wedge\{d(a,b) \vee d(b,c) : b \in A \cap C\}$$

for any $a \in A$ and $c \in C$. As was the case in **3.3.9**, the above relation is in fact fulfilled with equality, since $d(a,c) \leq d(a,b) \vee d(b,c)$.

(5) *The sets A and C are in general position iff*

$$\mathbf{V}^{(B)} \models (A \cap C)^\sim = A^\sim \cap C^\sim.$$

◁ It should be remarked that $(A\cap C)^\sim = \pi_X((A\cap C)^\wedge) = \pi_X(A^\wedge \cap C^\wedge)$ and $A^\sim \cap C^\sim = \pi_X(A^\wedge)\cap \pi_X(C^\wedge)$. Hence, the inclusion $(A\cap C)^\sim \subset A^\sim \cap C^\sim$ is always valid, while $A^\sim \cap C^\sim \subset (A\cap C)^\sim$ is equivalent to the formula

$$(\forall a \in A^\wedge)(\forall c \in C^\wedge)(a \sim c \to (\exists b \in (A\cap C)^\wedge)(b\sim a \wedge b \sim c)).$$

Writing out the Boolean truth-value of the last formula and making use of the equality $[a^\wedge \sim c^\wedge] = d(a,c)^*$, we get

$$\bigwedge_{a\in A, c\in C} d(a,c)^* \Rightarrow \left(\bigvee_{b\in A\cap C} d(a,b)^* \wedge d(b,c)^*\right) = 1.$$

It is now evident that $[A^\sim \cap C^\sim \subset (A\cap C)^\sim] = \mathbf{1}$ iff for any $a\in A$ and $c \in C$ we obtain

$$d(a,c) \vee \left(\bigwedge_{b\in A\cap C} d(a,b) \vee d(b,c)\right)^* = \mathbf{1}.$$

It is this result that implies that A and C are in general position. ▷

3.4.13. Theorem. *Let (X,d_X) and (Y,d_Y) be certain B-sets and Φ be a contractive correspondence from X to Y. Then inside $\mathbf{V}^{(B)}$ there is a unique correspondence $\Phi^\sim$ from $X^\sim$ to $Y^\sim$ such that*

$$\mathrm{dom}(\Phi^\sim) = \mathrm{dom}(\Phi)^\sim,$$
$$[\Phi^\sim(\pi_X x^\wedge) = \pi_Y(\Phi(x)^\wedge] = \mathbf{1} \quad (x \in \mathrm{dom}\,\Phi).$$

In this case the following relations are valid:

(1) *if the sets $A\subset X$ and* $\mathrm{dom}\,\Phi$ *are in general position, then*

$$\mathbf{V}^{(B)} \models \Phi(A)^\sim = \Phi^\sim(A^\sim);$$

(2) *the superposition $\Psi\circ\Phi$ of contractive correspondences Φ and Ψ is contractive, and if $\Psi\circ\Phi = \mathrm{dom}\Phi$ and the sets $\mathrm{dom}\,\Psi$ and $\Phi(x)$ are in a general position for all $x\in\mathrm{dom}\,\Phi$, then*

$$\mathbf{V}^{(B)} \models (\Psi\circ\Phi)^\sim = \Psi^\sim \circ \Phi^\sim;$$

(3) $\mathbf{V}^{(B)} \models (I_X)^{\sim} = I_{X^{\sim}}$.

◁ As is known from **3.1.5**, $\mathbf{V}^{(B)} \models$ '$\Phi^{\wedge}$ is a correspondence from $X^{\wedge}$ to $Y^{\wedge}$'. Let us put $\Phi^{\sim} := \pi_Y \circ \Phi^{\wedge} \circ \pi_X^{-1}$. It is obvious that $\mathbf{V}^{(B)} \models$ '$\Phi^{\sim}$ is a correspondence from $X^{\sim}$ to $Y^{\sim}$ and $\operatorname{dom}\Phi^{\sim} = \pi_X(\operatorname{dom}(\Phi^{\wedge})) = \pi_X((\operatorname{dom}(\Phi)^{\wedge}) = (\operatorname{dom}\Phi)^{\sim}$'. Let us now show that for any $x \in Z := \operatorname{dom}\Phi$ and $y \in \mathbf{V}^{(B)}$ the Boolean truth-values $b_1 := [y \in \Phi^{\sim} \circ \pi_X(x^{\wedge})]$ and $b_2 := [y \in \pi_Y \circ \Phi^{\wedge}(x^{\wedge})]$ coincide. Indeed,

$$
\begin{aligned}
b_1 &:= [(\exists s \in Z^{\wedge})(\exists t \in Y^{\wedge})(y = \pi_Y(t) \wedge t \in \Phi^{\wedge}(s) \wedge \pi_X(s) = \pi_X(x^{\wedge}))] \\
&= \bigvee_{s \in Z} \bigvee_{t \in Y} [t^{\wedge} \in \Phi(s)^{\wedge}] \wedge [y = \pi_Y(t^{\wedge})] \wedge [\pi_X(s) = \pi_X(x^{\wedge})] \\
&\geq \bigvee_{t \in Y} [y = \pi_Y(t^{\wedge})] \wedge [t^{\wedge} \in \Phi(x)^{\wedge}] \\
&= [(\exists t \in Y^{\wedge})(y = \pi_Y(t) \wedge t \in \Phi^{\wedge}(x^{\wedge}))] = b_2.
\end{aligned}
$$

On the other hand, making use of the equalities

$$
\begin{aligned}
&d_X(s,x)^{*} = [\pi_X(s^{\wedge}) = \pi_X(x^{\wedge})], \\
&\bar{d}_Y(\Phi(x), \Phi(s))^{*} = [\pi_Y(\Phi(x)^{\wedge}) = \pi_Y(\Phi(s)^{\wedge})]
\end{aligned}
$$

and taking it into account that the correspondence Φ is contractive, we infer

$$
\begin{aligned}
b_1 &\leq \bigvee_{s \in Z} \bigvee_{t \in Y} [\pi_Y(\Phi(s)^{\wedge}) = \pi_Y(\Phi(x)^{\wedge})] \wedge = [t^{\wedge} \in \Phi(s)^{\wedge}] \\
&\wedge [y = \pi_Y(t^{\wedge})] \leq \bigvee_{s \in Z} [y \in \pi_Y(\Phi^{\wedge}(x^{\wedge}))] = b_2.
\end{aligned}
$$

Therefore, $b_1 = b_2$, which immediately implies the validity of the defining relation $[\pi_Y(\Phi(x)^{\wedge}) = \Phi^{\sim}(\pi_X(x^{\wedge}))] = \mathbf{1}$ $(x \in Z)$. Hence, the relation

$$
\mathbf{V}^{(B)} \models (\forall x \in (\operatorname{dom}\Phi)^{\wedge})\, \Phi^{\sim}(\pi_X x) = \pi_Y \Phi^{\wedge}(x)
$$

holds, which fact, in turn, implies that $\Phi^{\sim}$ is unique, since $\operatorname{dom}(\Phi^{\sim}) = (\operatorname{dom}\Phi)^{\sim} = \pi_X((\operatorname{dom}(\Phi)^{\wedge})$.

(1) Using **3.4.12** (5), we can easily note that

$$
\Phi^{\sim}(A^{\sim}) = \Phi^{\sim}(A^{\sim} \cap \operatorname{dom}(\Phi^{\sim})) = \Phi^{\sim}((A \cap \operatorname{dom}(\Phi)^{\sim}).
$$

On the other hand, $\Phi(A)^{\sim} = \Phi(A \cap \operatorname{dom}\Phi)^{\sim}$ and, hence, without loss of generality, it can be assumed that $A \subset \operatorname{dom}\Phi$. In this case, however, by the defining property of $\Phi^{\sim}$, we can write inside $\mathbf{V}^{(B)}$ the chain of equalities:

$$\Phi^{\sim}(A^{\sim}) = \bigcup_{a\in A^{\sim}} \Phi^{\sim}(a) = \bigcup_{a\in A^{\wedge}} \Phi^{\sim}(\pi_X a)$$
$$= \bigcup_{a\in A^{\wedge}} \pi_Y(\Phi^{\wedge}(a)) = \pi_Y(\Phi^{\wedge}(A^{\wedge})) = \pi_Y(\Phi(A)^{\wedge}) = \Phi(A^{\sim}).$$

(2) Let Ψ be a contraction correspondence from Y to U. Let us choose $x_1, x_2 \in Z$, $y_1 \in \Phi(x_1)$ and $u_1 \in \Psi(y_1)$. Then, according to **3.4.7** (1),

$$\mathrm{dist}(u_1, \Psi \circ \Phi(x_2)) \le \wedge\{\mathrm{dist}(u_1, \Psi(y)) : y \in \Phi(x_2)\}$$
$$\le \wedge\{d(y_1, y) : y \in \Phi(x_2)\} = \mathrm{dist}(y_1, \Phi(x_2)) \le d(x_1, x_2).$$

Therefore, since x_1, x_2, y_1 and u_1 are arbitrary, we conclude that the correspondence $\Psi \circ \Phi$ is contractive. Then, making use of (1), **3.1.5** (2) and the defining relations for $(\Psi \circ \Phi)^{\sim}$, $\Psi^{\sim}$ and $\Phi^{\sim}$, we can write ($x \in Z$):

$$(\Psi^{\sim} \circ \Phi^{\sim})(\pi_X x^{\wedge}) = \Psi^{\sim}(\Phi(x)^{\sim}) = \Psi(\Phi(x))^{\sim}$$
$$= \pi_Y((\Psi \circ \Phi)(x)^{\wedge}) = \pi_Y((\Psi \circ \Phi)^{\wedge}(x^{\wedge})) = (\Psi \circ \Phi)^{\sim}(\pi_X x^{\wedge}).$$

Hence, $[(\Psi \circ \Phi)^{\sim} = \Phi^{\sim} \circ \Phi^{\sim}] = \mathbf{1}$, since $Z^{\sim} = \mathrm{dom}(\Psi^{\sim} \circ \Phi^{\sim})$.

(3) The result follows obviously from **3.1.5** (4). $\triangleright$

3.4.14. Theorem *For any contraction mapping $f: X \to Y$ there is a unique element $f^{\sim} \in \mathbf{V}^{(B)}$ such that*

$$[f^{\sim}: X^{\sim} \to Y^{\sim}] = [f^{\sim} \circ \pi_X = \pi_Y \circ f^{\wedge}] = \mathbf{1}.$$

In this case the following statements are valid:

(1) $\mathbf{V}^{(B)} \models f(A)^{\sim} = f^{\sim}(A^{\sim})$ *for every* $A \subset X$;

(2) *if* $g: Y \to Z$ *is a contraction mapping, then* $g \circ f$ *is a contraction mapping and* $\mathbf{V}^{(B)} \models (g \circ f)^{\sim} = g^{\sim} \circ f^{\sim}$;

(3) $\mathbf{V}^{(B)} \models$ '$f^{\sim}$ *is injective' iff* f *is a* B*-isometry;*

(4) $\mathbf{V}^{(B)} \models$ '$f^{\sim}$ *is surjective iff* $\vee\{d(f(x), y) : x \in X\} = \mathbf{1}$ *for every* $y \in Y$.

3.4.15. Let us consider categories BSet$_*$ and CBSet$_*$. The objects of these categories are nonempty B-sets and nonempty extended B-sets, respectively, The composition of morphisms is the superposition of correspondences. The subcategories of the categories BSet$_*$ and CBSet$_*$ consisting of the same objects and of contraction mappings will be denoted by BSet and CBSet, respectively. Let $\mathfrak{F}^{\sim}$ be the function assigning to the object X and morphism Φ of the category BSet the elements $\mathfrak{F}^{\sim}(X) := X^{\sim}$ and $\mathfrak{F}^{\sim}(\Phi) := \Phi^{\sim}$.

3.4.16. The mapping $\mathfrak{F}^{\sim}$ is a covariant functor from the category BSet to the category $\mathfrak{V}^{(B)}$.

3.4.17. Remarks

(1) The concept of a Boolean metric appeared at the beginning of the 1950s as a result of studying various 'distances' on abstract sets with the values in ordered systems (see [15, 43, 216]. There has, however, been discovered no particularly rich geometry associated with this concept, which fact accounts for B-metrics being unpopular in the years to follow. The reason of this curiosity can be understood through theorems **3.4.13** and **3.5.4**.

The geometry of Boolean metrics is much more meaningful and interesting when combined with topological and functional-analytic structures. In this case the presence of a duly agreed B-metric shows it to be expedient to study the considered structure by the method of Boolean-valued models.

(2) The mapping $[\cdot = \cdot]: X^2 \to B$ is called a *Boolean-valued equality*, provided it obeys conditions **2.2.8** (1,3,4). Such mappings are widely used for Boolean-valued interpretations of first-order theories (see [56]). The notion of a Boolean-valued equality is easily seen to be just a 'mirror mapping' of the idea of a Boolean metric, since conditions **2.2.8** (1,3,4) are fulfilled only iff the mapping $(x,y) \to [x = y]^*$ is a Boolean metric. In this context the idea of a Boolean metric proves to be quite fruitful.

(3) Definitions **3.4.1** adopted in this section are motivated by the fact that in analysis the structures under study are often related to a certain B-(semi)metric, while a B-valued equality is to be introduced artificially.

(4) We can demonstrate that the statement converse to **3.4.6** is also true. Namely, if ψ is an ultrafilter on $B^{\wedge}$ inside $\mathbf{V}^{(B)}$, then the mapping $\rho_\psi : B \to B$ determined by the formula $\rho_\psi(b) := [b^{\wedge} \in \psi]$ is an automorphism of $B^{\wedge}$. In this case $\rho_{\psi_\rho} = \rho$ and $[\psi_{\rho_\psi} = \psi] = 1$.

(5) As to statements **3.4.13** (1,2), one can apply to them the same remarks as in **3.3.15** (5).

3.5. Interrelations of Basic Functors

Among the basic functors described in the preceding four paragraphs, there are interesting and quite expedient for applications interrelations. Their study is the contents of the present section.

3.5.1. It should be recalled that for an arbitrary $X \in \mathcal{P}(\mathbf{V}^{(B)})$ the set $(x_\xi) \subset X$ consists of all possible mixings $\mathrm{mix}(b_\xi x_\xi)$ of $(x_\xi) \subset X$ families relative to any partitions of unity $(b_\xi) \subset B$ (see **3.2.8**). Let us expand mix onto extensional correspondences.

Let X *and* Y *be subsets of the class* $\mathbf{V}^{(B)}$, *while* Φ *be an extensional correspondence from* X *to* Y. *There is a unique fully extensional correspondence* Ψ *from* $\mathrm{mix}(X)$ *to* $\mathrm{mix}(Y)$, *for which*

$$\Psi(x) = \mathrm{mix}\,\Phi(x) \quad (x \in \mathrm{dom}\Phi).$$

$\lhd$ Indeed, we should set $\Psi := \Phi\uparrow\downarrow$ and make use of statements **3.3.12** (1) and **3.4.7** (2). From **3.2.13** and **3.3.3** (1) we see that $\mathrm{Gr}(\Psi) = \mathrm{mix}\,\mathrm{Cr}(\Phi)$. $\rhd$

By definition, we put $\mathrm{mix}\Phi := \Psi$. If Θ is another extensional correspondence and $\mathrm{dom}\Theta \subset Y$, then, by **3.2.13** (3) and **3.3.4** (8) we have $\mathrm{mix}(\Theta \circ \Phi) = \mathrm{mix}(\Theta) \circ \mathrm{mix}(\Phi)$ iff $(\Theta \circ \Phi)\uparrow = \Theta\uparrow \circ \Phi\uparrow$. Moreover, it is obvious that $(I_X) = I_{\mathrm{mix}(X)}$.

3.5.2. Let us choose a nonempty set X. By the symbol $B_0(X)$ we shall denote a set of all partitions of unity in B of the type $(b_x = b(x))_{x \in X}$:

$$b \in B_0(X) \leftrightarrow (b \in B^X \wedge (\forall x \in X)(\forall y \in X)(x \neq y) \to b(x) \wedge b(y) = 0)).$$

Assign to an element $y \in X$ the partition of unity $\iota y := \iota_X y := (b_x)_{x \in X}$, where $b_x = 1$ for $x = y$ and $b_x = 0$ for $x \neq y$. It is obvious that ι_X is an injection from X to $B_0(X)$. For elements $u, v \in B_0(X)$ let us, according to the definition, put

$$d(u,v) := \wedge\{u(x)^* \vee v(x)^* : x \in X\}.$$

It can easily checked that d is a B-metric on $B_0(X)$. Moreover, $(B_0(X),d)$ is an extended B-set. The last fact is, in essence, established by the same considerations as in **3.2.8**. Hence, $B_0(\cdot)$ is a mapping from $\mathbf{V}$ to CBSet.

Let us extend this mapping onto correspondences. Choose a correspondence $\Phi := (F,Y,Y)$ and determine $B_0(\Phi) := (G, B_0(X), B_0(Y))$, where

$$G := \{(u,v) \in B_0(X) \times B_0(Y) \leftrightarrow$$
$$\leftrightarrow (\forall x \in X)(\forall y \in Y)(u(x) \wedge v(y) \neq 0 \rightarrow (x,y) \in F)\}.$$

If Φ is unique, then $B_0(\Phi)$ is unique. Directly from definitions we can deduce:

$$B_0(I_X) = I_{B_0(X)},$$
$$B_0(\Psi \circ \Phi) = B_0(\Psi) \circ B_0(\Phi),$$
$$\Phi = \iota_Y^{-1} \circ B_0(\Phi) \circ \iota_X.$$

Hence, the mapping $B_0(\cdot)$ is a covariant functor from $\mathfrak{V}_*$ to CBSet$_*$.

3.5.3. Some interrelations between basic operations of Boolean-valued analysis have been earlier presented in the form of the rules of arrow cancellation. Let us now supply these rules with functorial formulations.

(1) *The descent functor $\mathfrak{F}^{\downarrow}$ and the ascent functor $\mathfrak{F}^{\uparrow}$ establish a homomorphism between the categories $\mathfrak{V}^{(B)}$ and $\mathfrak{CPV}^{(B)}$. This implies that $\mathfrak{F}^{\uparrow} \circ \mathfrak{F}^{\downarrow}$ and $\mathfrak{F}^{\downarrow} \circ \mathfrak{F}^{\uparrow}$ coincide with identity functors on $\mathfrak{V}^{(B)}$ and $\mathfrak{CPV}^{(B)}$, respectively.*

◁ The identity for the functor $\mathfrak{F}^{\uparrow} \circ \mathfrak{F}^{\downarrow}$ is prompted by the rules for 'descent-ascent' **3.3.3** (2) and **3.3.12** (3), while that of the functor $\mathfrak{F}^{\downarrow} \circ \mathfrak{F}^{\uparrow}$ from the rule of 'ascent-descent' **3.3.3** (1) and **3.3.12** (1). ▷

Let us recall some notions of the theory of categories. Let us choose categories $\mathfrak{H}, \mathfrak{K}$ and covariant functors $\mathfrak{F} : \mathfrak{H} \rightarrow \mathfrak{K}$, $\mathfrak{G} : \mathfrak{K} \rightarrow \mathfrak{H}$. *The natural transform* $\Theta : \mathfrak{F} \rightarrow \mathfrak{G}$ of the functor $\mathfrak{F}$ to the functor $\mathfrak{G}$ is a mapping $\Theta : \mathrm{Ob}\,\mathfrak{H} \rightarrow \mathrm{Mor}\,\mathfrak{K}$ such that

(a) $\Theta_a := \Theta(a) \in \mathfrak{K}(\mathfrak{F}(a), \mathfrak{G}(a))$ for every $a \in \mathrm{Ob}\,\mathfrak{H}$, and

(b) $\mathfrak{G}(\alpha)\Theta_a = \Theta_b \mathfrak{F}(a)$ for any morphism $\alpha : a \rightarrow b$.

The functors $\mathfrak{F}$ and $\mathfrak{G}$ are termed *isomorphic* and are marked as $\mathfrak{F} \sim \mathfrak{G}$, provided there is a natural transform $\Theta : \mathfrak{F} \rightarrow \mathfrak{G}$ such that Θ_a is an isomorphism for every $a \in \mathrm{Ob}\,\mathfrak{H}$. The categories $\mathfrak{H}$ and $\mathfrak{K}$ are said to be *equivalent* provided there is such a pair of functors

$\mathfrak{F}:\mathfrak{H}\to\mathfrak{K}$ and $\mathfrak{G}:\mathfrak{K}\to\mathfrak{H}$ that $\mathfrak{F}\circ\mathfrak{G}\sim I_{\mathfrak{K}}$ and $\mathfrak{G}\circ\mathfrak{F}\sim I_{\mathfrak{H}}$. In this case one often uses the phrase 'the functors $\mathfrak{F}$ and $\mathfrak{G}$ establish equivalence for the categories $\mathfrak{H}$ and $\mathfrak{K}$'.

The category $\mathfrak{H}^0$ *dual to* $\mathfrak{H}$ consists, by definition, of the same objects and morphisms as $\mathfrak{H}$, but in the definition of $\mathfrak{H}^0$ the mappings D and R swap their places and the order of morphisms in the composition is reversed (see **2.5.19**). The *product* of the categories $\mathfrak{H}$ and $\mathfrak{K}$ is determined by the relations

$$\mathrm{Ob}\,\mathfrak{H}\times\mathfrak{K}:=\mathrm{Ob}\,\mathfrak{H}\times\mathrm{Ob}\,\mathfrak{K};$$
$$\mathfrak{H}\times\mathfrak{K}((a,b),(a',b')):=\mathfrak{H}(a,a')\times\mathfrak{K}(b,b');$$
$$(\alpha',\beta')\circ(\alpha,\beta):=(\alpha'\alpha,\beta',\beta),$$

where $a,a'\in\mathrm{Ob}\,\mathfrak{K}$; $b,b'\in\mathrm{Ob}\,\mathfrak{K}$; $\alpha,\alpha'\in\mathrm{Mor}\,\mathfrak{H}$; $\beta,\beta'\in\mathrm{Mor}\,\mathfrak{K}$.

Let us now introduce the notion of a conjugate functor. Again consider the functors $\mathfrak{F}:\mathfrak{H}\to\mathfrak{K}$ and $\mathfrak{G}:\mathfrak{K}\to\mathfrak{H}$. Then determine two new functors $\mathfrak{F}^h$ and $\mathfrak{G}_h$ from the category $\mathfrak{H}^0\times\mathfrak{K}$ to the category Set of sets and mappings. For any $(a,b)\in\mathrm{Ob}\,\mathfrak{H}\times\mathrm{Ob}\,\mathfrak{K}$, $\alpha\in\mathfrak{H}(a,a')$, $\beta\in\mathfrak{K}(b,b')$, set

$$\mathfrak{F}^h(a,b):=\mathfrak{K}(\mathfrak{F}(a),b),\ \mathfrak{F}_h(a,b):=\mathfrak{H}(a,\mathfrak{G}(b)),$$
$$\mathfrak{F}^h(\alpha,\beta):=f\to\beta f\mathfrak{F}(\alpha),\ \mathfrak{F}_h(\alpha,\beta):g\to\mathfrak{G}(\beta)g\alpha,$$

where $f\in\mathfrak{K}(\mathfrak{F}(a,b))$, $g\in\mathfrak{H}(a,\mathfrak{G}(b))$. If the functors $\mathfrak{F}^h$ and $\mathfrak{G}_h$ are isomorphic, then we say that $\mathfrak{G}$ is *right-conjugate* to the functor $\mathfrak{F}$, while $\mathfrak{F}$ is *left-conjugate* to the functor $\mathfrak{G}$, or, to put it short, that $\mathfrak{F}$ and $\mathfrak{G}$ is a *conjugate pair of functors.* The isomorphism $\Theta:\mathfrak{F}^h\to\mathfrak{G}_h$ is referred to as *conjugation*, while the inverse isomorphism Θ^{-1}, as *co-conjugation.*

Let $\mathfrak{K}$ be a subcategory of the category $\mathfrak{H}$. An object $b\in\mathrm{Ob}\,\mathfrak{K}$ is called an $\mathfrak{K}$-reflector of the object $a\in\mathrm{Ob}\,\mathfrak{H}$ provided there is such a morphism $\delta:a\to b$ that any morphism $(\alpha:a\to c)\in\mathfrak{H}$, $c\in\mathrm{Ob}\,\mathfrak{K}$ is presentable as $\alpha=\beta\delta$ for a uniquely determined morphism $\beta:b\to c$. If for any object of the category $\mathfrak{H}$ there exists a $\mathfrak{K}$-reflector, then $\mathfrak{K}$ is said to be a reflective subcategory. It should, finally, be remarked that the subcategory $\mathfrak{K}$ is reflective iff the functor of an identical embedding $\mathfrak{K}\to\mathfrak{H}$ has a left conjugate functor $R:\mathfrak{H}\to\mathfrak{K}$. The functor R is called an $\mathfrak{K}$-reflector of the category $\mathfrak{H}$.

(2) *The functor* mix$:\mathcal{PV}^{(B)}\to\mathfrak{C}\mathcal{PV}^{(B)}$ *coincides with the superposition* $\mathfrak{F}^{\uparrow}\circ\mathfrak{F}^{\downarrow}$ *and is a* $\mathfrak{C}\mathcal{PV}^{(B)}$*-reflector of the category* $\mathcal{PV}^{(B)}$*. In particular,* $\mathfrak{C}\mathcal{PV}^{(B)}$ *is a reflective subcategory in* $\mathcal{PV}^{(B)}$.

⊲ The equality mix$:=\mathfrak{F}^{\uparrow}\circ\mathfrak{F}^{\downarrow}$ results from **3.3.3** (1) and **3.3.12** (2). Let us choose nonempty sets $A,C \in \mathcal{P}(\mathbf{V}^{(B)})$, and let C be cyclic. Then any extensional mapping $g: A \to C$ allows a unique extensional extension $\bar{g} = g \uparrow \downarrow: \operatorname{mix} A \to C$ (see **3.2.12**, **3.3.11** and **3.3.12** (2)). Therefore, the mapping of restriction $\theta_{A,C}: h \to h \uparrow A$ is a bijection of $\mathfrak{CPV}^{(B)}(\operatorname{mix} A, C)$ on $\mathfrak{PV}^{(B)}(A,C)$. Let us denote the family of the mappings $\theta_{A,C}$ by θ. Then θ is a conjugation from mix to the functor of the identical embedding of $\mathfrak{CPV}^{(B)}$ to $\mathfrak{PV}^{(B)}$. Indeed, if $A',C' \in \mathcal{P}(\mathbf{V}^{(B)})$ and C' is cyclic, then for any extensional mappings $f: \operatorname{mix} A \to C$, $g: A' \to A$, $h: C \to C'$ we get $(f \circ \operatorname{mix}(g)) | A' = (f | A) \circ g$. And obviously valid is the equality

$$h \circ (f \circ \operatorname{mix}(g))) | A' = h \circ (f | A) \circ g,$$

or, which is equivalent,

$$\theta_{A',C'}(h \circ f \circ \operatorname{mix}(g)) = h \circ \theta_A(f) \circ g. \ \triangleright$$

(3) *The superposition of a functor of the canonical embedding and that of descent is naturally isomorphic to the functor* B_0 *or, in symbols,* $\mathfrak{F}^{\downarrow} \circ \mathfrak{F}^{\wedge} \sim B_0$.

$\triangleleft$ For any set X the mapping

$$\theta_X: (b_X)_{x \in X} \to \operatorname{mix}_{x \in X}(b_X x^{\wedge}) \quad ((b_x)_{x \in X} \in B_0(X))$$

is a bijection of $B_0(X)$ on $X^{\wedge} \downarrow$. The mapping $\theta: X \to \theta_X (X \in \operatorname{Ob} \mathcal{V}_*)$ is an isomorphism of the functors B_0 and $\mathfrak{F}^{\downarrow} \circ \mathfrak{F}^{\wedge}$. To this end it suffices to remark that for $u \in B_0(X)$ and $v \in B_0(Y)$, $a = \theta_X(u)$ and $b := \theta_Y(v)$ we get $(a,b) \in \Phi^{\wedge} \downarrow$ iff $(x,y) \in \Phi$ whenever $u(x) \wedge v(x) \neq 0$. $\triangleright$

3.5.4. Theorem. *Let* (X,d_X) *be a* B*-set and* $X' := X^{\sim} \downarrow$. *Then the following statements are valid:*

(1) *there is an injection* $\iota_X: X \to X'$ *such that*

$$d_X(x_1,x_2) = [\iota_X x_1 \neq \iota_X x_2] \quad (x_1, x_2 \in X);$$

(2) *for any* $x' \in X'$ *there is a partition of unity* (b_ξ) *and a family* $(x_\xi) \subset X$ *such that* $x' = \operatorname{mix}(b_\xi \iota(x_\xi))$;

(3) *if* Φ *is a contraction correspondence from* X *to a* B*-set* Y, $Y' := Y^{\sim} \downarrow$ *and*

$\Phi' := \Phi^{\sim}\downarrow$, *then* $\Phi^{\sim}$ *is a unique fully extensional correspondence from* X' *to* Y', *for which* $\operatorname{dom}\Phi' = \operatorname{mix}\iota_X(\operatorname{dom}\Phi)$,

$$\Phi'(\iota_X x) = \operatorname{mix}\iota_Y(\Phi(x)) \quad (x \in \operatorname{dom}\Phi).$$

$\lhd$ (1) According to the definition of $X^{\sim}$ and π_X (see **3.4.12** (1-3)), for any $x \in X$ we have $[\pi_X x^{\wedge} \in X^{\sim}] = \mathbf{1}$, and therefore, there is a unique element $x' \in X'$ such that $[x' = \pi_X x^{\wedge}] = \mathbf{1}$. Let us put $\iota_X x := x'$. We thus defined the mapping $\iota := \iota_X : X \rightarrow X'$, in which case $[\iota x_1 = \pi_X x^{\wedge}] = \mathbf{1}\,(x \in X)$. Using the last relation and equality **3.4.12** (4), we deduce, for arbitrary $x_1, x_2 \in X$:

$$[\iota x_1 \neq \iota x_2] = [\pi_X x_1^{\wedge} = \pi_X x_2^{\wedge}]^* = [x_1 \sim x_2]^* = d_X(x_1, x_2),$$

which, in particular, implies that ι is injective.

(2) Let us first of all observe that the formula $[(\operatorname{im}\iota)\uparrow = \pi_X(X^{\wedge})] = \mathbf{1}$ holds. Indeed, for $t \in \mathbf{V}^{(B)}$ by the definition of the injection ι we have

$$[t \in (\operatorname{im}\iota)\uparrow] = \bigvee_{x \in X} [t = \iota x] = \bigvee_{x \in X} [t = \pi_X x^{\wedge}] = [t \in \pi_X(X^{\wedge})].$$

Now, making use of the cancellation rule **3.3.3** (1), we get

$$X' = \pi_X(X^{\wedge})\downarrow = (\operatorname{im}\iota)\uparrow\downarrow = \iota(X)\uparrow\downarrow = \operatorname{mix}\iota(X).$$

(3) Since $\Phi^{\sim}$ is a correspondence from $X^{\sim}$ to $Y^{\sim}$ inside $\mathbf{V}^{(B)}$, therefore, Φ' is a fully extensional correspondence from X' to Y' (see **3.4.7** (2)). Making use of the property of descent **3.2.13** (1), we write for arbitrary $x \in X$ and $y \in Y$:

$$\iota_Y y \in \Phi'(\iota_X x) \leftrightarrow [\iota_Y y \in \Phi^{\sim}(\iota_X x)] = \mathbf{1}.$$

In the right-hand part of this equivalence $\iota_X x$ can, by the construction of ι_X, be replaced with $\pi_X x^{\wedge}$. Then, by theorem **3.4.13**,

$$[\iota_Y y \in \Phi^{\sim}(\pi_X x^{\wedge})] = [\iota_Y y \in \pi_Y(\Phi(x)^{\wedge})].$$

All these facts imply that $\iota_Y y \in \Phi'(\iota_X x)$ iff $\iota_Y y \in \pi_Y(\Phi(x)^{\wedge})\downarrow$, which results in the required relation. Indeed, the facts proved in (1) and (2) allow one to conclude that $A^{\sim}\downarrow = \pi_Y(A^{\wedge})\downarrow = \operatorname{mix}\iota_Y(A)$ for any $A \subset Y$. Taking into account rule **3.2.13** (1) as well, we deduce:

$$\Phi'(\iota_X x) = \Phi^{\sim}\downarrow(\iota_X x) = \Phi^{\sim}(\pi_X(x^{\wedge}))\downarrow = \pi_Y(\Phi(x)^{\wedge}) = \operatorname{mix}\iota_Y(\Phi(x)),$$

where $x \in \operatorname{dom}\Phi$. Let us put $X_1 := \operatorname{im}\iota_X, Y_1 := \operatorname{im}\iota_Y$ and $\Phi_1 := \iota_Y^{-1} \circ \Phi' \circ \iota_X$. Then Φ_1 is an extensional correspondence from X_1 to Y_1 and the following equalities hold:

$$X' = \operatorname{mix}X_1, \quad Y' = \operatorname{mix}Y_1, \quad \Phi'(x) = \operatorname{mix}\Phi_1(x) \quad (x \in \operatorname{dom}\Phi_1).$$

Hence, $\Phi' = \operatorname{mix}\Phi_1$, and thus Φ' is unique. ▷

3.5.5. Let us describe modified descents and ascents of correspondences.

(1) Let X be a nonempty B-set, Y is an arbitrary element of $\mathbf{V}^{(B)}$ such that $[Y \neq \varnothing] = \mathbf{1}$. Let us consider an $\Phi \in \mathbf{V}^{(B)}$, for which $\mathbf{V}^{(B)} \models {}'\Phi = (F, X^{\sim}, Y)$ is a correspondence from $X^{\sim}$ to Y'. By theorem **3.2.13**, $\Phi\downarrow$ is a correspondence from $X' := X^{\sim}\downarrow$ to $Y\downarrow$. By definition, we put $\Phi \mathrel{\rlap{\downarrow}{\,\_}} := \Phi\downarrow \circ \iota_X$. The correspondence $\Phi\downarrow$ is termed the *modified descent of the correspondence* Φ. By virtue of theorems **3.2.13** and **3.5.4**, $\Phi\downarrow$ is the only fully contractive correspondence from X to $Y\downarrow$, for which

$$[y \in \Phi \mathrel{\rlap{\downarrow}{\,\_}} (x) \leftrightarrow [y \in \Phi(\iota_X x)]] = \mathbf{1} \quad (x \in X).$$

It should be also remarked that $\Phi \mathrel{\rlap{\downarrow}{\,\_}} = (F\downarrow^{-}, X, Y\downarrow)$, where

$$F\downarrow^{-} := \{(x, y) \in X \times Y\downarrow : (\iota_X x, y)^B \in F\}.$$

(2) Let us now assume that $\Psi := (F, X, Y\downarrow)$ is a contraction correspondence. The operation of ascent of **3.3** cannot be directly applied to Ψ. The correspondence $\Psi \circ \iota_X$ is, however, seen to be extensional and the ascent can be applied to it. Define $\Psi \mathrel{\rlap{\uparrow}{\,\_}} := (\Psi \circ \iota_X)\uparrow$ and call $\Psi \mathrel{\rlap{\uparrow}{\,\_}}$ the *modified ascent of the correspondence* Ψ. By virtue of theorems **3.3.10** and **3.5.4**, $\Psi \mathrel{\rlap{\uparrow}{\,\_}}$ is a unique correspondence from $X^{\sim}$ to Y inside $\mathbf{V}^{(B)}$ such that

$$[\operatorname{dom}\Psi \mathrel{\rlap{\uparrow}{\,\_}} = (\operatorname{dom}\Psi)^{\sim}] = \mathbf{1}, \ [\Psi \mathrel{\rlap{\uparrow}{\,\_}} (\iota_X x) = \Psi(x)\uparrow] = \mathbf{1} \quad (x \in \operatorname{dom}\Psi).$$

It should be again remarked that $\Psi \mathrel{\rlap{\uparrow}{\,\_}} = (F_{-}\uparrow, X^{\sim}, Y)$, where

$$F_{-} := \{(\iota_X x, y)^B : (x, y) \in F\}.$$

(3) Let us now assume that X is a discrete B-set. Then $\Phi \mathrel{\rlap{\downarrow}{\,\_}}$ is a correspondence from

X to $Y\downarrow$ and it is uniquely determined by the relation

$$y \in \Phi \Downarrow (x) \leftrightarrow [y \in \Phi(x^\wedge)] = \mathbf{1} \quad (x \in X).$$

On the other hand, in this case any correspondence Ψ from X to $Y\downarrow$ is contractive, so that there is a unique correspondence $\Psi \Uparrow$ from $X^\wedge$ to Y, for which

$$[\Psi \Uparrow (x^\wedge) = \Psi(x)\uparrow] = \mathbf{1} \quad (x \in X).$$

3.5.6. Theorem. *Let $[X^\sim, Y]$ be a set of elements , for which $[\Phi$ is a correspondence from $X^\sim$ to $Y] = \mathbf{1}$, while $[X, Y\downarrow]$ be a set of all fully contractive correspondences from X to $Y\downarrow$. The modified descent and ascent are mutually inverse mappings implementing a bijection between $[X^\sim, Y]$ and $[X, Y\downarrow]$.*

◁ Let us, for simplicity, denote $\iota := \iota_X$. By virtue of **3.5.4** (2) and **3.3.3** (1), $X^\sim = (\operatorname{im}\iota)\uparrow$. Hence, in line with **3.3.10** (3), we get $I_{X^\sim} = (I_{\operatorname{im}\iota})\uparrow$. Then, applying the rules of cancelling arrows for correspondences, we conclude that inside $\mathbf{V}^{(B)}$ the following equalities hold:

$$\Phi\downarrow\uparrow = \big((\Phi\downarrow \circ \iota)\circ \iota^{-1}\big)\uparrow = (\Phi\downarrow \circ I_{\operatorname{im}\iota})\uparrow = \Phi\downarrow\uparrow \circ (I_{\operatorname{im}\iota})\uparrow = \Phi \circ I_{X^\sim} = \Phi.$$

On the other hand, for fully contractive Ψ we have

$$\Psi \Uparrow \Downarrow (x) = (\Psi \circ \iota^{-1})\uparrow\downarrow(\iota x) = \operatorname{mix}\Psi \circ \iota^{-1}(\iota x)$$
$$= \operatorname{mix}\Psi(x) = \Psi(x)\ (x \in \operatorname{mix}\operatorname{dom}\Psi = \operatorname{dom}\Psi).\ ▷$$

3.5.7. Theorem. *The functor $\mathfrak{F}^\downarrow$ of descent is right-conjugate to the immersion functor $\mathfrak{F}^\sim$. In this case the modified descent $\Downarrow$ is a conjugation, while the modified ascent $\Uparrow$ is a co-conjugation.*

◁ Let us consider functors $\mathfrak{H}^\sim$ and $\mathfrak{H}^\downarrow$ from the category $\mathrm{BSet} \times \mathcal{V}^{(B)}$ to the category $\mathcal{V}$ determined by the relations

$$\mathfrak{H}^\sim(X,Y) := \mathcal{V}^{(B)}(X^\sim, Y), \quad \mathfrak{H}^\downarrow(X,Y) := \mathrm{BSet}_0(X, Y\downarrow);$$
$$\mathfrak{H}^\sim(\alpha,\beta) := \Phi' \leftrightarrow V^{(B)} \models \Phi' = \beta \circ \Phi \circ \alpha^\sim;$$
$$\mathfrak{H}^\downarrow(\alpha,\beta) := \beta\downarrow \circ \Psi \circ \alpha,$$

where $X \in \mathrm{Ob}\,\mathrm{BSet}$, $Y \in \mathrm{Ob}\,\mathbb{V}^{(B)}$, $\alpha \in \mathrm{BSet}(X_1, X)$, $\alpha \in \mathbb{V}^{(B)}(Y, Y_1)$, $\Phi \in \mathfrak{H}^{\sim}(X,Y)$, $\Psi \in \mathfrak{H}^{\downarrow}(X,Y)$.

The statement to be proved is that the modified descent $\Downarrow$ is an isomorphism of the functors $\mathfrak{H}^{\sim}$ and $\mathfrak{H}^{\downarrow}$. By virtue of theorem **3.5.6**, we only have to establish that $\Downarrow$ is a functor morphism of the functor $\mathfrak{H}^{\sim}$ to the functor $\mathfrak{H}^{\downarrow}$ or, in other words, that the following diagram is commutative

$$
\begin{array}{ccc}
\mathfrak{H}^{\sim}(X,Y) & \xrightarrow{\ \Downarrow\ } & \mathfrak{H}^{\downarrow}(X,Y) \\
{\scriptstyle \mathfrak{H}^{\sim}(\alpha,\beta)}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathfrak{H}^{\downarrow}(\alpha,\beta)} \\
\mathfrak{H}^{\sim}(X_1,Y_1) & \xrightarrow[\ \Downarrow\ ]{} & \mathfrak{H}^{\downarrow}(X_1,Y_1)
\end{array}
$$

for any X, X_1, Y, Y_1, α and β given above. The last result is equivalent to the fact that the equality $(\mathfrak{H}(\alpha,\beta)\Phi)\Downarrow = \mathfrak{H}^{\downarrow}(\alpha,\beta)(\Phi\Downarrow)$ is valid for every $\Phi \in \mathfrak{H}^{\sim}(X,Y)$ or, making use of the definition of $\mathfrak{H}^{\sim}$ and $\mathfrak{H}^{\downarrow}$, that the following conditions are compatible:

$$
\begin{aligned}
&\Psi \in \mathfrak{H}^{\downarrow}(X,Y), \quad [\Psi = \beta \circ \Phi \circ \alpha^{\sim}] = \mathbf{1}, \\
&(\beta\downarrow) \circ (\Phi\Downarrow) \circ \alpha = \Psi\Downarrow.
\end{aligned}
$$

The last equalities are fulfilled only if

$$
[\beta \circ \Phi \circ \alpha^{\sim} = (\beta\downarrow \circ (\Phi\Downarrow) \circ \alpha)\Uparrow] = \mathbf{1}.
$$

As is, however, seen from the rules of cancelling arrows and the definitions of modified descents and ascents, inside $\mathbf{V}^{(B)}$ the following equalities hold:

$$
\begin{aligned}
&(\beta\downarrow \circ (\Phi\Downarrow) \circ \alpha)\Uparrow = (\beta\downarrow \circ (\Phi\downarrow) \circ \iota \circ \alpha \circ \iota^{-1})\uparrow \\
&= \beta\downarrow\uparrow \circ (\Phi\downarrow\uparrow) \circ (\iota \circ \alpha \circ \iota^{-1})\uparrow = \beta \circ \Phi \circ (\iota \circ \alpha \circ \iota^{-1})\uparrow.
\end{aligned}
$$

Now to prove the theorem, we only have to remark that $[(\iota \circ \alpha \circ \iota^{-1})\uparrow = \alpha^{\sim}] = \mathbf{1}$. $\triangleright$

3.5.8. Let us consider some important corollaries to theorem **3.5.4** (preserving the assumptions and notation assumed in it).

(1) *If (X, d_X) is an extended B-set, then ι_X is a bijection between X and X'.*

$\triangleleft$ We only have to remark that when $x = \mathrm{mix}(b_\xi x_\xi)$, for a partition of unity (b_ξ) and a family $(x_\xi) \subset X$ we have $\iota_X x = \operatorname{mix}(b_\xi \iota_X x_\xi)$. ▷

(2) *For any B-set (X, d_X) there is a triplet (X', d'_X, ι_X) termed a B-extension of (X, d_X) and obeying the following conditions:*

(a) (X', d'_X) *is an extended B-set, while ι_X is an isometric mapping of X to X'*;

(b) $X' = \operatorname{mix}(\operatorname{im} \iota_X)$;

(c) *for any contractive correspondence Φ from X to an extended B-set Y there is a unique fully contractive correspondence Φ' from X' to Y such that* $\operatorname{dom}\Phi' = \operatorname{mix} \iota(\operatorname{dom}\Phi)$ *and*

$$\operatorname{mix}\Phi(x) = \Phi'(\iota_X x) \quad (x \in \operatorname{dom}\Phi);$$

(d) *if a triplet (X'', d''_X, ι''_X) obeys* (a) - (c), *then there is a B-isomorphism ι between X' and X'', for which $\iota \circ \iota_X = \iota'_X$.*

◁ For the proof one is to use an extended B-set instead of Y in **3.5.4** (3), and make use of corollary (1). ▷

(3) *If $X \in \operatorname{Ob} \mathcal{V}^{(B)}$, then there is a $j_X \in \mathbf{V}^{(B)}$ such that $[\![j_X$ is an isomorphism (in the category $\mathcal{V}^{(B)}$) of X on $X\downarrow^{\sim}]\!] = 1$.*

◁ Indeed, if $Y := X\downarrow$, then, setting $j_X := \iota_Y \uparrow$, we get that j_X is an isomorphism between $Y\uparrow = X$ and $Y^{\sim} = X\downarrow^{\sim}$, since ι_Y is an isomorphism between Y and $Y^{\sim}\downarrow$. ▷

(4) *If X and Y are extended B-sets, and Φ is a correspondence from $X^{\sim}$ to $Y^{\sim}$ inside $\mathbf{V}^{(B)}$, then there is a unique fully contractive correspondence Ψ from X to Y such that $\Psi^{\sim} = \Phi$.*

◁ Indeed, $\Phi' := \Phi\downarrow$ is a fully extensional correspondence from $X' := X^{\sim}\downarrow$ to $Y' := Y^{\sim}\downarrow$. Hence, $\Psi := \iota_Y^{-1} \circ \Phi' \circ \iota_X$ is a fully contractive correspondence from X to Y. If $\Psi' := \Psi^{\sim}\downarrow$, then by **3.5.4** (3), we get $\iota_Y^{-1} \circ \Psi \circ \iota_X = \iota_Y^{-1} \circ \Psi' \circ \iota_X$. If we take account of (1), then $\Psi = \Psi'$ and, hence, $\Phi = \Phi'\uparrow = \Psi'\uparrow = \Psi\uparrow$. ▷

(5) *If X and Y are extended B-sets, then the mapping $\Phi \to \Phi^{\sim}$ sets a bijection between the sets of morphisms* $\operatorname{CBSet}_*(X,Y)$ *and* $\mathcal{V}_*^{(B)}(X^{\sim}, Y^{\sim})$.

3.5.9. *Let X and Y be arbitrary B-sets and F be a fully contractive correspondence from X to Y. Then for any set $A \subset \operatorname{dom}\Phi$ we have*

$$\mathbf{V}^{(B)} \models \pi_\Phi(A)^\sim = \pi_{\Phi^\sim}(A^\sim).$$

$\triangleleft$ It should be observed that the relations $(\forall a \in A^\wedge)(y \in \Phi^\sim(\pi_X a)$ and $y \in \pi_{\Phi^\sim}(A^\sim)$ are equivalent, since $A^\sim = \pi_X(A^\wedge)$. Making use of theorem **3.4.13** and of the fact that Φ is fully contractive, we can write out the following equivalences ($y \in Y$):

$$\begin{aligned} & y \in \pi_{\Phi^\sim}(A^\sim) \leftrightarrow \wedge\{[y \in \Phi^\sim(\pi_X a^\wedge)] : a \in A\} = 1 \\ & \leftrightarrow (\forall a \in A)[y \in \pi_Y(\Phi(a)^\wedge)] = 1 \leftrightarrow (\forall a \in A)(y \in \Phi(a)^\sim \downarrow) \\ & \leftrightarrow (\forall a \in A)\, y \in Y \cap \operatorname{mix} \iota_Y(\Phi(a)) = \iota_Y(\pi_\Phi(A)). \end{aligned}$$

Hence,

$$\pi_{\Phi^\sim}(A^\sim) = \iota_Y(\pi_\Phi(A))\uparrow = \pi_\Phi(A)^\sim. \ \triangleright$$

3.5.10. *The functors $\mathfrak{F}^\sim$ and $\mathfrak{F}^\downarrow$ establish the equivalence of the categories* CBSet$_*$ *and* $\mathcal{V}_*^{(B)}$. *In particular, $\mathfrak{F}^\sim$ and $\mathfrak{F}^\downarrow$ are mutually conjugate complete univalent functors preserving inductive and projective limits (for the given categories).*

$\triangleleft$ It suffices to substantiate the validity of the following two statements:

(1) the functor $\mathfrak{F}^\downarrow \circ \mathfrak{F}^\sim$ is naturally isomorphic to the identical functor on the CBSet$_*$, while the isomorphism is implemented by the mappings $\iota_X : X \to X'$ ($X \in$ CBSet$_*$);

(2) the functor $\mathfrak{F}^\sim \circ \mathfrak{F}^\downarrow$ is naturally isomorphic to the identical functor on $\mathcal{V}_*^{(B)}$, the isomorphism is accomplished by the mappings $j_X \in \mathcal{V}^{(B)}(X, X\downarrow^\sim)$ ($X \in \mathcal{V}_*^{(B)}$). To prove (1) we should use corollary **3.5.8** (1) and remark that, by virtue of **3.5.4** (3), for $X, Y \in$ Ob CBSet$_*$ and $\Phi \in$ CBSet$_*(X,Y)$, the following diagram is commutative

$$\begin{array}{ccc} X & \xrightarrow{\ \iota_X\ } & X^\sim\downarrow \\ {\scriptstyle\Phi}\big\downarrow & & \big\downarrow{\scriptstyle\Phi^\sim\downarrow} \\ Y & \xrightarrow[\ \iota_Y\ ]{} & Y^\sim\downarrow \end{array}$$

Then, it follows from **3.5.8** (3,4) that for any $X, Y \in \operatorname{Ob} \mathcal{V}_*^{(B)}$ and $\Phi \in \mathcal{V}_*^{(B)}(X,Y)$ the diagram

$$\begin{array}{ccc} X & \xrightarrow{\;j_X\;} & X^{\sim}\downarrow \\ \Phi \Big\downarrow & & \Big\downarrow \Phi^{\sim}\downarrow \\ Y & \xrightarrow[\;j_Y\;]{} & Y^{\sim}\downarrow \end{array}$$

is commutative. Hence, from we get (2). ▷

3.5.11. *For any* $X \in \mathrm{Ob}\,\mathrm{CBSet}_*$ *and* $Y \in \mathrm{Ob}\,\mathcal{V}_*^{(B)}$ *the following relations are valid:*

$$(j_Y)\downarrow = \iota_{Y\downarrow}, \quad \mathbf{V}^{(B)} \models (\iota_X)^{\sim} = j_{X^{\sim}}.$$

◁ The first equality results directly from the definitions $(j_Y)\downarrow = (\iota_{Y\downarrow})\uparrow\downarrow = \iota_{Y\downarrow}$. In order to prove the second equality, assign

$$b := [(\iota_X)^{\sim} = j_{X^{\sim}}], \; b_x := [\iota_{X^{\sim}}\pi_X x^{\wedge} = j_{X^{\sim}}\pi_X x^{\wedge}] \quad (x \in X).$$

It should be observed that $b = \wedge\{b_x : x \in X\}$, and, hence, we should prove that $b_x = \mathbf{1}$ for every $x \in X$. If, however, $x \in X$ then, by virtue of **3.4.13** and according to the definition of j_X we get $b_x = [\pi_{X^{\sim}\downarrow}(\iota_X x)^{\wedge} = (\iota_{X^{\sim}\downarrow})\uparrow \circ \pi_X(x^{\wedge})]$. Finally, applying the equalities

$$[\pi_X x^{\wedge} = \iota_X x] = [\pi_{X^{\sim}\downarrow} y^{\wedge} = \iota_{X^{\sim}\downarrow} y] = \mathbf{1} \, (x \in X, y \in X^{\sim}\downarrow),$$

and letting $y = \iota_X x$, we obtain, according to **3.4.13**,

$$b_x = [\pi_{X^{\sim}\downarrow}(\iota_X x)^{\wedge} = \iota_{X^{\sim}\downarrow}(\iota_X x)] = \mathbf{1},$$

which completes the proof. ▷

CHAPTER 4

BOOLEAN-VALUED ANALYSIS OF ALGEBRAIC SYSTEMS

In any Boolean-valued universe there are, in particular, sets of various structures: groups, rings, algebras, etc.. Applying the descent functor to algebraic systems in the Boolean-valued model singles out structures with new properties and results in discovering new facts about their structure and interrelations. Such a technique, called direct Boolean-valued interpretation, allows one to produce new theorems or, to be more exact, to extend the semantical volume of the theorems available by way of straightforward translating. The information arising in such a way, however, not always proves to be really new, expedient or interesting, so that the unsophisticated Boolean-valued interpretation sometimes becomes an aimless game.

In this respect it would be interesting to ask a question: what practically important mathematical structures can be obtained from a Boolean-valued interpretation of well-studied structures? What transfer principles are valid in this case? It is obvious that here we are speaking about a Boolean-valued implementation of specially structured sets, which is impossible for arbitrary objects. In the preceding chapter we have proved that an abstract B-set can be immersed in a Boolean-valued universe in such a way that the Boolean distance between objects becomes the Boolean truth-value of their noncoincidence. The corresponding element of the universe $\mathbf{V}^{(B)}$ is, by definition, a Boolean-valued implementation of the B-set under consideration. If the B-set is in some way *apriori* structured, one can try to furnish its Boolean-valued interpretation with an analogous structure in order to use the technique of descents and ascents to study the initial structure. Therefore, the questions formulated above can be treated as a problem of searching qualified Boolean-valued realizations of structured B-sets.

The present section will be devoted to an analysis of the aforementioned problem for general algebraic objects. The central notion under discussion will be that of an algebraic B-system, which is a nonempty B-set with contraction operations, having a certain quantity of B-predicates, i.e., B-valued contraction mappings, on it. The Boolean-valued realization of an algebraic B-system appears to be a conventional two-valued algebraic system of the same type. This implies that an appropriate extension of any algebraic B-system coincides with the descent of a two-valued algebraic system inside $\mathbf{V}^{(B)}$. On the other hand, a two-valued algebraic system can be turned into an algebraic B-system provided a complete Boolean algebra of congruences is singled out in it. In this case it is important to find out which formulas remain true under transition from a B-system to a two-valued one, and vice versa.

In other words, here some variations of the transfer principle or the 'principle of preserving relations' arise. General facts are illustrated by concrete examples of algebraic systems in which complete Boolean algebras of congruences are related with the relations of order and disjointness.

4.1. Algebraic B-Systems

Let us introduce a class of algebraic systems suitable for Boolean-valued interpretations of first-order languages. Such systems arise as B-sets furnished with contraction operations and predicates.

4.1.1. It should be recalled that a *signature* is a triplet $\sigma := (F, P, \mathfrak{a})$, where F and P are some (possibly, empty) sets, while $\mathfrak{a}$ is a mapping from $F \cup P$ to ω. An *n-ary operation* or an *n-ary predicate* on a B-set A is contraction mappings $f: A^n \to A$ or $p: A^n \to B$, respectively. By definition, the mappings f and p are contractions provided

$$d(f(a_0,\dots,a_{n-1}), f(a_0',\dots,a_{n-1}')) \le \bigvee_{k=0}^{n-1} d(a_k, a_k'),$$

$$d_s(p(a_0,\dots,a_{n-1}), p(a_0',\dots,a_{n-1}')) \le \bigvee_{k=0}^{n-1} d(a_k, a_k')$$

for all $a_0, a_0', \dots, a_{n-1}, a_{n-1}' \in A$, where d is the B-metric of the sets A, and d_s is the symmetric difference on B, i.e., $d_s(b_1, b_2) := b_1 \Delta b_2$ (see **1.1.4**).

An *algebraic B-system of signature* σ is a pair (A, v), where A is a nonempty B-set, while v is a mapping such that $\operatorname{dom}(v) = F \cup P$, and $v(f)$ is an $\mathfrak{a}(n)$-ary operation on A for all $f \in F$, while $v(p)$ is an $\mathfrak{a}(p)$-ary predicate on A for every $p \in P$. A contraction mapping from A^n to B is also termed a *B-predicate*, or a *B-valued predicate*. The mapping v is sometimes referred to as *interpretation* and presented, for convenience, as f^v and p^v instead of $v(f)$ and $v(p)$. The signature of an algebraic B-system $\mathfrak{A} := (A, v)$ will be often denoted by $\sigma(\mathfrak{A})$, while the carrier set A by $|\mathfrak{A}|$. As long as $A^0 = \{\varnothing\}$, then nullary operations and predicates on A are mappings from $\{\varnothing\}$ to the set A and to the algebra B, respectively. Let us identify the mapping $g: \{\varnothing\} \to A \cup B$ with an element $g(\varnothing)$. As a result, nullary operations on A are chosen A elements, while a set of all nullary predicates on A is a Boolean algebra B. If $F := \{f_1, \dots, f_n\}$ and $P := \{p_1, \dots, p_m\}$, then the algebraic B-system of signature σ is often represented as $(A, v(f_1), \dots, v(f_n), v(p_1), \dots, v(p_m))$, and even $(A, f_1, \dots, f_n, p_1, \dots, p_m)$, while instead of $\sigma(F, P, \mathfrak{a})$ use is made of the notation $\sigma = (f_1, \dots, f_n, p_1, \dots, p_m)$.

4.1.2. If B is the two-element Boolean algebra $\{0,1\}$, then instead of an algebraic B-system we speak about a *two-valued system* or simply about an *algebraic system.* In this case an arbitrary set can be treated as a B-set, while an n-ary operation and predicate on a B-set A are specialized, respectively, as an arbitrary mapping from A^n to A and a characteristic function $p: A^n \to \{0,1\}$ identified with the set $\{x \in A^n : p(x) = 1\}$. Therefore, an algebraic system of signature σ is a pair (A,v), where A is a nonempty set, while v is a function from $\mathrm{dom}(v) = F \cup P$ to $\mathbf{V}$ such that

$$v(f): A^{\mathfrak{a}(f)} \to A, \quad v(p) \subset A^{\mathfrak{a}(p)} \quad (f \in F, p \in P).$$

On the other hand, if (A,v) is an algebraic system of signature σ and $A \subset \mathbf{V}^{(B)}$, then, viewing A as a B-set (with the B-metric $d(a,a') := [a = a']^* = [a \neq a']$ $(a,a' \in A)$), for every $p \in P$ we can determine an $n := \mathfrak{a}(p)$-ary B-predicate $v'(p)$ on A by the following formula (see **3.4.5**):

$$v'(p) := (a_0,\dots,a_{n-1}) \to \mathrm{dist}((a_0,\dots,a_{n-1}), v(p)).$$

It is obvious that the mapping $v'(p): A^n \to B$ is a contraction. Let, besides, $v(f)$ be a contraction mapping for every $f \in F$. Let us put $v'(f) := v(f), f \in F$. Then (A,v') is an algebraic B-system.

4.1.3. An algebraic B-system $\mathfrak{A} := (A, v)$ is called *extended (decomposable)* provided A is an extended (decomposable) B-set (**3.4.3**). A B-valued predicate p on the set A is called *assertive* if there exists such an element $x \in A$ that $p(x) = 1$.

(1) *A contraction mapping p from an extended set A to B is an assertive B-valued predicate iff* $\mathbf{1} = \vee\{p(x) : x \in A\}$.

◁ Indeed, if the above condition is fulfilled, then there is a family $(x_\xi) \subset A$ and a partition of unity $(b_\xi) \subset B$ such that $p(x_\xi) \geq b_\xi$. If $x := \mathrm{mix}(b_\xi x_\xi)$, then $p(x) = \mathbf{1}$. ▷

Every algebraic B-system $\mathfrak{A}$ can be related to an algebraic system $\overline{\mathfrak{A}}$ with the same carrier set $|\overline{\mathfrak{A}}| := |\mathfrak{A}|$, whose interpretation $\bar{v}$ is determined in the following way. If f is a function symbol, then $\bar{v}(f) := v(f)$; while if p is a predicate symbol and $n = \mathfrak{a}(p)$, then $\bar{v}(p) := \{(x_0,\dots,x_{n-1}) \in A^n : p(x_0,\dots,x_{n-1}) = \mathbf{1}\}$. It is obvious that the predicate $\bar{v}(p)$ can prove to be empty for a certain p. The algebraic system $\overline{\mathfrak{A}}$ is said to be a *purification* or *reduct* of $\mathfrak{A}$, or $\overline{\mathfrak{A}}$ is said to be obtained from $\mathfrak{A}$ by the *purification* or *reduction.*

(2) *If* (A,v) *is an algebraic* B*-system and* $(A,\overline{v})$ *is its purification, then for every assertive predicate* p^v *we have*

$$p^v: x \to \operatorname{dist}(x, \overline{v}(p))^* \quad (x \in A^{a(p)}).$$

◁ By virtue of the theorem on Boolean-valued realization of B-sets (see **3.5.8**), a B-set A admits an extension $A' \subset \mathbf{V}^{(B)}$, while p^v admits a unique extension $v'(p)$ to a B-valued predicate on A'. In this case $v'(p)(x) = \operatorname{dist}(x, \operatorname{mix}(\overline{v}(p)))^* = \operatorname{dist}(x, \overline{v}(p))^* = [x \in p^\gamma \uparrow]$ $(x \in A^{\mathfrak{a}(p)})$. This yields the required result, since the assumption $A \subset A'$ limits no generality. ▷

Proposition (2) makes it possible to identify an algebraic B-system with assertive predicates and some algebraic system, namely, its purification. It would be natural to ask a question: what algebraic systems are obtainable by purification of decomposable (extended) algebraic B-systems? The answer to this question will be formulated in terms of congruences of an algebraic system.

4.1.4. Let us consider an arbitrary algebraic system $\mathfrak{A} := (A, v)$ of signature $\sigma := (F, P, \mathfrak{a})$. An equivalence relation ρ on the set A is termed a *congruence* of the system $\mathfrak{A}$ provided for every $f \in F$ and for any $x_0, \dots, x_{n-1}, y_0, \dots, y_{n-1} \in A$, $n = \mathfrak{a}(f)$ the relations $(x_0, y_0) \in \rho, \dots, (x_{n-1}, y_{n-1}) \in \rho$ yield $(f^v(x_0, \dots, x_{n-1}), f^v(y_0, \dots, y_{n-1})) \in \rho$. The set of all congruences on the algebraic system $\mathfrak{A}$ is denoted by the symbol $\operatorname{Cong}(\mathfrak{A})$. Let us introduce an order relation in $\operatorname{Cong}(\mathfrak{A})$ by the formula

$$\rho_1 \le \rho_2 \leftrightarrow \rho_1 \subset \rho_2 \quad (\rho_1, \rho_2 \in \operatorname{Cong}(\mathfrak{A})).$$

The identity congruence $I_A := \{(x,x) : x \in A\}$ and the trivial congruence $A \times A$ are obviously the least and the greatest elements of $\operatorname{Cong}(\mathfrak{A})$.

(1) Theorem. *The ordered set* $\operatorname{Cong}(\mathfrak{A})$ *is a complete lattice. The greatest lower bound of the set* $\mathcal{P} \subset \operatorname{Cong}(\mathfrak{A})$ *coincides with the intersection* $\cap\{\rho : \rho \in \mathcal{P}\}$. *The least upper bound of the set* $\mathcal{P} \subset \operatorname{Cong}(\mathfrak{A})$ *is the union of all possible composits* $\rho_1 \circ \dots \circ \rho_n$, *where* $\{\rho_1, \dots, \rho_n\}$ *is an arbitrary finite subset in* $\mathcal{P}$.

The congruence $\rho_1 \vee \rho_2$, for ρ_1 and $\rho_2 \in \operatorname{Cong}(\mathfrak{A})$, as is seen from this theorem, coincides with the union of all possible relations of the form $\rho_1 \circ \rho_2 \circ \rho_1 \circ \dots \circ \rho_1 \circ \rho_2$. Hence, if ρ_1 and ρ_2 commute, i.e., $\rho_1 \circ \rho_2 = \rho_2 \circ \rho_1$, then $\rho_1 \vee \rho_2 = \rho_1 \circ \rho_2$. Conversely, if $\rho_1 \vee \rho_2 = \rho_1 \circ \rho_2$, then the congruences ρ_1 and ρ_2 commute.

A set of congruences Λ on the algebraic system $\mathfrak{A}$ is called *independent (finitely independent)* if for any family (finite family) $(\lambda_\xi)_{\xi\in\Xi}$ in Λ and $(a_\xi)_{\xi\in\Xi}$ in A there is such an element $a \in A$ that $(a, a_\xi) \in \lambda_\xi$ for all $\xi \in \Xi$. The set Λ is termed *complete* provided:

(a) $\inf(\Lambda) := \bigcap(\Lambda) = I_A$ and

(b) for any $p \in P$ and an arbitrary n-tuple $(x_0,\dots,x_{n-1}) \in A^n, n = \mathfrak{a}(p)$ the relation $(x_0,\dots,x_{n-1}) \notin v(p)$ yields the existence of such a congruence $\lambda \in \Lambda$ that $(y_0,\dots,y_{n-1}) \notin v(p)$ as soon as $(x_0,y_0) \in \lambda,\dots,(x_{n-1},y_{n-1}) \in \lambda$ (see [170]).

In the definition of a complete set of congruences it is convenient to formulate the condition (b) in terms of mixing. Let us consider a family $(a_\lambda)_{\lambda\in\Lambda}$ in a set A. If for some $a \in A$ and all $\lambda \in \Lambda$ we have $(a, a_\lambda) \in \lambda$, then it is natural to say that a is the *mixing of the family* (a_λ) *relative to* Λ. A set $U \subset A^n$ will be called *stable under* Λ*-mixing*, if for any family $((a_\lambda^0,\dots,a_\lambda^{n-1}))$ in U we have $(a_0,\dots,a_{n-1}) \in U$, where a_k is the mixing of (a_λ^k) relative to Λ.

(2) *An independent set of congruences* Λ *of an algebraic system* $\mathfrak{A}$ *is complete iff* $\inf(\Lambda) = I_A$ *and any predicate* $v(p)$, $p \in P$ *is stable under* Λ*-mixing.*

◁ Indeed, let us assume that all the predicates are stable under Λ-mixing. Let $p \in P$, $n = \mathfrak{a}(p)$, $(x_0,\dots,x_{n-1}) \notin v(p)$ and, nonetheless, for any $\lambda \in \Lambda$ net there exist such $(y_\lambda^0,\dots,y_\lambda^{n-1}) \in v(p)$ that $(x_k, y_\lambda^k) \in \lambda$ $(k = 0,\dots,n-1)$. Let y_k be the mixing of a family $(y_{\lambda,k})_{\lambda\in\Lambda}$ relative to Λ. Then $(y_0,\dots,y_{n-1}) \in v(p)$. At the same time, $(x_k, y_k) \in \lambda$ for all $\lambda \in \Lambda$. Hence, $x_k = y_k$ $(k = 0,\dots,n-1)$, since $\bigcap\Lambda = I_A$, and we thus come to a contradiction.

Assume conversely that Λ is a complete set. Choose a $p \in P$ and a family of n-tuples $(a_{\lambda,0},\dots,a_{\lambda,n-1})$ contained in $v(p)$. Let a_k be the mixing of a family $(a_{\lambda,k})_{\lambda\in\Lambda}$ relative to Λ. If $(a_0,\dots,a_{n-1}) \notin v(p)$, then, since Λ is complete, there is a congruence $\lambda \in \Lambda$ for which $(a_{\lambda,0},\dots,a_{\lambda,n-1}) \notin v(p)$. This, however, contradicts the choice of $(a_{\lambda,0},\dots,a_{\lambda,n-1})$ and, hence, $v(p)$ is stable under mixing. The necessity is seen to be true without the assumption that Λ is independent. ▷

4.1.5. Let us refer as *Boolean algebra of congruences* to any Boolean algebra $\mathcal{B} \subset \mathrm{Cong}(\mathfrak{A})$ if in $\mathcal{B}$ the least upper bounds of arbitrary sets are inherited from the lattice of $\mathrm{Cong}(\mathfrak{A})$ and the least congruence I_A serves as zero in $\mathcal{B}$. It should be emphasized that the Boolean extension ρ^* of an element $p \in \mathcal{B}$ can also not be an extension ρ in the lattice of $\mathrm{Cong}(\mathfrak{A})$, i.e., the least upper bound of ρ and ρ^* in $\mathrm{Cong}(\mathfrak{A})$ may be less than $A \times A$.

A *base of an algebraic system* $\mathfrak{A}$ will be any complete Boolean algebra of congruences $\mathcal{B} \subset \mathrm{Cong}(\mathfrak{A})$ provided each predicate $v(p)$ $(p \in P)$ is stable under Λ^*-mixing for any partition of unity $\Lambda \subset \mathcal{B}$, where $\Lambda^* := \{b^* : b \in \Lambda\}$. An algebraic system with base $\mathscr{B}$ will be termed *extended (decomposable)* provided for any (any finite, respectively) partition of unity $\Lambda \subset \mathscr{B}$ the set of congruences Λ^* is independent. Then the following obvious statement is valid.

An algebraic system $\mathfrak{A}$ has a base $\mathscr{B}$ isomorphic to a complete Boolean algebra B iff there exists an injective mapping $h: B \to \mathrm{Cong}(\mathfrak{A})$ obeying the following conditions:

(a) *h preserves the greatest lower bounds of any sets and $h(0) = I_A$;*

(b) *any predicate $v(p)$ $(p \in P)$ is stable under $h(\Lambda^*)$-mixing for any partition of unity $\Lambda \subset B$.*

In this case $\mathfrak{A}$ is extended (decomposable) iff the set $h(\Lambda^*)$ is independent for every (for any finite) partition of unity $\Lambda \subset B$.

4.1.6. The algebraic B-system $\mathfrak{A}$ will be called *full* provided for any $0 \neq b \in B$ there are elements $x, y \in A$, $x \neq y$ such that $d(x,y) \leq b$. It is obvious that a decomposable B-system is full, but the converse statement is not valid in general.

Theorem. *An algebraic system $\mathfrak{A}$ is obtained by purification from a certain full algebraic B-system $\mathfrak{A}'$ iff $\mathfrak{A}$ has a base isomorphic to B. In this case both $\mathfrak{A}$ and $\mathfrak{A}'$ are extended (decomposable) or not simultaneously.*

$\triangleleft$ Let $\mathfrak{A}'$ be a full algebraic B-system. To every $b \in B$ let us put into correspondence the relation $h(b) := \{(x,y) \in A^2 : d(x,y) \leq b\}$. Since $v(t)$ is a contraction mapping for every $f \in F$; therefore, $h(b)$ is a congruence on A. It is obvious that $h(0) = I_A$ and that h preserves the greatest lower bounds. Since $\mathfrak{A}$ is full, h is injective. Let us assume that the algebraic system $\mathfrak{A}$ is obtained from $\mathfrak{A}'$ by purification. It should be remarked that a set of the type $\{z \in A : p(z) = 1\}$ is stable under any mixing in the B-set A. It is now seen, by virtue of **4.1.5**, that $\mathfrak{A}$ has a base isomorphic to B.

Conversely, let an algebraic system $\mathfrak{A}$ have a base $\mathscr{B}$ and let us assume that there exists a Boolean isomorphism h from B to $\mathscr{B}$. According to the definition, assign

$$d(x,y) := \wedge\{b \in B : (x,y) \in h(b)\} \quad (x,y \in A).$$

If $b_1, b_2 \in B$ are such that $(x,z) \in h(b_1)$ and $(z,y) \in h(b_2)$, then $(x,y) \in h(b_2) \circ h(b_1)$. However, $h(b_2) \circ h(b_1) \subset h(b_1 \vee b_2)$ and, hence, $d(x,y) \leq b_1 \vee b_2$. Taking the infimum over the given b_1 and b_2, and making use of the distributive law **1.1.5** (1), we get $d(x,y) \leq d(x,z) \vee d(z,y)$. It is now evident that d is a Boolean semimetric on A. Since h

preserves its greatest lower bounds; therefore,

$$h(d(x,y)) = \cap\{h(b): b \in B \wedge (x,y) \in h(b)\}.$$

From here we deduce that $d(x,y) \le b$ iff $(x,y) \in h(b)$. In particular, $d(x,y) = 0$ implies that $x = y$, while for $0 \ne b \in B$ we can find such $x, y \in A$ that $x \ne y$ and $d(x,y) \le b$.

Now it remains to show that if Λ is a partition of unity in B, then for a family $(a_b)_{b\in\Lambda} \subset A$ the mixing relative to $h(\Lambda^*)$ coincides with that in the sense of the B-metric d, i.e., with $\text{mix}_{b\in\Lambda}(ba_b)$. This fact, however, follows trivially from the above: $(a, a_b) \in h(b^*) \leftrightarrow d(a,a_b) \le b^* \leftrightarrow b \wedge d(a,a_b) = 0$. Let us now determine $\mathfrak{A}' := (A', v')$, setting $A' := A$, $v'(f) = v(f)$, $f \in F$ and

$$v'(p) := x \to \text{dist}(x, v(p)) \quad (p \in P, x \in A^{\mathfrak{a}(p)}).$$

If $f \in F$ and $n = \mathfrak{a}(f)$, then for any $b \in B$ and elements $x_0, y_0, \dots, x_{n-1}, y_{n-1} \in A$ the relations $(x_k, y_k) \in h(b), k < n$ implies that $(f^v(x_0,\dots,x_{n-1}), f^v(y_0,\dots,y_{n-1})) \in h(b)$, which gives

$$d(f^v(x_0,\dots,x_{n-1}), f^v(y_0,\dots,y_{n-1})) \le b.$$

Taking the greatest lower bound over b and observing that

$$\wedge\{b: (x_k,y_k) \in h(b), k < n\} = \bigvee_{k=0}^{n-1} d(x_k,y_k),$$

we conclude that the mappings $f^v = v(t)$ are contractions. Let us choose $p \in P$, $\mathfrak{a}(p) = m$ and elements $x := (x_0,\dots,x_{m-1})$ and $y := (y_0,\dots,y_{m-1})$ from A^m. Then

$$d(x,y) \wedge \text{dist}(x, v(p)) \le \text{dist}(y, v(p)),$$

which implies that $v'(p)$ is a contraction. Besides, since $v(p)$ is stable (see **4.1.3** (2)), we get $v(p) = \{x \in A^m : v'(p)(x) = 1\}$. Hence, $\mathfrak{A}$ is a purification of the full algebraic B-system $\mathfrak{A}'$. The fact that the systems $\mathfrak{A}$ and $\mathfrak{A}'$ are extended implies that Λ^*, where Λ is a partition of unity in B, is an independent set and that in (A,d) there are any mixings, the last two statements being, however, equivalent. ▷

4.1.7. Let us consider some concrete examples of algebraic B-systems. An associative ring R is called a *Boolean ring* if its every element is idempotent, i.e., if $(\forall x \in R)\ (x \cdot x = 0)$. A Boolean ring with unity turns into a Boolean algebra if the operations in the former are determined as follows:

$$x \wedge y := x \cdot y,\ x \vee y := x + y - x \cdot y,\ x^* := 1 - x \quad (x, y \in R).$$

Conversely, any Boolean algebra is a Boolean ring with unity under the following definition of ring operations:

$$x + y := x \Delta y := (x \Leftrightarrow y)^*, \quad x \cdot y = x \wedge y \quad (x, y \in R).$$

In both cases zero and unity of the ring coincide with Boolean zero and unity, respectively.

(1) Let B_0 be a Boolean algebra and X be a unital module over the Boolean ring B_0. Let B be the completion of the algebra B_0, while j be an isomorphism of B_0 on a dense subalgebra in B. According to definition, assign

$$d_j(x,y) := \wedge\{j(b) : b^* x = b^* y \wedge b \in B_0\} \quad (x, y \in X).$$

As is easily seen, d_j is a B-semimetric on X. Let us check, for instance, the triangle inequality. If $b^* x = b^* y$ and $c^* z = c^* y$, then for $c := b^* \wedge c^* = (b \vee c)^*$ we have $cx = cz$ and $cy = cz$. Therefore, $d_j(x,y) \le j(b \vee c) = j(b) \vee j(c)$ and, since b and c are arbitrary, we get $d_j(x,y) \le d_j(x,z) \vee d_j(z,y)$. Call the module X *laterally exact* if for any partition of unity (b_ξ) in B_0 from $(b_\xi x = 0)$ we get $x = 0$ for any element $x \in X$. No doubt that for a laterally exact unital B_0-module X the semimetric d_j is a metric. By analogy with the the triangle inequality for d_j, we can check that the module operations are contractions:

$$d_j(x + u, y + v) \le d_j(x,y) \vee d_j(u,v) \quad (x, y, u, v \in X),$$
$$d_j(bx, cy) \le d_j(x,y) \vee d_s(b,c) \quad (x, y \in X; b, c \in B).$$

The last inequality, in particular, implies

$$d_j(bx, by) \le d_j(x,y) \quad (b \in B; x, y \in X).$$

Besides, it is obvious that $d_j(-x,-y) = d_j(x,y)$. Therefore, the set X with the operations $+,-$ and unary operations of multiplication by $b \in B_0$ is an algebraic B-system.

(2) Let R be a commutative ring with unity, and consider a set of all of its idempotent elements $B_0 := \{e \in R : e \cdot e = e\}$. Then B_0 is a Boolean ring with unity and R is a module over B_0. If B and j are the same as in (1), there arises a B-semimetric d_j on R. It is obvious that R is laterally exact over B_0. By virtue of (1) we deduce that the commutative ring R with unity, laterally exact over the subring of its idempotents B_0 is an algebraic B-

system of signature $(+,-,\cdot,1)$.

(3) Let C be a Boolean algebra, and ι be a homomorphism from a Boolean algebra B_0 to C. As long as $\iota(B_0)$ is a subring of the Boolean ring C, we can readily determine on C the structure of the unital module over B_0. If B and j are the same as in (1), then the B-semimetric d_j has the form

$$d_j(x,y) := \wedge\{j(b) : \iota(b^*)x = \iota(b^*)y\}.$$

The module C is laterally exact if ι is a full monomorphism. In view of the relation between Boolean and ring operations discussed above, the Boolean algebra C is an algebraic B-system of signature $(\vee, \wedge, *, 0, 1)$ in the case of the full monomorphism ι. This system will be extended if, for instance, B_0 and C are complete Boolean algebras.

4.1.8. Let us now turn our attention to B-valued interpretation of first-order languages. Let us consider an algebraic B-system $\mathfrak{A} := (A, v)$ of signature $\sigma := \sigma(\mathfrak{A}) := (F, P, \mathfrak{a})$.

Let $\varphi(x_0,\dots,x_{n-1})$ be a formula of signature σ with n free variables, and $a_0,\dots,a_{n-1} \in A$. We can readily determine the truth-value $|\varphi|^{\mathfrak{A}}(a_0\dots,a_{n-1}) \in B$ of the formula φ in the system $\mathfrak{A}$ for the given values $a_0,\dots,a_{n-1}$ of the variables $x_0,\dots,x_{n-1}$. The definition is, as usual, given by induction on the complexity of φ. Let us set for propositional connectives and quantifiers

$$|\varphi \wedge \psi|^{\mathfrak{A}}(a_0,\dots,a_{n-1}) := |\varphi|^{\mathfrak{A}}(a_0,\dots,a_{n-1}) \wedge |\psi|^{\mathfrak{A}}(a_0,\dots,a_{n-1});$$
$$|\varphi \vee \psi|^{\mathfrak{A}}(a_0,\dots,a_{n-1}) := |\varphi|^{\mathfrak{A}}(a_0,\dots,a_{n-1}) \vee |\psi|^{\mathfrak{A}}(a_0,\dots,a_{n-1});$$
$$|\neg\varphi|^{\mathfrak{A}}(a_0,\dots,a_{n-1}) := |\varphi|^{\mathfrak{A}}(a_0,\dots,a_{n-1})^*;$$
$$|(\forall x_0)\varphi|^{\mathfrak{A}}(a_1,\dots,a_{n-1}) := \bigwedge_{a_0 \in A} |\varphi|^{\mathfrak{A}}(a_0,\dots,a_{n-1});$$
$$|(\exists x_0)\varphi|^{\mathfrak{A}}(a_1,\dots,a_{n-1}) := \bigvee_{a_0 \in A} |\varphi|^{\mathfrak{A}}(a_0,\dots,a_{n-1}).$$

Now the case of atomic formulas remains to be considered. Let $p \in P$ be an m-ary predicate symbol, $q \in P$ be a nullary predicate symbol, and $t_1,\dots,t_{m-1}$ be terms of signature σ assuming values $b_0,\dots,b_{m-1}$ for the given values $a_0,\dots,a_{n-1}$ of the variables $x_0,\dots,x_{n-1}$. By definition, assign

$$|\varphi|^{\mathfrak{A}}(a_0,\dots,a_{n-1}) := v(q) \text{ if } \varphi := q^v;$$
$$|\varphi|^{\mathfrak{A}}(a_0,\dots,a_{n-1}) := d(b_0,b_1)^* \text{ if } \varphi := (t_0 = t_1);$$
$$|\varphi|^{\mathfrak{A}}(a_0,\dots,a_{n-1}) := p^v(b_0,\dots,b_{m-1}) \text{ if } \varphi := p^v(t_0,\dots,t_{m-1}),$$

where d is a B-metric on the set A.

We say that $\varphi(x_0,\dots,x_{n-1})$ *is assertive in the system* $\mathfrak{A}$ *for the given values* $a_0,\dots,a_{n-1}\in A$ *of the variables* $x_0,\dots,x_{n-1}$ (or, to put it short, that $\varphi(a_0,\dots,a_{n-1})$ is *assertive in* $\mathfrak{A}$) and write $\mathfrak{A}\models\varphi(a_0,\dots,a_{n-1})$ if $|\varphi|^{\mathfrak{A}}(a_0,\dots,a_{n-1})=\mathbf{1}_B$. When $B:=\{\mathbf{0},\mathbf{1}\}$, we get the conventional definition of validity for a formula in an algebraic system (see [46, 170]).

It should be recalled that a closed formula φ of signature σ is termed *identically assertive* if it is fulfilled on any algebraic $\mathbf{2}$-system of signature σ.

4.1.9. Theorem. *Let* $\mathfrak{A}$ *be an arbitrary algebraic* B*-system. Then the following statements are valid:*

(1) *any theorem of first-order predicate calculus with equality is assertive in* $\mathfrak{A}$;

(2) *any identically assertive closed formula of signature* $\sigma(\mathfrak{A})$ *is assertive in* $\mathfrak{A}$.

$\lhd$ (1) Here we have to make sure that the axioms of first-order predicate calculus with equality are assertive in $\mathfrak{A}$, while the rules of inference do not violate validity in $\mathfrak{A}$. To this end, we have to check the corresponding calculations of Boolean truth-values (see [10, 46, 114, 121, 248, 249]).

(2) If the closed formula φ is not satisfied in $\mathfrak{A}$, then $b:=|\varphi|^{\mathfrak{A}}<\mathbf{1}_B$. Let $h: B\to \mathbf{2}:=\{\mathbf{0},\mathbf{1}\}$ be a full homomorphism, in which case $h(b)=\mathbf{0}$. Such an h does exist, since the ideal $[\mathbf{0},b]$ can be extended to a maximal ideal $h^{-1}(\mathbf{0})$. If v is an interpretation of $\mathfrak{A}$, then let us set $v'(f):=f^v$ for functional symbols and $v'(p):=h\circ p^v$ for predicate symbols. Then $\mathfrak{A}':=(|\mathfrak{A}|,v')$ is an algebraic $\mathbf{2}$-system and $|\varphi|^{\mathfrak{A}'}=h(b)=\mathbf{0}$, i.e., φ is not satisfied in $\mathfrak{A}'$ and cannot be identically assertive.

4.1.10. Let us consider algebraic $\mathscr{B}$-systems $\mathfrak{A}:=(A,v)$ and $\mathfrak{C}:=(C,\mu)$ of the same signature σ. The mapping $h: A\to C$ is called a *homomorphism of the* $\mathscr{B}$*-system* $\mathfrak{A}$ *to the algebraic* $\mathscr{B}$*-system* $\mathfrak{C}$, provided for any $a_0,\dots,a_{n-1}\in A$ we have:

(1) $d_B(h(a_1),h(a_2))\le d_A(a_1,a_2)$;

(2) $h(f^v)=f^\mu$, $\mathfrak{a}(f)=0$;

(3) $h(f^{\nu}(a_0,\dots,a_{n-1})) = f^{\mu}(h(a_0),\dots,h(a_{n-1})),\quad 0 \neq n := \mathfrak{a}(f)$;

(4) $p^{\nu}(a_0,\dots,a_{n-1}) \leq p^{\mu}(h(a_0),\dots,h(a_{n-1})),\quad n := \mathfrak{a}(p)$.

The homomorphism h is called *strong* if

(5) for an arbitrary $p \in P, \mathfrak{a}(p) := n \neq 0$, and for any $b_0,\dots,b_{n-1} \in B$ the following inequality holds:

$$p^{\mu}(b_0,\dots,b_{n-1}) \leq \bigvee_{a_0,\dots a_{n-1} \in A} \{p^{\nu}(a_0,\dots,a_{n-1}) \wedge d_C(b_0, h(a_0)) \wedge \dots \wedge d_C(b_{n-1}, h(a_{n-1}))\}.$$

If the homomorphism h is injective, and conditions (1) and (4) are fulfilled with equality holding, then h is said to be a homomorphism from $\mathfrak{A}$ to $\mathfrak{C}$. Undoubtedly, any surjective isomorphism h and, in particular, the mapping $I_A : \mathfrak{A} \to \mathfrak{A}$ are strong homomorphisms. The composition of (strong) homomorphisms is a (strong) homomorphism. If h is a homomorphism and there is a mapping h^{-1} which is also a homomorphism, then h is an isomorphism.

It should be again observed that in the case of the two-element Boolean algebra $B := \{0,1\}$ conventional notions of a homomorphism, strong homomorphism, and isomorphism arise (see 46, 170]).

4.1.11. Let us consider a certain set Φ of formulas of the same fixed signature σ. Let us determine the category B – AS(Φ) in the following way. The class Ob B – AS(Φ) consists of all algebraic B-systems of signature σ, on each of which all the formulas of Φ are assertive. The class Mor B – AS(Φ) is a class of all homomorphisms of algebraic B-systems of Ob B – AS(Φ) with the conventional composition as the composition of morphisms. The isomorphism in the category B – AS(Φ) is obvious to be a B-isometric strong homomorphism. By the symbol B – CAS(Φ) let us denote a complete subcategory of the category B – AS(Φ), the objects in which are extended algebraic B-systems.

4.1.12. According to **4.1.5** and **4.1.6**, the structure of an algebraic B-system is restored with the help of a complete Boolean algebra of congruences. On the other hand, one of most common methods of generating complete Boolean algebras is associated with disjointness. Let us consider some simplest interrelations among these notions, starting with some facts to be recalled.

Let us choose sets X and Y. Let Φ be a correspondence from X to Y. As before, $\pi_{\Phi}(A)$ and $\pi_{\Phi}^{-1}(C)$ are the polar of $A \subset X$ and the inverse polar of $C \subset Y$ relative to the correspondence Φ, respectively. A set $K \subset Y$ is called a *Φ-component* (or simply a *component* of Φ, when it is clear what Φ is meant), provided $K = \pi_\Phi(\pi_\Phi^{-1}(K))$ or, which is equivalent, $K = \pi_\Phi(A)$ for some $A \subset X$. The collection of all the Φ-components is denoted by the symbol $\mathfrak{K}_\Phi(Y)$. The least component containing a given set $C \subset Y$ is denoted by $[C]$, in which case $[C] = \pi_\Phi(\pi_\Phi^{-1})(C)$.

(1) Theorem. *The set $\mathfrak{K}_\Phi(Y)$ ordered by inclusion is a complete lattice. The supremum and infimum of a family $(K_\xi)_{\xi \in \Xi} \subset \mathfrak{K}_\Phi(Y)$ are calculated by the formulas*

$$\bigwedge_{\xi \in \Xi} K_\xi = \bigcap_{\xi \in \Xi} K_\xi, \quad \bigvee_{\xi \in \Xi} K_\xi = \Big[\bigcap_{\xi \in \Xi} K_\xi\Big].$$

The taking of the inverse polar $K \to \pi_\Phi^{-1}(K)$ is an antitonic bijection of $\mathfrak{K}_\Phi(Y)$ on $\mathfrak{K}_{\Phi^{-1}}(X)$.

(2) A correspondence Δ from X to X is termed a *disjointness relation* or *disjointness* (on the set X) provided the following conditions are met:

(a) $\Delta = \Delta^{-1}$, i.e., Δ is symmetric;

(b) $\Delta \cap I_X \subset \Theta \times \Theta$, where $\Theta := \pi_\Delta(X)$ is the least Δ-component;

(c) $[x] \cap [y] \subset \Theta \to (x, y) \in \Delta$.

A disjointness Δ is called *simple* if it obeys an additional requirement

(d) $(x, y) \in \Delta \to x \in \Theta \vee y \in \Theta$.

Since Δ is symmetric, the lattices $\mathfrak{K}_\Delta(X)$ and $\mathfrak{K}_{\Delta^{-1}}(X)$ coincide. If $A \subset X$, then the polar $\pi_\Delta(A)$ is called the *disjoint complement* of A and is also denoted by $A^\perp$. The relations $x \in \pi_\Delta(A)$ and $C \subset \pi_\Delta(A)$ are presented as $x^\perp A$ and $C^\perp A$. It should be also observed that $A^{\perp\perp} := (A^\perp)^\perp = [A]$.

(3) Theorem. *The set $\mathfrak{K}_\Delta(X)$ of all components of the disjointness Δ ordered by inclusion, is a complete Boolean algebra. The Boolean complement of a component coincides with its disjoint complement.*

$\triangleleft$ As mentioned in (1), $\mathfrak{K}_\Delta(X)$ is a complete lattice. The zero and unity of this lattice are the sets Θ and X, respectively. Applying elementary rules of operating polars and making use of distributivity of set-theoretic operations for arbitrary components K, L, M, can can write the following chain of equalities:

$$(K \vee L) \wedge M = \left((K \vee L)^{\perp} \cup M^{\perp}\right)^{\perp} = \left((K^{\perp} \cap L^{\perp}) \cup M^{\perp}\right)^{\perp}$$
$$= \left((K^{\perp} \cup M^{\perp}) \cap (L^{\perp} \cup M^{\perp})\right)^{\perp} = \left[(K^{\perp} \cup M^{\perp})^{\perp} \cup (L^{\perp} \cup M^{\perp})^{\perp}\right]$$
$$= (K^{\perp\perp} \cap M^{\perp\perp}) \vee (L^{\perp\perp} \cap M^{\perp\perp}) = (K \wedge M) \vee (L \wedge M).$$

Hence, the lattice $\mathfrak{K}_{\Delta}(X)$ is distributive. Obviously, $K \cap K^{\perp} = \Theta$. On the other hand,

$$K \vee K^{\perp} = \left[K \cup K^{\perp}\right] = (K^{\perp} \wedge K)^{\perp} = \Theta^{\perp} = X,$$

i.e., $K^{\perp}$ is the complement of K in the lattice $\mathfrak{K}_{\Delta}(X)$. ▷

4.1.13. Let us consider a set X with fixed disjointness Δ. Let j be an isomorphism of $\mathfrak{K}_{\Delta}(X)$ onto a complete Boolean algebra B. Let us introduce a mapping $s: X \to B$ by the formula $s(x) := j([x])$ $(x \in X)$. Let us assume that the least component is a singleton, i.e., $\Theta := \{\theta\} = [\theta]$ for some $\theta \in X$. We say that the B-metric d and disjointness Δ on the set X *are consistent* provided

$$d(x, \theta) = s(x) \quad (x \in X).$$

Let us consider another mapping

$$\delta: (x, y) \to (s(x) \wedge s(y))^{*} \quad (x, y \in X).$$

Theorem. *Let a set X be furnished with disjointness and a B-metric d consistent with it. Then the triplet $\mathfrak{X} := (X, \delta, \theta)$ is an algebraic B-system on which the axioms of simple disjointness (a) - (d) of* **4.1.12** (2) *are fulfilled.*

◁ First of all observe that

$$d(x,y)^{*} \wedge s(x) = d(x,y)^{*} \wedge d(x,\theta)$$
$$\leq d(x,y)^{*} \wedge (d(x,y) \vee d(y,\theta)) \leq d(y,\theta) = s(y).$$

Now this is obvious that s is a contraction mapping. Therefore, the mapping δ will be also a contraction and, hence, $\mathfrak{X}$ is an algebraic B-system with a binary predicate δ and a distinguished element θ. By definition **4.1.8**, we get

$$|x \delta y|^{\mathfrak{X}} = \delta(x,y), \quad |x \neq \theta|^{\mathfrak{X}} = s(x) \quad (x, y \in X).$$

Let us check the axioms of disjointness for δ. Obviously, δ is symmetric. The fact that $\{\theta\}$ is the least component is evident from the following relations:

$$|x \in \pi_\delta(X) \to x = \theta|^{\lambda} = (\bigwedge_{y \in X} \delta(x,y)) \Rightarrow s(x^*)$$
$$= \bigvee_{y \in X} (s(x) \wedge s(y)) \vee s(x)^* \vee \bigvee_{y \in X} s(y) = 1.$$

It is also obvious that for $x, y \in X$ we have

$$\delta(x,x) = |x\delta y|^{\lambda} = s(x^*) = |x = \theta|^{\lambda}.$$

Therefore, condition (b) of the definition of disjointness is fulfilled. It should then be observed that

$$|u \in [x]|^{\lambda} = s(u) \Rightarrow s(x) \quad (x, u \in X).$$

On the basis of this fact, we calculate:

$$|[x] \cap [y] = \{\theta\}|^{\lambda} = \Big(\bigwedge_{u \in X} (s(u) \Rightarrow s(x)) \wedge s(u) \Rightarrow s(y))\Big)$$
$$\Rightarrow s(u)^* = \bigwedge_{u \in X} s(u)^* \vee (s(x) \wedge s(y))^* = \delta(x,y).$$

Therefore, $|[x] \cap [y] = \{\theta\} \to x\delta y|^{\lambda} = 1$ and δ is a disjointness. The fact that δ is simple implies that for any $x, y \in X$ we have

$$|x\delta y \to x = \theta \vee y = \theta|^{\lambda} = 1,$$

or, which is equivalent,

$$\delta(x,y) \Rightarrow s(x)^* \vee s(y)^* = 1,$$

the last resulting from the definition of δ. ▷

Let $\mathfrak{A} := (A, v)$ be an algebraic B-system, while Δ be the same as in **4.1.13**. Assume that all the operations of the system $\mathfrak{A}$ preserve disjointness, i.e., for any functional symbol f and any elements $a \in A$, $x_0, \dots, x_{n-1} \in A$ $(n := \mathfrak{a}(f))$ the relations $x_k \perp a (k := 0, 1, \dots, n-1)$ yield $f^v(x_0, \dots, x_{n-1}) \perp a$. If, moreover, the B-metric and disjointness Δ are consistent, then the triplet (A, v, Δ) is called an *algebraic B-system with disjointness*.

4.1.14. Remarks

(**1**) In our proving the Stone theorem **1.2.4**, we find that a Boolean algebra B is isomorphic to the algebra of continuous functions $C(\mathfrak{C}(B), \mathbf{2})$, where $\mathfrak{C}(B)$ is an extremally disconnected compact space. In this statement one can try to substitute the two-element field **2** with an arbitrary universal algebra. This path leads us to an important example of an algebraic B-system, the Boolean power of a universal algebra introduced by P.F.Arens and I.Kaplanski [6] (see also [54, 55]).

(**2**) In the sequel we shall, as in the case of the present section, discuss only the problems pertaining to Boolean-valued realization of algebraic B-systems, to the specific methods of descents and ascents in the situation under consideration. Logical-algebraic aspects of algebraic B-systems are discussed in more detail elsewhere [8, 57].

4.2. Descents of Algebraic Systems

In the present section the operation of a descent is extended onto general algebraic systems, followed by some concrete examples.

4.2.1. Let $\sigma := (F, P, \mathfrak{a})$ be a signature. It follows from the general properties of the canonical embedding of sets $\mathbf{V}$ into the universe $\mathbf{V}^{(B)}$ (see **3.1.6, 3.1.9**) that $\mathbf{V}^{(B)} \models$ '$\mathfrak{a}^{\wedge}$ is a mapping from $F^{\wedge} \cup P^{\wedge}$ into the set of natural numbers'. Besides, $\mathbf{V}^{(B)} \models \sigma^{\wedge} = (F^{\wedge}, \Gamma^{\wedge}, \mathfrak{a}^{\wedge})$ and, hence,

$$\mathbf{V}^{(B)} \models \text{'}\sigma^{\wedge} \text{ is a signature'}.$$

If σ is a signature inside $\mathbf{V}^{(B)}$, then $\sigma\downarrow$ is not, in general, a signature in the conventional sense of the word. Indeed, let $\sigma = (F, P, \mathfrak{a})^B \in \mathbf{V}^{(B)}$ for certain $F, P, \mathfrak{a}^B \in \mathbf{V}^{(B)}$, in which case $[\mathfrak{a} : F \cup P \to \omega^{\wedge}] = \mathbf{1}$. Then, for every $u \in F\downarrow \cup P\downarrow$, we can find a countable partition of unity $(b_n)_{n \in \omega} \subset B$ such that $\mathfrak{a}\downarrow(u) = \operatorname{mix}(b_n n^{\wedge})$. Therefore, under the descent of a system of arbitrary signature functional and predicate symbols of a 'mixed arity' arise. It goes without saying that it is possible to study a more general case of operations and predicates of a mixed arity, no principal difficulties arising on this way. Another direction of generalization is associated with algebraic systems having operations and predicates of infinite arity. These problems, however, will not be a subject for the discussions to follow.

4.2.2. Before giving general definitions, let us consider the descent of a very simple but important algebraic system, the two-element Boolean algebra. Let us choose two arbitrary elements, $\mathbf{0},\mathbf{1}\in\mathbf{V}^{(B)}$, for which $[0\neq 1]=\mathbf{1}_B$. We can for instance, assume that $0:=\mathbf{0}_B{}^{\wedge}$ and $1:=\mathbf{1}_B{}^{\wedge}$.

The descent C of the two-element Boolean algebra $\{0,1\}^B\in\mathbf{V}^{(B)}$ is a complete Boolean algebra isomorphic to B. The isomorphism $\chi\colon B\to C$ can be chosen in such a way that

$$[\chi(b)=1]=b,\quad [\chi(b)=0]=b^*\quad (b\in B).$$

$\lhd$ Since $0,1\in C$, for every $b\in B$ the mixing $c:=\operatorname{mix}(b1,b^*0)$ also belongs to C, in which case $[c=1]\geq b$ and $[c=0]\geq b^*$. On the other hand,

$$[c=1]\wedge[c=0]=[c=1\wedge c=0]\leq[0=1]=\mathbf{0}_B,$$

and, hence, $[c=1]=b$ and $[c=0]=b^*$. Setting $\chi(b):=c$, we get a mapping $\chi\colon B\to C$. It is obvious that χ is injective. Now let us check that χ is surjective. Indeed, if $c\in C$ then for $b:=[c=1]$ we have

$$[\chi(b)=0]=b^*=[c=0],\quad [\chi(b)=1]=b,$$

and, hence,

$$[\chi(b)=c]\geq[\chi(b)=1]\wedge[c=1]=b.$$

Analogously, $[\chi(b)=c]\geq b^*$ and, hence, $\chi(b)=c$.

Let us now implement the descent of the Boolean operations of $\{0,1\}^{(B)}$. In this case for any $x,y,z\in C$ the following equalities are valid:

$$z=x\wedge y\leftrightarrow[z=1\leftrightarrow x=1\wedge y=1]=\mathbf{1},$$
$$z=x\vee y\leftrightarrow[z=0\leftrightarrow x=0\wedge y=0]=\mathbf{1},$$
$$z=y^*\leftrightarrow[x=1\leftrightarrow y=0]=\mathbf{1}.$$

Using these relations, we can easily prove that C is a Boolean algebra, while χ is a Boolean isomorphism. Let us, for instance, show that χ preserves the least upper bounds of any pair of elements. Let $b_1,b_2\in B$, $b_0:=b_1\vee b_2$ and $c_l:=\chi(b_l)$ for $l=0,1,2$. Then, by definition, we get

$$[c_l=1]=b_l,\quad [c_l=0]=b_l^*\quad (l=0,1,2),$$

and, hence,

$$[c_0 = 0] = b_0^* = b_1^* \wedge b_2^* = [c_1 = 0] \wedge [c_2 = 0],$$

or, which is the same, $[c_0 = 0 \leftrightarrow c_1 = 0 \wedge c_2 = 0] = 1$. Therefore, $c_0 = c_1 \vee c_2$, or $\chi(b_0) = \chi(b_1) \vee \chi(b_2)$. In an analogous way we can deduce that the greatest lower bounds and complements are preserved, too. ▷

4.2.3. Let us now consider an algebraic system $\mathfrak{A}$ of signature $\sigma^\wedge$ inside $\mathbf{V}^{(B)}$, and let $[\mathfrak{A} = (A,v)^B] = 1$ for some $A,v \in \mathbf{V}^{(B)}$. The descent of the algebraic system is the pair $\mathfrak{A}\downarrow := (A\downarrow,\mu)$, where μ is a function determined by the relations

$$\mu: f \to (v \Downarrow (f))\downarrow \quad (f \in F),$$
$$\mu: p \to \chi^{-1} \circ (v \Downarrow (p))\downarrow \quad (p \in P).$$

Here χ is the isomorphism of the Boolean algebras B and $\{0,1\}^B$ determined in **4.2.2.**

In more detail, the modified descent $v \Downarrow$ is a mapping with the domain of definition $\mathrm{dom}(v\Downarrow) = F \cup P$. For every $f \in F$ we have $[\mathfrak{a}(f)^\wedge = a^\wedge(f^\wedge)] = 1$, $[v \Downarrow (f) = v(f^\wedge)] = 1$ and, hence,

$$\mathbf{V}^{(B)} \models v \Downarrow (p): A^{\mathfrak{a}(f)^\wedge} \to \{0,1\}^B.$$

It is now obvious that $(v \Downarrow (p))\downarrow: A^{\mathfrak{a}(f)} \to \{0,1\}^B \downarrow$ and we can put $\mu(p) := \chi^{-1} \circ (v \downarrow (p))\downarrow$.

Let $\varphi(x_0,\dots,x_{n-1})$ be a fixed formula of signature σ in n free variables. Let us construct a formula $\Phi(x_0,\dots,x_{n-1},\mathfrak{A})$ in the language of set theory which formalizes the statement $\mathfrak{A} \models \varphi(x_0,\dots,x_{n-1})$. It should be recalled that the relation $\mathfrak{A} \models \varphi(x_0,\dots,x_{n-1})$ determines an n-ary predicate in A or, which is equivalent, a mapping from A^n to $\{0,1\}$. Then, by the maximum and transfer principles, there is a unique element $|\varphi|^{\mathfrak{A}} \in \mathbf{V}^{(B)}$ such that

$$[|\varphi|^{\mathfrak{A}} : A^{n^\wedge} \to \{0,1\}^B] = 1,$$
$$[|\varphi|^{\mathfrak{A}}(a\uparrow) = 1] = [\Phi(a(0),\dots,a(n-1),\mathfrak{A})] = 1$$

for every $a: n \to A\downarrow$. Henceforth instead of $|\varphi|^{\mathfrak{A}}(a\uparrow)$ we shall write $|\varphi|^{\mathfrak{A}}(a_0,\dots,a_{n-1})$, where $a_l := a(l)$. Therefore, the relation

$$\mathbf{V}^{(B)} \models \text{`}\varphi(a_0,\dots,a_{n-1}) \text{ holds in the model'}$$

is valid iff $[\Phi(a_0,\dots,a_{n-1},a)] = \mathbf{1}$.

4.2.4. Theorem. *Let $\mathfrak{A}$ be an algebraic system of signature $\sigma^\wedge$ inside $\mathbf{V}^{(B)}$. Then $\mathfrak{A}\downarrow$ is an extended algebraic B-system of signature σ. In this case for any formula φ of signature σ we have*

$$\chi \circ |\varphi|^{\mathfrak{A}\downarrow} = |\varphi|^{\mathfrak{A}}\downarrow.$$

$\triangleleft$ As we already know, $A\downarrow$ is an extended B-set. Then, the modified descent v' of the element $v \in \mathbf{V}^{(B)}$ is a mapping, in which case $\mathrm{dom}(v') = F \cup P$ (see **3.5.5** (3)). Besides,

$$[v'(f): A^{\mathfrak{a}(f)^\wedge} \to A] = \mathbf{1} \quad (f \in F),$$
$$[v'(p): A^{\mathfrak{a}(p)^\wedge} \to \{0,1\}] = \mathbf{1} \quad (p \in P).$$

The above relations show $v'(f)\downarrow$ and $v'(p)\downarrow$ to be contraction mappings from $A^{\mathfrak{a}(f)}$ to A and from $A^{\mathfrak{a}(p)}$ to $C := \{0,1\}^B\downarrow$, respectively. Hence, $(A\downarrow,\mu)$ is an extended algebraic B-system. Let now φ be a formula of signature σ and let us show that

$$[|\varphi|^{\mathfrak{A}}(a_0,\dots,a_{n-1}) = 1] = |\varphi|^{\mathfrak{A}\downarrow}(a_0,\dots,a_{n-1})$$

for any $a_0,\dots,a_{n-1} \in A\downarrow$. Use induction on the complexity of the formula φ. At first let φ be an atomic formula. If $q \in P$ and $\mathfrak{a}(q) = 0$, then $[v(q^\wedge) = 0 \vee v(q^\wedge) = 1] = \mathbf{1}$, so that $v'(q) \in C$ and $\mu(q) = \chi^{-1}(v'(q)) \in B$. According to **4.2.2**, $\mu(q) = [\chi \circ \mu(q) = 1] = [1 = v(q^\wedge)]$. Now consider the terms $t_0,\dots,t_{m-1}$ of signature σ assuming the values $b_0,\dots,b_{m-1}$ for the values $a_0,\dots,a_{n-1}$ of the variables $x_0,\dots,x_{n-1}$. Let $p \in P$ and $\mathfrak{a}(p) = m$. If $\varphi(x_0,\dots,x_{n-1}) := p(t_0,\dots,t_{m-1})$, then

$$[|\varphi|^{\mathfrak{A}}(a_0,\dots,a_{n-1}) = 1] = [v\downarrow(p)(b_0,\dots,b_{m-1}) = \mathbf{1}]$$
$$[\chi \circ p^{\mu}(b_0,\dots,b_{m-1}) = 1] = p^{\mu}(b_0,\dots,b_{m-1}).$$

Whereas if $\varphi(x_0,\dots,x_{n-1}) := (t_0(x_0,\dots,x_{n-1}) = t_1(x_0,\dots,x_{n-1}))$, then

$$[|\varphi|^{\mathfrak{A}}(a_0,\dots,a_{n-1}) = 1] = [b_0 = b_1] = d(b_0,b_1)^*.$$

Let us now assume that φ_1 and φ_2 have the forms $\varphi \wedge \psi$ and $(\forall x_0)\varphi$, respectively. In this case

$$[|\varphi_1|^{\mathfrak{A}}(a_0,\ldots,a_{n-1})=1]=[|\varphi|^{\mathfrak{A}}(a_0,\ldots,a_{n-1})=1\wedge\psi|^{\mathfrak{A}}(a_0,\ldots,a_{n-1})=1]$$
$$=[|\varphi|^{\mathfrak{A}}(a_0,\ldots,a_{n-1})=1]\wedge[|\psi|^{\mathfrak{A}}(a_0,\ldots,a_{n-1})=1]=|\varphi_1|^{\mathfrak{A}\downarrow}(a_0,\ldots,a_{n-1})=1;$$
$$[|\varphi_2|^{\mathfrak{A}}(a_0,\ldots,a_{n-1})=1]=[(\forall x_0\in A)|\varphi|^{\mathfrak{A}}(a_0,\ldots,a_{n-1})=1]$$
$$=\bigwedge_{a_0\in A\downarrow}[|\varphi|^{\mathfrak{A}}(a_0,\ldots,a_{n-1})=1]=|\varphi_2|^{\mathfrak{A}\downarrow}(a_0,\ldots,a_{n-1}).$$

The case of the universal quantifier and the remaining propositional connectives is considered analogously. ▷

4.2.5. Theorem. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be algebraic systems of the same signature $\sigma^\wedge$ inside $\mathbf{V}^{(B)}$. Assign $\mathfrak{A}':=\mathfrak{A}\downarrow$ and $\mathfrak{B}':=\mathfrak{B}\downarrow$. Then, if h is a homomorphism (strong homomorphism) inside $\mathbf{V}^{(B)}$ from the system $\mathfrak{A}$ to the system $\mathfrak{B}$, then $h':=h\downarrow$ is a homomorphism (strong homomorphism) of the B-systems $\mathfrak{A}'$ and $\mathfrak{B}'$. Conversely, if $h':\mathfrak{A}'\to\mathfrak{B}'$ is a homomorphism (strong homomorphism) of algebraic B-systems, then $h:=h'\uparrow$ is a homomorphism (strong homomorphism) inside $\mathbf{V}^{(B)}$ from the system $\mathfrak{A}$ to the system $\mathfrak{B}$.*

◁ Let us confine ourselves to substantiating **4.1.10** (3) of the definition of homomorphism, i.e., let us consider only the case of a non-nullary functional symbol, the considerations for other symbols of signature σ being analogous. Let $\mathfrak{A}:=(A,v)^B$ for some $A,v\in\mathbf{V}^{(B)}$ and $\mathfrak{A}'=(A',v')$. Let us assume that $\mu\in\mathbf{V}^{(B)}$ and $\mu'\in\mathbf{V}$ be interpreting mappings of the systems $\mathfrak{B}$ and $\mathfrak{B}'$, respectively. Consider a functional symbol f of the arity $n=\mathfrak{a}(f)$ and elements $a_0,\ldots,a_{n-1}\in A'$. As before, the presentation $t=g(a_0,\ldots,a_{n-1})$ for $g\in\mathbf{V}^{(B)}$ will denote the formula $t=g(a)$, where $a\in\mathbf{V}^{(B)}$ is such an element from $\mathbf{V}^{(B)}$ that $[a:n^\wedge\to A]=\mathbf{1}$ and $a\downarrow(l)=a_l$ $(l<n)$. If $h\in\mathbf{V}^{(B)}$ is a homomorphism inside $\mathbf{V}^{(B)}$ from $\mathfrak{A}$ to $\mathfrak{B}$, then

$$[h(v(f^\wedge)(a_0,\ldots,a_{n-1}))=\mu(f^\wedge)(h(a_0),\ldots,h(a_{n-1}))]=\mathbf{1}.$$

Besides, in accord with the definition of descents (see **3.5.5** (3)),

$$[v(f^\wedge)=v\downarrow(f)]=[\mu(f^\wedge)=\mu\downarrow(f)]=\mathbf{1};$$
$$[v\downarrow(f)(a_0,\ldots,a_{n-1})=v'(f)(a_0,\ldots,a_{n-1})]=\mathbf{1};$$
$$[\mu\downarrow(f)(b_0,\ldots,b_{n-1})=\mu'(f)(b_0,\ldots,b_{n-1})]=\mathbf{1};$$
$$[h(t)=h'(t)]=\mathbf{1}\quad(t\in A').$$

Summing up all the above relations and bearing in mind that $\mathbf{V}^{(B)}$ is separated, we get

$$h'(v'(f)(a_0,\dots,a_{n-1})) = \mu'(f)(h(a_0),\dots,h(a_{n-1})).$$

Assume, conversely, that the last equality holds. By replacing in it h' with $h := h'\uparrow$, we get a formula which is assertive inside $\mathbf{V}^{(B)}$. Substituring in it $v\downarrow(f)$ for $v'(f)$ and $v(f^\wedge)$ for $v\downarrow(f)$, and then $\mu\downarrow(f)$ for $\mu'(f)$ and $\mu(f^\wedge)$ for $\mu\downarrow(f)$, we get a new formula assertive inside $\mathbf{V}^{(B)}$. It is this new formula that is the required property inside $\mathbf{V}^{(B)}$. ▷

Corollary. *In terms of theorem* **4.2.5** [*h is a homomorphism between the algebraic systems* $\mathfrak{A}$ *and* $\mathfrak{B}$ |] = **1** *iff* h' *is an isomorphism between the algebraic* B*-systems* $\mathfrak{A}'$ *and* $\mathfrak{B}'$.

4.2.6. As observed in **4.1.3**, an extended algebraic B-system $\mathfrak{A} := (A, v)$ can be viewed as a conventional (i.e., {0,1}-valued) algebraic system $\mathfrak{A}' := (A, v')$ of the same signature provided the B-valued predicates p^v are replaced with the sets $v'(p) := \{(x_0,\dots,x_{n-1}) \in A^n : p^v(x) = \mathbf{1}\}$. This, however, does not at all mean that if $\mathfrak{A}$ is a B-model of the formula φ of signature $\sigma(\mathfrak{A})$, then $\mathfrak{A}'$ is a $\{\mathbf{0},\mathbf{1}\}$-valued model, i.e., a model in the conventional sense of the word for the same formula φ. Moreover, for some formulas this is exactly the case. This problem will be considered in more detail in the section to follow, while here we shall confine ourselves to some concrete examples of algebraic B-systems obtained with the help of descent. If the formula φ is the conjunction of the axioms of a group , a ring, a module, etc., and the algebraic system $\mathfrak{A}$ is a two-valued model for φ, then, as usual, $\mathfrak{A}$ is said to be a group, a ring, a module, etc.. Whereas if $\mathfrak{A}$ is a B-model for φ, then $\mathfrak{A}$ is said to be a B-group, a B-ring, a B-module, etc..

Let us consider an arbitrary group G. An endomorphism $\pi : G \to G$ is termed a *projection* if $\pi \circ \pi = \pi$. We say that $\mathbf{B}$ is a Boolean algebra of projections in the group G, if $\mathbf{B}$ consists of mutually commuting projections in G and forms a Boolean algebra with zero, $\mathbf{0}_{\mathbf{B}} := 0$, and unit, $\mathbf{1}_{\mathbf{B}} := I_G$ if furnished with:

$$\pi_1 \vee \pi_2 := \pi_1 + \pi_2 - \pi_1 \circ \pi_2, \quad \pi_1 \wedge \pi_2 := \pi_1 \circ \pi_2, \pi^* := 1 - \pi$$
$$(\pi_1, \pi_2, \pi \in \mathbf{B}).$$

The order in $\mathbf{B}$ is such that $\pi_1 \le \pi_2$ iff $\pi_1(G) \subset \pi_2(G)$. A group G with a Boolean algebra of projections $\mathbf{B}$ will be called *extended* if $\mathbf{B}$ is order-complete and for any partition of unity $(\pi_\xi) \subset \mathbf{B}$ there is a unique element $x \in G$ such that $\pi_\xi x_\xi = \pi_\xi x$ for all ξ. Let $(G, \mathbf{B})$ and

$(G',\mathbb{B}')$ be groups with Boolean algebras of projections (or, to put it short, groups with projections). A group isomorphism $h: G \to G'$ is called a *homomorphism of groups with projections*, if there is a Boolean isomorphism $j: \mathbb{B} \to \mathbb{B}'$ such that $h \circ \pi = j(\pi) \circ h$ for all $\pi \in \mathbb{B}$.

Let now R be a ring, and let the additive group of this ring have a Boolean algebra of projections $\mathbb{B}$. If, moreover, each projection $\pi \in \mathbb{B}$ is a ring homomorphism, then $(R,\mathbb{B})$ is said to be a *ring with a Boolean algebra of projections*, or a *ring with projections*. The projection $[x] := \wedge\{\pi \in \mathbb{B} : \pi x = x\}$ will be termed the *carrier of an element* $x \in R$.. It is obvious that if the carriers $[x]$ and $[y]$ are disjoint (as elements of the Boolean algebra $\mathbb{B}$), then $x \cdot y = 0$, the converse statement being, generally speaking, invalid. If $x \cdot y = 0$, then x and y are said to *be orthogonal.* An element is called *regular* if it is orthogonal only to the zero element. A *zero divisor* is any element orthogonal to a nonzero element.

Let S be a multiplicative subset of a ring with unity K, i.e., $1 \in S$ and $xy \in S$ for any x and $y \in S$. In the set $K \times S$ let us introduce the equivalence relation, by letting

$$(x,s) \sim (x',s') \leftrightarrow (\exists t \in S) \quad (t(sx' - s'x) = 0).$$

Let $S^{-1}K := K \times S/\sim$, and $(x,s) \to x/s$ be the canonical factor-mapping. The set $S^{-1}K$ can be endowed with a ring structure using the equalities

$$(x/s) + (y/t) := (tx + sy)/st, \quad (x/s)(y/t) := (xy)/(st).$$

The mapping $x \to x/1$, $x \in K$, is a homomorphism from K to $S^{-1}K$ called *canonical*. The ring $S^{-1}K$ is termed the *ring of fractions of K relative to S*.

4.2.7. Theorem. *Let $\mathcal{G}$ be a group inside $\mathbf{V}^{(B)}$ and $G := \mathcal{G}\downarrow$. Then G is a group and in it there is a complete Boolean algebra of projections $\mathbb{B}$ and an isomorphism $j: B \xrightarrow{on} \mathbb{B}$ such that*

$$b \le [x = 0] \leftrightarrow j(b)x = 0 \quad (x \in G, b \in B).$$

Moreover, $(G,\mathbb{B})$ is an extended group with projections, and the following equivalences hold:

(1) $\mathbf{V}^{(B)} \models$ *'$\mathcal{G}$ is commutative' $\leftrightarrow$ 'G is commutative';*

(2) $\mathbf{V}^{(B)} \models$ *'$\mathcal{G}$ is torsion-free' $\leftrightarrow$ 'G is torsion-free'.*

⊲ By theorem **4.2.4**, $\mathcal{G}\downarrow$ is an extended algebraic B-system, being a B-group. Let us determine the descent of addition + by the same symbol. Let us show that G is a group, limiting ourselves to the case when there are inverse elements. Let $\varphi := (\forall x)(\exists! y)(x + y = 0)$. Then, according to **4.1.8**,

$$|\varphi|^G := \bigwedge_{x\in G} \bigvee_{y\in G} |x + y = 0|^G = \mathbf{1}.$$

Since the B-set G is extended, for every $x \in G$ there is a $y \in G$ such that

$$\mathbf{1} = |x + y = 0|^G = d(x + y, 0)^* = [x + y = 0],$$

and, hence, $x + y = 0$. If $x + z = 0$ for a certain $z \in G$, then $|x + z = 0|^G = \mathbf{1}$. As G is a B-group, we have

$$\mathbf{1} = |x + y = 0 \wedge x + z = 0|^G \Rightarrow |y = z|^G,$$

and, hence, $|y = z|^G = [z = y] = \mathbf{1}$ and $z = y$.

The congruences of the group G are exactly the equivalences determined by its different normal subgroups. Therefore, by virtue of theorem **4.16**, there is an isomorphism j from B on a certain complete Boolean algebra $\mathfrak{B}'$ of normal subgroups of the group G such that

$$b \le [x = 0] \leftrightarrow x \in j(b^*) \quad (b \in B, x \in G).$$

If $b \in B$, then $j(b) \cap j(b^*) = 0$. On the other hand, for every $x \in G$ there are $x_1 := \operatorname{mix}\{bx, b^* 0\}$, $x_2 := \operatorname{mix}\{b^* x, b0\}$ and, since $b^* \le [x_1 = 0]$, $b \le [x_2 = 0]$, we have $x_1 \in j(b)$, $x_2 \in j(b^*)$. Besides, $[x = x_1 + x_2] \ge [x_1 = x] \wedge [x_2 = 0] \ge b$ and $[x = x_1 + x_2] \ge [x_1 = 0] \wedge [x_2 = x] \ge b^*$, which gives $x = x_1 + x_2$. Therefore, any subgroup of the type $j(b)$ is singled out into a direct addent and is corresponded to by the operator of projecting π_b on $j(b)$ along the complementary subgroup $j(b^*)$. To be more exact, π_b is determined by the conditions: $\pi_b x = x$ for all $x \in j(b)$ and $\pi_b x = 0$ for $x \in j(b^*)$. Let the same letter j denote the isomorphism $b \to \pi_b, b \in B$ and put $\mathfrak{B} := j(B)$. Obviously, $\mathfrak{B}$ and j obey the required conditions. The fact that the group G is extended is equivalent to the fact the corresponding B-set is also extended, as $x = \operatorname{mix}(b_\xi x_\xi)$ iff $j(b_\xi)x = j(b_\xi)x_\xi$ for all ξ.

Let us assume that $\mathcal{G}$ is torsion-free. Then

$$[(\exists x \in \mathcal{G})(\exists n \in \omega^\wedge)(nx = 0) \wedge (0 \ne x) \wedge (0 < n)] = \mathbf{1},$$

and, hence, there is an element $0 \ne x \in G$ and a partition of unity $(b_n)_{n\in\omega}$ in B such that $b_n \le [n^\wedge x = 0]$ for all $n \in \omega$. It should be observed that $[n^\wedge x = nx] = \mathbf{1}$ and, hence,

$b_n \leq [x \neq 0]$, $b_n \leq [nx = 0]$ and $j(b_n)(nx) = nj(b_n)x = 0$. For at least one $0 \neq n \in \omega$ the projection $j(b_n)$ is non-zero which implies that G is not torsion-free. Conversely, if $nx = 0$ for some $0 \neq x \in G$ and $n \in \omega$, then $[(\exists n \in \omega^{\wedge})(nx = 0) \wedge (n > 0)] = \mathbf{1}$, i.e., [$\mathcal{G}$ is not free-torsion] $= \mathbf{1}$. The statement referring to commutativity is obvious. ▷

4.2.8. Theorem. *Let $\mathfrak{R}$ be a ring inside $\mathbf{V}^{(B)}$ and $K := \mathfrak{R}\downarrow$. Then $\mathfrak{R}$ is an extended ring with Boolean algebra of projections $\mathbb{B}$ and there is an isomorphism $j: K \xrightarrow{on} \mathbb{B}$ such that*

$$b \leq [x = 0] \leftrightarrow j(b)x = 0 \quad (x \in K, b \in B).$$

In this case the following equivalences are valid:

(1) $\mathbf{V}^{(B)} \models$ '$\mathfrak{R}$ *is commutative (semiprimitive)'* $\leftrightarrow$ *'K is commutative (semiprimitive)'* ;

(2) $\mathbf{V}^{(B)} \models$ '$\mathfrak{R}$ *has no zero divisors'* $\leftrightarrow$ *'any two elements of K are orthogonal only if their carriers are disjoint'*;

(3) $\mathbf{V}^{(B)} \models \mathfrak{C}$ *is a multiplicative subset of* $\mathfrak{R}$' $\leftrightarrow$ '$S := \mathfrak{C}\downarrow$ *is a multiplicative subset in K', in which case* $(\mathfrak{C}^{-1}K)\downarrow \simeq S^{-1}K$ *(here $\simeq$ denotes a ring isomorphism)*;

(4) $\mathbf{V}^{(B)} \models$ '$\mathfrak{R}$ *is a field'* $\leftrightarrow$ *'K is semiprimitive, the orthogonality of the elements of K is equivalent to disjointness of their carriers and any regular element in it is invertible'*;

(5) $\mathbf{V}^{(B)} \models$ '$\mathbf{R}$ *is the radical of a ring with unity* $\mathfrak{R}$' $\leftrightarrow$ '$\mathfrak{R}\downarrow$ *is the radical of the ring with unity* $\mathfrak{R}$' *; in other words, if* $\mathfrak{R}$ *has unity, then* $\mathfrak{R}(\mathfrak{R})\downarrow = \mathbf{R}(K)$;

(6) $\mathbf{V}^{(B)} \models$ '$(\mathfrak{R}, \mathfrak{D})$ *is a ring with projections'* $\leftrightarrow$ *'the mapping* $\pi \to \pi\downarrow$ $(\pi \in \mathfrak{D}\downarrow)$ *is an isomorphism* $\mathfrak{D}\downarrow$ *on a certain Boolean algebra of projections D to K, in which case* $\mathbb{B}$ *is a regular subalgebra in D, and* (K, D) *is a ring with projections'.*

◁ According to theorem **4.2.7**, K is an extended group with projections, and there is an isomorphism j from B onto the complete Boolean algebra $\mathbb{B}$ of additive projections obeying the necessary condition. Let us supply K with the operation of multiplication in accordance with the general definition **4.2.3**, i.e., for elements $x, y \in K$, we have $[x, y \in \mathfrak{R}] = \mathbf{1}$ and, hence, in the model $\mathbf{V}^{(B)}$ there is a product z of these elements: $[z \in \mathfrak{R}] = [z = x \cdot y] = \mathbf{1}$. Let us assume z to be the product of x and y in K. Therefore,

$$z = x \cdot y \leftrightarrow [z = x \cdot y] = \mathbf{1} \quad (x, y, z \in K).$$

The fact that in this case we get a ring is easily deduced using theorem **4.2.4**. Let us choose an arbitrary element $b \in B$ and show that the projection $j(b)$ is a ring homomorphism. Indeed, the operation of multiplication in K is the descent of the corresponding operation in $\mathcal{R}$ and, hence, extensional, and thus it preserves mixing. Therefore, by the definition of the projection $j(b)$ (see **4.2.7**), for any $x, y \in K$ we get

$$j(b)xy = \operatorname{mix}\{bxy, b*0\} =$$
$$= \operatorname{mix}\{bx, b*0\} \cdot \operatorname{mix}\{by, b*0\} = j(b)x \cdot j(b)y.$$

Let us now turn our attention to proving statements (1) - (6).

(1) The proof is conducted by analogy with **4.2.7** (1).

(2) The statement $\mathbf{V}^{(B)} \models$ '$\mathcal{R}$ *has no zero divisors*' is equivalent to the fact that for any $x, y \in \mathcal{R}\downarrow$ we have $b: [xy = 0] = [x = 0] \vee [y = 0]$. If the latter relation is fulfilled and $xy = 0$, then $b = \mathbf{1}$ and, hence, for $e := [x = 0]$ and $c := [y = 0]$ we have $e^* \wedge c^* = 0$. Besides, $j(e^*)x = x$ and $j(c^*)y = y$ and, therefore, $[x] \leq j(e^*)$ and $[y] \leq j(c^*)$. The carriers $[x]$ and $[y]$ are thus seen to be disjoint. If, however, $[x] \circ [y] = 0$, then, as has been earlier remarked in **4.2.6**, $x \cdot y = 0$.

Conversely, assume that the equality $xy = 0$ is equivalent to the fact that the carriers $[x]$ and $[y]$ are disjoint. Then for $b := [xy = 0]$ the equalities $0 = j(b)xy = (j(b)x) \cdot (j(b)y)$ yield that the projections $\pi := [j(b)x]$ and $\rho := [j(b)y]$ are disjoint. It should be observed that $j(b) \circ \pi * x = 0$ and $j(b) \circ \rho * y = 0$ and, hence,

$$[x = 0] \vee [y = 0] \geq (b \wedge j^{-1}(\pi^*)) \vee (b \wedge j^{-1}(\rho^*)) = b.$$

(3) The statement concerning multiplication is evident. Let us prove that the descent of a ring of fractions is a ring of fractions. Let us first remark that $(\mathcal{T} \times K)\downarrow = S \times K$. Let us consider an equivalence relation $\mathcal{P} \in \mathbf{V}^{(B)}$ such that for $x, x' \in K$ and $s, s' \in S$ we have

$$\mathbf{V}^{(B)} \models (x, s)\mathcal{P}(x', s') \leftrightarrow (\exists t \in \mathcal{T})(t(sx' - s'x) = 0).$$

If $P := \mathcal{P}\downarrow$, then P is an equivalence relation in $K \times S$, in which case

$$(x, s)P(x', s') \leftrightarrow (\exists t \in S) \quad (t(sx' - s'x) = 0).$$

Then, the descent of the factor-set $\mathcal{T} \times \mathcal{R} / \mathcal{P}$ is bijective with the set $KS \times K / P$. And, finally, for $x, y \in K$ and $s, t \in S$ the equalities

$$(x/s) + (y/t) = (tx + sy)/st, \ (x/s)(y/t) = (xy/st)$$

are valid iff they are true inside $\mathbf{V}^{(B)}$. All we have to do now is to correlate the above-proved with the definition of a ring of fractions.

(4) Let us assume that $[\mathcal{R}$ is a field$] = \mathbf{1}$. In this case K is semiprimitive and $xy = 0$ yields that $[x] \circ [y] = 0$ for all $x, y \in K$ (see (1) and (2)). For any regular element $x \in K$ we get $j(b)xy = 0 \to j(b)y = 0$ whatever $b \in B$ and $y \in K$ are. In this case, however, $[xy = 0] \le [y = 0]$, i.e., $[x \ne 0] = \mathbf{1}$. Therefore, there is an element $u \in K$ such that $[xu = ux = 1] = \mathbf{1}$ and, hence, $xu = ux = 1$, i.e., x is invertible in the ring K. Conversely, let K be semiprimitive, any regular element in it be invertible and the orthogonality of the elements in K be equivalent to the disjointness of their carriers. Then $\mathbf{V}^{(B)} \models$ '$\mathcal{R}$ is a commutative ring', and, hence, $[\mathcal{R}$ is a field$] = [(\forall x)(x \in \mathcal{R} \wedge x \ne 0 \to$ 'x is invertible ')$] = \wedge\{[\exists x^{-1}] : x \in K \wedge \wedge [x \ne 0] = \mathbf{1}\}$. Therefore, it suffices to show that if $[x \ne 0] = \mathbf{1}$, then $[x$ is convertible $] = \mathbf{1}$, whatever an $x \in K$. Let us assume that $[x \ne 0] = \mathbf{1}$ and $xy = 0$ for some $y \in K$. Then for $\pi := [x]$ and $\rho = [y]$ we have $\pi \circ \rho = 0$. On the other hand, $j(b)x = 0$ implies $b \le [x = 0] = [x \ne 0]^* = \mathbf{1}^* = \mathbf{0}$, and, hence, $\rho = j(1) = I_K$. Therefore, we get $\pi \le \rho^* = 0$, or $y = 0$, and, hence the element x is invertible in the ring K. This immediately results in the relation $[x$ is invertible in $\mathcal{R}\,] = \mathbf{1}$.

(5) The element x is in the radical of a ring iff for any y the element $1 - yx$ is left-invertible. Now we have to remark that $1 - yx$ is left-invertible in K iff $[1 - yx$ is left-invertible in $\mathcal{R}\,] = \mathbf{1}$.

(6) If $[(\mathcal{R}, \mathcal{D})$ is a ring with projections $] = \mathbf{1}$ and $\pi \in \mathcal{D}\downarrow$, then, by **4.2.7**, $\pi\downarrow : K \to K$ is a homomorphism. On the other hand, $[\pi \circ \pi = \pi] = \mathbf{1}$ and, hence, $(\pi\downarrow) \circ (\pi\downarrow) = (\pi \circ \pi)\downarrow = \pi\downarrow$, i.e., $\pi\downarrow$ is a projection. The fact that D is a Boolean algebra will be established in **4.2.9**. Therefore, (K, D) is a ring with projections. By definition, $\mathfrak{B} = \{\pi\downarrow : \pi \in \{\mathbf{0}_{\mathcal{D}}, \mathbf{1}_{\mathcal{D}}\}^B\downarrow\}$ (see **4.2.7**) and, hence, $\mathfrak{B} \subset D$. The converse implication is established analogously. ▷

4.2.9. Theorem. *Let $\mathcal{D}$ be a complete Boolean algebra inside $\mathbf{V}^{(B)}$ and $D := \mathcal{D}\downarrow$. Then D is a complete Boolean algebra and there is a full monomorphism $\iota : B \to D$ such that*

$$b \le [x \le y] \leftrightarrow \iota(b)x \le \iota(b)y$$

for all $x, y \in D$ and $b \in B$.

◁ By virtue of **4.2.4**, D is an extended algebraic B-system of signature $(\vee, \wedge, *, 0, 1)$. The fact that D is a Boolean algebra also follows from **4.2.4**. Let us temporarily denote

Boolean operations in D by $\tilde{\vee}$, $\tilde{\wedge}$, and check distributivity, for instance. Let us consider the terms $t_1(x,y,z) := (x \wedge y) \vee z$, $t_2(x,y,z) := (x \vee z) \wedge (x \vee y)$ and the formula $\Psi := (\forall x,y,z)\varphi(x,y,z)$, where $\varphi(x,y,z) := (t_1(x,y,z) = t_2(x,y,z))$. In this case

$$[|\Psi|^{\mathfrak{D}} = 1] = |\Psi|^D = \bigwedge_{a,b,c \in D} |\varphi|^D(a,b,c) = \mathbf{1},$$

and, hence, $|\varphi|^D(a,b,c) = \mathbf{1}$ for all $a,b,c \in D$. Then

$$\mathbf{1} = |\varphi|^D(a,b,c) = d(t_1(a,b,c), t_2(a,b,c))^* \\ = [t_1(a,b,c) = t_2(a,b,c)] = [(a \tilde{\wedge} b) \tilde{\vee} c = (a \tilde{\vee} c) \tilde{\wedge} (b \tilde{\vee} c)].$$

Hence, since $\mathbf{V}^{(B)}$ is separated, we get $(a \tilde{\wedge} b) \tilde{\vee} c = (a \tilde{\vee} c) \tilde{\wedge} (b \tilde{\vee} c)$. In the same way we check the validity of the remaining axioms of Boolean algebras. Therefore, D is a Boolean algebra.

The completeness of D is not a first-order property and, hence, it cannot be deduced by using the above scheme. Let $\le \in \mathbf{V}^{(B)}$ be the conventional order relation in $\mathfrak{D}$, i.e.,

$$\mathbf{V}^{(B)} \models (\forall x \in \mathfrak{D})(\forall y \in \mathfrak{D})(x \le y \leftrightarrow x \wedge y = x).$$

Let us put $\tilde{\le} := (\le)\downarrow$. Then for $x,y \in D$ $x \tilde{\le} y$ is fulfilled iff $x \tilde{\wedge} y = x$. Consider a correspondence $\Phi := (\tilde{\le}, D, D)$. It is obvious that Φ is a full contraction. Then, if $A \subset D$, then $\pi_\Phi(A)$ ($\pi_\Phi^{-1}(A)$) is a set of all upper (lower, respectively) bounds of the set A (relative to the order $\tilde{\le}$). Therefore,

$$\{\sup(A)\} = \pi_\Phi(A) \cap \pi_\Phi^{-1}(\pi_\Phi(A)),$$

provided $\sup(A)$ exists. If $\Psi := (\le, \mathfrak{D}, \mathfrak{D})^B$, then Ψ is a correspondence inside $\mathbf{V}^{(B)}$ and $\Phi = \Psi\downarrow$. Since $\mathfrak{D}$ is complete, there is such an element $a \in D$ such that $[a = \sup(A)] = \mathbf{1}$ or $[\pi_\Psi(A) \cap \pi_\Psi^{-1}(\pi_\Psi(A)) = \{a\}] = \mathbf{1}$. Employing the rule for descending polars (see **3.2.13** (2)), we can easily calculate

$$\{a\} = [\pi_{\Psi^{-1}}(\pi_\Psi(A\uparrow)) \cap \pi_\Psi(A\uparrow)]\downarrow = \\ = \pi_{\Psi^{-1}}(\pi_\Psi(A\uparrow\downarrow)) \cap \pi_\Phi(A\uparrow\downarrow) = \sup(\mathrm{mix}(A)) = \sup(A).$$

Therefore, $a = \sup(A)$ and we have substantiated the fact that D is complete. Let $\lambda \in \mathbf{V}^{(B)}$ be the identical embedding of the algebra $\{\mathbf{0}_\mathfrak{D}, \mathbf{1}_\mathfrak{D}\}^B$ in $\mathfrak{D}$ inside $\mathbf{V}^{(B)}$. Let us put $\iota_1 = \lambda\downarrow$ and $\iota := \iota_1 \circ \iota_2$, where ι_2 is an isomorphism of B on $\{\mathbf{0}_\mathfrak{D}, \mathbf{1}_\mathfrak{D}\}^B\downarrow$. In this case ι is a monomorphism. The monomorphism ι is full since for $A \subset B$ we have

$\iota(\pi_{\Phi'}(A)) \subset \pi_{\Phi}(\iota(A))$, where $\Phi' := \iota^{-1} \circ \Phi \circ \iota$.

Then, by virtue of the obvious relation

$$\mathbf{V}^{(B)} \models (\forall x, y \in \mathfrak{D})(\forall c \in \{\mathbf{0}_{\mathfrak{D}}, \mathbf{1}_{\mathfrak{D}}\})(\lambda(c)x = \lambda(c)y \leftrightarrow (c = \mathbf{0}_{\mathfrak{D}}) \vee (c = \mathbf{1}_{\mathfrak{D}} \wedge x = y))$$

for any $x, y \in D$ and $b \in B$ we have

$$[\iota(b)x = \iota(b)y] = b^* \vee (b \wedge [x = y]).$$

Hence, we get

$$\iota(b)x = \iota(b)y \leftrightarrow b \leq [x = y],$$

and, therefore,

$$d(x,y)^* = [x = y] = \vee\{b \in B: \iota(b)x = \iota(b)y\}.$$

It is now evident that if $\varphi(x, y) := x \leq y$, then

$$|\varphi|^D(x,y) = \vee\{b \in B: \iota(b)x \leq \iota(b)y\},\ [|\varphi|^D(x,y) = 1] = [x \leq y],$$

which yields the equivalence in question. ▷

4.3. Immersion of Algebraic B-systems into a Boolean-Valued Universe

In the present section the immersion functor studied in **3.4** is extended onto the category of algebraic B-systems.

4.3.1. Let $\mathfrak{A} := (A, v)$ be an algebraic B-system of signature $\sigma := (F, P, \mathfrak{a})$. Let us consider a mapping $v': F \cup P \to \mathbf{V}^{(B)}$ operating by the rule

$$v': s \to v(s)^{\sim} := \mathfrak{F}^{\sim}(v(s)) \quad (s \in F \cup P),$$

where $\mathfrak{F}^{\sim}$ is the descent functor (see **3.4.12** - **3.4.16**). In accordance with the general definition of correspondences **3.4.13**, for every $f \in F, \mathfrak{a}(f) = n$ the mapping $\lambda'(f): (A^{\sim})^{n^{\wedge}} \to A^{\sim}$ inside $\mathbf{V}^{(B)}$ is governed by the relation

$$[v'(f)(\iota_A(x_0),\dots,\iota_A(x_{n-1})) = \iota_A \circ v(f)(x_0,\dots,x_{n-1})] = \mathbf{1},$$

where ι_A is the canonical embedding of A to $A' := A^{\sim}\downarrow$ (see **3.5.4**). Analogously, for $p \in P, \mathfrak{a}(p) = m$ the element $v'(p) \in \mathbf{V}^{(B)}$ is such a mapping from $(A^{\sim})^{m^{\wedge}}$ to $\{0,1\}^B \in \mathbf{V}^{(B)}$ that

$$[v'(p)(\iota_A(x_0),\dots,\iota_A(x_{m-1})) = \iota_B \circ v(p)(x_0,\dots,x_{m-1})] = \mathbf{1}.$$

The modified ascent $\mu := (v')\Uparrow$ of the mapping $v' : F \cup P \to \operatorname{im}(v')$ is seen to be an interpretation inside $\mathbf{V}^{(B)}$. The pair $(A^{\sim},\mu)$ or the element $(A^{\sim},\mu)^B \in \mathbf{V}^{(B)}$ is termed the *Boolean-valued realization of the algebraic B-system* $\mathfrak{A}$ and denoted by the symbol $\mathfrak{A}^{\sim}$.

4.3.2. Theorem. *For any algebraic B-system $\mathfrak{A}$ of signature σ its Boolean-valued realization $\mathfrak{A}^{\sim}$ is an algebraic system of signature $\sigma^{\wedge}$ inside $\mathbf{V}^{(B)}$. In this case for any formula φ of signature σ with n free variables and for arbitrary $a_0,\dots,a_{n-1} \in A := |\mathfrak{A}|$ we have*

$$|\varphi|^{\mathfrak{A}}(a_0,\dots,a_{n-1}) = [\![\, |\varphi|^{\mathfrak{A}^{\sim}}(\iota_A(a_0),\dots,\iota_A(a_{n-1})) = 1]\!].$$

$\lhd$ It should be recalled that while considering an arbitrary set as a B-set we mean the discrete B-metric in it. Therefore, $\sigma^{\sim} = \sigma^{\wedge}$ (see **3.4.12**). By virtue of **3.5.5**, it holds that

$$\mathbf{V}^{(B)} \models \text{`}\mu \text{ is a function and } \operatorname{dom}(\mu) = F^{\wedge} \cup P^{\wedge}\text{'}.$$

By theorem **3.4.14**, $\mathbf{V}^{(B)} \models \text{`}\mu(f^{\wedge})$ is a mapping from $(A^{\sim})^{\mathfrak{a}(f)^{\wedge}}$ to $A^{\sim}\text{'}$ for all $f \in F$ and $\mathbf{V}^{(B)} \models \text{`}\mu(p)$ is a mapping from $(A^{\sim})^{\mathfrak{a}(p)^{\wedge}}$ to $\{0,1\}\text{'}$ for every $p \in P$. Hence, $\mathbf{V}^{(B)} \models \text{`}\mathfrak{A}^{\sim}$ is an algebraic system of signature $\sigma^{\wedge}\text{'}$.

Let us now consider a formula φ of signature σ. By theorem **3.5.5** (3) for $f \in F$ and $p \in P$ we have

$$\iota_A \circ f^{v}(a_0,\dots,a_{n-1}) = \mu(f^{\wedge})\downarrow(\iota_A(a_0),\dots,\iota_A(a_{n-1})) \quad (a_l \in A),$$
$$\iota_B \circ p^{v}(a_0,\dots,a_{n-1}) = \mu(p^{\wedge})\downarrow(\iota_A(a_0),\dots,\iota_A(a_{n-1})) \quad (a_l \in A).$$

Using the above equalities, we deduce by induction on the complexity of the formula φ:

$$|\varphi|^{\mathfrak{A}}(a_0,\dots,a_{n-1}) = |\varphi|^{\mathfrak{A}'}(\iota_A(a_0),\dots,\iota_A(a_{n-1})) \quad (a_0,\dots,a_{n-1}\in A),$$

where $\mathfrak{A}' := \mathfrak{A}^{\sim}\downarrow$. Now we have to use theorem **4.2.4**. ▷

4.3.3. Theorem. *Let now $\mathfrak{A} := (A, v)$ be an algebraic B-system of signature σ. Then there are such $\mathcal{A}$ and $\mu \in \mathbf{V}^{(B)}$ that the following conditions are fulfilled:*

(2) $\mathbf{V}^{(B)} \models$ '$(\mathcal{A}, \mu)$ *is an algebraic system of signature* $\sigma^{\wedge}$';

(2) *if $\mathfrak{A}' := (A', v')$ is the descent of the system $(\mathcal{A}, \mu)$, then $\mathfrak{A}'$ is an extended algebraic $\mathcal{B}$-system of signature σ;*

(3) *there is an isomorphism ι from $\mathfrak{A}$ to $\mathfrak{A}'$ such that* $A' = \mathrm{mix}(\iota(A))$;

(4) *for any formula φ of signature σ in n free variables it holds that*

$$|\varphi|^{\mathfrak{A}}(a_0,\dots,a_{n-1}) = |\varphi|^{\mathfrak{A}'}(\iota(a_0),\dots,\iota(a_{n-1}))$$
$$= \chi^{-1}\circ\left(|\varphi|^{\mathfrak{A}^{\sim}}\right)\downarrow(\iota(a_0),\dots,\iota(a_{n-1}))$$

for all $a_0,\dots,a_{n-1} \in A$.

◁ Assign $\mathcal{A} := A^{\sim}$, $\iota := \iota_A$, while letting μ be determined as in **4.3.1**. Then the required statements result from **3.5.5** (3), **4.2.4** and **4.3.2**. ▷

4.3.4. Theorem. *Let us consider algebraic B-systems $\mathfrak{A}$ and $\mathcal{B}$ of the same signature.*

(1) *Let h be a contraction mapping from $|\mathfrak{A}|$ to $|\mathcal{B}|$. Then h is a homomorphism (strong homomorphism, isomorphism) iff* $\mathbf{V}^{(B)} \models$ '$h^{\sim}$ *is a homomorphism (strong homomorphism, isomorphism) from $\mathfrak{A}^{\sim}$ to $\mathcal{B}^{\sim}$*'. *The monomorphism $h^{\sim}$ is surjective inside $\mathbf{V}^{(B)}$ iff* $|\mathcal{B}| = \mathrm{mix}(h(|\mathfrak{A}|))$.

(2) *Let $g \in \mathbf{V}^{(B)}$ and $\mathbf{V}^{(B)} \models$ '$g : \mathfrak{A}^{\sim} \to \mathcal{B}^{\sim}$ is a homomorphism of algebraic B-systems'. If in this case $\mathcal{B}$ is an extended algebraic B-system, then there is a unique homomorphism $h : \mathfrak{A} \to \mathcal{B}$ such that $g = h^{\sim}$.*

◁ (1) If $h' := h^{\sim}\downarrow$, $\mathfrak{A}' := \mathfrak{A}^{\sim}\downarrow$, $\mathcal{B}' := \mathcal{B}^{\sim}\downarrow$, $i := \iota_{|\mathfrak{A}|}$ and $j := \iota_{|\mathcal{B}|}$, then $h'\circ i = j\circ h$

(see **3.5.4** (3)). Let us show that h is a homomorphism iff h' is a homomorphism. In this case we shall limit ourselves to substantiating **4.1.10** (3) with $n = 1$. In other words, we must show that h and h' either preserve or do not preserve unary operations simultaneously. Let $v, \lambda, \mu(v)$ and $\mu(\lambda)$ be interpretations of the systems $\mathfrak{A}, \mathfrak{B}, \mathfrak{A}^{\sim}$ and $\mathfrak{B}^{\sim}$. If h is a homomorphism, then $h \circ f^{v} = f^{\lambda} \circ h$. Besides, $i \circ f^{v} = (f^{\mu(v)}{\downarrow}) \circ i$ and $j \circ f^{\lambda} = (f^{\mu(\lambda)}{\downarrow}) \circ j$ and, hence,

$$h' \circ (f^{\mu(v)}{\downarrow}) \circ i = j \circ h \circ f^{v} = j \circ f^{\lambda} \circ h = (f^{\mu(\lambda)}{\downarrow}) \circ h' \circ i.$$

Making use of the relation $|\mathfrak{A}^{\sim}{\downarrow}| = \mathrm{mix}(i(|\mathfrak{A}|))$, we get $h' \circ (f^{\mu(v)}{\downarrow}) = (f^{\mu(\lambda)}{\downarrow}) \circ h'$. Conversely, if the last equality holds, then, reasoning in the opposite direction, we find $h \circ f^{v} = f^{\lambda} \circ h$. The case of arbitrary operations, as well as that of arbitrary predicates is more cumbersome but causes no principal difficulties. Therefore, h is a homomorphism, a strong homomorphism or an isomorphism between the algebraic B-systems $\mathfrak{A}$ and $\mathfrak{B}$ iff the mapping h' from $\mathfrak{A}'$ to $\mathfrak{B}'$ has the corresponding property. Therefore, the required property results from **4.2.5** and **4.3.3**. ▷

4.3.5. Let us note some corollaries to theorems **4.3.3** and **4.3.4**.

(1) Theorem. *If $\mathfrak{A}$ is an algebraic system of finite signature σ, then $\mathbf{V}^{(B)} \models {}'\mathfrak{A}^{\wedge}$ is an algebraic system of signature $\sigma^{\wedge}{}'$. In this case for any formula of signature φ with n free variables we have*

$$\mathfrak{A} \models \varphi(a_0,\dots,a_{n-1}) \leftrightarrow [\mathfrak{A}^{\wedge} \models \varphi(a_0^{\wedge},\dots,a_{n-1}^{\wedge})] = \mathbf{1},$$

for all $a_0,\dots,a_{n-1} \in A$.

◁ To prove the theorem, we only have to remark that if $\mathfrak{A} := (A, f_0, \dots, f_{k-1}, p_0, \dots, p_{m-1})$, then the statement $\mathfrak{A} \models \varphi(a_0,\dots,a_{n-1})$ is written as a bounded formula of set theory $\Psi(A^{\wedge}, f_0^{\wedge}, \dots, f_{m-1}^{\wedge}, a_0^{\wedge}, \dots, a_{n-1}^{\wedge})$ and refer to **2.2.9**. ▷

(2) Theorem. *For any algebraic B-system $\mathfrak{A}$ there is an extended algebraic B-system $\mathfrak{A}'$ of signature $\sigma(\mathfrak{A})$ and an isomorphism ι from $\mathfrak{A}$ to $\mathfrak{A}'$ such that*

(a) $|\mathfrak{A}'| = \mathrm{mix}(\iota(|\mathfrak{A}|))$;

(b) *if h is a homomorphism from $\mathfrak{A}$ to the extended algebraic B-system $\mathfrak{B}$, then there is a unique homomorphism $h' : \mathfrak{A}' \to \mathfrak{B}$ such that $h' \circ \iota = h$;*

(c) *if* $\mathfrak{A}''$ *is an extended algebraic* B*-system, and the isomorphism* $\iota' : \mathfrak{A} \to \mathfrak{A}''$ *obeys condition (a) (with* $\mathfrak{A}'$ *substituted for* $\mathfrak{A}''$*), then there is a unique isomorphism* h *from* $\mathfrak{A}'$ *on* $\mathfrak{A}''$ *such that* $h \circ \iota = \iota'$.

◁ Let $(\mathcal{A}, \mu)$ be a Boolean-valued realization of the algebraic B-system $\mathfrak{A}$. Then the descent $\mathfrak{A}' := (\mathcal{A}, \mu)\downarrow$ obeys all the required conditions. Indeed, by virtue of **4.3.3** (3,4) the canonical embedding $\iota := \iota_{|\mathfrak{A}|}$ is an isomorphism, in which case condition (a) is fulfilled. If h and $\mathcal{B}$ are the same as in (b), then by theorem **4.3.4**, $g := h^{\sim}\downarrow$ is a homomorphism from $\mathfrak{A}'$ to $\mathcal{B}' := \mathcal{B}^{\sim}\downarrow$. Since $\mathcal{B}$ is extended, the canonical mapping $j := \iota_{|\mathcal{B}|}$ is an isomorphism 'onto'. It is obvious that $h' := j^{-1} \circ g$ is the sought homomorphism. It is expedient to remark that if $a \in |\mathfrak{A}'|$ and $a = \mathrm{mix}(b_\xi \iota(a_\xi))$, then $h'(a) = \mathrm{mix}(b_\xi h \circ \iota(a_\xi))$. Statement (c) results from (a) and theorem **4.3.4**. ▷

Any pair $(\mathfrak{A}', \iota)$ where $\mathfrak{A}'$ is an extended algebraic B-system, while ι is an isomorphism from $\mathfrak{A}$ to $\mathfrak{A}'$ obeying condition (a) of theorem (2), can be naturally called a *maximal extension* of $\mathfrak{A}$. Then theorem (2) yields the following statement.

(3) *Any algebraic* B*-system has a maximal extension unique up to isomorphism.*

Let us choose a full homomorphism π from B to a complete Boolean algebra C. Let $\mathfrak{A} := (A, f_0, \dots, f_{k-1}, p_0, \dots, p_{m-1})$ is an algebraic system of finite signature inside $\mathbf{V}^{(B)}$. Let us denote

$$\pi^*(\mathfrak{A}) := (\pi^*(A), \pi^*(f_0), \dots, \pi^*(p_{m-1}))^C, \quad \pi^*(\mathfrak{A}) \in \mathbf{V}^{(C)},$$

where $\pi^* : \mathbf{V}^{(B)} \to \mathbf{V}^{(C)}$ is a mapping associated with π (see **2.2**).

(4) **Theorem.** *An element* $\pi^*(\mathfrak{A})$ *is an algebraic system of finite signature* $\sigma(\mathfrak{A})$ *inside* $\mathbf{V}^{(C)}$. *The mapping* $a \to \pi^*(a)\,(a \in A\downarrow)$ *is a homomorphism from* $\mathfrak{A}\downarrow$ *to* $\pi^*(\mathfrak{A})\downarrow$. *For any formula* φ *of signature* $\sigma(\mathfrak{A})$ *with* n *free variables and for arbitrary* $a_0, \dots, a_{n-1} \in |\mathfrak{A}\downarrow|$ *the following formula is fulfilled*

$$\mathfrak{A}\downarrow \models \varphi(a_0, \dots, a_{n-1}) \to \pi^*(\mathfrak{A})\downarrow \models \varphi(\pi^*(a_0), \dots, \pi^*(a_{n-1})).$$

In particular, if $\mathcal{B}$ *is an algebraic* B*-system of finite signature and* $\mathfrak{A} = \mathcal{B}^{\sim}$, *then for* $a_0, \dots, a_{n-1} \in |\mathcal{B}|$ *we have*

$$\mathcal{B} \models \varphi(a_0, \dots, a_{n-1}) \to \pi^*(\mathfrak{A})\downarrow \models \varphi(\pi^* \circ \iota(a_0), \dots, \pi^* \circ \iota(a_{n-1}))$$

where $\iota := \iota_{|\mathcal{B}|}$. *If* π *is an isomorphism, then* π^* *is an isomorphism from* $\mathfrak{A}\downarrow$ *to*

$\pi^*(\mathfrak{A})\downarrow$ and in the above formulas the reverse implication is also valid. If π is an isomorphism, then π^* is an isomorphism of algebraic B-systems.

◁ To prove this fact we must combine **2.2.4**, **2.2.5**, **4.1.10** and **4.2.5** and make use of the considerations of (1). ▷

(5) *For any algebraic system $\mathfrak{A}$ inside $\mathbf{V}^{(B)}$ we have $[\mathfrak{A}\downarrow^{\sim}$ is isomorphic to $\mathfrak{A}] = 1$.*

(6) **Theorem.** *The Boolean-valued realization $(\mathcal{A},\mu,\delta)$ of an algebraic B-system with disjointness (A,v,Δ) is an algebraic system with simple disjointness inside $\mathbf{V}^{(B)}$. If $(A',v') := (\mathcal{A},\mu)\downarrow$ and $\Delta' := \{(x,y)\in A'\times A' : \delta\downarrow(x,y) = 1\}$, then (A',v',Δ') is an extended algebraic B-system with disjointness, and for any $x,y\in A$ the following equivalences are valid:*

$$x\perp y \leftrightarrow \iota x\perp \iota y \leftrightarrow [\iota x = \theta \vee \iota y = \theta] = 1,$$

where $\iota = \iota_A : A \to A'$ is the canonical embedding.

◁ It suffices to use **4.1.13** and **4.3.3**. ▷

4.3.6. Let us now consider in more detail the important problem mentioned in **4.2.6**. Choose an algebraic B-system $\mathfrak{A}$ of signature σ. For the formula φ of the same signature and elements $a_0,\dots,a_{n-1}\in|\mathfrak{A}|$ we shall temporarily employ a more informative presentation $\mathfrak{A}\models_B \varphi(a_0,\dots,a_{n-1})$ instead of $\mathfrak{A}\models\varphi(a_0,\dots,a_{n-1})$. Using the B-system $\mathfrak{A}$ let us evolve the two-valued algebraic system $\overline{\mathfrak{A}}$ through the procedure of purification described in **4.1.3.** We can say that $\varphi(a_0,\dots,a_{n-1})$ is assertive both in $\mathfrak{A}$ and $\overline{\mathfrak{A}}$ as $|\mathfrak{A}| = |\overline{\mathfrak{A}}|$ and $\sigma(\overline{\mathfrak{A}}) = \sigma$. There arises a natural question about an interrelation between the statements $\mathfrak{A}\models_B \varphi(a_0,\dots,a_{n-1})$ and $\overline{\mathfrak{A}}\models\varphi(a_0,\dots,a_{n-1})$. Theorems **4.2.7** and **4.2.8** provide examples of such formulas φ for which $\overline{\mathfrak{A}}\models\varphi$ results from $\mathfrak{A}\models_B\varphi$. On the other hand, we can easily give an example violating this implication. Indeed, let $B := \mathcal{P}([0,1])$ and $A := \mathbf{R}^{[0,1]}$ be the set of all real functions on the interval $[0,1]$ with the B-metric

$$d(f,g) := \{t\in[0,1] : f(t)\neq g(t)\}\quad (f,g\in A).$$

Let us introduce a B-valued binary predicate $[\cdot\leq\cdot]$ on A by the formula

$$[f\leq g] = \{t\in[0,1] : f(t)\leq g(t)\}\quad (f,g\in A).$$

Then $\mathfrak{A} := (A,[\cdot \leq \cdot])$ is an algebraic B-system and $\mathfrak{A} \models_B \varphi$, where $\varphi := (\forall x)(\forall y)(x \leq y \vee y \leq x)$. Besides, it is obvious that $\overline{\mathfrak{A}} := (A,\leq)$ is a purification of $\mathfrak{A}$ if we set

$$f \leq g \leftrightarrow (\forall t \in [0,1])\ f(t) \leq g(t).$$

Obviously, $\overline{\mathfrak{A}} \models \neg\varphi$. Therefore, if $\mathfrak{T}^B(\mathfrak{A})$ and $\mathfrak{T}(\overline{\mathfrak{A}})$ are sets of all the formulas (with the constants ranging over $|\mathfrak{A}|$) true in the systems $\mathfrak{A}$ and $\overline{\mathfrak{A}}$, respectively, then none of these two sets is not, generally speaking, a subset of the other. It stands to reason, therefore, to expect that for a certain class Φ of formulas of signature σ there exist only relations of the type $\mathfrak{T}^B(\mathfrak{A}) \cap \Phi (?) \mathfrak{T}(\overline{\mathfrak{A}}) \cap \Phi$. For exact formulations a definite syntactic analysis of texts is needed.

4.3.7. **(1)** Let us introduce a class of generic and strictly generic formulas. The definition is given by recursion over the length of a formula. The rules of formulation are as follows.

(a) Any atomic formula is strictly generic.

(b) If φ and ψ are strictly generic formulas, then $\varphi \wedge \psi$, $(\exists x)\varphi$, $(\forall x)\varphi$ are also strictly generic.

(c) A strictly generic formula is generic.

(d) If φ and ψ are generic formulas, then $\varphi \wedge \psi$, $(\exists x)\varphi$, $(\forall x)\varphi$ are also generic.

(e) If φ is a strictly generic formula, then $\neg\varphi$ is a generic formula.

(f) If φ is a strictly generic formula, and ψ is a generic formula, then $\varphi \rightarrow \psi$ is a generic formula.

(2) *A basic Horn formula* is a disjunction $\theta_1 \wedge \ldots \wedge \theta_n$, where at most one of the formulas θ_k is basic, while the rest formulas are negations of atomic formulas. A formula is termed *a Horn one* if it is built of basic Horn formulas with the connectives $\wedge, \exists$ and $\forall$.

(3) *Any generic formula is logically equivalent to a Horn formula and vice versa.*

4.3.8. Examples.

(**1**) Let φ be a formula of signature $\{\le\}$ with the only predicate symbol. If φ are axioms of a lattice-ordered set (= lattice; see **1.1.1**), then φ is a generic formula. In the signature in question distributivity is not written as a generic formula. If, however, we choose a signature $\sigma := \{\wedge, \vee\}$, where $\wedge$ and $\vee$ are binary functional symbols, then the formula $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ is atomic and, hence, strictly generic. Moreover, the property of being a distributive lattice is a strictly generic formula of signature $\{\wedge, \vee\}$.

(**2**) Let us choose formulas φ and ψ of signature $\{\wedge, \vee, *, 0, 1\}$. Let φ be axioms of a Boolean algebra (see **1.1.2**), while $\psi :=$' there exists at least one atom', i.e.,

$$\psi := (\exists x)(\forall y)(x \neq 0 \wedge x \wedge y = y \rightarrow x = y \vee y = 0).$$

Then φ is a strictly generic formula, then ψ is not generic.

(**3**) Let $\sigma := \{+, 0\}$, where $+$ is a binary functional symbol and 0 is a constant symbol. If φ are the axioms of a group (associativity of a group operation, the axiom of zero, existence of an inverse element), then φ is a strictly generic formula of signature σ.

(**4**) Let $\sigma := \{+, \cdot, 0, 1\}$, where $+$ and $\cdot$ are binary functional symbols, and 0 and 1 are constant symbols. Let φ be the axioms of a ring, while ψ be the axioms of an integral domain, i.e., $\psi := \varphi \wedge \theta$, where

$$\theta := (\forall x)(\forall y)(x \cdot y = 0 \rightarrow x = 0 \vee y = 0).$$

Then φ is a strictly generic formula, while ψ is a generic formula.

4.3.9. Theorem. *Let $\mathfrak{A}$ be an extended algebraic B-system, while φ be a formula of signature $\sigma(\mathfrak{A})$.*

(**a**) *if φ is strictly generic, then*

$$\mathfrak{A} \models_B \varphi(a_0, \dots, a_{n-1}) \leftrightarrow \bar{\mathfrak{A}} \models \varphi(a_0, \dots, a_{n-1}) \ (a_0, \dots, a_{n-1} \in |\mathfrak{A}|).$$

(**b**) *if φ is generic, then*

$$\mathfrak{A} \models_B \varphi(a_0, \dots, a_{n-1}) \rightarrow \bar{\mathfrak{A}} \models \varphi(a_0, \dots, a_{n-1}) \ (a_0, \dots, a_{n-1} \in |\bar{\mathfrak{A}}|).$$

$\lhd$ The proof is carried out by induction on length of the formula φ. In relation with theorem **4.3.3**, one can assume $\mathfrak{A} = \mathcal{A}{\downarrow}$, where $\mathcal{A}$ is an algebraic system of signature $\sigma^{\wedge}$

inside $\mathbf{V}^{(B)}$.

(a) If φ is an atomic formula, then the statement directly follows from the definition of purification, since for a predicate symbol $p = \sigma(\mathfrak{A})$, $\varkappa(p) = n$ we have

$$p^{v}(a_0,\ldots,a_{n-1}) = 1 \leftrightarrow (a_0,\ldots,a_{n-1}) \in \bar{v}(p)\ (a_0,\ldots,a_{n-1} \in |\mathfrak{A}|).$$

For the conjunction $\varphi := \psi \wedge \theta$ we have, making use of definition **4.1.8** and the induction hypothesis,

$$|\psi \wedge \theta|^{\mathfrak{A}} = 1 \leftrightarrow |\psi|^{\mathfrak{A}} = 1 \wedge |\theta|^{\mathfrak{A}} = 1 \leftrightarrow \bar{\mathfrak{A}} \models \psi \wedge \bar{\mathfrak{A}} \models \theta \leftrightarrow \bar{\mathfrak{A}} \models \psi \wedge \theta.$$

Analogous is the case of the universal quantifier $\varphi := (\forall x)\,\psi$:

$$|(\forall x)\varphi|^{\mathfrak{A}} = 1 \leftrightarrow (\forall a \in |\mathfrak{A}|)\, |\psi(a)|^{\mathfrak{A}} = 1$$
$$\leftrightarrow (\forall a \in |\mathfrak{A}|)\, \bar{\mathfrak{A}} \models \psi(a) \leftrightarrow \bar{\mathfrak{A}} \models (\forall x)\psi.$$

Let us consider the case of the existential quantifier $\varphi := (\exists x)\,\psi$. By virtue of the maximum principle, there is an element $z \in \mathbf{V}^{(B)}$ such that

$$[\mathcal{A} \models (\exists x)\psi\,] = [z \in |\mathcal{A}| \wedge \mathcal{A} \models \psi(z)].$$

According to theorem **4.3.3**, this formula can be rewritten as

$$[z \in |\mathcal{A}|] \wedge |\psi(z)|^{\mathfrak{A}} = |(\exists x)\,\psi|^{\mathfrak{A}}.$$

Hence, in line with the induction hypothesis, we can deduce that the following equivalences are valid:

$$|(\exists x)\psi|^{\mathfrak{A}} = 1 \leftrightarrow (\exists z \in |\mathfrak{A}|)\, |\psi(z)|^{\mathfrak{A}} = 1$$
$$\leftrightarrow (\exists z \in |\bar{\mathfrak{A}}|)\, \bar{\mathfrak{A}} \models \psi(z) \leftrightarrow \bar{\mathfrak{A}} \models (\exists x)\psi,$$

since, by definition **4.2.3** $|\mathfrak{A}| = |\mathcal{A}|\!\downarrow$. Therefore, in each of the cases considered the induction step is realizable and the proof is completed by referring to items (a) and (b) of definition **4.3.7** (1).

(b) The case of the connectives $\wedge$, $\exists$ and $\forall$ is considered in the same way as above in (a). Let $\varphi := \neg\,\psi$, where ψ is a strictly generic formula. If $\varphi|^{\mathfrak{A}} = 1$, then $|\varphi|^{\mathfrak{A}} = \mathbf{0}$ and, by virtue of what has been proved in (a), φ cannot be true in $\bar{\mathfrak{A}}$. In this case, however,

$\overline{\mathfrak{A}} \models \varphi$. And finally, let us consider a formula of the type $\varphi := \theta \to \psi$, where θ is a strictly generic formula, while ψ is a generic formula. Let us assume that $|\theta \to \psi|^{\mathfrak{A}} = 1$. If $\overline{\mathfrak{A}} \models \theta$, then the facts proved in (a) yield $|\theta|^{\mathfrak{A}} = 1$ and, hence, $|\psi|^{\mathfrak{A}} = 1$. By the induction hypothesis we get $\overline{\mathfrak{A}} \models \theta \to \psi$. ▷

It should be remarked that the Jech theorem makes it possible to replace the proofs of some fragments of theorems **4.2.7** - **4.2.9** with a syntactical analysis of corresponding sentences. It goes without saying that a general fact of the kind can also be formulated.

(2) **Corollary.** *Let $\mathcal{A}$ and $\overline{\mathfrak{A}}$ be some Boolean-valued realization and purification of an extended algebraic B-system, respectively. For any Horn sentence φ we have*

$$[\mathcal{A} \models \varphi] = 1 \to \overline{\mathfrak{A}} \models \varphi .$$

4.3.10. Let Φ be a certain set of formulas of the same signature σ. Let us introduce the category $\mathrm{AS}^{(B)}(\Phi)$ in the following way:

$$\mathrm{ObAS}^{(B)}(\Phi) := \{\mathfrak{A} \in \mathrm{V}^{(B)} : [\mathfrak{A} \text{ is an algebraic system of signature } \sigma^{\wedge} \text{ and } \mathfrak{A} \models \Phi] = 1\};$$

$$\mathrm{AS}^{(B)}(\mathfrak{A}, \mathcal{B}) := \{h \in \mathrm{V}^{(B)} : [h \text{ is a homomorphism from } \mathfrak{A} \text{ to } \mathcal{B}] = 1\};$$
$$\mathrm{Com}(f, g) = h \leftrightarrow [h = g \circ f] = 1.$$

The fact that the above conditions do determine a category results from the principle of transfer, that of maximum, theorem **4.3.2**, as well as from the other properties of the embedding functor. As before, by the symbols $\mathfrak{F}^{\sim}$ and $\mathfrak{F}^{\downarrow}$ we shall denote, respectively, the mappings of the embedding and descent, which operate in the categories of algebraic systems: $\mathfrak{F}^{\sim} : B - \mathrm{AS}(\Phi) \to \mathrm{AS}^{(B)}(\Phi)$, $\mathfrak{F}^{\downarrow} : \mathrm{AS}^{(B)}(\Phi) \to B - \mathrm{AS}(\Phi)$.

Theorem. *The following statements are valid*:

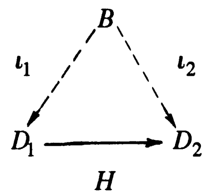
(1) *the mapping $\mathfrak{F}^{\downarrow}$ is a covariant functor from the category $\mathrm{AS}^{(B)}(\Phi)$ into the category $B - \mathrm{CAS}^{(B)}(\Phi)$;*

(2) *the mapping $\mathfrak{F}^{\sim}$ is a covariant functor from the category $B - \mathrm{AS}(\Phi)$ (as well as from $B - \mathrm{CAS}(\Phi)$) into the category $\mathrm{AS}^{(B)}(\Phi)$;*

(3) *the functors* $\mathfrak{F}^{\downarrow}$ *and* $\mathfrak{F}^{\sim}$ *establish the equivalence of the categories* $\mathrm{AS}^{(B)}(\Phi)$ *and* $B-\mathrm{CAS}(\Phi)$.

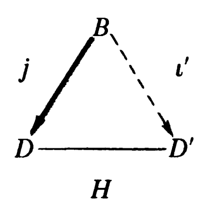
Let us now note some corollaries for rings with projections and Boolean algebras whose proofs are in essence contained in **4.2.5**, **4.2.7**, **4.2.8** and **4.3.2.** We consider rings with projections K_1 and K_2, and let j_1 and j_2 be isomorphisms of B on the Boolean algebras of projections in K_1 and K_2, respectively. The homomorphism $h: K_1 \to K_2$ will be termed *B-homogeneous* if $h \circ j_1(b) = j_2(b) \circ h$ $(b \in B)$. We shall also say that *K_l is a ring with Boolean algebra of projections B and that h commutes with projections of B.*

4.3.11. (1) **Theorem.** *Let* $\mathfrak{D}_1$ *and* $\mathfrak{D}_2$ *be complete Boolean algebras inside* $\mathbf{V}^{(B)}$. *Let us set* $D_k := \mathfrak{D}_k \downarrow$ *and let* $\iota_k: B \to D_k$ *be a canonical monomorphism for* $k = 1,2$ *(see* **4.2.9**). *If* $h \in \mathbf{V}^{(B)}$ *is an isomorphism of* $\mathfrak{D}_1$ *on* $\mathfrak{D}_2$ *inside* $\mathbf{V}^{(B)}$, *then there is an isomorphism* H *of the algebra* D_1 *on* D_2, *for which the following diagram is commutative:*



Conversely, if $H: D_1 \to D_2$ *is an isomorphism of Boolean algebras such that the above diagram is commutative, then the algebras* $\mathfrak{D}_1$ *and* $\mathfrak{D}_2$ *are isomorphic inside* $\mathbf{V}^{(B)}$.

(2) **Theorem.** *Let* D *be a complete algebra and* $j: B \to D$ *be a full monomorphism. Then there is a complete Boolean algebra* $\mathfrak{D}$ *inside* $\mathbf{V}^{(B)}$, *and an isomorphism* H *from* D *on* $D' := \mathfrak{D} \downarrow$ *such that the following diagram is commutative:*



where ι' *is the canonical monomorphism from* B *to* D'.

4.3.12. (1) **Theorem.** *Let* $\mathfrak{R}_1$ *and* $\mathfrak{R}_2$ *be rings with algebra of projections* $\mathfrak{D}$ *inside* $\mathbf{V}^{(B)}$. *Assign* $D := \mathfrak{D} \downarrow$, $K_l := \mathfrak{R}_l \downarrow$ *and* $l := 1,2$. *Then* K_1 *and* K_2 *are rings with algebra of projections* D, *and if inside* $\mathbf{V}^{(B)}$ *it is true that* h *is a homomorphism from the ring* $\mathfrak{R}_1$ *to* *the ring $\mathfrak{K}_2$ commuting with projections of $\mathfrak{D}$, then $h\downarrow$ is a homomorphism of the ring K_1 to the ring K_2 commuting with projections of D. If h is an isomorphism of $\mathfrak{K}_1$ on $\mathfrak{K}2$, then $h\downarrow$ is an isomorphism of K_1 on K_2.*

(2) Theorem. *Let (K,D) be a ring with projections and let $j: B \to D$ be a full homomorphism. Then there is a ring with projections $(\mathfrak{K},\mathfrak{D})$ inside $\mathbf{V}^{(B)}$, and an isomorphism h of the ring K to the ring $K' := \mathfrak{K}\downarrow$ such that for any $b \in B$ the following diagram is commutative:*

$$\begin{array}{ccc} & h & \\ K & \dashrightarrow & K' \\ j(b)\downarrow & & \downarrow \iota'(b) \\ K & \longrightarrow & K' \\ & h & \end{array}$$

where ι' is the canonical monomorphism from B to D'.

Analogous results are also valid for groups with projections.

4.3.13. Remarks

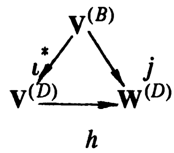
(1) Let C and D be Boolean algebras, while P and Q be their Stone spaces. Determine the tensor product $C \otimes D$ of the algebras C and D as the Boolean algebra of clopen subsets of the Cartesian product $P \times Q$ (see **1.1.6** (6) and **1.2.6** (8)). Let $C \hat{\oplus} D$ be the completion of the Boolean algebra $C \otimes D$ (see **1.1.6** (7) and **1.2.6.** (9)). If D is a Boolean algebra, and an element $\mathfrak{D} \in \mathbf{V}^{(B)}$ is such that $\mathbf{V}^{(B)} \models$ "$\mathfrak{D}$ is the completion of the Boolean algebra $D^\wedge$", then the algebras $\mathfrak{D}\downarrow$ and $B \hat{\oplus} D$ are isomorphic (see **[236]**).

(2) The Solovay-Tennenbaum theorems (see **4.3.11**) can form the cornerstone of iterating the construction of a Boolean-valued model. Let $\mathfrak{D} \in \mathbf{V}^{(B)}$ and $\mathbf{V}^{(B)} \models$ "$\mathfrak{D}$ is a complete Boolean algebra". According to scheme **2.1**, inside $\mathbf{V}^{(B)}$ it is possible to construct $\mathbf{V}^{(B)}$-classes, i.e., the Boolean-valued universe $(\mathbf{V}^{(B)})^{\mathfrak{D}}$, corresponding Boolean truth-values $[\cdot = \cdot]^{\mathfrak{D}}$ and $[\cdot \in \cdot]^{\mathfrak{D}}$, as well as the canonical embedding $(\cdot)^\wedge$ of the universal class $\mathbf{U}_B$ in $(\mathbf{V}^{(B)})^{\mathfrak{D}}$. Let us set $D := \mathfrak{D}\downarrow$, $\mathbf{W}^{(D)} := (\mathbf{V}^{(B)})^{\mathfrak{D}}\downarrow$, $[\cdot = \cdot]^D := \left([\cdot = \cdot]^{\mathfrak{D}}\right)\downarrow$, $[\cdot \in \cdot]^D := \left([\cdot \in \cdot]^{\mathfrak{D}}\right)\downarrow$, $j := (\cdot)^\wedge\downarrow$. Let $\iota: B \to D$ be the canonical monomorphism, while $\iota^*: \mathbf{V}^{(B)} \to \mathbf{V}^{(D)}$ be the corresponding injection (see **2.2**).

Then there is a unique bijection $h: \mathbf{V}^{(D)} \to \mathbf{W}^{(D)}$ such that

$$[x = y]^D = [h(x) = h(y)]^{\mathfrak{D}}, \ [x \in y]^D = [h(x) \in h(y)]^{\mathfrak{D}},$$

whatever x and $y \in \mathbf{V}^{(D)}$ might be. In this case the diagram



is commutative (for details see [**236**]).

(**3**) Further iterations of the construction presented above result in a transfinite sequence of Boolean-valued extensions. In this way there appears an efficient method, iterated forcing, which has been used to establish relative consistency of the Suslin hypothesis with ZFC (see **236**]).

4.4. Ordered Algebraic Systems

A complete Boolean algebra of congruences necessary for Boolean-valued realization of an algebraic system is often generated by a relation of order. This peculiarity brings about the possibility of the Boolean-valued realization for ordered algebraic systems. In the next section we shall present some results in this direction. All the necessary additional information can be found in [**13, 110, 12, 59**].

4.4.1. *An ordered group* is an algebraic system $(G,+,0,\leq)$ for which the following conditions are met:

(**1**) $(G,+,0)$ is a group;

(**2**) $(G,\leq)$ is a (partially) ordered set;

(**3**) the group structure and the order structure are compatible, which means that group translations are isotonic mappings, i.e., G is a model for

$$(\forall x)\,(\forall y)\,(\forall a)\,(\forall b)\quad (x \le y \leftrightarrow a + x + b \le a + y + b).$$

(An additive presentation of the group operation does not imply that it is commutative). We say that G is *a linearly ordered group* in the case when in addition to (**1**) - (**3**) the following condition is also met:

(**4**) $(G,\le)$ is a linearly ordered set, i.e., the formula $(\forall x)\,(\forall y)\quad (x \le y \vee y \le x)$ holds on G. The element $x \in G$ is termed *positive* if $x \ge 0$. A set of all positive elements is called *the positive cone* and denoted by G^+. A subset K of the group G is a positive cone of some order on G provided the following conditions are met:

(a) $K \cap (-K) = \{0\}$;

(b) $K + K = K$;

(c) $x + K = K + x \quad (x \in G)$.

In this case the cone K and the order corresponding to it are related as follows:

$$x \le y \leftrightarrow y - x \in K \leftrightarrow -x + y \in K.$$

The group G is linearly ordered iff

(d) $G = G^+ \cup (-G^+)$.

The cone of positive elements is called *reproducing* provided $G = G^+ - G^+$. When this condition is fulfilled, G is also said to be *a directed group.* By definition, an ordered group G is *integrably-closed (Archimedean)* iff for any $x, y \in G$ it follows from the inequalities $nx \le y$, $n \in \omega$ (respectively, $nx \le y$, $\pm n \in \omega$) that $x \le 0$ (respectively, $x = 0$). The homomorphism $h{:}\,G \to G'$ of ordered groups is positive if $h(x) \ge 0$ for every $0 \le x \in G$.

4.4.2. *A lattice-ordered group* is an ordered group G, in which every nonempty finite set $\{x_0,\dots,x_{n-1}\} \subset G$ has a least upper bound $x_0 \vee \dots \vee x_{n-1} := \sup\{x_0,\dots,x_{n-1}\}$ and a greatest lower bound $x_0 \wedge \dots \wedge x_{n-1} := \inf\{x_0,\dots,x_{n-1}\}$. For any element x of the lattice-ordered group G determined are elements $|x| := x \vee (-x)$, $x^+ := x \vee 0$ and $x^- := (-x)^+ = -x \wedge 0$ called, respectively, *the modulus, the positive part and the negative part* of x. In any lattice-ordered group the following relations hold:

(**1**) $x = x^+ - x^-$, $|x| = x^+ + x^-$, $x^+ \wedge x^- = 0$;

(**2**) $(x + y)^+ \le x^+ + y^+$, $(x + y)^- \le x^- + y^-$;

(3) $(nx)^+ = nx^+$, $(nx)^- = nx^-$, $|nx| = n|x| (n \in \omega)$;

(4) $|x + y| \le |x| + |y| + |x|$;

(5) $|x + y - x| = x + |y| - x$; $(x + y - x)^- = x + y^- - x$;

(6) $u \wedge x = 0 \wedge u \wedge y = 0 \to u \wedge (x + y) = 0$.

The lattice-ordered group G is commutative iff instead of (4) we have $|x + y| \le |x| + |y|$ for all $x, y \in G$. Among the other properties of the group G it should be recalled that it is torsion-free, is a distributive lattice and the following relations are valid:

$$a + (\vee x_a) + b = \vee(a + x_a + b),$$
$$a + (\wedge x_a) + b = \wedge(a + x_a + b).$$

A subgroup G_0 of a lattice-ordered group is called an *o-ideal* (or *a convex subgroup*) if for any x and y it follows from $|x| \le |y|$ and $y \in G_0$ that $x \in G_0$. If, moreover, the subgroup G_0 is normal, than it is termed an *l-ideal.*

4.4.3. Let us from now on assume G to be a lattice-ordered group and introduce in it the disjointness $\perp$ by the following formula:

$$\perp := \{(x, y) \in G \times G : |x| \wedge |y| = 0\}.$$

There is no doubt that $\perp$ obeys all the axioms of disjointness of **4.1.2.**(2). The complete Boolean algebra of $\perp$-components $\mathfrak{K}_\perp(G)$ is called *the base* of G and is denoted by $\mathfrak{B}(G)$. Let us assume that a component $K \in \mathfrak{B}(G)$ is singled out by a direct addent of the group G. Then the corresponding projection π_K is a positive endomorphism in G, in which case $\pi_K x \le x$ for all $0 \le x \in G$. If any component in K is singled out by a direct addent, then the set $\mathfrak{Pr}(G)$ of all projections of the kind $\pi_K (K \in \mathfrak{B}(G))$ is a complete Boolean algebra isomorphic to $\mathfrak{B}(G)$. Under these circumstances G is said to be *a group with projections on components*or a *group with band projections.* A lattice-ordered group G' with projections on components is said to be *extended* or *orthogonally complete* or *laterally complete* provided it is extended relative to the algebra of projections $\mathfrak{Pr}(G)$. The *maximal or universal extension of the lattice-ordered group G* is an extended lattice-ordered group G' combined with an o-isomorphism $\iota : G \to G'$ such that $G' = \text{mix}(\iota(G))$ and for any $0 < x' \in G'$ we can be find $0 < x \in G$, $\iota(x) \le x'$ (here mix is calculated relative to the Boolean algebra $\mathfrak{Pr}(G)$).

It should be recalled that $[x]$ denotes the least component that contains x. The properties listed in **4.4.2** yield the following conclusions.

(1) *The following relations are valid:*

$$[x+y]=[x\vee y]=[x]\vee[y] \quad (x,y\in G^{+});$$
$$[x]=[|x|]=[x^{+}]\vee[x^{-}] \quad (x\in G);$$
$$[x+y-x]=x+[y]-x \quad (x,y\in G);$$
$$x\perp y\rightarrow x+y=y+x \quad (x,y\in G).$$

(2) *Any component* $K\in\mathfrak{B}(G)$ *is an* o*-ideal.*

◁ Indeed, if x and $y\in A^{\perp}$ for some $A\subset G$, then, by virtue of the second relation of **(1)** and **4.4.2** (4), we can write

$$\{x+y\}^{\perp}\supset\{x\}^{\perp}\wedge\{y\}^{\perp}\wedge\{x\}^{\perp}\supset A,$$

and, hence, $x+y\in\{x+y)^{\perp\perp}\subset A^{\perp}$. We thus have established that if $y\in A^{\perp}$ and $|x|\leq|y|$, then $\{x\}^{\perp}\supset\{y\}^{\perp}\supset A$ and, hence, $x\in\{x\}^{\perp\perp}\subset A^{\perp}$, which completes the proof. ▷

4.4.4. If the group G is not commutative, then the components in it are not necessarily normal subgroups, i.e., they are not, generally speaking, l-ideals. Therefore, the following notion can be introduced. The component $K\in\mathfrak{B}(G)$ is called *invariant* if $x+K-x\subset K$ for any $x\in G$. By virtue of **4.4.3** (2) it is equivalent to the fact that K is an l-ideal. The set of all invariant components will be denoted by the symbol $\mathfrak{B}_{i}(G)$.

(1) *The set of all invariant components* $\mathfrak{B}_{i}(G)$ *is a regular subalgebra of the Boolean algebra of all the components.*

◁ It is obvious that the intersection of any set of invariant components will be an invariant component. It is therefore sufficient to prove that the disjoint complement of each invariant component is an invariant component. Let us choose $K\in\mathfrak{B}_{i}(G)$ and $x\in K^{\perp}$. Then for any $y\in K$ and $a\in G$ we have $0=(a+|y|-a)\wedge|x|=-a+(a+|y|-a)\wedge|x|+a=$ $=|y|\wedge(-a+|x|+a)$ and, hence, $-a+|x|+a\in K^{\perp}$, which is the proof that the component $K^{\perp}$ is invariant. ▷

(2) *The following statements are valid for a lattice-ordered group* G:

(a) *any component is invariant, i.e.,* $\mathfrak{B}(G)=\mathfrak{B}_{i}(G)$;

(b) *for any* $x,y\in G$ *the following equality holds:*

$$\{x\}^{\perp} = y + \{x\}^{\perp} - y;$$

(c) *if an element* $x \in G$ *is disjoint from any of its conjugates* $y + x - y$, *then* $x = 0$.

$\lhd$ The condition (b) is an obvious corollary to (a). Let us assume that (b) is fulfilled and $x^{\perp}(y + x - y)$ for some x and $y \in G$. Then

$$x \in \{y + x - y\}^{\perp} = y + \{x\}^{\perp} - y = \{x\}^{\perp},$$

which immediately yields $x = 0$. Let, finally, (c) be fulfilled and the component K have the form $A^{\perp}$, $A \subset G$. Let us arbitrarily choose $x \in K$, $y \in G$, $a \in A$ and set $z := (y + |x| - y) \wedge |a|$. It is obvious that $0 \le z \wedge (-y + z + y) \le |x| \wedge |a| = 0$, so that $z = 0$, which does imply that $|y + x - y| = y + |x| - y \in A^{\perp} = K$, i.e., that $y + K - y \subset K$. $\rhd$

Introduce a symmetric relation Δ in G by the formula

$$\Delta := \{(x,y) \in G \times G : (\forall a,b \in G)\,(a + |x| - a) \wedge (b + |y| - b) = 0\}.$$

If for some $x, y \in G$ it is invalid that $x \Delta y$, then there are such $a_0, b_0 \in G$ that $u_0 := (a_0 + |x| - a_0) \wedge (b_0 + |y| - b_0) \neq 0$. It is obvious that $u_0 \in \{a_0 + |x| - a_0\}^{\Delta\Delta}$ but, on the other hand, $\{a_0 + |x| - a_0\}^{\Delta\Delta} = \{x\}^{\Delta\Delta}$. Therefore, $u_0 \in \{x\}^{\Delta\Delta}$ and, analogously, $u_0 \in \{y\}^{\Delta\Delta}$. It should be also remarked that the least Δ-component is $\{0\}$ and $\Delta \cap I_G \subset \perp \cap I_G = \{(0,0)\}$. Hence, Δ is a disjointness on G (see **4.1.12** (2)).

(3) *The set of all* Δ*-components coincides with the complete Boolean algebra of invariant* $\perp$*-components:* $\mathfrak{K}_{\Delta}(G) = \mathfrak{B}_i(G)$.

4.4.5. Now assume that the group G has invariant base, i.e., its all components are invariant. This exactly implies that $\Delta = \perp$. The commutative lattice-ordered group obviously has an invariant base. Under these circumstances G can be turned into an algebraic B-system. Let now j be an isomorphism of a complete Boolean algebra B on the (invariant) basis $\mathfrak{B}(G)$. By definition, assign

$$p(x) := j^{-1}(\{x^{-}\}^{\Delta}) \quad (x \in G).$$

The mapping $p: G \to B$ has a number of important properties.

For any $x, y \in G$ the following relations are valid:

(1) $0 \le x \to p(x) = 1$;

(2) $p(x) \wedge p(-x) = j^{-1}(\{x\}^{\perp})$;
(3) $p(x) \wedge p(y) \leq p(x+y)$;
(4) $p(x) = p(y+x-y)$;
(5) $p(x) \vee p(-x) = 1$.

◁ The first statement is obvious. To prove (2), we should remark that $\{x\}^{\perp} = \{x^{+}\}^{\perp} \wedge \{x^{-}\}^{\perp} = \{x^{-}\}^{\perp} \wedge \{(-x)^{-}\}^{\perp}$ due to the fact that x^{+} and x^{-} are disjoint. It is then clear that $j^{-1}(\{x\}^{\perp}) = j^{-1}(\{x^{\cdot}-\}^{\perp}) \wedge j^{-1}(\{-x\}^{-}) = p(x) \wedge p(-x)$. Analogous considerations are used to establish (3) provided we begin with relations 4.4.2 (2,6). Relation (4) results from 4.4.2 (5) since the components are invariant. Taking it again into account that the elements x^{+} and x^{-} are disjoint, we can write

$$(\{x^{+}\}^{\perp} \vee \{x^{-}\}^{\perp})^{\perp} = \{x^{+}\}^{\perp\perp} \wedge \{x^{-}\}^{\perp\perp} = \{0\}.$$

Hence, we deduce $\{x^{+}\}^{\perp} \vee \{x^{-}\}^{\perp} = G$, which is equivalent to the required result. ▷

Let us introduce two mappings, σ and $d := G \times G \rightarrow B$ by the following formulas:

$$\sigma(x,y) := p(y-x), \quad d(x,y) := j^{-1}(\{x-y\}^{\Delta}) \quad (x,y \in G).$$

From (1) - (5) the following properties of the mapping σ are straightforward:

(a) $\sigma(x,x) = 0$ (symmetry);
(b) $\sigma(x,y) \wedge \sigma(y,z) \leq \sigma(x,z)$ (transitivity);
(c) $\sigma(x,y) = \sigma(a+x-b, a+y-b)$ (invariance);
(d) $\sigma(x,y) \wedge \sigma(y,x) = d(x,y)^{*}$ (reflexivity).

By virtue of (d), $d(x,y) = \sigma(x,y)^{*} \vee \sigma(y,x)^{*}$ and, hence, d is a B-metric on G invariant relative to left and right transitions, while σ is a B-predicate. Finally, it is obvious that $d(x,0) = j^{-1}(\{x\}^{\perp\perp})$, i.e., the B-metric d agrees with the disjointness $\perp$ (see 4.1.13).

4.4.6. Theorem. *Let G be a lattice-ordered group with invariant base. Then G, when considered with a B-predicate σ and with the corresponding B -metric d, is an algebraic B-system of signature $(+,0,\leq)$, on which the axioms of a linearly ordered group are fulfilled.*

◁ As has been remarked above, a B-metric d is invariant under translations. Taking this fact into consideration, we can write

$$d(x+y, u+v) = d(x, -y+u+v) \leq d(x,u) \vee d(u, -y+u+v),$$

$$d(u,-y+u+v)=d(u+y-u,v)\leq d(v,y)\vee d(u+y-u,y),$$
$$d(u+y-u,y)=d(u+y,u+y)=0.$$

These relations show that $d(x+y,u+v)\leq d(x,u)\vee d(y,v)$, i.e., the operation of the summation is a contraction mapping. Then, by virtue of **4.4.5** (3), the definition of d gives

$$d(x,y)^{*}\wedge p(x)=p(x)\wedge p(x-y)\wedge p(y-x)\leq p(y),$$

whatever $x,y\in G$ might be. From here we can easily deduce that $\sigma(x,y)\wedge d(x,u)^{*}\wedge d(y,v)^{*}\leq\sigma(u,v)$, which implies that the mapping σ is contractive. Hence, $(G,+,0,\leq,\sigma)$ is an algebraic B-system of signature $(+,0,\leq)$. In this case the following interpretation of the symbol $\leq$ is implied: if $x,y\in G$, then $|x\leq y|^{G}:=\sigma(x,y)$. Then the unary B-predicate p on G will, obviously, be an interpretation of the property of being a positive element, i.e., $|0\leq x|^{G}=p(x)$. The fact that G is a B-model for the axioms of a linearly ordered group is just a different interpretation of properties **4.4.5** (1-5). Let us check, for instance, that the order of σ agrees with group structure and is total.

If φ is a closed formula from **4.4.1** (3), then presenting the Boolean truth-values for quantifiers according to **4.1.8**, we get

$$|\varphi|^{G}=\bigwedge_{x,y,a,b\in G}|x\leq y\rightarrow a+x+b\leq a+y+b|^{G}.$$

Then, making use of the fact that σ serves as an interpretation of the symbol $\leq$, we write:

$$|x\leq y\rightarrow a+x+b\leq a+y+b|^{G}=\sigma(x,y)\Rightarrow\sigma(a+x+b,a+y+b).$$

By virtue of **4.4.5** (4), however, we have

$$\sigma(a+x+b,a+y+b)=p(a+y+b-(a+x+b))$$
$$=p(a+(y-x)-a)=p(y-x)=\sigma(x,y).$$

Therefore, $\mathbf{1}=\sigma(x,y)\Rightarrow\sigma(a+x+b,a+y+b)$ and, hence, $|\varphi|^{G}=\mathbf{1}$.

Let now φ be the axiom expressing the linearity of order **4.4.1** (4). Let us again use rules **4.1.8** and write

$$|\varphi|^{G}=\bigwedge_{x,y\in G}|x\leq y\vee y\leq x|^{G}=\bigwedge_{x,y\in G}\sigma(x,y)\vee\sigma(y,x).$$

Observe that by virtue of **4.4.5** (5) we have

$$\sigma(x,y)\vee\alpha(y,x)=p(y-x)\vee p(x-y)=\mathbf{1},$$

and, hence, $|\varphi|^G = \mathbf{1}$. ▷

4.4.7. Let us now consider the case of lattice-ordered rings. An algebraic system $(A,+\,,;0,\le)$ is called *an ordered ring* if the following statements are valid:

(1) $(K,+,0,\le)$ is a commutative ordered group;

(2) $(K,+,;0)$ is a ring (not necessarily commutative or associative);

(3) the multiplication of the ring K is compatible with the order in such a way that $0 \le x, y \in K$ yields $0 \le xy$, i.e., K is a model for $(\forall x)\,(\forall y)\,(x \ge 0 \wedge y \ge 0 \rightarrow x \cdot y \ge 0)$.

Therefore, an ordered ring is a ring such that its additive group is ordered and, moreover, the ring homotheties corresponding to positive elements are positive endomorphisms of the ordered group in question. We shall often ascribe to a ring the properties of the corresponding ordered group. Thus, for instance, the notions of a lattice- or linearly ordered ring, of the positive cone, etc., refer to the ordered group of a ring and need no further specifications. The order of K is called a *ring order* provided it obeys all the conditions from **(1)** to **(3)**.

The ordered ring K is termed *commutative* if alongside with **(1)** - **(3)** the following axiom is also fulfilled

(4) $(\forall x)\,(\forall y)\,(xy = yx)$.

A subset P of the ring K is the positive cone of a certain ring order iff

$$P \cap (-P) = \{0\};\quad P + P \subset P;\quad P \cdot P \subset P.$$

In the lattice-ordered ring K alongside with the relations considered in **4.4.2**, the following relations also hold: $(xy)^+ \le x^+y^+ + x^-y^-$; $(xy)^- \le x^+y^- + x^-y^+$; $|xy| \le |x| \cdot |y|$.

4.4.8. Any lattice-ordered ring K can be transformed into an ordered B-group, in which case K will not, in general, be a B-ring. The point is that the ring multiplication cannot obligatory be a contractive operation relative to the corresponding B-metric. In order to exclude this undesirable phenomenon, a more close interrelation between the multiplication and order is required. The lattice-ordered ring K is called an *f-ring* provided it satisfies the following condition: if $x, y \in K$ and $x \wedge y = 0$, then $(ax) \wedge y = 0$ and $(xa) \wedge y = 0$ for any $a \in K$. It should be observed that in any f-ring the following condition is fulfilled: $|x| \wedge |y| = 0 \rightarrow xy = 0$. If in an f-ring there are no nilpotent elements, then the converse statement is also valid or, as it is often said, the *f-ring is exact.* In particular, an f-ring without zero divisors is linearly ordered, and a linearly ordered ring without nilpotent elements is known to contain no zero divisors. Among the other properties of f-rings let us recall the following:

$$(x \vee y)z = (xz) \vee (yz); \quad z(x \vee y) = (zx) \vee (zy);$$
$$(x \wedge y)z = (xz) \wedge (yz); \quad z(x \wedge y) = (zx) \wedge (zy);$$
$$|xy| = |x||y|.$$

For any lattice-ordered ring K the following statements are equivalent:

(1) *K is an f-ring;*

(2) $\{xy\}^{\perp\perp} \le \{x\}^{\perp\perp} \wedge \{y\}^{\perp\perp}$;

(3) $d(xy,uv) \le d(x,u) \vee d(y,v)$.

◁ Assume that K is an f-ring. If $|x| \wedge |u| = 0$ or $|y| \wedge |u| = 0$, then $|xy| \wedge |u| = (|x||y|) \wedge |u| = 0$. Therefore, either $u \in \{x\}^{\perp}$ or $u \in \{y\}^{\perp}$ affords $u \in \{x \cdot y\}^{\perp}$, i.e., $\{x\}^{\perp} \cup \{y\}^{\perp} \subset \{xy\}^{\perp}$. Hence, $\{xy\}^{\perp\perp} \le (\{x\}^{\perp} \cup \{y\}^{\perp})^{\perp} = \{x\}^{\perp\perp} \wedge \{y\}^{\perp\perp}$. Let now condition (2) be fulfilled. It should be observed that $|xy - uv| = |x(y - v) + (x - u)v| \le$ $\le |x||y - v| + |x - u||v|$ and, hence,

$$\{xy - uv\}^{\perp\perp} \le \{y - v\}^{\perp\perp} \vee \{x - u\}^{\perp\perp}.$$

This inequality is equivalent to (3). Let us, finally, assume that the mapping $(x,y) \to xy$ is a contraction. In (3) let us put $u = 0$, $v = -y$ and rewrite it as $\{x \cdot y\}^{\perp\perp} \subset \{x\}^{\perp\perp} \vee \{0\}^{\perp\perp} = \{x\}^{\perp\perp}$, or $\{xy\}^{\perp} \supset \{x\}^{\perp}$. Analogously, $\{xy\}^{\perp} \supset \{y\}^{\perp}$ and, hence, K is an f-ring. ▷

4.4.9. *Any (associative, commutative) f-ring K together with a B-predicate σ and corresponding B-metric d is an algebraic B-system which is a B-model for the axioms of an (associative, commutative) totally ordered ring. In this case the element $0 \ne e \in K$ is a ring unity of the B-ring under discussion iff e is an order and ring unity of the ring K.*

◁ As has been proved in **4.4.6**, K is a linearly ordered B-group with σ and d specified above. Adjoin to this group a contraction mapping $(x,y) \to xy$ and prove that the the algebraic B-system obtained is an f-ring. Associativity, commutativity and distributivity in the B-system of K trivially follow from the corresponding properties of the ring K. Let us check the compatibility condition **4.4.7** (3). To this end let us observe that by virtue of **4.4.7** and **4.4.8** (2), we have

$$\{(xy)^-\}^\perp \geq \{x^+y^-\}^\perp \wedge \{x^-y^+\}^\perp \geq \{x^-\}^\perp \wedge \{y^-\}^\perp .$$

According to the definition of d, we conclude that $p(x) \wedge p(y) \leq p(xy)$. The task now is to calculate the Boolean truth-values by rules **4.1.8** :

$$\begin{aligned} &|(\forall x)(\forall y)(x \geq 0 \wedge y \geq 0 \rightarrow xy \geq 0)|^K \\ &= \bigwedge_{x,y \in K} |x \geq 0|^K \wedge |y \geq 0|^K \Rightarrow |xy \geq 0|^K \\ &= \bigwedge_{x,y \in K} p(x) \wedge p(y) \Rightarrow p(x \cdot y) = \mathbf{1}. \end{aligned}$$

It should be further observed that for $e \in K$ the equality $\mathbf{1} := |\theta < e|^K = |e \geq 0 \wedge e \neq 0|^K$ implies $p(e) \wedge d(e,0) = \mathbf{1}$, i.e., $e \geq 0$ and e is not an order unit. On the other hand,

$$|(\forall x)(xe = ex = x)|^K = \bigwedge_{x \in K} d(x,ex)^* \wedge d(x,xe)^*,$$

and, hence, e is a unit of the B-ring iff e is an order unit in K and for any $x \in K$ we have $d(xe,x) = d(ex,x) = 0$. The last fact implies that $x = ex = xe$, which is the required proof. ▷

4.4.10. Theorem. *Let $\mathcal{G}$ be an ordered group in the model* $\mathbf{V}^{(B)}$ *and* $G := \mathcal{G}\downarrow$. *Then G is an ordered group extended relative to the Boolean algebra of projections $\mathcal{B}$, and there is an isomorphism j from $\mathcal{B}$ to B such that*

$$b \leq [0 \leq x] \leftrightarrow 0 \leq j(b)x \quad (x \in G, b \in B).$$

In this case the following equivalences hold:

(1) $\mathbf{V}^{(B)} \models$ *"$\mathcal{G}$ is directed (integrally-closed, Archimedean)" $\leftrightarrow$ "G is directed (integrally-closed, Archimedian)";*

(2) $\mathbf{V}^{(B)} \models$ *"$\mathcal{G}$ is lattice-ordered (order-complete)" $\leftrightarrow$ "G is lattice-ordered (order-complete)";*

(3) $\mathbf{V}^{(B)} \models$ *"$\mathcal{G}$ is an ordered ring" $\leftrightarrow$ "G is an extended ordered ring with the Boolean algebra of projections $\mathcal{B}$";*

(4) $\mathbf{V}^{(B)} \models$ *"$\mathcal{G}$ is a linearly ordered skew field" $\leftrightarrow$ "G is an extended f-ring without*

nilpotent elements, $\mathcal{B}$ is an algebra of projections on all possible components of $\mathcal{G}$, and any regular element in G is invertible".

⊲ The fact that G is an extended group with complete Boolean algebra of projections $\mathcal{B}$ has been established in **4.2.7**. Let $\mathcal{G}^+$ be the positive cone of the group $\mathcal{G}^+$ inside $\mathbf{V}^{(B)}$. Then

$$[\mathcal{G}^+ + \mathcal{G}^+ \subset \mathcal{G}^+] = [\mathcal{G}^+ \cap -\mathcal{G}^+ = \{0\}] =$$
$$[((\forall x) \in \mathcal{G})(x + \mathcal{G}^+ = \mathcal{G}^+ + x)] = \mathbf{1}.$$

Assign $G^+ := \mathcal{G}^+ \downarrow$ and observe that, according to the rules of descents of intersection and image, $G^+ + G^+ \subset G^+$, $G^+ \cap -G^+ = \{0\}$. Then, for any $x \in G$ we have $[x + \mathcal{G}^+ = \mathcal{G}^+ + x] = \mathbf{1}$, i.e., $x + \mathcal{G}^+ = \mathcal{G}^+ + x$. In this case, however,

$$(x + G^+) = (x + \mathcal{G}^+)\downarrow = (\mathcal{G}^+ + x)\downarrow + x.$$

Therefore, G is an ordered group with positive cone G^+. In **4.2.7** we have proved that the isomorphism $j: B \to \mathcal{B}$ does exist, in which case the relations $b \le [x = y]$ and $j(b)x = j(b)y$ are equivalent. Let us choose $x \in G$ and remark that $[0 \le x \leftrightarrow (\exists y \in \mathcal{G}^+)(x = y)] = \mathbf{1}$. This implies that $b \le [0 \le x]$ iff $b \le [(\exists y \in \mathcal{G}^+)(x = y)]$. The last statement is equivalent to the existence of an $y \in \mathcal{G}^+\downarrow =: G^+$ such that either $b \le [x = y]$ or $j(b)x = j(b)y \ge 0$. Let us now prove equivalences **(1)** - **(4)**.

(1) If $\mathcal{G}$ is directed, then $[\mathcal{G}^+ - \mathcal{G}^+ = \mathcal{G}] = \mathbf{1}$, which is equivalent to the fact that G is directed, since $(\mathcal{G}^+ - \mathcal{G}^+)\downarrow = \mathcal{G}^+\downarrow - \mathcal{G}^+\downarrow = G^+ - G^+$. When $\mathcal{G}$ is integrally-closed, this implies nothing but

$$\wedge\{[x \le 0] : [(\exists y \in \mathcal{G})(\forall n \in \omega^\wedge)(nx \le y)] = \mathbf{1}\} = \mathbf{1}.$$

Hence, $\mathcal{G}$ is integrally-closed iff for every $x \in G$ the following implication is valid:

$$(\exists y \in G)([(\forall n \in \omega^\wedge)(nx \le y)] = \mathbf{1} \to [x \le 0] = \mathbf{1}),$$

or

$$((\exists y \in G)(\forall n \in \omega)[n^\wedge x \le y] = \mathbf{1}) \to [x \le 0] = \mathbf{1}.$$

The last line is an equivalent presentation of the fact that the group G is integrally-closed. The statement that G is Archimedean is proved analogously.

(2) Let $\mathcal{G}$ be lattice-ordered. Let us prove that on the algebraic system G the closed

formula $(\forall x)(\forall y)(\exists z)(z = \sup\{x, y\})$ is valid, i.e., in G for any two elements there is a least upper bound. If x and $y \in G$, then $[\{x,y\} \subset \mathscr{G}] = \mathbf{1}$, and, therefore, $[(\exists u \in \mathscr{G})(u = \sup\{x, y\})] = \mathbf{1}$. By virtue of the maximum principle, there is a $z \in \mathbf{V}^{(B)}$ such that

$$[z \in \mathscr{G}] \wedge [z = \sup\{x, y\}] = \mathbf{1}.$$

This implies that, on the one hand, $z \in G$, while on the other,

$$|z = \sup\{x,y\}|^{\mathscr{G}\downarrow} = \mathbf{1}.$$

By virtue of the definition of order, we deduce from here that $z = x \vee y$. Analogous considerations make us conclude that there is a least upper bound $x \wedge y$.

Assume now that [$\mathscr{G}$ is an order-complete group] = $\mathbf{1}$. Let us show that in this case G is also order-complete. First we recall the following equivalent definition of the least upper bound $\sup(A)$ of a set A in an arbitrary ordered set

$$\{\sup(A)\} = \pi_{\leq}(A) \cap \pi_{\leq}^{-1}(\pi_{\leq}(A)).$$

Let us now choose in the system $\mathscr{G}\downarrow$ an arbitrary subset A bounded from above. This implies that $\pi_{\leq}(A) \neq \varnothing$. In this case, however, according to the rules of descents and ascents of polars, $[\pi_{\leq}(A\uparrow) \neq \varnothing] = \mathbf{1}$, or, which is equivalent, [$A\uparrow$ is a subset bounded from above in $\mathscr{G}$] = $\mathbf{1}$. Hence, according to the maximum principle, we deduce that for a certain $a \in \mathscr{G}\downarrow$ we have

$$[\{a\} = \{\sup(A\uparrow)\} = \pi_{\leq}(A\uparrow) \cap \pi_{\leq}^{-1}(\pi_{\leq}(A\uparrow))] = \mathbf{1}.$$

Applying now the required rules of descents and ascents again, we deduce $a = \sup(\mathrm{mix}(A))$. Finally, making use of the fact that the relation $\leq$ is fully extensional, we conclude $\sup(\mathrm{mix}(A)) = \sup(A)$. Therefore, A has a least upper bound and, hence, G is an order-complete group.

(3) This statement follows from **4.2.8** and the properties of G established earlier.

(4) Let $\mathbf{V}^{(B)} \models$ "$\mathscr{G}$ is a linearly ordered skew field". By virtue of (3) and **4.2.8**, we can conclude that G is an ordered extended associative ring with the Boolean algebras of positive projections $\mathcal{B}$, having no nilpotent elements. Since $\mathscr{G}$ is a model for $(\forall x)(\forall y)(x \wedge y = 0 \rightarrow x = 0 \vee y)$, for any $x, y \in G$ we have $[x \wedge y = 0] \leq (x = 0) \vee \vee (y = 0)$. If $x \wedge y = 0$, then $b^* \leq [x = 0]$ and $b \leq [y = 0]$, or $j(b)x = x$ and $j(b)y = 0$ for a suitable $b \in B$. Hence, we easily deduce that $\mathcal{B}$ is a Boolean algebra of projections on

components. In this case, however, the fact that G is orthogonally complete is equivalent to the statement that G is extended relative to $\mathscr{B}$. Since the projections $j(b)$ $(b \in B)$ are multiplicative (see **4.2.8**), the kernel of any projections is a ring ideal. This directly leads us to the conclusion that the characteristic property of an f-ring is valid in G (see **4.4.8** (2)). Conversely, if G obeys the conditions specified in (4), then, by virtue of (2), $[\mathscr{G}$ is a lattice-ordered ring$] = \mathbf{1}$. As is readily seen, $\mathscr{G}$ is also an f-ring without nilpotent elements inside $\mathbf{V}^{(B)}$. In this case, however, for $x, y \in G$ it follows from $[xy = 1] = \mathbf{1}$ that $[|x| \wedge |y| = 0] = \mathbf{1}$, or $|x| \wedge |y| = 0$, and, hence, there is such an element $b \in B$ that $j(b)x = 0$ and $j(b^*)y = 0$. Therefore, $b \leq [x = 0]$ and $b^* \leq [y = 0]$ and, hence, $[x = 0 \vee y = 0] \geq b \vee b^* = \mathbf{1}$. We thus established that $\mathbf{V}^{(B)} \models$ "$\mathscr{G}$ has no zero divisors". An f-ring with no zero divisors is, however, known to be linearly ordered, so that $\mathbf{V}^{(B)} \models$ "$\mathscr{G}$ is linearly ordered". Finally, by virtue of **4.2.8**, the nonzero elements of $\mathscr{G}$ are invertible and, hence, $\mathbf{V}^{(B)} \models$ "$\mathscr{G}$ is a linearly ordered skew field". ⊳

4.4.11. Therefore, both linearly ordered groups and f-rings are in a certain way transformed into B-groups and B-rings. By virtue of **4.3**, this implies that they have Boolean-valued realizations which are linearly ordered groups and rings, respectively. Hence, any information on the structure of linearly ordered groups and rings can be used for studying more general classes of groups and rings. Let us demonstrate this statement with examples of the well-known facts presented below (see **[13, 59]**).

(1) **Hölder theorem.** *Any Archimedean linearly ordered group is isomorphic to a subgroup of the additive group of real numbers.*

(2) *Any Archimedean directed group is commutative.*

(3) **Theorem.** *An Archimedean linearly ordered ring is either zero (i.e., the product of any two elements is zero), or order and algebraically isomorphic to a uniquely determined subring of the field of real numbers.*

4.4.12. Theorem. *Let G be an Archimedean lattice-ordered group with its base isomorphic to a Boolean algebra B. Then in the Boolean-valued model $\mathbf{V}^{(B)}$ there is a subgroup $\mathscr{G}$ of the additive group of the field of real numbers such that the lattice-ordered group $G' := \mathscr{G} \downarrow$ is the maximal extension of the group G.*

◁ By **4.4.6**, the group G can be transformed into an ordered B-group. Let $\mathcal{G}$ be a Boolean-valued realization of this algebraic B-system. Then, by **4.3.3**, $\mathcal{G}$ is a linearly ordered group inside $\mathbf{V}^{(B)}$. In line with theorem **4.4.10**, $G' := \mathcal{G}\downarrow$ is a lattice-ordered group, in which case, as is known, $G' = \operatorname{mix}(\iota(G))$, where ι is the canonical isomorphism from G to G'. If $b \in B$, while $L_b \in \mathfrak{B}(G)$ and $\pi_b \in \mathfrak{Pr}(G')$ are the component and projection, respectively, then the conditions $x \in L_b$ and $(I - \pi_b)(\iota(x)) = 0$ are equivalent for any $x \in G$. Indeed, by the definition of a B-metric on G (see **4.4.5**), $\pi_b\iota(x) = \iota(x)$ is fulfilled iff $b^* \le [\iota(x) = 0]$. In this case, as is known,

$$[\iota(x) = 0] = [\iota(x) \ne 0]^* = d(x,0)^*.$$

We thus established that the correspondence $L' \mapsto \iota^{-1}L' \cap G$, $L' \in \mathfrak{B}(G')$ is an isomorphism of the bases $\mathfrak{B}(G')$ and $\mathfrak{B}(G)$. Let us now choose $0 < x \in G'$. If $x = \operatorname{mix}(\pi_\xi \iota(x_\xi))$, then $0 < \pi_\xi \circ \iota(x_\xi) \le \iota(x_\xi)$ for a certain ξ. According to the isomorphism of bases presented above, there is a $0 < z \in G$, for which $z \in \{\pi_\xi \circ \iota(x_\xi)\}^{\perp\perp}$. Now for $x_0 := x_\xi \wedge z$ we have

$$0 < \iota(x_0) \le \iota(z) \wedge \pi_\xi \circ \iota(x_\xi) \le \pi_\xi \circ \iota(x_\xi) \le x.$$

Therefore, $\iota(G)$ is minorant in G'. Let us now assume that for some $x, y \in G'$ it is fulfilled that $n|x| \le y$ $(n \in \omega)$. Let $y = \operatorname{mix}(\pi_\xi \iota(y_\xi))$ and $x = \operatorname{mix}(\pi_\xi \iota(x_\xi))$ for some families (x_ξ) and (y_ξ) in G and a partition of unity (π_ξ) in $\mathfrak{Pr}(G')$. Designate $\Xi_0 := \{\xi \in \Xi : \pi_\xi \circ \iota(|x_\xi|) = 0\}$. Since $\iota(G)$ is minorant, for any $\xi \in \Xi \setminus \Xi_0$ there is a $0 < u_\xi \in G$, for which $\iota(u_\xi) \le \pi_\xi(\iota|x_\xi|)$. Then, for the same ξ and for all $n \in \omega$ we get

$$\iota(nu_\xi) \le \pi_\xi \circ \iota(n|x_\xi|) = \pi_\xi(n|x|) \le \pi_\xi y = \pi_\xi \circ \iota(y_\xi) \le \iota(y_\xi),$$

or $nu_\xi \le y_\xi$. Since G is Archimedean, we get $u_\xi = 0$, which implies that $\Xi_0 = \Xi$ and, hence, $x = 0$. Therefore, the group G' is Archimedean and, by **4.4.10**, $[\mathcal{G}$ is Archimedean$] = \mathbf{1}$. In line with the Hölder theorem **4.4.11** (1) $\mathcal{G}$ is isomorphic to an additive subgroup of the group of real numbers $\mathcal{R}$. According to theorem **4.3.4**, $\mathcal{G}$ can be assumed to be a linearly ordered subgroup in $\mathcal{R}$. ▷

4.4.13. Theorem. *Let K be an Archimedean f-ring. Then in K there are two components K_0 and K_1 complementary to each other such that if the bases $\mathfrak{B}(K_0)$ and $\mathfrak{B}(K_1)$ are isomorphic to the Boolean algebras B_0 and B_1, respectively, then the following statements are valid:*

(1) *in the Boolean-valued model $\mathbf{V}^{(B)}$ there is a subgroup $\mathcal{R}_0$ of the group of real*

numbers such that the lattice-ordered group $K_0' := \mathcal{R}_0 \downarrow$ *with zero multiplication is the maximal extension of the* f*-ring* K_0*;*

(2) *in the Boolean-valued model* $\mathbf{V}^{(B)}$ *there is a subring* $\mathcal{R}_1$ *of the ring of real numbers such that an* f*-ring* $K_1' := \mathcal{R}_1' \downarrow$ *is the maximal extension of* K*. In this case an* f*-ring* $K_0' \oplus K_1'$ *is the maximal extension of the* f*-ring* K*.*

$\lhd$ As we have seen in **4.4.12**, the realization of the additive group of the f-ring K in the model $\mathbf{V}^{(B)}$, $B = \mathfrak{B}(K)$, will be a subgroup of the additive group of real numbers. According to **4.4.9**, however, K is a B-ring, while by theorem **4.3.3**, $[\mathcal{R}$ is a ring$] = \mathbf{1}$. Let us set $b_0 := [\mathcal{R}$ is a zero ring$]$ and $b_1 := [\mathcal{R}$ is a subring of the ring of real numbers$]$. By the transfer principle and theorem **4.4.11** (3), $b_0 \vee b_1 = \mathbf{1}$. On the other hand, $b_0 \wedge b_1 = \mathbf{0}$, since a ring cannot be simultaneously both zero and a subring of the ring of real numbers. Let K_0 and K_1 be components in K corresponding to the elements b_0 and b_1, i.e., K_0 and K_1 are determined by the conditions

$$x \in K_l \leftrightarrow d(x,0) \leq b_l \quad (l = 0,1),$$

where d is the B-metric of the B-system K. Assign $B_l := [0,b_l]$ and observe that the base $\mathfrak{B}(K_l)$ is isomorphic to B_l, in which case b_l is the unity of the algebra B_l. Let us introduce the denotation $\mathcal{R}_l := \pi_l^*(\mathcal{R}) \in \mathbf{V}^{(B_l)}$, where $\pi_l : b \to b \wedge b_l$, $b \in B$. Since π_l is an isomorphism of B on B_l; therefore, $\mathbf{V}^{(B_0)} \models$ "$\pi_0^*(\mathcal{R})$ is a subgroup of the additive group of real numbers" and $\mathbf{V}^{(B_1)} \models$ "$\pi_1^*(\mathcal{R})$ is a subring of the ring of real numbers". By theorem **4.4.12**, $K' := K \downarrow$ is an extension of the ordered group K. As far as $b_l = [\pi_l^*(\mathcal{R}) \simeq \mathcal{R}]$, where $K_l' := \mathcal{R}_l \downarrow \simeq j(b_l)(K_l)$ and, hence, $K' \simeq K_0' \oplus K_1'$. Therefore, K' is the maximal extension of K. $\rhd$

CHAPTER 5

BOOLEAN REPRESENTATIONS IN FUNCTIONAL ANALYSIS

We have already convinced ourselves that the Boolean-valued universe $\mathbf{V}^{(B)}$ associated with a fixed Boolean algebra B is one of the arenas where mathematical events occur. Indeed, by virtue of the transfer and maximum principles, in $\mathbf{V}^{(B)}$ there are numbers and groups, Lebesque and Riemann integrals, the Radon-Nikodym theorems are fulfilled, and the Jordan expansion of a matrix is implementable. The elementary technique of descents and ascents, which we got acquainted with when considering algebraic systems, shows each of mathematical objects in $\mathbf{V}^{(B)}$ to be a realization of an analogous classical object with an additional structure determined by the algebra B. In particular, this consideration refers to functional-analytical objects as well.

In the present chapter facts associated with Boolean-valued realization of the last objects are considered. The most important and remarkable achievement of Boolean-valued analysis is the fact that it establishes a inseparable immanent interrelation of the realizations discussed above with the concepts of the theory of ordered vector spaces and, above all, with K-spaces introduced by L.V.Kantorovich at the beginning of the thirties.

5.1 Vector Lattices

In this section the basic notions of the theory of vector lattices are briefly presented, a more detailed presentation can be found elsewhere [**1, 103, 159, 279**].

5.1.1. Let $\mathbf{F}$ be a linearly ordered field. An *ordered vector space* over $\mathbf{F}$ is an algebraic system E of signature $(+,0,\le\ \lambda)$, where λ ranges over the set of elements of the field $\mathbf{F}$, and denotes, when λ is fixed, a unary operation on F such that the following conditions are met:

(1) $(E,+,0,\le)$ is a commutative ordered group;

(2) $(E,+,0,\lambda)$ is a vector space over $\mathbf{F}$;

(3) every positive element of the field $\mathbf{F}$ determines the respective positive endomorphism of the ordered group $(E,+,0,\leq)$ by its action of scalar multiplication.

Therefore, an ordered vector space can be defined as a pair $(E,\leq)$, where E is a vector space over the field $\mathbf{F}$, while $\leq$ is a *vector order* in E, i.e., a relation of order in E which agrees with the structure of a vector space. The last, speaking informally, implies that inequalities in E "can be added and multiplied by the positive elements of the field $\mathbf{F}$". Introducing a vector order in a vector space E over the field $\mathbf{F}$ is equivalent to defining a set (*positive cone*) $E^+ \subset E$ with the following properties: $E^+ + E^+ \subset E^+$; $\lambda E^+ \subset E^+$ $(0 \leq \lambda \in \mathbf{F})$; $E^+ \cap E^+ = \{0\}$. In this case the order $\leq$ and the cone E^+ are connected by the relation

$$x \leq y \leftrightarrow y - x \in E^+ \quad (x,y \in E).$$

The notions and results of the theory of ordered groups are undoubtedly applicable to ordered vector spaces. It is, for instance, obvious that for an ordered vector space the notions of being Archimedean, linearly ordered, of the σ-ideal, etc., refer to the corresponding ordered group.

5.1.2. *A vector space* is an ordered vector space which is a lattice-ordered group. It should be recalled that disjointness of the vector lattice E is introduced through the formula

$$\perp := \{(x,y) \in E \times E : |x| \wedge |y| = 0\}.$$

A component (or *band*) of the vector lattice E is a set of the type

$$M^{\perp} := \{x \in E : (\forall y \in M) x \perp y\},$$

where M is an arbitrary nonempty set in E. The totality of all components of the vector lattice ordered by inclusion forms a complete Boolean algebra $\mathfrak{B}(E)$, with its Boolean operations having the following form:

$$L \wedge K = L \cap K, \quad L \vee K = (L \cup K)^{\perp\perp}, \quad L^* = L^{\perp} \quad (l,k \in \mathfrak{B}(E)).$$

The algebra $\mathfrak{B}(E)$ is called the *base of E*.

Let K be a component (or band) of the vector lattice E and $0 \leq x \in E$. If in E there is an element $\sup\{u \in K: 0 \in u \leq x\}$, then it is termed the *projection* of x on the component K and denoted by $[K]x$ (or $\mathrm{Pr}_K x$). For an arbitrary $x \in E$ we set $[K]x$: $[K]x^+ - [K]x^-$. The projection of an element $x \in E$ on the component K exists iff the expansion $x = y + z$ is valid, where $y \in K$ and $z \in K^{\perp}$, in which case $y = [K]x$ and $z = [K^{\perp}]x$. Let us assume that any element $x \in E$ has a projection on K. Then the operator $x \to [K]x$ $(x \in E)$ is linear, idempotent, and $0 \le [K]x \le x$ for all $0 \le x \in E$. The vector lattice is said *to admit projections on components* (on principal components) if for any component (principal component) K the operator of projection $[K]$ is determined. If the vector lattice allows projections on components and any disjoint set of positive elements in it has a supremum, then it is called *extended* (or *universally complete*).

5.1.3. An element $\mathbf{1} \in E$ is called an (order) *unit* if $\{\mathbf{1}\}^{\perp\perp} = E$, i.e., if in E there are no nonzero elements disjoint from $\mathbf{1}$. Let for a certain $0 \le e \in E$ we have $e \wedge (\mathbf{1} - e) = 0$. In this case we say that e is a *unit element* (relative to $\mathbf{1}$). The set $\mathfrak{E}(\mathbf{1}) := \mathfrak{E}(E)$ of all unit elements with the order induced from E is a Boolean algebra. The lattice operations in $\mathfrak{E}(\mathbf{1})$ are inherited from E, while the Boolean extension has the form $e^{*} = \mathbf{1} - e$ $(e \in \mathfrak{E}(\mathbf{1}))$.

Henceforth, unless the field $\mathbf{F}$ is explicitly indicated, the vector lattice over the linearly ordered field of real numbers $\mathbf{R}$ is implied. In the ideal $I(u) := \bigcup_{n=1}^{\infty} [-nu, nu]$ generated by an element $0 \le u \in E$ the following seminorm can be introduced:

$$\|x\|_u := \inf\{\lambda \in \mathbf{R} : |x| \le \lambda u\} \quad (x \in I(u)).$$

If $I(u) = E$, then we say that u is a *strong unit*, while E is the *vector lattice* of *bounded elements*. The seminorm $\|\cdot\|_u$ is a norm iff E is Archimedean.

An element $x \ge 0$ of the vector lattice is called *discrete* if $[0, x] = [0,1]x$, i.e., it follows from $0 \le y \le x$ that $y = \lambda x$ for some $0 \le \lambda \le 1$. A vector lattice E is termed *discrete* if for any $0 < y \in E$ there is a discrete element $x \in E$ such that $0 < x \le y$. In the case when E has no nonzero discrete elements, it is said to be *continuous*.

5.1.4. *A Kantorovich space*, or, in brief, a *K-space* is a vector lattice such that every order-bounded nonempty subset has exact bounds. Sometimes, instead of this term, K-space, a more expanded term is used, i.e., a conditionally order-complete vector lattice. If in a vector lattice there are exact bounds only of countable bounded sets, then it is called a K_{σ}-*space*. Any K_{σ}-space and, moreover, any K-space is Archimedean.

The set of the projections on all possible components in E is denoted by the symbol $\mathfrak{Pr}(E)$. For projections π and ρ we set $\pi \le \rho$ iff $\pi x \le \rho x$ at all $0 \le x \in E$.

Theorem. *Let E be an arbitrary K-space. Then projecting onto a component determines an isomorphism $K \to [K]$ of the Boolean algebras $\mathfrak{K}(E)$ and $\mathfrak{Pr}(E)$. If there a unit in E, then the mapping $\pi \to \pi\mathbf{1}$ from $\mathfrak{Pr}(E)$ to $\mathfrak{E}(E)$ and $e \to \{e\}^{\perp\perp}$ from $\mathfrak{E}(E)$ to $\mathfrak{K}(E)$ are also* *isomorphisms of Boolean algebras.*

The projection π_u on a component of the type $\{u\}^{\perp\perp}$, where $0 \le u \in E$, can be obtained by a simpler rule than that in **5.1.2**, namely,

$$\pi_e x = \sup\{x \wedge (nu) : n \in \mathbb{N}\} \quad (0 \le x \in E).$$

In particular, in an K_σ-space there is a projection of any element on any principal component.

Let E be a K_σ-space with a unit $\mathbf{1}$. The projection of the unit on the component $\{x\}^{\perp\perp}$ is called the *trace* of the element x and denoted by the symbol e_x. Therefore, $e_x := \sup\{\mathbf{1} \wedge (n|x|) : n \in \mathbb{N}\}$. The trace e_x serves both as a unit in $\{x\}^{\perp\perp}$ and a unit element in E. For every real number λ the trace of the positive part of an element $\lambda\mathbf{1} - x$ is denoted by e_λ^x, i.e., $e_\lambda^x := e_{(\lambda\mathbf{1}-x)^+}$. The function $\lambda \to e_\lambda^x$ $(\lambda \in \mathbf{R})$ arising in this case is called the *spectral function* or *characteristic* of the element x.

5.1.5. **(1)** *An ordered algebra over* $\mathbf{F}$ (or *an ordered* $\mathbf{F}$*-algebra*) is any ordered ring E with operations such that the external law has the field $\mathbf{F}$ as the set of its operators and, together with addition given in E, determines the structure of an order vector space in E. Therefore, an ordered algebra can be defined as an algebraic system of signature $(+, 0, \le, \lambda, \cdot)$, where λ runs over the set of elements of the field $\mathbf{F}$ and denotes a unary operation for every λ. In this case the following conditions are met:

(a) $(E, +, 0, \le, \lambda)$ is an ordered vector space;
(b) $(E, +, \mathbf{0}, \le, \cdot)$ is an ordered ring.

Let us say that E is a *lattice-ordered algebra* (*f-algebra*) if E is an ordered algebra and the corresponding ordered ring is lattice-ordered (is an f-ring). *Exact* is such an f-algebra wherein for any two elements x and y it follows from $x \cdot y = 0$ that $x \perp y$. It can be easily proved that an f-algebra is exact iff there no nonzero nilpotent elements in it. The fact that an f-algebra is exact is equivalent to the absence of positive elements whose square is nonzero (see **4.4.8**).

(2) *A complex vector lattice* is the complexification $E \oplus iE$ (i is the imaginary unity) of a real vector lattice E. In this case it is often additionally required that any element $z \in E \oplus iE$ should have the modulus

$$|z| := \sup(\mathrm{Re}(e^{i\theta} z) : 0 \le \theta \le \pi\}.$$

For a K-space this requirement is excessive, so that *a complex K-space* is the complexification of a real K-space. Speaking about the order properties of a complex vector lattice $E\oplus iE$, we mean its real part E. The notions of a sublattice, ideal, projection band, etc., are naturally extended to the case of a complex vector lattice by way of appropriate complexification.

5.1.6. The order relation in a vector lattice is related with various types of convergence. Let $(A,\leq)$ be a set filtered upwards, and let us consider a net $(x_\alpha):=(x_\alpha)_{\alpha\in A}$ in E, which is called *increasing (decreasing)* if $x_\alpha\leq x_\beta$ $(x_\beta\leq x_\alpha)$ for $\alpha\leq\beta$ $(\alpha,\beta\in A)$.

The net (x_α) is said to be *o-convergent* to an element $x\in E$ if in E there is a decreasing net $(e_\alpha)_{\alpha\in A}$ with the properties $\inf_{\alpha\in A}e_\alpha=0$ and $|x-x_\alpha|\leq e_\alpha$ $(\alpha\in A)$. In this case x is termed the *o-limit* of the net (x_α) and we write $x=o-\lim x_\alpha$, or $x_\alpha\xrightarrow{(o)}x$. In a K-space for an order-bounded net E also introduces the *upper and lower o-limits* (or *limit superior* and *limit interior*) through the formulas:

$$\limsup_{\alpha\in A} x_\alpha:=\overline{\lim_{\alpha\in A}}\, x_\alpha:=\inf_{\alpha\in A}\sup_{\beta\geq\alpha}x_\beta,$$
$$\liminf_{\alpha\in A} x_\alpha:=\underline{\lim}_{\alpha\in A}\, x_\alpha:=\sup_{\alpha\in A}\inf_{\beta\geq\alpha}x_\beta.$$

There is an evident relation between these objects:

$$x=o-\lim x_\alpha\leftrightarrow\limsup x_\alpha=x=\liminf x_\alpha.$$

The net $(x_\alpha)_{\alpha\in A}$ is said *to r-converge* (*to converge with regulator*) to $x\in X$ if there is an element $0\leq u\in E$ termed the *regulator of convergence*, and a numerical net $(\lambda_\alpha)_{\alpha\in A}\subset\mathbf{R}$ with the properties $\lim\lambda_\alpha=0$ and $|x-x_\alpha|\leq\lambda_\alpha u$ $(\alpha\in A)$. In this case x is termed the *r-limit* of the net (x_α) and we write $x=r\text{-}\lim x_\alpha$, or $x_\alpha\xrightarrow{(r)}x$. As is seen, the convergence with regulator is that in the normed space $(I(u),\|\cdot\|_u)$.

The presence of o-convergence in a K-space allows one to determined the sum of an infinite family $(x_\xi)_{\xi\in\Xi}$. Indeed, for $\theta:=\{\xi_1,\dots,\xi_n\}\in\mathcal{P}_{\text{fin}}(\Xi)$ let us denote $y_\theta:=x_{\xi_1}+\dots+x_{\xi_n}$. Then there arises a net $(y_\theta)_{\alpha\in\Theta}$, where $\Theta:=\mathcal{P}_{\text{fin}}(\Xi)$ is naturally ordered by inclusion. If there is an $x=o-\lim_{\theta\in\Theta}y_\theta$, then the element x is termed the *o-sum* of the family (x_ξ) and we write $x=\sum_{\xi\in\Xi}x_\xi$. It is obvious that, given $x_\xi\geq 0$ $(\xi\in\Xi)$, for the o-sum of the family (x_ξ) to exist it is necessary and sufficient that the net $(y_\theta)_{\theta\in\Theta}$ be order-bounded, in which case $o\text{-}\sum_{\xi\in\Xi}x_\xi=\sup_{\theta\in\Theta}y_\theta$. If the elements of the family (x_ξ) are pairwise disjoint, then

$$o\text{-}\sum_{\xi\in\Xi}x_\xi=\sup_{\xi\in\Xi}x_\xi^+-\sup_{\xi\in\Xi}x_\xi^-.$$

Any K-space E is *o-complete* in the following sense. If the net $(x_\alpha)_{\alpha\in A}$ in E satisfies the condition $\limsup |x_\alpha - x_\beta| = \inf_{\gamma\in A} \sup_{\alpha,\beta\geq\gamma} |x_\alpha - x_\beta| = 0$, then there is such an element $x\in E$, that $x = o - \lim x_\alpha$.

5.1.7. Examples

(1) Let $(E_\alpha)_{\alpha\in A}$ be a family of vector lattices (f-algebras) over the same ordered field $\mathbf{F}$. Then the Cartesian product $E := \prod_{\alpha\in A} E_\alpha$, considered with the coordinatewise operations and order, is a vector lattice (f-algebra) over the field $\mathbf{F}$. In this case the lattice E is order-complete, extended or discrete iff all the cofactors E_α have this property. The basis $\mathfrak{E}(E)$ is isomorphic to the product of the family of the Boolean algebras $(\mathfrak{E}(E))_{\alpha\in A}$. An element $e\in E$ is a unit iff $e(\alpha)$ is a unit in E_α for all $\alpha\in A$. In particular, the set $\mathbf{R}^A$ ($\mathbf{C}^A$) of all real (complex) functions on a nonempty set A is an extended discrete K-space (complex K-space). It should be emphasized that an extended K-space is often referred to as *universally complete* in the western literature.

(2) Any ideal and, therefore, the foundation of a vector lattice (of a K-space) is a vector lattice (a K-space). The base of a vector lattice is isomorphic to that of each of its foundations (a foundation is recalled to be an order-dense o-ideal). In particular, $l_p(A)$ is a K-space for any $1\leq p\leq\infty$ (see **(1)**).

(3) Let N be an o-ideal of a vector lattice E. Then the factor-space $\tilde{E} := E/N$ is also a vectors lattice, provided the order relation in it is determined by the positive cone $\varphi(E^+)$, where $\varphi: E\to\tilde{E}$ is the canonical factor-homomorphism. The vector lattice $\tilde{E}$ is Archimedean iff N is closed relative to convergence with regulator. If E is an f-algebra, and the o-ideal N is a ring ideal as well, then $\tilde{E}$ is an f-algebra. If E is a K_σ-space and N is sequentially o-closed, then $\tilde{E}$ is a K_σ-space, while the homomorphism φ is sequentially o-continuous. The bases of the vector lattice $\tilde{E}$ is isomorphic to the complete Boolean algebra of Δ-components $\mathfrak{K}_\Delta(E)$, where $\Delta := \{(x,y)\in E\times E : |x|\wedge|y|\in N\}$.

(4) Let $(\Omega,\mathfrak{A})$ be a measurable space, i.e., Ω is a nonempty set and $\mathfrak{A}$ is a σ-algebra of its subsets. Let us denote by $\mathfrak{M}(\Omega,\mathfrak{A})$ the set of all real (complex) measurable functions on Ω with the operations and order induced from $\mathbf{R}^\Omega$ (from $\mathbf{C}^\Omega$). Let us choose a σ-ideal $\mathfrak{N}$ of the algebra $\mathfrak{A}$. Let N be a set of such functions $f\in\mathfrak{M}(\Omega,\mathfrak{A})$ that $\{t\in\Omega : f(t)\neq 0\}\in\mathfrak{N}$, and assign $M(\Omega,\mathfrak{A},\mathfrak{N}) := \mathfrak{M}(\Omega,\mathfrak{A})/\mathfrak{N}$. Then $\mathfrak{M}(\Omega,\mathfrak{A})$ and $M(\Omega,\mathfrak{A},\mathfrak{N})$ are real (complex) K_σ-spaces and and f-algebras at the same time. Let us assume that $\mu:\mathfrak{A}\to\mathbf{R}\cup\{+\infty\}$ is a countably additive positive measure. The vector lattice

$M(\Omega,\mathfrak{A},\mu) := M(\Omega,\mathfrak{A},\mu^{-1}(0))$ will be an extended K-space provided the measure μ is either finite or σ-finite. Generally speaking, the order completeness of the lattice $M(\Omega,\mathfrak{A},\mu)$ is related with the direct sum property for the measure μ **[159, 227, 279]**. For simplicity, however, we shall confine ourselves to a σ-finite measure μ. The space $M(\Omega,\mathfrak{A},\mu)$ is continuous iff μ has no atoms. It should be recalled that the ***atom of a measure*** is a set $A \in \mathfrak{A}$ such that $0 < \mu(A)$ and if $A' \in \mathfrak{A}$, $A' \subset A$, then either $\mu(A') = 0$ or $\mu(A') = \mu(A)$. If $M(\Omega,\mathfrak{A},\mu)$ is discrete then the measure μ is *purely atomic*, i.e., any set of nonzero measure contains an atom of μ. The class of equivalence of a function which is identically equal to unity will be an order and ring unit in $M(\Omega,\mathfrak{A},\mu)$.

The base of the K-space $M(\Omega,\mathfrak{A},\mu)$ is isomorphic to the Boolean algebra $\mathfrak{A}/\mu^{-1}(0)$ of measurable sets modulo sets of zero measure. By virtue of (2), the spaces $L_p(\Omega,\mathfrak{A},\mu)$ $(1 \le p \le \infty)$, which are foundations of $M(\Omega,\mathfrak{A},\mu)$, are also K-spaces.

(5) Let H be a complex Hilbert space and $\mathfrak{U}$ be a strongly-closed commutative algebra of selfadjoint bounded operators in H. By the letter B we shall denote the set of all orthoprojections in H belonging to the algebra $\mathfrak{U}$. Then B is a complete Boolean algebra. Let $\mathfrak{U}_\infty$ be a set of all densely defined selfadjoint operators a in H such that the spectral function $\lambda \to e^a_\lambda$ $(\lambda \in \mathbf{R})$ of the operator a assumes its values in B. Let $\overline{\mathfrak{U}}_\infty$ be a set of densely defined normal operators a in H such that if $a = u|a|$ is the polar decomposition of a, then $|a| \in \mathfrak{U}_\infty$. In the sets $\mathfrak{U}_\infty$ and $\overline{\mathfrak{U}}_\infty$ the structure of an ordered vector space is introduced in a natural way. Thus, for a and $b \in \mathfrak{U}_\infty$ the sum $a+b$ and the product $a \cdot b$ are determined as the unique selfadjoint extensions of the operators $h \to ah + bh$, and $h \to a \cdot bh$, $h \in \mathfrak{D}(a) \cap \mathfrak{D}(b)$, where $\mathfrak{D}(c)$ is the domain of definition for c. Moreover, for $a \in \mathfrak{U}_\infty$ we set $a \ge 0$ iff $\langle ah,h \rangle \ge 0$ for all $h \in \mathfrak{D}(a)$. The operations and order in $\overline{\mathfrak{U}}_\infty$ are obtained by complexifying $\mathfrak{U}_\infty$.

The sets $\mathfrak{U}_\infty$ and $\overline{\mathfrak{U}}_\infty$ with the operations and order discussed above are an extended K-space and a complex extended K-space, respectively, with the base of unit elements B. In this case $\mathfrak{U}$ is a K-space of bounded elements in $\mathfrak{U}_\infty$.

(6) Let Q be a topological space, while $\mathcal{B}(Q) := \mathcal{B}(Q,\mathbf{R})$ be the set of all Borel functions from Q to $\mathbf{R}$ with pointwise operations of addition and multiplication, as well as with the pointwise order relation. Then $\mathcal{B}(Q,\mathbf{R})$ is a K_σ-space. By N we denote the set of such Borel functions $f \in \mathcal{B}(Q)$ that $\{t \in Q : f(t) \ne 0\}$ is a meager set (i.e., a set of the first category). Let $B(Q)$ be the factor-space of $\mathcal{B}(Q)/N$ with the operations and order induced from $B(Q)$. Then $B(Q)$ is a K-space whose base is isomorphic to the Boolean algebra of Borel subsets Q modulo the first-category sets. If the topological space Q is *Baire* (i.e., any open set in Q is not meager), then the base $\mathfrak{E}(B(Q))$ is isomorphic to the Boolean algebra of all regular open (or regular closed) subsets Q. Each of the spaces $\mathcal{B}(Q)$ and $B(Q)$ is an exact f-algebra. The function identically equal to unity serves as an order and ring unit in them. Having replaced $\mathbf{R}$ with $\mathbf{C}$, we get the complex K-space $B(Q)$.

(7) Let Q be again a topological space, while $C(Q)$ be the space of all continuous real functions on Q. Then $C(Q)$ is a sublattice and a subalgebra in $\mathcal{B}(Q)$. In particular, $C(Q)$ is an exact Archimedean f-algebra. Generally speaking, $C(Q)$ is not a K-space. The order completeness of $C(Q)$ is associated with the total disconnectedness of the space Q (see **1.2.5**) For a uniformizable topological space Q the base of the vector space $C(Q)$ is isomorphic to the algebra of regular open sets.

Let now $LSC(Q)$ be the set of (the equivalence classes of) lower semicontinuous functions $f: Q \to \overline{\mathbf{R}} := \mathbf{R} \cup \{\pm\infty\}$ such that $f^{-1}(-\infty)$ is nowhere dense, while the interior of the set $f^{-1}([-\infty,\infty))$ is dense in Q. As usual, two functions are considered equivalent if their values differ only on a meager set. The sum $f+g$ (the product $f \cdot g$) of the elements $f, g \in LSC(Q)$ will be determined as the lower semicontinuous regularization of the pointwise sum $t \to f(t) + g(t)$ $(t \in Q_0)$ (of the pointwise product $t \to f(t) \cdot g(t)$ $(t \in Q_0)$), where Q_0 is a dense subset of Q on which f and g are finite. Therefore, $LSC(Q)$ turns into an extended K-space and an f-algebra, in which case the base of $LSC(Q)$ is isomorphic to the algebra of regular open sets. Therefore, when Q is Baire, the K-spaces $B(Q)$ and LSC(Q) are isomorphic, while if Q uniformizable, then $C(Q)$ is a (order) dense sublattice in $LSC(Q)$.

5.1.8. A special role in the theory of vector lattices is played by spaces of continuous functions assuming infinite values on a nowhere dense set. In order to introduce such a space, some additional facts are to be employed. For an arbitrary function $f: Q \to \overline{\mathbf{R}}$ and a number $\lambda \in \overline{\mathbf{R}}$, we shall denote

$$\{f < \lambda\} := \{t \in Q : f(t) < \lambda\}, \quad \{f \le \lambda\} := \{t \in Q : f(t) \le \lambda\}.$$

(1) *Let Q be an arbitrary topological space, Λ be a dense set in $\overline{\mathbf{R}}$ and $\lambda \to U_\lambda$ $(\lambda \in \Lambda)$ be an increasing mapping from Λ to the set $\mathcal{P}(Q)$ ordered by inclusion. Then the following statements are equivalent:*

(a) *there is a unique and continuous function $f: Q \to \overline{\mathbf{R}}$ such that*

$$\{f < \lambda\} \subset U_\lambda \subset \{f \le \lambda\} \quad (\lambda \in \Lambda),$$

(b) *for any $\lambda, \mu \in \Lambda$, it follows from $\lambda < \mu$ that*

$$\mathrm{cl}(U_\lambda) \subset \mathrm{int}(U_\lambda).$$

◁ The implication (a) ⇒ (b) is trivial. Let us prove (b) ⇒ (a). For every $t \in Q$, put $f(t) := \inf\{\lambda \in \Lambda : t \in U_\lambda\}$. We, thus, have defined the function $f: Q \to \overline{\mathbf{R}}$ and can now easily prove that $\{f < \lambda\} \subset U_\lambda \subset \{f \le \lambda\}$. It is also obvious that

$$\{f < \lambda\} = \cup\{U_\mu : \mu < \lambda, \mu \in \Lambda\}, \quad \{f \le \lambda\} = \cap\{U_v : \mu < v, v \in \Lambda\}.$$

It should be observed that so far we have made use of only the isotonicity of the mapping $\lambda \to U_\lambda$. Let us also consider the mappings

$$\lambda \to V_\lambda := \operatorname{int}(U_\lambda),\ \lambda \to W_\lambda := \operatorname{cl}(U_\lambda) \quad (\lambda \in \Lambda).$$

These mappings are seen to increase as well and, hence, by what was above proved above, there are such functions g and $h: Q \to \overline{\mathbf{R}}$ that

$$\{g < \lambda) \subset V_\lambda \subset \{g \le \lambda\},\ \{h < \lambda) \subset W_\lambda \subset \{h \le \lambda\} \quad (\lambda \in \Lambda).$$

It follows from the definition of W_λ that for $\mu < \lambda$ $U_\mu \subset W_\lambda$. Since Λ in $\mathbf{R}$ is dense, for any $t \in Q$ and $v > f(t)$ there are such $\lambda, \mu \in \Lambda$ that $f(t) < \mu < \lambda < v$ and, hence, $t \in U_\mu \subset W_\lambda$ and $h(t) < \lambda < v$. Tending v to $f(t)$, we get $h(t) \le f(t)$, the same inequality being obvious for $f(t) = +\infty$, too. Writing the relation (b) as $W_\mu \subset V_\lambda$ $(\mu < \lambda)$, we again conclude, using the considerations presented above, that $g(t) \le h(t)$ for all $t \in Q$. Therefore, $f = g = h$. The fact that f is continuous follows from the inequalities

$$\{f < \lambda\} = \{g < \lambda\} = \cup\{V_\mu : \mu < \lambda, \mu \in \Lambda\},$$
$$\{f \le \lambda\} = \{h \le \lambda\} = \cap\{W_\mu : \mu < \lambda, \mu \in \Lambda\},$$

since V_μ is open, while W_μ is closed for all $\mu \in \Lambda$. ▷

(2) Let Q be a totally disconnected compactum, i.e., Q is a compact topological space wherein the closure of every open set is open (and closed). Let Q_0 be an open dense subset of Q and let $f: Q_0 \to \mathbf{R}$ is a continuous function. Then there is a unique continuous function $\bar{f}: Q_0 \to \overline{\mathbf{R}}$ such that $f(t) = \bar{f}(t)$ $(t \in Q_0)$.

◁ Indeed, if U_μ: $\operatorname{cl}\{f < \mu\}$, then the mapping $\mu \to U_\mu$ $(\mu \in \mathbf{R})$ increases and meets the condition (b) of (1). Therefore, there is a unique function $\bar{f}: Q \to \overline{\mathbf{R}}$ with the properties $\{\bar{f} < \mu\} \subset U_\mu \subset \{\bar{f} \le \mu\}$ $(\mu \in \mathbf{R})$. It is obvious that in this case $\bar{f} \upharpoonright Q_0 = f$. ▷

(3) Let us denote by the symbol $C_\infty(Q)$ the set of all continuous functions $x: Q \to \overline{\mathbf{R}}$ that can assume the values $\pm\infty$ only on a nowhere dense set. Introduce into $C_\infty(Q)$ a relation of order by setting $x \le y$ iff $x(t) \le y(t)$ for all $t \in Q$. Then, choose $x, y \in C_\infty(Q)$ and set $Q_0 := \{|x| < +\infty\} \cap \{|y| < +\infty\}$. In this case Q_0 is open and dense in Q. According to (2), there is a unique continuous function $z: Q \to \overline{\mathbf{R}}$ such that $z(t) = x(t) + y(t)$ for $t \in Q_0$. It this function of z that we shall assume to be the sum of the elements x and y. In an analogous way one can determine the product of each pair of elements. Identifying the number λ with the function identically equal to λ on Q, we get the product of any $x \in C_\infty(Q)$ and $\lambda \in \mathbf{R}$.

One can easily see that $C_\infty(Q)$ with the operations and order introduced above is a vector lattice and an exact f-algebra simultaneously. Below we shall prove that $C_\infty(Q)$ is an extended K-space. The function identically equal to unity is a ring and order unit. The base of the vector space of $C_\infty(Q)$ is isomorphic to the Boolean algebra of all clopen subsets of the compact set Q.

5.2. Boolean-Valued Analysis of Vector Lattices

In the present section we prove that Archimedean vector lattices are realizable as subgroups of an additive group of real numbers in an appropriate Boolean-valued model. Such an approach enables one to deduce the basic structural properties of vector lattices, such as functional calculus, integral presentation of elements, presentation by spaces of functions, etc..

5.2.1. Let $\mathbf{R}$ be the linearly ordered field of real numbers, while $\mathbf{R}^\wedge$ be its image under the canonical embedding of the class of all sets into the universe $\mathbf{V}^{(B)}$ (see **2.2.7**). Since $\mathbf{R}$ is an algebraic system of signature $\sigma := (+,\cdot,0,1,\leq)$, then, by virtue of corollary **4.3.5** (1), $\mathbf{R}^\wedge$ is an algebraic system of signature $\sigma^\wedge$ inside $\mathbf{V}^{(B)}$. Moreover, for any formula $\varphi(u_0,\dots,u_{n-1})$ of signature σ and for any $x_0,\dots,x_{n-1} \in \mathbf{R}$, $\varphi(x_0,\dots,x_{n-1})$ is fulfilled on $\mathbf{R}$ iff inside $\mathbf{V}^{(B)}$ $\varphi(x_0^\wedge,\dots,x_{n-1}^\wedge)$ is fulfilled on $\mathbf{R}^\wedge$. In particular, we can choose as φ the axioms of an Archimedean linearly ordered field. Hence, $\mathbf{V}^{(B)} \models$ "$\mathbf{R}^\wedge$ is an Archimedean linearly ordered field". However, $\mathbf{R}^\wedge$ cannot be claimed to be the field of real numbers inside $\mathbf{V}^{(B)}$. The point is that the axiom of completeness for the field of real numbers is not expressed by a bounded formula. Here is one of the equivalent formulations of the axiom of completeness:

$$(\forall A)\,(A \subset \mathbf{R} \wedge \neq \varnothing \wedge \pi_{\leq}(A) \neq \varnothing \rightarrow (\exists x \in \mathbf{R})(x = \sup(A))),$$

i.e., any nonempty set of real numbers bounded above has a least upper bound as well. In this axiom the universal quantifier runs over the powerset of $\mathbf{R}$.

It should be recalled (see **3.1.1**) that $B_0(\mathbf{R}) := \mathbf{R}^{\wedge\downarrow}$ consists of all mixings of the type $\underset{t \in \mathbf{R}}{\operatorname{mix}}(b_t t^\wedge)$, where $(b_t)_{t \in \mathbf{R}}$ is a partition of unity in B. According to theorem **4.4.10**, $B_0(\mathbf{R})$ is an extended exact f-ring. The f-ring $B_0(\mathbf{R})$ can be identified with the f-ring of all continuous functions x from the Stone compactum Q of the algebra B to the set

$\overline{\mathbf{R}} := \mathbf{R} \cup \{\pm\infty\}$ with the discrete topology, which assume the values $\pm\infty$ on a nowhere dense set. Obviously, $B_0(\mathbf{R})$ is indeed an f-algebra, since we can assume $\mathbf{R} \subset B_0(\mathbf{R})$ when identifying the number λ with the function identically equal to λ on Q.

5.2.2. By virtue of the principles of transfer and maximum, there is such an element $\mathfrak{R} \in \mathbf{V}^{(B)}$ that $\mathbf{V}^{(B)} \models$ "$\mathfrak{R}$ is an ordered field of real numbers". It is obvious that inside $\mathbf{V}^{(B)}$ the field $\mathfrak{R}$ is unique up to isomorphism, i.e., in $\mathfrak{R}'$ is another field of real numbers inside $\mathbf{V}^{(B)}$, then $\mathbf{V}^{(B)} \models$ "$\mathfrak{R}$ and $\mathfrak{R}'$ are isomorphic". As has been pointed out above, $\mathbf{R}^\wedge$ is an Archimedean ordered field inside $\mathbf{V}^{(B)}$ and, hence, we can assume that $\mathbf{V}^{(B)} \models$ "$\mathbf{R}^\wedge \in \mathfrak{R}$ and $\mathfrak{R}$ is the (metric) completion of the field $\mathbf{R}^\wedge$". In this case for the unit 1 of the field $\mathbf{R}^\wedge$ we get $\mathbf{V}^{(B)} \models$ "$1 := 1^\wedge$ is a unit of the field $\mathfrak{R}'$".

Let us now consider the descent $\mathfrak{R}\downarrow$ of the algebraic system $\mathfrak{R} := (|\mathfrak{R}|, +, \cdot, 0, 1, \leq)$. In other words, the descent of the carrier of the system $\mathfrak{R}$ is considered together with the descended operations and order in $\mathfrak{R}$. In more detail, addition, multiplication and order in $\mathfrak{R}$ are introduced by the following rules (see **4.2.3**):

$$\begin{aligned}
&x + y = z \leftrightarrow [x + y = z] = \mathbf{1},\\
&xy = z \leftrightarrow [xy = z] = \mathbf{1},\\
&x \leq y \leftrightarrow [x \leq y] = \mathbf{1},\\
&\lambda x = y \leftrightarrow [\lambda^\wedge x = y] = \mathbf{1},\\
&(x, y, z \in \mathfrak{R}\downarrow, \lambda \in \mathbf{R}).
\end{aligned}$$

Gordon theorem. *Let $\mathfrak{R}$ be the field of real numbers in the model $\mathbf{V}^{(B)}$. The algebraic system $\mathfrak{R}\downarrow$ (i.e., a set $|\mathfrak{R}|\downarrow$ with the operations and order descended) is an extended K-space. In this case there is an isomorphism χ from the Boolean algebra B onto the Boolean algebra of projections $\mathfrak{Pr}(\mathfrak{R}\downarrow)$ (or of unit elements $\mathfrak{E}(\mathfrak{R}\downarrow)$) such that the following equivalences are valid:*

$$\begin{aligned}
&\chi(b)x = \chi(b)y \leftrightarrow b \leq [x = y],\\
&\chi(b)x \leq \chi(b)y \leftrightarrow b \leq [x \leq y]
\end{aligned}$$

for all $x, y \in \mathfrak{R}\downarrow$ and $b \in B$.

◁ The proof of this result can actually be found in **4.4.10**. Indeed, by **4.4.10** (2,4), $\mathfrak{R}\downarrow$ is an extended and order-complete f-ring with unit $\mathbf{1} := 1^\wedge$. The mapping $\lambda \to \lambda^\wedge \cdot 1$ is an isomorphism of the field $\mathbf{R}$ in $\mathfrak{R}\downarrow$. Letting $\lambda x := \lambda^\wedge x$ $(x \in \mathfrak{R}\downarrow, \lambda \in \mathbf{R})$, we obtain the required vector structure on $\mathfrak{R}\downarrow$. Therefore, $\mathfrak{R}\downarrow$ is an extended K-space. ▷

5.2.3. With the same notations as in **5.2.2**, let us elucidate the sense of some statements in terms of the K-space $\mathfrak{R}\downarrow$.

(1) *Let* $(b_\xi)_{\xi\in\Xi}$ *be a partition of unity in* B *and* $(x_\xi)_{\xi\in\Xi}$ *be an arbitrary family in* $\mathfrak{R}\downarrow$. *Then*

$$\underset{\xi\in\Xi}{mix}(b_\xi x_\xi) = o\text{-}\sum_{\xi\in\Xi}\chi(b_\xi)x_\xi.$$

◁ Indeed, if $x = mix(b_\xi x_\xi)$, then the definition of mixing, with theorem **5.2.2** taken into account, implies that $\chi(b_\xi)x = \chi(b_\xi)x_\xi$ for every ξ. Summing this relation over ξ, we get the required result. ▷

(2) *For a set* $A\subset\mathfrak{R}\downarrow$ *and arbitrary* $a\in\mathfrak{R}\downarrow$ *and* $b\in B$ *the following equivalence is valid:*

$$\chi(b)a = \sup(\chi(b)(A)) \leftrightarrow b\le[a = \sup(A\uparrow)].$$

◁ Indeed, by virtue of **5.2.2**, the equality $\chi(b)a = \sup\{\chi(b)x : x\in A\}$ holds iff $b\le[x\le a]$ for all $a\in A$ and for every $y\in\mathfrak{R}\downarrow$ the relation $(\forall x\in A)\times(b\le[x\le y])$ implies $b\le[a\le y]$. The last statement can be easily seen to be a different expression of the relation $b\le[\sup(A\uparrow) = a]$. ▷

(3) *Let us consider a net* $s:\mathrm{A}\to\mathfrak{R}\downarrow$, *where* A *is a directed set. Then the modified ascent* $s\uparrow:\mathrm{A}^\wedge\to\mathfrak{R}$ *is a net inside* $\mathbf{V}^{(B)}$, *in which case for any* $x\in\mathfrak{R}\downarrow$ *and* $b\in B$ *we have*

$$\chi(b)x = o\text{-}\lim(\chi(b)\circ s) \leftrightarrow b\le[x = \lim(s\uparrow)].$$

◁ The relation $\chi(b)x = o\text{-}\lim(\chi(b)\circ s)$ is equivalent to the existence of a net $r:\mathrm{A}\to\mathfrak{R}\downarrow$ such that $r(\alpha)\le r(\beta)$ for $\alpha\le\beta$, $\inf\{r(\alpha):\alpha\in\mathrm{A}\} = 0$ and $|\chi(b)x - \chi(b)s(\alpha)|\le\chi(b)r(\alpha)$ for all $\alpha\in\mathrm{A}$. By virtue of **5.2.3** (2) and the equality $r(\mathrm{A})\uparrow = r\uparrow(\mathrm{A}^\wedge)$, the last three relations imply that the following inequalities are valid:

$$b\le[(\forall\alpha\in\mathrm{A}^\wedge)(|x - s\uparrow(\alpha)|\le r(\alpha))],$$
$$b\le[\inf(r\uparrow(\mathrm{A}^\wedge) = 0)],$$
$$b\le[(\forall\alpha,\beta\in\mathrm{A}^\wedge)(\alpha\le\beta\to r\uparrow(\alpha)\le r\uparrow(\beta)],$$

or, in short, $b\le[x = \lim(s\uparrow)]$, which was required. ▷

The following proposition is proved in an absolutely analogous way.

(4) *Let* s *and* $\mathrm{A}\in\mathbf{V}^{(B)}$ *be such that* $[s:\mathrm{A}\to\mathfrak{R}$ *is a net* $] = 1$. *Then the descent*

$s{\downarrow}: A{\downarrow} \to \mathfrak{R}{\downarrow}$ *is a net, in which case, for any* $x \in \mathfrak{R}{\downarrow}$ *and* $b \in B$, *we have*

$$\chi(b)x = o - \lim(\chi(b) \circ s{\downarrow}) \leftrightarrow b \le [x = \lim(s)].$$

(5) *For every element* $x \in \mathfrak{R}{\downarrow}$ *the following equalities are valid:*

$$e_x = \chi([x \ne 0]), \quad e_\lambda^x = \chi([x < \lambda]) \quad (\lambda \in \mathbf{R}).$$

◁ It should be observed that a real number t is distinct from zero iff the least upper bound of the set $\{1 \wedge (n|t|): n \in \omega\}$ is 1. Hence, according to the principle of transfer, for $x \in \mathfrak{R}{\downarrow}$ we have $[x \ne 0] = [\sup\{1^\wedge \wedge (n|x|): n \in \omega^\wedge\} = 1]$. If $A := \{1 \wedge (h|x|): n \in \omega\}$, then $[\sup(A{\uparrow}) = \sup\{1^\wedge \wedge (n|x|): n \in \omega^\wedge\}] = 1$ and $e_x = \sup(A)$. Therefore, $b := [x \ne 0] = [e_x = 1]$ and, analogously, $b^* = [e_x = 0]$. Making use of the properties of χ, we deduce $e_x = \chi(b)$. Let us now choose an arbitrary number $\lambda \in \mathbf{R}$ and note that $\lambda^\wedge = \lambda^\wedge 1$ and, hence, $e_\lambda^x = e_{(\lambda^\wedge - x)^+}$. Using the facts proved earlier, we get

$$\chi^{-1}(e_\lambda^x) = [(\lambda^\wedge - x) \vee 0 \ne 0] = [(\lambda^\wedge - x) > 0] = [x < \lambda^\wedge]. \ ▷$$

5.2.4. Theorem. *Let* X *be an Archimedean vector lattice with base* $B := \mathfrak{E}(X)$. *Let* $\mathfrak{R}$ *be the field of real numbers in the model* $\mathbf{V}^{(B)}$. *Then there is a linear and lattice isomorphism* i *from* X *into an extended* K*-space* $\mathfrak{R}{\downarrow}$ *such that the following conditions are met:*

(1) *the isomorphism* i *preserves the least upper and greatest lower bounds of any nonempty bounded sets;*

(2) *the order ideal* $J(i(X))$ *generated by a set* $i(X)$ *is a foundation of* $\mathfrak{R}{\downarrow}$;

(3) *for any* $y \in J(i(X))$ *the following equalities are valid:*

$$\inf\{i(x): x \in X \wedge i(x) \ge y\} = y = \sup\{i(x): x \in X \wedge i(x) \le y\};$$

(4) *for* $x \in X$ *and* $b \in B$, *we have* $b \le [i(x) = 0]$, *iff* $x \in b^\perp$.

◁ In theorem **4.4.12** we have already proved that there is a subgroup $\mathcal{X}$ of the additive group of the field of real numbers $\mathfrak{R} \in \mathbf{V}^{(B)}$ as well as an additive and lattice isomorphism $i := \iota_X$ from X to $\mathcal{X}$. Let e be a nonzero positive element of the group $\mathcal{X}$. Replacing, when necessary, $\mathcal{X}$ with the group $e^{-1}\mathcal{X}$ isomorphic to it, we can assume $e = 1 \in \mathcal{X}$. It should be recalled that $X^\wedge$ is a vector space over the field $\mathbf{R}^\wedge$. It is obvious that under these circumstances the factor-mapping $\varphi := \varphi_X : X^\wedge \to \mathcal{X}$ is $\mathbf{R}^\wedge$-linear. In particular, $[\varphi((\lambda x)^\wedge) = \lambda^\wedge \varphi(x^\wedge)] = \mathbf{1}$ $(\varphi \in \mathbf{R}, x \in X)$. Therefore, $[i(\lambda x) = \lambda^\wedge i(x)] = \mathbf{1}$, or $i(\lambda x) = \lambda i(x)$ (see **5.2.2**). Now for $\mathbf{1} = \operatorname{mix}(b_\xi i(e_\xi))$, $(e_\xi) \subset X$, and for $\lambda \in \mathbf{R}$ we can write

$$b_\xi \le [\lambda^\wedge = \lambda^\wedge \cdot ie_\xi] \wedge [\lambda^\wedge \cdot ie_\xi = i(\lambda e_\xi)] \wedge [i(\lambda e_\xi) \in \mathcal{X}] \le [\lambda^\wedge \in \mathcal{X}].$$

Therefore, $\lambda^\wedge \in \mathcal{X}$ and, hence, $[\mathbf{R}^\wedge \subset \mathcal{X} \subset \mathfrak{R}] = \mathbf{1}$. Moreover, $\mathbf{V}^{(B)} \models$ "$\mathcal{X}$ is a vector sublattice of the field $\mathfrak{R}$ considered as a vector lattice over $\mathbf{R}^\wedge$". In this case, however, $\mathcal{X}\downarrow$ is a vector sublattice of the extended K-space $\mathfrak{R}\downarrow$, while i can be considered as an embedding of X in $\mathfrak{R}\downarrow$. The task now is to check whether (1) - (4) are valid.

(1) Let us choose such $A \subset X$ and $a \in X$ that $a = \sup(A)$. Let $z = \sup(i(A))$, where the supremum is calculated in $\mathfrak{R}\downarrow$.

From an obvious relation $[\mathcal{X}$ is minorant in $\mathfrak{R}] = \mathbf{1}$ we can easily deduce that $\mathcal{X}\downarrow$ is minorant in $\mathfrak{R}\downarrow$. In this case, however, $i(X)$ is also minorant in $\mathfrak{R}\downarrow$ (see **4.4.12**). If $i(a) > z$, then for a certain $0 < x \in X$ we have $i(x) \le i(a) - z$, or $z \le i(a - x)$, which implies that $a - x$ is an upper bound of the set A and, by virtue of the equality $a = \sup(A)$ we must get $a - x > a$ or $x \le 0$. This contradiction shows that $z = i(a)$.

(2) As far as $z = i(a)$ i(X) is minorant in $\mathfrak{R}\downarrow$, we have $\mathfrak{R}\downarrow = i(X)^{\perp\perp}$. Moreover, the equality $\mathfrak{R}\downarrow = J(i(X))^{\perp\perp}$ holds, where $J(i(X))$ is the order ideal generated by the set $i(X)$.

(3) The relation $[\mathbf{R}^\wedge \subset \mathcal{X} \subset \mathfrak{R}] = \mathbf{1}$ allows one to conclude that $\mathbf{V}^{(B)} \models$ "$\mathcal{X}$ is a dense subgroup in $\mathfrak{R}$". Hence, for any $x \in \mathfrak{R}\downarrow$ inside $\mathbf{V}^{(B)}$ we have

$$\inf\{x' \in X : x' \ge x\} = x = \sup\{x' \in X : x' \le x\}.$$

Applying **5.2.3** (2), we directly deduce from the above relation

$$\inf\{x' \in \mathcal{X}\downarrow : x' \ge x\} = x = \sup\{x' \in \mathcal{X}\downarrow : x' \le x\}$$

and all we have to do now to complete the proof is to take into account the fact that $i(X)$ is minorant in $\mathcal{X}\downarrow$.

(4) This statement has been proved in **4.4.12**. ▷

5.2.5. Notice some corollaries to the theorem proved above.

(1) *Let X be an Archimedean vector lattice, the base of which, $\mathfrak{B}(X)$, is isomorphic to the Boolean algebra B. There is an element $\mathcal{X} \in \mathbf{V}^{(B)}$ which obeys the following conditions:*

(a) $\mathbf{V}^{(B)} \models$ "*$\mathcal{X}$ is a vector sublattice of the field of real numbers $\mathfrak{R}$ considered as a vector space over $\mathbf{R}^{\wedge}$*";

(b) *$X' := \mathcal{X}\downarrow$ is an extended vector lattice with projections, which is an r-dense sublattice of the K-space $\mathfrak{R}\downarrow$;*

(c) *There is a linear and lattice isomorphism $\iota : X \to X'$ preserving suprema and infima, in which case for any $x \in X'$ there is a partition of unity $(\pi_\xi)_{\xi \in \Xi}$ in $\mathfrak{B}(X')$, and a family $(x_\xi)_{\xi \in \Xi}$ in X such that*

$$x = o - \sum_{\xi \in \Xi} \pi_\xi \circ i(x_\xi).$$

$\lhd$ All the statements above can, in essence, be found in **5.2.4.** Let us, for instance, prove that X' is r-dense in $\mathfrak{R}\downarrow$. If $x \in \mathfrak{R}\downarrow$, then $\mathbf{V}^{(B)} \models$ "x is a real number and it can be approximated within any accuracy by the elements of $\mathcal{X}$". In other words, the following equality is valid:

$$[\![(\forall \varepsilon \in \mathbf{R}^{\wedge})(\varepsilon > 0 \to (\exists \lambda \in \mathcal{X})(|\lambda - x| < \varepsilon))]\!] = \mathbf{1}.$$

Writing out Boolean truth-values for the quantifiers, for any $\varepsilon > 0$ we can find a $\lambda \in X'$ such that $|\lambda - x| \leq \varepsilon \mathbf{1}$, which completes the proof. $\rhd$.

(2) *If X is a K-space, then $\mathcal{X} = \mathfrak{R}$, while $i(X)$ is a foundation in $\mathfrak{R}\downarrow$. The whole of $\mathfrak{R}\downarrow$ serves as the image of X under the isomorphism i iff X is an extended K-space.*

$\lhd$ A proof results from **5.2.2** and **5.2.4** (2,3). $\rhd$

(3) *Extended K-spaces are order-isomorphic iff they have isomorphic bases.*

$\lhd$ Indeed, if X and Y are extended K-spaces, while h is an order isomorphism of X on Y, then the correspondence $K \to h(K)$ $(K \in \mathfrak{B}(X))$ is an isomorphism of the bases. Vice versa, if $\mathfrak{B}(X)$ and $\mathfrak{B}(Y)$ are isomorphic to a Boolean algebra B then, making use of (2), X and Y are order-isomorphic to the extended K-space $\mathfrak{R}\downarrow$. $\rhd$

(4) An *extension* of a K-space X is a pair (Y,i), where Y is also a K-space, while i is an isomorphism of X on a certain foundation in Y. Let us introduce a preorder in the class $\mathrm{Ext}(X)$ of all extensions of the K-space as follows. For (Y,i) and $(Z,j) \in \mathrm{Ext}(X)$ we shall set $(Y,i) \prec (Z,j)$ provided there is an isomorphism h of the space Y on a certain foundation in Z such that $h \circ i = j$. The maximal elements of the preordered class $\mathrm{Ext}(X)$ are called *maximal extensions* of X (sometimes the term *universal completion* is used). The following result ensues from (1) and (2).

Any K-space has a maximal extension. Such a maximal extension is unique up to an order isomorphism and presents an extended K-space.

(5) *Let X be an extended K-space with a fixed order unit* $\mathbf{1}$.. *In this case it is a unique possibility of determining multiplication in X in such a manner that X becomes an exact f-ring, while* $\mathbf{1}$ *becomes the identity of multiplication.*

$\lhd$ Let us identify a number $\lambda \in \mathbf{R}$ with the element $\lambda \cdot \mathbf{1}$. By virtue of (2), X is isomorphic to $\mathfrak{R}\downarrow$ and under such an isomorphism $\mathbf{1}$ is transformed to $1 := 1^{\wedge} \in \mathfrak{R}\downarrow$, since $[1^{\wedge}$ is a unit of the field $\mathfrak{R}\downarrow] = \mathbf{1}$. The descent of the operation of multiplication in $\mathfrak{R}$ supplies us with the required multiplicative structure. If $\times$: $X^2 \to X$ is another multiplication in X satisfying the above conditions, then it is extensional and its ascent $(\times)\uparrow$ is a multiplication in $\mathfrak{R}$ with the unit $\mathbf{1}$. It is obvious that in this case $\times = \cdot$, since the multiplicative structure of the field $\mathfrak{R}$ is unique. $\rhd$

(6) For any Archimedean vector lattice X there is a K-space Y, unique up to linear and lattice isomorphism, as well as a linear isomorphism j: $X \to Y$ preserving suprema and infima such that

$$\sup\{j(x) : x \in X, j(x) \le y\} = y = \inf\{j(x) : x \in X, j(x) \ge y\}.$$

$\lhd$ Let $\mathfrak{R}$ and $J(i(X))$ be the same as in **5.2.4**. Then a pair $(J(i(X)), i)$ obeys all the properties required. If (Y,j) is a pair with the same properties, then the bases $\mathfrak{B}(Y)$ and $\mathfrak{B}(\mathfrak{R}\downarrow)$ are mutually isomorphic and, hence, by virtue of (2), the K-spaces $m\,Y$ and $\mathfrak{R}\downarrow$ are also isomorphic. Therefore, we can assume that $i(X) \subset Y \subset \mathfrak{R}\downarrow$, in which case Y is a foundation of $\mathfrak{R}\downarrow$. Then $(J(i(X)) \subset Y$. For every $y \in Y$, however, there must exist such x' and $x'' \in X$, that $i(x') \le y \le i(x'')$, i.e., there must be $Y \subset j(i(X))$. $\rhd$

5.2.6. Theorem. *Let X be an arbitrary K_σ-space with order unit* 1. *The spectral function $\lambda \to e_\lambda^x$ $(\lambda \in \mathbf{R})$ of an element $x \in X$ has the following properties*:

(1) $e_\lambda^x \le e_\mu^x$ for $\lambda \le \mu$;

(2) $e^x_{+\infty} := \bigvee_{\mu \in \mathbf{R}} e^x_\mu = 1, \quad e^x_{-\infty} := \bigwedge_{\mu \in \mathbf{R}} e^x_\mu = 0 \quad ;$

(3) $\bigvee_{\mu < \lambda} e^x_\mu = e^x_\lambda \quad (\lambda \in \mathbf{R});$

(4) $x \le y \leftrightarrow (\forall \lambda \in \mathbf{R})(e^y_\lambda \le e^x_\lambda);$

(5) $e^{x+y}_\lambda = \vee\{e^x_\mu \cdot e^y_v\colon \mu, v \in \mathbf{R}, \mu + v = \lambda\};$

(6) $e^{xy}_\lambda = \vee\{e^x_\mu \cdot e^y_v\colon 0 \le \mu, v \in \mathbf{R}, \mu v = \lambda\} \, (x \ge 0, y \ge 0);$

(7) $e^{-x}_\lambda = \vee\{1 - e^x_{-\mu}\colon \mu \in \mathbf{R}, \mu < \lambda\} = (1 - e^x_{-\lambda}) \cdot e_{x + \lambda 1};$

(8) $x = \inf(A) \leftrightarrow (\forall \lambda \in \mathbf{R})(e^x_\lambda = \vee\{e^a_\lambda\colon a \in A\});$

(9) $e^{x \vee y}_\lambda = e^x_\lambda \cdot e^y_\lambda;$

(10) $e^{cx}_\lambda = c e^x_\lambda + c^*$ for $\lambda > 0$,
$e^{cx}_\lambda = c e^x_\lambda$ for $\lambda \le 0$ $(c \in \mathfrak{S}(X))$.

When calculating exact bounds in (2), (3) *and* (5)-(7), *one can consider* μ *and* v *to assume values in a certain dense subfield* $\mathbf{P} \subset \mathbf{R}$.

◁ By virtue of theorem **5.2.4**, we can assume, without loss of generality, that $X = \mathfrak{R}\downarrow$. In this case, however, the required statements are easily deduced from **5.2.3** (5) and properties of numbers. Let us prove, for instance, (6) and (8). Let $x \ge 0$, $y \ge 0$ and assume that there is a product $x \cdot y$. Then x and y are nonnegative numbers inside $\mathbf{V}^{(B)}$. According to **5.2.3** (5), $e^x_\lambda = \chi([\![x \cdot y < \lambda^\wedge]\!])$, $e^x_\lambda = \chi([\![x < \lambda^\wedge]\!])$ and $e^y_\lambda = \chi([\![y < \lambda^\wedge]\!])$. Inside $\mathbf{V}^{(B)}$, however, we have

$$(\forall x, y \in \mathfrak{R})(x \ge 0 \wedge y \ge 0 \rightarrow (x \cdot y < \lambda$$
$$\leftrightarrow (\exists 0 < \mu, v \in \mathbf{P}^\wedge)(x < \mu) \wedge (y < v) \wedge (\lambda = \mu v))),$$

and, hence,

$$[\![x \cdot y < \lambda^\wedge]\!] = \bigvee_{\substack{0 < \mu, v \in \mathbf{P} \\ \lambda = \mu v}} \{[\![x < \mu^\wedge]\!] \wedge [\![y < v^\wedge]\!]\},$$

whence we get the required result.

Let now $A \subset X$ and let us assume that $x = \inf(A)$. Then $e^x_\lambda = \chi([\![x < \lambda^\wedge]\!]) = \chi([\![\inf(A\uparrow) < \lambda^\wedge]\!])$ (see **5.2.3** (1,5)). However, $A\uparrow$ is a certain nonempty set of real numbers inside $\mathbf{V}^{(B)}$, and, hence,

$$\mathbf{V}^{(B)} \models \inf(A\uparrow) < \lambda^{\wedge} \leftrightarrow (\exists a \in A\uparrow)(a < \lambda^{\wedge}).$$

Calculating Boolean truth-values, we find

$$[x < \lambda^{\wedge}] = \bigvee_{a \in A} [a < \lambda^{\wedge}],$$

and, hence,

$$e_{\lambda}^{x} = \vee\{\chi([a < \lambda^{\wedge}]) : a \in A\} = \vee\{e_{\lambda}^{a} : a \in A\}.$$

Conversely, let us assume that e_{λ}^{x} is the supremum of the set $\{e_{\lambda}^{a} : a \in A\}$, for $\lambda \in \mathbf{R}$. Then

$$[x < \lambda^{\wedge}] = [(\exists a \in A\uparrow)\,(a < \lambda^{\wedge})] = [\inf(A\uparrow) < \lambda^{\wedge}]$$

for every $\lambda \in \mathbf{R}$ and, hence,

$$[(\forall \lambda \in \mathbf{R}^{\wedge})\,(x < \lambda \leftrightarrow \inf(A\uparrow) < \lambda)] = \mathbf{1}.$$

The preceding expression implies $[x = \sup(A\uparrow)] = \mathbf{1}$ and, applying **5.2.3** (2), we get $x = \inf(A)$.

The last claim of the theorem results from the fact that if $\mathbf{P}$ is a dense subfield of $\mathbf{R}$, then $\mathbf{V}^{(B)} \models$ "the field $\mathbf{P}^{\wedge}$ is dense in $\mathfrak{R}$". ▷

5.2.7. Now establish the following three useful characteristics of o-convergence.

(1) *Let X be again a K-space with order unit* $\mathbf{1}$.. *Choose an order-bounded net $(x_\alpha)_\alpha \in A$ of positive elements in X. Then (x_α) o-converges to zero iff for any* $0 < \varepsilon \in \mathbf{R}$ *the net of unit elements* $\left(e_{\varepsilon}^{x_\alpha}\right)$ *o-converges to* $\mathbf{1}$.

Indeed, by theorem **5.2.4**, x_α can be considered as positive elements of the K-space $\mathfrak{R}\downarrow$. The mapping $s: \alpha \to s_\alpha$ has the modified ascent $\delta := s\uparrow$, which is a set in $\mathfrak{R}$, i.e., a numerical net inside $\mathbf{V}^{(B)}$. According to **5.2.3** (3), $o-\lim(x_\alpha) = 0$ iff $[\lim(\delta) = 0] = \mathbf{1}$, which can be rewritten in an equivalent form as

$$\mathbf{V}^{(B)} \models (\forall \varepsilon \in \mathbf{R}^{\wedge})(\varepsilon > 0 \to (\exists \alpha \in \mathrm{A}^{\wedge})(\forall \beta \in \mathrm{A}^{\wedge})(\beta \geq \alpha \to x_\beta < \varepsilon)).$$

Writing out Boolean truth-values for quantifiers, we find another equivalent presentation:

$$(\forall \varepsilon > 0)(\exists (b_\alpha))(\forall \beta \in \mathrm{A})(\alpha \le \beta \to b_\alpha \le [x_\beta < \varepsilon^\wedge]),$$

where (b_α) is a partition of unity in B. Finally, applying **5.2.3** (5), we get

$$(\forall \varepsilon > 0)(\exists (b_\alpha)_{\alpha \in \mathrm{A}})(\forall \beta \in \mathrm{A})(\alpha \le \beta \to \chi(b_\alpha) \le e_\varepsilon^{x_\beta})$$

or

$$(\forall \varepsilon > 0)(\exists (b_\alpha)_{\alpha \in \mathrm{A}})(\chi(b_\alpha) \le \wedge \{e_\varepsilon^{x_\beta} \ge \alpha\}).$$

Since $\vee(b_\alpha) = \mathbf{1}$, the expression o-$\lim x_\alpha = 0$ is seen to be equivalent to the following statement: for every $\varepsilon > 0$ we have

$$o - \lim(e_\varepsilon^{x_\alpha}) = \lim\inf(e_\varepsilon^{x_\alpha}) = \bigvee_{\alpha \in \mathrm{A}} \wedge \{e_\varepsilon^{x_\beta} : \beta \ge \alpha\} = \mathbf{1}. \ \triangleright$$

(2) *An order-bounded net (x_α) in a K-space X with order unit o-converges to an element $x \in X$ iff for any $\varepsilon > 0$ there is a partition of unity $(\pi_\alpha)_{\alpha \in \mathrm{A}}$ in $\mathfrak{Pr}(X)$ such that*

$$\pi_\alpha |x - x_\beta| \le \varepsilon \mathbf{1} \quad (\alpha, \beta \in \mathrm{A}, \beta \ge \alpha).$$

$\triangleleft$ For the proof we again use **5.2.4.** Let s, δ be the same as in (1). Reasoning as above, we find out that $x_\alpha \xrightarrow{o} x$ is equivalent to the following condition: for any $\varepsilon > 0$ there is a partition of unity $(b_\alpha)_{\alpha \in \mathrm{A}}$ in B such that

$$b_\alpha \le [|x_\beta - x| \le \varepsilon^\wedge] \quad (\alpha, \beta \in \mathrm{A}, \beta \ge \alpha).$$

If $\pi_\alpha := \chi(b_\alpha)$ (see **5.2.2.**), then the last relation implies that

$$\pi_\alpha |x_\beta - x| \le \varepsilon \mathbf{1} \quad (\alpha, \beta \in \mathrm{A}, \beta \ge \alpha). \ \triangleright$$

(3) *An order-bounded net (x_α) in the K-space X with order unit o-converges to an element $x \in X$ iff for any $\varepsilon > 0$ there is an increasing net of projections (ρ_α) such that $o - \lim(\rho_\alpha) = I_X$ and*

$$\rho_\alpha |x - x_\beta| \le \varepsilon \mathbf{1} \quad (\alpha, \beta \in \mathrm{A}, \beta \ge \alpha).$$

$\triangleleft$ Indeed, this is true if we put $\rho_\alpha := \vee\{\pi_\beta : \beta \ge \alpha\}$ in (2). $\triangleright$

5.2.8. Let us now turn our attention to results on functional realization of vector spaces.

(1) Let B be a complete Boolean algebra. A *resolution of unity* (not to be mixed with partition!) in the algebra B is a mapping $e: \mathbf{R} \to B$ having the properties **5.2.6** (1-3) of a spectral function. The set of all resolutions of unity in B will be denoted by the symbol $\mathfrak{S}(B)$. In this set let us introduce addition, multiplication by real numbers and the order according to the following rules (cf. **5.2.6** (4-6)):

$$\begin{aligned}
&(e_1 + e_2)(\lambda) := \vee\{e_1(\mu)\cdot e_2(v): \mu, v \in \mathbf{R};\ \ \mu + v = \lambda\};\\
&(\alpha e)(\lambda) := e(\lambda/\alpha) \quad (\alpha > 0);\\
&(-e)(\lambda) := \bigvee_{\mu<\lambda} 1 - e(-\mu) = 1 - \bigwedge_{\mu<\lambda} e(-\mu);\\
&(0\cdot e)(\lambda) := \mathbf{0}(\lambda) := \begin{cases} 1, \text{if} & \lambda > 0,\\ 0, \text{if} & \lambda \le 0;\end{cases}\\
&e_1 \le e_2 \leftrightarrow (\forall \lambda \in \mathbf{R})\, e_1(\lambda) \ge e_2(\lambda).
\end{aligned}$$

The set $\mathfrak{S}(B)$ with the operations and order introduced above is an extended K-space isomorphic to $\mathfrak{R}\downarrow$.

$\lhd$ In line with **5.2.2**, without loss of generality, we can assume B to be the base of unit elements of the K-space $\mathfrak{R}\downarrow$. Let us put in correspondence to an element $X \in \mathfrak{R}\downarrow$ its spectral function $\lambda \to e_\lambda^x$ $(\lambda \in \mathbf{R})$. Thus, we have obtained an injective lattice homomorphism from $\mathfrak{R}\downarrow$ to $\mathfrak{S}(B)$, as is seen from theorem **5.2.6**. It is now necessary to substantiate the fact that this homomorphism is surjective. Let us choose an arbitrary resolution of unity $e: \mathbf{R} \to B$. Let Σ be a set of all partitions of the numerical straight lime, i.e., $\sigma \in \Sigma$ if $\sigma: \mathbf{Z} \to \mathbf{R}$ is a strictly increasing function, $\lim_{n\to\infty} \sigma(n) = \infty$ and $\lim_{n\to\infty} \sigma(-n) = -\infty$ (as usual, $\mathbf{Z}$ is the set of integer numbers). In an extended K-space $\mathfrak{R}\downarrow$ there is a sum $x_\sigma := \sum_{n\in\mathbf{Z}} \overline{\sigma}(n+1)b_{n\sigma}$, where $b_{n\sigma} := e(\sigma(n+1)) - e(\sigma(n))$. Let us set $A := \{x_\sigma : \sigma \in \Sigma\}$ and $x = \inf A$. The infimum does exist since $x_\sigma \ge \sum_{n\in\mathbf{Z}} \overline{\sigma}(n)b_{n\sigma}$ for a fixed partition $\overline{\sigma} \in \Sigma$. Let us also remark that $x_\sigma = \operatorname{mix}(b_{n\sigma}\sigma(n)^\wedge)$ and

$$[x_\sigma < \lambda^\wedge] = \vee\{b_{n\sigma} : \sigma(n) < \lambda\} = \vee\{e(\sigma(n)) : \sigma(n) < \lambda\}.$$

Since $[x = \inf(A\uparrow)] = \mathbf{1}$, the following calculations are valid:

$$\begin{aligned}
&[x < \lambda^\wedge] = [(\exists a \in A\uparrow)\, a < \lambda^\wedge]\\
&= \bigvee_{a\in A} [a < \lambda^\wedge] = \bigvee_{\sigma\in\Sigma} \bigvee_{\sigma(n)<\lambda} b_{n\sigma}\\
&= \bigvee_{\sigma\in\Sigma} \bigvee_{\sigma(n)<\lambda} e(\sigma(n)) = \bigvee_{\mu<\lambda} e(\mu) = e(\lambda).
\end{aligned}$$

Therefore, e is the spectral function of the element x. ▷

(2) **Theorem.** *Let Q be the Stone space of a complete Boolean algebra B, while $\mathfrak{R}$ be the field of real numbers in the model $\mathbf{V}^{(B)}$. The vector lattice $C_\infty(Q)$ is an extended K-space which is linear and lattice isomorphic to $\mathfrak{R}$t. An isomorphism can be established by assigning the function $\hat{x}\colon Q \to \overline{\mathbf{R}}$ to an element $x \in \mathfrak{R}\downarrow$ by the formula*

$$\hat{x}(q) := \inf\{\lambda \in \mathbf{R}\colon [x < \lambda^\wedge] \in q\}.$$

◁ We have already established in (1) that the K-space $\mathfrak{R}\downarrow$ is isomorphic to the space of all B-valued spectral functions, in which case the function $\lambda \to [x < \lambda^\wedge]$ $(\lambda \in \mathbf{R})$ corresponds to an element $x \in \mathfrak{R}\downarrow$. Let an element $[x < \lambda^\wedge] \in B$ be assigned to a clopen set U_λ of the Stone space Q. Then, by virtue of **5.8.1** (2), to every element $x \in \mathfrak{R}\downarrow$ there corresponds a unique continuous function $\hat{x}\colon Q \to \overline{\mathbf{R}}$ such that $\{\hat{x} < \lambda\} \subset U_\lambda \subset \{\hat{x} \leq \lambda\}$. In this case, however, $\hat{x}(q) = \inf\{\lambda \in \mathbf{R}\colon q \in U_\lambda\} = \;= \inf\{\lambda \in \mathbf{R}\colon [x < \lambda^\wedge] \in q\}$. The relations $\wedge\{[x < \lambda^\wedge]\} = 0$ and $\vee\{[x < \lambda^\wedge]\} = 1$ imply that the closed set $\cap\{U_\lambda\colon \lambda \in \mathbf{R}\}$ has an empty interior, while the set $\cup\{U_\lambda\colon \lambda \in \mathbf{R}\}$ is dense in Q. Therefore, the function $\hat{x}$ can assume the values $\pm\infty$ only on a nowhere dense set and, hence, $\hat{x} \in C_\infty(Q)$. An elementary checking if $\hat{x}$ is a linear and lattice isomorphism is omitted. ▷

5.2.9. Let us note some corollaries to the theorem just proved.

(1) *Let X be an arbitrary K-space and $\{e_\xi\}_{\xi \in \Xi}$ be a complete set of pairwise disjoint positive elements in X Let Q be the Stone space of the Boolean algebra of bands of $\mathfrak{E}(X)$. Then there is a unique linear and lattice isomorphism of X on a foundation of the K-space $C_\infty(Q)$ such that e_ξ transforms into the characteristic function of a certain clopen set $Q_\xi \subset Q$, This isomorphism puts the function $\hat{x}\colon Q \to \overline{\mathbf{R}}$ into correspondence to an element $x \in X$ by the rule*

$$\hat{x}(q) := \inf\{\lambda \in \mathbf{R}\colon \{e_\lambda^\xi\}^{\perp\perp} \in q\} \quad (q \in Q_\xi),$$

where (e_λ^ξ) is the characteristic of the band projection of x on $\{e_\lambda^\xi\}^{\perp\perp}$ with respect to the unit e_ξ.

(2) *The space X is an extended (K-space of bounded elements) iff under the given isomorphism its image is all $C_\infty(Q)$ (the subspace $C(Q)$ of all continuous finite functions on the compactum Q).*

(3) *Any Archimedean vector lattice (f-algebra) X is both linearly and lattice-isomorphic to a vector sublattice (and a subalgebra) of the space $C_\infty(Q)$, where Q is the Stone space of the base $\mathfrak{E}(X)$.*

By the symbol $C_\infty(Q,S\mathbf{Z})$ let us denote the subset of the functions of $C_\infty(Q)$ assuming integer values on an clopen set $S \subset Q$. It is obvious that $C_\infty(Q,S\mathbf{Z})$ is an extended f-ring.

(4) *A complete lattice-ordered group G is isomorphic to a foundation of the extended lattice-ordered group $C_\infty(Q,S\mathbf{Z})$, where Q is the Stone space of the base $\mathfrak{E}(G)$.*

$\lhd$ If $\mathcal{G}$ is the Boolean-valued realization of G then, by virtue of **4.4.10** and **4.4.12**, $\mathcal{G}$ is a complete totally ordered group. In this case, however, $\mathcal{G}$ is either isomorphic to $\mathfrak{R}$ or is an infinite cyclic group. Therefore, there is such a $b \in B$ that $b = [\![\mathcal{G} \cong \mathbf{Z}^\wedge]\!]$ and $b^* = [\![\mathcal{G} \cong \mathfrak{R}]\!]$. In the same way as in **4.4.13** we establish that G is expanded into a direct sum of two components, one of which is implemented as $\mathfrak{R}$ in $\mathbf{V}^{([0,b^*])}$, while the other as $\mathbf{Z}$ in $\mathbf{V}^{([0,b])}$. Now we have to apply theorem (1) and observe that $\mathbf{Z}^\wedge\downarrow \cong B_0(\mathbf{Z}) \cong C_\infty(Q,S\mathbf{Z})$, where S is an clopen set in Q corresponding to an element $b \in B$. $\rhd$

In an analogous way we can deduce the following statement.

(5) *Any f-ring is o-isomorphic to the product of two f-rings K_1 and K_2 such that K_1 is a foundation and subring of an extended f-ring $C_\infty(Q_1,S_1\mathbf{Z})$, while K_2 is a foundation of the extended group $C_\infty(Q_2,S_2\mathbf{Z})$ with zero multiplication, where Q_l is the Stone space of the algebra $\mathfrak{E}(K_l)$ and $S_l \in \mathfrak{E}(Q_l)$ $(l = 1,2)$.*

5.2.10. Let us construct an integral of Stieltjes type over a spectral measure. Let Ω be an arbitrary nonempty set, while Σ be a certain σ-algebra of subsets of Ω. Let us consider a Boolean algebra B of unit elements of a fixed K_σ-space X. A *spectral measure* is a σ-continuous Boolean homomorphism μ from Σ to B. Here σ-continuity means that for any sequence $(e_n)_{n\in\omega}$ of elements of the σ-algebra Σ we have $\mu\left(\bigvee_{n=0}^{\infty}\right) = \bigvee_{n=0}^{\infty} \mu(e_n)$.

Let us choose a measurable function $f\colon \Omega \to \mathbf{R}$. For an arbitrary partition of a numerical straight line $\Lambda := (\lambda_k)_{k\in\mathbf{Z}}$, $-\infty \leftarrow \ldots \lambda_{-1} < \lambda_0 < \lambda_1 < \ldots \to +\infty$ let us compose the integral sums

$$\underline{\sigma}(f,\Lambda) = \sum_{-\infty}^{\infty} \lambda_n \mu(e_n), \quad \overline{\sigma}(f,\Lambda) = \sum_{-\infty}^{\infty} \lambda_{n+1}\mu(e_n),$$

where the sums are calculated in X. For any choice of $t_n \in e_n$ $(n \in \mathbf{Z})$ we obviously have

$$\underline{\sigma}(f,\Lambda) \le \sum_{-\infty}^{\infty} f(t_n)\mu(e_n) \le \overline{\sigma}(f,\Lambda).$$

It is also clear that, while partitions are refined, $\underline{\sigma}(f,\Lambda)$ increases and $\overline{\sigma}(f,\Lambda)$ decreases. If there is such an element $x \in X$, that $\underline{\sigma}(f,\Lambda) = x = \inf \overline{\sigma}(f,\Lambda)$, where the exact bounds are taken over all possible partitions $\Lambda := (\lambda_l)_{l\in\mathbf{Z}}$ of the numerical straight line as $\delta(\Lambda) := \sup\{\lambda_n - \lambda_{n-1}\} \to 0$, then the function f is said *to be integrable with respect to the spectral measure* and we write

$$I(f) := I_\mu(f) := \int_\Omega f(t)\, d\mu(t) := x.$$

Since $0 \le \overline{\sigma}(f,\Lambda) - \underline{\sigma}(f,\Lambda) \le \sum_{n=-\infty}^{\infty} \delta\mu(e_k) = \delta\mathbf{1}$, where $\delta := \delta(\Lambda)$, for the integral $I_\mu(f)$ to exist it is necessary and sufficient that there exist $\overline{\sigma}(f,\Lambda)$ and $\underline{\sigma}(f,\Lambda)$ for at least one partition of Λ. In particular, a bounded measurable spectral function is integrable.

(1) *Let* $X = \mathfrak{R}\!\downarrow$ *and* μ *be a spectral measure with values in* $B := \mathfrak{B}(X)$. *Then for any measurable function* f *we have*

$$[I_\mu(f) < \lambda^\wedge)] = \mu(\{f < \lambda\}) \quad (\lambda \in \mathbf{R}),$$

in which case $I_\mu(f)$ *is the only element of the* K*-space* X *obeying this condition.*

⊲ Let us choose an arbitrary number $\lambda \in \mathbf{R}$, and let $b \le [\lambda^\wedge \le I_\mu(f)]$. According to theorem **5.2.2**, for any partition Λ we have $b\lambda \le bI_\mu(f) \le b\overline{\sigma}(f,\Lambda)$. If the partition $\Lambda := (\lambda_l)_{l\in\mathbf{Z}}$ is such that $\lambda_0 = \lambda$ and $c_n := \{u \in \Omega : \lambda_n \le f(u) < \lambda_{n+1}\}$, then for $n < -1$ we get $\lambda b \wedge \mu(c_n) \le \lambda_{n+1} b \wedge \mu(c_n)$ and, hence, $b \wedge \mu(c) = 0$, or $b \le \mu(c)^* = \mu(\Omega - c) = \mu(\{f \ge \lambda\})$. Therefore, $[I_\mu(f) \ge \lambda^\wedge] = \mu(\{f < \lambda\})$, which is equivalent to the relation sought.

Let us assume that $[x < \lambda^\wedge] = \mu(\{f < \lambda\})$ for some $x \in X$. By virtue of the above-stated property of $I_\mu(f)$ we can write

$$[(\forall \lambda \in \mathbf{R}^\wedge)\,(I_\mu(f) < \lambda \leftrightarrow x < \lambda)]$$
$$= \bigwedge_{\lambda\in\mathbf{R}} [I_\mu(f) < \lambda] \Leftrightarrow [x < \lambda^\wedge] = \mathbf{1}.$$

By the density of $\mathbf{R}^\wedge$ in $\mathfrak{R}$, we get $x = I_\mu(f)$. ⊳

(2) *Under the conditions of proposition* (2) *the function* $\lambda \to \mu(\{f<\lambda\})$, $\lambda \in \mathbf{R}$ *is the characteristic of the element* $I_\mu(f)$.

5.2.11. Theorem. *Let* X *be an extended* K_σ*-space, while* $\mu: \Sigma \to B = \mathfrak{S}(X)$ *be a spectral measure. The spectral integral* $I_\mu(\cdot)$ *is a sequentially* o*-continuous (linear, multiplicative and lattice) homomorphism from the* f*-algebra of measurable functions* $\mathfrak{M}(\Omega,\Sigma)$ *to* X.

◁ Without loss of generality, we can assume $X \subset \mathfrak{R}\downarrow$. The sums $\bar{\sigma}(f,\Lambda)$ and $\underline{\sigma}(f,\Lambda)$ do exist, since pairwise disjoint elements are summed, while the space X is extended. This fact, as has been remarked earlier, implies that there exists the integral $I_\mu(f)$. It is obvious that the operator I_μ is both linear and positive. Let us prove that it is sequentially o-continuous. Choose a decreasing sequence $(f_n)_{n\in\omega}$ of measurable functions for which $\lim_{n\to\infty} f_n(t) = 0$ for all $t \in \Omega$. Assign $x_n := I_\mu(f_n)$ $(n \in \omega)$ and $0 < \varepsilon \in \mathbf{R}$. If we denote $c_n := \{t \in \Omega : f_n(t) < \varepsilon\}$, then $\Omega = \bigcup_{n=0}^{\infty} c_n$. By virtue of propositions **5.2.3** (5) and **5.2.10** (2), we can write

$$o-\lim_{n\to\infty} e_\varepsilon^{x_n} = o-\lim_{n\to\infty} \mu(c_n) = \bigvee_{n\in\omega} \mu(c_n) = \mu(\Omega) = \mathbf{1}.$$

By virtue of the test for o-convergence **5.2.7** (1), we get $o-\lim_{n\to\infty} x_n = 0$. Then, for any measurable functions f and $g: \Omega \to \mathbf{R}$ it follows from **5.2.6** (9) and **5.2.10** (2) that

$$\begin{aligned} e_\lambda^{f\vee g} &= \mu(\{f\vee g<\lambda\}) = \mu(\{f<\lambda\}\cap\{g<\lambda\}) \\ &= \mu(\{f<\lambda\}) \wedge \mu(\{g<\lambda\}) = e_\lambda^{I(f)} \wedge e_\lambda^{I(g)} = e_\lambda^{I(f)\vee I(g)}. \end{aligned}$$

Therefore, $I(f\vee g) = I(f) \vee I(g)$, which implies that $I := I_\mu$ is a lattice homomorphism. Analogously, for $f \geq 0$ and $g \geq 0$ it follows from **5.2.6** (6) and **5.2.8** (2) that for $\lambda \in \mathbf{Q}$ we get

$$\begin{aligned} e_\lambda^{I(fg)} &= \mu(\{fg<\lambda\}) = \vee\{\mu(\{f<\aleph\}) \wedge (\{g<\upsilon\}) : \lambda = \upsilon\aleph, 0 \leq \aleph, \upsilon \in \mathbf{Q}\} \\ &= \vee\{e_\aleph^{I(f)} \cdot e_\upsilon^{I(g)} : 0 \leq \aleph, \upsilon \in \mathbf{Q}, \upsilon\aleph = \lambda\} = e_\lambda^{I(f)\cdot I(g)}. \end{aligned}$$

Hence, $I(f)\cdot I(g) = I(fg)$. For arbitrary f and g the last equality follows from the earlier established properties of the spectral integral:

$$I_\mu(fg) = I_\mu(f^+g^+) + I_\mu(f^-g^-) - I_\mu(f^+g^-) - I_\mu(f^-g^+)$$
$$= I_\mu(f)^+ I_\mu(g)^+ + I_\mu(f)^- I_\mu(g)^- - I_\mu(f)^- I_\mu(g)^+ - I_\mu(f)^+ I_\mu(g)^-$$
$$= I_\mu(f)\cdot I_\mu(g). \ \triangleright$$

5.2.12. *Let* $e_0,\dots,e_{n-1}:\mathbf{R}\to B$ *be an arbitrary finite set of spectral functions with values in a* σ*-algebra* B. *Then there is a unique* B*-valued spectral measure* μ *defined on the Borel* σ*-algebra* $\mathscr{B}(\mathbf{R}^n)$ *of the space* $\mathbf{R}^n$ *such that*

$$\mu\left(\prod_{l=0}^{n-1}(-\infty,\lambda_l)\right) = \bigwedge_{l=0}^{n-1} e_l(\lambda_l),$$

for arbitrary $\lambda_0,\dots,\lambda_{n-1}\in\mathbf{R}$.

$\triangleleft$ Without loss of generality, we can assume that $B = \mathfrak{A}(Q)$, where Q is the Stone space of B. According to **5.2.8** (2), there are continuous functions $x_l: Q\to\overline{\mathbf{R}}$ ($l := 0,\dots,n-1$) such that $e_l(\lambda) = \{x_\lambda < \lambda\}$ for all $\lambda\in\mathbf{R}$ and $l = 0,\dots,n-1$. Assign $f(t) := (x_0(t),\dots,x_{n-1}(t))\in\mathbf{R}^n$, if all $x_l(t)$ are finite and $f(t) = \infty$ provided $x_l(t) = +\infty$ for at least one index l. We, thus, have defined a continuous mapping $f: Q\to\mathbf{R}^n\cup\{\infty\}$ (a filter base of neighbourhoods of the point ∞ consists of all the complements of arbitrary balls centred at zero). It is obvious that f is measurable relative to the Borel algebras $\mathscr{B}(Q)$ and $\mathscr{B}(\mathbf{R}^n)$. Let $\mathfrak{E}_\sigma(Q)$ be the σ-algebra of the Q subsets generated by the algebra $\mathfrak{A}(Q)$, while Δ be the σ-ideal in $\mathfrak{E}_\sigma(Q)$ consisting of meager sets. In this case there is an isomorphism h of the factor-algebra $\mathfrak{E}_\sigma(Q)/\Delta$ on a σ-algebra $B := \mathfrak{E}(Q)$. Let us denote by $[A]_\Delta$ the class of equivalence of a set $A\in\mathfrak{E}_\sigma(Q)$. Now we determine $\mu:\mathscr{B}(\mathbf{R}^n)\to B$ by the formula

$$\mu(A) := h([f^{-1}(A)]_\Delta) \quad (A\in\mathscr{B}(\mathbf{R}^n)).$$

It is obvious that μ is a spectral measure. If $A = \prod_{l=0}^{n-1}(-\infty,\lambda_l)$, then

$$f^{-1}(A) = \bigcap_{l=0}^{n-1}\{x_l<\lambda_l\} = \bigwedge_{l=0}^{n-1} e_l(\lambda_l),$$

and, hence, $\mu(A) = e_0(\lambda_0)\wedge\dots\wedge e_{n-1}(\lambda_{n-1})$. If μ' is another spectral measure with the same properties as μ, then the set $\mathscr{B} := \{A\subset\mathbf{R}^n : \mu(A) = \mu'(A)\}$ is a σ-algebra and contains all sets of the type $(-\infty,\lambda_0)\times\dots\times(-\infty,\lambda_{n-1})$. Therefore, $\mathscr{B}(\mathbf{R}^n)\subset\mathscr{B}$ and $\mu = \mu'$. $\triangleright$

Now let us choose elements $x_0,\dots,x_{n-1}$ of a K_σ-space X with unit $\mathbf{1}$. Let

$e^{x_l}: \mathbf{R} \to B := \mathfrak{S}(1)$ be the characteristic of the element x_l. In accord with the proposition proved above, there is a spectral measure $\mu: \mathcal{B}(\mathbf{R}^n) \to B$ for which

$$\mu\left(\prod_{l=0}^{n-1}(-\infty,\lambda_l)\right) = \bigwedge_{l=0}^{n-1} e^{x_l}(\lambda_l).$$

The integral of a measurable function $f: \mathbf{R}^n \to \mathbf{R}$ with respect to measure μ will be denoted by $I(f,\mathfrak{x}) := I(f,x_0,\dots,x_{n-1})$, where $\mathfrak{x} := (x_0,\dots,x_{n-1})$. It should be recalled that $\mathcal{B}(\mathbf{R}^n,\mathbf{R})$, which is the space of all Borel functions from $\mathbf{R}^n$ to $\mathbf{R}$, is a K_σ-space and an exact f-algebra.

5.2.13. Theorem. *For any ordered set $\mathfrak{x} := (x_0,\dots,x_{n-1})$ of elements of an extended K_σ-space X the mapping $f \to I(f,\mathfrak{x})$ $(f \in \mathcal{B}(\mathbf{R}^n,\mathbf{R}))$ is a homomorphism of the f-algebra $\mathcal{B}(\mathbf{R}^n,\mathbf{R})$ in X which obeys the following conditions:*

(1) *$I(d\lambda_l,\mathfrak{x}) = x_l$ $(l < n)$, where $d\lambda_l: \mathbf{R}^n \to \mathbf{R}$ is the l-th coordinate function $(\lambda_0,\dots,\lambda_{n-1}) \to \lambda_l$;*

(2) *if a sequence $(f_k) \subset \mathcal{B}(\mathbf{R}^n,\mathbf{R})$ is such that $\lim\limits_{n\to\infty} f_k(t) = f(t)$ for all $t \in \mathbf{R}^n$, then $o\text{-}\lim\limits_{n\to\infty}(f_k,\mathfrak{x}) = I(f,\mathfrak{x})$.*

◁ According to theorem **5.2.11**, it suffices to prove only statement (1). For simplicity, we limit ourselves to the case when $n = 1$.

So, let $x \in X$, and μ be a spectral measure associated with the characteristic $(e_\lambda^x)_{\lambda\in\mathbf{R}}$ of the element x. Prove then that

$$x = \int_{\mathbf{R}} \lambda \, d\mu(\lambda) := \int_{\mathbf{R}} \lambda \, de_\lambda^x.$$

Choose an arbitrary number $\varepsilon > 0$. Let a partition $\Lambda := (\lambda_l)$ of the numerical straight line be such that $\lambda_{l+1} - \lambda_l < \varepsilon$ for all $l \in \mathbf{Z}$. Assign

$$\sigma := \sum_{-\infty}^{\infty} \xi_n \mu([\lambda_{n-1},\lambda_n)) = \sum_{-\infty}^{\infty} \xi_n (e_{\lambda_n}^x - e_{\lambda_{n-1}}^x),$$

where $\xi_n \in [\lambda_{n-1},\lambda_n)$. By virtue of **5.2.3** (2),

$$b_n := e_{\lambda_n}^x - e_{\lambda_{n-1}}^x = e_{\lambda_n}^x \wedge (e_{\lambda_{n-1}}^x)^* = [\lambda_{n-1}^\wedge \le x < \lambda_n^\wedge].$$

It should be observed that $b_n = [\xi_n^\wedge = \sigma]$ (see **5.2.2**). On the other hand,

$$b_n = [\lambda_{n-1}^\wedge \leq x < \lambda_n^\wedge] \wedge [\lambda_n^\wedge - \lambda_{n-1}^\wedge \leq \varepsilon^\wedge]$$
$$\wedge [\lambda_{n-1}^\wedge \leq \xi_n < \lambda_n^\wedge] \leq [|x - \xi_n^\wedge| \leq \varepsilon^\wedge],$$

and, hence, $[|x - \sigma| \leq \varepsilon^\wedge] = \mathbf{1}$, or $|x - \sigma| < \varepsilon \mathbf{1}$. This implies that x is the r-limit of the integral sums in question.

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Nonstandard Methods of Analysis

Mathematics and Its Applications

Volume 291

Nonstandard Methods of Analysis

by

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SPRINGER SCIENCE+BUSINESS MEDIA, B.V.

A C.I.P. Catalogue record for this book is available from the Library of Congress

ISBN 978-94-010-4497-4 ISBN 978-94-011-1136-2 (eBook)
DOI 10.1007/978-94-011-1136-2

This is the translation of the original work
Nonstandard Methods of Analysis, Novosibirsk, Nauka © 1990
Translated from the Russian and typeset by L. Iouzina-Tarvainen.

Printed on acid-free paper

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INTRODUCTION

Nonstandard methods of analysis in the modern sense of this term consist in using two different models, i.e., 'standard' and 'nonstandard', of the theory of sets for studying concrete mathematical objects and problems. Such methods have been substantially developed in the last thirty years and have resulted in several trends, the principal ones of which, named *infinitesimal* analysis and *Boolean-valued* analysis, are the subject of the present monograph.

The former of these trends is often referred to, following its founder A.Robinson, by an expressive though slightly awkward term, *nonstandard* analysis (nowadays, however, they often say *classical* or *Robinson nonstandard* analysis). Infinitesimal analysis is characterized by a wide use of long-known in natural sciences but long-prohibited in the mathematics of the 20th century concepts pertaining to the notions of actually infinitely large and actually infinitely small magnitudes. The Robinson nonstandard analysis has been rapidly developing and has already introduced dramatic changes in the system of general mathematical notions. This is first of all associated with the fact that this method has both offered a new understanding of infinitesimal methods of indivisibles which stem from the ancient time and implemented a synthetic approach to the differential and integral calculuses originating from its founders. Nowadays infinitesimal analysis is being widely used and has penetrated all branches of modern mathematics, especially such as nonsmooth analysis, probability theory, the qualitative theory of differential equations and mathematical economics.

Boolean-valued analysis is characterized by extensive use of such terms as descents and ascents, cyclic envelopes and mixings, and B-sets and presentation of objects as models. The development of this trend which originated from the famous P.J.Cohen elaborations of the continuum hypothesis, has resulted in principally new ideas and data in a number of branches of functional analysis, and primarily in the theory of Kantorovich spaces, in the theory of von Neumann algebras, in convex analysis and the theory of vector measures.

When writing this monograph, we basically oriented ourselves to the reader who is striving, like ourselves, to acquire new methods for solving analytical problems, which has resulted in some peculiarities both in the structure and way of presenting the material. Taking into consideration the difference in the level of formal requirements for acquiring infinitesimal and Boolean-valued ideas, we found it necessary to divide the monograph into two parts, having ensured their relative independence, which fact is stressed by the independent enumeration of both parts. Our primary intention was to embrace synthetic nonstandard methods and to essentially increase the number of concrete analytical applications. The rapidly increasing volume of the monograph made us, however, shelve these ideas for the time being. In the reference section, which in no way claims completeness, we, however, preserved the references that pertain to the previous intentions. We also did our best to include all the references known to us which contain both historical and reference data of importance.

Studying nonstandard methods of analysis has given us plenty of positive emotions, which accounts for the fact that when presenting the material we sometimes try to avoid traditional dry formalized way of doing it. We bring our apologies beforehand to those who might be irritated by such passages. As an excuse, however, we would like to point out that emotional presentation not only leaves the mathematical facet of the subject intact, but sometimes, in our opinion, even facilitates its better understanding.

We wish to express gratitude to all our colleagues and friends whose interest and valuable advice helped us in our work on this monograph.

A.G.Kusraev
S.S.Kutateladze

PART I

INFINITESIMAL ANALYSIS

The idea of an infinitesimal, i.e., of an infinitely small magnitude, can be traced as far back as the ancient time. Nowadays, after approximately half a century of neglect, modern mathematics has started paying rapt attention to infinitesimal notions. Infinitely big or infinitely small numbers, mathematical atoms, i.e., 'indivisible' monads, are oftener encountered in various publications, becoming part and parcel of mathematical practice. A turning point in the evolution of infinitesimal concepts is associated with an outstanding achievement of A.Robinson who created nonstandard analysis.

For a prolonged period of time nonstandard analysis has been viewed as a quite subtle and even exotic logical technique designed for substantiating the method of actually infinitely large and infinitely small numbers. This technique has also been considered to be of restricted applicability and, in any case, to be principally unable to result in any serious reconsiderations of general mathematical notions. By the end of the 1970s, after the E.Nelson internal set theory (and, some time later, the external set theories propounded by K.Hrbaček and T.Kawai) was made public, the views on the place and role of nonstandard analysis had been dramatically enriched and changed. In the light of new discoveries it became possible to view nonstandard elements not as 'imaginary, ideal or surd entities' appended to common sets by considerations of formal convenience, but as indispensable parts of any routine mathematical objects. There arose a doctrine the essence of which was that every set is composed of both standard and nonstandard elements. Standard sets, in turn, form a kind of a frame of reference net densely located in the totality of all objects investigated by mathematics. It was discovered that in this case such objects of nonstandard mathematical analysis as monads of filters, standard parts of numbers and vectors, operator shadows, etc., form 'Cantorian' sets, which are not described by any of the canonized pictures drawn by known formal set theories. *The von Neumann universe does not exhaust the world of classical mathematics*, that was one of the obvious conclusions of the new approaches. Therefore, traditional views on nonstandard analysis started requiring at least a revision, a reconsideration of the infinitesimal concepts.

An important advantage of the new approaches originated was the axiomatic approach which made it possible to master the apparatus of nonstandard mathematical analysis without preliminary studying either the techniques of ultraproducts or Boolean-valued models, or their analogues. The axioms put forward are simple and precisely motivated at the semantic level within the framework of the 'naive' set-theoretical stances analysis commonly used. At the same time, they essentially broaden the number of mathematical objects, open up possibilities for developing a new formal apparatus, and enable one to significantly decrease dangerous gaps between the presentations, methodical doctrines and levels of rigour adopted in mathematics and its applications to natural and social sciences. In other words, the axiomatic set-theoretic foundation is of a general scientific significance.

In 1947 K.Gödel wrote: "There might exist axioms so abundant in their verifiable consequences, shedding so much light upon a whole discipline, and furnishing such powerful methods for solving given problems (and even solving them, as far as that is possible, in a constructive way) that quite irrespective of their intrinsic necessity they would have to be assumed at least in the same sense as any well established physical theory." [61, p.521]. This prediction by K.Gödel is becoming true.

The purpose of the present work is to make new roads to nonstandard analysis more accessible. To this end, we start with presenting the semantic qualitative views on standard and nonstandard objects, on the apparatus of nonstandard analysis at the 'naive' level of rigour, which is absolutely sufficient for effective applications without appealing to a logical formalism. Then we give a brief but sufficiently complete reference material pertaining to modern axiomatic constructions of nonstandard analysis within the Cantor doctrine. In this case we found it appropriate to pay special attention to the ideological and historic facets of the matter, which determined the peculiarities of our working plan. The historic data, the qualitative motivations of the principles of nonstandard analysis and the discussion of their simplest corollaries for differential and integral calculuses presented in the first and second chapters form the 'naive' foundation of infinitesimal analysis. Formal details of the corresponding apparatus of nonstandard set theory are given in the third chapter. A weighty argument in favour of a certain concentricity in our presenting the material are the remarkable words by N.N.Luzin: "Mathematical analysis is far from being an absolutely completed science, as it is sometimes inclined to be viewed, with once and forever found principles which are good only for further corollaries to be drawn from... Mathematical analysis is not different from any other science, and it has its own motion of ideas which is not only translational but also rotational, constantly returning to a group of former ideas which are always, however, newly lit" [160, p.389].

In the fourth and fifth chapters concluding the first part of the monograph we present infinitesimal methods under general topology and differential calculus. The choice of these topics from the variety of modern applications of nonstandard analysis has been basically determined by the personal preferences of the authors.

CHAPTER 1

EXCURSUS INTO THE HISTORY OF MATHEMATICAL ANALYSIS

The ideas of differential and integral calculuses can be traced as far back as the ancient time and are associated with the most fundamental mathematical concepts. Any detailed presentation of the history of the evolution of interpreting mathematical objects, of calculation and measurement processes determining modern views on infinitesimals, would require special efforts which are beyond our possibilities and intentions. The situation is significantly complicated by the fact that the history of mathematics has been subjected to well-known negative processes which arise under constant attempts to apologize some up-to-date views. In particular, the evolution of the apparatus of a mathematical analysis is far from always being presented sufficiently completely and objectively. One-sided views on the essence of a differential and an integral, hypertrophying the role played by the notion of limit, and neglect of actually infinitely large and infinitely small numbers have been so widely spread during the last fifty years that their existence cannot be shammed.

It has become a truism to say that "the very foundations of analysis have been for a long time surrounded with mystery as a result of unwillingness to admit that the notion of a limit enjoys an exclusive right to be a source of new methods." [30, p.562]. However, as has been justly noted by L.S.Pontryagin: "In a historical sense, integral and differential calculuses had already been well-developed fields of mathematics before the theory of limits appeared. The latter originated as a superstructure over the existing theory. Many physicists opine that the so-called rigorous definition of derivatives and integrals is not at all necessary for satisfactory comprehension of differential and integral calculuses. I share their viewpoint." [213, pp. 64-65].

Taking into account all these facts, we found it necessary to brief the reader about some turning points of the history of analysis, as well as about the ideas expressed by classics in the evolution of modern views. The choice of the corresponding fragments is certain to be subjective. We nevertheless hope that it will be sufficient to develop a critical attitude to one-sided and distorted delineations of the evolution of infinitesimal methods.

1.1 G.W.Leibniz and I.Newton

The ancient name for differential and integral calculuses is 'infinitesimal analysis'. It is this title that was used for the first textbook on mathematical analysis published as far back as 1696. The textbook was written by G. de l'Hôpital as a result of his contacts with I.Bernoulli (senior), one of the most famous disciples of G.W.Leibniz.

"Of all theoretical achievements of cognition hardly any other but the invention of calculus of the infinitely small in the second half of the XVII century can be considered the highest triumph of human spirit. If we are to have any pure and exclusive creation of human spirit somewhere, it is in this sphere", that was a high estimate given by F.Engels [44, p.582] to this new branch of mathematics.

The scientific legacy, the creative work and personal relations between the founders of mathematical analysis, G.W.Leibniz and I.Newton, have been subjected to a detailed, one may say, thorough examination. An attempt to restore the train of thought of the men of genius, to elucidate the ways resulted in the discovery of new truths, is both justified and understandable. One, however, should bear in mind that there is a principal difference between draft papers and notes, personal letters to colleagues and the works especially designed for publications. It is, therefore, first of all necessary to consider the 'official' presentations of the G.W.Leibniz's and I.Newton's views on infinitely small we are interested in.

The first work on differential calculus ever published was the paper by G.W.Leibniz "A new method of maxima and minima, as well as of tangents which is hampered by neither fractional or irrational quantities, and a special type of calculus for it" (see [152]). In this paper published in the Leipzig magazine 'Acta Eruditorum' in 1684, more than three centuries ago, Leibniz gives the following definition of differential. Considering a curve YY and an interval of the tangent passing through a fixed point of the curve Y, which corresponds to a chosen coordinate X on the axis AX, and denoting the point of intersection of the tangent with the axis considered by D, he writes: "Now some straight line selected arbitrarily is called dx, while another one whose ratio to dx is the same as that of $...y...$ to XD, will be called...dy, or difference (differentia) of $...y...$." This text is accompanied by a drawing whose essential features (with Leibniz's written explanations) are reproduced here (Fig.1).

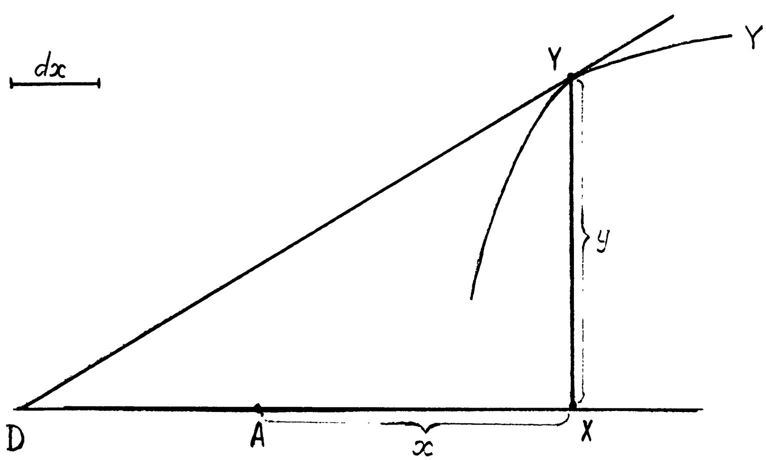


Fig.1

Therefore, according to Leibniz, at a point x, given an arbitrary dx, for the function $x \to y(x)$, we obtain

$$dy := \frac{YX}{XD} dx$$

In other words, the differential is determined as a corresponding linear mapping, i.e., in the manner which is readily recognized by the majority of modern specialists.

G.W.Leibniz is a thorough thinker who considered the "invention of the syllogistic form to be one the most beautiful and even important discoveries of the human spirit. This is a sort of *universal mathematics* the significance of which has not yet been completely realized. It can be said to contain the art of being faultless..." [151, pp. 492-493]. Realizing undoubtedly that the description and substantiation of the algorithm of differential calculus (in that way G.W.Leibniz referred to the rules of differentiation) suggested by him required the notion of a tangential to be refined, he further explains: "...to find a tangent means to draw a line that connects two points of the curve at the infinitely small distance, or the continual side of a polygon with an infinite number of angles which for us takes the place of the curve." Hence, Leibniz rests his calculus on the appeal to the structure of curves 'in the small'.

At that time there were practically two points of view as regards the status of infinitesimals. According to one of them, which seemed to be shared by G.W.Leibniz, an infinitely small number was thought to be less that any other 'arbitrarily given quantity'. Actually existing 'indivisible' elements which form quantities and figures are the images corresponding to the concept of the infinitely small in question. For G.W.Leibniz the existence of "simple substances incorporated into the structure of complex ones", i.e., monads, is doubtless. "It is these monads that are true atoms of nature, or, to put it short, elements of things" [150, p.413].

For another founder of the analysis, I.Newton, infinitesimals are related to the idea of vanishing quantities [194, 277]. He viewed indeterminate quantities "not as made up of indivisible particles but as described by a continuous motion", "... as increasing or decreasing by a perpetual motion, in their nascent or evanescent state". The famous 'method of prime and ultimate ratios' in his classical treatise "Mathematical Principles of Natural Philosophy" (1687) is formulated as follows.

"Quantities, and the ratios of quantities, which in any finite time converge continuously to equality, and before the end of that time approach nearer to each other than by any given difference, become ultimately equal" [277, p.101].

Pursuing the ideas which are nowadays closely associated with the theory of limits, I.Newton exhibited insight, prudence and wisdom in estimating the concurrent views inherent in true scientific workers. He wrote: "... to institute an analysis after this manner in finite quantities and investigate the prime or ultimate ratios of these finite quantities when in their nascent or evanascent state is consonant to the geometry of the ancients, and I was willing to show that in the method of fluxions there is no necessity of introducing figures infinitely small into geometry. Yet the analysis may be performed in any kind of figure, whether finite or infinitly small, which are imagined similar to the evanescent figures, as likewise in the figures, which, by the method of indivisibles, use to be reckoned as infinitely small provided you proceed with due caution" [194, p.169].

G.W.Leibniz's views were as much pliable and in-depth dialectic. In his well-known letter to P.Varignion of February 2, 1702 [277], stressing the idea that "... it is unnecessary to make mathematical analysis depend on metaphysical controversies", he points out to the entity of concurrent views on the objects of the new apparatus: "... if any opponent tries to contradict this proposition, it follows from our calculus that the error will be less than any possible assignable error, since it is in our power to make this incomparably small magnitude small enough for this purpose, inasmuch as we can always take a magnitude as small as we wish. Perhaps this is what you mean, Sir, when you speak on the inexhaustible, and the rigorous demonstration of the infinitesimal calculus which we use undoubtedly is to be found here. ... So it can also be said that infinites and infinitesimals are grounded in such a way that everything in geometry, and even in nature, takes place as if they were perfect realities. Witness not only our geometrical analysis of transcendental curves but also my law of continuity, by virtue of which we may consider rest as infinitely small motion (that is, as equivalent to a particular instance of its own contradictory), coincidence as infinitely small distance, equality as the limit of inequalities, etc."

Similar views were expressed by G.W.Leibniz in the quotation to follow, the italicized end of which is, according to A.Robinson [222, pp. 260-261], often cited in works on nonstandard analysis: "... it is not necessary here to view the infinite in the strict sense of the word, but only in the sense in which in optics they say that the rays of the sun are emitted from an infinitely distant point and can therefore be considered parallel. When there are different orders of the infinite and infinitely small, they are understood in the same sense in which Earth is considered to be a dot as compared with the distance to the fixed stars, while a ball in our hands is viewed as a dot as compared with the semi-diameter of our planet, so that the distance from the fixed stars is infinitely infinite or the infinity of the infinity as regards the ball diameter. The point is that, instead of the infinite and infinitely small, the values are chosen to be as big or as small as it is necessary for the mistake to be less than the given one and, therefore, *the difference from Archimedes style is only in expressions which in our method are more straightforward and more applicable to the art of invention*" [152, p.190].

1.2 Karl Marx on Mysterious Differential Calculus

The high level of the requirements on the soundness and rigour of their methods characteristic of the works by G.Leibniz and I.Newton was not, unfortunately, taken up by their followers who contributed a lot to make the mysterious mist surrounding the nontrivial as they were abstract ideas even thicker. It suffices to say that in the textbook by de l'Hôpital mentioned above the definition II says: "... an infinitely small part, which a variable value increases or decreases by is called its differential."

One can see that a colossal step back from the initial definition by G.Leibniz was made here. It is not incidentally that K.Marx called the mathematical analysis of the XVII century, while making first acquaintance with it, the 'mysterious differential calculus'.

Of interest is the fact that when interpreting the draft papers by K.Marx [176], a somewhat dramatized criticism of the actually infinitely small is sometimes derived from them. For instance, in one of the textbooks on the fundamentals of nonstandard analysis a known statement is quoted: "The consolation some rationalizing mathematicians are tightly holding at, that in the qualitative sense dy and dx are in fact only infinitely small, that their ratio] is only approaching $\frac{0}{0}$, is a chimera..." [176, p.33]. In this case omitted is the following principal observation: "...in fact $\frac{dy}{dx}$ does not denote the extravagant $\frac{0}{0}$, but, on the contrary, it is a festive uniform for $\frac{dy}{dx}$ when the latter is functioning as a ratio of infinitely small differences ..." [176, p. 71].

Therefore, in order to elucidate the details, let us give a complete quotation of one of the abstracts of K.Marx's manuscripts often cited but with blanks distorting the sense.

"We therefore have nothing else to do but to view increments of the variable h as infinitely small and to ascribe to them, as such, *independent existence*, in, for instance, symbols $\dot{x}$, $\dot{y}$, etc., or dx, dy [etc.]. Infinitely small magnitudes, however, are as much magnitudes as infinitely large ones are (the word infinitely [small] denotes in fact only indefinitely small); these dx, dy, etc., or $\dot{x}$, $\dot{y}$, [etc.] are also functioning in calculus as common algebraic magnitudes, and in the equation given above,

$$(y+k)-y \text{ or } k = 2xdx + dxdx,$$

the term $dxdx$ has the same right to exist as $2xdx$ does. But the most astonishing statement is the one by virtue of which this term is forced to be omitted due to relativity of the notion of the infinitely small; $dxdx$ is omitted since it is infinitely small as compared with dx and, hence, with $2xdx$ or $2x\dot{x}$. Or, if in

$$\dot{y} = \dot{u}z + \dot{z}u + \dot{u}\dot{z}$$

[the term] $\dot{u}\dot{z}$ is omitted since it is infinitely small as compared with $\dot{u}z$ or $\dot{z}u$, then mathematically this can be justified only by referring to the fact that $\dot{u}z + \dot{z}u$ has for us an approximate value which can be thought to be any close to the exact one. A maneuver of the kind can be encountered in common algebra as well. In this case, however, we face a greater wonder: due to this method for an arbitrary function [in] x we obtain not approximate but absolutely exact values (though, as above, only symbolically correct), as in the example $\dot{y} = 2x\dot{x} + \dot{x}\dot{x}$. Neglecting here $\dot{x}\dot{x}$, we obtain

$$\dot{y} = 2x\dot{x},$$
$$\frac{\dot{y}}{\dot{x}} = 2x,$$

which is the correct first derivative function of x^2 which has already been proved by the binom[ial theorem].

But this miracle is far from being a miracle. On the contrary, it would be a miracle if the forced omission of $\dot{x}\dot{x}$ *did not give an exact result. It is a certain mistake in calculations that is omitted*, which is an inevitable consequence of the method making it possible to introduce an indefinite increment, for instance, h, of the variable immediately as the differential dx or x, as a ready-made operation symbol and, therefore, differential calculus also immediately appears to be an independent way of calculus, other than conventional algebra" [176, pp.151-153].

1.3 L.Euler

In the history of mathematical analysis the eighteenth century is rightfully called L.Euler's century. Everyone who will get acquainted with his textbooks will be staggered by his subtle technique and in-depth penetration into the essence of the matter. One can recall that an outstanding engineer-scientist A.N.Krylov was in rapture seeing in the famous Euler formula $e^{i\pi} = -1$ a symbol of the entity of the whole of mathematics. He, in particular, noted that "... here -1 presents arithmetics, i algebra, π geometry, and e analysis".

L.Euler is characterized by a many-sided approach, a 'systemic' one, as might be put today, to studying mathematical problems, he widely used the whole of the apparatus developed by that time. It should be stressed that he was constantly making effective and productive use of the infinitesimal concepts, and, first of all, infinitely large and infinitely small numbers. L.Euler gave a sufficiently detailed explanations of the methodological foundation of his ideas, called 'calculus of zeros'. There is a tendency to look for spots on the sun (other than the existing ones) and, analogously, for weak points in men of genius. For many years L.Euler had been incriminated 'incorrect' treatment of diverging series, until his ideas were accepted. At present you can encounter such a phrase: "As to the problem of diverging series, Euler was sharing quite an up-to-date point of view... ." It would be more correct to topsy-turvy the phrase and say that modern mathematicians have finally caught it up with Euler's ideas. As will be shown in the sections to follow (see **2.2**, **2.3**), the opinion that "... we cannot admire the way Euler corroborates his analysis by introducing zeros of various orders" is as self-assured as the statement that "... the giants of science, mainly, Euler and Lagrange, have laid false foundations of analysis." It must be admitted once and for ever that Euler was in full possession of analysis and completely aware what he had created.

1.4 Statement by G.Berkeley

In their general form the ideas of analysis greatly affected the character of the ideological outlook in the XYIII century. Vivid examples of the depth of penetration of the notions of infinitely large and infinitely small quantities into the cultural media of that time are, in particular, "Gulliver's Travels" by Jonathan Swift published in 1726 (Lilliput and Brobdingnag!) and the famous "Micromegas 1752" written by bright and venomous F.M.Arouer, i.e., Voltaire. Of interest is the fact that as an epigraph for his classical treatise [222], A.Robinson chose the beginning of the following speech of Micromegas:

"Now I see clearer than ever that nothing can be judged by its visible magnitude. Oh, my God, who granted reason to creatures of such tiny sizes! An infinitely small thing is equal to an infinitely large one when facing you; if living beings still smaller than those were possible, they could have reason exceeding the intellect of those magnificent creatures of yours which I can see in the sky, and one foot of which could cover the earth" [266, p.154].

A serious 'reverse' impact on the development of mathematical analysis was made in 1734 by a publication of a great church figure and theologian, bishop G.Berkeley, a pamphlet "The Analyst, or a Discourse Addressed to an Infidel Mathematician, wherein it is examined whether the object, principles and inferences of the modern analysis are more deduced than religious mysteries and points of faith" [11, pp.396-422]. The antihumanistic spirit of the paper by G.Berkeley is combined with aphoristicity, subtle observations and killing precision of expression. "... By inconsistent supposition truth may be obtained, but not science though", this is the basic idea of his criticism of analysis. G.Berkeley's challenge was addressed to the whole of natural science: "I have no controversy about your conclusions, but only about your logic and method. How do you demonstrate? What objects are you conversant with, and whether you conceive them clearly? What principles you proceed upon; how sound they may be; and how you apply them?"

V.I.Lenin, exposing the venomous plot of G.Berkeley, wrote: "Assume the external world, nature to be a 'combination of perceptions' brought to our minds by a god. Admit it, refuse the idea of looking for 'the basis' of these perceptions outside your consciousness, outside the human being, and I will accept all natural sciences, all the significance and validity of its conclusions within the framework of my idealistic cognition theory. It is this framework and only this framework that I need for my deductions in favour of 'peace and religion'." This is Berkeley's idea" [153, p.22].

G.Berkeley's challenge could not but be answered by advanced representatives of the scientific thought of the XVIII century, the encyclopedians.

1.5 J.D'Alambert and L.Carnot

A turning point in the history of formulating the basic notions of analysis is associated with the ideas and activities of J.D'Alambert, one of the initiators and leading authors of the immortal masterpiece of the thought of the Age of Enlightment, "Encyclopedia or Explanatory Dictionary of Sciences, Arts and Crafts". In the article "Differential" he wrote: "Newton never considered differential calculus to be calculus of the infinitely small, but he viewed it as the method of the first and last relations" [277, p.157]. D'Alambert was the first mathematician who declared that he had found the proof that the infinitely small "... do exist neither in the nature nor in the assumptions of geometricians" (a quotation from his article "Infinitesimal" of 1759). It is the J.D'Alambert's viewpoint presented in "Encyclopedia ..." that contributed much to the formulation by the end of the XVIII century of the understanding of the infinitely small as a magnitude tending to zero. In all probability, one should recall in this respect a paper by L.Carnot "Considerations on metaphysics of the infinitely small" where he remarks "... the notion of infinitesimal is less clear than that of limit since it is nothing else but the difference between such a limit and the quantity the ultimate value of which it is."

1.6 B.Bolzano, O.Cauchy and K.Weierstrass

The XIX century was the century of the foundation of analysis by means of the theory of limits. An outstanding contribution to this process was made by B.Bolzano, O.Cauchy and K.Weierstrass, their achievements mirrored in any traditional textbook on differential and integral calculus. The new canon of rigour forwarded by B.Bolzano, the definition given by O.Cauchy to an infinitely small quantity as a variable with zero limit and, finally, the $\varepsilon - \delta$-technique suggested by K.Weierstrass are indispensable part of the history of mathematical thought, part and parcel of modern culture. It should be emphasized (see 2 [76]) that, while giving a verbal characteristic of continuity, both O.Cauchy and K.Weierstrass used practically the same expressions:

> "an infinitely small increment of a variable always results in that of the function itself",
>
> O.Cauchy;
>
> "infinitely small variations in the argument correspond to those of the function",
>
> K.Weierstrass.

The coincidence under discussion emphasizes the necessity shared by the cited authors and worthy of respect to interrelate the new ideas with the viewpoints of their great predecessors.

Speculating about significance of the reconsiderations of analytical views in the XIX century, one should bear in mind the important observation made by F.Severi [229, p.113] in this respect: "The reconsideration that has been mostly completed nowadays has, however, no definite value most scientists believe in. Rigour itself is, in fact, a function of the total volume of knowledge at every historic period which corresponds to the way of the scientific treatment of the truth."

1.7 N.N.Luzin

The beginning of the XX century in mathematics was marked by a further growth of distrust to the concept of infinitesimal. This tendency grew stronger as mathematics was being reconsidered on the basis of the set theoretical credo which gained the key strongholds in the thirties.

In the first edition of the "Great Soviet Encyclopedia" in 1934, N.N.Luzin wrote: "As to a constant infinitely small quantity distinct from zero, modern mathematical analysis, without discarding the formal possibility of determining the idea of a constant infinitesimal (for instance, as corresponding length in non-Archimedean geometry), views this idea as absolutely fruitless, since it has proved to be impossible to introduce such an infinitesimal in calculus" [160, pp. 293-294]. At that time the publication in the USSR of a textbook by M.Ya.Vygodskii "Fundamentals of Infinitesimal Calculus" became a noticeable event and gave rise to a serious and sharp criticism. M.Ya.Vygodskii tried to preserve the concept of infinitesimals by appealing to the historical practice. He, in particular, remarked: "If it were only the problem of creating a logical apparatus which could work by itself then, having eliminated infinitesimals from considerations and having driven differentials out of mathematics, one could celebrate a victory over the difficulties that have been hampering the way of mathematicians and philosophers during the last two centuries. Infinitesimal analysis, however, originated from practical needs, its relations with natural sciences and technology (and, later, with social sciences) becoming increasingly strong and fruitful in the course of time. Complete neglect of infinitesimals would incidentally make these relations extremely difficult, not to say impossible" [271, p. 160].

Giving his estimate to the textbook by M.Ya.Vygodskii, N.N.Luzin wrote in the forties: "This course, being internally both integrated and lit by the great idea the author remains faithful to, goes beyond the framework of the style in which modern mathematical analysis has been developed for 150 years and which is now nearing its completion" [160, p.398].

The attitude of N.N.Luzin to infinitesimals deserves a special attention as an important manifestation of a hidden drama the history of any profound idea inspiring people is usually filled with. N.N.Luzin had a unique gift of penetration into the essence of the most intricate mathematical problems, and he might be said to possess a remarkable gift of foresight [147, 148, 259]. The idea of actual infinitesimals seemed to be extremely appealing to him psychologically, as he emphasized: "... the idea about them has never been successfully driven out of mind. There are, obviously, some deeply hidden reasons still not completely revealed that make our minds to be predisposed to treat infinitesimals seriously" [160, p.396]. In another of his publications N.N.Luzin remarks sorrowfully: "When the mind starts getting acquainted with analysis, i.e., during its spring season, it is the actually small, which can be termed the 'elements' of quantity, that it starts with. Gradually, however, step by step, as it is accumulating knowledge, theories, surfeit of abstraction and tiredness, the mind starts forgetting its primary intentions, it starts smiling at their 'childishness'. To put it short, when the mind is in its autumn season, it allows one to be convinced in the uniqueness of the correct foundation by means of limits" [264]. The latter point of view was energetically developed by N.N.Luzin in his textbook "Differential calculus", where he in particular emphasized the idea that "to understand correctly the very *essence of the matter*, the student should first of all understand that an infinitesimal is always, *by its very definition*, a variable quantity and, therefore, no constant number, however small it be, is *ever* infinitely small. The student should beware of using comparisons or resemblances of such, for instance, kind as 'One centimeter is a magnitude infinitely small as compared with the diameter of the sun'. This phrase is absolutely incorrect. Both magnitudes, i.e., a centimeter and the diameter of the sun, are constant values and, hence, they are *finite*, one being much smaller than the other, though. Incidentally, a centimeter is not a small value at all when, for instance, compared with the 'thickness of a hair', while for a moving microbe a centimeter is a vast space. In order to get rid of any risky comparisons and random subjective resemblances, the student *must remember that no constant value is infinitesimal, as well as no number however small it might be*. Therefore, it would be more correct to use not the term "*an infinitesimal*", but the term '*an infinitely decreasing variable*', as it expresses the *idea of variability* most vividly" [161, p. 61].

1.8 A.Robinson

The seventh (posthumous) edition of the textbook by N.N.Luzin under discussion was published in 1961 simultaneously with the A.Robinson's "Nonstandard Analysis" which laid an up-to-date foundation for the method of the actually infinitely small. A.Robinson based his work on the local theorem by A.I.Mal'tsev, stressing its significance as that of the 'fundamental importance for our theory' [222, p. 13] and giving direct references to an A.I.Mal'tsev's work dated as far back as 1936. The Robinson's discovery both elucidates the ideas of the founders of differential and integral calculus and gives a further confirmation of the dialectical character of the evolution of mathematics.

CHAPTER 2

NAIVE FUNDAMENTALS OF INFINITESIMAL METHODS

One of the most widely spread prejudices as regards the infinitely small and the constructions related to them through is the opinion that the apparatus of nonstandard analysis is extremely difficult to master. Moreover, it is usually emphasized that nonstandard analysis is based on the advanced sections of the modern formalized set theory and mathematical logic. Actually, the existence of this relation, although irrefutable, in no way hampers either understanding or handling infinitesimals. The purpose of the present chapter is to corroborate the above statement by way of presenting the methodology of nonstandard analysis at the level of rigour accepted in the modern system of mathematical education which is based on the ideas of the naive set-theoretic stance proposed by G.Cantor. Alongside with elucidating the essence of the concepts of nonstandard set theory and the principles of transfer, idealization and standardization adopted in it, certain attention will be also paid to paralleling quite recent ideas on the objects of elementary mathematical analysis and the approaches of classics. In doing so, we would like to confirm the continuity in the evolution of the ideas of differential and integral calculuses which nonstandard analysis sheds new light upon.

2.1. The Notion of Set in Nonstandard Analysis

In this section we will expose a fragment of the foundations of nonstandard methods at the level of rigour close to that adopted in introducing the elements of mathematical analysis.

2.1.1. Modern courses of mathematical analysis are often based on the notion of set.

2.1.2. Examples

(1) *L.Schwartz "Analysis"*: "A set is the collection of certain objects.

Examples:

the set of graduates in one class,

the set of points on a plane,

the set of nondegenerate second-order surfaces in a three-dimensional space,
the set **N** of nonnegative integers,
the set **Z** of arbitrary integers,
the set **Q** of rational numbers,
the set **R** of real numbers,
the set **C** of complex numbers" [228, p. 9].

(2) *V.A.Il'yin, V.A.Sadovnitchii, Bl.Kh.Sendov "Mathematical Analysis"*:

"...for studying real numbers the notion of set is of importance. It should be emphasized that we view a set a an initial notion not determined from other ones.

In this section we shall study sets of an arbitrary nature. They are often called abstract sets, which implies that the objects comprising the set in question or, as they say, elements of the given set, do not necessarily have to be real numbers. Elements of an abstract set can, for instance, be functions, letters of the alphabet, figures on a plane, etc." [89, p.69].

(3) *Yu.G.Reshetnyak "Introduction to Mathematical Analysis"*:

"The notion of a set belongs to primary mathematical notions and cannot be determined by reduction to any other simpler notions.

A set is any collection of subjects considered as a whole entity. The word 'set' is a mathematical term used to denote some objects denoted in everyday speech by such words as a totality, a pack, a system, a cortege, an assembly, a family, etc.. For instance, we can speak about the set of solutions to an equation, about the set of pictures in a museum, the set of points of an interval, etc.

The objects comprising one set or another are called its elements. A set is considered to be given if for any object it is possible to determine whether it is an element of the set or not" [217, p.3].

(4) *V.A.Zorich "Mathematical Analysis"*:

"The basic suppositions of the Cantor (or, conditionally speaking, 'naive') theory of sets are as follows:

1^* a set can be composed of any different objects;
2^* a set is uniquely determined by the collection of objects comprising it;
3^* any property defines the set of objects that have this property.

If x is an object, P is a property, $P(x)$ is the notation that x has the property P, by $\{x | P(x)\}$ we shall denote the whole class of objects with the property P.

The objects comprising a class or set are termed *elements* of the class or set.

In the framework of naive set theory the terms 'class', 'family', 'union', 'collection' are treated as synonyms of the word 'set'.

The following examples demonstrate the application of this terminology:

the set of letters 'a' in the word 'I';
the set of Adam's wives;
the set of ten digits;
the legume set;
the set of grains of sand on Earth;
the totality of the points on a plane equidistant from its two given points;
a family of sets;
the set of all sets.

The difference in the possible degree of definiteness in constituting sets leads one to the conclusion that the notion of a set is not as simple and harmless as it might seem.

Indeed, the notion of the set of all sets is, for instance, just contradictory" [284, pp.17-18].

2.1.3. Nonstandard analysis or, in more detail, nonstandard mathematical analysis is a branch of mathematical analysis, and, therefore, it obviously adopts the accepted views on sets. In other words, *nonstandard analysis views as sets only those collections that the classical 'standard' theory operates with.* It should be emphasized that this statement is also valid when reformulated in the following way: nonstandard analysis does not view as sets only those unions that conventional mathematics does not accept as such. At the same time, *nonstandard analysis is related to refined views on sets*, i.e., it is, as they often say, constructed within the framework of nonstandard set theory.

2.1.4. Naive set theory is based on the classical G.Cantor's formulation: "A set is any many which can be thought of as one, that is every totality of definite elements which can be united to a whole through a law", and a set is "every collection to a whole of definite, well-differentiated objects of our intuition or our thought" [102, p.173]. Such concepts are well-known to be far too broad, this drawback by-passed by a certain detalization of the difference between sets and nonsets. For instance, in order to define inappropriate, 'too big' collections of sets the term 'class' is being used, just to denote the fact that the class does not have to be a set. In other words, when formalizing the notions of naive set theory, the procedures allowing one to introduce any 'Cantorian' set into mathematical operations are more fully and thoroughly regulated. All the sets allowed into mathematics enjoy the same rights. This in no case implies that they all are equal or have no distinctions. This implies that the sets are of one type, that they share a common status since they are elements of the 'class of all sets'.

2.1.5. The cornerstone of nonstandard set theory, its underlying principle is extremely simple: *sets can be different: standard and nonstandard.* It would therefore be more correct to speak not of the nonstandard theory of sets, but of the theory of sets, standard and nonstandard. The intuitive concept revealed by the phrase 'a set A is standard' implies that A has been described in plain and definite terms, has become an 'artefact' of the cognition activity of human beings. The term 'standard' draws a borderline between the objects determined from explicit mathematical constructions, using, for instance, theorems of existence and uniqueness, and called standard sets, and the objects originating in the course of investigation in an implicit, indirect way, those called nonstandard sets.

Such objects as the numbers π, e, $\sin 81$ that are given unambiguously as the sets of natural and real numbers that have been described in detail, are standard objects. Within the framework of the set-theoretic stance, however, an arbitrary 'abstract' real number arises in an indirect manner, being introduced as an element of the set of all real numbers mentioned earlier. A similar method of introducing objects into consideration is widely spread: a vector is an element of a vector space, a filter is a set of subsets of a given set with, incidentally, specific properties, etc.. Hence, among real numbers there are standard and nonstandard ones, there are standard and nonstandard vectors and filters, and, generally speaking, there are standard and nonstandard sets.

As an example, let us consider the multitude of sand grains on Earth. In his classical work "Psammiths" Archimedes wrote: "... of the numbers named by me and given in the work which I sent to Zeuxippes, some exceed not only the numbers of the mass of sand equal in magnitude to Earth filled up in the way described but also that of a mass equal in magnitude to the universe" [5, p.358]. Therefore, the number of the sand grains on Earth is a concrete natural number. It is, however, impossible either to give a direct definition of this number or to determine it. A sequential count of the sand grains is obvious to be impossible for implementation. Hence, the number of the sand grains on Earth is expressed by a '*nonrealizable*', '*nondefinable*' nonstandard natural number and, thus, the multitude of the sand grains is nonstandard.

It goes without saying that the presented *views on the difference between standard and nonstandard sets are of an auxiliary value* for mastering the rules of applying them into practice. One can find here an analogy with the situation in geometry where the intuitive visual presentation of spatial forms helps in elaborating the skills of using axioms which, in the long run, result in strict definitions of points, straight lines, planes, etc.. According to A.D.Alexandrov, one should remark that "axioms by themselves need no substantiation , they only sum up other data and give rise to a logical construction of a theory" [3, p. 51]. Therefore, the formal introduction of the axioms of nonstandard set theory should be preceded by their qualitative discussion.

As we already know, the nonstandard theory of sets begins with the primary observation: sets can be different, i.e., standard and nonstandard. Alongside with this postulate, the following ones are adopted (or, to be more exact, variations of the following postulates).

2.1.6. Transfer principle: *a conventional mathematical statement claiming the existence of a certain set simultaneously determines a standard set.*

In other words, the theorems of existence and uniqueness accepted in classical mathematics are considered to be direct explicit definitions of mathematical objects. An equivalent reformulation of this principle, elucidating the essence of its name, is as follows: *in order to prove a statement on all sets, it suffices to prove it only for standard ones.* An intuitive substantiation of the transfer principle is the evident fact that the statements pertaining to arbitrary sets are made when operating with the sets which have been in fact described, i.e., with standard sets.

If you think about the essence of the transfer principle, you will see that it deals with two aspects of presenting standard objects. The first is that *new standard objects arise from existing ones in a deductive way*, using descriptions of the kind of the existence and uniqueness theorems. This peculiarity can be expressed by the assertion that in a standard nonempty set there is a standard element, and that the object which is being constructed from or determined by existing standard elements is standard itself. The second aspect of the idea of being standard expressed by the transfer principle is continuously related with the former one and implies *presentability of the world of standard objects in the universe of all sets under investigation.* One can say that here postulated is the possibility of induction, i.e., of the cognition of ideal constructions by studying really accessible standard objects.

2.1.7. Idealization principle: *in every infinite set there is a nonstandard element.*

This statement certainly conforms with general ideas about infinity. The idealization principle is often given below in stronger forms which reflect the inexhaustible variety of ideal objects. For instance, the idea that *all standard sets are elements of a finite set* is often accepted. The number of the elements of such a 'universal' set is huge and, which is most important, it is not 'realizable', i.e., nonstandard, the nonstandardness of the universal set itself being, therefore, no surprise.

It should be emphasized that when working with the first two postulates presented above (not only with them though) one should be careful. Thus, for instance, by virtue of the transfer principle, a standard set is uniquely determined with its standard elements in the medium of its family, i.e., standard sets. The set under study is not however reduced to, generally speaking, the standard elements belonging to it. There can exist other nonstandard sets containing all the stock of the standard elements of the initial set and having no other standard elements. There is another peculiarity worth mentioning: *the notion 'statement' should be used with caution*, as though is the case in the conventional set theory as well. The transfer principle should deal with common mathematical propositions not appealing to the new property of sets which has been described at the semantic level, i.e., to be or not to be standard. In the opposite case we would, stemming from the fact that all standard sets are standard, come to the conclusion that an arbitrary set is standard which contradicts the idealization principle. Hence, the statement that a certain set is standard is not a conventional one.

2.1.8. Standardization principle: *any standard set and any property determine a new standard set which is a subset of the initial set, with its standard elements possessing the property under study.*

In more detail: let A be a standard set in question, and let P be its property. The standardization principle states that there is a standard set commonly denoted by ${}^*\{x \in A: P(x)\}$ and obeying the relation

$$y \in {}^*\{x \in A: P(x)\} \leftrightarrow y \in \{x \in A: P(x)\}$$

for any standard y. The set ${}^*\{x \in A: P(x)\}$ is often referred to as *standardization*, omitting the parameters that participate in its definition. An intuitive substantiation of the standardization principle lies in the fact that, having at our disposal implicit descriptions of mathematical objects, we can operate with new quite concrete sets composed of them by some definite laws. *Standardization extends the conventional method of formation of subsets by selecting elements with a given property* When thinking over the standardization principle you would reasonably pay attention to the fact that it says nothing about the nonstandard elements of the arising set. It is not by chance since such elements can possess or cannot possess the property under discussion. It should also be emphasized that the standardization principle must be used with due caution. Attempts to standardize a 'universal' set containing all standard sets would result in an immediate contradiction.

2.1.9. The postulates considered above give grounds for the axiomatic presentation of set theory. They will be discussed in detail below, and meanwhile we would like to share the V.A.Zorich point of view: "As a whole, any of the existing axiomatics is such that, on the one hand, it eliminates the known contradictions of the naive theory, and, on the other hand, it ensures freedom of operation with concrete sets arising in various sections of mathematics, and, before all, in mathematical analysis understood in the broad sense of the word" [284, pp. 18-19].

2.2. Simplest Properties of Standard and Nonstandard Real Numbers

Let us now go over to getting acquainted with new properties of the classical real axis which can be studied with the principles of nonstandard analysis.

2.2.1. For a set A we shall write $a \in {}^{\circ}A$ instead of the expression 'a is a standard element of A'.

2.2.2. *The following statements are valid:*

(1) *the principle of induction over standard natural numbers is valid, i.e., if A is a set for which $1 \in A$ and for $n \in {}^{\circ}\mathbf{N}$ $n \in A \to n+1 \in A$, then A contains all standard natural numbers*: ${}^{\circ}\mathbf{N} \subset A$;

(2) *a finite set (i.e.,one not allowing one-to-one mappings on its own subsets) composed of standard elements is standard*;

(3) *a standard finite set has only standard elements*;

(4) *if a set has only standard elements, it is finite*;

(5) *for an infinite standard set A (i.e., not finite, (see* **(2)**)*) the union ${}^{\circ}A$ is not a set.*

$\lhd$ (1) Using the standardization principle, let us form the following (standard) subset of the set of natural numbers:

$$B := {}^{*}\{n \in \mathbf{N} : n \notin A\}.$$

Let us assume $B \neq \varnothing$. Then B has the least element m standard by transfer. By condition, $m \neq 1$ (since $1 \in A$). Moreover, $m \notin A$, and hence, $m-1 \notin A$. According to the transfer principle, $m-1 \in {}^{\circ}\mathbf{N}$, i.e., $m-1 \in B$. We get $m-1 \geq m$, a contradiction. Hence, $B = \varnothing$, i.e., $(\forall n \in {}^{\circ}\mathbf{N})\, n \in A$, which implies the inclusion ${}^{\circ}\mathbf{N} \subset A$.

(2) An obvious corollary to the transfer principle.

(3) A singleton standard set has the only (and, hence, standard) element. The number of the elements of a finite standard set A is standard. Moreover, $A = (A - \{a\}) \cup \{a\}$ for every $a \in A$. Making use of the fact that the number of elements $A - \{a\}$ is also standard, we can apply the induction principle (1).

(4) A direct corollary to the idealization principle.

(5) Let us assume that $^{\circ}A$ is a set. By virtue of (4) we conclude that $^{\circ}A$ is finite. According to (2) $^{\circ}A$ is a standard set. In line with the transfer principle $A = {^{\circ}A}$ and, hence, A is finite, which is a contradiction. ▷

2.2.3. *A natural number N is nonstandard (i.e., nonrealizable) iff N is greater than any standard natural number. Or, in symbols,*

$$N \in \mathbf{N} - {^{\circ}\mathbf{N}} \leftrightarrow (\forall n \in {^{\circ}\mathbf{N}}) \quad N > n.$$

◁ It suffices to remark that by virtue of, for instance, **2.2.2**, the condition $n > N$ yields for a realilizable number n that N is realizable: $N \in {^{\circ}\mathbf{N}}$. ▷

2.2.4. According to **2.2.3**, nonstandard natural numbers are termed *actually infinitely large* or, to put it shortly, *infinite.*

Despite the following widely spread statement: "Euler claimed quite light-mindedly that 1/0 stands for infinity, though he did not find it necessary to define what the infinity is, but only introduced its notation, ∞", in fact L.Euler directly pointed out [48, p. 89]: "...an infinite number and a number greater than any arbitrarily given one, are synonyms."

The fact that the number N is infinite is expressed by a symbol $N \approx \infty$ or, in more detail, $N \approx +\infty$. It should be emphasized that the use of the term 'an infinite number' can result in a contradiction. Indeed, if we strictly follow the set-theoretic stance, we will see that in the set-theoretic sense the corresponding set N is finite (compare with 2.2.2 (2)). The use of the phrase 'N is an infinite number' must, therefore, be in no case associated with the infiniteness of N as a set. In fact, N is a finite set whose number of elements is nonstandard. It is this and only this sense that is inferred in the notion of an infinitely large natural number N (within set-theoretic stance).

2.2.5. *The following statements are valid:*

(1) $N \approx +\infty, M \approx +\infty \rightarrow N + M \approx +\infty, NM \approx +\infty$;

(2) $N \approx +\infty, n \in {^{\circ}\mathbf{N}} \rightarrow N + n \approx +\infty, N - n \approx +\infty, nN \approx +\infty$;

(3) *for every* $n \in {^{\circ}\mathbf{N}}$,

$$N \approx +\infty \leftrightarrow N^{n} \approx +\infty \;;$$

(4) *an infinite large composite number has infinitely large divisors*;

(5) $N \approx +\infty, M \geq N \rightarrow M \approx +\infty$;

(6) "*...if* $\frac{1}{0}$ *denotes an infinitely large number, then, since* $\frac{2}{0}$ *is undoubtedly a doubled* $\frac{1}{0}$, *it is clear that a number, be it even infinitely large, can become still two or several times greater*" (L.Euler [176, p.620]);

(7) *let* t *be a real positive number. The integral part of* t *is infinitely small iff so is* t (i.e., $(\forall_r \in {}^\circ\mathbf{R})\, t \geq |r|$);

(8) *let* $\psi: \mathbf{N} \rightarrow \mathbf{N}$ *be a strictly increasing standard function. Then for* $N \in \mathbf{N}$ *we have*

$$N \approx +\infty \leftrightarrow \psi(N) \approx +\infty .$$

$\triangleleft$ Let us prove only (7) and (8), since the other statements are easier to check.

(7) If the integral part s of a number t is infinitely small and $(\exists_r \in \mathbf{R})\, t \leq r$, $t \leq n$ for a certain $n \in {}^\circ\mathbf{N}$. Hence, we get $n + 2 \leq s \leq t \leq n$, which is ridiculous. Therefore, $t \approx +\infty$. But if $t \approx +\infty$, then $s + 1 \geq t$, where s is the integral part of t. Hence, $s + 1 \approx +\infty$, which yields, by virtue of **2.2.5** (2), $s \approx +\infty$.

(8) $\leftarrow$ Let first $\psi(N) \approx +\infty$ and $n \in {}^\circ\mathbf{N}$. Then the number $\psi(n)$ is realizable, i.e., $\psi(n) \in {}^\circ\mathbf{N}$ and, hence, $\psi(N) > \psi(n)$. Since ψ is strictly monotonic, we deduce: $N > n$, i.e., $N \approx +\infty$.

$\rightarrow$ Let us assume that $N \approx +\infty$. Then for $n \in {}^\circ\mathbf{N}$ we get $N > n$ and, hence, $\psi(N) > \psi(n) \geq n$. Thus, $\psi(N) \approx +\infty$. $\triangleright$

2.2.6. Let $\overline{\mathbf{R}}$ be the *extended numerical straight line*, i.e., $\overline{\mathbf{R}} := \mathbf{R} \cup \{-\infty, +\infty\}$, where $-\infty, +\infty$ are the largest and the least elements adjoined to R. It would be convenient to term the set $\infty := \{+\infty, -\infty\}$ the (symbolic) *potential infinity*, or to speak about $+\infty$ (or $-\infty$) as about the positive (or negative), respectively, (symbolic) infinity.

The number $t \in \mathbf{R}$ is called *finite*, *limited*, or *accessible* provided there is a standard number $n \in {}^\circ\mathbf{N}$, for which $|t| \leq n$. The condition that t is finite is presented in symbols as follows: $t \in {}^{\approx}\mathbf{R}$. The elements of $\mathbf{R}$ which are not finite are called *infinite*, or, in more detail, *actually infinite numbers*. For $t \notin {}^{\approx}\mathbf{R}$ and $t > 0$ we write $t \approx +\infty$. The presentation $t \approx -\infty$ is understood in an analogous way. The notation $t \approx \infty$ means that t is infinite.

Frequently used are also the convention $t \approx +\infty \leftrightarrow t \in \mu(+\infty)$ and the phrases like 'the number belongs to the *monad* of an infinitely distant point (in the monad of plus-infinity)'.

The number $t \in \mathbf{R}$ is termed *infinitesimal*, or *infinitely small* or, in more detail, *actually infinite small* if for any realizable $n \in {}^\circ\mathbf{N}$ we have $|t| \leq 1/n$. In this case we write $t \approx 0$, or $t \in \mu(\mathbf{R})$ and say that t is in the *monad* of zero. (The symbol $\mu(\mathbf{R})$ is used alongside with the notation $\mu(0)$ to stress an obvious relation with the only separated vector topology on $\mathbf{R}$.) Positive infinitely small numbers are often called *infinitesimals*, the unsuccessful term *differentials* being of a more restricted use.

2.2.7. The term *monad* ($\mu o \nu \alpha \zeta$) goes as far back as the ancient times and has been traditionally translated, with no sufficient ground for that, as a unit. By the primary definition of Book VII of the Euclid's "Elements", a monad "...is <that> by virtue of which each of the things that exist is called one" [47, p.9].

Let us recall here some quantitative elucidation for views of the structure of a monad expressed by Sextus Empiricus:

"...Pythagoras used to say that the origin of the existing is a monad, by relation to which each of the existing is referred to as a unit" [230, p.361];

"...a point is structured as a monad, since as a monad is a certain origin of numbers, so a point is a certain origin of lines" [230, p.364];

"...a whole, as it is whole, is indivisible, and a monad, since it is a monad, is not divisible. Or, if it is partitioned into many parts, it becomes a union of many monads, not just a [simple] monad" [230. p.367].

Below we shall study monads, their status and structure in detail. Let us start with considering the elementary properties of the infinitesimals or, which is equivalent, the monad of the infinitely small.

2.2.8. *The following statements are valid*:

(1) $s \approx 0,\, t \approx 0 \rightarrow s + t \approx 0$;

(2) $t \approx 0,\ s \in {}^\bullet\mathbf{R} \rightarrow st \approx 0$;

(3) $z \approx 0 \leftrightarrow 1/z \approx \infty$ (for $z \neq 0$):

"*...if z becomes a quantity less than any quantity that might be given, i.e., infinitely small, then the value of the fraction $\frac{1}{z}$ must become greater than any quantity that might be given, i.e., an infinitely large quantity*" (L.Euler [48, p.93]).

(4) ($t \approx 0$ *and* t *is standard*) $\rightarrow t = 0$.

◁ (1) Let $n \in {}^{\circ}\mathbf{N}$. Obviously, $|s| \le (2n)^{-1}$ and $|t| \le (2n)^{-1}$. Hence, $|s+t| \le |s| + |t| \le (2n)^{-1} + (2n)^{-1} = n^{-1}$, i.e., $s+t$ is infinitely small.

(2) Let $s \in {}^{\blacksquare}\mathbf{R}$ and $s \ne 0$ (otherwise there is nothing to prove). Let us assume $n \in {}^{\circ}\mathrm{N}$. By condition, for some $m \in {}^{\circ}\mathrm{N}$ we have $|s| \le m$. Therefore, $|t| \le (nm)^{-1}$, and, hence, $|st| \le |s||t| \le m(nm)^{-1} = n^{-1}$, i.e., $st \approx 0$.

(3) Let z be a finite nonzero number, i.e., $0 < |z| \le |n|$, where $n \in {}^{\circ}\mathbf{N}$. Obviously, we get $|1/z| \ge 1/n$, i.e., $1/z$ is not an infinitely small number. On the contrary, if $z \approx \infty$, then for any finite n we get $|z| \ge n$, which implies $z^{-1} \approx 0$.

(4) We have $|t| \le 2^{-1}|t|$ iff t is standard, which is impossible for $|t| \ge n$. Hence, $t = 0$.
▷

2.2.9. *The monad* $\mu(\mathbf{R})$ *is not a set.*

◁ Let us assume that the opposite is valid. Then $\mu(\mathbf{R})$ is a subset of $\mathbf{R}$. For every $t > 0$, $t \in {}^{\circ}\mathbf{R}$ we get $t \ge \mu(\mathbf{R})$. Hence, $t \ge s := \sup \mu(\mathbf{R})$. The number s is, obviously, infinitely small. Moreover, $2s \ge s \to s = 0$, but this contradicts the existence of nonstandard (actual) infinitely small numbers. ▷

2.2.10. When we work with real numbers, it is convenient to single out various cases of their mutual location.

For $s,t,r \in \mathbf{R}$ we write $s =_s t$, or $s \approx t (\mathrm{mod}\, r)$ provided $(s-t)/r \approx 0$ (here $r \ne 0$). In this case the numbers s and r are called r-close, or *infinitely close modulo* r. When $r = 1$, we simply write $s \approx t$ and say that s and t are *infinitely close.*

The founders of infinitesimal analysis often made no distinction between the numbers infinitely close to a certain number and the number itself. L.Euler expressed this in the following way: "...an infinitely small quantity is exactly zero" [48, p.92]. That was why for $x \in \mathbf{R}$, $y \in {}^{\circ}\mathbf{R}$ the expression $x = y$ used to be employed instead of $x \approx y$. In this respect G.W.Leibniz remarked: "...I consider equal not only those quantities the difference between which is absolutely nothing, but also those the difference between which is incomparably small" [152, p.188], emphasizing that "...the error is undetectable and cannot be found by means of whatever construction" [278, p.195].

For $s,t \in \mathbf{R}$ we write $s = O(t)$ for $s/t \in {}^{\blacksquare}\mathbf{R}$; if $s = O(t)$ and $t = O(s)$, then we say that s and t have *the same order*; if $s/t \approx 0$, then we write $s = o(t)$ and say that s has *higher order of smallness* than t; and, finally, if $s - t = o(t)$ and $s - t = o(s)$, then s and t are called *equivalent* and we write $s \sim t$.

Presenting his views on higher order infinitesimals, G.W.Leibniz wrote: "I would like to add one more remark in order to prevent all arguments against the reality of differences of any orders, and, namely, that they can always be represented as conventional intervals of a straight line proportional to them... . I have already explained how to present first-order differences with conventional straight intervals of a straight line, when first presenting the elements of this calculus in 'Acta' in October 1684" (see [278, pp.188-190, cf. **1.1**).

2.2.11. *Let us introduce for* $s,t \in \mathbf{R}$ *the following natural abbreviations*:

$$s \in O := O(t) \leftrightarrow s = O(t);\ s \in o := o(t) \leftrightarrow s = o(t).$$

The E.Landau rules hold:

$$O + O \subset o;\quad O + o \subset O;\quad o + o \subset O;$$
$$Oo \subset O;\quad OO \subset O;\quad oo \subset o.$$

$\triangleleft$ Let us check, for definiteness, the relation $O + o \subset O$. So, denote $s = O(t)$ and $r = o(t)$. Then $s / t \in {}^{\approx}\mathbf{R}$ and $r / t \approx 0$. Hence, $(s + r) / t \in {}^{\approx}\mathbf{R}$, i.e., $(s + t) = O(t)$. $\triangleright$

2.2.12. *For the numbers* $s,t \in \mathbf{R}$ *the following statements are equivalent*:

(1) s *and* t *are equivalent*;

(2) $s - t = o(t)$ *or* $t - s = o(s)$;

(3) $s / t \approx 1$ *or* $t / s \approx 1$;

(4) $s / t \approx 1$ *and* $t / s \approx 1$.

$\triangleleft$ It is clear that (1) $\rightarrow$ (2). If, for instance, $t - s = o(s)$, $(t - s) / s \approx 0$, i.e., $t / s - 1 \approx 0$. Hence, for $\varepsilon > 0$ and $\varepsilon \in {}^{\circ}\mathbf{R}$ we have $1 - \varepsilon \le t / s \le 1 + \varepsilon$. Therefore, $(1 - \varepsilon)^{-1} \ge s / t \ge (1 + \varepsilon)^{-1}$ and $\varepsilon/(1 - \varepsilon) \ge s / t - 1 \ge \varepsilon/(1 + \varepsilon)$, i.e., $s / t \approx 1$. Hence, (2) $\rightarrow$ (3) $\rightarrow$ (4), and the implication (4) $\rightarrow$ (1) is obvious. $\triangleright$

2.2.13. *Let* $N \in \mathbf{N}$ *and* $\alpha_k, \beta_k \in o(1)$ *be infinitely small, and* $\alpha_k \sim \beta_k$ *for* $k := 1, \ldots, N$. *The following statements are valid*:

(1) $\sum\limits_{k=1}^{N} \alpha_k \sim \sum\limits_{k=1}^{N} \beta_k$ *for* $\alpha_k, \beta_k \geq 0$;

(2) $\sum\limits_{k=1}^{N} = |\alpha_k| O(1)$, *(i.e., if the sum under discussion is in* ${}^{\bullet}\mathbf{R}$*), then*

$$\sum_{k=1}^{N} \alpha_k \approx \sum_{k=1}^{N} \beta_k.$$

◁ To prove this, let us remark that by virtue of **2.2.12**, for every standard $\varepsilon > 0$ we get $\varepsilon\alpha_k + \alpha_k \leq \beta_k \leq \alpha_k + \varepsilon\alpha_k$, which yields (1). Moreover, if $t := \sum\limits_{k=1}^{N} |\alpha_k| \in {}^{\bullet}\mathbf{R}$, then

$$|\sum_{k=1}^{N} (\alpha_k - \beta_k)| \leq \sum_{k=1}^{N} |\alpha_k - \beta_k| \leq \frac{\varepsilon}{n} \sum_{k=1}^{N} |\alpha_k| \leq \varepsilon,$$

as soon as a standard $n \in {}^{\circ}\mathbf{N}$ is such that $|t| \leq n$. ▷

2.2.14. *There is such a natural number* N *that for any standard number* t *in* $\mathbf{R}$ *the product* Nt *is infinitely close to a certain natural number.*

◁ Let us choose in $\mathbf{R}$ a finite subset $\{x_1, \ldots, x_n\}$ containing all standard real numbers, and an infinitely small positive number $\varepsilon > 0$, $\varepsilon \approx 0$. In the theory of numbers there is a theorem, 'the Dirichlet principle for sets', which states: for any $\varepsilon > 0$ and arbitrary $x_1, \ldots, x_n \in \mathbf{R}$ there is such an integer $N \in \mathbf{N}$ that the numbers $N_{x_1}, \ldots, N_{x_n}$ differ from integers by for most ε. Now one has to apply this theorem to the parameters in question. ▷

2.2.15. It would be useful to remark that infinite closeness (as well as equivalence) of numbers cannot be called a subset of the product $\mathbf{R} \times \mathbf{R}$. Indeed, in the opposite case the image of the element zero under this relation, i.e., the monad $\mu(\mathbf{R})$, would become a set. We, however, have already established the monad $\mu(\mathbf{R})$ not to be a set. It should be also emphasized that the monad $\mu(\mathbf{R})$ is *indivisible* in the following implicit sense: for every standard n we have: $n^{-1}\mu(\mathbf{R}) = \mu(\mathbf{R})$.

When thinking over the role of the monad $\mu(\mathbf{R})$ in constructing the system of integers, for us it would be advisable to address Definition **2** of Book VII of the Euclid's "Elements": "A number is a set composed of monads" [47, p.9]. Analogously, the whole 'nonstandard' extended number straight line $\overline{\mathbf{R}}$ and, which is most nontrivial, its accessible part ${}^{\bullet}\mathbf{R}$ are ensembles of monads located at standard points. A more rigorous formulation of this statement rests on the following fundamental fact, the proof of which is essentially based on the standardization principle.

2.2.16. *For every finite number there is a standard (and, moreover, unique) number infinitely close to it.*

◁ By virtue of the standardization principle one can, given $t \in {}^{\approx}\mathbf{R}$, organize a standard set $A := {}^{*}\{a \in \mathbf{R}\colon a \le t\}$. Obviously, $A \ne \varnothing$ and $A \le n$, where the standard number $n \in {}^{\circ}\mathbf{N}$ is such that $-n \le t \le n$. Indeed, for every standard $a \in A$ we have $a \le t \le n$. By the transfer principle, we deduce $A \le n$. Since $\mathbf{R}$ is complete, we have $s := \sup A \in \mathbf{R}$. Obviously, s is a standard number. Let us show that $s \approx t$. In the opposite case we get, for a certain standard $\varepsilon > 0$, $|s - t| > \varepsilon$. If $s \ge t$, then for every standard $a \in A$ we get $s \ge t + \varepsilon$, i.e., $s \ge a + \varepsilon$. In this case, however, we would get $s \ge s + \varepsilon$, which is impossible. The remaining possibility, $s < t$ also results in a contradiction, since then we would get $t > s + \varepsilon$, and, again, $s \ge s + \varepsilon$. ▷

2.2.17. The standard number infinitely close to a finite number $t \in {}^{\approx}\mathbf{R}$ is called the *standard part*, or the *shadow* of t, and is denoted by $\mathrm{st}(t)$ or ${}^{\circ}t$. For the sake of convenience it is also assumed that ${}^{\circ}t = \mathrm{st}(t) = +\infty$ if $t \approx +\infty$ and, respectively, ${}^{\circ}t = \mathrm{st}(t) = -\infty$ for $t \approx -\infty$ (under the obligatory supposition that $+\infty \approx +\infty$ and $-\infty \approx -\infty$). Therefore, every (standard) $t \in \overline{\mathbf{R}}$ is put into correspondence with its monad $\mu(t)$, i.e., all the elements s of $\mathbf{R}$ for which $s \approx t$.

Thus, in nonstandard analysis an extended straight line should be visualized in connection with the scheme presented in Figure 2. Choosing a standard number ${}^{\circ}t$ on the axis $\mathbf{R}$, we draw a big dot, a blob, a monad $\mu({}^{\circ}t)$, which is an 'indivisible implicit presentation of ${}^{\circ}t$'. If we view the region of the point ${}^{\circ}t$ with a powerful microscope, we will see a smeared small cloud with a blurred boundary, which is the image of $\mu({}^{\circ}t)$.

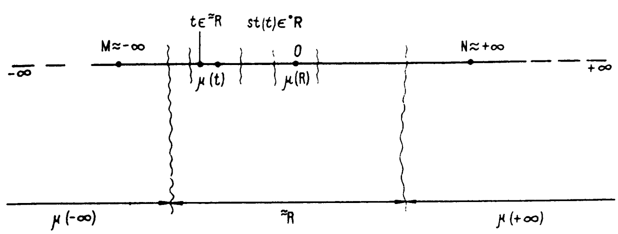


Fig. 2

When using a more powerful microscope, the portion of the 'point-monad' under observation would enlarge, reveal more details and partially disappear from the view. In this case, however, we again would deal with the same standard real number, which is, if you please, described by the process of 'studying the microstructure of a physical straight line' presented above.

2.2.18. *The following statements are valid:*

(1) *for* $s \in \mathbf{R}$, $t \in {}^{\approx}\mathbf{R}$,

$$\mathrm{st}(s+t) = \mathrm{st}(s) + \mathrm{st}(t); \quad \mathrm{st}(st) = \mathrm{st}(s)\mathrm{st}(t);$$

(2) *if* $s,t \in \mathbf{R}$ *and* $s \le t$, *then* ${}^\circ s \le {}^\circ t$;

(3) *for* $s,t \in \overline{\mathbf{R}}$ *we have*

$$(\exists t' \approx t)\ t' \ge s \leftrightarrow {}^\circ s \le {}^\circ t \leftrightarrow (\forall \varepsilon > 0,\ \varepsilon \in {}^\circ\mathbf{R})\quad s \le {}^\circ t + \varepsilon;$$
$$(\forall t' \approx t)\ t' \ge s \leftrightarrow {}^\circ s < {}^\circ t \quad (t \in {}^{\approx}\mathbf{R});$$

(4) *the transition from a real number to its standard part is not a set (nor a function, in particular).*

$\lhd$ (1) Let us, for instance, prove that the transition to the standard part is multiplicative. We have $s \approx \mathrm{st}(s) \to ts \approx t\,\mathrm{st}(s)$. Besides, $t \approx \mathrm{st}(t) \to \mathrm{st}(s)t \approx \mathrm{st}(t)\mathrm{st}(s)$. Finally, $st \approx \mathrm{st}(s)\mathrm{st}(t)$. It now remains to recall that the product of standard numbers is standard.

(2) Let $s < t$ (otherwise everything is obvious). If $s \approx t$, then $\mathrm{st}(s) = \mathrm{st}(t)$. In the opposite case the monads $\mu(s)$ and $\mu(t)$ do not intersect. Hence, ${}^\circ s < {}^\circ t$.

(3) In the initial equivalence the implication to the right is obvious, while the reverse one is ensured by the fact that for $s \le {}^\circ t$ we get $s \le {}^\circ t + s - {}^\circ s$. Besides, $s < t + \varepsilon \to \mathrm{st}(s) \le \mathrm{st}(t) + \mathrm{st}(\varepsilon) = {}^\circ t + \varepsilon$ for every $\varepsilon > 0$, $\varepsilon \in {}^\circ\mathbf{R}$. By transfer this implies that for an arbitrary positive ε we get ${}^\circ s \le {}^\circ t + \varepsilon$, and, hence, ${}^\circ s \le {}^\circ t$. In turn, if ${}^\circ s < {}^\circ t$, then, making use of the fact that the monads $\mu\,{}^\circ(s)$ and $\mu({}^\circ t)$ are disjoint, we deduce $s < {}^\circ t + \varepsilon$ for any $\varepsilon > 0$, $\varepsilon \in {}^\circ\mathbf{R}$.

In order to check the arrow to the right in the lower equivalence, let us note that s does not lie in the monad $\mu(t)$ of the number t. Hence, the whole of the monad of s lies to the left from the monad of t, i.e., $\mu(s) < \mu(t)$. Therefore, ${}^\circ s < {}^\circ t$. And, finally, to prove the remaining implication let us remark that for ${}^\circ s = -\infty$ we get either $\mu(t) > {}^\circ s$ or $t \in {}^{\approx}\mathbf{R}$. But if ${}^\circ s \in {}^\circ\mathbf{R}$, then $\mu({}^\circ s) < {}^\circ t$. Hence, for $t' \approx t$ the condition $t' \ge s$ is met.

(4) If the law $t \to \mathrm{st}(t)$ were a set, then the monad $\mu(\mathbf{R})$ would also be a set (since $t \in \mu(\mathbf{R}) \leftrightarrow {}^{\circ}t = 0$). Now we have to make use of **2.2.9.** ▷

2.3. Primary Notions of Mathematical Analysis on a Real Line

Let us now discuss the fundamental notions pertaining to differential and integral calculus of functions in a single real variable.

2.3.1. Nonstandard criteria of limits. *For a standard sequence (a_n) and a standard number $a \in \mathbf{R}$ the following statements are valid:*

(1) *the number a is a partial limit of (a_n) iff for a certain infinitely large N the condition $a = {}^{\circ}a_N$ is met;*

(2) *the number a is the limit of (a_n) iff for all infinitely large numbers N the term a_N is infinitely close to a, i.e.,*

$$a = \lim a_n \leftrightarrow (\forall N \approx +\infty)\, a_N \approx a.$$

◁ These statements are checked in an analogous way. Therefore, let us prove one of them, for instance, (2). So, let $a_n \to a$, and let, for the sake of definiteness, assume $a \in \mathbf{R}$ (the cases $a = +\infty$ and $a = -\infty$ are proved by the same scheme). By condition, for an arbitrary positive number $\varepsilon > 0$ and a certain $n \in \mathbf{N}$ we have $|a_N - a| \le \varepsilon$ as soon as $N \in \mathbf{N}$ and $n \le N$. Hence, by the transfer principle, for a standard $\varepsilon > 0$ there is a standard n with the same property Every infinitely large N majorizes the obtained n, i.e., $|a_N - a| \le \varepsilon$. Since ε is arbitrary, this implies $a_N \approx a$.

Let it, in turn, be known that for all $N \approx +\infty$ we have ${}^{\circ}a_N = a$. For the sake of definiteness and diversity, let us assume $a := -\infty$. Let us choose an arbitrary standard number $n \in {}^{\circ}\mathbf{N}$. Obviously, for all $N \ge M$, where M is an infinitely large number, we have $a_N \le -n$. Hence, for every standard n we have proved 'something' (namely, 'something': $= (\exists M)(\forall N \ge M)\ (a_N \le -n)$). By the transfer principle, this 'something' is valid for every $n \in \mathbf{N}$. It is the latest fact that, as is well known, signifies $a_n \to -\infty$. ▷

2.3.2. Let us emphasize the merits of the criteria obtained. We have seen that the partial limits of a standard sequence are exactly those 'realizable' numbers which correspond to infinitely large indices. In other words, the partial limit is an 'observable' value of a certain infinitely far term of the sequence. The statements under consideration have an implicit intuitive foundation and differ drastically from conventional definitions of partial limit as a number to which a subsequence of the initial sequence is tending, or as such an element of a straight line that every interval containing it intersects with any remainder, i.e., 'any tail', of the sequence in question.

It would be useful to get acquainted with a clarification of the notion of partial limit for a [generalized] sequence with which N.N.Luzin furnished the formulation of the conventional definition (see [162], [169]). "At first the reader will undoubtedly view this formulation as cumbersome and abstract. But the sense of ambiguity would disappear if the reader recalls the conventional notions of a 'variable' and 'time'. Indeed, what is the intention of this formulation when translated into the language of a 'variable' and 'time'? In order to understand this, let us consider a variable x which runs through a given numerical sequence M, going over from preceding numbers to succeeding ones ... in the language of a variable and time this formulation means that the ([partial]) *limit* of a numerical sequence M is such a number a that the variable x cannot get completely detached from, since 'at certain periods of time' the value of the variable x gets 'any close' to a."

Using the same images, in nonstandard analysis we can still be more presentable and clear: 'if at a certain infinitely distant moment of time a variable x is infinitely small different from a, then a is the [partial] limit of M'.

Going now over to considering the nonstandard criterion of the limit of a sequence, let us recall the following statements by R.Courant.

"**Motivation of the implicit definition of limit.** It is no wonder that nobody who hears the abstract definition of the limit of a sequence for the first time can at once understand it completely. The definition of limit mimics a game between two persons, A and B: A demands that a constant value a be approximately presented by a value a_n in such a way that the deviation be less than an arbitrary bound $\varepsilon = \varepsilon_1$, set by A himself. B fulfills his requirements by proving the existence of such an integer number $N = N_1$, that all a_n, starting from the element a_N , obey the requirement of ε_1. Then A wants to set a new, smaller bound, $\varepsilon = \varepsilon_2$, while B, in turn, fulfills this requirement by finding a new integer number $N = N_2$ (possibly, much greater), etc.. If B is always ready to meet the requirements set by A, whatever small bound A could set, then we have the situation that is symbolically expressed as $a_n \to a$.

There is, no doubt, a psychological difficulty in mastering this exact definition of passage to the limit. Our visual presentation imposes upon us a 'dynamical' idea of passage to the limit as a result of motion, i.e., we 'run' through the sequence of numbers $1,2,3,\ldots,n,\ldots$ observing on our way the behavior of the sequence a_n. We have a feeling that while 'running' we must observe this approximation. This 'natural' formulation, however, does not allow a rigorous mathematical one. In order to get an exact definition, the order of consideration should be reversed: instead of first watching the argument n followed by considering the dependent variable a_n associated with it, we base our definition on steps which make it possible to subsequently check the statement $a_n \to a$. Under such an approach one has to first choose whatever small interval surrounding a, and then to check if the condition is fulfilled by choosing the variable n to be sufficiently large. In this way we come to the exact definition of limit denoting the expressions 'whatever small boundary' and 'a sufficiently large n' by ε and N" [31, pp.66-67].

It goes without saying that the criterion formulated in **2.3.1** (2): "if for all infinitely large N the general term a_N is impossible to be distinguished from a standard number a, then a is declared (and, in fact, is) the limit of (a_n)" successfully expresses the dynamic idea of passage to the limit.

In this case one should always bear in mind that the nonstandard criterion of limit is applicable only to standard sequences and is not, generally speaking, valid for nonstandard, ill-described sequences. Thus, if $a_n : N / n$, where $N \approx +\infty$, then $a_n \to 0$ and, at the same time, $a_N = 1$. In other words, criterion **2.3.1** supplements the up-to-date views on limit without rejecting or abolishing them. Or, more exactly, when determining the limit only for standard sequences, we therefore automatically form the standard set of all converging sequences by employing the standardization principle. To state it otherwise, both the conventional $\varepsilon - N$-definition and the unusual definition with the actual infinitely large and infinitely small are closely interrelated, tightly interwoven.

It would be useful to put special emphasis on the fact that in concrete applications (in particular, in physics) one often has to face real, explicitly described, i.e., standard sequences. Moreover, under such conditions 'the infinitely large' has an exact (physical) sense, i.e., the horizon, or the boundary beyond which the numbers are declared indiscernible, is given overtly. Bearing in mind that in practice the problems of existence are also solved on the ground of semantic considerations (if there is no physical speed, it is not worth while looking for), there arises a problem of recognizing the limit of a standard sequence known beforehand. Nonstandard analysis gives a simple receipt: "Take a general term of your sequence with any (no matter which) infinitely large index; it is the number determined by this term (up to infinitesimal) that is the sought limit". In this respect more understandable becomes the background of the methods of the founders of differential and integral calculuses who were seeking the answers to the problems on the exact values of concrete 'standard' objects: areas of figures, equations of tangents to 'named' curves, integrals of implicitly written analytical expressions, etc..

2.3.3. An important new contribution of nonstandard analysis is the formulation of the notion of limit for a *finite sequence* $a[N] := (a_1,\dots,a_N)$, where N is an infinitely large natural number. The intuitive idea melted in the foundation of of the definition to follow, well reflects practical techniques of finding numerical characteristics of indiscernible discrete unions, such as thermodynamic parameters of fluid media, estimates for the population demand, etc..

2.3.4. The number a is called *the microlimit* or *the nearlimit value* of a sequence $a[N]$ provided, for all infinitely large M less than N, we have $a_M \approx a$. In this case we also say that $a[N]$ *converges nearly* to a. When a is a finite number, the standard part $^\circ a$ is termed the *limit* (or the *S-limit*) of the sequence $a[N]$ and we write either $^\circ a = {}^{\approx}\lim a[N]$ or $^\circ a = S - \lim\limits_{n<N} a_n$. Therefore,

$$^\circ a = {}^{\approx}\lim a[N] \leftrightarrow a \in {}^{*}\mathbf{R} \wedge (\forall M \approx +\infty, M \le N)\ a_M \approx a.$$

2.3.5. *Let* (a_n) *be a standard sequence,* $N \approx +\infty$ *and* $a \in {}^{*}\mathbf{R}$. *The following statements are equivalent:*

(1) a *is the microlimit of* $a[N]$;

(2) *the sequence* (a_n) *converges to* $^\circ a$.

◁ The implication (2) → (1) is contained in **2.3.1** (2). To prove (1) → (2), let us choose an arbitrary standard $\varepsilon > 0$ and consider the set

$$A := \{m \in \mathrm{N} : (\forall n)(m \le n \le N) \to |a_n - {}^\circ a| \le \varepsilon\}.$$

The set A is nonempty as $N \in A$. Hence, A contains the least element m. If $m \approx +\infty$, then $m - 1 \approx +\infty$, and, by condition, $m - 1 \in A$. Therefore, m is standard. Moreover, if $n \ge m$ and n is standard, then $n \le N$ and $|a_n - {}^\circ a| \le \varepsilon$. Hence, $(\forall \varepsilon \in {}^\circ\mathbf{R}, \varepsilon > 0)$ $(\exists m \in {}^\circ\mathbf{N})$ $(\forall n \in {}^\circ\mathbf{N})$ $n \ge m \to |a_n - {}^\circ a| \le \varepsilon$. Using the transfer principle, we deduce that (a_n) converges to $^\circ a$. ▷

2.3.6. The criterion just stated gives an exact foundation for the *principle of a granted horizon*, which states that in concrete investigations one indicates a 'physical' or 'economic' actually infinitely large number which serves both as a measure of presentability of the collection under study and its natural bound from above.

2.3.7. Examples

(1) $\lim \dfrac{n-1}{n} = 1.$

◁ Let us choose an infinitely large i. We have ${}^{\circ}\left(\frac{i-1}{i}\right) = {}^{\circ}\left(1 - \frac{1}{i}\right) = 1$. In more detail (L.Euler): "Since i is an infinitely large number, $\frac{i-1}{i} = 1$; and indeed, obviously, the greater number is put instead of i, the closer the value of $\frac{i-1}{i}$ will be to 1; if i becomes greater than any given number, then the fraction $\frac{i-1}{i}$ will become equal to 1" [50, p.116]. ▷

(2) $\lim \frac{n}{2^n} = 0$.

◁ For every infinitely large N we have $2^N = (1+1)^N \geq N(N-1)/2$, i.e., $0 \leq N/2^N \leq 2/(N-1) \approx \mathbf{0}$. Hence, $N/2^N \approx 0$. ▷

(3) $\lim \sin(2\pi n!e) = 0$.

◁ For any natural n, we have

$$0 < e - \sum_{k=1}^{n} \frac{1}{k!} < \frac{3}{(k+1)!}.$$

From here we deduce, for an infinitely large N;

$$0 \leq N!\left(e - \sum_{k=1}^{N} \frac{1}{k!}\right) \leq \frac{3N!}{(N+1)!} = \frac{3}{N+1} \approx 0.$$

Let $x := 2\pi N!n$ and $y := 2\pi N! \sum_{k=1}^{N} 1/k!$. Then $x \approx y$, in which case $y = 0$. Obviously,

$$|\sin x - \sin y| = 2\left|\cos\frac{x+y}{2}\, \sin\frac{x-y}{2}\right| \leq |x-y|,$$

i.e., $\sin x \approx 0$. ▷

(4) *Let (a_n) be such that the sequences (a_{2n}), (a_{2n+1}) and (a_{3n}) converge. Then (a_n) converges, too.*

◁ One can consider (a_n) to be a standard sequence. For an infinitely large N we have $2N \approx +\infty$, $2N+1 \approx +\infty$ and $3N \approx +\infty$, i.e., $a_{2N} \approx a$, $a_{2N+1} \approx b$, $a_{3N} \approx c$ for some standard numbers a, b, c, respectively. In particular, $a_{6N} \approx a \approx c$ and $a_{6N+1} \approx b \approx c$. Hence, $a = b = c$, which was required. ▷

(5) *Let* (a_n) *vanish. Then*

$$\lim \frac{a_1+\ldots+a_n}{n} = 0.$$

◁ By transfer, the sequence (a_n) can be considered standard. Let us choose $N \approx +\infty$. Let M be the integral part of $\sqrt{N}$. It is obvious that M is an infinitely large number. In this case for every standard $n \in {}^\circ\mathbf{N}$ we have $|a_N| \le n^{-1}$, and, hence,

$$s_N := \left|\frac{a_1+\ldots+a_N}{N}\right| \le \left|\frac{a_1+\ldots+a_M}{N}\right| + \left|\frac{a_{M+1}+\ldots+a_N}{N}\right|$$
$$\le \frac{M}{N} \sup_{n \in N} |a_n| + \frac{1}{n}\,\frac{N-M-1}{N}.$$

Since $1/N \approx 0$ and $\sup_{n\in N} |a_n| \in {}^\circ\mathbf{R}$, we come to the conclusion that the number s_N is infinitely small. ▷

(6) *There is a Banach limit, i.e., such a continuous linear functional* l *in the space* $l_\infty := l_\infty(\mathbf{N},\mathbf{R})$, *that for any sequence* $a := (a_n)$ *of* l_∞ *we have*:

$$(\exists \lim a_n) \to l(a) = \lim a_n;$$
$$\liminf a_n \le l(a) \le \limsup a_n;$$
$$('a)(n) := a_{n+1} \to l(a) = l('a).$$

◁ To prove this statement, choose an infinitely large natural number N. For every standard sequence a of l_∞ the value

$$f(a) := \frac{1}{N} \sum_{k=N}^{2N-1} a_k$$

is finite. Indeed, as a is standard, the value $\|a\|_\infty := \sup_{n\in N} |a_n|$ is standard. Moreover,

$$|f(a)| \le \frac{1}{N} \sum_{k=N}^{2N-1} |a_k| \sum_{k=N}^{2N-1} \|a\|_\infty \le \|a\|_\infty.$$

Let us now consider (making use of the fact that the set $l_\infty \times \mathrm{R}$ is standard) the standardization

$$l := {}^*\{(a,t) \in l_\infty \times \mathbf{R} : t = {}^\circ f(a)\}.$$

First we prove that l is the sought object, starting with the statement that l is a function. We have to show that

$$(\forall a \in l_\infty)(\forall t_1,t_2 \in \mathbf{R})\,(a,t_1) \in l \wedge (a,t_2) \in l \to t_1 = t_2.$$

It suffices, according to the transfer principle, to check this property for standard a,t_1,t_2. In this case, however, by the definition of standardization, we have $t_1 = {}^\circ f(a)$ and $t_2 = {}^\circ f(a)$. Now we recall (see **2.2.16**) that the standard part is unique. The linearity of l is checked by a similar consideration. It is also obvious that $a \geq 0 \to l(a) \geq 0$, i.e., that l is positive.

Let a be a standard sequence converging to $\bar{a}$. Then, by virtue of **2.3.1** (2) for any standard $\varepsilon > 0$ we have $|a_N - \bar{a}| \leq \varepsilon, \dots, |a_{2N-1} - \bar{a}| \leq \varepsilon$, as for $M \geq N$ all a_M are infinitely close to $\bar{a}$. Hence,

$$|f(a) - \bar{a}| = \Big|\frac{1}{N}\sum_{k=N}^{2N-1}(a_k - \bar{a})\Big| \leq \varepsilon,$$

i.e., $\bar{a} = {}^\circ f(a)$. It is the obtained property combined with the positiveness of l that ensures the sought estimates.

We now have to establish that the functional constructed is invariant under shifts, i.e., that $l('a) = l(a)$ for all $a \in l_\infty$. And again, the sequence a can be considered standard, in which case the element $'a$ is also standard and, hence,

$$\begin{aligned} l('a) &= {}^\circ\Big(\frac{1}{N}\sum_{k=N}^{2N-1} a_{k+1}\Big) = \operatorname{st}(N^{-1}(a_{N+1} + a_{N+2} + \dots + a_{2N})) \\ &= \operatorname{st}\Big(\frac{1}{N}\sum_{k=N}^{2N-1} a_k + \frac{1}{N}a_2N - \frac{1}{N}a_N\Big) = {}^\circ(f(a) + N^{-1}a_{2N}) - N^{-1}a_N) \\ &= {}^\circ f(a) + (N^{-1}a_{2N}) -{}^\circ (N^{-1}a_N) = {}^\circ f(a) = l(a). \end{aligned}$$

Here account has been made of the finiteness of the numbers a_{2N} / N and a_N / N, as well as of **2.2.18**. ▷

2.3.8. Nonstandard criterion for continuity. *Let f be a standard real function and x be a standard point of its standard domain of definition,* dom(f). *Then the following statements are equivalent:*

(1) *f is continuous at the point x;*

(2) *f maps the points infinitely close to x into the points infinitely close to $f(x)$, i.e.,*

$$x' \approx x,\ x' \in \operatorname{dom}(f) \to f(x') \approx f(x).$$

◁ (1) → (2). Let $\varepsilon > 0$ be a standard number. There is a $\delta > 0$, such that for $|x' - x| \leq \delta$ and $x' \in \mathrm{dom}(f)$ we have $|f(x') - f(x)| \leq \varepsilon$. By transfer, there also is a standard δ with the same property. If $x' \approx x$ and $x' \in \mathrm{dom}(f)$, then, obviously, $|x' - x| \leq \delta$ (since $\delta \in {}^{\circ}\mathbf{R}$ and, hence, $|f(x) - f(x')| \leq \varepsilon$). Since $\varepsilon \in {}^{\circ}\mathbf{R}$ is arbitrary, this implies $f(x') \approx f(x)$.

(2) → (1) Let us choose an arbitrary $\varepsilon > 0$. Now we have to find a δ which is also participating in the '$\varepsilon - \delta$-definition'. By transfer, it suffices to find such a δ only for the standard ε. In the latest case, however, instead of δ we can take any actually small positive number. ▷

2.3.9. According to **2.3.8** (2), the function $f{:}\mathrm{dom}(f) \to \mathbf{R}$ is called *microcontinuous at the point x of* dom$f$ provided for $x' \in \mathrm{dom}\, f$ and $x' \approx x$, we have $f(x') \approx f(x)$.

2.3.10. When discussing the nonstandard criterion obtained, i.e. that 'for standard functions at a standard point both continuity and microcontinuity coincide', we can repeat the argumentation that was given in **2.3.2**. One should emphasize, following R.Courant, that "as was the case for the limit of a sequence, the Cauchy definition rests, so to say, on the reversion of the intuitively acceptable order in which we would like to consider variables. Instead of first considering the independent and then dependent variable, we first direct out attention to 'the boundary of accuracy' of ε, and then try to to restrict the corresponding 'arena' of δ for the independent variable" [31, p.73]. The nonstandard criterion makes us free from the unpleasant reversal of quantifiers for all functions and points accessible for us, i.e., standard. At the same time, the $\varepsilon - \delta$-definition in its full scope is only indirectly restored through microcontinuity at a point, using the standardization procedure. In this way the standard and nonstandard approaches again, as has to be expected, demonstrate their intricate but genuine unity. A new mathematical property, microcontinuity of the function at a point, seems to be an interesting acquisition. The following statements will facilitate our understanding of microcontinuity on a greater scale.

2.3.11. Examples

(1) *The function* $x \to x^2$ *is not microcontinuous at every infinitely large point* $t \in \mathbf{R}$.

◁ Indeed, $t + t^{-1} \approx t$ and, at the same time, $(t + t^{-1})^2 - t^2 \approx 2$. ▷

(2) *Let* δ *be a strictly positive infinitely small number. Let us consider the function*

$x \to \delta \sin x^{-1}$ *which is additionally defined to be zero at zero . This function is discontinuous at zero and microcontinuous.*

◁ It suffices to remark that $\sin x \in {}^{\approx}\mathbf{R}$ for $x \in \mathbf{R}$ and refer to the properties of infinitely small numbers in **2.2.8.** ▷

2.3.12. Nonstandard criterion of uniform continuity. *For a standard numerical function f determined on a standard set* $\mathrm{dom}(f)$ *the following statements are valid:*

(1) *f is microcontinuous, i.e., f is microcontinuos at every point of* $\mathrm{dom}(f)$ *or, in symbols,*

$$(\forall x, x' \in \mathrm{dom}(f))\, x' \approx x \to f(x') \approx f(x);$$

(2) *f is uniformly continuous.*

◁ (1) → (2). Let $\varepsilon > 0$ be a standard number, and let $\delta > 0$ be infinitely small. Obviously, for $|x - x'| \le \delta$ we have $x \approx x'$. Therefore,

$$(\forall \varepsilon \in {}^{\circ}\mathrm{R}, \varepsilon > 0)\,(\exists \delta > 0)\,(\forall x, x' \in \mathrm{dom}(f))\quad |x - x'| \le \delta$$
$$\to |f(x) - f(x')| \le \varepsilon.$$

Applying the transfer principle, we see that f is uniformly continuous.

(2) → (1) By the transfer principle, for every standard $\varepsilon > 0$ and some standard $\delta > 0$ we get $|x - x'| \le \delta \to |f(x) - f(x')| \le \varepsilon$ for all $x, x' \in \mathrm{dom}(f)$. By remarking that $x \approx x' \to |x - x'| \le \delta$, we obtain the required result. ▷

2.3.13. Nonstandard criterion of a derivative. *Let f be a standard function given in a standard neighbourhood of a standard point x of* $^{\circ}\mathbf{R}$. *The following statements are equivalent:*

(1) *f is differentiable at the point x and $f'(x) = t$;*

(2) *for every nonzero infinitely small number h we have:*

$$t = \mathrm{st}((f(x+h) - f(x)) / h).$$

◁ The result required is a direct corollary to **2.3.8.** ▷

2.3.14. *Let y be a standard function given in a neighbourhood of a standard point x and differentiable at this point. Let, then, dx be an arbitrary nonzero infinitesimal. Let us denote (following G.W.Leibniz) by the symbol dy the differential of the function y at the point x applied to the element dx. In this case*

$$dy \approx y(x+dx) - y(x), \quad \frac{dy}{dx} \approx y'(x).$$

◁ By the Leibniz definition, making use of **2.3.9**, we get:

$$dy = y'(x)dx, \quad y'(x) = \operatorname{st}\left(\frac{y(x+dx) - y(x)}{dx}\right).$$

Hence,

$$dy \approx \frac{y(x+dx) - y(x)}{dx} dx = y(x+dx) - y(x),$$

which proves the first part of the statement. The second part follows from **2.3.10.** ▷

2.3.15. The nonstandard considerations of the role of infinitesimals when determining derivatives, differentials and increments given in **2.3.13** and **2.3.14** complement the following statement by L.Euler.

"I have already remarked that in differential calculus the problem of finding differentials should be understood in relative rather than absolute sense; this means that if y is a function of x, then it is not its differential but rather its relation to the differential dx that should be determined . Indeed, since all differentials are exactly equal to zero, then, whatever the function y of the quantity x might be, dy is always equal to zero; therefore, in the absolute sense there is nothing here distinct than that to be sought for. The correct formulation of the problem is as follows: x gets an increasingly small, i.e., vanishing [=evanescent, the actual number that 'is exactly zero'] increment dx; the task is to determine what is the relation to dx of the increment that the function y finally gets. Though both increments are zero, there is a certain relation between them which is found out duely by differential calculus. Since if $y = x^2$, then, as is proved in differential calculus, $\frac{dy}{dx} = 2x$, and this ratio of the increments is valid only if the increment dx which dy is generated by, is considered to be equal zero. Nevertheless, after this warning on the true notion of differential has been made, it is allowed to make use of conventional expressions which treat the differential as in the absolute sense, provided, though, the truth is constantly borne in mind. For instance, we have the right to say: if $y = x^2$, then $dy = 2xdx$. In fact, if somebody said that $dy = 3xdx$ or that $dy = 4xdx$, then it would also be true, since even these equalities are valid as $dx = 0$ and $dy = 0$. Only the first equality, however, agrees with the true ratio $\frac{dy}{dx} = 2x$" [49, p.9].

It would be useful to remark that L.Euler used the sign '=' at the place where we write '$\approx$' (see **2.2.10**). Moreover, it should be emphasized that he was seeking for the differential that was considered as existing, while working with concrete (differentiable) functions. Therefore, it would be quite legitimate to use for finding the differential any infinitely small dx chosen in any way,.

Therefore, L.Euler had no doubts that the differential dy (calculated for an infinitely small dx) "is exactly zero", the differential dy is exactly the increment, i.e., "the absolute differential", and at the same time the differential dy is "the fourth proportional" for infinitely small increments, i.e., in up-to-date notation:

$$
{}^\circ dy = 0, \quad {}^\circ(dy - (y(x + dx) - y(x)) = 0;
$$

$$
{}^\circ\left(\frac{dy}{dx} - \frac{y(x + dx) - y(x)}{dx}\right) = 0.
$$

The analysis carried out demonstrates the soundness of the ideas and techniques used by L.Euler when working with explicitly set, i.e., standard, objects. Repeating the aphoristic expression by F.Engels, one can say that "... dx is infinitely small but is, nonetheless, active and productive of everything" [44, p.580].

In the limelight of the above-said the following statements by R.Courant should be treated at a certain critical angle: "...if we would like to understand the essence of differential calculus, we should beware of viewing a derivative as a ratio of two actually existing (actual) 'infinitely small values'. The point is that first we must always form a ratio of the increments $\Delta y / \Delta x$, where the difference Δx is not equal to zero. Then we should imagine that either by way of transforming this ratio, or by some other way passage to the limit has been accomplished. But in no case you should imagine that *at first* there is a transition from Δx to an infinitely small value dx, which is in any case is other than zero, and from Δy to dy, followed by dividing these 'infinitely small values' one by the other. Such a viewpoint on the derivative is absolutely incompatible with the requirement of the mathematical clarity of the notions, and it has hardly any sense at all" [31, p.126-128]. The excessive rigidity of the last phrase is but partially smoothed down by the clarification that follows: "A physicist, a biologist, a technician or any other specialist who has to deal with these notions in practice, has, therefore, the right to identify, within the accuracy required, the derivative with the ratio of the increments...

...'physically infinitely small' values have an exact sense. They are, undoubtedly, finite and distinct from zero values, chosen though to be sufficiently small in the problem under discussion, less, for instance, than a certain portion of the wavelength, or smaller than the distance between two electrons in an atom, etc., generally speaking, less than a certain required degree of accuracy" [31, p.135].

2.3.16. Nonstandard presentation of the Riemann integral. *Let $f:[a,b] \to \mathbf{R}$ be a standard continuous function, and let $a = x_1 < x_2 < \ldots < x_N < x_{N+1} = b$ be a partition of $[a,b]$, such that $\xi_k \in [x_k, x_{k+1}]$ and $x_k \approx x_{k+1}$ for $k := 1,\ldots,N$. In this case the following equality is valid:*

$$\int_a^b f(x)dx = {}^\circ\left(\sum_{k=1}^{N} f(\xi_k)(x_{k+1} - x_k)\right).$$

◁ It should be first of all observed that N is infinitely large, and use should be made of the definition of the integral, as well as of the nonstandard criteria on **2.3.1** and on that of uniform continuity **2.3.12**. ▷

2.3.17. Basic principle of integral calculus. "*...When calculating the sum of an infinitely large number of infinitely small addends (of the same sign) it is possible to subtract from every addent a higher-order infinitesimal.*"

◁ Consider a sum ${}^\circ\sum_{k=1}^{N} \alpha_k = t$, where $\alpha_k \approx 0$. By condition, we have $\beta_k := \alpha_k - o(\alpha_k)$. By **2.2.13** (2), we deduce $\beta_k \sim \alpha_k$, and, hence,

$$t = {}^\circ\left(\sum_{k=1}^{N} \alpha_k\right) = {}^\circ\left(\sum_{k=1}^{N} \beta_k\right).$$ ▷

2.3.18. The above statements give a formal foundation of presenting the integral as a finite sum of infinitely small elements, i.e., it justifies the position of viewing integration as a specific process of summation, which dates back to the ancient time. In this respect it would be useful to quote here the following definition of the integral ('with a variable upper limit') suggested by L.Euler:

"Integration is usually defined in the following way. It is said to be a summation of all the values of the differential expression Xdx provided the variable x sequentially assumes all the values that differ by the difference dx, starting from a certain given value up to x, this difference being considered infinitely small... . From the method presented it is, in any case, clear that integration can be obtained, within any accuracy, from summation; it is also obvious that integration can be accomplished in no other way but by setting these differences to be infinitesimals, i.e., zeros" [49, p.163].

It is worthwhile again to emphasize that in order to find the integral of a standard continuous function one should, by virtue of the facts presented above, calculate the exact value (= the standard part) of only one finite sum of an infinitely large number of infinitely small addends, in which the higher-order small values can be neglected. For nonstandard functions this technique does not work in general. In other words, we again discover, as has been repeatedly the case in the material presented above, that nonstandard ideas on the objects of mathematical analysis are supplementing, refining and developing (but in no case abolishing) their classical analogues.

2.3.19. All these facts manifest the nonstandard analysis in its contemporary forms to be a direct descendent of calculus of infinitesimals. That is why the term 'infinitesimal analysis' has been lately gaining in popularity; it is more exact in presenting the essence of the matter than a somewhat extravagant term 'nonstandard analysis' which often causes an irritation quite understandable in the long run.

Special attention should be paid to the fact that the concept of actually infinitely large and infinitely small quantities has never been abandoned as a working tool in natural sciences but was just absent from mathematics for nearly thirty years. This makes it possible for us not to go into details as regards the importance of nonstandard methods.

CHAPTER 3

SET-THEORETIC FORMALISMS OF NONSTANDARD ANALYSIS

The 'naive'-level discussion of differences between the standard, i.e., realizable, and nonstandard, i.e., indirect, methods of introducing objects conducted in Chapter **2**, has enriched our notions of actually infinitely large and actually infinitely small numbers with a sense agreeable with intuition. A remarkable acquisition is that the understanding of the methods of consideration adopted when formulating mathematical analysis has been extended. At the same time, we face serious complications even in simple examples. First of all, the method of distinguishing standard objects from nonstandard ones is still open to argument, which makes us admit the possibility of incorrectly applying the principles of nonstandard analysis. Growing alarm has been caused by the appearance of objects formed with mathematical constructions which, at first sight, seem to be quite legitimite but cannot unconditionally be given the status of the 'naive' sets. Such are, for instance, all kinds of monads, families of standard elements, and objects like $\approx$, $^{\sim}\mathbf{R}, O, o$, etc.. Still more unpleasant is the fact that the 'mathematical law' $x \to {}^\circ x$, acting from $\overline{\mathbf{R}}$ to $\overline{\mathbf{R}}$, is not a function. The point is that the notion of function had been formed in mathematics long before the set-theoretic stance was developed. Thus, as far back as in 1775 L.Euler wrote: "When certain quantities depend on other ones in such a way that when the latter are changed the former are also subject to changes, then the former are called functions of the latter. The application of this term is extremely wide; it embraces all the methods by which one quantity can be determined with the help of others. Therefore, ... all quantities which are dependent on x in one or another way, i.e., determined by x, are called functions of x" [48, p.38]. The dynamic idea of transforming some objects into other ones is not embraced completely by the 'stationary' set theoretical view of a function as a set reigning now. This presentation is a "formal set-theoretic model of the intuitive idea of a function, *a model*, which embraces only one aspect of the idea but not its importance as a whole" [63, p.32]. It should be recalled in this respect that for $s,t \in [0,1]$ we have

$$^\circ(s+t) = {}^\circ s + {}^\circ t, \quad {}^\circ 0 = 0, \quad {}^\circ 1 = 1,$$

and, moreover, ${}^\circ t = 0$ in a certain subinterval $t \in [0,h]$, where h is a strictly positive number (any actually infinitely small number). The presence of such a 'numerical' function is a sure sign of a contradiction or, to put it delicately, implies the presence of antinomes.

All these circumstances require immediate and implicit elucidation of the concepts and means we are using, outlining the grounds which they are built on.

As has been earlier remarked, nonstandard analysis is substantiated within the set-theoretic stance. To be more exact, it appears that the ideas of the 'naive' nonstandard set theory developed above can be placed on the same solid foundation on which the Cantor theory or, strictly speaking, axiomatic set theories on the same foundation, rest.

In order to bring into focus the relations between mathematical analysis and set theory, the following statements are worth comparing.

> "... analysis ... is the science of infinity by itself"
>
> (G.W.Leibniz),

> "Leibniz is the founder of mathematics of the infinite"
>
> (F.Engels),

> "... mathematical analysis is just the science of the infinite. This old definition has been living through ages ..."
>
> (N.N.Luzin),

> "SETS, THEORY OF, a branch of mathematics studying general properties of sets, basically of infinite ones"
>
> (The Soviet Encyclopedic Dictionary).

The very notion, 'the infinity', is seen to be tightly associated with set theory. One should not forget, however, that the classical works by G.Cantor appeared two centuries after the invention of mathematical analysis. The laying of the set-theoretic foundation under mathematics could be compared with a modern method of erecting buildings, rack mounting, when a house is assembled starting with upper stores, 'from attic to cellar'. Of interest is the fact that in this case the foundation of the building is laid beforehand. The initial foundation of mathematical analysis was likewise laid by practical activities of people.

The present day mathematics is essentially resting on set theory, or, to be more exact, the set-theoretic foundation has been floated under the main stores of modern mathematics. What is going to happen next will be shown by the future. By now we can only state that the process of erecting the building of the future mathematics is going on, and that this process is fraught with dramatic changes. This accelerated development has been revealing itself through aggravation of the situation, collision of opinions, a fierce struggle of ideas. The set of citations to follow (far from claiming for completeness) will illustrate the process of polarization now in progress.

Pro:

"After the initial period of incredulity there began a triumphal march of the created theory of sets in all branches of mathematics. Its influence on the mathematics of our century manifests itself both in the choice of modern problems and in the methods of their solution. Set theory is being applied everywhere."

K.Kuratowski and A.Mostowski, [112, p. 7]

"One of the creations of George Cantor is the theory of sets, some of whose elements are nowadays taught in senior grades of high schools, and even earlier. It is one more field of mathematics which used to be thought of as having no practical application whatsoever. What a fallacy! Elements of set theory are now being used even by the authors of detective novels. Well known is the use of set theory in making programs for computers, the latter servicing an innumerable host of practical projects."

L.Young [276, p.154]

Contra:

"... it is claimed that the theory of sets is important for scientific-technological progress and is one of the newest achievements in mathematics. In fact the theory of sets has got nothing to do with scientific-technological progress and is not one of the newest achievements of mathematics."

L.S.Pontryagin [212, p.6]

"The mathematics based on Cantor set theory has turned into that of Cantor set theory... . Therefore, modern mathematics is studying a construction whose relation to the real world is at least problematic... . This questions the validity of mathematics as a scientific and useful method. The role of mathematics can be reduced to a simple game played in a certain specific artificial world. This is not a danger facing mathematics in the future but an outright crisis of modern mathematics".

P.Vopenka [268. p.14]

By way of concluding the preliminary discussion it should be emphasized that only now, having dispelled the illusion that it is possible to provide some final 'absolute' foundation of nonstandard analysis (as well as the whole of mathematics) by set-theoretic stance, we can get down to implementing this project.

3.1. The Language of Set Theory

Axiomatic set theories regulate legitimate ways of set formation in exact terms. Figuratively speaking, axiomatics describe worlds, i.e., universes, of sets which are to serve adequate presentations of our intuitive ideas about the 'Cantor paradise', the universe of the naive theory of sets. The axiomatics of interest for us are built and studied as formal theories. It should be specially emphasized that, despite obvious limitations (mathematics cannot be reduced to the syntax of its texts) and in many respects thanks to them (singling out semiotic aspects clarifies the problem of meaning), the formal approach has proved ito be exclusively fruitful (the Gödel theorem, independence of the continuum-hypothesis and of the axiom of choice, Boolean-valued analysis, etc.).

The cornerstone of a formal theory is its language. The exact description and study of such a language is, if required, carried out by means of a certain, generally speaking, different language which is usually called a metalanguage. A metalanguage is commonly a collection of fragments of natural languages limited and regulated in a certain way and enriched with various technical terms. The means allowed in a metalanguage are important from the mathematical point of view. Taking into account the fact that we are interested not in mathematical but in applied set-theoretic aspects of formal set theory, we do not impose extremely rigid constraints on the metalanguage. In particular, conventional expressive means and the level of rigour employed by conventional meaningful mathematics will be widely used further on.

3.1.1. Any axiomatic set theory is a *formal system*. The components of any system of the kind are its alphabet, formulas, axioms and rules of inference. As an alphabet, a fixed set A of symbols of arbitrary nature, i.e., a Cantor set, taken. Finite sequences of elements of A are called expressions and, sometimes, texts. If in some way (by prescriptions, algorithms, etc.) a certain set of 'well-formed' expressions $\Phi(A)$ has been chosen, then we say that a language with alphabet A is given, the chosen expressions called formulas. After that fixed are certain finite (or infinite) families of formulas called axioms, and implicitly described are the admissible rules of inference, i.e., relations in $\Phi(A)$. Formulas obtained from axioms in a finite number of steps by rules of inference are called theorems. There is a more free and convenient way of expression which is often used (we will do the same). Namely, they say that the theorems of a formal system comprise the least set of formulas which contains all the axioms and is closed relative to the rules of inference.

3.1.2. Of interest for us will be a special type of formal language, i.e., *a first-order language* (of the predicate logic) (with equality). The signature σ is a triplet (F,P,a), where F and P are some sets called the set of operation symbols and that of predicate symbols, respectively, while a is a mapping of $F \cup P$ into a set of natural numbers. They say that $u \in F \cup P$ is an *n-ary symbol* provided $a(u) = n$. The alphabet of a first-order language of signature σ consists of the following terms:

(1) *a set of symbols* of signature σ, i.e., the set $F \cup P$;

(2) *a set of variables*: small or capital Latin letters; possibly, with indices;

(3) *propositional connectives*: $\wedge$ is conjunction, $\vee$ is disjunction, $\rightarrow$ is implication, $\neg$ is negation;

(4) *quantifiers*: $\forall$ is the universal quantifier, $\exists$ is the existential quantifier;

(5) *equality sign*, $=$;

(6) *auxiliary symbols*: (is the opening parenthesis,) is the closing parenthesis, , is a comma.

3.1.3.

(1) *A term* of signature σ is an element of the least set of expressions of the language (of the same signature) obeying the following conditions:

(a) any variable is a term;
(b) any nullary operation symbol is a term;
(c) if $f \in F, a(f) = n$ and $t_1,\dots,t_n$ are terms, then $f(t_1,\dots,t_n)$ is a term.

(2) *Atomic (= atom) formulas* of signature σ are all expressions of the kind

$$t_1 = t_2, \quad p(y_1,\dots,y_n), q,$$

where $t_1, t_2, y_1,\dots,y_n$ are terms of signature σ, p is a certain n-ary predicate symbol, and q is a nullary predicate symbol.

(3) *Formulas of signature* σ constitute the least set of expressions obeying the following conditions:

(a) the atomic formulas of signature σ are formulas of signature σ;
(b) if φ and ψ are formulas of the signature σ, then $(\varphi \wedge \psi), (\varphi \vee \psi), (\varphi \rightarrow \psi), \neg\varphi$ are formulas of signature σ, too;
(c) if φ is a formula of signature σ, and x is a variable, then $(\forall x)\varphi$ and $(\exists x)\varphi$ are formulas of signature σ, too

The occurrence of the variable x in the formula φ is *bound* in φ, or belongs to the domain of a quantifier, provided x is incorporated into a subformula of φ of the kind $(\forall x)\varphi$ or $(\exists x)\varphi$. In the opposite case the occurrence of x into φ is free. They say that *x is free* (bound) in φ if all occurrences of x in φ are free. When we like to stress that only the variables $x_1,\dots,x_n$ are free in the formula φ, we write $\varphi = \varphi(x_1,\dots,x_n)$, or simply $\varphi(x_1,\dots,x_n)$. A formula with no free variables is called a *sentence*.

3.1.4. *The language of set theory* is a first-order language whose signature contains but one binary predicate symbol $\in$ and has neither other predicate nor functional symbols. We usually write $x \in y$ instead of $\in(x,y)$, and say that x is an *element of* y. Therefore, the formulas of set theory are formal texts composed of atomic formulas $x \in y$ and $x = y$ by way of propositional connectives and quantifiers.

Set theory is built on the basis of the laws of classical logics. In other words, it accepts conventional logical axioms and rules of inference of the propositional calculus with equality, which can be found in nearly every manual on mathematical logic (see, for instance, [27, 45, 232]). Besides, accepted are some nonlogical and special axioms reflecting the adopted presentation of sets and classes. When being varied within reasonable limits, special axioms result in axiomatic systems for set theory, different in their expressive forces. In the present chapter we shall describe three systems of the kind: the set theory of Zermelo-Fraenkel, the theory of internal sets by Nelson, and the theory of external sets by Kawai-Hrbacek. The theory of classes by von Neumann-Gödel-Bernays will be presented for discussion in the first chapter of the second part of the present monograph.

3.1.5. One of the most important functions of a metalanguage is the introduction of new abbreviating symbols and the establishment of new corresponding syntax. The point is that formalization of even simple fragments of meaningful mathematics results in cumbersome texts, the writing and reading of which are problematic by both physical and psychological reasons. That is why we have to introduce a great number of abbreviations and, in fact, build a more convenient abridged variation of the initial symbolic language. A necessary requirement is in this case the principal possibility of a one-to-one translation of a reduced presentation into a formalized language. In accordance with our intentions, we will not expatiate on the ways of introducing reductions, exact descriptions, functional expressions, etc.. For instance, henceforth, as before, we use the term *the assignment operator* or *definor*: $:=$, without going into corresponding subtleties.

3.1.6. We will give some examples of abbreviating formal texts in set theoretic language, semantic expressions of such texts appealing to intuitive naive presentations of sets. First of all we recall the following conventional abbreviations:

$$(\exists!\, x)\ \varphi(x) := (\exists x)\ \varphi(x) \wedge (\forall x)(\forall y)(\varphi(x) \wedge \varphi(y) \rightarrow x = y);$$
$$(\exists x) \in y)\ \varphi := (\exists x)\,(x \in y \wedge \varphi);$$
$$(\forall x) \in y)\ \varphi := (\forall x)\,(x \in y \rightarrow \varphi),$$

where φ is a formula. We also set $x \neq y := \neg (x = y)$ and $x \notin y := \neg (x \in y)$. For the simplest set-theoretic operations the following conventional abbreviation are adopted:

$$x \subset y := (\forall z)(z \in x \rightarrow z \in y);$$
$$u = \cup x = \cup(x) := (\forall z)(z \in u \leftrightarrow (\exists y \in x)\, z \in y);$$
$$u = \cap x = \cap(x) := (\forall z)(z \in u \leftrightarrow (\forall y \in x)\, z \in y);$$
$$u = y - x = y \setminus x := (\forall z)(z \in u \leftrightarrow (z \in y \wedge z \notin x)).$$

If φ is a formula, then a family $\mathcal{P}_\varphi(x)$ of all x subsets obeying the condition φ is described by the expression

$$u = \mathcal{P}_\varphi(x) := (\forall z)(z \in u \leftrightarrow (z \subset x) \wedge (\varphi(z)).$$

The empty set $\varnothing$ contains no elements, so

$$u = \varnothing := (\forall x)(x \in u \leftrightarrow x \neq x).$$

In the texts presented above use has been made of one of the most wide-spread methods of abbreviation, the removal of part of the parentheses.

3.1.7. The statement that x is an *unordered pair* of elements y and z is formalized as follows:

$$(\forall u)(u \in x \leftrightarrow u = y \vee u = z).$$

In this case we set $\{y, z\} := x$. It should be remarked that braces do not belong to the initial alphabet and thus they are metasymbols.

An ordered pair and an ordered n-tuple are introduced by the Kuratowski trick:

$$(x, y) := \langle x, y \rangle := \{x\{x\}, \{x, y\}\};$$
$$(x_1, \ldots, x_n) := \langle x_1, \ldots, x_n \rangle := \langle\langle x_1, \ldots, x_{n-1} \rangle, x_n \rangle,$$

where $\{x\} := \{x, x\}$. The overuse of round brackets is inevitable and should not be regarded as a pretext for introducing new symbols.

Using the agreements made, a formalized sense can be attributed to the expression 'X is the Cartesian product of $Y \times Z$'. Namely, according to the definition, we assume that $X := \{(y, z) : y \in Y, z \in Z\}$.

3.1.8. Let us consider the following statement:

(1) $\mathrm{Rel}(X)$, i.e., X is *a correspondence* (= *relation*);

(2) $Y = \mathrm{dom}(X)$;

(3) $Z = \mathrm{im}(X)$.

Therefore, in (1) - (3) we state that elements of X are ordered pairs, Y being the domain of definition of X, Z being the range of values or the image of X.

The corresponding formal texts are as follows:

(1) $(\forall u)(u \in X \to (\exists v)(\exists \omega)\, u = (v,\omega)$;

(2) $(\forall u)(u \in Y \leftrightarrow (\exists v)(\exists \omega)\, \omega = (u,v) \wedge \omega \in X)$;

(3) $(\forall u)((u \in Z \leftrightarrow (\exists v)(\exists \omega)\, \omega = (v,u) \wedge \omega \in X)$.

The functionality of X, or $\mathrm{Un}(X)$, is expressed by the formula

$$\mathrm{Un}(X) := (\forall u)(\forall v_1)(\forall v_2)((u,v_1) \in X \wedge (u,v_2) \in X \to v_1 = v_2).$$

A single-valued relation is termed *functional*. We set $\mathrm{Fnc}(X) := \mathrm{Func}(X) := \mathrm{Un}(x) \wedge \mathrm{Rel}(X)$. In this case, in order to express $(u,v) \in X$, we write $v = X(u)$, $X: u \to v$, etc.. Then the phrase F *is a mapping* (=*function*) from X in Y implies that F is a subclass of $X \times Y$, F is functional, and the domain of F coincides with X:

$$F: X \to Y := F \subset X \times Y \wedge \mathrm{Func}(F) \wedge \mathrm{dom}(F) = X.$$

The *restriction* of X to U is, by the definition, $X \cap (U \times Z)$, and is denoted by $X|U$.

If there is, and the only one, Z for which $(Y,Z) \in Z$, then we set $X'Y := Z$. In all the remaining cases we set $X'Y := \varnothing$. And, finally, by definition, $X''Y := \mathrm{im}(X|Y)$. Instead of $X''\{z\}$ we write $X(x)$, or even Xx if it does not result in misunderstanding. It is worth emphasizing that henceforth we adopt a free point of view on placing and removing parentheses. In other words, both their introduction and elimination are as a rule governed by considerations of convenience and by requirements on the level of formalization of the fragment of the text under discussion.

3.1.9. *The superposition* (or *composition*) of relations X and Y, denoted by the symbol $Y \circ X$, consists exactly of ordered pairs (z,x) such that $(x,y) \in X$ and $(y,z) \in Y$ for a certain y:

$$(\forall u)(u \in Y \circ X \leftrightarrow (\exists x)(\exists y)(\exists z) \quad (x,y) \in X \wedge (y,z) \in Y \wedge u = (x,z)).$$

The relation X^{-1}, inverse to X, is defined as

$$(\forall u) \quad (u \in X^{-1} \leftrightarrow (\exists x)(\exists y) \quad (x,y) \in X \wedge u = (y,z)).$$

The symbol I_X denotes *the identity relation* on X, i.e.,

$$(\forall u) \quad (u \in I_X \leftrightarrow (\exists x) \quad (x \in X \wedge u = (x,x))).$$

3.1.10. When $\mathrm{Rel}(X) \wedge ((X \cap Y^2) \circ (X \cap Y^2) \subset X)$, we say that X is a *transitive* relation on Y. This X is called *reflexive* (over Y) if $\mathrm{Rel}(X) \wedge (I_Y \subset X)$. And, finally, if $\mathrm{Rel}(X) \wedge ((X \cap X^{-1}) \cap Y^2 \subset I_Y)$ the term "X is an *antisymmetric* relation on Y' is used. Here, of course, a standard abbreviation $Y^2 := Y \times Y$ is used. An antisymmetric, reflexive and transitive relation X on Y is termed an *order* (or order relation). Another terminology standard under these conditions is also used. It should, in particular, be recalled that an order X on Y is called *linear*, while Y itself is called a *chain* (relative to X) provided $Y^2 \subset X \cup X^{-1}$. If any nonempty subset of the set Y has the least (relative to the order of X) element, then we say that X *well-orders* Y, or that Y *is well-ordered* (by the understood order of X).

3.1.11. Quantifiers are termed *restricted* or *bounded*, if they appear in the text as $(\forall x \in y)$ or $(\exists x \in y)$. There is a classification of the formulas of set theory (and, generally speaking, of any first-order theory) based on the way of using restricted or unrestricted (i.e., those not restricted) quantifiers. In our further discussions of special importance will be two classes of formulas: restricted or bounded formulas (= Σ_0-formulas) and Σ_1-formulas. We say that the formula φ is *restricted* if any quantifier encountered in it either has the form $(\forall x \in y)$ or the form $(\exists x \in y)$ (see abbreviations **3.1.6**). The formula φ is of class Σ_1 and is called an Σ_1-*formula*, if it is constructed from atomic formulas and their negations, using only logical operations $\wedge$, $\vee$, $(\forall x \in y)$ and $(\exists x)$. Clearly, every restricted formula belongs to the class Σ_1. Not every Σ_1-formula is, however, restricted, and there are formulas not belonging to the class Σ_1. Let us consider corresponding examples, starting with restricted formulas.

3.1.12. The proposition $z = \{x, y\}$ is equivalent to the restricted formula

$$x \in z \wedge y \in z \wedge (\forall u \in z) \quad (u = x \vee u = y).$$

An ordered pair is seen from here to be introduced by a restricted formula. The same is true as regards a Cartesian product, since $Z = X \times Y$ can be written as

$$(\forall z \in Z)(\exists x \in X)(\exists y \in Y) \quad (z = (x,y))$$
$$\wedge (\forall z \in X)(\forall y \in Y)((\exists z \in Z) \quad (z = (x,y)).$$

One more restricted formula results from the notion 'a mapping F from X to Y' (see **3.1.8**). Indeed, it follows from the above that $F \subset X \times Y$ is a restricted formula, and that, moreover, the expressions $\mathrm{dom}(F) = X$ and $\mathrm{Un}(F)$, which are equivalent to the respective formulas

$$(\forall x \in X)(\exists y \in Y)(\exists z \in F) \quad z = (x,y),$$
$$(\forall z_1 \in F)(\forall z_2 \in F)(\forall x \in X)(\forall y_1 \in Y)(\forall y_2 \in Y)$$
$$z_1 = (x,y_1) \wedge z_2 = (x,y_2) \to y_1 = y_2,$$

are restricted, too.

3.1.13. The statement 'the sets x and y *are equipotent* or have the same cardinality or, in other words, 'there is a bijection between x and y', or, symbolically, $x \simeq y$, is presented by the following $\sum_1$-formula:

$$(\exists f)(f : x \to y \wedge \mathrm{im}(f) = y \wedge \mathrm{Un}(f^{-1})).$$

This peculiarity is not, however, expressed by a restricted formula. One more $\sum_1$-formula can be given by the notion of a relation:

$$\mathrm{Rel}(X) := (\forall u \in X)(\exists v)(\exists \omega) \quad u = (v,\omega).$$

The following formula, stating that the set y is equal to none of its elements, is not of class Σ_1:

$$(\forall x \in y) \quad \neg(x \simeq y).$$

Remarks.

(1) It goes without saying that not only special axioms of a first-order theory (see **3.1.4**) can be varied but also its logical part, i.e., logical axioms and rules of inference. A great number of theorems obtained in this way can essentially differ from each other. Thus, for instance, eliminating the law of the excluded middle from the axioms of propositional calculus, we obtain the intuitionistic calculus of propositions. The intuitive calculus of predicates (see [63,97]) is constructed in an analogous way.

(2) The modern formal logic has been with great difficulties formed in the course of the evolution of philosophical and mathematical thought. The classical calculus of predicates stems from the Aristotle syllogistic, while the origin of the intuitionistic logic is related to other philosophical ideas. Logical systems essentially different from both systems mentioned were created in various periods of times for various purposes. Thus, the ancient Indian logic had three types of negation: something has never been and cannot be, something has been but is now absent, something that is now but will soon disappear.

(3) As is seen from **3.1.6** and **3.1.7**, abbreviations can be encountered in formulas, in abbreviations, in abbreviations of abbreviations, etc.. The invention and introduction of abbreviating symbols is an art of its own, and as any art, it cannot be completely formalized. Nevertheless, systematization and codification of the rules of determining abbreviations is necessary from both theoretical and practical points of view. Some of such systems of rules (exact descriptions, introduction of functional letters, etc.) can be found in literature [26, 27, 60].

3.2. Zermelo-Fraenkel Set Theory

As has been noted in **3.1.4**, the axioms of set theory include the general-logical axioms of first-order theories with equality, which fix the classical rules of logical inference. Below we shall enumerate special axioms of set theory ZF_1-ZF_6 and AC. If ZF_1-ZF_6 are assumed as special axioms, then the arising axiomatic system is called the Zermelo-Fraenkel set theorem (or system) and is denoted by ZF. When ZF is modified with the axiom of choice AC then there arises a wider theory which is conventionally denoted by ZFC. It should be remarked that the parallel verbal formulations of the axioms are motivated by the Cantor ideas on sets.

3.2.1. In studying ZFC the terms *'a property'* and *'a class'* are often used. Let us elucidate their formal status. Consider a formula $\varphi = \varphi(\mathrm{x})$ of ZFC (in symbols, $\varphi \in$ (ZFC)). Instead of the text $\varphi(y)$ we write $y \in \{x : \varphi(x)\}$. Therefore, the so-called *Church scheme* for classification works:

$$y \in \{x\colon \varphi(x)\} := \varphi(y).$$

In the language of ZFC the expression $y \in \{x\colon \varphi(x)\}$ implies that y has the property φ, or that y lies in the class $\{x: \varphi(x)\}$. In this sense the property, the formula and the class mean the same in ZFC. The Church schema has been already practically used in **3.1.6** and **3.1.7**. When working with ZFC, it would be convenient for us to make use of many widely-spread abbreviations and, in particular, of the following:

$\mathrm{V} := \{x: x = x\}$ is the universe of discourse, or the class of all sets;
$\{x: \varphi(x)\} \in \mathrm{V} := (\exists z)\,(\forall y)\ \varphi(y) \leftrightarrow y \in z$;
$x \cup y := \cup\{x, y\}, \quad x \cap y \cap z := \cap\{x, y, z\} \ldots$.

Let us now go over to formulating special axioms of ZFC.

3.2.2. The Axiom of Extensionality, ZF_1: *two sets are equal iff they consist of the same elements*:

$$(\forall x)(\forall y)(\forall z) \quad (z \in x \leftrightarrow x \in y) \leftrightarrow x = y.$$

It should be remarked that the last equivalence can be replaced by $\rightarrow$ without loss of scope, since the reverse implication is the theorem of predicate calculus with equality.

3.2.3. The Axiom of Union, ZF_2: *a union of a set of sets is also a set*:

$$(\forall x)(\exists y)(\forall z)(\exists u)\ (u \in z \wedge z \in x) \leftrightarrow z \in y.$$

Using the abbreviations of **3.1.6** and **3.2.1**, the axiom ZF_2 can be presented as

$$(\forall x) \ \cup x \in \mathrm{V}.$$

3.2.4. The Axiom of Powers (of Powersets), ZF_3: *all subsets of a given set constitute a certain set, i.e.*,

$$(\forall x)(\exists y)(\forall z)\,(z \in y \leftrightarrow (\forall u)\,(u \in z \rightarrow u \in x)),$$

or, in an abbreviated form,

$$(\forall x)\ \mathcal{P}(x) \in \mathbf{V}.$$

3.2.5. The Axiom of Replacement, ZF_4^φ: *an arbitrary one-to-one image of a set is a set, too*:

$$(\forall x)(\forall y)(\forall z)\,(\varphi(x,y)\wedge\varphi(x,z)\rightarrow y=z)\rightarrow$$
$$\rightarrow(\forall a)(\exists b)\,((\exists s\in x)(\exists t)\,\varphi(s,t)\leftrightarrow t\in y).$$

And now in a contracted form:

$$(\forall z)(\forall y)(\forall z)\;\varphi(x,y)\wedge\varphi(x,z)\rightarrow y=z)\rightarrow$$
$$\rightarrow(\forall a)\,(\{v\colon(\exists u\in a)\,\varphi(u,v)\}\in \mathrm{V}).$$

Here φ is a formula of ZFC containing no free occurrences of a. It should be remarked that ZF_4^φ is a schema for an infinite set of axioms, since for any appropriate $\varphi\in$(ZFC) its own axiom is formed. Nevertheless, for the sake of brevity and uniformity, we speak about the axiom of replacement, bearing in mind the peculiarity mentioned above.

Let us now formulate some useful corollaries to ZF_4^φ.

3.2.6. Let $\psi=\psi(z)$ be a formula of ZFC. Given any set x, we can compose its subset, by choosing the elements of x with the property ψ, i.e.,

$$(\forall x)\,\{z\in x\colon\psi(z)\}\in\mathbf{V}.$$

This statement is the axiom ZF_4^φ, where the formula $\psi(u)\wedge(u=v)$ is used instead of φ. The situation under discussion is often called *the axiom of separation*, or *comprehension*.

3.2.7. Applying the axiom ZF_4^φ to the formula

$$\varphi(u,v):=(u=\varnothing\rightarrow v=x)\wedge(u\neq\varnothing\rightarrow v=y)$$

of a set $z:=\mathcal{P}(\mathcal{P}(\varnothing))$, we make sure that the unordered pair $\{x,y\}$ of two sets (cf. **3.1.7**) is also a set. The preceding statement is often referred to as *the axiom of pairing*.

3.2.8. The Axiom of Infinity, ZF_5: *there is at least one infinite set*:

$$(\exists x)\,(\varnothing\in x\wedge(\forall y)\,(y\in x\rightarrow y\cup\{y\}\in x)).$$

Therefore, there is such a set x, that $\varnothing \in x$, $\{\varnothing\} \in x$, $\{\varnothing,\{\varnothing\}\} \in x$, $\{\varnothing,\{\varnothing\},\{\varnothing,\{\varnothing\}\}\} \in x$, etc..

3.2.9. The Axiom of Foundation (of Regularity), ZF_6: *any nonempty set has an element non-intersecting with it*

$$(\forall x)\ (x \neq \varnothing \rightarrow (\exists y)(y \in x \wedge y \cap x = \varnothing)).$$

Applying the axiom ZF_6 to a unielement set $x := \{y\}$, we get $y \notin y$. It should be remarked, speaking a bit beforehand, that by an analogous reason (if we take $x := \{x_1, \dots, x_n\}$) there are no infinitely decreasing $\in$-sequences $x_1 \ni x_2 \ni \dots \ni x_n \ni \dots$.

3.2.10. The Axiom of Choice (of Multiplication), AC: *the product of a set of nonempty sets is nonempty:*

$$(\forall x)\ (\exists f)\,(\mathrm{Func}(f) \wedge x \subset \mathrm{dom}(f)) \wedge (\forall y \in x)\ y \neq \varnothing \rightarrow f(y) \in y.$$

The function f is, under the circumstances, called a *selector* for x.

There is a great number of mathematical statements equivalent, within the framework of the theory under consideration, to the axiom of choice (see [95]). Let us recall the formulations of the two most popular among them.

Zermelo theorem (the well-ordering principle). *Any set can be well-ordered.*

Kuratowski-Zorn lemma (the maximality principle). *Let M be a (partially) ordered set whose every chain has an upper bound. Then for every $x \in M$ there is a maximal element $m \in M$ such that $m \geq x$.*

3.2.11. On the basis of the axiomatics discussed above we get an exact presentation for the class of all sets as the 'von Neumann universe'. An initial object of construction is the empty set. An elementary step of introducing new sets from those already constructed consists in forming the union of sets of the subsets of available sets. Transfinite repetition of such steps exhausts the class of all sets. Classes (in the 'Platonic' sense) can be viewed as external objects relative to the elements of the von Neumann universe. Within this approach a class is a family of sets obeying a set-theoretic property described by a formula of Zermelo-Fraenkel theory. Therefore, the class consisting of elements of a certain set is (according to the axiom of replacement) also a set. A formally correct definition of the von Neumann universe requires preliminary acquaintance with the notions of ordinal and cumulative hierarchy. Below we give the minimum of information on these objects necessary for a 'naive' definition, a more detailed presentation can be found in the first chapter of the second part of the book.

3.2.12. A set x is called *transitive* if every element of x is a subset of x. A set x is called *an ordinal* if x is transitive and linearly ordered by the relation $\in$. These definitions in symbolic form are as follows:

$$\mathrm{Tr}(x) := (\forall y \in x)\,(y \subset x) := \text{`}x \text{ is a transitive set'};$$
$$\mathrm{Ord}(x) := \mathrm{Tr}(x) \wedge (\forall y \in x)(\forall z \in x)$$
$$(y \in z \vee z \in y \vee z = y) := \text{`}x \text{ is an ordinal'}.$$

Ordinals are commonly denoted by lower-case Greek letters. Every ordinal is considered with a natural order: for $\beta, \gamma \in \alpha$ we designate

$$\gamma \le \beta \leftrightarrow \gamma \in \beta \vee \gamma = \beta.$$

The class of all ordinals is denoted by the symbol On, and, thus, $\mathrm{On} := \{\alpha : \mathrm{Ord}(\alpha)\}$.

The ordinal is a well-ordered set; i.e., it is linearly ordered and its every subset has the least element (which is ensured by the axiom of foundation). We can easily see that

$$\alpha \in \mathrm{On} \wedge \beta \in \mathrm{On} \rightarrow \alpha \in \beta \vee \alpha = \beta \vee \beta \in \alpha;$$
$$a \in \mathrm{On} \wedge \beta \in \alpha \rightarrow \beta \in \mathrm{On};$$
$$a \in \mathrm{On} \rightarrow \alpha \cup \{\alpha\} \in \mathrm{On};$$
$$\mathrm{Ord}(\varnothing).$$

The ordinal $\alpha + 1 := \alpha \cup \{\alpha\}$ is called *the successor of* α. An ordinal which is not equal to zero and not a successor is termed a *limit* ordinal. The following notation is used:

$$K_I := \{\alpha \in \mathrm{On} : (\exists \beta)\ \mathrm{Ord}(\beta) \wedge a = \beta + 1 \vee \alpha = \varnothing\};$$
$$K_{II} := \{\alpha \in \mathrm{On} : \alpha \text{ is a limit ordinal}\};$$
$$0 := \varnothing,\quad 1 := 0 + 1,\quad 2 := 1 + 1, \ldots,$$
$$\omega := \{0, 1, 2, \ldots\}.$$

3.2.13. It should be remarked that within ZFC one can prove the possibility of using well-known (at a 'naive' level) properties of ordinals, and, in particular, the validity of transfinite induction and recursive definitions. Let us define the von Neumann universe, so far purposefully omitting formal validation for such definitions. For every ordinal α,

$$V_\alpha := \bigcup_{\beta < \alpha} \mathcal{P}(V_\beta),$$

i.e., $V_\alpha = \{x : (\exists\beta)(\beta \in \alpha \wedge x \subset V_\beta)\}$. Or, in more detail,

$$V_0 := \varnothing ;$$
$$V_{\alpha+1} := \mathcal{P}(V_\alpha);$$
$$V_\beta := \bigcup_{\alpha<\beta} V_\alpha, \text{ provided } \beta \in K_{II}.$$

Of principal importance is the following theorem ensured by the axiom of foundation:

$$(\forall x)(\exists\alpha)(\mathrm{Ord}(\alpha) \wedge x \in V_\alpha),$$

which can be presented in the form

$$\mathbf{V} = \bigcup_{\alpha\in\mathrm{On}} V_\alpha .$$

In words this can be expressed as *'the class of all sets is the von Neumann universe'*, or, using the terminology stemming from Mirimanoff, *'any set is well-founded'*.

Graphically the von Neumann universe can be depicted as shown in Figure **3**, the 'lower' levels of the universe being as follows:

$$V_0 = \varnothing, \quad V_1 = \{\varnothing\}, \quad V_2 = \{\varnothing, \{\varnothing\}\}, \dots,$$
$$V_\omega = \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}, \dots\}, \dots .$$

3.2.14. The representation of the universe $\mathbf{V}$ as the *'cumulative hierarchy'* of sets $(\mathrm{V}_\alpha)_{\alpha\in\mathrm{On}}$ makes it possible to relate rank to any set:

$$\mathrm{rank}\,(x) := \text{a least ordinal } \alpha \text{ such that } x \in V_{\alpha+1}.$$

We can readily prove that

$$a \in b \rightarrow \mathrm{rank}\,(a) < \mathrm{rank}\,(b);$$
$$\mathrm{Ord}(\alpha) \rightarrow \mathrm{rank}\,(\alpha) = \alpha;$$
$$(\forall x)(\forall y)\ \mathrm{rank}\,(x) \rightarrow (\varphi(y) \rightarrow \varphi(x)) \rightarrow (\forall x)\ \varphi(x),$$

where φ is a formula of ZFC. The preceding theorem (or, more precisely, the schema of theorems) is called *the principle of induction on rank.*

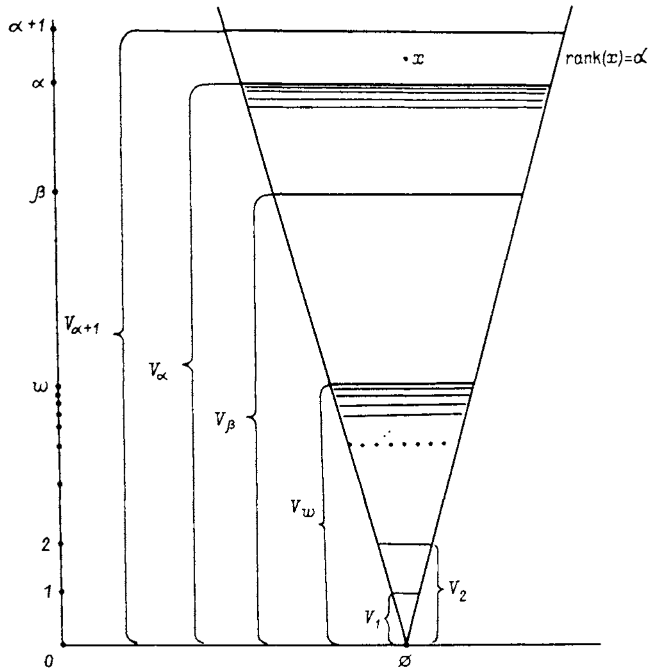


Fig. 3

3.2.15. Remarks.

(**1**) The first system of axioms for set theory (together with the B.Russel theory of types) suggested by E.Zermelo in 1908, essentially coincides with ZF_1 - ZF_3, ZF_5, **3.2.5**, **3.2.6**. The axiom of extensionality ZF_1 and union ZF_2 were proposed earlier by G.Freget (1883) and G.Cantor (1899), respectively. The idea of the axiom of infinity ZF_5 belongs to R.Dedekind.

(**2**) The axiom of choice AC had, in all probability, been implicitly used for a long time before it was noticed by G.Peano in 1890 and B.Levy in 1902. This axiom was introduced by E.Zermelo in 1904 and for many years remained most disputable. The axiom of choice is the cornerstone of many important fragments of modern mathematics and there is no wonder that it is being accepted by the majority of the mathematicians working at present. Discussions of the place and role of the axiom of choice can be found elsewhere [28, 56, 61, 95, 98].

(3) The system of ZFC was completely elaborated at the beginning of the 1920s. By that time the formalization of the set theoretic language had been completed, which made it possible to clarify the vague description of the type of properties admissible in the axiom of comprehension. On the other hand, Zermelo axioms do not yield as a corollary the Cantor statement that each one-to-one image of a set is a set. This drawback was eliminated by A.Fraenkel in 1922 and T.Scolem in 1923, who suggested variations of the axiom of replacement.

(4) The axiom of foundation ZF_6 was in fact suggested by von Neumann in 1925. This axiom is independent of the other axioms of ZFC.

(5) The system of the axioms of ZFC is not infinite, as noted in **3.2.4**. Nonfinite axiomatizability for ZFC was proved by R.Montague in 1960 (see [75, 56, 155]).

3.3. Nelson Internal Set Theory

The preliminary analysis of the properties of standard and nonstandard sets carried out above showed that in the von Neumann universe there is a place for infinitely small numbers but there is no place for the whole of their union. In other words, nonstandard analysis points out that Zermelo-Fraenkel theory, while describing the classical world of 'standard' mathematics, singles out a proper internal part of the universe of 'naive' sets. In order to emphasize this peculiarity, in the nonstandard theory of sets, the elements of the von Neumann universe are called *internal sets*. Therefore, a set in the sense of Zermelo-Fraenkel theory and an internal set are synonyms. A convenient foundation for the nonstandard analysis is given by internal set theory IST suggested by E.Nelson.

3.3.1. The alphabet of formal IST is obtained by adjoining the only one new symbol to that of ZFC, the symbol of the unary predicate St that expresses the property of being a *standard set*. In other words, the number of admissible fragments of the texts of IST is enlarged by expressions of the type $\mathrm{St}(x)$, or, in more detail, 'x is standard', or, finally, 'x is a standard set'. Therefore, the semantic domain of definition for variables of IST is the Zermelo-

Fraenkel universe, i.e., the von Neumann universe, in which standard and nonstandard sets are now distinguished.

3.3.2. The formulas of IST are defined by a conventional procedure. In this case the number of atom formulas is modified by the texts $\mathrm{St}(x)$, where x is a variable. Each of the formulas of ZFC is a formula of IST, the converse statement being obviously not valid. In order to distinguish the formulas, the following terminology is used: the formulas of ZFC are called *internal*, the formulas of IST that are not ZFC are called *external*. Hence, the text 'x is standard' is an external formula of IST.

Sometimes, in our further discussions it would be convenient to use the following abbreviations: we shall write $\varphi \in(\mathrm{IST})$ instead of φ is a formula of IST and, respectively, $\varphi \in(\mathrm{ZFC})$ instead of φ is a formula of ZFC, i.e., φ is an internal formula of IST.

3.3.3. The difference between the formulas of IST results in singling out external and internal classes. If φ is an external formula of IST, then the text $\varphi(y)$ is described with the following words: '*y is an element of the external class $\{x: \varphi(x)\}$*'. The term *internal class* is used in the same sense as the term *class* in Zermelo-Fraenkel theory. In the cases in which this does not result in misunderstanding, both external and internal classes are called simply classes.

3.3.4. The external classes composed of elements of an internal set are termed *external sets*, or, in more detail, external subsets of the given set. It would useful to recall again that the internal class composed of elements of an internal set is also an internal set. Alongside with the abbreviations assumed in ZFC, in internal set theory some additional agreements are concluded. A list of them follows:

$\mathbf{V}^{\mathrm{st}} := \{x : \mathrm{St}(x)\}$ is an external class of standard sets;

$x \in \mathbf{V}^{\mathrm{st}} := x$ is standard $= (\exists y)\ \mathrm{St}(y) \wedge y = x$;

$(\forall^{\mathrm{st}} x)\ \varphi := (\forall x)\ (x$ is standard $\rightarrow \varphi)$;

$(\exists^{\mathrm{st}} x)\ \varphi := (\exists x)\ (x$ is standard $\wedge\ \varphi)$;

$(\forall^{\mathrm{st\,fin}} x)\ \varphi := (\forall^{\mathrm{st}} x)\ (x$ is finite $\rightarrow \varphi)$;

$(\exists^{\mathrm{st\,fin}} x)\ \varphi := (\exists^{\mathrm{st}} x)\ (x$ is finite $\wedge\ \varphi)$;

${}^{\circ}x := \{y \in x : y$ is standard$\}$.

The external set ${}^{\circ}x$ is often called the *standard core of x*.

The collision of notation arising by virtue of the traditional terminology (for $x \in {}^{*}\mathbf{R}$ the symbol ${}^{\circ}x$ denotes the standard part, $\mathrm{st}(x)$, of this number as well) results in no significant controversy.

3.3.5. The axioms of IST are obtained by supplementing the list of axioms of ZFC with three new schemata, bearing, as we have remarked earlier, the name of the principles of nonstandard set theory:

(1) the transfer principle,

$$(\forall^{st} x_1)(\forall^{st} x_2)\dots(\forall^{st} x_n)((\forall^{st} x)\varphi(x,x_1,\dots,x_n) \\ \to (\forall x)\varphi(x,x_1,\dots,x_n))$$

for every internal formula φ;

(2) the idealization principle,

$$(\forall x_1)(\forall x_2)\dots(\forall x_n)((\forall^{st\,fin} z)(\exists x)(\forall y \in z)\ \varphi(x,y,x_1,\dots,x_n) \\ \leftrightarrow (\exists x)(\forall^{st} y)\ \varphi(x,y,x_1,\dots,x_n)),$$

where $\varphi \in$(ZFC) is an arbitrary internal formula;

(3) the standardization principle,

$$(\forall x_1)\dots(\forall x_n)((\forall^{st} x)(\exists^{st} y)(\forall^{st} z)\ z \in y \leftrightarrow z \in x \wedge \varphi(z,x_1,\dots,x_n))$$

for every formula φ.

3.3.6. Powell theorem. IST *is a conservative extension of* ZFC.

◁ A proof is given in [187]. ▷

3.3.7. The above theorem implies that the internal theorems of internal set theory are theorems of Zermelo-Fraenkel theory. In other words, when proving 'standard' theorems on sets we are rightful to use the formalism of IST to the same degree of validity we enjoy while working within ZFC. One should not, however, forget that ZFC is, in the long run, substantiated with its practical infallibility and semantical justification.

3.3.8. When thinking about the sense of the formal expression of the axioms of IST, one cannot help but notice that the presentation of the idealization principle is somewhat cumbersome. While the refined rules of transfer and standardization presented above quite adequately portray the naive concepts put forward earlier, the position of the idealization principle formulation is controversial. Therefore, let us first of all prove that the idealization principle **3.3.5** (2) guarantees the presence of nonstandard elements.

3.3.9. *There is a finite internal set, among whose elements every standard set is encountered.*

◁ Let us consider the following formula: $\varphi := (x \text{ is finite }) \wedge (y \in x)$. Note that $\varphi \in$ (ZFC). Obviously, for every standard finite z we can find an x such that for all $y \in z$ we have $\varphi(x, y)$. If we choose z itself as such an x and use the idealization principle, then the proof will be completed. ▷

3.3.10. When applying the idealization principle, it is useful for us to bear in mind that standard finite sets are exactly the sets whose every element is standard, which fact has been proved earlier (**2.2.2**). It would be instructive to consider its formal inference based on the idealization principle.

3.3.11. *For an internal set* A,

$$A = {}^{\circ}A \leftrightarrow (A \textit{ is standard}) \wedge (A \textit{ is finite}).$$

◁ Let us construct the formula $\varphi := x \in A \wedge x \neq y$. No doubt, $\varphi \in$(ZFC). Then, by virtue of the idealization principle,

$$(\forall^{\mathrm{st\,fin}} z)\,(\exists x)\,(\forall y \in z)\quad \varphi(x, y, A) \leftrightarrow (\exists x)(\forall^{\mathrm{st}} y)\, x \in A \wedge x \neq y$$
$$\leftrightarrow (\exists x \in A)\quad (x \text{ is nonstandard}) \leftrightarrow A \setminus {}^{\circ}A \neq \varnothing.$$

In other words, we get

$$A = {}^{\circ}A \leftrightarrow (\exists^{\mathrm{st\,fin}} z)(\forall x)(\exists y \in z)\, x \notin A \vee x = y$$
$$\leftrightarrow (\exists^{\mathrm{st\,fin}} z)(\forall x \in A)(\exists y \in z)\, x = y \leftrightarrow (\exists^{\mathrm{st\,fin}} z)\, A \in z. \;▷$$

3.3.12. *Let* X, Y *be standard sets, and* $\varphi = \varphi(x,y,z)$ *be a formula of* IST. *The rule of introduction of standard functions* (= construction principle) *is valid:*

$$(\forall^{st}x)(\exists^{st}y)\quad (x\in X\to y\in Y\wedge \varphi(x,y,z))$$
$$\leftrightarrow (\exists^{st}y(\cdot))\,(\forall^{st}x)(y(\cdot)\ \textit{is a function from}\ X\ \text{to}\ Y\wedge (x\in X\to \varphi(x,y(x),z)).$$

◁ Let us consider the standardization $\overline{F}(x) :=^{*} \{y\in Y\colon \varphi(x,y,z)\}$. Applying **3.3.5** (3) again, we form a standard set

$$F :=^{*} \{(x,A)\in X\times \mathcal{P}(Y)\colon \overline{F}(x)=A\}$$

(here we use the fact that $\mathcal{P}(Y)$ is standard, ensured by the supposition that Y is standard). By hypothesis, $(\forall^{st}x\in X)\ \overline{F}(x)\neq\varnothing$. In this case, by the definition of F, $F(x)=\overline{F}(x)$. Hence, by virtue of the transfer principle, we get

$$(\forall^{st}x\in X)\ F(x)\neq\varnothing\to(\forall x\in X)\ F(x)\neq\varnothing.$$

Using now the axiom of choice, we can conclude:

$$(\exists y(\cdot))\ (y(\cdot)\ \text{is a function from}\ X\ \text{in}\ Y)\wedge(\forall x\in X)\,(y(x)\in F(x)).$$

Applying the transfer principle, we deduce that there is a standard function $y(\cdot)$, determined on X with the values in Y, for which $y(x)\in F(x)$ for all $x\in X$. If we recall the definition of F once again, we shall see that $y(\cdot)$ is the sought function. ▷

3.3.13. From now on, as above, it would be convenient to use some symbolic presentations of the rules deduced, making certain infringements to the agreements concluded earlier. Thus, the rules for introducing standard functions from **3.3.12** can be conveniently rewritten as follows:

(1) $(\forall^{st}x)(\exists^{st}y)\,\varphi(x,y)\leftrightarrow(\exists^{st}y())(\forall^{st}x)\,\varphi(x,y(x))$,

(2) $(\exists^{st}x)\,(\forall^{st}y)\ \varphi(x,y)\leftrightarrow(\forall^{st}y(\cdot))(\exists^{st}x)\ \varphi(x,y(x))$,

where $\varphi\in$(IST), i.e., an arbitrary formula of IST. In other words, we neglect the requirements of possible presence of free variables in φ and of necessary assumption of the 'limitedness', which implies that x and y are considered ranging over given standard sets. In the same way, if $\varphi=\varphi(x_1,\dots,x_n)$ and $\psi=\psi(y_1,\dots,y_n)$, then we shall write $\varphi\to\psi$ when

$$(\forall^{st}x_1)\dots(\forall^{st}x_n)\,(\forall^{st}y_1)\dots(\forall^{st}y_n)\quad \varphi(x_1,\dots,x_n)\leftrightarrow\psi(y_1,\dots y_n),$$

and say that the formulas φ and ψ are equivalent (though if one of the formulas φ and ψ is external, the formulas $\varphi(x_1,\dots,x_n)$ and $\psi(y_1,\dots y_n)$ can be not equivalent for a certain choice of variables!) Using the new means, we can present the transfer principle in a reduced form by the symbols:

(3) $(\forall^{st}x)\ \varphi(x) \leftrightarrow (\forall x)\ \varphi(x)$,

(4) $(\exists^{st}x)\ \varphi(x) \leftrightarrow (\exists x)\ \varphi(x)$,

always keeping in mind that the formula φ must be internal in such a presentation: $\varphi \in$(ZFC). It would be useful to give here some elementary rules valid for any formula φ,

(5) $(\forall x)(\forall^{st}y)\ \varphi(x,y) \leftrightarrow (\forall^{st}y)(\forall x)\ \varphi(x,y)$,

(6) $(\exists x)(\exists^{st}y)\ \varphi(x,y) \leftrightarrow (\exists^{st}y)(\exists x)\ \varphi(x,y)$,

as well as new presentations of the idealization principle:

(7) $(\forall^{stfin}z)(\exists x)(\forall y \in z)\ \varphi(x,y) \leftrightarrow (\exists x)(\forall^{st}y)\ \varphi(x,y)$,

(8) $(\exists^{stfin}z)(\forall x)(\exists y \in z)\ \varphi(x,y) \leftrightarrow (\forall x)(\exists^{st}y)\ \varphi(x,y)$,

pertaining, obviously, only to internal formulas $\varphi \in$(ZFC).

3.3.14. The rules discussed above enable one to transfer many (though, obviously, not all) notions and suppositions of the nonstandard analysis into equivalent mathematical definitions and statements appealing to standardness. In other words, the formulas of IST expressing 'something unusual' about standard objects can be transformed into equivalent formulas of ZFC, which are conventional mathematical records of the propositions under consideration. The procedure by means of which we get the described result is called *the Nelson algorithm*, rules **3.3.13** (1) - **3.3.13** (8) being parts of this procedure. In qualitative terms the essence of the 'decoding' algorithm is as follows: introducing standard functions, applying idealization and permutations of quantifiers, we reduce the statement to a form adopted for the transfer. In the long run, the transfer is reducing a formula to the form convenient for eliminating the external notion of standardness. It should be emphasized that in all cases of practical application of any statements **3.3.13**, the requirements mentioned above which ensure the legitimacy of their application, must be set beforehand.

3.3.15. The Nelson algorithm consists of the following steps:

(**1**) a statement of the nonstandard analysis is presented as a formula of IST, i.e., all the abbreviations are decoded;

(**2**) the obtained formula of IST is reduced to a normal prenex form

$$(Q_1x_1)\ldots(Q_nx_n)\ \varphi(x_1,\ldots,x_n),$$

where φ is a ZFC formula, while $Q_k := \forall \vee \exists \vee \forall^{\text{st}} \vee \exists^{\text{st}}$ for $k := 1,\ldots,n$;

(**3**) if Q_n is *an 'internal' quantifier*, i.e., $\forall$ or $\exists$, then we set $\varphi := (Q_nx_n)\ \varphi(x_1,\ldots,x_n)$ and go over to step (**2**);

(**4**) if Q_n is *an 'external' quantifier*, i.e., $\forall^{\text{st}}$ or $\exists^{\text{st}}$, then there can be found the first internal quantifier when looking through the quantifier prefix $(Q_1x_1)\ldots(Q_nx_n)$ from right to left;

(**5**) if there are no internal quantifiers in step (**4**), then, by virtue of **3.3.13** (3) and **3.3.13** (4), the quantifier Q_n is replaced by the corresponding internal quantifier, and then we go over to step (**2**) (i.e., going from right to left, we step by step 'erase' the super index $^{\text{st}}$ over each quantifier);

(**6**) let Q_m be the first internal quantifier encountered. Let us assume Q_{m+1} to be an external quantifier of the same type as Q_m (i.e., $Q_m = \forall$ and $Q_{m+1} = \forall^{\text{st}}$, or $Q_m = \exists$ and $Q_{m+1} = \exists^{\text{st}}$). Now we use rules **3.3.13** (5) and **3.3.13** (6) and then return to (**2**);

(**7**) if all the quantifiers $Q_{m+1},\ldots,Q_n$ are of the same type, then we apply the idealization principle in the form **3.3.13** (7) or **3.3.13** (8), and go over to (**2**);

(**8**) if the quantifiers alternate, i.e., Q_{p+1} is of the same type as Q_m, while all quantifiers $Q_{m+1},\ldots,Q_p$ are of the opposite type, then we can apply **3.3.13** (1) or **3.3.13** (2), under the assumption that $x := (x_{m+1},\ldots,x_p)$, $y := x_{p+1}$. Then we go over to (**2**).

3.3.16. One should bear in mind that the same statement can be expressed in different ways, including the form absolutely blocking any comprehension. In this respect, when applying the Nelson algorithm, one should take into account concrete possibilities of reducing the procedure of 'dragging external quantifiers out'. In particular, it is not always expedient to consider the formulas reduced to a normal prenex form from the onset (i.e. to carry out step **(2)** of the algorithm to its completion).

3.3.17. Examples

(1) *In nonstandard analysis the principle of external induction is valid, i.e., for an arbitrary formula* $\varphi \in$ (IST),

$$(\varphi(1) \wedge ((\forall n \in {}^{\circ}\mathbf{N}) \quad \varphi(n)\varphi(n+1))) \rightarrow (\forall n \in {}^{\circ}\mathbf{N}) \quad \varphi(\mathrm{n}).$$

$\lhd$ We cannot directly apply the Nelson algorithm to the formal presentation of the principle under study, since the formula φ may be external. Therefore, let us consider the standardization $A := {}^*\{n \in \mathbf{N}: \varphi(n)\}$. Obviously, we get $1 \in A$, and for every standard $n \in A$ we have $n+1 \in A$. The task is to prove that ${}^{\circ}\mathbf{N} \subset A$. Let us write out the required formula and apply the Nelson algorithm to it:

$$(1 \in A \wedge (\forall^{\mathrm{st}} n \in \mathbf{N})(n \in A \rightarrow (n+1) \in A)) \rightarrow {}^{\circ}\mathbf{N} \subset A$$
$$\leftrightarrow (\forall^{\mathrm{st}} m)\ (\forall^{\mathrm{st}} n)(m \in \mathbf{N} \wedge n \in \mathbf{N} \wedge 1 \in A \wedge n \in A \rightarrow (n+1) \in A)$$
$$\rightarrow m \in A \leftrightarrow (1 \in A \wedge (\forall n \in \mathbf{N})(n \in A \rightarrow (n+1) \in A)) \rightarrow \mathbf{N} \subset A. \rhd$$

(2) *The sum of infinitesimals is an infinitesimal.*

$\lhd$

$$(\forall s \in \mathbf{R})\ (\forall t \in \mathbf{R})\ s \approx 0 \wedge t \approx 0 \rightarrow s+t \approx 0$$
$$\leftrightarrow (\forall s \in \mathbf{R})(\forall t \in \mathbf{R})\ (s \approx 0 \wedge t \approx 0 \rightarrow (\forall^{\mathrm{st}} \varepsilon > 0)\,|s+t| < \varepsilon)$$
$$\leftrightarrow (\forall^{\mathrm{st}} \varepsilon > 0)(\forall s \in \mathbf{R})(\forall t \in \mathbf{R})\ ((\forall^{\mathrm{st}} \delta_1 > 0)$$
$$\wedge(\forall^{\mathrm{st}} \delta_2 > 0)\ (|s| < \delta_1 \wedge |t| < \delta_2 \rightarrow |s+t| < \varepsilon)$$
$$\leftrightarrow (\forall^{\mathrm{st}} \varepsilon > 0)(\forall s \in \mathbf{R})(\forall t \in \mathbf{R})\ (\exists^{\mathrm{st}} \delta_1 > 0)\ (\exists^{\mathrm{st}} \delta_2 > 0)(|s| < \delta_1$$
$$\wedge |t| < \delta_2 \rightarrow |s+t| < \varepsilon) \leftrightarrow (\forall^{\mathrm{st}} \varepsilon)(\forall s)(\forall t)(\exists^{\mathrm{st}} \delta_1)(\exists^{\mathrm{st}} \delta_2)(\varepsilon > 0$$
$$\ldots \wedge \delta_2 > 0 \wedge |s| < \delta_1 \wedge |t| < \delta_2 \rightarrow |s+t| < \varepsilon)$$
$$\leftrightarrow (\forall^{\mathrm{st}} \varepsilon)(\forall s)(\forall t)(\exists^{\mathrm{st}} \delta_1)(\exists^{\mathrm{st}} \delta_2)(|s| < \delta_1 \wedge |t| < \delta_2 \rightarrow |s+t| < \varepsilon)$$
$$\leftrightarrow (\forall^{\mathrm{st}} \varepsilon)(\exists^{\mathrm{stfin}} \Delta_1)(\exists^{\mathrm{stfin}} \Delta_2)(\forall s)(\forall t)(\exists \delta_1 \in \Delta_1)(\exists \delta_2 \in \Delta_2)$$
$$(|s| < \delta_1 \wedge |t| < \delta_2 \rightarrow |s+t| < \varepsilon)$$
$$\leftrightarrow (\forall^{\mathrm{st}} \varepsilon)(\exists^{\mathrm{st}} \delta_1)\ (\exists^{\mathrm{st}} \delta_2)(\forall |s| < \delta_1)(\forall |t| < \delta_2)\ |s+t| \leq \varepsilon$$
$$\leftrightarrow (\forall \varepsilon > 0)(\exists \delta > 0)(\forall |s| < \delta)(\forall |t| < \delta)\ |s+t| \leq \varepsilon.\ \rhd$$

(3) Robinson lemma. *Let* a_n *be an internal sequence of numbers, and* $a_n \approx 0$ *for all* $n \in {}^{\circ}\mathbf{N}$. *Then there is an* $N \approx +\infty$, *such that* $a_n \approx 0$ *for any* $n \leq N$.

$\lhd$ Let us apply the Nelson algorithm to the required conclusion:

$$(\exists N \approx +\infty)(\forall n \leq N)\ a_n \approx o \leftrightarrow$$

$$\leftrightarrow (\exists N \in \mathbf{N})((\forall^{st} m \in \mathbf{N})\ N \geq m) \wedge (\forall n \in \mathbf{N})(n \leq N$$
$$\rightarrow (\forall^{st} \varepsilon > 0)\,|a_n| < \varepsilon) \leftrightarrow (\exists N)(\forall^{st} m)(\forall^{st} \varepsilon)$$
$$(\forall n)(N \geq m \wedge (n < N \rightarrow |a_n| < \varepsilon_k))$$
$$\leftrightarrow (\forall^{st}\{m_1,\dots,m_p\})(\forall^{st}\{\varepsilon_1,\dots,\varepsilon_p\})(\exists N)(\forall k = 1,\dots,p)$$
$$N \geq m_k \wedge n \leq N \rightarrow |a_n| < \varepsilon_k)$$
$$\leftrightarrow (\forall^{st} m)\ (\forall^{st} \varepsilon)(\exists N)\ (N \geq m \wedge (n \leq N \rightarrow |a_n| < \varepsilon)) \leftrightarrow$$
$$\leftrightarrow (\forall^{st} m)\ (\forall^{st} \varepsilon)(m \in \mathbf{N} \wedge \varepsilon > 0 \rightarrow |a_m| < \varepsilon).$$

Let us now apply the Nelson algorithm to the premises of the statement in question:

$$(\forall n \in {}^{\circ}\mathbf{N})\ a_n \approx 0 \leftrightarrow (\forall^{st} n)(n \in \mathbf{N} \rightarrow (\forall^{st} \varepsilon > 0)\,|a_n| < \varepsilon)$$
$$\leftrightarrow (\forall^{st} n)\ (\forall^{st} \varepsilon)(n \in \mathbf{N} \wedge \varepsilon > 0 \rightarrow |a_n| < \varepsilon).$$

Therefore, both the premises and the conclusion are equivalent. ▷

3.4. External Set Theories

The basic statements of nonstandard analysis are adequately reflected in the formal apparatus of Nelson internal set theory. The Powell theorem makes it possible to view IST as a technique of studying the von Neumann universe. At the same time, the presence of external objects completely undermines the widely-spread opinion that Zermelo-Fraenkel formalism ensures a sufficient operational freedom from the viewpoint of the naive set theory. Remaining within the approach of IST, we cannot ask even such, for instance, a question as: 'Is it possible to single out such numbers that every element of **R** could have a one-to-one presentation in the form of their certain combination with standard coefficients, since **R** can be considered a vector space over $^{\circ}\mathbf{R}$?' The quantity of such inadmissible questions, undoubtedly mathematical from the semantic point of view, is great to the extent that the necessity of extending the limits of IST becomes vitally important. The *a priori* prohibition against formulating problems is nothing but imposing arbitrary restrictions on the reason. The introduction of the *ad hoc* dogma, "the explicitly expressed prohibition against thinking" (as was aphoristically remarked by L.Feuerbach) is the way *a afortiori* unacceptable when searching for the truth. A practical solution of the problem of returning to the 'Cantor paradise' is, in particular, in finding a formalism which enables one to work with external, as regards the von Neumann universe sets, with conventional mathematical means. Now we are going to get acquainted with axiomatic approximations to studying external sets. The first variation of the appropriate formalism, EXT, was suggested by K.Hrbacek. A close variation, NST, was constructed later by T.Kawai. The nonstandard set theories mentioned above demonstrate, semantically speaking, the universe of external sets to be, from the viewpoint of a mathematical pragmatist-Philistine, constructed in the same way as the universe of naive sets. In other words, it allows classical set-theoretic operations, including singling out subsets using the properties (the axiom of comprehension) and putting arbitrary sets into complete order (the choice axiom). At the same time, among external sets there is a whole set of standard and nonstandard internal sets, which obey variations of the principles of transfer, idealization and standardization close to their intuitive formulations. In more strict terms, one may say that internal sets are included into the number of external ones by the definition.

From the standpoint of practical requirements of conventional (standard and nonstandard) mathematical analysis, both EXT and NST furnish us with practically equal means which are more than enough for a substantiated use of common analytical constructions. It is, however, necessary to go through details of the axiomatics of the external set theory under discussion, armed with attention and a certain degree of criticism, in order to avoid illusions pertaining to the euphoria of omnipotence. For instance, it should be emphasized that the universe of external sets is not the von Neumann universe (the axiom of foundation is absent, which is of importance). Moreover, the exact formulations of the principles of nonstandard analysis in EXT have technical differences from their analogs in IST. Therefore, EXT is not an extension of Nelson's IST, though EXT is a conservative extension of ZFC. This gap was filled by T.Kawai, whose NST enriches the formal apparatus of IST and, alongside with IST and EXT, provides a reliable technique for studying ZFC.

3.4.1. The alphabet of formal EXT is obtained by adjoining to the of alphabet IST the only new symbol, the symbol of a unary predicate, Int, which expresses the property of being an internal set. In other words, allowable for consideration are texts containing records of the type $\mathrm{Int}(x)$, or, in more detail, "x is internal', and finally, 'x is an internal set'. It is intuitively considered that the contextual domain of changing the variables of EXT is *the universe of all external sets* $\mathbf{V}^{\mathrm{Ext}} := \{x : x = x\}$, which contains both *the universe of standard sets* $\mathbf{V}^{\mathrm{St}} := \{x \in \mathbf{V}^{\mathrm{Ext}} : \mathrm{St}(x)\}$ and the *universe of internal sets* $\mathbf{V}^{\mathrm{Int}} := \{x \in \mathbf{V}^{\mathrm{Ext}} : \mathrm{Int}(x)\}$ that includes the universe $\mathbf{V}^{\mathrm{St}}$.

3.4.2. The conventions of EXT are analogous to those of ZFC and IST. In particular, we, by all means, are going to use in EXT the 'classifiers', i.e., braces (see **3.3.3**) and conventional symbols for denoting the simplest operations with classes of external sets. Following the previous samples, for a formula φ of EXT (or, symbolically, $\varphi \in$(EXT)) we shall write:

$$(\forall^{\mathrm{st}})\,\varphi := (\forall x) \quad (\mathrm{St}(x) \rightarrow \varphi) := (\forall x \in \mathbf{V}^{\mathrm{St}})\,\varphi,$$

$$(\exists^{\mathrm{Int}})\ \varphi := (\exists x)\ \ (\mathrm{Int}(x) \wedge \varphi) := (\exists x \in \mathbf{V}^{\mathrm{Int}})\ \varphi .$$

Rules of the kind, i.e., understandable from the context, will be further on used without further specifications. Besides, we will need a special new notion and a corresponding notation. We shall say that an external set A has *standard size* (symbolically, $A \in V^{size}$), if there is a standard set a and an external function f such that $(\forall X)\,(X \in A \leftrightarrow (\exists^{st} x \in a)\ X = f(x))$.

3.4.3. Let $\varphi \in$ (ZFC) be a formula of EXT which is also a formula of ZFC (i.e., φ contains no symbols St and Int). Let us replace every quantifier Q in the presentation of φ by Q^{st}. The formula obtained is denoted by φ^{St} and is termed *the standardization* of φ, or *the relativization* of φ to $\mathbf{V}^{\mathrm{St}}$. Analogously, replacing every quantifier Q with Q^{Int}, we obtain the formula φ^{Int} which is termed the *internalization* of φ or *relativization* of φ to $\mathbf{V}^{\mathrm{Int}}$. It should be emphasized that in this case nothing happens to free variables in φ. This rule is also valid for abbreviations. For instance, for external sets A and B we write:

$$A \subset^{\mathrm{St}} B := (\forall^{st} x)\,(x \in A \rightarrow x \in B)$$
$$:= ((\forall x)\,(x \in A \rightarrow x \in B))^{\mathrm{St}} := (A \subset B)^{\mathrm{St}};$$
$$A \in^{\mathrm{Int}} B := (A \in B)^{\mathrm{Int}} := A \in B := A \in^{\mathrm{St}} B := (A \in B)^{\mathrm{St}}.$$

3.4.4. Special axioms of EXT fall into three groups. The first group is *the rules of the formation of external sets*, the second group is *the axioms interrelating the universes* of sets $\mathbf{V}^{\mathrm{St}}$, $\mathbf{V}^{\mathrm{Int}}$ and $\mathbf{V}^{\mathrm{Ext}}$, and, finally, the third group is formed by *the principles of transfer, idealization and standardization.*

3.4.5. In EXT valid are the laws of *Zermelo set theory* (the theory Z), i.e., the following axioms of constructing external sets are accepted:

(1) the axiom of extensionality:

$$(\forall A)(\forall B)\,(A \subset B \wedge B \subset A) \leftrightarrow A = B;$$

(2) the axiom of pairing:

$$(\forall A)(\forall B)\,\{A, B\} \in \mathrm{V}^{\mathrm{Ext}};$$

(3) the axiom of union:

$$(\forall A)\ \cup A \in \mathrm{V}^{\mathrm{Ext}};$$

(4) the axiom of powersets:

$$(\forall A)\ \mathcal{P}(A) \in \mathrm{V}^{\mathrm{Ext}};$$

(5) the axiom schema of comprehension:

$$(\forall A)\,(\forall X_1)\ldots(\forall X_n)\{X \in A\colon \varphi(X, X_1,\ldots, X_n)\} \in \mathbf{V}^{\mathrm{Ext}}$$

for an arbitrary formula $\varphi \in$(EXT);

(6) the axiom of well-ordering: *every external set can be well-ordered.*

The last property, the Zermelo theorem, ensures, as is known (cf. (**3.2.10**)), the axiom of choice either in the conventional multiplicative form or in the form of the Kuratowski-Zorn lemma. It should be also remarked here that the axioms of Z commonly include the axiom of infinity, which in EXT will appear below.

3.4.6. The second group of the axioms of EXT contains the following statements:

(1) the modelling principle: *the universe of internal sets* $\mathbf{V}^{\mathrm{Int}}$ *is the von Neumann universe*, i.e., for every φ axiom of Zermelo-Fraenkel theory the internalization φ^{Int} is an axiom of EXT;

(2) the axiom of transitiveness:

$$(\forall x \in \mathbf{V}^{\mathrm{Int}})\ x \subset \mathbf{V}^{\mathrm{Int}};$$

i.e., internal sets are composed of only internal elements;

(3) the axiom of embedding:

$$\mathbf{V}^{\mathrm{St}} \subset \mathbf{V}^{\mathrm{Int}},$$

i.e., standard sets are internal.

3.4.7. The third group of the axioms of EXT includes the following statements:

(1) the transfer principle:

$$(\forall^{\mathrm{st}} x_1)\ldots(\forall^{\mathrm{st}} x_n)\ \varphi^{\mathrm{St}}(x_1,\ldots,x_n) \leftrightarrow \varphi^{\mathrm{Int}}(x_1,\ldots,x_n)$$

for every formula $\varphi \in$ (ZFC);

(2) the idealization principle:

$$(\forall^{\mathrm{Int}} x_1)\ldots(\forall^{\mathrm{Int}} x_n)(\forall A \in V^{size})(((\forall^{\mathrm{fin}} z)\ z \subset A$$
$$\to (\exists^{\mathrm{Int}} x)(\forall y \in z)\ \varphi^{\mathrm{Int}}(x,y,x_1,\ldots x_n))$$
$$\to (\exists^{\mathrm{Int}} x)(\forall^{\mathrm{Int}} y \in A)\ \varphi^{\mathrm{Int}}(x,y,x_1,\ldots x_n))$$

for an arbitrary $\varphi \in$ (ZFC);

(3) the standardization principle:

$$(\forall A)(\exists^{\mathrm{st}} a)(\forall^{\mathrm{st}} x)\ (x \in A \leftrightarrow x \in a),$$

i.e., for any external set A there is its standardization $^{*}A$.

3.4.8. A simplest useful corollary to the above axioms worth mentioning is that *the bounded formulas* ZFC *are absolute.* To be more precise, for $\varphi \in (\Sigma_0)$ we get

$$(\forall^{\mathrm{Int}} x_1)\ldots(\forall^{\mathrm{Int}} x_n)\ \varphi(x_1,\ldots x_n) \leftrightarrow \varphi^{\mathrm{Int}}(x_1,\ldots x_n),$$
$$(\forall^{\mathrm{st}} x_1)\ldots(\forall^{\mathrm{st}} x_n)\ \varphi^{\mathrm{St}}(x_1,\ldots x_n) \leftrightarrow \varphi^{\mathrm{Int}}(x_1,\ldots x_n) \leftrightarrow \varphi(x_1,\ldots x_n).$$

Hence, any 'bounded' property of standard sets can be safely expressed both in terms of external and internal or standard elements. For instance, $x \subset y \leftrightarrow x \subset^{\mathrm{St}} y \leftrightarrow x \subset^{\mathrm{Int}} y$ for standard sets x and y.

3.4.9. Hrbacek theorem. *EXT is conservative over* ZFC, *i.e., for every* $\varphi \in$ (ZFC) *we have*

(φ *is a theorem of* ZFC) $\leftrightarrow$ (φ^{Int} *is a theorem of* EXT) $\leftrightarrow$ (φ^{St} *is a theorem of* EXT).

$\lhd$ The proof of this theorem can be found in [85].

3.4.10. When thinking over the axiomatics presented above, it would be first of all useful to realize that EXT is not an extension of IST. In other words, the universe of internal sets $\mathbf{V}^{\mathrm{Int}}$ is not a model of the Nelson internal set theory, since the idealization and standardization principles in these theories are formulated in a different way. In the universe $\mathbf{V}^{\mathrm{Int}}$ standardization is allowed under essentially less strict propositions than that in IST. Thus, for any $\varphi \in (\mathrm{IST})$ and an arbitrary $A \in \mathbf{V}^{\mathrm{Int}}$ we can organize $^*\{x \in A\colon \varphi(x)\}$ since $\{x \in A\colon \varphi(x)\}$ is an external subset of A. In this case in EST an additional requirement that A be standard is, generally speaking, necessary since in IST one cannot standardize a set that contains all standard elements. In EXT, in turn, the union of all standard elements $\mathbf{V}^{\mathrm{St}}$ is not included in an external (to say nothing of an internal) set at all. Then, indeed, the following statement is valid.

3.4.11. *There is no external set, a element of* $\mathbf{V}^{\mathrm{Ext}}$, *which contains each standard set.*

◁ Let us, on the contrary, assume that for a certain $X \in \mathbf{V}^{\mathrm{Ext}}$ we have $\mathbf{V}^{\mathrm{St}} \subset X$. By virtue of the axiom of comprehension of **3.4.5** (5), for the formula $\varphi(x) = \mathrm{St}(x)$ we conclude that $\mathbf{V}^{\mathrm{St}}$ is an external set, i.e., $(\exists Y)(\forall Z)\, CZ \in Y \leftrightarrow \mathrm{St}(Z)$. If we consider the standardization $^*\mathbf{V}^{\mathrm{St}}$, it appears to be a standard finite set containing every standard set. The last is, obviously, impossible. ▷

3.4.12. The above proposition shows the idealization principle in EXT ('relativized' to $\mathbf{V}^{\mathrm{Int}}$) to be different from its analogue in IST not only in the form but also in the essence. At the same time, these differences should not be absolutized. The following facts will help to elucidate this statement.

3.4.13. *The following statements are valid:*

(1) *external natural numbers and standard natural numbers coincide*;

(2) *a finite external set is standard iff it consists of only standard elements*;

(3) *for an arbitrary external set A its standard core* $^\circ A := \{a \in A\colon \mathrm{St}(a)\}$ *is a set of standard size*;

(4) *every infinite internal set contains a nonstandard element.*

◁ (1) By the principle of induction on standard natural numbers (which is, obviously, valid in EXT (cf. **2.2.2** (1))), for a set $\mathbf{N}^{\mathrm{Ext}}$ of external natural numbers we have $\mathbf{N}^{\mathrm{Ext}} \supset {}^{\circ}\mathbf{N}$. Moreover, it is clear that ${}^{*}\varnothing = \varnothing$ and ${}^{*}1 = {}^{*}\{\varnothing\} = \{\varnothing\} = 1$. Hence, by virtue of induction on external natural numbers (a conventional theorem of Z), we get $\mathbf{N}^{\mathrm{Ext}} \supset {}^{\circ}\mathbf{N}$, and, finally, ${}^{\circ}\mathbf{N} = \mathbf{N}^{\mathrm{Ext}}$.

(2) A standard set is internal. Hence, making use of **3.4.6** (2), we can use the argumentation of the proof of **2.2.2** (3). According to **2.2.2** (2), a finite set composed of standard elements is standard.

(3) Let ${}^{*}A$ be the standardization of A. Let us set $f(a) := a$ for $a \in {}^{\circ}A$. Obviously, $(\forall X)(X \in {}^{\circ}A \leftrightarrow (\exists^{\mathrm{st}} x \in {}^{*}A)\, f(x) = X)$.

(4) Let us denote the internal set under discussion by A. By virtue of (3), ${}^{\circ}A$ is of standard size. Hence, we can apply the idealization principle for $\varphi(x,y) := y \neq x \wedge x \in A$. Since A is infinite, for every finite $z \subset {}^{\circ}A$ we, obviously, get $(\exists x \in A)(\forall y \in z)\ x \neq y$. And, finally, $(\exists x \in A)(\forall y \in {}^{\circ}A)\ x \neq y$. ▷

3.4.14. As regards **3.4.13** and **3.4.9**, it would be convenient a variation of INT which is a conservative extension of ZFC and such that EXT, in turn, is an extension of INT. The difference between INT and EXT in adopting the idealization and standardization principles is as follows:

$$(1)\ (\forall A)(\forall x_1)\ldots(\forall x_n)((\forall^{\mathrm{stfin}} z)\ z \subset A(\exists x)(\forall y \in z)\ \varphi(x,y,x_1,\ldots,x_n) \leftrightarrow (\exists x)(\forall^{\mathrm{st}} y \in A)\ \varphi(x,y,x_1,\ldots,x_n))$$

for every $\varphi \in$ (ZFC);

$$(2)\ (\forall A)(\exists^{\mathrm{st}}\ {}^{*}A)(\forall^{\mathrm{st}} x)\ (x \in A \leftrightarrow x \in {}^{*}A \wedge \varphi(x))$$

for an arbitrary $\varphi \in$ (INT).

It should be remarked that the Nelson algorithm is, in its essential features, operative in INT.

3.4.15. Let us now go over to the description of NST in a variation most close to EXT and IST (in fact, T.Kawai has constructed a somewhat different scheme, enabling one to consider the classes of the von Neumann-Gödel-Bernays theory as external sets).

3.4.16. The alphabet and conventions in formal NST coincide with those of EXT. Moreover, NST adopts all the axioms of constructing external sets, all the axioms interrelating the universes of sets, and the transfer principle of EXT. The differences between NST and EXT are in the ways of formulating the idealization and standardization principles, as well in the following supplementary postulate.

3.4.17. The axiom of acceptability, $\mathbf{V}^{\mathrm{St}} \in \mathbf{V}^{\mathrm{Ext}}$, *i.e., the universe of standard sets of the Kawai theory is an external set.*

In line with the axiom formulated, in NST the external set A is called *a set of appropriate size*, and we write $A \in \mathbf{V}^{\mathrm{a\text{-}size}}$ provided there is an external function f mapping $\mathbf{V}^{\mathrm{St}}$ on A. It should be emphasized that $\mathbf{V}^{\mathrm{St}}$ is of appropriate size. It should be also remarked that in the sequel the presentation $\mathrm{a\text{-}fin}(A)$ implies that there is a one-to-one external A mapping on a standard finite set.

3.4.18. The standardization principle in NST is as follows:

$$(\forall A)((\exists^{\mathrm{st}} X)\ A \subset X \rightarrow (\exists^{\mathrm{st}}\ {}^*\!A)(\forall^{\mathrm{st}} x)\ (x \in A \leftrightarrow x \in {}^*\!A)).$$

In other words, in NST only external subsets of standard sets can be standardized, not arbitrary external subsets, as is the case in EXT.

3.4.19. The idealization principle in NST is as follows:

$$\begin{aligned}&(\forall^{\mathrm{Int}} x_1)\dots(\forall^{\mathrm{Int}} x_n)(\forall A \in \mathbf{V}^{\mathrm{a\text{-}size}})\ (((\forall z)\ z \subset A \wedge a = \mathrm{fin}(z)\\ &\rightarrow (\exists^{\mathrm{Int}} x)(\forall y \in z)\ \varphi^{\mathrm{Int}}(x, y, x_1, \dots, x_n)\\ &\rightarrow (\exists^{\mathrm{Int}} x)(\forall^{\mathrm{Int}} y \in A)\ \varphi^{\mathrm{Int}}(x, y, x_1, \dots, x_n))\end{aligned}$$

for an arbitrary formula $\varphi \in (\mathrm{ZFC})$.

3.4.20. Kawai theorem. NST *is a conservative extension of* ZFC.

◁ The proof can be found in [104]. ▷

3.4.21. Let us again pay attention to the fact that the world of the internal sets V^{Int} in the universe of NST with the standardization, idealization and transfer principles relativized, is a model for IST. In other words, the technical means employed by NST to work with external sets arising in IST can be surely used to obtain the statements of 'standard' mathematics. It should be also remarked that the proof of the Kawai theorem, as well as that of the Hrbacek and Powell theorems is essentially based on the application of appropriate analogues of the Mal'tsev local theorem, or, to be more exact, on the technique of ultraproducts and ultralimits. A detailed presentation of the aforementioned apparatus is beyond the scope of the present book, the second part of this work, however, dealing with nonstandard methods of analysis which consider every possibility of understanding and constructing the proofs of the theorem in question.

3.4.22. Taking liberties with notations, we denote the universe of external sets by $\mathbf{V}^E$ (irrespective of whether NST or EXT is in question). Analogously, let us use the notation $\mathbf{V}^I$ (and, respectively, $\mathbf{V}^S$) to denote the world of internal (and, respectively, standard) sets. Repeating the scheme of constructing the von Neumann universe, i.e., iterating step-by-step the operations of taking unions and powersets of all external subsets of a given set, we see that the empty set can give rise to the world $\mathbf{V}^C$, i.e., *the universe of 'classical' sets.* In more detail, we set

$$V_\beta^C := \{x : (\exists^{\mathrm{st}} \alpha \in \beta) \quad x \in \mathcal{P}^{\mathrm{Ext}}(V_\alpha^C)\},$$
$$\mathbf{V}^C := \bigcup_{\beta \in \mathrm{On}^{\mathrm{St}}} V_\beta^C,$$

where $\mathrm{On}^{\mathrm{St}}$ is the class of all standard ordinals. Therefore, an empty set is 'classical', and every 'classical' set is composed of 'classical' elements' only.

3.4.23. Using recursion, i.e., a walk about the stores of the universe of 'classical' sets, one can determine the Robinson standardization or the $*$-map.

A standard set $*A$ is called *the Robinson standardization* or *the $*$-image* of a 'classical' set A iff every standard element of $*A$ is a $*$-image of a certain element of A. Or, symbolically.

$$*\varnothing := \varnothing, \quad *A := {}^*\{*a : a \in A\}.$$

It should be remarked that within EXT the legitimacy of applying the conventional standardization gives rise to no ambiguity. In NST the admissibility of employing this operation in determining the Robinson standardization results from the method of constructing $\mathbf{V}^C$. Analogous considerations (cf. **3.2.12**) prove that that the $*$-map identifies, and does it in a one-to-one manner, the worlds $\mathbf{V}^C$ and $\mathbf{V}^S$. Moreover, the Robinson standardization ensures the validity of the *transfer principle*

$$(\forall A_1 \in \mathbf{V}^C)\ldots(\forall A_n \in \mathbf{V}^C)\ \varphi^C(A_1,\ldots,A_n) \leftrightarrow \varphi^S({*A_1},\ldots,{*A_n})$$

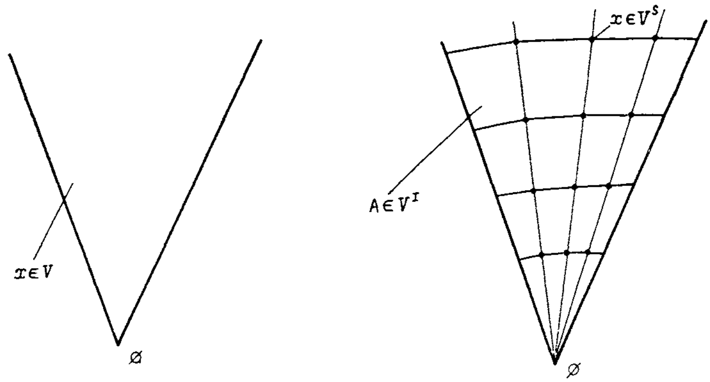
for an arbitrary formula φ of Zermelo-Fraenkel theory (as usual, φ^C and φ^S are φ relativizations on $\mathbf{V}^C$ and $\mathbf{V}^S$, respectively).

3.5. Set-Theoretic Stances on Nonstandard Analysis

The considerations carried out in the preceding paragraphs have enriched and extended the initial naive ideas of a set used in nonstandard analysis. We have passed from the conventional von Neumann universe, $\mathbf{V}$, to the world $\mathbf{V}^I$ of internal set theory with its points of reference, standard sets, comprising the class $\mathbf{V}^S$ (Fig. 4). Further analysis has shown $\mathbf{V}^I$ to belong to a new class, the universe $\mathbf{V}^E$ of external sets (comprising the Zermelo world). In $\mathbf{V}^E$ we have singled out the universe of 'classical' sets $\mathbf{V}^C$, which is another implementation of the world of standard sets, $\mathbf{V}^S$. To be more exact, we mean the Robinson $*$-standardization which identifies $\mathbf{V}^C$ and $\mathbf{V}^S$ in the elementwise manner. In this case, by virtue of the transfer principles, $\mathbf{V}^C$, $\mathbf{V}^S$ and $\mathbf{V}^I$ can be viewed as 'hypostases' of the von Neumann universe, $\mathbf{V}$ (Fig. 5).

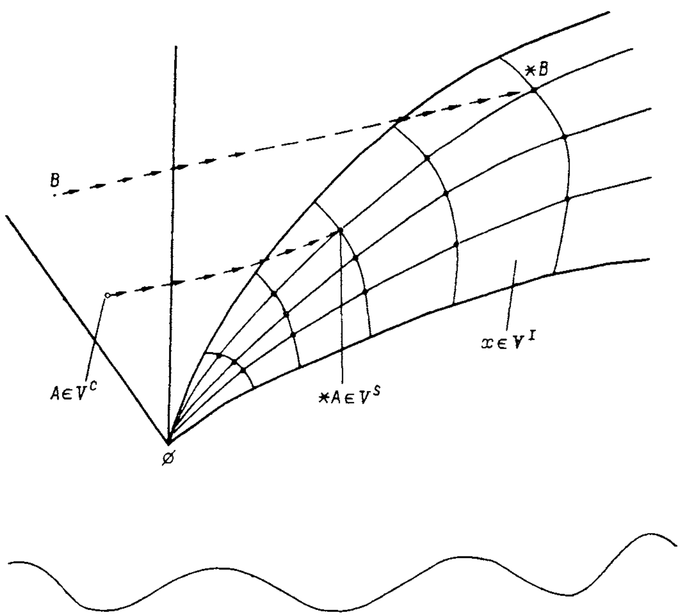
3.5.1. The presented picture of the location as well as the other known interrelations between the worlds $\mathbf{V}^E$, $\mathbf{V}^I$, $\mathbf{V}^S$ and $\mathbf{V}^C$ result in three general set-theoretic stances on nonstandard analysis. These stances, which are called classical, neoclassical and radical, fix the presentation of the subject and the means of its investigation. The acceptance of this or that stance determines, in particular, the way of presenting the mathematical results obtained with nonstandard methods. Therefore, acquaintance with the aforementioned stances are to be considered a must.

3.5.2. The classical stance on nonstandard analysis corresponds to the methods employed by its founder A.Robinson, the corresponding formalism being most widely-spread nowadays. Under this stance *the world of classical mathematics identified with the universe of 'classical' sets,* $\mathbf{V}^C$, *is declared to be the principal object to be studied.* The latter is considered to be 'a standard universe' (in practice, however, the object of investigation is the so-called 'superstructure', i.e., a sufficiently large fragment of $\mathbf{V}^C$ containing all necessary objects of study). As the technique of investigating the initial, i.e., standard, universe, viewed are the 'nonstandard universe' of internal sets, $\mathbf{V}^I$ (or its appropriate part) and the $*$-map, pasting conventional standard objects to their images in



Von Neumann universe   Universe of internal sets

Fig. 4



Universe of external sets

Fig. 5

the 'nonstandard universe'. It would be expedient to note a peculiar use of the words 'standard' and 'nonstandard' in the approach under discussion. Robinson standardizations, i.e. elements of the universe $\mathbf{V}^S$, are viewed as 'nonstandard' objects. A 'standard' set which is, by definition, an arbitrary representative of the world of 'classical' sets $\mathbf{V}^C$, is a member of the 'standard universe'. The $*$-map adds, as a rule, new 'ideal' elements into the set, which implies that $*A = \{*a\colon a \in A\}$ iff a 'classical', i.e., 'standard', set A is finite. For instance, putting $\mathbf{R}$ in $\mathbf{V}^C$ and studying its $*$-image $*\mathbf{R}$ in accordance with what have been discussed above, we see that $*\mathbf{R}$ plays the role of the field of real numbers in the sense of the universe of internal sets, i.e., 'in the inner sense of the nonstandard universe'. At the same time, $*\mathbf{R}$ is not reduced to the set of its standard elements, $^\circ(*\mathbf{R}) = \{*t\colon t \in \mathbf{R}\}$. Making use of the fact that $*\mathbf{R}$ is the 'internal set of real numbers $\mathbf{R}$', while $^\circ(*\mathbf{R})$ is its standard core, it would be a certain excessive liberty to assume $^\circ\mathbf{R} := \{*t\colon t \in \mathbf{R}\}$, or even $\mathbf{R} := \{*t\colon t \in \mathbf{R}\}$. The presence of 'new' elements in $*\mathbf{R}$ is expressed by the symbol $*\mathbf{R} - \mathbf{R} \neq \varnothing$, and we say about the construction of the system of 'hyperreal' numbers $*\mathbf{R}$ which extends the conventional field of real numbers $\mathbf{R}$. An analogous policy is pursued when considering an arbitrary classical set X. Namely, it is assumed that $X = \{*x\colon x \in X\}$ and, hence, $X \subset *X$. If X is infinite, then $*X - X \neq \varnothing$. In other words, through the Robinson standardization all infinite sets are saturated with new elements. Moreover, a large quantity of 'ideal' objects is added, since in $\mathbf{V}^I$ operates the idealization principle which is often called *the technique of concurrence and saturation* within the stance under consideration.

3.5.3. Let U be an arbitrary correspondence, and A and B be sets. We say that *U is concurrent from A to B* if for every nonempty finite subset A_0 in A there is an element $b \in B$ such that $(a_0, b) \in U$ for all $a_0 \in A_0$.

3.5.4. Weak concurrence principle. *For any correspondence U concurrent from A to B there is an element $b \in *B$, maintaining the relation $(*a, b) \in *U$ for every $a \in A$.*

3.5.5. It is obvious that the validity of the concurrence principle ensures, in turn, a natural equivalent of the weak idealization principle, i.e., that which is 'relativized for standard sets'. In this respect in applications distinguished are conservative extensions of the classical set theory using both the possibility of weak idealization mentioned earlier and conventional formulations ensuring additional possibilities of introducing nonstandard elements which are more adequate to the contents of the idealization principle in its complete expression.

3.5.6. Strong concurrence principle. *Let a correspondence U be such that $^*U$ is directed from A to $^*B$. Then there is an element $b \in {^*B}$ for which $(^*a, b) \in {^*U}$ for all $a \in A$.*

3.5.7. Saturation principle. *Let $A_1 \supset A_2 \supset \ldots$ be a decreasingly nested sequence of nonempty internal sets. Then $\bigcap_{n \in \mathrm{N}} A_n \neq \varnothing$.*

3.5.8. It would be expedient to recall that in the 'extended', 'nonstandard' world, i.e., in the universe of internal sets $\mathbf{V}^I$, there operates the transfer principle. Thus, making use of the Robinson standardization we may write

$$(\forall x_1 \in \mathbf{V}^C) \ldots (\forall x_n \in \mathbf{V}^C) \quad \varphi^C(x_1, \ldots, x_n) \leftrightarrow \varphi^I(^*x_1, \ldots, x_n)$$

for every formula φ of Zermelo-Fraenkel set theory. The transfer principle in such a form is often referred to as *the Leibniz principle.*

3.5.9. When working with the 'nonstandard universe', special use is sometimes made of the *'technique of internal sets'*, i.e., the way of proving based on the fact that external sets defined in the 'set-theoretic fashion' are internal. Here is one of the possibilities of applying this technique.

3.5.10. *Let A be an infinite set. For every set-theoretic property of φ the following statement is invalid:*

$$\{x\colon \varphi^I(x)\} = {^*A} - A.$$

$\lhd$ Let us assume that the opposite statement is valid. Then the class $\{x\colon \varphi^I(x)\}$ is an internal set $^*A$. Hence, A is an internal set. For a finite A, however, the external set $^*A - A$ is not internal. $\rhd$

3.5.11. Summarizing, one can say that under the classical stance two universes are being worked with, i.e., standard and nonstandard. There are formal possibilities of interrelating the properties of standard and nonstandard objects by the procedure of 'starring', i.e., using a $*$-map. In this case we enjoy the right of freely transferring the statements on objects from one world to another through the action of the Leibniz principle. The nonstandard world is abundant with ideal elements, and all possible transfinite constructions are actually implementable there since valid is the concurrence principle. The sets falling beyond the limits of the nonstandard universe are called external (here the peculiarity of the assumed terminology is revealing itself: internal sets are not external under the approach in question). The technique of internal sets is an expedient method of investigation.

The principal merit of the classical stance is the presence of the *-map, which makes it possible to apply the apparatus of nonstandard analysis to common arbitrary sets. For instance, one can claim that the function $f:[a,b] \to \mathbf{R}$ is uniformly continuous iff $^*f:{}^*[a,b] \to {}^*\mathbf{R}$ is microcontinuous, i.e., if $^*f$ does not loose infinite proximity of hyperreal numbers. The basic difficulty hampering the way to understanding the phenomena under discussion is related with the necessity to imagine a colossal host of new ideal objects joining common sets. Noticeable complications result from a natural desire to work (at least at initial stages) with two sets of variables pertaining, respectively, to the standard and nonstandard universes. (When constructing the internalization φ^I of the formula φ we, in fact, presuppose such a procedure). To put it short, *the bilingual basis and Robinson standardization that are indispensable attributes of the classical stance* determine all the peculiarities of the latter, as well as the merits and demerits of its apparatus.

3.5.12. The neoclassical stance on nonstandard analysis corresponds to the technique suggested by E.Nelson. Under this stance *the principal object of investigation is the world of mathematics viewed as a universe* $\mathbf{V}^I$ *lying in the medium of external sets that are elements of* $\mathbf{V}^E$. 'Classical' sets are not analysed separately. Standard and nonstandard elements are given in conventional mathematical objects comprising $\mathbf{V}^I$. For instance, as the field of real numbers used is $\mathbf{R}$ from the world $\mathbf{V}^I$ which, obviously, coincides with the field $^*\mathbf{R}$ of hyperreal numbers which is, in its turn, an 'ideal' object of the classical stance. The statements considered in Chapter **2** correspond to the neoclassical stance. Its basic advantage is the possibility of studying sets which are already well-known and to find new features in their construction using additional language means. As has been remarked by E.Nelson, "... really new in nonstandard analysis are not theorems or proofs but the notions, i.e., external predicates ..." [188, p. 134]. The drawbacks of the neoclassical stance are caused by necessity to implicitly transfer the definitions and properties from standard objects to internal ones, which peculiarity we have already observed.

3.5.13. The radical stance on nonstandard analysis is that *the object of mathematical investigation is the universe of external sets* in all the completeness and complexity of its structure. Under the radical approach the classical and neoclassical ideas of nonstandard analysis as of a technique for studying mathematics (based on the Zermelo-Fraenkel formalism) are declared to be 'narrow', 'shy' and brushed away. At first sight this approach seems to be very radical and not serious. Upon proper thinking, however, the ideas about the extremity of the radical stance on nonstandard analysis should be abandoned. This 'extremity' is just a seeming one. Both a widely-spread viewing mathematics as a science about forms and relations taken irrespective of their contents, and even less imposing classical set-theoretic stance stemming from G.Cantor, embrace, undoubtedly, the 'extreme' thoughts about the subject of the nonstandard analysis. Therefore, the most 'intrepid' views on sets arising as a result of rather laborious investigations have finally become a comprising part of the initial point, having enriched it with new contents. And the initial point for us was a modest statement that nonstandard analysis operates with exactly the same sets as the whole of mathematics (see **2.1.3**).

It would be expedient to recall here the aphoristic observations, made by V.I.Lenin, which pertain to the dynamics of the cognition process:

"Every shade of a thought is a circle on the great circle (spiral) of the development of the human thought in general" [154, p. 221].

"Cognition of a man is not (not goes along) a straight line, but a curve infinitely approaching a number of circles, a spiral" [154, p.322].

CHAPTER 4

MONADS IN GENERAL TOPOLOGY

Within the set-theoretic stance on mathematics, at the beginning of the XX century a universal approach was developed to study the structure of continuity and proximity which was formulated in general topology. When considering the microstructure of the numerical line we have already seen that from the viewpoint of nonstandard analysis a set of infinitesimals arises as a monad, i.e., the external intersection of standard elements of the filter of zero neighbourhoods of the only separated topology that agrees with the algebraic structure of the field of real numbers. One can say that through the notion of the monad of a filter a certain synthesis of general topological and infinitesimal ideas is implemented, the corresponding relations being the basic objects of investigation of the present chapter. We will focus our attention on the most elaborated ways of studying classical topological concepts and constructions that group around compactness which is allowed into the nonstandard set theory through idealization. The contribution of the new approach to the problem under discussion is basically associated with the elaboration of a new principally important notion, that of a nearstandard point. The corresponding criterion of compactness of a standard space, i.e., the nearstandardness of its every point, demonstrates the value and essence of the concept of nearstandardness which carries out a certain individualization for the points of the conventional notion of compactness pertaining to sets. Similar techniques of individualization comprise an important and characteristic part of the arsenal of the nonstandard methods of analysis.

4.1. Monads and Filters

A simplest example of a filter is, as is known, the family of supersets of a certain nonempty set. Nonstandard analysis makes it similarly possible to study an arbitrary standard filter as the standardization of the filter of external supersets of an appropriate external set, i.e., as the monad of this filter. The method of introducing such monads and their simplest properties are to be considered in the present section.

4.1.1. Let X be a standard set and $\mathcal{B}$ be a standard filter base in X. Therefore, $\mathcal{B} \neq \varnothing$, $\mathcal{B} \subset \mathcal{P}(X)$, $\varnothing \notin \mathcal{B}$ and $B_1, B_2 \in \mathcal{B} \rightarrow (\exists B \in \mathcal{B})\ B \subset B_1 \cap B_2$. The symbol $\mu(\mathcal{B})$ denotes the *monad* of $\mathcal{B}$, i.e., the external set determined by the relation

$$\mu(\mathcal{B}) := \bigcap\{B : B \in {}^{\circ}\mathcal{B}\}.$$

4.1.2. *An internal set is a superset of a certain standard element of a standard filter base* $\mathcal{B}$ *iff it contains the monad* $\mu(\mathcal{B})$.

◁ If $A \supset B$ and $B \in {}^{\circ}\mathcal{B}$, then $A \supset \mu(\mathcal{B})$ by definition. Conversely, if $A \supset \mu(\mathcal{B})$ then by virtue of the idealization principle there is an internal set $B \in \mathcal{B}$ for which $B \subset \mu(\mathcal{B})$, and we deduce $A \supset B$. ▷

4.1.3. *Every standard filter* $\mathcal{F}$ *is the standardization of the principal external filter of supersets of the monad* $\mu(\mathcal{B})$.

◁ Symbolically, we have to establish

$$(\forall^{\mathrm{st}} A)\ A \in \mathcal{F} \leftrightarrow A \supset \mu(\mathcal{F}).$$

The preceding relation is obviously contained in **4.1.2.** ▷

4.1.4. *The monad of a filter* $\mathcal{F}$ *is an internal set iff the former set is standard. In this case the initial standard* $\mathcal{F}$ *is the filter of supersets of* $\mu(\mathcal{F})$.

◁ If $\mu(\mathcal{F})$ is an internal set, then, taking into account **4.1.3** and the idealization principle, we get

$$(\exists A)\,(\forall^{\mathrm{st}} F)\quad (F \in \mathcal{F} \leftrightarrow F \supset A) \leftrightarrow (\forall^{\mathrm{st\,fin}} \mathcal{U})\,(\exists A)$$
$$(\forall F \in \mathcal{U})\quad (F \in \mathcal{F} \leftrightarrow F \supset A) \leftrightarrow (\forall^{\mathrm{st}} U)\,(\exists A)\ (U \in \mathcal{F} \leftrightarrow U \supset A).$$

Using the transfer principle, we come to the conclusion that $\mathcal{F}$ is the filter of supersets of a certain set A. Since such a set A is unique, $A = \mu(\mathcal{F})$ with A being standard here. ▷

4.1.5. For a standard filter base $\mathcal{B}$ the elements of $\mu(\mathcal{B})$ are termed *infinitesimal* or *distant*, or *remote*, or *astray* (relative to $\mathcal{B}$). Analogously, an element $B \in \mathcal{B}$ such that $B \subset \mu(\mathcal{B})$ is also called infinitesimal or distant, or remote, or astray. The union of all infinitesimal sets of $\mathcal{B}$ is denoted by ${}^a\mathcal{B}$.

4.1.6. Examples.

(1) The monad $\mu(\mathbf{R})$ is the monad of the filter of zero neighbourhoods for the natural topology on $\mathbf{R}$.

(2) Let $\mathcal{B}$ be a filter base, and $\tilde{\mathcal{B}}$ be the filter generated by $\mathcal{B}$, i.e., the collection of supersets of the elements of $\mathcal{B}$, or, symbolically,

$$\tilde{\mathcal{B}} := \{F \subset X: \quad (\exists B \in \mathcal{B})\, B \subset F\}.$$

In line with the transfer principle, if $\mathcal{B}$ is a standard filter (in a standard set X), then $\tilde{\mathcal{B}}$ is also a standard (base of the) filter. In this case $\mu(\mathcal{B}) = \mu(\tilde{\mathcal{B}})$. It should be remarked that further on it would be more convenient to operate with the monad of an arbitrary internal filter $\mathfrak{F}$ which is determined in an obvious way: $\mu(\mathfrak{F}) := \cap\, {}^\circ\mathfrak{F}$. It should be emphasized that in a standard set X the monad of the filter $\mathfrak{F}$ is obligatory an external superset of an internal element of $\mathfrak{F}$.

(3) Let Ξ be a standard direction, i.e., a nonempty directed set. According to the idealization principle, in Ξ there are internal elements majorizing all standard points of Ξ. Such Ξ elements are called *distant*, *remote*, *infinitely large*, or *astray* in Ξ. Let us consider a standard base of the tail filter $\mathcal{B} := \{[\xi, \rightarrow) := \{\eta \in \Xi: \eta \geq \xi\}: \xi \in \Xi\}$. By definition, $\eta \in \mu(\mathcal{B}) \leftrightarrow (\forall^{\mathrm{st}} \xi \in \Xi)\ \eta \geq \xi$, i.e., the monad of the tail filter is, as was to be expected, composed of distant elements of the direction considered. We shall use the notation ${}^a\Xi := \mu(\mathcal{B})$.

(4) Let $\mathcal{E}$ be a standard cover of a standard set X, i.e., $X \subset \cup\mathcal{E}$. Let us consider the family $\Xi(\mathcal{E})$ of standard finite unions of elements of $\mathcal{E}$. Therefore, $\Xi(\mathcal{E}) := {}^*\{\cup\mathcal{E}_0: \mathcal{E}_0 \in \mathcal{P}_{\mathrm{st\,fin}}(\mathcal{E})\}$, where $\mathcal{P}_{\mathrm{st\,fin}}(\mathcal{E})$ is the set of standard finite subsets of $\mathcal{E}$. The external collection of distant elements of $\Xi(\mathcal{E})$ is termed the monad of $\mathcal{E}$ and is denoted by $\mu(\mathcal{E})$. Hence,

$$\mu(\mathcal{E}) = \cup\{E: E \in {}^\circ\mathcal{E}\}.$$

One can analogously determine the monad of any family of sets filtered upwards.

(5) Let $f \subset X \times Y$ and $\mathfrak{F}$ is a (base of a) filter in X such that f meets $\mathfrak{F}$, i.e.,

$(\forall F \in \mathfrak{F})\ \mathrm{dom} f \cap F \neq \varnothing$. Let us, as usual, assume

$$f(\mathfrak{F}) := \{B \subset Y : (\exists F \in \mathfrak{F})\ B \supset f(F)\}.$$

Therefore $f(\mathfrak{F})$, a filter in Y, is the image of $\mathfrak{F}$ under the correspondence f. In 'standard entourage', i.e., assuming that X, Y, f and $\mathfrak{F}$ are standard objects and employing the idealization principle, we get

$$\begin{aligned}
& y \in \mu(f(\mathfrak{F})) \leftrightarrow (\forall^{\mathrm{st}} B \in f(\mathfrak{F}))\ y \in B \leftrightarrow (\forall^{\mathrm{st}} F \in \mathfrak{F})\ y \in f(F) \\
& \leftrightarrow (\forall^{\mathrm{st}} F \in \mathfrak{F})(\exists x)\ x \in F \wedge y \in f(x) \\
& \leftrightarrow (\forall^{\mathrm{st\,fin}} \mathfrak{F}_0 \subset \mathfrak{F})(\exists x)(\forall F \in \mathfrak{F}_0)\ x \in F \wedge y \in f(x) \\
& \leftrightarrow (\exists x)(\forall^{\mathrm{st}} F \in \mathfrak{F})\ x \in F \wedge y \in f(x) \\
& \leftrightarrow (\exists x \in \mu(\mathfrak{F}))\ y \in f(x) \leftrightarrow y \in f(\mu(\mathfrak{F})).
\end{aligned}$$

Therefore, *the image of the monad of a filter is the monad of the image of this filter*:

$$\mu(f(\mathfrak{F})) = f(\mu(\mathfrak{F})).$$

Let now $\mathcal{B}$ be a base of the filter in Y, and let f^{-1} meet $\mathcal{B}$. Let us consider the preimage or inverse image $f^{-1}(\mathcal{B})$ of the filter $\mathcal{B}$ f (i.e., the image of this filter under the correspondence f^{-1}). Obviously, in line with the above-proved, $\mu(f^{-1}(\mathcal{B})) = f^{-1}(\mu(\mathcal{B}))$. It would be expedient to remark that the last relation can be proved without using 'saturation'. Indeed, strictly by definition, we deduce

$$\mu(f^{-1}(\mathcal{B})) = \bigcap_{G \in {}^{\circ}\mathcal{B}} f^{-1}(G) = f^{-1}\Big(\bigcap_{G \in {}^{\circ}\mathcal{B}} G\Big) = f^{-1}(\mu(\mathcal{B})),$$

i.e., the monad of the preimage of a filter is the preimage of the monad of the initial filter. It is worth to emphasize that when deducing this statement we have made use of the fact that the correspondence f allows one to define external preimages of external sets Y as well.

4.1.7. *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two standard filter bases in a certain standard set. In this case*

$$\tilde{\mathcal{B}}_1 \supset \tilde{\mathcal{B}}_2 \leftrightarrow \mu(\mathcal{B}_1) \subset \mu(\mathcal{B}_2).$$

◁ → If B_2 is standard and $B_2 \supset \mu(\mathcal{B}_2)$, then, by **4.1.2**, $B_2 \in \tilde{\mathcal{B}}_2$ and, hence, $B_2 \in \tilde{\mathcal{B}}_1$. Therefore, $B_2 \supset \mu(\mathcal{B}_1)$, and finally, $\mu(\mathcal{B}_1) \subset \mu(\mathcal{B}_2)$.

← Let F_2 be a standard element of $\tilde{\mathcal{B}}_2$, i.e., a superset of a certain standard $B_2 \in \mathcal{B}_2$.

By hypothesis, B_2 contains the monad $\mu(\mathcal{B}_1)$. Hence, by virtue of **4.1.2**, $B_2 \in \tilde{\mathcal{B}}_1$ and, thus, $F_2 \in \tilde{\mathcal{B}}_1$. Now we are to use the transfer principle. ▷

4.1.8. *Let $f: X \to Y$, and let $\mathfrak{A}$ be a base of a filter in X, while $\mathcal{B}$ be such of that in Y. If the parameters are standard, then the following statements are valid:*

(1) $f(\mathfrak{A}) \supset \tilde{\mathcal{B}}$;

(2) $f^{-1}(\mathcal{B}) \subset \tilde{\mathfrak{A}}$;

(3) $\mu(f(\mathfrak{A})) \subset \mu(\mathcal{B})$;

(4) $f(\mu(\mathfrak{A})) \subset \mu(\mathcal{B})$.

◁ The following argument proves the equivalence (1) ↔ (2):

$$
\begin{aligned}
&f(\mathfrak{A}) \supset \tilde{\mathcal{B}} \leftrightarrow (\forall B \in \mathcal{B})(\exists A \in \mathfrak{A}) \quad f(A) \subset B \\
&\leftrightarrow (\forall B \in \mathcal{B})(\exists A \in \mathfrak{A}) \quad A \subset f^{-1}(B) \leftrightarrow f^{-1}(\mathcal{B}) \subset \tilde{\mathfrak{A}} .
\end{aligned}
$$

Equivalence between (1) and (3) is ensured by **4.1.7**. To complete the proof, it should be remarked that by **4.1.6** (5) we get

$$
\begin{aligned}
&f(\mu(\mathfrak{A})) \subset \mu(\mathcal{B}) \leftrightarrow \mu(\mathfrak{A}) \subset f^{-1}(\mu(\mathcal{B})) \\
&\leftrightarrow \mu(\mathfrak{A}) \subset \mu(f^{-1}(\mathcal{B})) \leftrightarrow f^{-1}(\mathcal{B}) \subset \tilde{\mathfrak{A}} . \triangleright
\end{aligned}
$$

4.1.9. Assuming the classical stance, we can reduce the formulation of **4.1.8**. Namely, we can omit the words 'if the parameters are standard', presenting **4.1.8** (4) as $*f(\mu(\mathfrak{A})) \subset \mu(\mathcal{B})$, where $*$ is the Robinson standardization. Common practice is to silently accept $f := *f$, which results in the most presentable and easily memorizable formulation. The same formulation is also often used under neoclassical and radical doctrines. In other words, if nonstandard analysis is used as a technique of studying the von Neumann universe, the 'given' parameters are, if not otherwise stated, considered standard sets, while the term an 'internal' set is replaced with a more common one, 'a set'. This convenient agreement, obviously, correlates with the qualitative ideas on standard objects. From now on, we shall continue sharing this free point of view, omitting, wherever possible, indications as to the type of the sets arising in all cases when it should not result in any serious misunderstanding.

4.1.10. *The following statements are valid:*

(1) *filters* $\mathfrak{F}_1$ *and* $\mathfrak{F}_2$ *have the least upper bound iff* $\mu(\mathfrak{F}_1) \cap \mu(\mathfrak{F}_2) \neq \varnothing$;

(2) *for any set of filters* $\mathfrak{E}$ *bounded from above we have*

$$\mu(\sup \mathfrak{E}) = \cap\{\mu(\mathfrak{F}) : \mathfrak{F} \in {}^\circ\mathfrak{E}\},$$

i.e., the monad of the intersection of filters is the intersection of their monads.

◁ Statement (1) results immediately from **4.1.7**. To prove (2), let us first remark that for $\mathfrak{F} \in {}^\circ\mathfrak{E}$ we have $\mathfrak{F} \le \sup \mathfrak{E}$ and, hence, $\mu(\sup \mathfrak{E}) \subset \mu(\mathfrak{F})$. This ensures the inclusion $\mu(\sup \mathfrak{E}) \subset \cap\{\mu(\mathfrak{F}) : \mathfrak{F} \in {}^\circ\mathfrak{E}\}$. Let now $F \in {}^\circ\sup \mathfrak{E}$. By virtue of the properties of the filter, there is a standard finite set $\mathfrak{E}_0 \subset \mathfrak{E}$ such that $F \in \sup \mathfrak{E}_0$. According to **4.1.3**, with (1) taken into account, we deduce $F \supset \mu(\sup \mathfrak{E}_0) = \cap\{\mu(\mathfrak{F}) : \mathfrak{F} \in \mathfrak{E}_0\}$. And, finally,

$$\mu(\sup \mathfrak{E}) \supset \cap\{\mu(\mathfrak{F}) : \mathfrak{F} \in \mathfrak{E}_0, \mathfrak{E}_0 \in \mathcal{P}_{\mathrm{stfin}}(\mathfrak{E})\} = \cap\{\mu(\mathfrak{F}) : \mathfrak{F} \in^\circ \mathfrak{E}\}. \ \triangleright$$

4.1.11. *Let* $\mathfrak{A}$ *be an ultrafilter, i.e., one maximal by inclusion in the set of filters* $\mathfrak{F}(X)$ *of the set* X *under discussion, and* $\mathfrak{F}$ *be a filter:* $\mathfrak{F} \in \mathfrak{F}(X)$. *Then either* $\mu(\mathfrak{A}) \cap \mu(\mathfrak{F}) = \varnothing$, *or* $\mu(\mathfrak{A}) \subset \mu(\mathfrak{F})$.

◁ If $\mu(\mathfrak{A}) \cap \mu(\mathfrak{F}) \neq \varnothing$, then, by **4.1.10** (1), there is a least upper bound $\mathfrak{A} \vee \mathfrak{F} = \mathfrak{A}$. Hence, $\mathfrak{F} \subset \mathfrak{A}$, and, by **4.1.7**, valid is $\mu(\mathfrak{A}) \subset \mu(\mathfrak{F})$. ▷

4.1.12. Nonstandard criterion for an ultrafilter. *A filter* $\mathfrak{F}$ *in* X *is an ultrafilter iff its monad is easy to catch, i.e., for any standard subsets* A *and* B *in* X *such that* $A \cup B = X$, *then we have either* $\mu(\mathfrak{F}) \subset A$ *or* $\mu(\mathfrak{F}) \subset B$.

◁ → As soon as $\mu(\mathfrak{F}) \subset A \cup B$, we can assume $\mu(\mathfrak{F}) \cap A \neq \varnothing$. Since $A = \mu(\{\tilde{A}\})$, therefore, by **4.1.11**, $\mu(\mathfrak{F}) \subset A$.

← Let $\mathfrak{G} \supset \mathfrak{F}$. Then, according to **4.1.7**, $\mu(\mathfrak{G}) \subset \mu(\mathfrak{F})$. If A is standard and $A \supset \mu(\mathfrak{G})$, then either $A \supset \mu(\mathfrak{F})$, or $A' := X - A \supset \mu(\mathfrak{F})$, by hypothesis. The case $A' \supset \mu(\mathfrak{F})$ is impossible, since in that case we would get $\mu(\mathfrak{F}) \cap \mu(\mathfrak{G}) \subset A \cap A' = \varnothing$.

Hence, $A \supset \mu(\mathfrak{F})$, i.e., $A \in \mathfrak{F}$ (by **4.1.2**). Therefore, for any standard $A \in \mathfrak{G}$ we have $A \in \mathfrak{F}$. In line with the transfer principle, $\mathfrak{G} \subset \mathfrak{F}$; i.e., $\mathfrak{F}$ is an ultrafilter. ▷

4.1.13. Standard criterion for an ultrafilter. *A filter $\mathfrak{F}$ is an ultrafilter iff $A \cup B \in \mathfrak{F} \to A \in \mathfrak{F} \vee B \in \mathfrak{F}$.*

◁ → If $A \cup B \in \mathfrak{F}$, then the monad is caught; $\mu(\mathfrak{F}) \subset A \cup B$. If $\mu(\mathfrak{F}) \cap A \neq \varnothing$, then $\mu(\mathfrak{F}) \subset A$ and $A \in \mathfrak{F}$. If $\mu(\mathfrak{F}) \cap B \neq \varnothing$, then $\mu(\mathfrak{F}) \subset B$ and $B \in \mathfrak{F}$.

← Let $A \cup B = X$. If $A \in \mathfrak{F}$, then $A \supset \mu(\mathfrak{F})$. If $B \in \mathfrak{F}$, then $B \supset \mu(\mathfrak{F})$, i.e., the monad is easily caught. ▷

4.1.14. *Every limit of a filter is one of its adherent points. An adherent point of an ultrafilter is one of its limits.*

◁ It suffices to work in 'standard entourage'. Obviously, $\mathfrak{F} \to x \leftrightarrow \mu(\mathfrak{F}) \subset \mu(x) := \mu(\tau(x))$. Besides, $x \in \mathrm{cl}\, \mathfrak{F} := \cap\{\mathrm{cl}\, F : F \in \mathfrak{F}\} \leftrightarrow (\forall F \in \mathfrak{F})\ (\forall U \in \tau(x))\ U \cap F \neq \varnothing \leftrightarrow \mu(\mathfrak{F}) \cap \mu(x) \neq \varnothing$, by **4.1.10** (1). Therefore, the first part of the statement has been proved. If now $\mathfrak{F}$ is an ultrafilter, and $x \in \mathrm{cl}\, \mathfrak{F}$, then $\mu(\mathfrak{F}) \cap \mu(x) \neq \varnothing$. Based on the alternative described in **4.1.11**, we deduce $\mu(\mathfrak{F}) \subset \mu(x)$, i.e., $\mathfrak{F} \to x$. ▷

4.1.15. *Let $\mathfrak{E}$ be the cover of X. The following statements are equivalent:*

(1) *there is a standard finite subcover $\mathfrak{E}_0$ in, such $\mathfrak{E}_0 \in \mathcal{P}_{\mathrm{stfin}}(\mathfrak{E})$ that $X \subset \cup \mathfrak{E}_0$;*

(2) *the monad $\mu(\mathfrak{E})$ coincides with X;*

(3) *the monad $\mu(\mathfrak{E})$ is a standard set;*

(4) *the monad $\mu(\mathfrak{E})$ is an internal set;*

(5) *for every standard ultrafilter $\mathfrak{F}$ in X there is an $E \in \mathfrak{E}$ lying in $\mathfrak{F}$.*

◁ Implications (1) → (2) → (3) → (4) are obvious. If $\mu(\mathfrak{E})$ is an internal set, then by **4.1.6** (4) and **4.1.4** we conclude that $\mu(\mathfrak{E})$ is standard, i.e., there is a standard finite $\mathfrak{E}_0 \subset \mathfrak{E}$ such that $\mu(\mathfrak{E}) = \cup \mathfrak{E}_0 \supset X$. Hence, (4) → (1). Implication (1) → (5) is obvious. To prove (5) → (1), let us assume that, on the contrary, $(\forall^{\mathrm{stfin}} \mathfrak{E}_0) \cup \mathfrak{E}_0 \neq X$.

Let us consider $\mathfrak{E}' := \{E' := X - E : E \in \mathfrak{E}\}$. The family $\mathfrak{E}'$ can obviously be considered as generating a filter base in X. Let $\mathfrak{F}$ be an ultrafilter containing this base. In this case there is an $E \in \mathfrak{E}$ such that $E \in \mathfrak{F}$. Besides, by construction, $E' \in \mathfrak{F}$. Thus, we come to a contradiction. ▷

4.1.16. By way of concluding the present section, let us give some useful characteristic features based on the 'technique of internal sets'.

4.1.17. Cauchy principle. *Let $\mathfrak{F}$ be a standard filter in a standard set. Let, then, $\varphi := \varphi(x)$ be a certain internal property (i.e., $\varphi = \varphi^{I}$ for a set-theoretic formula φ). If for every remote element x we have $\varphi(x)$, then there is a standard set such that $(\forall x \in F)\ \varphi(x)$.*

◁ There is an internal set F with the required property (such is any distant element of the filter $\mathfrak{F}$). Hence, in line with the transfer principle, there is a standard F sought. ▷

4.1.18. Principle of a granted horizon. *Let X and Y be standard sets, $\mathfrak{F}$ and $\mathscr{G}$ be standard filters in X and Y, respectively, in which case $\mu(\mathfrak{F}) \cap {}^{\circ}X \neq \varnothing$. Let us now fix a remote set, a 'horizon', F in ${}^{a}\mathfrak{F}$. For a standard correspondence $f \subset X \times Y$ meeting $\mathfrak{F}$, the following statements are equivalent:*

(1) $f(\mu(\mathfrak{F}) - F) \subset F) \subset \mu(\mathscr{G})$;

(2) $(\forall F' \in {}^{a}\mathfrak{F})\, f(F' - F) \subset \mu(\mathscr{G})$;

(3) $f(\mu(\mathfrak{F})) \subset \mu(\mathscr{G})$.

◁ Obviously, (3) → (1) → (2). Hence, we have only to establish the implication (2) → (3).

Choose a $G \in {}^{\circ}\mathscr{G}$. Assume that for every standard F'' of ${}^{\circ}\mathfrak{F}$ there is an x of $F'' - F$, for which $f(x) \notin G$. According to the idealization principle, in this case there is an $x' \in \mu(\mathfrak{F})$ such that $x' \notin F$ and, at the same time, $f(x') \notin G$. Now consider $F' := F \cup \{x'\}$. Obviously, $F' \in {}^{a}\mathfrak{F}$, which results in a contradiction implying that for a certain standard $F'' \in \mathfrak{F}$ we have $f(F'' - F) \subset G$. Making use of the fact that there no standard elements X in F, we deduce:

$$(\forall^{\mathrm{st}} G \in \mathscr{G})\,(\exists^{\mathrm{st}} F \in \mathfrak{F})\,(\forall^{\mathrm{st}} x \in F)\quad f(x) \in G.$$

Now the transfer principle is to be used. ▷

4.2. Monads in Topological Spaces

In this paragraph we shall study the properties of the monads of the filters of neighbourhoods in topological spaces.

4.2.1. Let (X,τ) be a standard *pretopological space*. Therefore, to every (standard) x of X a (standard) filter $\tau(x)$ is assigned in X. Let us denote $\mu(x) := \mu_\tau(x) := \mu(\tau(x))$. The elements $\mu(x)$ are called *infinitely close* to x. Obviously, $\mu(x)$ is the monad of the neighbourhood filter $\tau(x)$ of the point x. The pretopological space (X,τ) is termed *topological* if every neighbourhood of a point in X contains an open neighbourhood of this point. In other words, any $x \in {}^\circ X$ has an infinitely small neighbourhood $U \in \tau(x)$, for which $\mu(x') \subset \mu(x)$ for all $x' \in U$.

4.2.2. Let G be a (external) set in a topological space (X,τ). Let us set $h(G) := \cup\{\mu(x) : x \in {}^\circ G\}$. The set $h(G)$ is called the *halo* of G in X. The set $G \cap h(G)$ is called the *autohalo* or *nearstandard part* of G and is denoted by $\operatorname{nst}(G)$. If $G \supset h(G)$, then G is called *saturated* or, in more detail, τ*-saturated.* If for any $x \in G$ we have $\mu(x) \subset G$, then G is called *well-saturated* (*well-*τ*-saturated*).

4.2.3. *A standard set is open iff it is saturated.*

◁ If G is open and $x \in {}^\circ G$, then $G \supset \mu(x)$. Hence, G contains its halo. On the contrary, if $G \supset h(G)$, then, choosing a distant element U_x of the filter $\tau(x)$ for $x \in {}^\circ G$, we see that $G \supset U_x$. Then by the transfer principle, G is open. ▷

4.2.4. A standard point x of X is called a *microlimit* point of U provided $\mu(X) \cap U \neq \varnothing$. A standard set formed by all microlimit points of U is termed the *microclosure* of U and is denoted by $\operatorname{cl}_{\approx} U$.

4.2.5. *The microclosure* $\operatorname{cl}_{\approx} U$ *of an arbitrary internal set* U *is closed. If* U *is a standard set, then the microclosure* $\operatorname{cl}_{\approx} U$ *coincides with the closure* $\operatorname{cl} U$ *of the set* U.

◁ Let $A := \mathrm{cl}_{\ast} U = {}^*\{x \in X : \mu(x) \cap U \neq \varnothing\}$ and $y \in \mathrm{cl}\, A$. The task is to establish that $y \in A$. According to the transfer principle, y can be considered as standard element. Let us choose a standard open neighbourhood V of the point y. By hypothesis, there is a standard point $x \in V$ such that $x \in A$. From the definitions of standardization and a monad, we deduce that $V \supset \mu(x)$ and $\mu(x) \cap U \neq \varnothing$. Hence, $(\forall^{\mathrm{st}} V \in \tau(y))\ V \cap U \neq \varnothing$. Then by the idealization principle we deduce $\mu(y) \cap U \neq \varnothing$, i.e., $y \in \mathrm{cl}_{\ast} U$.

Let now U be standard. Obviously, ${}^{\circ}U \subset \mathrm{cl}_{\ast} U$. Hence, in line with the above-proved, $U \subset \mathrm{cl}_{\ast} U$ and $\mathrm{cl}\, U \subset \mathrm{cl}_{\ast} U$. If we choose $y \in \mathrm{cl} U$, then $(\forall^{\mathrm{st}} V \in \tau(y))\ V \cap U \neq \varnothing$. Hence, by the idealization principle, $\mu(y) \cap U \neq \varnothing$, i.e., $y \in \mathrm{cl}_{\ast} U$. ▷

4.2.6. *For a point x and a nonempty set U the following statements are equivalent:*

(**1**) *x is an adherent point of U;*

(**2**) *x is a microlimit point of U;*

(**3**) *there is a standard filter $\mathfrak{F}$ whose monad $\mu(\mathfrak{F})$ lies in the monad $\mu(x)$;*

(**4**) *there is a standard net $(x_{\xi})_{\xi \in \Xi}$ of the U point such that its elements with infinitely large indices are infinitely close to x, i.e., $x_{\xi} \in \mu(x)$ for all $\xi \in {}^{a}\Xi$.*

◁ (1) → (2). If $x \in \mathrm{cl}\, U$, then there is a least upper bound $\tau(x) \vee \tilde{\{\tilde{U}\}}$. By **4.1.10** (1), we get

$$\varnothing \neq \mu(\tau(x) \vee \tilde{\{\tilde{U}\}}) = \mu(\tau(x)) \cap \mu(\tilde{\{\tilde{U}\}}) = \mu(x) \cap U,$$

the last implying $x \in \mathrm{cl}_{\ast} U$.

(2) → (3). If $x \in \mathrm{cl}_{\ast} U$, then $U \cap \mu(x) \neq \varnothing$. Hence, on the basis of **4.1.10** (1) we can construct a filter $\mathfrak{F}$ such that $A \in \mathfrak{F} \leftrightarrow A \supset U \cap \mu(x)$. Obviously, this is the filter sought.

(3) → (4). Let us set $\Xi := \tau(x)$ and $\xi_1 \leq \xi_2 \leftrightarrow \xi_1 \supset \xi_2$. Let us define x_{ξ} as an arbitrary point of a $F \in \mathfrak{F}$ such that $F \subset \xi$. Obviously, $(x_{\xi})_{\xi \in \Xi}$ is the sought net. Indeed, by construction, $x_{\xi} \in \mu(x)$ for $\xi \in {}^{a}\Xi$.

(4) → (1). Let V be a standard neighbourhood of x, and η be an arbitrary large index

of Ξ. Obviously, $x_\xi \in V$ for $\xi \geq \eta$, since $\mu(x) \subset V$ and $\xi \in^a \Xi$. Hence, $V \cap U \neq \varnothing$ (since, by hypothesis, $x_\xi \in U$).

4.2.7. Nonstandard criteria for continuity. *Let (X,τ) and (Y,σ) be standard topological spaces, $f: X \to Y$ be a standard mapping, and x be a standard point in X. The following statements are equivalent:*

(1) *f is continuous at the point x;*

(2) *the function f sends points infinitely close to x to points infinitely close to $f(x)$, i.e.,*

$$(\forall x')\, x' \in \mu_\tau(x) \to f(x') \in \mu_\sigma(f(x)).$$

⊲ It suffices to refer to **4.1.8.** ⊳

4.2.8. For a set A in X by the symbol $\mu(A)$ we shall denote the intersection of standard open sets containing A. The set $\mu(A)$ is termed the *monad* of A. It should be remarked that $\mu(\varnothing) = \varnothing$. If $A \neq \varnothing$, then $\mu(A)$ is the monad of the neighbourhood filter of the set A.

4.2.9. *Let (X,τ) be a standard topological space. Then*

(1) *(X,τ) is a separated ($= T_1$) space iff ${}^\circ\mu(x) = \{x\}$ for any point $x \in {}^\circ X$;*

(2) *(X,τ) is a Hausdorff ($= T_2$) space iff $\mu(x_1) \cap \mu(x_2) = \varnothing$ for $x_1, x_2 \in {}^\circ X$;*

(3) *(X,τ) is regular if it is T_1-spaced and satisfies axiom T_3: for every closed standard $A \subset X$ and a standard point $x \notin A$ the following relation holds $\mu(x) \cap \mu(A) = \varnothing$;*

(4) *(X,τ) is normal if it is separated and satisfies axiom T_4: for any two disjoint closed sets A and B in X we have $\mu(A) \cap \mu(B) = \varnothing$.*

4.2.10. *The following statements are valid:*

(1) *a standard set is well-saturated iff it is open;*

(2) *the monad of an arbitrary set is well-saturated*;

(3) *the monad of a standard filter $\mathfrak{F}$ is well-saturated iff $\mathfrak{F}$ has a base of open sets*;

(4) *the monad $\mu(A)$ of an arbitrary open A is the least well-saturated set containing A, in which case the presentation $\mu(A) = \bigcup\{\mu(a) : a \in A\}$ is valid.*

◁ (1) If A is standard and well-saturated, then it is saturated and, hence, A is open (see **4.2.3**). If we know beforehand that A is standard and open, then, by the definition of monad, for $a \in A$ we get $\mu(a) \subset A$, i.e., A is well-saturated.

(2) The monad of a set is, by definition, the intersection of standard open sets. Hence, with (1) taken into account, it is well-saturated.

(3) If $\mathfrak{F}$ has a base of open standard sets then the proof follows from (1). If $\mu(\mathfrak{F})$ is well-saturated and V is a standard $\mathfrak{F}$ element, then $V \supset \mu(\mathfrak{F}) \supset \bigcup\{U_a : a \in F\}$, where F is an infinitely distant $\mathfrak{F}$ element, and U_a is an infinitely small neighbourhood of the point a. Since $\bigcup\{U_a : a \in F\} \in \mathfrak{F}$, the required result follows from the transfer principle.

(4) By (2), $\mu(A)$ is well-saturated. Moreover, according to (3) well-saturated is $B := \bigcup\{\mu(a) : a \in A\}$. We have to check if $B = \mu(A)$. The inclusion $B \subset \mu(A)$ is obvious. Let us assume, contrary to what has been proved, that $B \neq \mu(A)$, i.e., there is an $x \in \mu(A)$ such that $x \notin B$. Therefore, for every $a \in A$ there is a standard neighbourhood U_a of the point a with the property $x \notin U_a$. In other words, $(\forall a \in A)(\exists^{\mathrm{st}} U_a)\, U_a \in \tau(a)$. Employing the idealization principle, we see that there is a standard finite set $\{a_1, \dots, a_n\} \subset A$ such that $A \subset U_{a_1} \cup \dots \cup U_{a_n}$. Hence, $x \in \mu(A) \subset U_{a_1} \cup \dots \cup U_{a_n}$, which is a contradiction. ▷

4.2.11. *Let (X,τ) be a separated topological space. The mapping $f : (X,\tau) \to (Y,\sigma)$ is continuous at the point x iff $f(\mu_\tau(x) \setminus U) \subset \mu_\sigma(f(x))$ for an infinitely small neighbourhood U of the point x.*

◁ By axiom T_1, $\mu_\tau(x) - U = \mu(x) - U$, where $\mu(x)$ is the monad of the filter $\tau(x)$ of the deleted neighbourhoods of x, i.e., $V \in \tau(x) \leftrightarrow V \cup \{x\} \in \tau(x)$. Obviously, $\mu(x) = \mu_\tau(x) - \{x\}$, in which case $U - \{x\}$ is an infinitely small element of $\tau(x)$. Using the principle of a granted horizon **4.1.18**, we see that $f(\mu(x) - U) \subset \mu_\sigma(f(x)) \leftrightarrow$ $\leftrightarrow f(\mu(x)) \subset \mu_\sigma(f(x)) \leftrightarrow f(\mu_\tau(x)) \subset \mu_\sigma(f(x))$. ▷

4.2.12. *Let $(Y_\xi, \sigma_\xi)_{\xi \in \Xi}$ be a family of topological spaces. Let, then, $(f_\xi : X \to Y_\xi)_{\xi \in \Xi}$ be* *a family of mappings, and* $\tau := \sup_{\xi \in \Xi} f_\xi^{-1}(\sigma_\xi)$ *be the initial topology in* X*, i.e., the weakest topology in which the mappings* f_ξ *are continuous for all* $\xi \in \Xi$*. In this case, for every standard point* $x \in X$,

$$\mu_\tau(x) = \bigcap_{\xi \in {}^\circ\Xi} f_\xi^{-1}(\sigma_\xi(f_\xi(x))).$$

◁ The required result is obtained from **4.1.8.** ▷

4.2.13. *A point* x' *of a Tychonoff product is infinitely close to the given point* x *provided the standard coordinates of* x' *are close to the corresponding standard coordinates of* x.

◁ Let, formally speaking, $(X_\xi, \tau_\xi)_{\xi \in \Xi}$ be a standard family of standard topological spaces. Let, then, $(\mathfrak{X}, \tau)$ be the Tychonoff product of $(X_\xi, \tau_\xi)_{\xi \in \Xi}$, i.e.,

$$\mathfrak{X} := \prod_{\xi \in \Xi} X_\xi; \quad \tau := \sup_{\xi \in \Xi} \mathrm{Pr}_\xi^{-1}(\tau_\xi),$$

where Pr_ξ is the projection operator of $\mathfrak{X}$ on X_ξ. Making use of **4.2.11** and **4.1.6** (5), for $x \in {}^\circ\mathfrak{X}$, we deduce

$$\mu(x) = \bigcap_{\xi \in {}^\circ\Xi} \mu(\mathrm{Pr}_\xi^{-1}(\tau_\xi(x_\xi))) = \bigcap_{\xi \in {}^\circ\Xi} \mathrm{Pr}_\xi^{-1}(\mu(\tau_\xi(x_\xi))).$$

It should be remarked that for $\xi \in {}^\circ\Xi$ we have $x' \in \mathrm{Pr}_\xi^{-1}(\mu(\tau_\xi(x_\xi))) \leftrightarrow \mathrm{Pr}_\xi\, x' \in \mu(\tau_\xi(x_\xi))$, i.e.,

$$\mathrm{Pr}_\xi^{-1}(\mu(\tau_\xi(x_\xi)) = \mu_{\tau_\xi}(x_\xi) \times \prod_{\eta \neq \xi} X_\eta.$$

Therefore, for every standard $\xi \in \Xi$ (cf. **4.1.6** (5)), we have

$$\mathrm{Pr}_\xi(\mu(x)) = \mu(\tau_\xi(x_\xi)),$$

which completes the proof. ▷

4.3. Nearstandardness and Compactness

Proximity to a standard point arising in topological spaces makes it possible to give convenient criteria of compact spaces. Obtaining these criteria is the basic topic of the present section.

4.3.1. A point x of the topological space (X,τ) is called *nearstandard* or, in more detail, τ-nearstandard, if $x \in \mathrm{nst}(X)$, i.e., if for a certain standard $x' \in {}^{\circ}X$ we have $x \in \mu(x')$.

4.3.2. *A point $x \in X$ is nearstandard iff for every standard open cover $\mathfrak{E}$ of the set X we have $x \in \mu(\mathfrak{E})$. In other words,*

$$\mathrm{nst}(X) = \bigcap\{\mu(\mathfrak{E}) : \mathfrak{E} \text{ is an open cover of } X\}.$$

◁ Let first $x \in \mathrm{nst}(X)$ and $x' \in {}^{\circ}X$ be such that $x \in \mu(x')$. For an open cover $\mathfrak{E}$ there is a standard element $E \in \mathfrak{E}$ such that $x' \in \mathfrak{E}$, i.e., $\mu(x') \subset E$ (see **4.2.3**). Therefore, $x \in \mu(x') \subset E \subset \mu(\mathfrak{E})$. Let now $x \notin \mathrm{nst}(X)$. Then for any $x' \in {}^{\circ}X$ we have $x \notin \mu(x')$. Hence, there is a standard open neighbourhood $U_{x'}$ of the point x' for which $x \notin U_{x'}$. The standardization $\mathfrak{E} := {}^{*}\{U_{x'} : x' \in {}^{\circ}X\}$ is an open cover of X for which $x \notin \mu(\mathfrak{E})$. ▷

4.3.3. *Every nearstandard point is infinitely close to the only standard point iff the space considered is Hausdorff.*

◁ If τ is a Hausdorff topology, and $x', x'' \in {}^{\circ}X$, then $\mu(x') \cap \mu(x'') \neq \varnothing \to x' = x''$. Let, on the contrary, $x \in \mu(x') \cap \mu(x'')$ for $x', x'' \in {}^{\circ}X$. Since x is nearstandard, $x' = x''$ by hypothesis. Hence, $x' \neq x'' \to \mu(x') \cap \mu(x'') = \varnothing$. ▷

4.3.4. Let us determine the external correspondence $\mathrm{st}(x) := \{x' \in {}^{\circ}X : x \in \mu(x')\}$. In the Hausdorff case st is a mapping of $\mathrm{nst}(X)$ on ${}^{\circ}X$.

4.3.5. *For every internal U, the following presentation holds $\mathrm{cl}_{\approx} U = {}^{*}\mathrm{st}(U)$. In particular, a standard set U is closed iff $U = {}^{*}\mathrm{st}(U)$.*

◁ The proof is in **4.2.5.** ▷

4.3.6. Nonstandard criteria for compactness. *For a standard space X the following statements are equivalent:*

(1) *X is compact;*

(2) *every point of X is nearstandard;*

(3) *the autohalo of X is an internal set.*

$\triangleleft$ (1) $\rightarrow$ (2). Let $\mathscr{E}$ be an open cover of X. The monad $\mu(\mathscr{E})$ coincides with X by **4.1.15** (since X is compact). By virtue of **4.3.2** we come to the conclusion: $\mathrm{nst}(X) = \bigcap_{\mathscr{E}} \mu(\mathscr{E}) = X$.

(2) $\rightarrow$ (3). This is obvious.

(3) $\rightarrow$ (1). Let $\mathscr{E}$ be an open cover of X. Since $(\forall x \in \mathrm{nst}(X))\,(\exists^{\mathrm{st}} E \in \mathscr{E})\; x \in E$, according to the idealization principle, $(\exists^{\mathrm{st\,fin}} \mathscr{E}_0 \subset \mathscr{E})\; \cup \mathscr{E}_0 \supset \mathrm{nst}(X) \supset {}^{\circ}X$. Hence, by the transfer principle, $\mathscr{E}_0$ is a cover of X. $\triangleright$

4.3.7. *Let C be a set in a topological space X. The following statements are equivalent:*

(1) *C is compact in the induced topology;*

(2) *C lies in the halo $h(C)$;*

(3) *the monad $\mu(C)$ coincides with the halo $h(C)$.*

$\triangleleft$ (1) $\rightarrow$ (2). As C is compact in the induced topology, we have $C \subset \mathrm{nst}(C) \subset h(C)$ (see **4.3.6**).

(2) $\rightarrow$ (3). It is obvious that we always have $h(G) = \cup\{\mu(x) : x \in {}^{\circ}G\} \subset \mu(G)$. By hypothesis, for every $x \in C$ there is a $y \in {}^{\circ}C$ which obeys the relation $x \in \mu(y)$. By **4.2.8** (2), $\mu(x) \subset \mu(y)$. Therefore, making use of **4.2.8** (4), we get $\mu(C) = \cup\{\mu(x) : x \in C\} \subset \cup\{\mu(y) : y \in {}^{\circ}C\} = h(C)$.

(3) $\rightarrow$ (1). Let $\mathscr{E}$ be a standard cover of C. By definition, $C \subset \mu(C) \subset h(C)$. Therefore (cf. **4.3.2**), $C \subset \mu(\mathscr{E})$. Hence, in line with **4.1.5**, there is a finite subcover of

C in $\mathscr{E}$. ▷

4.3.8. Nonstandard criterion for relative compactness. *For a regular space X and a set C in X the following statements are equivalent:*

(1) *C is relatively compact (i.e., $\operatorname{cl} C$ is compact);*

(2) *C lies in the nearstandard part of X.*

◁ (1) → (2). With no additional hypotheses, **4.3.7** obviously yields:

$$C \subset \operatorname{cl} C \subset h(\operatorname{cl} C) \subset h(X) = h(X) \cap X = \operatorname{nst}(X).$$

(2) → (1). Let us consider the closure $\operatorname{cl} C$, and let $\mathscr{E}$ be an open cover of $\operatorname{cl} C$. Hence, for every $c \in C$ there is an $E \in \mathscr{E}$ containing c. Let E_c be a closed neighbourhood of c contained in E. Obviously, the family $\mathscr{E}' := \{E_c : c \in C\}$ comprises a standard cover of $\operatorname{cl} C$. The family $\mathscr{E}' \cup \{X - \operatorname{cl} C\}$ forms a cover of X and, hence, from **4.3.1**, we deduce $C \subset \operatorname{nst}(X) \subset \mu(\mathscr{E}') \cup \{X - \operatorname{cl} C\}$. By virtue of **4.1.15**, there is a finite set $\mathscr{E}_0 \subset \mathscr{E}'$ cover C. Obviously, $\cup \mathscr{E}_0$ is closed, i.e., $\mathscr{E}_0$ is a cover of $\operatorname{cl} C$. Every element of $\mathscr{E}_0$ is, by construction, a subset of an element of $\mathscr{E}$. Therefore, it is possible to refine a finite subcover of $\operatorname{cl} C$ from the initial $\mathscr{E}$. ▷

4.3.9. Criterion **4.3.8** allows strengthening. Namely, the microclosure of an arbitrary internal subset of the nearstandard part of an arbitrary Hausdorff space proves to be compact.

4.3.10. *Let $\mathcal{X} := \prod_{\xi \in \Xi} X_\xi$ be a standard product of standard topological spaces. A point $x \in \mathcal{X}$ is nearstandard iff nearstandard are its standard coordinates $x_\xi \in \operatorname{nst}(X_\xi)$ for $\xi \in {}^\circ\Xi$.*

◁ If $x \in \operatorname{nst}(\mathcal{X})$, then, by **4.1.12**, for a certain $y \in {}^\circ\mathcal{X}$ and any $\xi \in {}^\circ\Xi$ we get $x_\xi \in \mu(y_\xi)$. Now we have to remark that, by the transfer principle, $y_\xi \in {}^\circ X_\xi$. Assume that we know beforehand that $x_\xi \in \operatorname{nst}(X_\xi)$ for $\xi \in {}^\circ\Xi$. Consider the external function $y : \xi \to \operatorname{st}(x_\xi)$ from ${}^\circ\Xi$ to $\cup_{\xi \in \Xi} {}^\circ X_\xi$. Obviously, by virtue of **4.1.12**, for the standardization ${}^*y$ we get ${}^*y \in {}^\circ\mathcal{X}$ and $x \in \mu({}^*y)$. ▷

4.3.11. Tychonoff theorem. *The Tychonoff product of compact sets is compact.*

◁ According to the transfer principle one can assume that a standard family of standard spaces is considered here. In this case, making use of **4.3.10**, we conclude that every point of the product is nearstandard. ▷

4.3.12. Further on we will, as a rule, consider Hausdorff compact spaces. In conventional terminology such spaces are referred to in brief as *compacta.*

4.4. Infinite Proximity in Uniform Spaces

In uniform spaces there arises a symmetric, reflexive and transitive relation between internal points, i.e., their infinite proximity. Let us now study the most important constructions that pertain to this notion.

4.4.1. Let $(X,\mathcal{U})$ be a uniform space. This implies that $U := \{\varnothing\}$ if $X = \varnothing$. When $X \neq \varnothing$, then $\mathcal{U}$ is a filter in X^2 with the following properties:

(1) $\mathcal{U} \subset \widetilde{\{I_X\}}$;

(2) $(\forall U \in \mathcal{U})\ U^{-1} \in \mathcal{U}$;

(3) $(\forall V \in \mathcal{U})\ (\exists U \in \mathcal{U})\, U \circ U \subset V$.

4.4.2. Luxemburg criterion. *The filter* $\mathcal{U}$ *in* X^2 *is a uniformity on a (nonempty) set X iff the monad* $\mu(\mathcal{U})$ *is an external equivalence.*

◁ We have

$$\mu(\mathcal{U}) = \cap{}^\circ\mathcal{U} = \bigcap_{U \in {}^\circ\mathcal{U}} U = \bigcap_{U \in {}^\circ\mathcal{U}} U^{-1} = \mu(\mathcal{U})^{-1};$$
$$\mu(\mathcal{U}) \supset I_X;$$
$$\mu(\mathcal{U}) = \cap\{U \circ U : U \in {}^\circ\mathcal{U}\} \supset \mu(\mathcal{U}) \circ \mu(\mathcal{U}) \supset \mu(\mathcal{U}) \circ I_X \supset \mu(\mathcal{U}).$$

Here we have made use of the fact that U^{-1} and $U \circ U$ are standard under the condition that U is standard. Besides, by the definition of monad, $U \supset \mu(\mathcal{U})$ for $U \in {}^\circ\mathcal{U}$.

By virtue of **4.1.4**, the filter $\mathcal{U}$ is the standardization of all the supersets of its monad, i.e.,

$$U \in {}^{\circ}\mathcal{U} \leftrightarrow U \supset \mu(\mathcal{U}).$$

This implies $\mathcal{U} \subset \widetilde{\{I_X\}}$ and $U \in \mathcal{U} \rightarrow U^{-1} \in \mathcal{U}$. Let us consider an infinitely small element W of the filter $\mathcal{U}$. By virtue of the above, $U := W^{-1} \cap W \in \mathcal{U}$. Moreover, $U \circ U \subset \mu(\mathcal{U}) \circ \mu(\mathcal{U}) = \mu(\mathcal{U})$. Hence, for every standard $V \in \mathcal{U}$, there is a $U \in \mathcal{U}$ such that $U \circ U \subset V$. By the transfer principle we conclude that $\mathcal{U}$ is a uniformity. ▷.

4.4.3. When applying the Luxemburg criterion it is expedient to bear in mind that not every equivalence relation on X^2 is a monad (i.e., it produces a uniformity in X). For instance, if we assume that $x, y \in \mathbf{R}$ are equivalent for $x - y \in {}^{\circ}\mathbf{R}$, then points equivalent to zero will compose the set ${}^{\circ}\mathbf{R}$ which is not a monad of any filter. This, in particular, implies that such an equivalence is produced by no standard uniformity.

4.4.4. If x, y are points of the space X with uniformity $\mathcal{U}$, then we call x and y *infinitely close* (relative to $\mathcal{U}$) and write $x \approx_{\mathcal{U}} y$, or simply $x \approx y$, provided $(x,y) \in \mu(\mathcal{U})$. For an arbitrary set A in X (possibly, external) the set $\mu_{\mathcal{U}}(A)$ is called the *microhalo* of the set A in X and denoted by ${}^{\approx}A$. If the set A is standard, then, taking the liberty of being inconsistent, the symbol ${}^{\approx}A$ can be also used to denote the halo $h(A)$ of the set A, meaning the equality $h(A) = {}^{\approx\circ}A$. It goes without saying that in this case the halo is calculated relative to the uniform topology $\tau_{\mathcal{U}}$ generated by $\mathcal{U}$. It should be remarked that in such a topology the monad of a standard point x consists, as it might be expected, of all the points infinitely close to it, i.e., it is the microhalo ${}^{\approx}x := {}^{\approx}\{x\}$ of this point. Sometimes, when a terminology less adequate to the essence of the matter is used, the microhalo ${}^{\approx}x$ of an internal point x is called the monad of this point.

4.4.5. A function f acting from a uniform space X into a uniform space Y and transferring infinitely close points into infinitely close ones, is termed *microcontinuous* over X.

4.4.6. *The following statements are valid:*

(1) *a standard function is microcontinuous iff it is uniformly continuous;*

(2) *a standard set consists of microcontinuous functions iff this set is (uniformly)* *equicontinuous.*

◁ (1) The uniform continuity of $f\colon X \to Y$ implies that $f^{\times}(\mathcal{U}_X) \supset \mathcal{U}_Y$, where $\mathcal{U}_X, \mathcal{U}_Y$ are the uniformities of X and Y, respectively, and $f^{\times}(x,x') := (f(x), f(x'))$ for $x, x' \in X$. Taking into account **4.1.8**, we deduce

$$f^{\times}(\mathcal{U}_X) \supset \mathcal{U}_Y \leftrightarrow \mu(f^{\times}(\mathcal{U}_X)) \subset \mu(\mathcal{U}_Y).$$

(2) The set $\mathcal{E} \subset Y^X$ is, as is known, termed equicontinuous, if $(\forall V \in \mathcal{U}_Y)$ $f^{\times -1}(V) = f^{-1} \circ V \circ f \in \mathcal{U}_X$. Therefore, for such an $\mathcal{E}$, by the transfer principle, we have $(\forall^{\mathrm{st}} V \in \mathcal{U}_Y) \times (\exists^{\mathrm{st}} U \in \mathcal{U}_X)\ (\forall f \in \mathcal{E})\ (\forall x, x' \in U)\ (f(x), f(x')) \in V$. In particular, if $x \approx x'$, then for every $f \in \mathcal{E}$ for any $V \in^{\circ} \mathcal{U}_Y$ we get $(f(x), f(x')) \in V$, i.e., $f(x) \approx f(x')$. Therefore, an equicontinuous standard set has only microcontinuous elements.

In order to prove the reverse implication, let us, for the sake of diversity, make use of the Cauchy principle (**4.1.17**). Indeed, for $V \in {}^{\circ}\mathcal{U}_Y$ and an arbitrary remote element $U \in {}^{a}\mathcal{U}_X$ we have $(\forall f \in \mathcal{E})\ f^{\times}(U) \subset V$. Hence, the same internal property is also valid for a certain standard $U \in \mathcal{U}_X$. Now we are to apply the transfer principle. ▷

4.4.7. *Let $(X, \mathcal{U}_X), (Y, \mathcal{U}_Y)$ be standard uniform spaces, and let f be an internal function; $f\colon X \to Y$. Let, then, ${}^{E}\mathcal{U}_X, {}^{E}\mathcal{U}_Y$ be the filters of external supersets of ${}^{\circ}\mathcal{U}_X, {}^{\circ}\mathcal{U}_Y$, respectively. In this case the following statements are valid:*

(1) *f is microcontinuous;*

(2) *$f\colon (X, {}^{E}\mathcal{U}_X) \to (Y, {}^{E}\mathcal{U}_Y)$ is uniformly continuous;*

(3) $(\forall^{\mathrm{st}} V \in \mathcal{U}_Y)\ (\exists^{\mathrm{st}} U \in \mathcal{U}_X)\quad f^{\times}(U) \subset V.$

◁ (1) → (3). Let $V \in {}^{\circ}\mathcal{U}_Y$. For any remote element $U \in {}^{\circ}\mathcal{U}_X$ we have $(x, x') \in U \to x \approx x' \to f(x) \approx f(x')$, i.e., $f^{\times}(U) \subset V$. By the Cauchy principle (**4.1.17**), there is a standard U with the same property.

(3) → (1). Let us set $x \approx x'$ and a standard element $V \in \mathcal{U}_Y$. By hypothesis, for a certain standard $U \in \mathcal{U}_X$, we obtain $f^{\times}(U) \subset V$. In particular, $(f(x), f(x')) \in V$. Hence, $f(x) \approx f(x')$.

(3) → (2). This is obvious. ▷

4.4.8. Examples

(1) Let X be a set, and d be a *semimetric* (= *deviation*) on X. In other words, there are (standard) objects X and $d: X^2 \to \mathbf{R}$ such that

$$d(x,x) = 0 \quad (x \in \mathbf{X});$$
$$d(x,y) = d(y,x) \quad (x,y \in X);$$
$$d(x,y) \leq d(x,z) + d(z,y) \quad (x,y,z \in X).$$

Let us consider the cylinders $\{d \leq \varepsilon\} := \{(x,y) \in X_2 : d(x,y) \leq \varepsilon\}$ and the family $\mathcal{U}_d := \{\{d \leq \varepsilon\} : \varepsilon \in \mathbf{R}, \varepsilon > 0\}$. Obviously, $\mathcal{U}_d$ provides X with the structure of a uniform space, i.e., the conventional uniformity of a *semimetric space* (X,d). *It should be remarked that the monad of this uniformity is defined by the following equivalence relation:*

$$x \approx_d y \leftrightarrow d(x,y) \approx 0 \leftrightarrow d(x,y) \in \mu(\mathbf{R}).$$

(2) Let $(X, \mathfrak{M})$ be *a multimetric space*, i.e., $\mathfrak{M}$ is a *multimetric* (= a nonempty set of semimetrics on X). The monad $\mu(\mathfrak{M})$ is defined as the intersection of the monads of (standard) uniform spaces (X,d), where $d \in {}^\circ\mathfrak{M}$. Namely,

$$x \approx_{\mathfrak{M}} y \leftrightarrow (\forall d \in {}^\circ\mathfrak{M})\, d(x,y) \approx 0.$$

The monad $\mu(\mathfrak{M})$ is, undoubtedly, the monad of the uniformity $\mathcal{U}_{\mathfrak{M}} := \sup\{\mathcal{U}_d : d \in \mathfrak{M}\}$ of the multimetric space $(X, \mathfrak{M})$ under consideration. It would be expedient to recall that every uniform space $(X, \mathcal{U})$ is multimetrizable, i.e., $\mathcal{U} = \mathcal{U}_{\mathfrak{M}}$ for a suitable multimetric $\mathfrak{M}$.

(3) Let $(X, \mathcal{U})$ be a uniform space. Let us endow the space $\mathcal{P}(X)$ with the *Vietoris uniformity*, whose neighbourhood filterbase is composed of the sets:

$$\{(A,B) \in \mathcal{P}(X)^2 : B \subset U(A), A \subset U(B)\},$$

where $U \in \mathcal{U}$. Obviously, the monad $\mu_v := \mu_v(\mathcal{U})$ of the Vietoris uniformity has the form:

$$\mu_v = \{(A,B) : A \subset {}^{\bullet}B,\ B \subset {}^{\bullet}A\}.$$

(4) Let (X,τ) be a compactum, i.e., a Hausdorff compact space. This space is (uniquely) uniformizable, i.e., a filter of $\mathcal{U}$ such that the uniform topology $\tau_{\mathcal{U}}$ coincides with τ is the neighbourhood filter of the diagonal in X^2. Therefore, $\mu(\mathcal{U}) = \mu_{\tau\times\tau}(I_X)$. In other words, $x \approx x' \leftrightarrow \mathrm{st}(x) = \mathrm{st}(x')$, since $\mu_{\tau\times\tau}(x,x) = \mu_\tau(x) \times \mu_\tau(x)$ for a standard point

x (see **4.2.1**) and every point of X^2 is nearstandard (see **4.3.6**).

(5) Let X,Y be nonempty sets, $\mathcal{U}_Y$ be a uniformity in Y, and $\mathcal{B}$ be a family of subsets of X filtered upwards by inclusion. Let us consider the uniformity $\mathcal{U}$ in Y^X, which is called *the uniformity of uniform convergence on the sets of* $\mathcal{B}$. The family of $\mathcal{U}$ is a union of the supersets of the following elements:

$$V_{B,U} := \{(f,g) \in Y^X \times Y^X : g \circ I_B \circ f^{-1} \subset U\}$$

where $B \in \mathcal{B}$ and $U \in \mathcal{U}_X$. It is obvious that

$$(f,g) \in \mu(\mathcal{U}) \leftrightarrow (\forall^{st} B \in \mathcal{B})(\forall^{st} U \in U_Y)(\forall x \in B)\,(f(x),g(x) \in U$$
$$\leftrightarrow (\forall^{st} B \in \mathcal{B})(\forall x \in B)\; f(x) \approx g(x) \leftrightarrow (\forall x \in \mu(\mathcal{B}))\; f(x) \approx g(x),$$

where, as usual, $\mu(\mathcal{B}) := \cup\, {}^{\circ}\mathcal{B}$ is the monad of the family $\mathcal{B}$. If $\mathcal{B} = \{X\}$, then we speak about *the strong uniformity* $\mathcal{U}_s$ on X. The following relation is obvious:

$$(f,g) \in \mu(\mathcal{U}_s) \leftrightarrow (\forall x \in X)\; f(x) \approx g(x).$$

If $\mathcal{B} = \mathcal{P}_{\text{fin}}(X)$, then $\mu(\mathcal{B}) = {}^{\circ}X$ and, hence, for the corresponding *weak convergence* $\mathcal{U}_\omega$ (or, which is the same by definition, for *the uniformity of pointwise convergence*), we get

$$(f,g) \in \mu(\mathcal{U}_\omega) \leftrightarrow (\forall^{st} x \in X)\; f(x) \approx g(x).$$

4.4.9. A set A is called *infinitely small* (relative to the uniformity $\mathcal{U}$), provided $A^2 \subset \mu(\mathcal{U})$, i.e., if any two points of A are infinitely close.

4.4.10. *For a standard filter* $\mathcal{F}$ *in* $(X,\mathcal{U})$ *the following statements are valid:*

(1) *the monad* $\mu(\mathcal{F})$ *is infinitely small;*

(2) *the filter* $\mathcal{F}$ *is a Cauchy filter;*

(3) *for any* $U \in {}^{\circ}\mathcal{U}$ *there is an* $x \in {}^{\circ}X$ *such that* $\mu(\mathcal{F})^2 \subset U(x)$.

$\triangleleft$ (1) $\rightarrow$ (2). Let $\mu(\mathcal{F})^2 \subset \mu(\mathcal{U})$. Obviously, $\mu(\mathcal{F})^2 = \mu(\mathcal{F}^{\times})$, where $\mathcal{F}^{\times} := \{F^2 : F \in \mathcal{F}\}$, as

$$(x,y) \in \mu(\mathfrak{F}^{\times}) \leftrightarrow (\forall^{st} F \in \mathfrak{F})\, x \in F \wedge y \in F \leftrightarrow x \in \mu(\mathfrak{F}) \wedge y \in \mu(\mathfrak{F}).$$

Therefore, $\mu(\mathfrak{F}^{\times}) \subset \mu(\mathcal{U})$, i.e., $\mathfrak{F}^{\times} \supset \mathcal{U}$. The last result implies that $\mathfrak{F}$ is a Cauchy filter.

(2) → (3). For $U \in {}^{\circ}\mathcal{U}$ there is a standard element $F \in \mathfrak{F}$ for which $F \times F \subset U$. If $x \in {}^{\circ}F$, then $(\forall^{st} y \in F)\; y \in U(x)$. Hence, $F \subset U(x)$, and, moreover, $\mu(\mathfrak{F}) \subset U(x)$.

(3) → (1). Applying idealization, we see that $(\exists x \in X)\; \mu(\mathfrak{F}) \subset {}^{\approx}x$. Hence, $\mu(\mathfrak{F})$ is infinitely small. ⊳

4.4.11. *A Cauchy filter converges iff its monad contains a nearstandard point.*

⊲ → If $\mathfrak{F}$ is the filter under consideration, then $\mu(\mathfrak{F}) \subset \mu(x)$ as soon as $\mathfrak{F} \to x$. Any point of $\mu(\mathfrak{F})$ is nearstandard.

← Let $\mu(\mathfrak{F}) \cap {}^{\approx}x \neq \varnothing$. For $y \in \mu(\mathfrak{F})$ and $z \in \mu(\mathfrak{F}) \cap {}^{\approx}x$ we get $y \approx z \approx x$, i.e., $y \approx x$. Hence, $\mu(\mathfrak{F}) \subset \mu(x)$ and now we have to appeal to **4.1.7.** ⊳

4.5. Pre-Neartandardness, Completeness and Total Boundedness

As is known, in uniform spaces a convenient indication of compactness, the classical Hausdorff criterion, is ensured. In the present section we consider its nonstandard analogues and the criteria of prenearstandardness pertaining to them in spaces of continuous functions.

4.5.1. *For an (internal) point of a (standard) uniform space X the following statements are equivalent:*

(1) *the microhalo of x is a monad of a certain (standard) filter in X;*

(2) *the microhalo of X is a monad of a certain Cauchy filter in X;*

(3) *the microhalo of x coincides with a monad of a minimal (by inclusion) Cauchy filter;*

(4) *the microhalo of x contains a certain (infinitely small) monad;*

(5) *There is a standard generalized sequence $(x_\xi)_{\xi\in\Xi}$ of X elements microconverging to x, i.e. such that for all remote elements $\xi \in {}^a\Xi$ we have $x_\xi \approx x$.*

$\lhd$ (1) $\to$ (2). If ${}^{-}x = \mu(\mathfrak{F})$ for a certain standard filter $\mathfrak{F}$, then $\mu(\mathfrak{F})$ is infinitely small (since the microhalo ${}^{-}x$ is infinitely small).

(2) $\to$ (3). Let ${}^{-}x = \mu(\mathfrak{F})$, $\mathfrak{F}'$ is a Cauchy filter and $\mathfrak{F}' \subset \mathfrak{F}$. Then, by virtue of **4.1.17**, $\mu(\mathfrak{F}') \supset \mu(\mathfrak{F}) = {}^{-}x$. If $y \in \mu(\mathfrak{F}')$, then since $\mu(\mathfrak{F}')$ is infinitely small, we get $y \approx x$, i.e., $\mu(\mathfrak{F}') = \mu(x) = \mu(\mathfrak{F})$. Hence, $\mathfrak{F}' = \mathfrak{F}$ (see **4.1.4**).

(3) $\to$ (4). It is obvious.

(4) $\to$ (1). Let us assume that ${}^{-}x \supset \mu(\mathfrak{F})$ and the filter $\mathfrak{F}$ is a Cauchy filter. Let us set $\mathfrak{F}' := \{U(F) : U \in \mathcal{U}_X, F \in \mathfrak{F}\}$. For $\mathcal{U} := \mathcal{U}_X$ we have the following relations:

$$
\begin{aligned}
{}^{-}\mu(\mathfrak{F}) &= \mu(\mathcal{U})(\mu(\mathfrak{F})) = \mu(\mathcal{U})\Big(\bigcap_{F\in{}^\circ\mathfrak{F}} F\Big) = \bigcap_{F\in{}^\circ\mathfrak{F}} \mu(\mathcal{U})(F) \\
&= \bigcap_{F\in{}^\circ\mathfrak{F}} \bigcap_{U\in{}^\circ\mathcal{U}} U(F) = \bigcap\{F' : F' \in {}^\circ\mathfrak{F}'\} = \mu(\mathfrak{F}').
\end{aligned}
$$

Obviously, ${}^{-}\mu(\mathfrak{F}) \supset {}^{-}x$. Hence, $\mu(\mathfrak{F}) = {}^{-}x = \mu(\mathfrak{F}')$.

(4) $\to$ (5). If $\mathfrak{F}$ is a filter and $\mu(\mathfrak{F}) \subset {}^{-}x$, then, choosing in a conventional way one point from each standard $F \in {}^\circ\mathfrak{F}$ and applying standardization, we construct the sequence required. Conversely, if $(x_\xi)_{\xi\in\Xi}$ microconverges to x, then the monad of the tail filter of this sequence is contained in the microhalo ${}^{-}x$. $\rhd$

4.5.2. A point x obeying one (and, hence, all) of the equivalent conditions **4.5.1** (1) - (4), is called *prenearstandard* in X. The external set of all prenearstandard points in X is denoted by $\mathrm{pst}(X)$.

4.5.3. *Nearstandard points (relative to the uniform topology) are prenearstandard.*

$\lhd$ Let $x \in \mathrm{nst}(X)$ for the space under study $(X, \mathcal{U})$. Hence, for a certain $y \in {}^\circ X$ we have $x \in {}^{-}y$. Therefore, ${}^{-}x \supset {}^{-}y = \mu(\tau_{\mathcal{U}}(y))$. In line with **4.5.1**, $x \in \mathrm{pst}(X)$. $\rhd$

4.5.4. *The image of a prenearstandard point under a uniformly continuous mapping is prenearstandard.*

◁ Let $\mathfrak{F}$ be a Cauchy filter and $\mu(\mathfrak{F}) \subset {}^{\approx}x$. Obviously, $f(\mathfrak{F})$ is a Cauchy filter in the image of X under the mapping f. Hence, $\mu(f(\mathfrak{F})) \subset {}^{\approx}f(x)$, i.e., $f(x)$ is a prenearstandard point (see **4.5.2**). ▷

4.5.5. *A point of the Tychonoff product of uniform spaces is prenearstandard iff its standard coordinates are prenearstandard.*

◁ Let $\mathfrak{X} := \prod_{\xi\in\Xi} X_\xi$ and $\mathcal{U}_{\mathfrak{X}} := \sup_{\xi\in\Xi} \Pr_\xi^{x-1}(\mathcal{U}_\xi)$ be the Tychonoff product of standard spaces $(X_\xi, \mathcal{U}_\xi)_{\xi\in\Xi}$. Let us choose an $x \in \mathrm{pst}(X)$. According to **4.5.1**, in $(\mathfrak{X}, \mathcal{U}_{\mathfrak{X}})$ there is a Cauchy filter $\mathfrak{F}$ such that ${}^{\approx}x = \mu(\mathfrak{F})$. For any standard $\xi \in \Xi$ we get, since $\Pr_\xi$ is continuous and by virtue of **4.4.6**, $\Pr_\xi({}^{\approx}x) \subset {}^{\approx}x_\xi$, i.e., ${}^{\approx}x_\xi \supset \Pr_\xi(\mu(\mathfrak{F})) = \mu(\Pr_\xi(\mathfrak{F}))$. Therefore, x_ξ is a prenearstandard point in X_ξ for $\xi \in {}^\circ\Xi$.

If for any $\xi \in {}^\circ\Xi$ we have ${}^{\approx}x_\xi = \mu(\mathfrak{F}_\xi)$ for a suitable choice of the filter $\mathfrak{F}_\xi$, then we can consider a filter

$$\mathfrak{F} := \sup_{\xi\in\Xi} \Pr_\xi^{-1}(\mathfrak{F}_\xi).$$

Obviously, the filter $\mathfrak{F}$ is standard and

$$\mu(\mathfrak{F}) = \bigcap_{\xi\in{}^\circ\Xi} \mu(\Pr_\xi^{-1}(\mathfrak{F}_\xi)) = \bigcap_{\xi\in{}^\circ\Xi} \Pr_\xi^{-1}(\mu(\mathfrak{F}_\xi))$$
$$= \bigcap_{\xi\in{}^\circ\Xi} \Pr_\xi^{-1}({}^{\approx}x_\xi) = \{y \in \mathfrak{X} : (\forall_\xi \in {}^\circ\Xi)\, y_\xi \approx x_\xi\} = {}^{\approx}x. \quad ▷$$

4.5.6. Nonstandard criterion for completeness. *A standard space is complete iff each of its prenearstandard points is nearstandard.*

◁ → Let X be a complete space, i.e., such that every Cauchy filter in X converges. Let us choose an $x \in \mathrm{pst}(X)$. By **4.5.2**, for a certain Cauchy filter $\mathfrak{F}$ we have $\mu(\mathfrak{F}) = {}^{\approx}x$. According to completeness, there is a $y \in {}^\circ X$ such that $\mu(y) \supset \mu(\mathfrak{F})$. Hence, ${}^{\approx}y = \mu(y) \supset \mu(\mathfrak{F}) \supset {}^{\approx}x$. Therefore, ${}^{\approx}y = {}^{\approx}x$, i.e., $x \in \mathrm{nst}(X)$.

← Let $\mathrm{nst}(X) = \mathrm{pst}(X)$ and $\mathfrak{F}$ be a Cauchy filter in X. Let us choose an $x \in \mu(\mathfrak{F})$. Then ${}^{\approx}x \supset \mu(\mathfrak{F})$ (as $\mu(\mathfrak{F})$ is an infinitely small set). By **4.5.2**, $x \in \mathrm{pst}(X)$. Hence,

$x \in \mathrm{nst}(X)$. Now we have to use **4.4.11.** ▷

4.5.7. *The Tychonoff product of complete uniform spaces is complete.*

◁ By virtue of the transfer principle, it suffices to consider the case of standard parameters. If standard factors are complete, then their every prenearstandard point is nearstandard (see **4.5.5**). We now have to recall that nearstandard points are those with nearstandard standard coordinates (see **4.3.10**), while prenearstandard points are those with prenearstandard standard coordinates (by **4.5.5**). Besides, we make use of the fact that the uniform topology of a product is the product of the uniform topologies of factors. ▷.

4.5.8. *The space of functions acting into a complete space becomes complete when endowed with the strong uniformity.*

◁ Let $(Y,\mathcal{U})$ be a complete standard uniform space, X be a standard set. Choose a prenearstandard point $f \in Y^X$. By virtue of **4.5.2** and **4.4.8**, this implies that there is a standard sequence $(f_\xi)_{\xi \in \Xi}$ of the elements of Y^X, for which

$$(\forall \xi \in {}^a\Xi)\,(\forall x \in X)\quad f_\xi(x) \approx f(x).$$

According to **4.5.7**, f is nearstandard in the weak uniformity, i.e., there is a standard element $g \in Y^X$ such that

$$(\forall \xi \in {}^a\Xi)\,(\forall^{\mathrm{st}} x \in X)\quad f_\xi(x) \approx g(x).$$

Hence, for every standard $x \in X$ the sequence $(f_\xi(x))_{\xi \in \Xi}$ converges to $g(x)$. By the transfer principle, $(\forall x \in X)\ f_\xi(x) \to g(x)$. Hence, $(\forall U \in {}^\circ\mathcal{U})\,(\forall x \in X)\ (f(x),g(x)) \in U$, which insures the fact that f is infinitely close to g in the strong uniformity. The proof is completed by referring to **4.5.6** and the transfer principle. ▷

4.5.9. *Let E be a set in a uniform space $(X,\mathcal{U})$. The following statements are equivalent:*

(1) *the set E is totally bounded, i.e., for every $U \in \mathcal{U}$ there is a finite set $E_0 \subset E$ such that $E \subset U(E_0)$ (for every $U \in \mathcal{U}$ there is a finite U-net);*

(2) *there is an internal finite cover of E by infinitely small internal sets;*

(3) *the set* E *has a finite skeleton, i.e., there is an internal finite set* E_0 *in* X *such that* E *lies in the microhalo* $^{\approx}E_0$;

(4) *the set* E *lies in the microhalo of an internal totally bounded set.*

◁ (1) ⇔ (2). Using the definition and principle of idealization, we deduce:

$$(\forall^{st} U \in \mathcal{U})(\exists E_0)(E_0 \subset E \wedge E_0 \in \mathcal{P}_{fin}(X) \wedge E \subset U(E_0))$$
$$\leftrightarrow (\forall^{st fin} \mathcal{U}_0 \subset \mathcal{U})(\exists E_0)(\forall U \in \mathcal{U}_0)(E_0 \subset E \wedge E_0 \in \mathcal{P}_{fin}(X) \wedge E$$
$$\subset U(E_0)) \leftrightarrow (\exists E_0)(\forall^{st} U \subset \mathcal{U})(E_0 \subset E \wedge E_0 \in \mathcal{P}_{fin}(X) \wedge E \subset U(E_0))$$
$$\leftrightarrow (\exists E_0 \subset E)(E_0 \in \mathcal{P}_{fin}(X) \wedge E \subset {}^{\approx}E_0).$$

(1) ↔ (3). Obviously, E is totally bounded iff for any standard $U \in \mathcal{U}$ there is a finite cover $\{E_1,\dots,E_n\}$ of the set E such that $E_k \times E_k \subset U$ (i.e., E_k is small of order U) for $k = 1,\dots,n$. Now the idealization principle is to be used.

(3) ↔ (4). This is selfevident.

(4) ↔ (1). Let U be a standard entourage. There is a symmetric element $V \in {}^{\circ}\mathcal{U}$ for which $V \circ V \subset U$. Obviously, for a finite E' in X we have $V(E') \supset E_0$, where E_0 is a given totally bounded set with the property $^{\approx}E_0 \supset E$. Hence, $U(E') \supset V \circ V(E') \supset V(E_0) \supset E$. ▷

4.5.10. *In every standard uniform space there is a universal finite skeleton, i.e., a common internal finite skeleton for all totally bounded standard sets of the initial space.*

◁ Recalling that the union of a finite number of totally bounded sets is totally bounded and making use of **4.5.9**, for a space X, a finite standard family $\mathcal{E}$ of totally bounded sets and a standard finite family $\mathcal{U}_0 \subset \mathcal{U}_X$ one can choose a common finite set in X which serves as a U-net of any $E \in \mathcal{E}$ for any $U \in \mathcal{U}_0$. Now employ idealization. ▷

4.5.11. Nonstandard criteria for total boundedness. *For a uniform space* X *the following statements are equivalent:*

(1) X *is totally bounded;*

(2) *every point of* X *is prenearstandard;*

(3) *the set* $\operatorname{pst}(X)$ *is internal;*

(4) *the set* X *has a finite skeleton.*

◁ (1) → (2). Let $x \in X$. For any standard $U \in \mathcal{U}$ there is a standard point $x' \in {}^\circ X$ for which $x \in U(x')$ is an element of the finite standard U-net for X. Let us set $\mathfrak{F} := {}^*\{U(x') : U \in {}^\circ\mathcal{U}\}$. Obviously, F is a Cauchy filter (see **4.4.10**). In this case, by construction, $x \in \mu(\mathfrak{F})$, i.e., $x \in \mathrm{pst}(X)$.

(2) → (3). This is obvious.

(3) → (1). Let us assume that for a certain standard $U \in \mathcal{U}$ and any finite standard $E \subset X$ the inclusion $\mathrm{pst}(X) \subset U(E)$ is not valid. According to the idealization principle this implies that there is an internal point $x \in \mathrm{pst}(X)$ with the property $x \notin U(y)$ for any $y \in {}^\circ X$. By definition **4.5.2**, we have ${}^{\approx}x = \mu(\mathfrak{F})$ for a suitable Cauchy filter $\mathfrak{F}$. Let us choose $F \in {}^\circ\mathfrak{F}$ in such a way that $F \times F \subset U$. Then for any $y \in {}^\circ F$ we get $x \in \mu(\mathfrak{F}) \subset U(y)$, which contradicts the assumption. Therefore, $(\forall^{\mathrm{st}} U \in \mathcal{U})(\exists^{\mathrm{st\,fin}} E \subset X)\ \ U(E) \supset \mathrm{pst}(X)$. Now we have to recall that $\mathrm{pst}(X) \supset {}^\circ X$.

(1) → (4). For the proof see **4.5.9**. ▷

4.5.12. Hausdorff criterion. *A uniform space is compact iff it is complete and totally bounded.*

◁ → If a space X is compact (and standard), then every point in it is nearstandard and, hence, prenearstandard (by **4.5.3**), X is totally bounded (by **4.5.11**), and X is complete (by **4.5.6**).

Since X is totally bounded; therefore, according to **4.5.11**, $X = \mathrm{pst}(X)$. Since X is complete, by **4.6.6**, $\mathrm{pst}(X) = \mathrm{nst}(X)$. And, finally, $X = \mathrm{nst}(X)$, i.e., X is compact (see **4.3.6**). ▷

4.5.13. *Let* X *be an arbitrary set,* Y *be a uniform space, and* $f : X \to Y$ *be a (standard) function. The following statements are equivalent:*

(1) f *is a totally bounded mapping, i.e.,* $\mathrm{im} f$ *is totally bounded in* Y;

(2) *there is an internal finite cover* $\mathfrak{E}$ *of the set* X *such that* $f(E)$ *is infinitely small for every* $E \in \mathfrak{E}$, *i.e.,* f *is a nearstep function relative to* $\mathfrak{E}$;

(3) *there is an internal* $n \in \mathbf{N}$ *and a set* $\{X_1,\dots,X_n\}$ *of external mutually disjoint sets such that* $X_1 \cup \dots \cup X_n = X$ *and* $f(x) \approx x'$ *for all* $x, x' = X_k$ *and every* $k := 1,\dots,n$.

◁ (1) → (2). In line with **4.5.9**, there is an internal finite cover $\mathcal{E}$ of the set im$f$ such that $E \in \mathcal{E} \to E_2 \subset \mu(\mathcal{U}_Y)$. Let us set $\mathcal{E}' := \{f^{-1}(E) : E \in \mathcal{E}\}$. Obviously, $\mathcal{E}'$ is the sought cover of X.

(2) → (3). This is obvious.

(3) → (1). Let us choose $y_k \in f(X_k)$ and set $E := \{y_k : k = 1,\dots,n\}$. Obviously, E is a finite internal set. By hypothesis, E is a skeleton of $f(X)$. Hence, according to **4.5.9**, im$f$ is totally bounded. ▷.

4.5.14. *The space* $CB(X,Y)$ *of totally bounded mappings from* X *to* Y *is complete in the strong uniformity.*

◁ By virtue of **4.5.8**, it suffices to establish that $CB(X,Y)$ is closed. Hence, let a standard $f : X \to Y$ be such that for a certain totally bounded function g we have $(\forall x \in X)\ f(x) \approx g(x)$. Obviously, im$f \subset {}^{\approx}img$. Making use of the fact that im$g$ is completely limited, and taking into account **4.2.5** and **4.5.9**, we deduce: $f \in \mathrm{cl}\, CB(X,Y) \to f \in CB(X,Y)$. ▷

4.5.15. A finite cover $\mathcal{E}$ of the set X is called *tiny* if it is refined into every standard finite cover $\mathcal{E}_0$ of the standard set X, i.e., if every set of $\mathcal{E}$ is contained in a certain set of $\mathcal{E}_0$. The mapping which acts from X into a uniform space and is nearstep relative to every tiny cover of X, is called *microstep* on X.

4.5.16. Criterion for prenearstandardness in $CB(X,Y)$. *A function* $f : X \to Y$, *where* Y *is a complete uniform space, is prenearstandard in* $CB(X,Y)$ *(relative to the strong uniformity) iff* f *is microstep on* X *and the image of* f *is composed of nearstandard points of* Y.

◁ On the basis of **4.5.11** and **4.5.6** we conclude that f is nearstandard in the strong uniformity. Therefore, for a certain $g \in {}^{\circ}CB(X,Y)$ we get $f(x) \approx g(x)$ for all $x \in X$. Obviously, im$f \subset {}^{\approx}$im$g$. Besides, im$g \subset \mathrm{pst}(Y)$ (see **4.5.13**). If now $\mathcal{E}$ is a tiny cover, then, making use of the definition of total boundedness, for every standard $V \in \mathcal{U}_Y$ one can find a standard finite cover $\mathcal{E}'$ in X such that $g(E)^2 \subset V$ for any $E \in \mathcal{E}'$. Therefore,

$(\forall E \in \mathfrak{E})$ $g(E)^2 \subset V$, i.e., g is nearstep on $\mathfrak{E}$. Hence, for $E \in \mathfrak{E}$ and $x, x' \in E$ we have $g(x) \approx f(x) \approx f(x') \approx g(x')$, i.e., f is also nearstep relative to $\mathfrak{E}$. Since $\mathfrak{E}$ is arbitrary, the mapping f is microstep.

$\leftarrow$ Since $\operatorname{im} f \subset \operatorname{nst}(Y)$, we have $(\forall x \in X)\,(\exists^{\mathrm{st}} y \in Y)\,(\forall^{\mathrm{st}} W \in \tau(y))\; f(x) \in W$. Applying the rule for introducing standard functions, we get $(\forall^{\mathrm{st}} W(\cdot))\,(\forall x \in X)$ $(\exists^{\mathrm{st}} y \in Y)$ $f(x) \in \;\in W(y)$. According to the idealization principle, we conclude $(\forall^{\mathrm{st}} W(\cdot))\,(\exists^{\mathrm{st}}$ $\{y_1, \dots, y_n\})$ $(\forall x \in X)\,(\exists k)\; f(x) \in W(y_k)$. Let us now choose $V \in \mathcal{U}_Y$. By hypothesis, for every tiny cover $\mathfrak{E}$ of the set X and for $E \in \mathfrak{E}$ we have $f(E)^2 \subset V$. Applying the Cauchy principle **4.1.17** (taking it into account that tiny covers are remote elements of the directed set of finite covers), we see that there is a standard finite cover $\mathfrak{E}_V$ such that $f(E)^2 \subset V$ for $E \in \mathfrak{E}_V$.

Let us choose a corresponding standard cover $\mathfrak{E}_V$ and a standard finite set Y_0 of Y elements, for which $\operatorname{im} f \subset V(Y_0)$.

Using $\mathfrak{E}_V$ and Y_0, we can easily construct a standard step function f_V such that $(\forall x \in X)$ $(f_V(x), f(x)) \in V$. Obviously, for $U \in {}^{\circ}\mathcal{U}_Y$ obeying the conditions $U = U^{-1}$ and $U \circ U \subset V$ we get $(f_{V'}(x), f_{V''}(x)) \in V' \circ V''^{-1} \subset U \circ U \subset V$ for any $V', V'' \subset U$. Therefore, the standard net $(f_V)_{V \in \mathcal{U}_Y}$ (in more detail, ${}^*\{f_V\colon V \in {}^{\circ}\mathcal{U}_Y\}$) is fundamental. Let us denote by g its standard limit in $CB(X,Y)$. As above, we have $(\forall^{\mathrm{st}} V \in \mathcal{U}_Y)\,(\forall_x \in X)\,(g(x), f(x)) \in V$. And, finally, $g \approx f$ in the strong uniformity. Therefore, f is nearstandard, and, hence, also prenearstandard by virtue of completeness of $CB(X,Y)$ discussed in **4.5.14**. $\triangleright$

4.5.17. Nonstandard criteria for relative compactness. *In a complete separated space X the following statements are equivalent for a set E:*

(1) *E is relatively compact;*

(2) *E is precompact (i.e., the completion of E is compact);*

(3) *E is totally bounded;*

(4) $E \subset \operatorname{pst}(X)$;

(5) $E \subset \operatorname{nst}(X)$;

(6) *E lies in the microhalo of a finite set;*

(7) $\operatorname{cl} U$ *has a finite skeleton.*

◁ Since X is complete, by **4.5.6**, pst(X) = nst(X). Therefore, (5) → (1) → (4) (see **4.3.8**). Obviously, (7) → (6) → (3) → (1) → (2). If (2) is fulfilled, then cl E is complete and totally bounded by the Hausdorff criterion. Making use of **4.5.11**, we deduce (2) → (7). ▷

4.5.18. Criteria for prenearstandardness in $C(X,Y)$. *Let X be a compact set, Y be a complete uniform space, and $C(X,Y)$ be the space of continuous functions from X to Y endowed with strong uniformity. For an internal element $f \in C(X,Y)$ the following statements are equivalent:*

(1) *f is prenearstandard;*

(2) *f is nearstandard;*

(3) *f is microcontinuous and transforms standard points into nearstandard points.*

◁ (1) → (2). Obviously, f is prenearstandard in Y^X in the strong uniformity by virtue of **4.5.4**, while by **4.5.8** and **4.5.6** f is nearstandard in Y^X, i.e., there is a standard $g \in Y^X$, for which $f(x) \approx g(x)$ for all $x \in X$. Let $(f_\xi)_{\xi \in \Xi}$ be a standard sequence in $C(X,Y)$ microconverging to f. Let us choose an $x' \approx x$ and remark that $f_\xi(x') \approx f_\xi(x)$ for all standard $\xi \in \Xi$ (as f_ξ is continuous and X is compact). Then (cf. **3.3.17** (3)) for a certain $\eta \in {}^a\Xi$ we get $f_\eta(x') \approx f_\eta(x)$. Hence, we deduce $g(x') \approx f(x') \approx f_\eta(x') \approx f_\eta(x) \approx f(x) \approx g(x)$. Therefore, the standard function g is microcontinuous and, hence, by **4.4.6**, $g \in C(X,Y)$.

(2) → (3). By hypothesis, for a certain continuous function g it is fulfilled that $g(x) \approx f(x)$ for all $x \in X$. Therefore, $f({}^\circ X) \subset {}^{\approx}g({}^\circ X) \subset {}^{\approx}g(X) \subset \mathrm{nst}(Y)$. Moreover, by **4.5.6**, g is microcontinuous and, hence, for $x' \approx x$ we get $f(x) \approx g(x) \approx g(x')$.

(3) → (1). By **4.5.3**, we are to make sure that (3) → (2). Let us choose a microcontinuous f for which $f({}^\circ X) \subset \mathrm{nst}(X)$. According to the construction principle, there is a standard g such that $g(x) \in {}^\circ f(x)$. Let us check that g is uniformly continuous. To this end, choose a standard entourage $V \in \mathcal{U}_Y$, and a standard $W \in \mathcal{U}_Y$ from the condition $W \circ W \circ W \subset V$. Making use of **4.5.7**, find a standard U of the unique uniformity $\mathcal{U}_X$ (see **4.4.8** (4)), so that $f^\times(U) \subset W$. At $(x,x') \in U$ for standard $x,x' \in {}^\circ X$ we get $(f(x), f(x')) \in W, (f(x'),g(x')) \in W, (g(x), f(x)) \in W$. Therefore, $(g(x), g(x')) \in W \circ W \circ W \subset V$. And, finally,

$$(\forall^{\mathrm{st}} V \in \mathcal{U}_Y)\,(\exists^{\mathrm{st}} U \in \mathcal{U}_X)\,(\forall^{\mathrm{st}} x, x' \in U)\,(g(x), g(x')) \in V.$$

Therefore, by the transfer principle, we get $g \in C(X,Y)$. Now for an arbitrary $x \in X$ we deduce $f(x) \approx f(x') \approx g(x') \approx g(x)$, where x' is the only standard point infinitely close to x. Hence, f is infinitely close to g in the strong uniformity. ▷

4.5.19. Ascoli-Arzela theorem. *Let X be a compact set, Y be a complete separated uniform space, and $E \subset C(X,Y)$. The set E is relatively compact in the strong uniformity iff E is equicontinuous and uniformly (totally) bounded (i.e., for a certain totally bounded C in Y we have $f(X) \subset C$ for all $f \in F$).*

◁ The proof follows from **4.5.18**, **4.5.17** and **4.4.6** (2). ▷

CHAPTER 5

INFINITESIMALS AND SUBDIFFERENTIALS

The nonstandard methods of analysis have been applied to various fields of mathematics. In the present chapter we shall consider the use of infinitesimals in *subdifferential calculus*, one of the new branches of functional analysis which originated from evolution of the theory of extremal problems. When studying optimization problems, a significant attention is paid to the search for convenient convex approximations to rather arbitrary functions and sets. The point is that for convex problems a quite powerful and effective technique of theoretical analysis has been developed and the corresponding calculation algorithms have been constructed. The ways of local approximation to sets and functions being developed in subdifferential calculus are related to constructing quite complex and often cumbersome formulas. The arising notions such as hypertangents, Rockafeller limits, and Clarke derivatives, seem to be difficult to understand when first encountered, since it is too complicated to comprehend the sense of their formal definitions.

Nonstandard analysis offers effective simplifying procedures since the use of the external notions legalized by it *'kills quantifiers'* essentially simplifies the complexity of perception of the standard constructions described. Below we shall basically study the evolution and examples of these operations for classifying one-sided tangents to arbitrary functions and sets.

It should be emphasized that many constructions described in the present chapter have a wider range of applicability than subdifferential calculus in the context of which the presentation below will be given.

5.1. Topologies in Vector Spaces

Studies of local approximations in vector spaces are associated with the characteristic features of the monads that introduce topologies agreeable with the structure available. It is these topologies that we shall study in the present section.

5.1.1. *Let U be a star-like set in a vector space, i.e.,* $[0,1]U \subset U$. *The set U absorbs a set*

V iff for any (and then for all) positive infinitesimal α we have $\alpha V \subset U$.

$\lhd$ As U absorbs V, by definition, there is a $\beta > 0$ for which $\beta V \subset U$. By the transfer principle, making use of the fact that both U and V are standard, we can conclude that $(\exists^{st}\beta > 0)\ \beta V \subset U$. If now $\alpha > 0$ and $\alpha \approx 0$, then $\alpha V = \alpha/\beta(\beta V) \subset \alpha/\beta U \subset U$. The remaining part of the statement is obvious. $\rhd$

5.1.2. Let x be a standard element of the standard vector space X under consideration. An external set $\{\alpha x\colon \alpha > 0, \alpha \approx 0\}$ is called *the radius-monad of x*, or *the infinitely small pointer to x*, or, finally, *the direction of x*. The family of the radius-monads of all the standard elements of X is termed *the direction monad* of this space and is denoted by $\mathrm{md}(X)$.

5.1.3. *A standard star-like set is absorbing in X iff it contains the direction monad* $\mathrm{md}(X)$ *of the space X.*

5.1.4. Nonstandard criterion for a vector topology. *Let X be a standard vector space over the basic field* $\mathbf{F}$, *and $\mathcal{N}$ be a standard filter in X. There is a vector topology τ on X such that $\mathcal{N} = \tau(0)$ iff the monad $\mu(\mathcal{N})$ of the filter $\mathcal{N}$ contains the direction monad* $\mathrm{md}(X)$ *and, besides, constitutes an external* ${}^{\approx}\mathbf{F}$*-submodule of X.*

(Here, as usual, ${}^{\approx}\mathbf{F} := \{t \in F\colon (\exists^{st} n \in \mathbf{N})\ |t| \le n\}$ is the finite part of the field $\mathbf{F}$ endowed with the natural structure of an external ring.)

$\lhd$ $\rightarrow$ Since addition is continuous at zero, $\mu(\mathcal{N}) + \mu(\mathcal{N}) = \mu(\mathcal{N})$; i.e., $\mu(\mathcal{N})$ is an external subgroup of X. Let $\alpha \in {}^{\approx}\mathbf{F}$ and $\mathcal{B}$ be a base of $\mathcal{N}$ consisting of balanced sets. If $n \in {}^{\circ}\mathbf{N}$ is such that $|\alpha| \le n$, then for $G \in {}^{\circ}\mathcal{B}$ and $x \in \mu(\mathcal{N})$ we have $\alpha/n\, x \in G$. Therefore, $\alpha/n\, x \in \cap\{G\colon G \in {}^{\circ}\mathcal{B}\} = \mu(\mathcal{B}) = \mu(\mathcal{N})$; hence, $\alpha x \in n\mu(\mathcal{N}) = \mu(\mathcal{N})$; and, finally, $\alpha\mu(\mathcal{N}) = \mu(\mathcal{N})$ for $\alpha \in {}^{\approx}\mathbf{F}$. We should observe that $\mathcal{N}$ has a base of absorbing sets and recall **5.1.3** in order to conclude $\mu(\mathcal{N}) \supset \mathrm{md}(X)$.

$\leftarrow$ Let us choose a $U \in {}^{\circ}\mathcal{N}$. By **4.1.4**, this implies $U \supset \mu(\mathcal{N})$. If W is an infinitely small element of $\mathcal{N}$, then its balanced hull V is also infinitely small (as $V \subset \mu(\mathcal{N})$). Besides, $V + V \subset \mu(\mathcal{N}) + \mu(\mathcal{N}) \subset \mu(\mathcal{N}) \subset U$. Hence,

$$(\forall^{st} U \in \mathcal{N})(\exists V \in \mathcal{N})\quad (V \text{ is balanced} \wedge V + V \subset U).$$

By the transfer principle, we conclude that $\mathcal{N} + \mathcal{N} = \mathcal{N}$ and, moreover, that $\mathcal{N}$ has a base of balanced sets. By virtue of **5.1.3**, we also obsreve that $\mathcal{N}$ is composed of balanced standard sets. Therefore, $\mathcal{N}$, indeed, determines a vector topology on X. ▷

5.1.5. *For every point x of the monad $\mu(X) := \mu(\tau(0))$ of a topological vector space there is an infinitely large natural number $N \in \mathbf{N} - {}^{\circ}\mathbf{N}$ such that $Nx \in \mu(X)$.*

◁ If V is a standard neighbourhood of zero and $n \in {}^{\circ}\mathbf{N}$, then (see **5.1.4**) the set $A(n,V) := \{m \in \mathbf{N}: m \geq n \wedge mx \in V\}$ is nonempty (since $\mu(X) \subset V$). According to the transfer principle, there is an element N for which $(\forall^{\mathrm{st}} n \in \mathbf{N})(\forall^{\mathrm{st}} U \in \tau(0))\ \ N \in A(n,V)$. It is obvious that the element N is the sought one. ▷

5.1.6. In applications it is sometimes convenient to consider *nearvector topologies.* Such a topology τ in the space X is characterized by the following properties: first, multiplication of the vectors of X by every scalar of the basic field is continuous and, second, addition is also continuous in both variables. The pair (X,τ), as well as X itself, is in this case called *a neartopological vector space.* This notion is natural, which can be easily understood from the following obvious statement.

5.1.7. Nonstandard criterion for a nearvector topology. *Let X be a vector space over $\mathbf{F}$. There is a nearvector topology τ on X such that $\tau(0)$ coincides with a fixed filter $\mathcal{N}$ iff the monad $\mu(\mathcal{N})$ is an external vector space over the external field of standard scalars ${}^{\circ}\mathbf{F}$.*

5.1.8. In relation to **5.1.7** we should remark that the monad of the neighbourhood filter of zero of a nearvector space is an external convex set. An internal convex set U contains, obviously, arbitrary convex combinations of its elements, i.e., for finite sets $\{\alpha_1,\dots,\alpha_N\}$ of positive scalars comprising unity as a sum, and a set $\{u_1,\dots \mu_N\}$ of the U elements we have $\Sigma_{k=1}^{N}\alpha_k u_k \in U$. Here N is an arbitrary (internal !) element of $\mathbf{N}$. The formulated property termed *hyperconvexity* is not valid for external convex sets (since in the world of external sets the induction on internal natural numbers completely fails). The examples confirming the above statement can be easily found by making use of the following expedient proposition.

5.1.9. Nonstandard criterion for a locally convex topology. *A vector topology is locally convex iff the monad of its filter of zero neighbourhoods is hyperconvex.*

◁ → Standard neighbourhoods of a locally convex topology contain standard convex and, hence, hyperconvex neighbourhoods. The intersection of hyperconvex external sets is also hyperconvex.

$\leftarrow$ Every standard neighbourhood of zero of the topology in question τ contains the convex hull of an infinitely small neighbourhood (this hull as a whole lies in the monad $\tau(0)$ due to its hyperconvexity). By the transfer principle, we conclude that any neighbourhood in $\tau(0)$ contains a convex neighbourhood of zero. ▷

5.1.10. By way of concluding this section, let us, deviating a bit from the basic direction of presenting the material, remark that the nonstandard analysis of topological vector spaces and operators in them is associated with studying the location of points of different types. In this case, alongside with nearstandard and prenearstandard points we have already encountered, an important role is played by specific notions of a 'bornological type'. Let us recall here only the simplest notions of the kind.

5.1.11. *Let (X,τ) be a locally convex space, and x be an internal point of X. The following statements are equivalent:*

(1) *for any infinitely small $\alpha \in \mathbf{F}$ the relation $\alpha x \approx_\tau 0$ is valid;*

(2) $x \in \bigcap_{V \in \tau(0)} \bigcup_{n \in {}^\circ\mathbf{N}} nV$;

(3) *for any standard continuous seminorm p (an element of the spectrum of τ) we have $p(x) \in {}^{\approx}\mathbf{F}$.*

◁ (1) ↔ (2). Let us make use of the Nelson algorithm:

$$(\forall \alpha \in F) \quad (\alpha \approx 0 \to \alpha x \approx 0)$$
$$\leftrightarrow (\forall^{st} V \in \tau(0))\, (\forall \alpha)\, ((\forall^{st} n \in \mathbf{N}) \quad |\alpha| \le n^{-1} \to \alpha x \in V)$$
$$\leftrightarrow (\forall^{st} V \in \tau(0))\, (\forall \alpha)\, (\exists^{st} n \in \mathbf{N}) \quad (|\alpha| \le n^{-1} \to \alpha x \in V)$$
$$\leftrightarrow (\forall^{st} V \in \tau(0))\, (\exists^{st} n \in \mathbf{N}) \quad x \in nV.$$

(1) ↔ (3). If p is a continuous seminorm, then for any $t \in {}^{\approx}\mathbf{R}$ we have $|t|\, p(x) = p(|t|\, x) \approx 0$ (see **4.2.7**). Hence, $p(x) \in {}^{\approx}\mathbf{R}$.

(3) ↔ (1). For every standard continuous seminorm p we have $p(\alpha x) = |\alpha| p(x) \approx 0$ as soon as $|\alpha| \approx 0$. Now we have to obsrve that the last fact implies that αx is infinitesimal in the topology τ. ▷

5.1.12. A point x satisfying the equivalent conditions **5.1.11** (1)-(3) is called *finite* (or *accessible*) *in* (X,τ). In this case we write $x \in \mathrm{fin}(X,\tau)$, or simply $x \in \mathrm{fin}(X)$ if there is no necessity to indicate the topology, and say that x belongs to *the finite* (or *accessible*) *part* of the space X.

5.1.13. Nonstandard criterion for boundedness. *Let X be a standard locally convex space. A standard set U in X is limited iff it is composed of finite points of X, i.e., if $U \subset \mathrm{fin}(X)$.*

$\lhd$ $\rightarrow$ If U is bounded, then there is a standard $t \in {}^{\circ}\mathbf{R}$ such that $p(U) \le t$ for a chosen continuous seminorm $p \in \mathfrak{M}_{\tau}$. Therefore, for $a \approx 0$ and $x \in U$ we get $p(\alpha x) \le t\alpha$, i.e., $\alpha x \approx 0$.

Let us now make use of the sequential criterion of boundedness (for diversity). Hence, let (α_n) be a standard sequence of scalars convergent to zero, and (u_n) be a standard sequence of the points of U. We have to show that $\alpha_n u_n \rightarrow 0$ Let N be an infinitely large number. Then $\alpha_N \approx 0$ and, hence, by condition **5.1.11** and (1), we get $\alpha_N u_N \approx 0$. $\rhd$

5.1.14. A point x of a space X is called *bounded* and we write $x \in \mathrm{bd}(X)$ provided there is a standard bounded set containing x.

5.1.15. Nonstandard criteria for seminormability. *Let X be a (separeted) locally convex space. The following statements are equivalent:*

(**1**) *X is normable*;

(**2**) $\mathrm{bd}(X) = \mathrm{fin}(X)$;

(**3**) $\mu(X) \subset \mathrm{bd}(X)$.

$\lhd$ (1) $\rightarrow$ (2). Obviously, $\mathrm{bd}(X) = \mathrm{fin}(X)$ with no hypotheses on X. If X is normable, then $\mathrm{fin}(X) = \{x \in X: \|x\| \in {}^{\bullet}\mathbf{R}\}$, where $\|\cdot\|$ is a suitable norm. Therefore, $\mathrm{fin}(X)$ lies, for instance, in the ball $B_X := \{x \in X: \|x\| \le 1\}$.

(2) $\rightarrow$ (3). Since $\mu(X)$ always lies in $\mathrm{fin}(X)$, the required result is obvious.

(3) $\rightarrow$ (1). Let U be an infinitely small neighbourhood in X. By condition, for every

$x \in U$ there is a standard set V such that V is bounded and $x \in V$. Therefore, according to the idealization principle, U lies in a certain bounded set. All we have to do now is to recall the Kolmogorov classical criterion. ▷

5.1.16. The above statement demonstrates, in particular, that in a general (not necessarily normable) case in a space there are more finite points than those bounded, while in a normable space X we, obviously, have $\operatorname{fin}(X) = \operatorname{bd}(X)$.

5.2. Classical Approximating and Regularizing Cones

In nonsmooth analysis there has been intensive search of convenient ways for local one-sided approximation to arbitrary functions and sets. A principal starting point of this search was the definition of subdifferential for a Lipshitz function given by F.Clarke [121]. Tangent cones and the corresponding derivatives constructed and studied in this respect are often defined by cumbersome and bulky formulas. Here we shall apply the nonstandard analysis as a method of 'killing quantifiers', i.e., simplifying complex formulas. Under a conventional supposition of standard entourage (in case when the free variables are standard (see **4.1.9**)) the Bouligand, Clarke and Hadamard cones and the regularizing cones pertaining to them prove to be determined by explicit infinitesimal constructions which appeal directly to infinitely close points and directions.

5.2.1. Let X be a real vector space. In this space let us, alongside with a fixed nearvector topology $\sigma := \sigma_X$ with the neighbourhood filter of zero $\mathcal{N}_\sigma := \sigma(0)$, single out a nearvector topology τ with a filter $\mathcal{N}_\tau := \tau(0)$. Following common practice, we introduce a relation of an infinite proximity associated with the corresponding uniformity: $x_1 \approx_\sigma x_2 \leftrightarrow x_1 - x_2 \in \mu(\mathcal{N}_\sigma)$, an analogous rule acting for τ. Below, if not otherwise stated, τ is considered to be a vector topology. In this case the monad of the neighbourhood filter $\sigma(x)$ will be denoted by $\mu(\sigma(x))$, while the monad $\mu(\sigma(0))$ simply by $\mu(\sigma)$.

5.2.2. In subdifferential calculus for fixed sets F in X and a point $x' \in X$ the following *Hadamard, Clarke, and Bouligand cones* are, in particular, considered:

$$\operatorname{Ha}(F, x') := \bigcup_{\substack{U \in \sigma(x') \\ \alpha'}} \operatorname{int}_\tau \bigcap_{\substack{x \in F \cap U \\ 0 < \alpha \le \alpha'}} \frac{F - x}{\alpha};$$

$$\mathrm{Cl}(F,x') := \bigcap_{V \in \mathfrak{N}_\tau} \bigcup_{\substack{U \in \sigma(x') \\ \alpha'}} \bigcap_{\substack{x \in F \cap U \\ 0<\alpha\le\alpha'}} \left(\frac{F-x}{\alpha} + V\right);$$

$$\mathrm{Bo}(F,x') := \bigcap_{\substack{U \in \sigma(x') \\ \alpha'}} \mathrm{cl}_\tau \bigcup_{\substack{x \in F \cap U \\ 0<\alpha\le\alpha'}} \frac{F-x}{\alpha},$$

where, as usual, $\sigma(x') := x' + \mathfrak{N}$. If $h \in \mathrm{Ha}(F,x')$ then we sometimes say that F *is epi-Lipshitz* in x' along h. Obviously,

$$\mathrm{Ha}(F,x') \subset \mathrm{Cl}(F,x') \subset \mathrm{Bo}(F,x').$$

5.2.3. We also distinguish the *cone of hypertangents, a cone of feasible directions and the contingency of F* at the point x' by the following relations:

$$H(F,x') := \bigcup_{\substack{U \in \sigma(x') \\ \alpha'}} \bigcap_{\substack{x \in F \cap U \\ 0<\alpha\le\alpha'}} \frac{F-x}{\alpha};$$

$$\mathrm{Fd}(F,x') := \bigcap_{\alpha'>0} \frac{F-x'}{\alpha'};$$

$$K(F,x') := \bigcap_{\alpha'} \mathrm{cl}_\tau \bigcap_{0<\alpha\le\alpha'} \frac{F-x'}{\alpha}.$$

For the sake of economizing words it is expedient to assume $x' \in F$. For instance, one can obviously say that the cones $F(F,x')$ and $K(F,x')$ are the Hadamard and Bouligand cones, respectively, for the case in which τ or σ is the discrete topology. Therefore, below we always assume $x' \in F$ with the following abbreviations taken to save the space:

$$(\forall^{\bullet} x)\varphi := (\forall x \approx_\sigma x')\ \varphi := (\forall x)\ (x \in F \wedge x \approx_\sigma x') \to \varphi,$$
$$(\forall^{\bullet} h)\varphi := (\forall h \approx_\tau h')\ \varphi := (\forall h)\ (h \in X \wedge h \approx_\tau h') \to \varphi,$$
$$(\forall^{\bullet} \alpha)\varphi := (\forall \alpha \approx 0)\ \varphi := (\forall \alpha)\,(\alpha > 0 \wedge \alpha \approx 0) \to \varphi.$$

The quantifiers $\exists^{\bullet} x$, $\exists^{\bullet} h$, $\exists^{\bullet} \alpha$ are determined in the natural way by duality, i.e., we assume that

$$(\exists^{\bullet} x)\varphi := (\exists x \approx_\sigma x')\ \varphi := (\exists x)\,(x \in F \wedge x \approx_\sigma x') \wedge \varphi,$$
$$(\exists^{\bullet} h)\varphi := (\exists h \approx_\tau h')\ \varphi := (\exists h)\ (h \in X \wedge h \approx_\tau h') \wedge \varphi,$$
$$(\exists^{\bullet} \alpha)\varphi := (\exists \alpha \approx 0)\ \varphi := (\exists \alpha)\,(\alpha > 0 \wedge \alpha \approx 0) \wedge \varphi.$$

Let us establish that the cones under discussion are defined by simple infinitesimal constructions.

5.2.4. *The Bouligand cone is the standardization of the* $\exists\exists\exists$*-cone*; *i.e., for a standard element* h' *we have*:

$$h' \in \mathrm{Bo}(F,x')\ (\exists x)\,(\exists \alpha)\,(\exists h)\ x + \alpha h \in F.$$

◁ The following equivalences follow from the definition of the Bouligand cone:

$$\begin{aligned}
&h' \in \mathrm{Bo}(F,x') \\
&\leftrightarrow (\forall U \in \sigma(x'))\,(\forall \alpha' \in \mathbf{R})\,(\forall V \in \mathcal{N}_\tau)\,(\exists x \in F \cap U) \\
&(\exists 0 < \alpha \le \alpha')\,(\exists h \in h' + V)\ x + \alpha h \in F \\
&\leftrightarrow (\forall U)\,(\forall \alpha')\,(\forall V)\,(\exists x)\,(\exists \alpha)\,(\exists h) \\
&(x \in F \cap U \wedge h \in h' + V \wedge 0 < \alpha \le \alpha' \wedge x + \alpha h \in F).
\end{aligned}$$

By virtue of the transfer principle we deduce:

$$\begin{aligned}
&h' \in \mathrm{Bo}(F,x') \leftrightarrow (\forall^{\mathrm{st}} U)\,(\forall^{\mathrm{st}} \alpha')\,(\forall^{\mathrm{st}} V)\ (\exists^{\mathrm{st}} x)\,(\exists^{\mathrm{st}} \alpha)\,(\exists^{\mathrm{st}} h) \\
&(x \in F \cap U \wedge h \in h' + V \wedge 0 < \alpha \le \alpha' \wedge x + \alpha h \in F).
\end{aligned}$$

Let us now use the weak idealization principle and obtain:

$$\begin{aligned}
&h' \in \mathrm{Bo}(F,x') \to (\exists x)\,(\exists \alpha)\,(\exists h)(\forall^{\mathrm{st}} U)\,(\forall^{\mathrm{st}} \alpha')\,(\forall^{\mathrm{st}} V) \\
&(x \in F \cap U \wedge h \in h' + V \wedge 0 < \alpha \le \alpha' \wedge x + \alpha h \in F) \\
&\to (\exists x \approx_\sigma x')\,(\exists \alpha \approx 0)\,(\exists h \approx_\sigma h')\ x + \alpha h \in F \\
&\to (\exists x)\,(\exists \alpha)\,(\exists h)\ x + \alpha h \in F.
\end{aligned}$$

Let, in turn, a standard element h' belong to the standardization of the '$\exists\exists\exists$-cone' Since standard elements of a standard filter contain the monads of this filter, we get

$$\begin{aligned}
&(\forall^{\mathrm{st}} U \in \sigma(x'))\,(\forall^{\mathrm{st}} \alpha' \in \mathbf{R})\,(\forall^{\mathrm{st}} V \in \mathcal{N}_\tau) \\
&(\exists x \in F \cap U)\,(\exists 0 < \alpha < \alpha')\,(\exists h \in h' + V)\ x + \alpha h \in F.
\end{aligned}$$

By virtue of the transfer principle, we conclude $h' \in \mathrm{Bo}(F,x')$. ▷

5.2.5. The just proved statement can be rewritten as

$$\mathrm{Bo}(F,x') = {}^*\{h' \in X\colon (\exists x)\,(\exists \alpha)\,(\exists h)\ x + \alpha h \in F\},$$

where, as usual, * is the symbol of standardization. In this respect the expressive notation is used:

$$\exists\exists\exists(F,x') := \mathrm{Bo}(F,x').$$

Further we shall use such notation without additional specification.

5.2.6. *The Hadamard cone is the standardization of the* $\forall\forall\forall$*-cone*:

$$\mathrm{Ha}(F,x') = \forall\forall\forall(F,x').$$

In other words, for standard h', F *and* x', *we have*

$$h' \in \mathrm{Ha}(F,x') \leftrightarrow (x' + \mu(\sigma)) \cap F + \mu(\mathbf{R}_+)(h' + \mu(\tau)) \subset F,$$

where $\mu(\mathbf{R}_+)$ *is the external set of positive infinitesimals.*

◁ The proof is obtained from **5.2.4** by duality, provided (which is by all means legitimate) we forget that F is present in $\exists x$. ▷

5.2.7. From the statements deduced above we can derive the following relations:

$$h' \in H(F,x') \leftrightarrow (\forall^{\bullet} x)(\forall^{\bullet} \alpha)\, x + \alpha h' \in F,$$
$$h' \in K(F,x') \leftrightarrow (\exists^{\bullet} x)(\exists^{\bullet} \alpha)\, x' + \alpha h \in F.$$

5.2.8. *For standard* h', F *and* x' *(under the conditions of weak idealization) the following statements are equivalent*:

(1) $h' \in \mathrm{Cl}(F,x')$;

(2) *there are infinitely small* $U \in \sigma(x')$, $V \in \mathcal{N}_\tau$ *and* $\alpha' > 0$ *such that*

$$h' \in \bigcap_{\substack{0<\alpha\le\alpha' \\ x\in F\cap U}} \left(\frac{F-x}{\alpha} + V\right);$$

(3) $(\exists U \in \sigma(x'))(\exists \alpha')(\forall x \in F \cap U)(\forall 0 < \alpha \le \alpha')(\exists h \approx_\tau h')\, x + \alpha h \in F$.

◁ Using obvious abbreviations, we can write

$$h' \in \mathrm{Cl}(F,x') \leftrightarrow (\exists V)(\exists U)(\exists \alpha')(\forall x \in F \cap U)(\forall 0 < \alpha \le \alpha')(\exists h \in h' + V)\; x + \alpha h \in F.$$

Applying the transfer principle and weak idealization, we get

$$\begin{aligned} &h' \in \mathrm{Cl}(F,x') \to (\forall^{st} V)(\exists^{st} U)(\exists^{st} \alpha')(\forall x \in F \cap V) \\ &(\forall 0 < \alpha \le \alpha')(\exists h \in h' + V)(x + \alpha h \in F \\ &\to (\forall^{st}\{V_1,\dots,Vn\})(\exists^{st} U)(\exists^{st} \alpha')(\exists^{st} V)(\forall k = 1,\dots,n) \\ &V_k \supset V \wedge (\forall x \in F \cap U)(\forall 0 < \alpha \le \alpha')(\exists h \in h' + V)\; x + \alpha h \in F \\ &\to (\exists U)(\exists \alpha')(\exists V)(\forall^{st} V')\; V' \supset V \wedge (\forall x \in F \cap U) \\ &(\forall 0 < \alpha \le \alpha')(\exists h \in h' + V)\; x + \alpha h \in F. \end{aligned}$$

Hence, we can obviously deduce that for some $V \in \mathfrak{N}_\tau$, $V \subset \mu(\tau)$ and $U \in \sigma(x')$, $U \subset \mu(\sigma) + x'$, as well as for an infinitesimal α we have (2) and, moreover, (3).

If, in turn, (3) is fulfilled, then, taking into account the definition of the relation $\approx$, we get

$$(\forall^{st} V)(\exists U)(\exists \alpha')(\forall x \in F \cap U)(\forall 0 < \alpha \le \alpha')(\exists h \in h' + V)\; x + \alpha h \in F.$$

Thus, by the transfer principle, $h' \in \mathrm{Cl}(F,x')$. ▷

5.2.9. *The Clarke cone is (under the conditions of strong idealization) the standardization of the* $\forall\forall\exists$*-cone*

$$\mathrm{Cl}(F,x') = {}^{*}\forall\forall\exists(F,x').$$

In other words,

$$h' \in \mathrm{Cl}(F,x') \leftrightarrow (\forall^{\cdot} x)(\forall^{\cdot} \alpha)(\exists^{\cdot} h)\; x + \alpha h \in F.$$

◁ Let first $h' \in \mathrm{Cl}(F,x')$. Choose arbitrarily $x \approx_\sigma x'$ and $\alpha > 0$, $\alpha \approx 0$. For any standard neighbourhood of V, which is an element of the filter $\mathfrak{N}_\tau$, there is, by virtue of the transfer principle, an element h for which $h \in h' + V$ and $x + \alpha h \in F$. Applying strong idealization, we obtain

$$\begin{aligned} &(\forall^{st} V)(\exists h)\,(h \in h' + V \wedge x + ah \in F) \to (\exists h)(\forall^{st} V)\; h \in h' \\ &+ V \wedge x + ah \in F \to (\exists^{\cdot} h)\; x + ah \in F, \end{aligned}$$

i.e., $h' \in \forall\forall\exists(F,x')$.

Let now $h' \in \forall\forall\exists(F,x')$. Choose an arbitrary standard neighbourhood V of the filter $\mathfrak{N}_\tau$. Let us fix an infinitesimal neighbourhood U of the point x' and a positive infinitesimal α'. Then, by condition, for a certain $h \approx_\tau h'$, we obtain

$$(\exists x \in F \cap U)\,(\forall 0 < \alpha \le \alpha')\; x + \alpha h \in F.$$

In other words,

$$(\forall^{\mathrm{st}} V)\,(\exists U)\,(\exists \alpha')\,(\forall x \in F \cap U)\,(\forall 0 < \alpha \le \alpha')\,(\exists h \in h' + V)$$
$$x + \alpha h \in F.$$

Then we apply the transfer principle and find $h' \in \mathrm{Cl}(F,x')$. ▷

5.2.10. Let us give an example of applying the obtained nonstandard criterion for the elements of the Clarke cone in order to deduce one of its basic (and well-known) properties. A more general statement will be derived below.

5.2.11. *The Clarke cone of an arbitrary set in a topological vector space is convex and closed.*

◁ By virtue of the transfer principle, it suffices to consider the situation in which the parameters (space, topology, set, etc.) are standard. Thus, take $h_0 \in \mathrm{cl}_\tau \mathrm{Cl}(F,x')$. Choose a standard neighbourhood V of $\mathfrak{N}_\tau$ and standard elements $V_1, V_2 \in \mathfrak{N}_\tau$ such that $V_1 + V_2 \subset V$. There is a standard element $h' \in \mathrm{Cl}(F,x')$ such that $h' - h_0 \in V'$. Besides, for any $x \approx_\tau x'$ and $\alpha > 0, \alpha \approx 0$ and for a certain h we have $h \in h' + V_2$ and $x + \alpha h \in F$. Obviously, $h \in h' + V_2 \subset h_0 + V_1 + V_2 \subset h_0 + V$ and, hence, $h_0 \in \mathrm{Cl}(F,x')$.

In order to prove that the Clarke cone is convex it suffices to observe that $\mu(\tau) + \mu(\mathbf{R}_+)\mu(\tau) \subset \mu(\tau)$, since the mapping $(x,\alpha,h) \to x + \alpha h$ is continuous. ▷

5.2.12. *Let θ be a vector topology and $\theta \ge \tau$ Then*

$$\forall\forall\exists(\mathrm{cl}_\theta F,x') \subset \forall\forall\exists(F,x').$$

Moreover, if $\theta \ge \sigma$, then

$$\forall\forall\exists(\mathrm{cl}_\theta F,x') = \forall\forall\exists(F,x').$$

◁ Let $h' \in \forall\forall\exists(\mathrm{cl}_\theta F, x')$ be a certain standard element of the cone in question. Let us choose elements $x \in F$ and $\alpha > 0$ such that $x \approx_\sigma x'$ and $\alpha \approx 0$. Obviously, $x \in \mathrm{cl}_\theta F$. Hence, for a certain $h \in_\tau h'$ we get $x + \alpha h \in \mathrm{cl}_\theta F$. Let us choose an infinitely small neighbourhood W from $\mu(\theta)$. The neighbourhood αW is also an element of $\theta(0)$ and, hence, for a certain $x'' \in F$ we have $x'' - (x + \alpha h) \in \alpha W$. Let us set $h'' := (x'' - x)/\alpha$. Obviously, $x + \alpha h'' \in F$ and, moreover, $\alpha h'' \in \alpha h + \alpha W$. Therefore, $h'' \in h + W \subset h' + \mu(\tau) + W \subset \ \subset h' + \mu(\tau) + \mu(\theta) \subset h' + \mu(\tau) + \mu(\tau) \subset h' + \mu(\tau)$, i.e., $h'' \approx {}_\tau h'$. Hence, $h' \in \forall\forall\exists(F, x')$.

Let now $\theta \geq \sigma$ and $h' \in \forall\forall\exists(F, x')$. Choose an arbitrary infinitely small α and an element $x \in \mathrm{cl}_\theta F$ such that $x \approx {}_\sigma x'$. Find an $x'' \in F$ for which $x - x'' \in \alpha W$, where $W \subset \mu(\theta)$ is an infinitely small symmetric neighbourhood of zero in θ. Since $\theta \geq \sigma$, we have $\mu(\theta) \subset \mu(\sigma)$, i.e., $x - x'' \in \mu(\theta) \subset \mu(\sigma)$ or, in other words, $x \approx_\sigma x' \approx_\sigma x''$. By definition (the element h', as usual, is considered standard!), for a certain $h \approx {}_\sigma h'$ we have $x'' + \alpha h \in F$. Let us set $h'' := (x'' - x)/\alpha + h$. Obviously, in this case we have

$$
\begin{aligned}
h'' &\in h + W \subset h + \mu(\theta) \subset h' + \mu(\theta) + \mu(\tau) \\
&\subset h' + \mu(\tau) + \mu(\tau) \subset h' + \mu(\tau),
\end{aligned}
$$

i.e., $h'' \approx {}_\tau h'$. Besides,

$$
x + \alpha h'' = x + (x'' - x) + \alpha h = x'' + \alpha h \in \mathrm{cl}_\theta F.
$$

And, finally, $h' \in \forall\forall\exists(\mathrm{cl}_\theta F, x')$. ▷

5.2.13. From the presentation obtained we can, in particular, deduce:

$$
\mathrm{Ha}(F, x') \subset H(F, x') \subset \mathrm{Cl}(F, x') \subset K(F, x') \subset \mathrm{cl}_\tau \mathrm{Fd}(F, x').
$$

Under the condition $\sigma = \tau$ for a convex F we obtain

$$
\mathrm{Fd}(F, x') \subset \mathrm{Cl}(F, x') \subset \mathrm{cl}\,\mathrm{Fd}(F, x');
$$

i.e.,

$$
\mathrm{Cl}(F, x') = K(F, x') = \mathrm{cl}\,\mathrm{Fd}(F, x').
$$

5.2.14. The nonstandard criteria of the Bouligand, Hadamard and Clarke cones presented above show these cones to be chosen from the list of eight possible cones with the infinitesimal prefix $(Qx)(Q\alpha)(Qh)$ (here Q is either $\forall$ or $\exists$). For a complete description of all these cones it obviously suffices to characterize $\forall\exists\exists$-cones and $\forall\exists\forall$-cones.

5.2.15. *The following presentation is valid:*

$$\forall\exists\exists(F,x') = \bigcap_{\substack{\alpha' \\ V\in\mathfrak{N}_\tau}} \bigcup_{U\in\sigma(x')} \bigcap_{x\in F\cap U} \left(V + \bigcup_{0<\alpha\le\alpha'} \frac{F-x}{\alpha}\right).$$

◁ In order to prove this statement one should first of all realize that the required equality is an abbreviated presentation of the following statement: for standard h', F, x', we have:

$$(\forall^\bullet x)\,(\exists^\bullet\alpha)\,(\exists^\bullet h)\; x+\alpha h\in F$$
$$\leftrightarrow (\forall V\in\mathfrak{N}_\tau)\,(\forall\alpha')\,(\exists U\in\sigma(x'))\,(\forall x\in F\cap U)$$
$$(\exists 0<\alpha\le\alpha')\,(\exists h\in h'+V)\; x+\alpha h\in F.$$

Therefore, for $h'\in\forall\exists\exists(F,x')$, and standard $V\in\mathfrak{N}_\tau$ and $\alpha>0$ we can choose an internal subset of the monad $\mu(\sigma(x'))$ as the required neighbourhood of U. The consecutive application of transfer and strong idealization affords

$$(\forall^{\mathrm{st}}V)\,(\forall^{\mathrm{st}}\alpha')\,(\forall x\approx_\sigma x')\,(\exists 0<\alpha\le\alpha')\,(\exists h\in h'+V)\; x+\alpha h\in F$$
$$\to (\forall x\approx_\sigma x')\,(\forall^{\mathrm{st}}\{V_1,\dots,V_n\})\,(\forall^{\mathrm{st}}\{\alpha_1',\dots,\alpha_n'\})$$
$$(\exists h)\,(\exists\alpha)\,(\forall k:=1,\dots,n)\,(0<\alpha\le\alpha'\wedge h\in h'+V_k\wedge x+\alpha h\in F)$$
$$\to (\forall x\approx_\sigma x')\,(\exists h)\,(\exists\alpha)\,(\forall^{\mathrm{st}}V)\,(h\in h'+V)\wedge(\forall^{\mathrm{st}}\alpha')\,(0<\alpha\le\alpha'\wedge x$$
$$+\alpha h\in F)\to(\forall^\bullet x)\,(\exists^\bullet h)\,(\exists\alpha\approx 0)\; x+\alpha h\in F$$
$$\to h'\in{}^*\{h' : (\forall^\bullet x)\,(\exists^\bullet\alpha)\,(\exists^\bullet h)\; x+\alpha h\in F\}\to h'\in\forall\exists\exists(F,x').$$

Therefore, the proof is complete. ▷

5.2.16. Alongside with the eight infinitesimal cones of the classical series discussed above, there are nine more pairs of cones containing the Hadamard cone and lying in the Bouligand one. Such cones are evidently generated by changing the order of quantifiers. Five out of these pairs are constructed in a complex way by the type of the $\forall\exists\forall$-cone, the remaining pairs generated by permutations and dualizations of the Clarke and $\forall\exists\exists$ cones. For instance, in natural notation we have

$$\forall\alpha\forall h\exists x(F,x') = \bigcap_{U\in\sigma(x')\,\alpha'} \bigcup \operatorname{int}_\tau \bigcap_{0<\alpha\le\alpha'} \bigcup_{x\in F\cap U} \frac{F-x}{\alpha},$$

$$\exists h\exists x\forall\alpha(F,x') = \bigcup_{\alpha'} \bigcap_{U\in\sigma(x')} \operatorname{cl}_\tau \bigcup_{x\in F\cap U} \bigcap_{0<\alpha\le\alpha'} \frac{F-x}{\alpha},$$

$$\exists h\forall x\forall\alpha(F,x') = \bigcap_{\substack{U\in\sigma(x') \\ \alpha'}} \operatorname{cl}_\tau \bigcup_{\substack{0<\alpha\le\alpha' \\ x\in F\cap U}} \frac{F-x}{\alpha}.$$

The last cone is narrower than the Clarke cone and is convex when $\mu(\sigma)+\mu(\mathbf{R}_+)\,\mu(\tau)\subset\mu(\sigma)$, in which case it is denoted by $\mathrm{Ha}^+(F,x')$. It should be observed that

$$\mathrm{Ha}(F,x')\subset\mathrm{Ha}^+(F,x')\subset\mathrm{Cl}(F,x').$$

Also convex is the $\forall\alpha\exists h\forall x$-cone denoted by the symbol $\mathrm{In}(F,x')$. Obviously,

$$\mathrm{Ha}^+(F,x')\subset\mathrm{In}(F,x')\subset\mathrm{Cl}(F,x').$$

5.2.17. When calculating tangents to the composition of correspondence, use is made of special *regularizing cones*.

Namely, if $F\subset X\times Y$, where the vector spaces X and Y have topologies σ_X,τ_X and σ_Y,τ_Y, respectively, and $a':=(x',y')\in F$, we set $\sigma:=\sigma_X\times\sigma_Y$ and

$$R^1(F,a'):=\bigcap_{V\in\mathfrak{N}_{\tau_Y}}\ \bigcup_{\substack{W\in\sigma(a')\\ \alpha'}}\ \bigcap_{\substack{a\in W\cap F\\ 0<\alpha\le\alpha'}}\left(\frac{F-a}{\alpha}+\{0\}\times V\right),$$

$$Q^1(F,a'):=\bigcap_{V\in\mathfrak{N}_{\tau_Y}}\ \bigcup_{\substack{W\in\sigma(a')\\ \alpha'\\ U\in\mathfrak{N}_\sigma}}\ \bigcap_{\substack{a\in W\subset F\\ 0<\alpha\le\alpha'\\ x\in U}}\left(\frac{F-a}{\alpha}+\{x\}\times V\right),$$

$$QR^2(F,a'):=\bigcup_{\substack{W\in\sigma(a')\\ \alpha'\\ U\in\mathfrak{N}_\sigma}}\ \bigcap_{\substack{a\in W\cap F\\ 0<\alpha\le\alpha'\\ x\in U}}\left(\frac{F-a}{\alpha}+(x,0)\right).$$

The cones $R^2(F,a')$, $Q^2(F,a')$ and $QR^1(F,a')$ are determined by duality. Moreover, analogous notation is used for the case of product of more than two spaces, bearing in mind that the upper index over the symbol of an approximating set denotes the number of the coordinate on which the condition of the corresponding type is imposed. It should be also remarked that in applications we usually consider pairwise coinciding topologies: $\sigma_X=\tau_X$ and $\sigma_Y=\tau_Y$. Let us give obvious nonstandard criteria for the regularizing cones described.

5.2.18. *For standard vectors $s'\in X$ and $t'\in Y$ we have:*

$$(s',t')\in R^1(F,a')$$
$$\leftrightarrow(\forall a\approx_\sigma a',\,a\in F)\,(\forall\alpha\in\mu(\mathbf{R}_+))\,(\exists t\approx_{\tau_Y}t')\ a+\alpha(s',t)\in F,$$
$$(s',t')\in Q^1(F,a')$$
$$(\forall a\approx_\sigma a',\ a\in F)\,(\forall\alpha\in\mu(\mathbf{R}_+))\,(\forall s\approx_{\tau_X}s')\,(\exists t\approx_{\tau_Y}t')\,(a+\alpha(s,t)\in F,$$
$$(s',t')\in QR^2(F,a')$$
$$\leftrightarrow(\forall a\approx_\sigma a',\,a\in F)\,(\forall\alpha\in\mu(\mathbf{R}_+))\,(\forall s\approx_{\tau_X}s')\ a+\alpha(s,t')\in F.$$

5.2.19. As seen from **5.2.18**, the cones of the type QR^j are variations of the Hadamard cone, while the cones R^j are particular cases of the Clarke cone. In this case the cones R^j are also obtained by specialization of cones of the type Q^j by appropriate choice of discrete topologies. Under conventional suppositions the cones under discussion are convex. Let us prove this statement only for the cone Q^j, which is quite sufficient by virtue of what has been said above.

5.2.20. *If the mapping* $(a,\alpha,b) \to a+\alpha b$ *is continuous as acting from* $(X\times Y,\sigma)\times(\mathbf{R},\tau_{\mathbf{R}})\times(X\times Y,\tau_X\times\tau_Y)$ *to* $(X\times Y,\sigma)$, *then the cones* $Q^j(F,a')$ *are convex for* $j:=1,2$.

◁ By transfer, the proof can be carried out in standard entourage, i.e., the parameters considered can be assumed to be standard, and use can be made of criterion **5.2.18**. So, let (s',t') and (s'',t'') lie in $Q^1(F,x')$. For $a\approx_\sigma a'$ and $a\in F$, for a positive $\alpha\approx 0$ and $s\approx_{\tau_X}(s'+s'')$, we get, by virtue of **5.2.18**, $a_1:=a+\alpha(s-s'',t_1)\in F$ for a certain $t_1\approx_{\tau_Y} t'$. By condition, $\mu(\sigma)+\alpha(\mu(\tau_X)\times\mu(\tau_Y))\subset\mu(\sigma)$. Therefore, $a_1\approx_\sigma a$ and $a_1\in F$. Applying **5.2.18** again, we find $t_2\approx_{\tau_Y} t''$, for which $a_1+\alpha(s'',t_2)\in F$. Obviously, for $t:=t_1+t_2$ we get $t\approx_{\tau_Y}(t'+t'')$ and $a+\alpha(s,t)=a+\alpha(s-s'',t_1)+\alpha(s'',t_2)=a_1+$ $+\alpha(s'',t_2)\in F$, which was required, since the homogeneity of $Q_1(F,a')$ is ensured by stability of the monads of nearvector topologies under multiplication by standard scalars (see **5.1.4**). ▷

5.2.21. The analysis conducted shows that it is worthwhile introducing into consideration the cones P^j and S^j that employ the following direct standardizations:

$$(s',t')\in P^2(F,a')$$
$$(\exists s\approx_{\tau_X}s')\,(\forall t\approx_{\tau_Y}t')\,(\forall a\approx_\sigma a',\ a\in F)\,(\forall\alpha\in\mu(\mathbf{R}_+))\ a+\alpha(s,t)\in F,$$
$$(s',t')\in S^2(F,a')\leftrightarrow(\forall t\approx_{\tau_Y}t')\,(\exists s\approx_{\tau_X}s')\,(\forall a\approx_\sigma a',\ a\in F)$$
$$(\forall\alpha\in\mu(\mathbf{R}_+))\ a+\alpha(s,t)\in F.$$

The explicit forms of the cones P^j and S^j can, in principle, be written out (this problem will be discussed in the section to follow). It is, however, little use of the arising formulas (especially of that for S^j) since they are enormously cumbersome. But, as we have already convinced ourselves, formulas of the type obscure analysis by hiding the transparent 'infinitesimal' essence of the constructions.

5.2.22. *For* $j := 1,2$ *we have*

$$\mathrm{Ha}(F,a') \subset P^j(F,a') \subset S^j(F,a') \subset Q^j(F,a') \subset R^j(F,a') \subset \mathrm{Cl}(F,a').$$

In this case the cones in question are convex as soon as $\mu(\sigma) + \alpha(\mu(\tau_X) \times \mu(\tau_Y)) \subset \mu(\sigma)$ *for all* $\alpha > 0, \alpha \approx 0$.

◁ The inclusions to be proved are obvious from the nonstandard definitions of the corresponding cones. We have already pointed out that the majority of these cones is convex. Let us, to make the picture complete, establish that $S^2(F,a')$ is convex.

The fact that $S^2(F,a')$ is stable under multiplication by positive standard scalars results from indivisibility of a monad. Let us check if $S^2(F,a')$ is a semigroup. Hence, for standard (s',t') and (s'',t'') of $S^2(F,a')$, let us choose $t \approx_{\tau_Y} (t' + t'')$. Then $t - t'' \approx_{\tau_Y} t'$ and there is an $s_1 \approx_{\tau_X} s'$ which serves $t - t''$ in accordance with the definition of $S^2(F,a')$. Let us choose an $s_2 \approx_{\tau_X} s''$ which serves t'' in the same obvious sense. It is clear that $(s_1 + s_2) \approx_{\tau_X} (s' + s'')$. In this case for any $a \in F$ and $\alpha > 0$ such that $a \approx_\sigma a'$ and $\alpha \approx 0$ we get $a_1 := a + \alpha(s_1, t - t'') \in F$. Since a_1 is seen to be infinitely close (in the sense of σ) to a', from the choice of s_2 we conclude that $a_1 + \alpha(s_2, t'') \in F$. Hence, we can directly deduce $a + \alpha(s_1 + s_2, t) \in F$, i.e., $(s' + s'', t' + t'') \in S^2(F,a')$.

An analogous straightforward consideration can prove that $P^j(F,a')$ is convex. ▷

5.2.23. From the proof of **5.2.22** one can deduce that it is possible to consider convex extensions of the cones P^j and S^j, i.e., cones P^{+j} and S^{+j} obtained by 'leapfrogging the quantifier $\forall\alpha$'. For instance, the cone $P^{+2}(F,a')$ is determined by the relation

$$(s',t') \in P^{+2}(F,a') \leftrightarrow \forall\alpha \in \mu(\mathbf{R}_+))(\exists s \approx_{\tau_X} s')(\forall t \approx_{\tau_Y} t')$$
$$(\forall a \approx_\sigma a',\ a \in F)\ a + \alpha(s,t) \in F.$$

Obviously, it is also expedient (see **5.2.19**) to use the regularizations obtained by the specialization of the cone Ha^+ when choosing discrete topologies, the corresponding explicit formulas omitted. The importance of regularizing cones is associated with their role in subdifferentiating composite mappings which will be discussed in Section **5.5**.

5.3. Kuratowski and Rockafeller Limits

In the preceding section we have seen many constructions of interest for us to be associated with the procedure of transposing quantifiers in infinitesimal constructions. Similar constructions arise in various problems and pertain to certain facts of principal importance. Now we are going to discuss those which are most often encountered when subdifferentiating. Let us start with general observations concerning the Nelson algorithm.

5.3.1. *Let $\varphi = \varphi(x,y) \in$(ZFC), i.e., φ is a certain formula of Zermelo-Fraenkel theory containing no free variables but x, y. Then*

$$(\forall x \in \mu(\mathfrak{F}))\, \varphi(x,y) \leftrightarrow (\exists^{\mathrm{st}} F \in \mathfrak{F})\, (\forall x \in F)\, \varphi(x,y),$$
$$(\exists x \in \mu(\mathfrak{F}))\, \varphi(x,y) \leftrightarrow (\forall^{\mathrm{st}} F \in \mathfrak{F})\, (\exists x \in F)\, \varphi(x,y)$$

(here, as usual, $\mu(\mathfrak{F})$ is the monad of a standard filter $\mathfrak{F}$).

◁ It suffices to prove the implication $\rightarrow$ in the first of the equivalences. By hypothesis, for any remote element F of the filter $\mathfrak{F}$ the internal property $\psi := (\forall x \in F)\, \varphi(x,y)$ is fulfilled. Hence, by the Cauchy principle, ψ is valid for a standard F. ▷

5.3.2. *Let $\varphi = \varphi(x,y,z) \in$(ZFC) and $\mathfrak{F}, \mathfrak{G}$ be certain standard filters (in some standard sets). In this case*

$$(\forall x \in \mu(\mathfrak{F}))\, (\exists y \in \mu(\mathfrak{G}))\, \varphi(x,y,z)$$
$$\leftrightarrow (\forall^{\mathrm{st}} G \in \mathfrak{G})\, (\exists^{\mathrm{st}} F \in \mathfrak{F})\, (\forall x \in F)\, (\exists y \in G)\, \varphi(x,y,z)$$
$$\leftrightarrow (\exists^{\mathrm{st}} F(\cdot))\, (\forall^{\mathrm{st}} G \in \mathfrak{G})\, (\forall x \in F(G))\, (\exists y \in G)\, \varphi(x,y,z),$$
$$(\exists x \in \mu(\mathfrak{F}))\, (\forall y \in \mu(\mathfrak{G}))\, \varphi(x,y,z)$$
$$\leftrightarrow (\exists^{\mathrm{st}} G \in \mathfrak{G})\, (\forall^{\mathrm{st}} F \in \mathfrak{F})\, (\exists x \in F)\, (\forall y \in G)\, \varphi(x,y,z)$$
$$\leftrightarrow (\forall^{\mathrm{st}} F(\cdot))\, (\exists^{\mathrm{st}} G \in \mathfrak{G})\, (\exists x \in F(G))\, (\forall y \in G)\, \varphi(x,y,z)$$

(here the symbol $F(\cdot)$ denotes a function from $\mathfrak{G}$ to $\mathfrak{F}$).

◁ The proof consists of appealing to the idealization and construction principles with use made of **5.3.1.** ▷

5.3.3. *Let $\varphi = \varphi(x,y,z,u) \in$(ZFC) and let $\mathfrak{F}, \mathfrak{G}, \mathfrak{H}$ be three standard filters. When the set u is standard, the following relations are fulfilled:*

$$
\begin{aligned}
&(\forall x \in \mu(\mathfrak{F}))(\exists y \in \mu(\mathscr{G}))(\forall z \in \mu(\mathfrak{H}))\, \varphi(x,y,z,u)\\
&\leftrightarrow (\forall G(\cdot))(\exists F \in \mathfrak{F})(\exists^{\mathrm{Fin}} \mathfrak{H}_0 \subset \mathfrak{H})(\forall x \in \mathfrak{F})\\
&(\exists H \in \mathfrak{H}_0)(\exists y \in G(H))(\forall z \in H)\, \varphi(x,y,z,u),\\
&(\exists x \in \mu(\mathfrak{F}))(\forall y \in \mu(\mathscr{G}))(\exists z \in \mu(\mathfrak{H}))\, \varphi(x,y,z,u)\\
&\leftrightarrow (\exists G(\cdot))(\forall F \in \mathfrak{F})(\forall^{\mathrm{Fin}} \mathfrak{H}_0 \subset \mathfrak{H})(\exists x \in \mathfrak{F})\\
&(\forall H \in \mathfrak{H}_0)(\forall y \in G(H))(\exists z \in H)\, \varphi(x,y,z,u),
\end{aligned}
$$

where $G(\cdot)$ is a function from $\mathfrak{H}$ to $\mathscr{G}$, and the superscript $^{\mathrm{Fin}}$ labelling a quantifier denotes its restriction to the class of nonempty finite sets.

⊲ By the Nelson algorithm, we deduce:

$$
\begin{aligned}
&(\forall x \in \mu(\mathfrak{F}))(\exists y \in \mu(\mathscr{G}))(\forall z \in \mu(\mathfrak{H}))\, \varphi\\
&\leftrightarrow (\forall x \in \mu(\mathfrak{F}))(\forall^{\mathrm{st}} G(\cdot))(\exists^{\mathrm{st}} H \in \mathfrak{H})(\exists y \in G(H))(\forall z \in H)\, \varphi\\
&\leftrightarrow (\forall^{\mathrm{st}} G(\cdot))(\forall x)(\exists^{\mathrm{st}} F \in \mathfrak{F})(\exists^{\mathrm{st}} H \in \mathfrak{H})\\
&(x \in F \to (\exists y \in G(H))(\forall z \in H)\, \varphi)\\
&\leftrightarrow (\forall^{\mathrm{st}} G(\cdot))(\exists^{\mathrm{stFin}} \mathfrak{F}_0)(\exists^{\mathrm{stFin}} \mathfrak{H}_0)(\forall x)(\exists F \in \mathfrak{F}_0)(\exists H \in \mathfrak{H}_0)\\
&(F \in \mathfrak{F} \wedge H \in \mathfrak{H} \wedge (x \in F \to (\exists y \in G(H))(\forall z \in H)\, \varphi))\\
&\leftrightarrow (\forall^{\mathrm{st}} G(\cdot))(\exists^{\mathrm{stFin}} \mathfrak{F}_0 \subset \mathfrak{F})(\exists^{\mathrm{stFin}} \mathfrak{H}_0 \subset \mathfrak{H})(\forall x)(\exists F \in \mathfrak{F}_0)\\
&(x \in F \to (\exists H \in \mathfrak{H}_0)(\exists y \in G(H))(\forall z \in H)\, \varphi)\\
&\leftrightarrow (\forall G(\cdot))(\exists^{\mathrm{Fin}} \mathfrak{F}_0 \subset \mathfrak{F})(\exists^{\mathrm{Fin}} \mathfrak{H}_0 \subset \mathfrak{H})(\forall x)\\
&((\forall F \in \mathfrak{F}_0)\, x \in F \to (\exists H \in \mathfrak{H}_0)(\exists y \in G(H))(\forall z \in H)\, \varphi)\\
&\leftrightarrow (\forall G(\cdot))(\exists^{\mathrm{Fin}} \mathfrak{F}_0 \subset \mathfrak{F})(\exists^{\mathrm{Fin}} \mathfrak{H}_0 \subset \mathfrak{H})(\forall x \in \cap \mathfrak{F}_0)\\
&(\exists H \in \mathfrak{H}_0)(\exists y \in G(H))(\forall z \in H)\, \varphi.
\end{aligned}
$$

Now we have to observe that for a nonempty finite $\mathfrak{F}_0$ lying in $\mathfrak{F}$ the relation $\cap \mathfrak{F}_0 \in \mathfrak{F}$ is valid by necessity. ⊳

5.3.4. The discussed statement makes it possible to characterize explicitly the $\forall\exists\forall$-cones and similar constructions. The arising standard descriptions are obviously cumbersome. Let us now discuss the constructions most important for applications and pertaining to the prefixes of the type $\forall\exists$, $\forall\forall$, $\exists\forall$ and $\exists\exists$. Let us start with certain means allowing one to use the conventional language of infinitesimals for analyzing such constructions.

5.3.5. Let Ξ be a *direction*, i.e., a nonempty directed set. In line with the idealization principle, in Ξ there are internal elements majorizing ${}^{\circ}\Xi$. Let us recall (**4.1.6** (3)) that they are called *remote* or *infinitely large* in Ξ. Let us consider a standard basis of the tail filter

$\mathcal{B} := \{\sigma(\xi) : \xi \in \Xi\}$, where σ is the order in Ξ. The monad of the tail filter is obviously composed of the remote elements of the direction considered. The following presentations are used: ${}^a\Xi := \mu(\mathcal{B})$ and $\xi \approx +\infty \leftrightarrow \xi \in {}^a\Xi$.

5.3.6. *Let* Ξ, H *be two directed sets, and* $\xi := \xi(\cdot) : \mathrm{H} \to \Xi$ *be a mapping. The following statements are equivalent:*

(1) $\xi({}^a\mathrm{H}) \subset {}^a\Xi$;

(2) $(\forall \xi \in \Xi)\,(\exists \eta \in \mathrm{H})\,(\forall \eta' \geq \eta)\ \xi(\eta') \geq \xi$.

◁ Indeed, (1) implies that the tail filter of Ξ is coarser than the image of the tail filter of H, i.e., that in each tail of the direction of Ξ lies an image of a tail of H. The last statement is the contents of (2). ▷

5.3.7. When equivalent conditions **5.3.6** (1) and **5.3.6** (2) are fulfilled, H is said to be a *subdirection* of Ξ (relative to $\xi(\cdot)$).

5.3.8. Let X be a certain set, and $x := x(\cdot) : \Xi \to X$ be a net of elements of X (we also write $(x_\xi)_{\xi \in \Xi}$ or simply (x_ξ)). Let, then, $(y_\eta)_{\eta \in \mathrm{H}}$ be another net of elements of X. We say that (y_η) is a *Moore subnet* of the net (x_ξ), or a *strict subnet* of (x_ξ), if H is a subdirection of Ξ relative to such a $\xi(\cdot)$ that $y_\eta = x_{\xi(\eta)}$ for all $\eta \in \mathrm{H}$, i.e., $y = x \circ \xi$. It should be emphasized that by virtue of **4.1.6** (5) we have $y({}^a\mathrm{H}) \subset x({}^a\Xi)$ fulfilled.

5.3.9. The last property of Moore subnets is a cornerstone of a more free definition of a subnet which is attractive by its direct relation with filters. Namely, the set $(y_\eta)_{\eta \in \mathrm{H}}$ of X elements is termed a *subnet* (or a *subnet in a broader sense*) of the set $(x_\xi)_{\xi \in \Xi}$ of X elements, provided

$$(\forall \xi \in \Xi)\,(\exists \eta \in \mathrm{H})\,(\forall \eta' \geq \eta)\,(\exists \xi' \geq \xi)\quad x(\xi') = y(\eta'),$$

i.e., in the case when every tail of the net x contains a certain tail of y. It goes without saying that in terms of monads the condition $y({}^a\mathrm{H}) \subset x({}^a\Xi)$ is fulfilled, or, in a more expressive form,

$$(\forall \eta \approx +\infty)\,(\exists \xi \approx +\infty)\quad y_\eta = x_\xi.$$

In this case, for the sake of expressiveness it is often written that $(x_\eta)_{\eta\in H}$ is a subset of the net $(x_\xi)_{\xi\in\Xi}$ (which can result in ambiguity). It would be expedient to emphasize that in a general case subnets are not obligatorily Moore subnets. It should also be stressed that two nets in one set are called *equivalent* if each of them is a subnet of the other, i.e., if their monads coincide.

5.3.10. If $\mathfrak{F}$ is a filter in X, and (x_ξ) is a net of elements in X, then we say that the considered net is *subordinate to* $\mathfrak{F}$ under the condition $\xi \approx +\infty \rightarrow x_\xi \in \mu(\mathfrak{F})$. In other words, the net (x_ξ) is subordinate to $\mathfrak{F}$ provided the tail filter is finer than $\mathfrak{F}$. In this case a certain abuse of language is allowed when we write $x_\xi \downarrow (\mathfrak{F})$ meaning an analogy with topological notations of convergence. It should also be remarked here that when $\mathfrak{F}$ is an ultrafilter, $\mathfrak{F}$ coincides with the tail filter of any net (x_ξ) subordinate to it, i.e. such a (x_ξ) net is itself *an ultranet.*

5.3.11. Theorem. *Let $\varphi = \varphi(x,y,z)$ be a formula of Zermelo-Fraenkel theory containing no free parameters but x, y, z, where z is a standard set. Let, then, $\mathfrak{F}$ be a filter in X, and $\mathfrak{G}$ be a filter in Y. The following statements are equivalent:*

(1) $(\forall G\in\mathfrak{G})(\exists F\in\mathfrak{F})(\forall x\in F)(\exists y\in G)\ \varphi(x,y,z)$;

(2) $(\forall x\in\mu(\mathfrak{F}))(\exists y\in\mu(\mathfrak{G}))\ \varphi(x,y,z)$;

(3) *for any set $(x_\xi)_{\xi\in\Xi}$ of elements of X subordinate to $\mathfrak{F}$ we can find a net $(y_\eta)_{\eta\in H}$ of elements of Y subordinate to $\mathfrak{G}$, and a strict subnet $(x_{\xi(\eta)})_{\eta\in H}$ of the net $(x_\xi)_{\xi\in\Xi}$ such that for all $\eta\in H$, we get $\varphi(x_{\xi(\eta)}, y_\eta, z)$, i.e., symbolically,*

$$(\forall x_\xi \downarrow \mathfrak{F})(\exists y_\eta \downarrow \mathfrak{G})\quad \varphi(x_{\xi(\eta)}, y_\eta, z);$$

(4) *for any net $(x_\xi)_{\xi\in\Xi}$ of elements of X subordinate to $\mathfrak{F}$ there is a net $(y_\eta)_{\eta\in H}$ of elements of Y subordinate to $\mathfrak{G}$, and a subnet $(x_\eta)_{\eta\in H}$ of the net $(x_\xi)_{\xi\in\Xi}$ such that for all $\eta\in H$ $\varphi(x_\eta, y_\eta, z)$; i.e., symbolically,*

$$(\forall x_\xi \downarrow \mathfrak{F})(\exists y_\eta \downarrow \mathfrak{G})\quad \varphi(x_\eta, y_\eta, z);$$

(5) *for any ultranet $(x_\xi)_{\xi\in\Xi}$ of elements of X subordinate to $\mathfrak{F}$ there is an ultranet $(y_\eta)_{\eta\in H}$ subordinate to , and an ultranet $(x_\eta)_{\eta\in H}$ equivalent to $(x_\xi)_{\xi\in\Xi}$ such that for all*

$\eta \in \mathrm{H}$ *we have* $\varphi(x_\eta, y_\eta, z)$.

$\lhd$ (1) → (2). Let $x \in \mu(\mathfrak{F})$. By the transfer principle, for every standard G there is a standard F such that $(\forall x \in F)(\exists y \in G)\ \varphi(x,y,z)$. Therefore, for $x \in \mu(\mathfrak{F})$ we get $(\forall G \in {}^{\circ}\mathcal{B})(\exists y \in G)\ \varphi(x,y,z)$. Applying the idealization principle, we deduce: $(\exists y)\ (\forall G \in {}^{\circ}\mathcal{B})\ y \in G \wedge \varphi(x,y,z)$. Hence, $y \in \mu(\mathcal{B})$ and $\varphi(x,y,z)$.

(2) → (3). Let $(x_\xi)_{\xi \in \Xi}$ be a standard net in X subordinate to $\mathfrak{F}$. For every standard G of $\mathcal{B}$ let us set

$$A_{(G,\xi)} := \{\xi' \geq \xi : (\forall \xi'' \geq \xi')(\exists y \in G)\ \varphi(x_{\xi''}, y, z)\}.$$

By **4.1.8**, ${}^{a}\Xi \subset A_{(G,\xi)}$. Since $A_{(G,\xi)}$ is an internal set, we use the Cauchy principle and conclude: ${}^{\circ}A_{(G,\xi)} \neq \varnothing$. Therefore, on the direction $\mathrm{H} := \mathcal{B} \times \Xi$ (with the natural order) there are standard mappings $\xi : \mathrm{H} \to \Xi$ and $y : \mathrm{H} \to Y$ such that $\xi(\eta) \in A_{(G,\xi)}$ and $y_\eta \in G$ for $G \in \mathcal{B}$ and $\xi \in \Xi$, such that $\eta \in (G, \xi)$. Obviously, for $\eta \approx +\infty$ we have $\xi(\eta) \approx +\infty$ and $y_\eta \in \mu(\mathcal{B})$.

(3) → (4). This is obvious.

(4) → (1). If (1) is not fulfilled, then, by hypothesis,

$$(\exists G \in \mathcal{B})(\forall F \in \mathfrak{F})(\exists x \in F)(\forall y \in G)\ \neg\varphi(x,y,z),$$

For $F \in \mathfrak{F}$ we choose $x_F \in F$ in such a way that we had $\neg\varphi(x,y,z)$ for all $y \in G$. It should be remarked that the obtained net $(x_F)_{F \in \mathfrak{F}}$ of the elements of X, as well as the set G, can be considered standard by virtue of the transfer principle. No doubts, $x_F \downarrow \mathfrak{F}$ and, hence, by virtue of (3), there is a direction H and a subnet $(x_\eta)_{\eta \in \mathrm{H}}$ of the net $(x_F)_{F \in \mathfrak{F}}$ such that for a certain net $(y_\eta)_{\eta \in \mathrm{H}}$ we get $\varphi(x_\eta, y_\eta, z)$ for any $\eta \in \mathrm{H}$. By definition **5.3.9**, for any infinitely large η, x_η coincides with x_F for a certain remote F, i.e., $x_\eta \in \mu(\mathfrak{F})$. By condition, $y_\eta \in \mu(\mathcal{B})$ and, moreover, $y_\eta \in G$. In this case it appears that $\varphi(x_\eta, y_\eta, x)$ and $\neg\varphi(x_\eta, y_\eta, x)$, which is impossible. The contradiction obtained proves that the assumption made is false. Therefore, (1) is fulfilled (as soon as (4) is valid).

(1) → (5). In order to prove the equivalence required, it suffices to remark that the equivalence becomes evident in the case when $\mathfrak{F}$ and $\mathcal{B}$ are ultrafilters. Now we have to remark that every monad is a union of the monads of ultrafilters. $\rhd$

5.3.12. In applications it is often convenient to consider concretizations of **5.3.11** corresponding to the cases when one of the filters is discrete. Thus, using natural notation, we deduce:

$$(\exists x \in \mu(\mathfrak{F}))\, \varphi(x,y) \leftrightarrow (\exists x_\xi \downarrow \mathfrak{F})\, \varphi(x_\xi, y);$$
$$(\forall x \in \mu(\mathfrak{F}))\, \varphi(x,y) \leftrightarrow (\forall x_\xi \downarrow \mathfrak{F})(\exists x_\eta \downarrow \mathfrak{F})\, \varphi(x_\eta, y).$$

5.3.13. Let $F \subset X \times Y$ be an internal correspondence from a standard set X to a standard set Y. Let us assume that in X there is a standard filter $\mathfrak{N}$, while in Y there is a topology τ. Let us set

$$\forall\forall(F) := {}^*\{y' : (\forall x \in \mu(\mathfrak{N}) \cap \operatorname{dom}(F))(\forall y \approx y') \quad (x,y) \in F\},$$
$$\exists\forall(F) := {}^*\{y' : (\exists x \in \mu(\mathfrak{N}) \cap \operatorname{dom}(F))(\forall y \approx y') \quad (x,y) \in F\},$$
$$\forall\exists(F) := {}^*\{y' : (\forall x \in \mu(\mathfrak{N}) \cap \operatorname{dom}(F))(\exists y \approx y') \quad (x,y) \in F\},$$
$$\exists\exists(F) := {}^*\{y' : (\exists x \in \mu(\mathfrak{N}) \cap \operatorname{dom}(F))(\exists y \approx y') \quad (x,y) \in F\},$$

where, as usual, * is the symbol of standardization, while the expression $y \approx y'$ means that $y \in \mu(\tau(y'))$. The set $Q_1Q_2(F)$ is called a Q_1Q_2*-limit* of F (here Q is one of the quantifiers, $\forall$ or $\exists$).

5.3.14. In applications they usually confine themselves to the case in which F is a standard correspondence determined on a certain element of the filter $\mathfrak{N}$, and then the $\exists\exists$-limit and the $\forall\exists$-limit are studied. The former is termed *the limit superior* or *upper limit*, the latter is called *the limit interior* or *lower limit* of F along $\mathfrak{N}$.

If we consider a net $(x_\xi)_{\xi \in \Xi}$ in the domain of the definition of F then, bearing in mind the tail filter of the net, we assign

$$\operatorname{Li}_{\xi \in \Xi} F := \liminf_{\xi \in \Xi} F(x_\xi) := \forall\exists(F),$$
$$\operatorname{Ls}_{\xi \in \Xi} F := \liminf_{\xi \in \Xi} F(x_\xi) := \forall\exists(F).$$

In such cases we speak about *Kuratowsky limits.*

5.3.15. *For a standard correspondence F we have the following presentations:*

$$\exists\exists(F) = \bigcap_{U \in \mathfrak{N}} \operatorname{cl} \bigcup_{x \in U} F(x);$$
$$\forall\exists(F) = \bigcap_{U \in \ddot{\mathfrak{N}}} \operatorname{cl} \bigcup_{x \in U} F(x),$$

where $\ddot{\mathcal{N}}$ *is the grill of* $\mathcal{N}$, *i.e., the family composed of all the subsets of* X *meeting the monad* $\mu(\mathcal{N})$. *In other words,*

$$\ddot{\mathcal{N}} = {}^*\{U' \subset X : U' \cap \mu(\mathcal{N}) \neq \varnothing\} = \{U' \subset X : (\forall U \in \mathcal{N})\, U \cap U' \neq \varnothing\}.$$

In this respect the following relations must be recalled:

$$\forall\exists(F) = \bigcap_{U \in \ddot{\mathcal{N}}} \operatorname{int} \bigcup_{x \in U} F(x),$$
$$\forall\forall(F) = \bigcap_{U \in \mathcal{N}} \operatorname{int} \bigcup_{x \in U} F(x).$$

5.3.16. Theorem **5.3.11** immediately yields a description for limits in the language of nets.

5.3.17. *An element* y *lies in the* $\forall\exists$*-limit of* F *iff for every net* $(x_\xi)_{\xi\in\Xi}$ *of elements of the* $\operatorname{dom}(F)$ *subordinate to* $\mathcal{N}$ *there is a subnet* $(x_\eta)_{\eta\in\mathrm{H}}$ *of the net* $(x_\xi)_{\xi\in\Xi}$ *and a net* $(y_\eta)_{\eta\in\mathrm{H}}$ *convergent to* y *such that* $(x_\eta, y_\eta) \in F$ *for all* $\eta \in \mathrm{H}$.

5.3.18. *An element* y *lies in the* $\exists\exists$*-limit of* F *iff there is a net* $(x_\xi)_{\xi\in\Xi}$ *of elements of the* $\operatorname{dom}(F)$ *subordinate to* $\mathcal{N}$, *and a net* $(y_\xi)_{\xi\in\Xi}$ *convergent to* y, *for which* $(x_\xi, y_\xi) \in F$ *for any* $\xi \in \Xi$.

5.3.19. *For any internal correspondence* F *we have*

$$\forall\forall(F) \subset \exists\forall(F) \subset \forall\exists(F) \subset \exists\exists(F).$$

In this case $\exists\exists(F)$, $\forall\exists(F)$ *are closed, while* $\forall\forall(F)$, $\exists\forall(F)$ *are open sets.*

$\triangleleft$ The inclusions sought are obvious. Therefore, taking into account the considerations of duality, let us, for the sake of definiteness, establish that the $\forall\exists$-limit is closed.

If V is a standard open neighbourhood of a point y' of $\operatorname{cl}\forall\exists(F)$, then there is a $y \in \forall\exists(F)$ for which $y \in V$. For an $x \in \mu(\mathcal{N})$ let us find an y'' so that we had $y'' \in \mu(\tau(y))$ and $(x, y'') \in F$. Obviously, $y'' \in V$ since V is a neighbourhood of y. Therefore,

$$(\forall x \in \mu(\mathcal{N}))\, (\forall V \in {}^\circ\tau(y'))\, (\exists y'' \in V)\, (x, y'') \in F.$$

Using now the idealization principle, we deduce: $y' \in \forall\exists(F)$. ▷

5.3.20. The general statements given above make it possible to characterize the elements of many approximating or regularizing cones in terms of nets which is common practice (see [121], [115]). It should be, in particular, observed that the Clarke cone $\mathrm{Cl}(F,x')$ of F in X is obtained by means of the Kuratowski limit:

$$\mathrm{Cl}(F,x') = \mathrm{Li}_{\tau(x')\times\tau_{\mathbf{R}^+}(0)}\Gamma_F,$$

where Γ_F is the *homothety* associated with F, i.e.,

$$(x,\alpha,h) \in \Gamma_F \leftrightarrow h \in \frac{F-x}{\alpha} \ (x,h \in X, \alpha > 0).$$

5.3.21. In convex analysis use if often made of special variations of Kuratowski limits pertaining to the epigraphs of functions which operate into the extended numerical straight line $\overline{\mathbf{R}}$. Let us, first of all, recall important characteristics of the upper and lower limits.

5.3.22. *Let $f: X \to \overline{\mathbf{R}}$ be a standard function defined on a standard X, and let $\mathfrak{F}$ be a standard filter in X. For every standard $t \in \mathbf{R}$, we have*

$$\sup_{F\in\mathfrak{F}} \inf f(F) \le t \leftrightarrow (\exists x \in \mu(\mathfrak{F}))\, {}^{\circ}f(x) \le t,$$
$$\inf_{F\in\mathfrak{F}} \sup f(F) \le t \leftrightarrow (\exists x \in \mu(\mathfrak{F}))\, {}^{\circ}f(x) \le t.$$

◁ Let us first check the first equivalence. Applying the transfer and idealization principles, we deduce:

$$\sup_{F\in\mathfrak{F}} \inf f(F) \le t \to (\forall F \in \mathfrak{F}) \inf f(F) \le t \to$$
$$(\forall F \in \mathfrak{F})\, (\forall \varepsilon > 0) \inf f(F) < t+\varepsilon \to (\forall \varepsilon)\, (\forall F)\, (\exists x \in F)\ f(x) < t +$$
$$+\,\varepsilon \to (\forall^{\mathrm{st}}\varepsilon)\, (\forall^{\mathrm{st}}F)\, (\exists x)\, (x \in F \wedge f(x) < t+\varepsilon)$$
$$\to (\exists x)\, (\forall^{\mathrm{st}}\varepsilon)\, (\forall^{\mathrm{st}}F)\, (x \in F \wedge f(x) < t+\varepsilon)$$
$$\to (\exists x \in \mu(\mathfrak{F}))\, (\forall^{\mathrm{st}}\varepsilon > 0)\ f(x) < t+\varepsilon \to (\exists x \in \mu(\mathfrak{F}))\, {}^{\circ}f(x) \le t$$

(here use was made of **2.2.18** (3)). Let us now remark that for any standard element F of the filter $\mathfrak{F}$ we have $x \in \mu(\mathfrak{F}) \subset F$. Hence, $\inf f(F) \le t$ (as $\inf f(F) \le f(x) < t+\varepsilon$ for every $\varepsilon > 0$). Therefore, by virtue of the transfer principle, for an internal F of $\mathfrak{F}$ we have $\inf f(F) \le t$, which was required.

Taking into account the above-proved statements and the fact that $-f$ and t are standard, we deduce

$$\begin{aligned}&\sup_{F\in\mathfrak{F}} \inf f(F) \ge t \leftrightarrow -\inf_{F\in\mathfrak{F}} \sup f(F) \le -t \leftrightarrow \sup_{F\in\mathfrak{F}} \inf (-f)(F) \ge t\\ &\leftrightarrow (\exists x \in \mu(\mathfrak{F}))\, {}^\circ(-f(x)) \le -t \leftrightarrow (\exists x \in \mu(\mathfrak{F}))\, {}^\circ f(x) \ge t.\end{aligned}$$

Therefore, we get

$$\begin{aligned}&\inf_{F\in\mathfrak{F}} \sup f(F) < t \leftrightarrow \neg\left(\inf_{F\in\mathfrak{F}} \sup f(F) \ge t\right)\\ &\leftrightarrow \neg((\exists x \in \mu(\mathfrak{F}))\, {}^\circ f(x) \ge t) \leftrightarrow (\forall x \in \mu(\mathfrak{F}))\, {}^\circ f(x) \le t.\end{aligned}$$

And, finally, from the above, we conclude

$$\begin{aligned}&\inf_{F\in\mathfrak{F}} \sup f(F) \le t \leftrightarrow (\forall \varepsilon > 0) \inf_{F\in\mathfrak{F}} \sup f(F) < t + \varepsilon\\ &\leftrightarrow (\forall^{\mathrm{st}} \varepsilon > 0)(\forall x \in \mu(\mathfrak{F}))\, {}^\circ f(x) < t + \varepsilon\\ &\leftrightarrow (\forall x \in \mu(\mathfrak{F}))(\forall^{\mathrm{st}} \varepsilon > 0)\, {}^\circ f(x) < t + \varepsilon \leftrightarrow (\forall x \in \mu(\mathfrak{F}))\, {}^\circ f(x) \le t,\end{aligned}$$

since the number ${}^\circ f(x)$ is standard. ▷

5.3.23. *Let X, Y be standard sets, $f: X \times Y \to \overline{\mathbf{R}}$ be a standard function, and $\mathfrak{F}, \mathcal{B}$ be standard filters in X and Y, respectively. For any standard real number t we have*

$$\sup_{G\in\mathcal{B}} \inf_{F\in\mathfrak{F}} \sup_{x\in F} \inf_{y\in G} f(x,y) \le t \leftrightarrow (\forall x \in \mu(\mathfrak{F}))(\exists y \in \mu(\mathcal{B}))\, {}^\circ f(x,y) \le t.$$

◁ Assign $f_G(x) := \inf\{f(x,y) : y \in G\}$. Observe that f_G is a standard function only if G is a standard set. Now successively make use of the transfer principle, proposition **5.3.22** and (strong) idealization, and deduce:

$$\begin{aligned}&\sup_{G\in\mathcal{B}} \inf_{F\in\mathfrak{F}} \sup_{x\in F} \inf_{y\in G} f(x,y) \le t \leftrightarrow (\forall G \in \mathcal{B}) \inf_{F\in\mathfrak{F}} \sup_{x\in F} f_G(x) \le t\\ &\leftrightarrow (\forall^{\mathrm{st}} G \in \mathcal{B}) \inf_{F\in\mathfrak{F}} \sup_{x\in F} f_G(x) \le t \leftrightarrow (\forall^{\mathrm{st}} G \in \mathcal{B})(\forall x \in \mu(\mathfrak{F}))\, {}^\circ f_G(x)\\ &\le t \leftrightarrow (\forall x \in \mu(\mathfrak{F}))(\forall^{\mathrm{st}} G \in \mathcal{B})(\forall^{\mathrm{st}} \varepsilon > 0) \inf_{y\in G} f(x,y) < t + \varepsilon\\ &\to (\forall x \in \mu(\mathfrak{F}))(\forall^{\mathrm{st}} \varepsilon > 0)(\forall^{\mathrm{st}} G \in \mathcal{B})(\exists y \in G)\, f(x,y) < t + \varepsilon\\ &\to (\forall x \in \mu(\mathfrak{F}))\ (\exists y \in \mu(\mathcal{B}))(\forall^{\mathrm{st}} \varepsilon > 0)\, f(x,y) < t + \varepsilon\\ &\to (\forall x \in \mu(\mathfrak{F}))\ (\exists y \in \mu(\mathcal{B}))\ {}^\circ f(x,y) \le t.\end{aligned}$$

For an internal element $F \subset \mu(\mathfrak{F})$ of the filter $\mathfrak{F}$ and a standard element G of the filter $\mathscr{B}$ the last relation yields (by virtue of the transfer principle):

$$\sup_{x\in F}\inf_{y\in G} f(x,y) \le t \to \inf_{F\in\mathfrak{F}}\sup_{x\in F}\inf_{y\in G} f(x,y) \le t$$
$$\to (\forall^{\mathrm{st}} G \in \mathscr{B}) \inf_{F\in\mathfrak{F}}\sup_{x\in F}\inf_{y\in G} f(x,y) \le t$$
$$\to (\forall G \in \mathscr{B}) \inf_{F\in\mathfrak{F}}\sup_{x\in F}\inf_{y\in G} f(x,y) \le t. \rhd$$

5.3.24. In relation with **5.3.23** the quantity

$$\limsup_{\mathfrak{F}} \inf_{\mathscr{B}} f := \sup_{G\in\mathscr{B}}\inf_{F\in\mathfrak{F}}\sup_{x\in F}\inf_{y\in G} f(x,y)$$

is often referred to as the *Rockafeller limit of* f.

If $f := (f_\xi)_{\xi\in\Xi}$ is a family of functions operating from the topological space (X,σ) in $\overline{\mathbf{R}}$, and if $\mathfrak{N}$ is a filter in Ξ, then we determine the *limit inferior* or *lower limit* at the point x' of X of the family f, and its *limit superior* or *upper limit*, or the Rockafeller limit

$$\mathrm{li}_{\mathfrak{N}} f(x') := \sup_{V\in\sigma(x')}\sup_{U\in\mathfrak{N}}\inf_{\xi\in U}\inf_{x\in V} f_\xi(x),$$
$$\mathrm{ls}_{\mathfrak{N}} f(x') := \sup_{V\in\sigma(x')}\inf_{U\in\mathfrak{N}}\sup_{\xi\in U}\inf_{x\in V} f_\xi(x).$$

The last limits are often termed *epilimits*. The essence of this term is revealed by the following obvious statement.

5.3.25. *The limit inferior and the limit superior of a family of epigraphs are the epigraphs of the respective limits of the family of functions under consideration.*

5.4. Approximations determined by a set of infinitesimals

In this section we shall study the problem of analysis of classical approximating cones of Clarke type by detalizing the contribution of infinitely small numbers participating in their definition. Such an analysis enables one to single out both new analogues of tangent cones and new descriptions for the Clarke cone.

5.4.1. Let us again consider a real vector space X with a linear topology σ and a nearvector topology τ. Let, then, in X there is a point x' of F. In line with **5.2**, these objects are considered as standard sets.

Let us fix a certain infinitesimal, a real number α for which $\alpha > 0$ and $\alpha \approx 0$. Let us set

$$\begin{aligned}
&\mathrm{Ha}_\alpha(F,x') := {}^*\{h' \in X : (\forall x \approx_\sigma x', x \in F)(\forall h \approx_\tau h')\, x + \alpha h \in F\},\\
&\mathrm{In}_\alpha(F,x') := {}^*\{h' \in X : (\exists h \approx_\tau h')(\forall x \approx_\sigma x', x \in F)\, x + \alpha h \in F\},\\
&\mathrm{Cl}_\alpha(F,x') := {}^*\{h' \in X : (\forall x \approx_\sigma x', x \in F)(\exists h \approx_\tau h')\, x + \alpha h \in F\},
\end{aligned}$$

where, as usual, * is the symbol of taking the standardization of an external set.

Let us now consider a nonempty and, generally speaking, external set of infinitesimals Λ, assigning

$$\begin{aligned}
&\mathrm{Ha}_\Lambda(F,x') := {}^* \bigcap_{\alpha \in \Lambda} \mathrm{Ha}_\alpha(F,x'),\\
&\mathrm{In}_\Lambda(F,x') := {}^* \bigcap_{\alpha \in \Lambda} \mathrm{In}_\alpha(F,x'),\\
&\mathrm{Cl}_\Lambda(F,x') := {}^* \bigcap_{\alpha \in \Lambda} \mathrm{Cl}_\alpha(F,x').
\end{aligned}$$

Let us pursue the same policy as regards notation for other types of the approximations introduced. As an example, it is worth emphasizing that by virtue of the definitions for a standard h' of X we have

$$\begin{aligned}
&h' \in \mathrm{In}_\Lambda(F,x')\\
&\leftrightarrow (\forall \alpha \in \Lambda)(\exists h \approx_\tau h')(\forall x \approx_\sigma x', x \in F)\, x + \alpha h \in F.
\end{aligned}$$

It is expedient to remark that when Λ is the monad of the corresponding standard filter $\mathfrak{F}_\Lambda$, where $\mathfrak{F}_\Lambda := {}^*\{A \subset \mathbf{R} : A \supset \Lambda\}$, then, say, for $\mathrm{Cl}_\Lambda(F,x')$ we get:

$$\mathrm{Cl}_\Lambda(F,x') = \bigcap_{V \in \mathfrak{N}} \bigcup_{\substack{U \in \sigma(x') \\ A \in \mathfrak{F}_\Lambda}} \bigcap_{\substack{x \in F \cap U \\ \alpha \in A, \alpha > 0}} \left(\frac{F - x}{\alpha} + V \right).$$

If Λ is not a monad (for instance, a singleton), then the implicit form of $\mathrm{Cl}_\Lambda(F,x')$ is associated with the model of analysis which is in fact under investigation. It should be emphasized that the ultrafilter $\mathcal{U}(\alpha) := {}^*\{A \subset \mathbf{R} : \alpha \in A\}$ has the monad not converging to the initial infinitesimal α, i.e., the set $\mathrm{Cl}_\alpha(F,x')$ is, generally speaking, broader than $\mathrm{Cl}_{\mu(\mathcal{U}(\alpha))}(F,x')$. At the same time, the introduced approximations prove to possess many advantages inherent to Clarke cones. When detalizing and corroborating the last statement, let us, without further specification, use, as was the case in **5.2**, the supposition that the mapping $(x,\beta,h) \to x+\beta h$ of the space $(X\times\mathbf{R}\times X, \sigma\times\tau_{\mathbf{R}}\times\tau)$ in X is continuous at zero (which 'in standard entourage' is equivalent to the inclusion $\mu(\sigma)+\mu(\mathbf{R}_+)\mu(\tau)\subset\mu(\sigma)$).

5.4.2. Theorem. *For every set* Λ *of positive infinitesimals the following statements are valid:*

(1) $\mathrm{Ha}_\Lambda(F,x')$, $\mathrm{In}_\Lambda(F,x')$, $\mathrm{Cl}_\Lambda(F,x')$ *are semigroups and, moreover,*

$$\mathrm{Ha}(F,x')\subset\mathrm{Ha}_\Lambda(F,x')\subset\mathrm{In}_\Lambda(F,x')\subset\mathrm{Cl}_\Lambda(F,x')\subset K(F,x'),$$
$$\mathrm{Cl}(F,x')\subset\mathrm{Cl}_\Lambda(F,x');$$

(2) *if* Λ *is an internal set, then* $\mathrm{Ha}_\Lambda(F,x')$ *is* τ*-open;*

(3) $\mathrm{Cl}_\Lambda(F,x')$ *is a* τ*-closed set and, for* F *convex, we have* $K(F,x')=\mathrm{Cl}_\Lambda(F,x')$ *as soon as* $\sigma=\tau$;

(4) *if* $\sigma=\tau$, *then the following equality is valid:*

$$\mathrm{Cl}_\Lambda(F,x')=\mathrm{Cl}_\Lambda(\mathrm{cl}F,x');$$

(5) *the Rockafeller formula holds*

$$\mathrm{Ha}_\Lambda(F,x')+\mathrm{Cl}_\Lambda(F,x')=\mathrm{Ha}_\Lambda(F,x');$$

(6) *if* x' *is a* τ*-boundary point of* F, *then, for* $F':=(X-F)\cup\{x'\}$,

$$\mathrm{Ha}_\Lambda(F,x')=-\mathrm{Ha}_\Lambda(F',x').$$

$\lhd$ (1) Let us make sure that $\mathrm{In}_\Lambda(F,x')$ is a semigroup. If standard h' and h'' belong to $\mathrm{In}_\Lambda(F,x')$, then, for every $\alpha\in\Lambda$ and a certain $h_1\approx_\sigma h'$, we get $x'':=x+\alpha h_1\in F$ as soon as $x\in F$ and $x\approx_\sigma x'$. By hypothesis, there is an $h_2\approx_\tau h''$ for which $x''+\alpha h_2\in F$, as $x''\approx_\sigma x$. Finally, $h_1+h_2\approx_\tau h'+h''$ and h_1+h_2 'serves' the membership $h'+h''\in\mathrm{In}_\Lambda(F,x')$.

If $h'\in\mathrm{Cl}_\Lambda(F,x')$ and h' is standard, then $x'+\alpha h\in F$ for some $\alpha\in\Lambda$ and $h\approx_\tau h'$, which implies $h'\in K(F,x')$. The rest of the inclusions written in (1) are obvious.

(2) If h' is a standard element of $\mathrm{Ha}_\Lambda(F,x')$, then

$$(\forall x\approx_\sigma x', x\in F)(\forall h\approx_\tau h')(\forall\alpha\in\Lambda)\, x+\alpha h\in F.$$

Taking into account **5.3.2** and using the fact that Λ is an internal set, we deduce:

$$(\exists^{\mathrm{st}} V \in \mathcal{N}_\tau)\,(\exists^{\mathrm{st}} U \in \sigma(x'))\,(\forall x \in U \cap F)\,(\forall h \in h' + V)\,(\forall \alpha \in \Lambda)\; x + \alpha h \in F\,.$$

Let us choose standard neighbourhoods $V_1, V_2 \in \mathcal{N}_\tau$ in such a way that we had $V_1 + V_2 \subset V$. In this case, for all standard $h'' \in h' + V_1$ it is fulfilled that

$$(\forall x \in U \cap F)\,(\forall h \in h'' + V_2)\,(\forall \alpha \in \Lambda)\; x + \alpha h \in F,$$

i.e., $h'' \in \mathrm{Ha}_\Lambda(F, x')$ for any $h'' \in h' + V_1$.

(3) Let now h' be a standard element of $\mathrm{cl}_\tau \mathrm{Cl}_\Lambda(F, x')$. Let us choose an arbitrary standard neighbourhood V of the point h' and again choose standard $V_1, V_2 \in \mathcal{N}_\tau$ by the condition $V_1 + V_2 \subset V$. By definition, there is an $h'' \in \mathrm{Cl}_\Lambda(F, x')$ such that $h'' \in h' + V_1$. By **5.4.1** and **5.3.2**, we have

$$(\forall \alpha \in \Lambda)\,(\exists^{\mathrm{st}} U \in \sigma(x'))\,(\forall x \in F \cap U)\,(\exists h \in h'' + V_2)\; x + \alpha h \in F.$$

In this case $h \in h'' + V_2 \subset h' + V_1 + V_2 \subset h' + V$. In other words,

$$(\forall^{\mathrm{st}} V \in \mathcal{N}_\tau)(\forall \alpha \in \Lambda)\,(\exists^{\mathrm{st}} U \in \sigma(x'))\,(\forall x \in F \cap U)\,(\exists h \in h' + V)\; x + \alpha h \in F\,.$$

Therefore, $h' \in \mathrm{Cl}_\alpha(F, x')$ for every $\alpha \in \Lambda$, i.e., $h' \in \mathrm{Cl}_\Lambda(F, x')$.

If now $h' \in \mathrm{Fd}_\Lambda(F, x')$ and h' is standard, then for a certain standard $\alpha' > 0$ we have, by the transfer principle, $x' + \alpha' h' \in F$. If $x \approx_\sigma x'$ and $x \in F$, then $(x - x')/\alpha' \approx_\sigma 0$. For $h := h' + (x - x')/\alpha'$ we get $h \approx_\tau h'$ and, moreover, $x + \alpha' h \in F$. Taking into account the fact that F is convex, we have $x + (0, \alpha'] h \in F$. In particular, $x + \Lambda h \subset F$. Hence, $(\forall x \approx_\sigma x', x \in F)\,(\forall \alpha \in \Lambda)\,(\exists h \approx_\tau h')\; x + \alpha h \in F$, i.e, $h' \in \mathrm{Cl}_\Lambda(F, x')$. Therefore,

$$\mathrm{Fd}(F, x') \subset \mathrm{Cl}_\Lambda(F, x') \subset K(F, x') \subset \mathrm{cl}\,\mathrm{Fd}(F, x')\,.$$

Taking into account the fact that $\mathrm{Cl}_\Lambda(F, x')$ is τ-closed, we conclude: $K(F, x') = \mathrm{Cl}_\Lambda(F, x')$.

(4) The proof is carried out as in **5.2.11**.

(5) For standard $k' \in \mathrm{Ha}_\Lambda(F, x')$ and $h' \in \mathrm{Cl}_\Lambda(F, x')$, for every $\alpha \in \Lambda$ and any $x \in F$ such that $x \approx_\sigma x'$, we get, having chosen h that enjoys the conditions $h \approx_\tau h'$ and $x + \alpha h \in F$,

$$x+\alpha(h'+k'+\mu(\tau)) = x+\alpha h+\alpha(k'+(h-h')+\mu(\tau))$$
$$\subset (x+\mu(\sigma))\cap F+\alpha(k'+\mu(\tau)+\mu(\tau))$$
$$\subset (x+\mu(\sigma))\cap F+\alpha(k'+\mu(\tau))\subset F,$$

which implies the membership of $h'+k'$ in $\mathrm{Ha}_\Lambda(F,x')$.

(6) Let $-h\notin \mathrm{Ha}_\Lambda(F',x')$. Then for a certain $\alpha\in\Lambda$ there is an $h\approx_\tau h'$, so that for an appropriate $x\approx_\sigma x'$, $x\in F$, we have $x-\alpha h\in F$. While if $h\in\mathrm{Ha}_\Lambda(F,x')$, in particular, $h\in\mathrm{Ha}_\alpha(F,x')$ and $x=(x-\alpha h)+\alpha h\in F$, since $x-\alpha h\approx_\sigma x$. Hence, $x\in F\cap F'$, i.e., $x=x'$. Besides, $(x'-\alpha h)+\alpha(h+\mu(\tau))\subset F$, since $h+\mu(\tau)\subset\mu(\tau(h'))$. Therefore, x' is a τ-internal point of F, which contradicts the condition. Hence, $h\notin\mathrm{Ha}_\Lambda(F,x')$, which ensures the inclusion $-\mathrm{Ha}_\Lambda(F,x')\subset\mathrm{Ha}_\Lambda(F',x')$. Substituting in the above considerations $F=(F')'$ for F', we come to the sought conclusion. ▷

5.4.3. It is important to emphasize that in many cases the described analogues of the Hadamard and Clarke cones are convex. In fact, the following propositions are valid.

5.4.4. *Let τ be a vector topology and $t\Lambda\subset\Lambda$ for a certain standard $t\in(0,1)$. Then* $\mathrm{Cl}_\Lambda(F,x')$ *is a convex cone. If in this case Λ is also an internal set, then* $\mathrm{Ha}_\Lambda(F,x')$ *is a convex cone as well.*

◁ Let us assume that we consider $\mathrm{Ha}_\Lambda(F,x')$ and that $h\in\mathrm{Ha}_\Lambda(F,x')$ is a standard element of this set. By virtue of **5.4.2** (2), $\mathrm{Ha}_\Lambda(F,x')$ is open in the topology τ. Moreover, $th\in\mathrm{Ha}_\Lambda(F,x')$, where t is the standard positive number mentioned in the hypothesis. ▷

5.4.5. *Let $t\Lambda\subset\Lambda$ for every standard $t\in(0,1)$. Then the sets* $\mathrm{Cl}_\Lambda(F,x')$, $\mathrm{In}_\Lambda(F,x')$ *and* $\mathrm{Ha}_\Lambda(F,x')$ *are convex cones.*

◁ Let us assume, for definiteness, that we consider $\mathrm{Cl}_\Lambda(F,x')$. Let h' be a standard vector of the set under discussion, and $0<t<1$ is a standard number. Let $x\approx_\sigma x'$, $x\in F$ and $\alpha\in\Lambda$. For x and $t\alpha\in\Lambda$ let us choose an h, for which $h\approx_\tau h'$ and $x+\alpha th\in F$. As long as $th\approx_\tau th'$ (see **5.1.7**), we have $th'\in\mathrm{Cl}_\alpha(F,x')$. In other words, by the transfer principle, $(0,1)\mathrm{Cl}_\Lambda(F,x')\subset\mathrm{Cl}_\Lambda(F,x')$. Now we are to recall **5.4.2** (1). ▷

5.4.6. A set Λ is called *representative*, provided $\mathrm{Ha}_\Lambda(F,x')$ and $\mathrm{Cl}_\Lambda(F,x')$ are (convex) cones, the propositions **5.4.4** and **5.4.5** giving examples of a representative Λ.

5.4.7. Let $f: X \to \overline{\mathbf{R}}$ be a function acting into the extended real line. For an infinitesimal α, a point x' of $\mathrm{dom}(f)$ and a vector $h' \in X$, we set:

$$f(\mathrm{Ha}_\alpha)(x')(h') := \inf\{t \in \mathbf{R}: (h',t) \in \mathrm{Ha}_\alpha(\mathrm{epi}(f),(x',f(x')))\},$$
$$f(\mathrm{In}_\alpha)(x')(h') := \inf\{t \in \mathbf{R}: (h',t) \in \mathrm{In}_\alpha(\mathrm{epi}(f),(x',f(x')))\},$$
$$f(\mathrm{Cl}_\alpha)(x')(h') := \inf\{t \in \mathbf{R}: (h',t) \in \mathrm{Cl}_\alpha(\mathrm{epi}(f),(x',f(x')))\}.$$

The derivatives $f(\mathrm{Ha}_\alpha)$, $f(\mathrm{In}_\alpha)$ and $f(\mathrm{Cl}_\alpha)$ are introduced in a natural way. It should be remarked that the derivative $f(\mathrm{Cl}) := f(\mathrm{Cl}_{\mu(\mathbf{R}_+)})$ is called the *Rockafeller derivative* and is denoted by the symbol $f^\uparrow$. Therefore, we write:

$$f_\alpha^\uparrow(x') := f(\mathrm{Cl}_\alpha)(x'), \quad f_\Lambda^\uparrow(x') := f(\mathrm{Cl}_\Lambda)(x').$$

If τ is the discrete topology, then $\mathrm{Ha}_\Lambda(F,x') = \mathrm{In}_\Lambda(F,x') = \mathrm{Cl}_\Lambda(F,x')$. In this case the Rockafeller derivative is termed the *Clarke derivative* and the following notation is used:

$$f_\alpha^\circ(x') := f_\alpha^\uparrow(x'), \quad f_\Lambda^\circ(x') := f_\Lambda^\uparrow(x').$$

For $\Lambda = \mu(\mathbf{R}_+)$, the indications of Λ are omitted.

When considering epiderivatives, the space $X \times \mathbf{R}$ is assumed to be endowed with the conventional product topologies $\sigma \times \tau_{\mathbf{R}}$ and $\tau \times \tau_{\mathbf{R}}$, where $\tau_{\mathbf{R}}$ is the conventional topology in $\mathbf{R}$. It is sometimes convenient to furnish $X \times \mathbf{R}$ with the pair of the topologies $\sigma \times \tau_0$ and $\tau \times \tau_{\mathbf{R}}$, where τ_0 is the trivial topology in $\mathbf{R}$. When using such topologies, we speak about *the Clarke and Rockafeller derivatives along effective domain* $\mathrm{dom}(f)$ and add the index d in the notation: f_d°, $f_{\Lambda,d}^\uparrow$, etc..

5.4.8. *The following statements are valid:*

$$f_\alpha^\uparrow(x')(h') \le t'$$
$$\leftrightarrow (\forall x \approx_\sigma x', t \approx f(x'), t \ge f(x))\, (\exists h \approx_\tau h')\, {}^\circ((f(x+\alpha h) - t)/\alpha) \le t';$$
$$f_\alpha^\circ(x')(h') < t'$$
$$\leftrightarrow (\forall x \approx_\sigma x', t \approx f(x'), t \ge f(x))\, (\forall h \approx_\tau h')\, {}^\circ((f(x+\alpha h) - t)/\alpha) < t';$$
$$f_{\alpha,d}^\uparrow(x')(h') \le t'$$
$$\leftrightarrow (\forall x \approx_\sigma x', x \in \mathrm{dom}(f))\, (\exists h \approx_\tau h')\, {}^\circ((f(x+\alpha h) - t)/\alpha) \le t';$$

$$f^{\circ}_{\alpha,d}(x')(h') < t'$$
$$\leftrightarrow (\forall x \approx_{\sigma} x', x \in \mathrm{dom}(f))\ (\forall h \approx_{\tau} h')\ {}^{\circ}((f(x+\alpha h) - t) / \alpha) < t'.$$

◁ For the proof, appeal to **2.2.18** (3). ▷

5.4.9. *If f is a lower semicontinuous function, then*

$$f^{\uparrow}_{\alpha}(x')(h') \le t'$$
$$\leftrightarrow (\forall x \approx_{\sigma} x', f(x) \approx f(x'))\ (\exists h \approx_{\tau} h')\ {}^{\circ}\left(\frac{f(x+\alpha h) - f(x)}{\alpha}\right) \le t;$$
$$f^{\circ}_{\alpha}(x')(h') < t'$$
$$\leftrightarrow (\forall x \approx_{\sigma} x', f(x) \approx f(x'))\ (\forall h \approx_{\tau} h')\ {}^{\circ}\left(\frac{f(x+\alpha h) - f(x)}{\alpha}\right) < t.$$

◁ It is only the implications to the right that have to be checked. Since such checks are identical, let us carry out the first of them. As f is lower semicontinuous, we can deduce: $x' \approx_{\sigma} x \rightarrow {}^{\circ}f(x) \ge f(x')$. Therefore, for x,t such that $t \approx f(x')$ and $t \ge f(x)$, we have ${}^{\circ}t \ge {}^{\circ}f(x) \ge f(x') = {}^{\circ}t$. In other words, ${}^{\circ}f(x) = f(x')$ and $f(x) \approx f(x')$. Choosing a suitable h from the conditions, we come to the conclusion

$${}^{\circ}(\alpha^{-1}(f(x+\alpha h) - t)) \le {}^{\circ}(\alpha^{-1}(f(x+\alpha h) - f(x))) \le t',$$

which ensures the sought result. ▷

5.4.10. *For a continuous function f the following equalities are valid:*

$$f^{\uparrow}_{\Lambda,d}(x') = f^{\uparrow}_{\Lambda}(x'), \quad f^{\circ}_{\Lambda,d}(x') = f^{\circ}_{\Lambda}(x').$$

◁ It suffices to remark that the continuity of f at a standard point yields $(x \approx_{\sigma} x', x \in \mathrm{dom}(f)) \rightarrow f(x) \approx f(x')$ (see **4.2.7**). ▷.

5.4.11. Theorem. *Let Λ be a monad. Then we have the following presentations:*

(1) *if f is a lower semicontinuous from function, then*

$$f^{\uparrow}_{\Lambda}(x')(h') = \lim_{\substack{x \to_f x' \\ \alpha \in \mathfrak{F}_{\Lambda}}} \sup \inf_{h \to h'} \frac{f(x+\alpha h) - f(x)}{\alpha},$$

$$f^{\circ}_{\Lambda}(x')(h') = \limsup_{\substack{x \to_f x' \\ \alpha \in \mathfrak{F}_\Lambda}} \frac{f(x+\alpha h') - f(x)}{\alpha},$$

where $x \to_f x'$ implies $x \to_\sigma x'$ and $f(x) \to f(x')$;

(**2**) *for a continuous function f we have*

$$f^{\uparrow}_{\Lambda,d}(x')(h') = \limsup_{\substack{x \to x' \\ \alpha \in \mathfrak{F}_\Lambda}} \inf_{h \to h'} \frac{f(x+\alpha h) - f(x)}{\alpha},$$

$$f^{\circ}_{\Lambda,d}(x')(h') = \limsup_{\substack{x \to x' \\ \alpha \in \mathfrak{F}_\Lambda}} \frac{f(x+\alpha h') - f(x)}{\alpha}.$$

⊲ For the proof we have to recall the criterion for the Rockafeller limit **5.2.23**, as well as **5.4.9** and **5.4.10**. ⊳

5.4.12. Theorem. *Let Λ be a representative set of infinitesimals. The following statements are valid:*

(1) *if f is a mapping directionally Lipshitz at the point x', i.e. such that* $\mathrm{Ha}(\mathrm{epi}(f), (x', f(x'))) \neq \varnothing$, *then*

$$f^{\uparrow}_{\Lambda}(x') = f^{\circ}_{\Lambda}(x');$$

if, moreover, f is continuous at the point x', then

$$f^{\uparrow}_{\Lambda}(x') = f^{\uparrow}_{\Lambda,d}(x') = f^{\circ}_{\Lambda,d}(x') = f^{\circ}_{\Lambda}(x');$$

(**2**) *if f is an arbitrary mapping, and the Hadamard cone of the effective domain of f at the point x' is nonempty, i.e.,* $\mathrm{Ha}(\mathrm{dom}(f), x') \neq \varnothing$, *then*

$$f^{\uparrow}_{\Lambda,d}(x') = f^{\circ}_{\Lambda,d}(x').$$

⊲ The proof of both statements sought is carried out in the same pattern as that of theorem **5.4.2**. Let us consider in detail the case when f is directionally Lipshitz.

Let us set $\mathfrak{A} := \mathrm{epi}(f)$, $a' := (x', f(x'))$. By hypothesis, both $\mathrm{Cl}_\Lambda(\mathfrak{A}, a')$ and $\mathrm{Ha}_\Lambda(\mathfrak{A}, a')$ are convex cones. In this case $\mathrm{Ha}_\Lambda(\mathfrak{A}, a') \supset \mathrm{Ha}(\mathfrak{A}, a')$ and, hence, $\mathrm{int}_{\tau \times \tau_{\mathbb{R}}} \mathrm{Ha}_\Lambda(\mathfrak{A}, a') \neq \varnothing$. On the basis of the Rockafeller formula we deduce:

$$\mathrm{cl}_{\tau\times\tau_{\mathbf{R}}}\mathrm{Ha}_{\Lambda}(\mathfrak{A},a') = \mathrm{Cl}_{\Lambda}(\mathfrak{A},a').$$

This equality implies the required statement. ▷

5.4.13. Theorem. *Let $f_1, f_2 : X \to \overline{\mathbf{R}}$ be arbitrary functions, and let $x' \in \mathrm{dom}(f_1) \cap \mathrm{dom}(f_2)$. In this case*

$$(f_1 + f_2)^{\uparrow}_{\Lambda,d}(x') \leq (f_1)^{\uparrow}_{\Lambda,d}(x') + (f_2)^{\circ}_{\Lambda,d}(x').$$

If, moreover, f_1 and f_2 are continuous at the point x', then

$$(f_1 + f_2)^{\uparrow}_{\Lambda}(x') \leq (f_1)^{\uparrow}_{\Lambda}(x') + (f_2)^{\circ}_{\Lambda}(x').$$

◁ Let a standard element h' be chosen in the following way:

$$h' \in \mathrm{dom}((f_2)^{\circ}_{\Lambda,d}) \cap \mathrm{dom}((f_1)^{\uparrow}_{\Lambda,d}).$$

If there is no such an h', the estimates sought are obvious.

Let us choose $t' \geq (f_1)^{\uparrow}_{\Lambda,d}(x')(h')$ and $s' > (f_2)^{\circ}_{\Lambda,d}(x')(h')$. Then, by virtue of **5.4.8**, for every $x \approx_{\sigma} x'$, $x \in \mathrm{dom}(f_1) \cap \mathrm{dom}(f_2)$ and any $\alpha \in \Lambda$ there is an h for which $h \approx_{\tau} h'$, and, moreover,

$$\delta_1 := {}^{\circ}((f_1(x + \alpha h) - f_1(x) / \alpha) \leq t';$$
$$\delta_2 := {}^{\circ}((f_2(x + \alpha h) - f_2(x) / \alpha) < s'.$$

Hence, we deduce: $\delta_1 + \delta_2 < t' + s'$, which ensures (1). If f_1 and f_2 are continuous at the point x, then we apply **5.4.10**. ▷

5.4.14. By way of concluding the present stage of discussion, let us consider special presentations of the Clarke cone which arise in a finite-dimensional space and pertain to the following remarkable result.

5.4.15. Cornet theorem. *In a finite-dimensional space the Clarke cone is the Kuratowski limit of contingencies:*

$$\mathrm{Cl}(F,x') = \mathrm{Li}_{\substack{x \to x' \\ x \in F}} K(F,x).$$

5.4.16. Corollary. *Let Λ be an (external) set of strictly positive infinitesimals, containing an (internal) sequence convergent to zero. Then the following equality is valid:*

$$\mathrm{Cl}_\Lambda(F,x') = \mathrm{Cl}(F,x').$$

⊲ By the Leibniz principle, we can work in standard entourage. Since the inclusion $\mathrm{Cl}_\Lambda(F,x') \supset \mathrm{Cl}(F,x')$ is obvious, let us choose a standard point h' from $\mathrm{Cl}_\Lambda(F,x')$ and establish that h' lies in the Clarke cone $\mathrm{Cl}(F,x')$.

Since use is made of **5.3.13**, the following presentation is valid:

$$\mathrm{Li}_{\substack{x\to x'\\ x\in F}} K(F,x) = {}^*\{h' : (\forall x \approx x', x \in F)(\exists h \approx h')\, h \in K(F,x)\};$$

let us make sure that for $x \approx x'$, $x \in F$ we have $h \in K(F,x)$ for a certain element h infinitely close to h'.

If (α_n) is a sequence of Λ elements convergent to zero then, by hypothesis,

$$(\forall n \in \mathbf{N})(\exists h_n)\ X + \alpha_n h_n \in F \wedge h_n \approx h'.$$

For any standard $\varepsilon > 0$ and the conventional norm $\|\cdot\|$, in $\mathbf{R}^n$ we have $\|\bar{h} - h'\| \le \varepsilon$. Therefore, taking into account finite dimensions, we can find sequences $(\bar{\alpha}_n)$ and $(\bar{h}_n)$ such that

$$\bar{\alpha}_n \to 0,\quad \bar{h}_n \to \bar{h},\quad \|\bar{h} - h'\| \le \varepsilon,\quad x + \bar{\alpha}_n \bar{h}_n \in F \quad (n \in \mathbf{N}).$$

Applying the strong idealization principle, we come to the conclusion that there are sequences $(\bar{\alpha}_n)$ and $(\bar{h}_n)$ serving simultaneously all standard positive numbers ε. Obviously, the corresponding limiting vector h is infinitely close to h', and, at the same time, by the definition of contingency, $h \in K(F, x)$. ⊳

5.4.17. In the theorem given above we can use as a set Λ the monad of any filter convergent to zero, for instance, of the tail filter of a fixed standard sequence (α_n) composed of strictly positive numbers and vanishing. Let us recall the characteristics of the Clarke cone pertaining to this case and supplementing those already considered. For the formulation let use the symbol $d_F(x)$ to denote the distance from the point x to the set F.

5.4.18. Theorem. *For a sequence (α_n) of strictly positive numbers convergent to zero* *the following statements are equivalent*:

(1) $h' \in \mathrm{Cl}(F,x')$,

(2) $\limsup\limits_{\substack{x \to x' \\ n \to \infty}} \dfrac{d_F(x+\alpha_n h') - d_F(x)}{\alpha_n} \le 0$,

(3) $\limsup\limits_{x \to x'} \limsup\limits_{n \to \infty} \alpha_n^{-1}(d_F(x+\alpha_n h') - d_F(x)) \le 0$,

(4) $\lim\limits_{\substack{x \to x' \\ x \in F}} \limsup\limits_{n \to \infty} \alpha_n^{-1} d_F(x+\alpha_n h') = 0$,

(5) $\limsup\limits_{x \to x'} \liminf\limits_{n \to \infty} \alpha_n^{-1}(d_F(x+\alpha_n h') - d_F(x)) \le 0$,

(6) $\lim\limits_{\substack{x \to x' \\ x \in F}} \liminf\limits_{n \to \infty} \dfrac{d_F(x+\alpha_n h')}{\alpha_n} = 0$.

◁ Let us first of all observe that for $\alpha > 0$ the following equivalence is valid:

$$^\circ(\alpha^{-1} d_F(x+\alpha h')) = 0 \leftrightarrow (\exists h \approx h')\, x + \alpha h \in F,$$

where $^\circ t$ is, as usual, the standard part of the number t.

Indeed, in order to establish the implication to the left, set $y := x + \alpha h'$. In this case

$$d_F(x+\alpha h') / \alpha = \|x + \alpha h' - y\| / \alpha \le \|h - h'\|.$$

When checking the reverse implication, we get, applying the strong idealization principle, the following result:

$$\begin{aligned} &^\circ(\alpha^{-1} d_F(x+\alpha h')) = 0 \to (\forall^{\mathrm{st}} \varepsilon > 0)\, d_F(x+\alpha h') / \alpha < \varepsilon \\ &\to (\forall^{\mathrm{st}} \varepsilon > 0)(\exists y \in F)\ \|x + \alpha h' - y\| / \alpha < \varepsilon \\ &\to (\exists y \in F)(\forall^{\mathrm{st}} \varepsilon > 0)\ \|h' - (y - x) / \alpha\| < \varepsilon \\ &\to (\exists y \in F)\, \|h - (y - x) / \alpha\| \approx 0. \end{aligned}$$

Setting now $h := (y - x) / \alpha$, we see that $h \approx h'$, in which case $x + \alpha h \in F$.

Let us now go over to the proof of the sought equivalences.

Since the implications (3) → (4) → (6) and (3) → (5) → (6) are obvious, we only establish that (1) → (2) → (3) and (6) → (1).

(1) → (2). Working in standard entourage, let us choose $x \approx x'$ and $N \approx +\in$. Choose

an $x'' \in F$ in such a way that we had $\|x = x''\| < d_F(x') + \alpha_N^2$. Since the inequality

$$d_F(x + \alpha_N h') - d_F(x'' + \alpha_N h') \leq \|x - x''\|,$$

is valid, we can deduce the following estimates:

$$(d_F(x + \alpha_N h') - d_F(x)) \,/\, \alpha_N \leq (d_F(x'' + \alpha_N h') + \|x - x''\| - \\ - d_F(x)) \,/\, \alpha_N \leq d_F(x'' + \alpha_N h') \,/\, \alpha_N + \alpha_N.$$

As $h' \in \mathrm{Cl}(F, x')$, and by the choice of x'' and N, for a certain $h \approx h'$ we get $x'' + \alpha_N h \in F$. Therefore, from the above, we infer ${}^\circ(d_F(x'' + \alpha_N h) \,/\, \alpha_N) = 0$. Hence,

$$(\forall x \approx x')\,(\forall N \approx +\infty)\; {}^\circ(\alpha_N^{-1}(d_F(x + \alpha_N h') - d_F(x))) \leq 0.$$

This is, by **5.3.22**, the nonstandard criterion for (2) to be valid.

(2) → (3). It suffices to remark that for $f\colon U \times V \to \overline{\mathbf{R}}$, as well as for the filters $\mathfrak{F}$ in U and $\mathscr{B}$ in V, we have

$$\begin{aligned}
&\limsup\nolimits_{\mathfrak{F}} \limsup\nolimits_{\mathscr{B}} f(x, y) \leq t \\
&\leftrightarrow (\forall x \in \mu(\mathfrak{F}))\; {}^\circ\!\limsup\nolimits_{\mathscr{B}} f(x, y) \leq t \\
&\leftrightarrow (\forall x \in \mu(\mathfrak{F}))\,(\forall^{\mathrm{st}} \varepsilon > 0) \inf_{G \in \mathscr{B}} \sup_{y \in G} f(x, y) < t + \varepsilon \\
&\leftrightarrow (\forall x \in \mu(\mathfrak{F}))\,(\forall^{\mathrm{st}} \varepsilon > 0)\,(\exists G \in \mathscr{B}) \sup_{y \in G} f(x, y) < t + \varepsilon \\
&\leftrightarrow (\forall x \in \mu(\mathfrak{F}))\,(\exists G \in \mathscr{B})\,(\forall^{\mathrm{st}} \varepsilon > 0) \sup_{y \in G} f(x, y)\, t + \varepsilon \\
&\leftrightarrow (\forall x \in \mu(\mathfrak{F}))\,(\exists G \in \mathscr{B})\,(\forall^{\mathrm{st}} \varepsilon > 0) \sup_{y \in G} f(x, y) \leq t + \varepsilon \\
&\leftrightarrow (\forall x \in \mu(\mathfrak{F}))\,(\exists G \in \mathscr{B})\,(\forall y \in G)\, {}^\circ f(x, y) \leq t.
\end{aligned}$$

Here, as usual, $\mu(\mathfrak{F})$ is the monad of the filter $\mathfrak{F}$.

(6) → (1). Let us start with the remark that in the notation of the preceding fragment of the proof, we have

$$\begin{aligned}
&\limsup\nolimits_{\mathfrak{F}} \limsup\nolimits_{\mathscr{B}} f(x, y) \leq t \\
&\leftrightarrow (\forall x \in \mu(\mathfrak{F})) \sup_{G \in \mathscr{B}} \inf_{y \in \mathscr{B}} f(x, y) \leq t \\
&\leftrightarrow (\forall x \in \mu(\mathfrak{F}))\,(\forall^{\mathrm{st}} \varepsilon > 0)\,(\forall G \in \mathscr{B}) \inf_{y \in G} f(x, y) \leq t + \varepsilon \\
&\leftrightarrow (\forall x \in \mu(\mathfrak{F}))\,(\forall G \in \mathscr{B})\,(\forall^{\mathrm{st}} \varepsilon > 0) \inf_{y \in G} f(x, y) < t + \varepsilon
\end{aligned}$$

$$\leftrightarrow (\forall x \in \mu(\mathfrak{F}))(\forall G \in \mathcal{B})(\forall^{\mathrm{st}} \varepsilon > 0)(\exists y \in G)\ f(x,y) < t + \varepsilon$$
$$\leftrightarrow (\forall x \in \mu(\mathfrak{F}))(\forall G \in \mathcal{B})(\exists y \in G)\ {}^\circ f(x,y) \le t.$$

Using the conditions, from the established characteristic we deduce:

$$(\forall x \approx x', x \in F)(\forall n)(\exists N \ge n\ {}^\circ(\alpha_N^{-1} d_F(x + \alpha_N h')) = 0.$$

In other words, for a certain h_N such that $h_N \approx h'$, we get $x + \alpha_N h_N \in F$. Taking into account the considerations presented, we can, as was the case in **5.4.16**, deduce that h' lies in the lower Kuratowski limit of the contingencies of the set F at the points close to x', i.e., in the Clarke cone $\mathrm{Cl}(F, x')$. ▷

5.5. Approximation to the Composition of Sets

Let us now go over to studying tangents of the Clarke type and compositions of correspondences. In this case we have to start with some topological considerations pertaining to open and nearly open operators.

5.5.1. Take, alongside with the vector space X with topologies σ_Y and τ_X under consideration, another vector space Y with topologies σ_Y and τ_Y. Let us consider a linear operator T from X to Y and study, first of all, the problem of the relation between the approximating sets F at the point x', where $F \subset X$, and the image $T(F)$ at the point Tx'.

5.5.2. *The following statements are valid:*

(1) *the inclusion*

$$T(\mu(\sigma_X(x')) \cap F) \supset \mu(\sigma_Y(Tx')) \cap T(F)$$

is equivalent to the relation

$$(\forall U \in \sigma_X(x'))(\exists V \in \sigma_Y(Tx'))\ T(U \cap F) \supset V \cap T(F),$$

which is the condition of (relative) preopenness, or condition (ρ_-) *(for the parameters* T, F *and* x'*);*

(2) *condition* (ρ_-) *combined with the requirement that* T *be continuous as a mapping from* (X,σ_X) *to* (Y,σ_Y) *is equivalent to the following condition* (ρ_-), *the condition of (relative) openness*:

$$T(\mu(\sigma_X(x'))\cap F)\supset \mu(\sigma_Y(Tx'))\cap T(F);$$

(3) *the operator* T *obeys the condition of (relative) near-openness, or condition* (ρ_-), *i.e.,*

$$(\forall U\in\sigma_X(x'))(\exists V\in\sigma_Y((Tx'))\quad \mathrm{cl}_{\tau_Y}T(U\cap F)\supset V\cap T(F)$$

iff

$$(\forall W\in\mathfrak{N}_{\tau_Y})\ T(\mu(\sigma_X(x'))\cap F)+W\supset \mu(\sigma_Y(Tx'))\cap T(F).$$

◁ Statements (1) and (2) are obtained by specialization of **5.3.2.**. To prove (3), let us denote

$$\mathcal{A}:=T(\sigma_X(x')\cap F),\quad \mathcal{B}:=\sigma_Y(Tx')\cap T(F),$$
$$\mathfrak{N}:=\{N\subset Y^2:\ (\exists W\in\mathfrak{N}_{\tau_Y})\quad N\supset\{(y_1,y_2):y_1-y_2\in W\}\},$$

i.e., $\mathfrak{N}$ is the uniformity in Y corresponding to the topology in question. Using the introduced notation and applying **5.3.2**, as well as the principles of idealization and transfer, we get:

$$(\forall N\in\mathfrak{N})\ N(\mu(\mathcal{A}))\supset\mu(\mathcal{B})$$
$$\leftrightarrow(\forall N\in\mathfrak{N})\ (\forall b\in\mu(\mathcal{B}))\ (\exists a\in\mu(\mathcal{A}))\ b\in N(a)$$
$$\leftrightarrow(\forall N\in\mathfrak{N})\ (\forall^{\mathrm{st}}A\in\mathcal{A})\ (\exists^{\mathrm{st}}B\in\mathcal{B})\ (\forall b\in B)(\exists a\in A)\ b\in N(a)$$
$$\leftrightarrow\ (\forall^{\mathrm{st}}A\in\mathcal{A})(\forall N\in\mathfrak{N})(\exists^{\mathrm{st}}B\in\mathcal{B})\ B\subset N(A)$$
$$\leftrightarrow\ (\forall^{\mathrm{st}}A\in\mathcal{A})(\exists^{\mathrm{st}}B\in\mathcal{B})(\forall N\in\mathfrak{N})\ B\subset N(A)$$
$$\leftrightarrow\ (\forall^{\mathrm{st}}A\in\mathcal{A})(\exists^{\mathrm{st}}B\in\mathcal{B})\ B\subset \mathrm{cl}A$$
$$\leftrightarrow\ (\forall A\in\mathcal{A})(\exists B\in\mathcal{B})\ B\subset \mathrm{cl}A,$$

where the closure is calculated in the corresponding uniform topology. ▷

5.5.3. Theorem. *The following statements are valid*:

(1) *if the operator* T *obeys condition* (ρ) *and is continuous as a mapping from* (X,τ_X) *to* (Y,τ_Y), *then*

$$T(\mathrm{Cl}_\Lambda(F,x'))\subset \mathrm{Cl}_\Lambda(T(F),Tx'),$$

$$T(\mathrm{In}_\Lambda(F,x')) \subset \mathrm{In}_\Lambda(T(F),Tx');$$

if, moreover, T is an open mapping of (X,τ_X) in (Y,τ_Y), then

$$T(\mathrm{Ha}_\Lambda(F,x')) \subset \mathrm{Ha}_\Lambda(T(F),Tx');$$

(2) *if τ_Y is a vector topology, the operator $T\colon (X,\tau_X) \to (Y,\tau_Y)$ is continuous and obeys condition $(\bar{\rho})$, then*

$$T(\mathrm{Cl}_\Lambda(F,x')) \subset \mathrm{Cl}_\Lambda(T(F),Tx').$$

◁ (1) Let us, for instance, check the second of the required inclusions. To this end, having fixed $h' \in \mathrm{In}_\Lambda(F,x')$, for $\alpha \in \Lambda$, we choose an $h \approx_{\tau_X} h'$ such that for all $x \approx_{\sigma_X} x'$, $x \in F$ we have $x + \alpha h \in F$. Obviously, $Th \approx_{\sigma_X} Th'$ and $Tx + \alpha Th \in T(F)$. Applying condition (ρ), we conclude: $Th' \in \mathrm{In}_\Lambda(T(F), Tx')$.

Let now be known that T obeys the additional condition of openness, i.e., on the basis of **5.5.2** (1), $T(\mu(\tau_X)) \supset \mu(\tau_Y)$. Combined with the continuity of T, this implies that the obtained monads coincide. If now $y \in T(F)$, $y \approx_{\sigma_Y} Tx'$, then by condition (ρ), we get $y = Tx$, where $x \in F$ and $x \approx_{\sigma_X} x'$. In this case for $z \approx_{\tau_Y} Th'$ we can find an $h \approx_{\tau_X} h'$, for which $z = Th$. Therefore, for all $\alpha \in \Lambda$ we have $x + \alpha h \in F$, i.e., $y + \alpha z = Tx + \alpha Th \in T(F)$ as soon as a standard h' is such that $h' \in \mathrm{Ha}_\Lambda(F,x')$.

(2) Let us choose an infinitesimal $\alpha \in \Lambda$ and any standard element $h' \in \mathrm{Cl}_\Lambda(F,x')$. Let W is a certain infinitely small zero neighbourhood of τ_Y. Then, by hypothesis, αW is also a zero neighbourhood. On the basis of $(\bar{\rho})$, having taken $y \approx_{\sigma_X} Tx'$, $y \in T(F)$, we find $x \in \mu(\sigma_X(x')) \cap F$ in such a way that $y = Tx + \alpha\omega$ and $\omega \approx_{\tau_Y} 0$. By the condition of the containment of h' in the Clarke cone, there is an element $h'' \approx_{\tau_Y} h'$ for which $x + \alpha h'' \in F$. Hence, $y + \alpha(Th'' - \omega) = y - \alpha\omega + \alpha Th'' = T(x + \alpha h'') \in T(F)$. Indeed, from here we deduce: $Th'' - \omega \in Th' + \mu(\tau_Y) - \omega \in Th' + \mu(\tau_Y) + \mu(\tau_Y) = Th' + \mu(\tau_Y)$. Therefore, we have established: $Th' \in \mathrm{Cl}_\alpha(T(F), Tx')$. ▷

5.5.4. Let us now consider vector spaces X,Y,Z furnished with topologies σ_X,τ_X; σ_Y,τ_Y, respectively. Let, then, $F \subset X \times Y$ and $G \subset X \times Z$ be two correspondences, and let the point $d' := (x',y',z') \in X \times Y \times Z$ be such that $a' := (x',y') \in F$ and $b' := (x',z') \in G$. Introduce the following notation: $H := X \times G \cap F \times Z$, $c' := (x',z')$. It should be remarked that $G \circ F = \mathrm{Pr}_{X \times Z} H$, where $\mathrm{Pr}_{X \times Z}$ is the operator of natural projection.

Let us introduce the following abbreviations:

$$\sigma_1 := \sigma_X \times \sigma_Y;\quad \sigma_2 := \sigma_Y \times \sigma_Z;\quad \sigma := \sigma_X \times \sigma_Z;\quad \bar{\sigma} := \sigma_X \times \sigma_Y \times \sigma_Z;$$
$$\tau_1 := \tau_X \times \tau_Y;\quad \tau_2 := \tau_Y \times \tau_Z;\quad \tau := \tau_X \times \tau_Z;\quad \bar{\tau} := \tau_X \times \tau_Y \times \tau_Z.$$

It would be expedient to recall that the operator $\mathrm{Pr}_{X\times Z}$ is continuous and open (when 'same-letter' topologies are used). Still fixed is a certain set Λ composed of infinitesimals. Let us also recall a property of monads we need.

5.5.5. *The monad of a composition is the composition of monads..*

◁ Let $\mathcal{A}$ is a filter in $X \times Y$, while $\mathcal{B}$ is a filter in $Y \times Z$. We have

$$\mathcal{B} \circ \mathcal{A} := \widetilde{\{B \circ A : A \in \mathcal{A}, B \in \mathcal{B}\}},$$

in which case the sets operating in the definition of $\mathcal{B} \circ \mathcal{A}$ can be considered to be nonempty. It is obvious that $B \circ A = \mathrm{Pr}_{X\times Z}(A \times Z \cap X \times B)$. Therefore, the filter under consideration, $\mathcal{B} \circ \mathcal{A}$, is the image $\mathrm{Pr}_{X\times Z}(\mathcal{C})$, where $\mathcal{C} := \mathcal{C}_1 \vee \mathcal{C}_2$ and $\mathcal{C}_1 := \mathcal{A} \times \{Z\}$, $\mathcal{C}_2 := \{X\} \times \mathcal{B}$. Since the monad of a product is the product of monads, and the monad of the least upper bound of filters is the intersection of their monads, then, making use of **4.1.6** (5), we come to the relation

$$\mu(\mathcal{B} \circ \mathcal{A}) = \mathrm{Pr}_{X\times Z}(\mu(\mathcal{A}) \times Z \cap X \times \mu(\mathcal{B})) = \mu(\mathcal{B}) \circ \mu(\mathcal{A}),$$

which was required. ▷

5.5.6. *The following statements are equivalent:*

(1) *for the operator* $\mathrm{Pr}_{X\times Z}$, *the correspondence* H *and a point* c', *condition* (ρ) *is fulfilled*;

(2) $G \circ F \cap \mu(\sigma(c')) = G \cap \mu(\sigma_2(b')) \circ F \cap \mu(\sigma_1(a'))$;

(3) $(\forall V \in \sigma_Y(y'))(\exists U \in \sigma_X(x'))(\exists W \in \sigma_Z(z'))$
$G \circ F \cap U \times W \subset G \circ I_V \circ F,$

where I_V is, as usual, the identity relation on V.

◁ Applying **5.3.2**, let us rewrite (3) in equivalent form:

$$(\forall V \in \sigma_Y(y'))(\exists O \in \sigma(c'))(\forall (x,z) \in O, (x,z) \in G \circ F)$$
$$(\exists y \in V)\,(x,y) \in F \wedge (y,z) \in G \leftrightarrow (\forall (x,z) \approx_{\sigma} c', (x,z) \in G \circ F)$$
$$(\exists y \approx_{\sigma_Y} y')\,(x,y) \in F \wedge (y,z) \in G \leftrightarrow \mu(\sigma(c')) \cap G \circ F$$
$$\subset \mu(\sigma_2(b')) \cap G \circ \mu(\sigma_1(a')) \cap F.$$

Now we have to observe that

$$\mathrm{Pr}_{X \times Z}(\mu(\overline{\sigma}(d')) \cap H)$$
$$= \{(x,z) \in G \circ F : x \approx_{\sigma_X} x' \wedge z \approx_{\sigma_Z} z' \wedge (\exists y \approx_{\sigma_Y} y')\,(x,y) \in F \wedge (y,z) \in G\}$$
$$= \mu(\sigma_2(b')) \cap G \circ \mu(\sigma_1(a')) \cap F. \ ▷$$

5.5.7. *The following statements are equivalent:*

(1) *for the operator* $\mathrm{Pr}_{X \times Z}$, *the correspondence H and a point c', condition $(\overline{p})$ is fulfilled;*

(2) $(\forall W \in \mathfrak{N}_\tau)\ \mu(\sigma_2(b')) \cap G \circ \mu(\sigma_1(a')) \cap F + W \supset \mu(\sigma(c')) \cap G \circ F;$

(3) $(\forall V \in \sigma_2(b'))(\forall U \in \sigma_1(a'))(\exists W \in \sigma(c'))$
$W \cap G \circ F \subset \mathrm{cl}_\tau(V \cap G \circ U \cap F);$

(4) $(\forall U \in \sigma_X(x'))(\forall V \in \sigma_Y(y'))(\forall W \in \sigma_Z(z'))(\exists V \in \sigma(c'))$
$O \cap G \circ F \subset \mathrm{cl}_\tau(G \circ I_V \circ F \cap U \times W);$

(5) *if* $\tau \geq \sigma$, *then*

$$(\forall V \in \sigma_Y(y'))(\exists U \in \sigma_X(x'))(\exists W \in \sigma_Z(z'))$$
$$G \circ F \cap U \times W \subset \mathrm{cl}_\tau(G \circ I_V \circ F),$$

i.e., condition $(\overline{\rho c})$ *is said to be fulfilled for the point* $d' := (x', y', z')$.

◁ From supposition **5.5.2** (3) and the proof of **5.5.2** (3) we directly conclude: (1) ↔ (2) ↔ (3).

In order to prove the equivalence (3) ↔ (4), it suffices to remark:

$$
\begin{aligned}
&(V \times W) \cap G \circ (U \times V) \cap F \\
&= \{(x,z) \in X \times Z \colon x \in U \wedge z \in W \wedge (\exists y \in V)\,(x,y) \in F \wedge (y,z) \in G\} \\
&= G \circ I_V \circ F \cap U \times W
\end{aligned}
$$

for any $U \subset X, V \subset Y, W \subset Z$.

Therefore, it now remains to be established that (4) ↔ (5), this implication, however, being obvious, since (5) is obtained by a specialization of (4) for $U := X$ and $W := Z$.

In order to check (5) ↔ (4) let us, having set $V \in \sigma_Y(y')$, choose an open neighbourhood $C \in \sigma(c')$, so that we had $G \circ F \cap C \subset \mathrm{cl}_\tau A$, where $A := G \circ U_V \circ F$. Having chosen open $U \in \sigma_X(x')$ and $W \in \sigma_Z(z')$, let us set $B := U \times W$ and $O := B \cap C$. Obviously, $G \circ F \cap O \subset (\mathrm{cl}_\tau A) \cap B$. Working in standard entourage, for an $a \in (\mathrm{cl}_\tau A) \cap B$ let us find a point $a \in A$ such that $a' \approx_\tau a$. Obviously, $a' \approx_\sigma a$, since $\mu(\tau) \subset \mu(\sigma)$ by condition. As B is σ-open, we get $a' \in B$, i.e., $a' \in A \cap B$ and $a \in \mathrm{cl}_\tau(A \cap B)$. Finally, $G \circ F \cap O \subset \mathrm{cl}_\tau(A \cap B)$, which was to be ensured. ▷

5.5.8. *The following inclusions are valid:*

(1) $\mathrm{Ha}_\Lambda(H,d') \supset X \times \mathrm{Ha}_\Lambda(G,b') \cap \mathrm{Ha}_\Lambda(F,a') \times Z$;

(2) $R^2_\Lambda(H,d') \supset X \times R^1_\Lambda(G,b') \cap R^2_\Lambda(F,a') \times Z$;

(3) $\mathrm{Cl}_\Lambda(H,d') \supset X \times Q^1_\Lambda(G,b') \cap \mathrm{Cl}_\Lambda(F,a') \times Z$;

(4) $\mathrm{Cl}_\Lambda(H,d') \supset X \times \mathrm{Cl}(G,b') \cap Q^2_\Lambda(F,a') \times Z$;

(5) $\mathrm{Cl}^2(H,d') \supset X \times P^2(G,b') \cap S^2(F,a') \times Z$, *where the cone* $\mathrm{Cl}^2(H,d')$ *is determined (in standard entourage) by the relation*

$$\mathrm{Cl}^2(H,d') := {}^*\{(s',t',r') \in X \times Y \times Z\colon\ (\forall d \approx_{\bar\sigma} d',\, d \in H)$$
$$(\forall \alpha \in \mu(\mathbf{R}_+))\ (\exists s \approx_{\tau_X} s')\,(\forall t \approx_{\tau_Y} t')\ (\exists r \approx_{\tau_Z} z')\ \ d + \alpha(s,t,r) \in H\}.$$

◁ Only (1) and (5) are to be checked, the remaining statements proved by the same scheme.

(1) Let the element (s',t',r') be standard and a member of the right-hand side of the relation under study. Let us choose a $d \approx_{\bar\sigma} d'$ and $\alpha \in \Lambda$, where $d := (x,y,z) \in H$. Obviously, $a := (x,y) \in F$ and $a \approx_{\sigma_1} a'$, while $b := (y,z) \in G$, $b \approx_{\sigma_2} b'$. In this respect, for $\alpha \in \Lambda$ and $(s,t,r) \approx_{\bar\tau} (s',t',r')$ we get $a + \alpha(s,t) \in F$ and $b + \alpha(t,r) \in G$. Hence,

$$d + \alpha(s,t,r) = (a + \alpha(s,t),\ z + \alpha r) \in F \times Z,$$
$$d + \alpha(s,t,r) = (x + \alpha s,\ b + \alpha(t,r)) \in X \times G,$$

i.e., $(s',t',r') \in \mathrm{Ha}_\Lambda(H,d')$.

(5) Let us take a standard element (s',t',r') from the right-hand side of (4). By definition, there is an element $s \approx_{\tau_X} s'$ such that for any $t \approx_{\tau_Y} t'$ for a certain $r \approx_{\tau_Z} r'$ and all $a \approx_{\sigma_1} a'$ and $b \approx_{\sigma_2} b'$, we have $a + \alpha(s,t) \in F$ and $b + \alpha(t,r) \in G$. Obviously, we get $d + \alpha(s,t,r) \in H$ as soon as $b \approx_{\bar\sigma} d'$ and $d \in H$. ▷

5.5.9. It should be emphasized that the mechanism of 'leapfrogging' demonstrated in **5.5.8**, can be modified in accord with the purposes of investigation. Such purposes include, as a rule, the estimates of the approximation to the composition of sets. In this case it would be most convenient to use the scheme based on the use of the method of general position [115, 121], as well as the results discussed above, both detalizing and generalizing this scheme. Let us formulate one of the possible results.

5.5.10.Theorem. *Let τ be a vector topology, $\tau \geq \sigma$, and correspondences $F \subset X \times Y$ and $G \subset Y \times Z$ be such that* $\mathrm{Ha}(F,a') \neq \varnothing$ *and the cones $Q_2(F,a') \times Z$ and $X \times \mathrm{Cl}(G,b')$ are in general position (relative to the topology $\bar\tau$), then*

$$\mathrm{Cl}(G \circ F, c') \supset \mathrm{Cl}(G,b') \circ \mathrm{Cl}(F,a'),$$

provided condition $(\overline{pc})$ *is fulfilled at the point* d'.

◁ The proof is carried out by the scheme of **5.3.13** in [121], and consists in verifying if the (primarily-set) conditions ensuring validity for the following statements:

$$
\begin{aligned}
&\mathrm{Cl}(G \circ F, c') = \mathrm{Cl}(\mathrm{Pr}_{X \times Z} H, \mathrm{Pr}_{X \times Z} d') \supset \mathrm{cl}_{\tau}\, \mathrm{Pr}_{X \times Z} \mathrm{Cl}(H, d') \\
&\supset \mathrm{Pr}_{X \times Z}\, \mathrm{cl}_{\bar{\tau}}(X \times \mathrm{Cl}(G, b') \cap Q^2(F, a') \times Z) \\
&= \mathrm{Pr}_{X \times Z}(\mathrm{cl}_{\bar{\tau}}(X \times \mathrm{Cl}(G, b')) \cap \mathrm{cl}_{\bar{\tau}}(Q^2(F, a') \times Z)) \\
&= \mathrm{Pr}_{X \times Z}(X \times \mathrm{Cl}(G, b') \cap \mathrm{Cl}(F, a') \times Z) = \mathrm{Cl}(G, b') \circ \mathrm{Cl}(F, a'). \ ▷
\end{aligned}
$$

5.6. Infinitesimal Subdifferentials

In the theory of extreme problems some attention is paid to the problem of taking into account the accuracy of observing optimality criteria in practical calculations. The adopted qualitative approach to the problem in question has been implemented in the so-called convex ε-programming which ensures an apparatus for estimating approximation to an optimum relative to a functional. The apparatus developed within this approach is quite specific and proves, in a certain sense, to be artificially complicated. At the same time, it is not quite well-correlated with the conventional techniques based on the search for the 'practical optimum' with the help of 'practically exact' satisfaction of complementary slackness, which correspond to the classical case $\varepsilon = 0$. As a result, we have to state a certain discrepancy, and even a gap between theoretical and practical viewpoints.

In this section we shall discuss an approach to overcome the present difficulties within the radical stance on nonstandard analysis. The approach is based on the introduction of the notion of an infinitesimally optimal solution, i.e., a feasible point, the value of the goal function in which is infinitely close to the ideal, i.e., to the value of the program not obligatory assumed. Therefore, infinitesimal optimum becomes a suitable challenger for the role of 'practical' optimum, since no feasible procedures are able to distinguish it from the conventional 'theoretical' optimum. The basic formulas of the calculus of infinitesimal subdifferentials corresponding to the concept of optimum presented above are given. The rules for external sets obtained coincide in form with their classical analogues of conventional convex analysis. In this case in the criteria of infinitesimal optimality there indeed arises approximately met complimentary slackness.

5.6.1. Let X be a vector space, $E^{\cdot}$ an ordered vector space with the greatest element $+\infty$ adjoined. Let us consider a convex operator $f: X \to E^{\cdot}$, and a point $\bar{x}$ of the effective domain $\mathrm{dom}(f) := \{x \in X: f(x) < +\infty\}$ of the operator f. For an element $\varepsilon \geq 0$ (of the cone of positive elements E^{+} of the space E) let us use the conventional definition of the ε-*subdifferential of f* at the point $\bar{x}$, i.e., the set

$$\partial_{\varepsilon} f(\bar{x}) := \{T \in L(X,E): (\forall x \in X)\ Tx - T\bar{x} \leq f(x) - f(\bar{x}) + \varepsilon\},$$

where $L(X,E)$ is the space of linear operators acting from X to E.

5.6.2. Let a family $\mathcal{E}$ of positive elements filtered upwards be chosen in E. Viewing E and $\mathcal{E}$ as standard sets, let us determine the monad $\mu(\mathcal{E})$ by the relation $\mu(\mathcal{E}) := \bigcap\{[0,\varepsilon]: \varepsilon \in {}^{\circ}\mathcal{E}\}$. The elements of $\mu(\mathcal{E})$ are termed (positive) *infinitely small* or *infinitesimal* (*relative to* $\mathcal{E}$). Henceforth it will be implied without further specifications that E is a K-space, the monad $\mu(\mathcal{E})$ is an external cone over ${}^{\circ}\mathbf{R}$ and, besides, $\mu(\mathcal{E}) \cap {}^{\circ}E = 0$. (In application, as a rule, $\mathcal{E}$ is the unit filter of E). Use also will be made of the relation of infinite proximity between the elements of E, i.e.,

$$e_1 \approx e_2 \leftrightarrow e_1 - e_2 \in \mu(\mathcal{E}) \wedge e_2 - e_1 \in \mu(\mathcal{E}).$$

5.6.3. *The following equality is valid*:

$$\bigcap_{\varepsilon \in {}^{\circ}\mathcal{E}} \partial_{\varepsilon} f(\bar{x}) = \bigcap_{\varepsilon \in \mu(\mathcal{E})} \partial_{\varepsilon} f(\bar{x}).$$

◁ For $T \in L(X, E)$ we deduce:

$$T \in \bigcap_{\varepsilon \in {}^{\circ}\mathcal{E}} \partial_{\varepsilon} f(\bar{x}) \leftrightarrow (\forall^{\mathrm{st}} \varepsilon \in \mathcal{E})\ (\forall x \in X)\ Tx - T\bar{x} \leq f(x) - f(\bar{x}) + \varepsilon$$
$$\leftrightarrow (\forall^{\mathrm{st}} \varepsilon \in \mathcal{E})\ f^{*}(T) := \sup_{x \in \mathrm{dom}(f)} (Tx - f(x)) \leq T\bar{x} - f(\bar{x}) + \varepsilon$$
$$\leftrightarrow (\forall^{\mathrm{st}} \varepsilon \in \mathcal{E})\ 0 \leq f^{*}(T) - (T\bar{x} - f(\bar{x})) \leq -\varepsilon \leftrightarrow f^{*}(T) - (T\bar{x} - f(\bar{x})) \approx 0$$
$$\leftrightarrow (\exists \varepsilon \in E^{+})\ \varepsilon \approx 0 \wedge f^{*}(T) = T\bar{x} - f(\bar{x}) + \varepsilon$$
$$\leftrightarrow T \in \bigcup_{\varepsilon \in \mu(\mathcal{E})} \partial_{\varepsilon} f(\bar{x}).\ ▷$$

5.6.4. The external set occurring in both parts of equality **5.6.3** is called the *infinitesimal subdifferential* of the function f at the point $\bar{x}$ and is denoted by $Df(\bar{x})$. The elements of

$Df(\bar{x})$ are called *infinitesimal subgradients* of f at the point $\bar{x}$. The possibility of ambiguity being in this case insignificant, no additional specifications of the set $\mathcal{E}$ are made.

5.6.5. Let the assumption of standard entourage be fulfilled, i.e., the parameters $X, f, \bar{x}$ are assumed to be standard sets. *The standardization of the infinitesimal subdifferential of f at the point $\bar{x}$ coincides with the (zero) subdifferential of f at $\bar{x}$, i.e.,*

$$^*Df(\bar{x}) = \partial f(\bar{x}).$$

◁ By virtue of the transfer principle, for a standard $T \in {}^\circ L(X,E)$ we have:

$$\begin{aligned}
&T \in {}^*Df(\bar{x}) \leftrightarrow T \in Df(\bar{x}) \\
&\leftrightarrow (\forall^{\mathrm{st}} \varepsilon \in \mathcal{E})\,(\forall x \in X)\; Tx - T\bar{x} \le f(x) - f(\bar{x}) + \varepsilon \\
&\leftrightarrow (\forall \varepsilon \in \mathcal{E})\,(\forall x \in X)\; Tx - T\bar{x} \le f(x) - f(\bar{x}) + \varepsilon \\
&\leftrightarrow T \in \partial f(\bar{x});
\end{aligned}$$

for $\inf \mathcal{E} = 0$ by virtue of the relation $\mu(\mathcal{E}) \cap {}^\circ E = 0$. ▷

5.6.6. *Let F be a standard K-space, and $g: E \to F$ be an increasing convex operator. If the sets $X \times \operatorname{epi}(g)$ and $\operatorname{epi}(f) \times F$ are in general position, then*

$$D(g \circ f)(\bar{x}) = \bigcup_{T \in Dg(f(\bar{x}))} D(T \circ f)(\bar{x}).$$

If, moreover, the parameters (except, possibly, the point $\bar{x}$) are standard, then the following presentation is valid for standard cores:

$$^\circ D(g \circ f)(\bar{x}) = \bigcup_{T \in {}^\circ Dg(f(\bar{x}))} {}^\circ D(T \circ f)(\bar{x}).$$

◁ It should be remarked that by condition the monad $\mu(\mathcal{E})$ is a normal external subsemigroup in F, i.e.,

$$\begin{aligned}
&\varepsilon \in \mu(\mathcal{E}) \to [0,\varepsilon] \subset \mu(\mathcal{E}), \\
&\mu(\mathcal{E}) + \mu(\mathcal{E}) \subset \mu(\mathcal{E}).
\end{aligned}$$

Taking into account this peculiarity and **5.6.3**, as well as the rules of calculating ε-subdifferentials, we get:

$$D(g \circ f)(\bar{x}) = \bigcup_{\varepsilon \in \mu(\mathcal{E})} \partial_\varepsilon (g \circ f)(\bar{x})$$

$$= \bigcup_{\varepsilon \in \mu(\mathbb{E})} \bigcup_{\substack{\varepsilon_1+\varepsilon_2=\varepsilon \\ \varepsilon_1 \geq 0, \varepsilon_2 \geq 0}} \bigcup_{T \in \partial_{\varepsilon_1} g(f(\bar{x}))} \partial_{\varepsilon_2}(T \circ f)(\bar{x})$$

$$= \bigcup_{\substack{\varepsilon_1 \geq 0, \varepsilon_2 \geq 0 \\ \varepsilon_1 \approx 0, \varepsilon_2 \approx 0}} \bigcup_{T \in \partial_{\varepsilon_1} g(f(\bar{x}))} \partial_{\varepsilon_2}(T \circ f)(\bar{x})$$

$$= \bigcup_{\varepsilon_1 \geq 0, \varepsilon_1 \approx 0} \bigcup_{T \in \partial_{\varepsilon_1} g(f(\bar{x}))} \bigcup_{\varepsilon_2 \geq 0, \varepsilon_2 \approx 0} \partial_{\varepsilon_2}(T \circ f)(\bar{x})$$

$$= \bigcup_{\varepsilon_1 \geq 0, \varepsilon_1 \approx 0} \bigcup_{T \in \partial_{\varepsilon_1} g(f(\bar{x}))} D(T \circ f)(\bar{x}).$$

Let now the assumption of standard entourage be fulfilled, and let $S \in {}^{\circ}D(g \circ f)(\bar{x})$. Then for a certain infinitely small ε we have

$$(g \circ f)^*(S) = \sup_{x \in \mathrm{dom}(g \circ f)} (Sx - g \circ f(x)) \leq S\bar{x} - g(f(\bar{x})) + \varepsilon.$$

By the change-of-variable formula for the Young-Fenchel transform, and making use of the transfer principle, we infer that there is a standard operator $T \in {}^{\circ}L(E,F)$ such that T is positive; i.e., $T \in L^+(E,F)$ and, moreover,

$$(g \circ f)^*(S) = (T \circ f)^*(S) + g^*(T).$$

This givess

$$\varepsilon \geq \sup_{x \in \mathrm{dom}(f)} (Sx - T \circ f(x)) + \sup_{e \in \mathrm{dom}(g)} (Te - g(e)) - S\bar{x} + g(f(\bar{x}))$$
$$= \sup_{x \in \mathrm{dom}(f)} (Sx - S\bar{x} - (T \circ f(x) - T \circ f(\bar{x})))$$
$$+ \sup_{e \in \mathrm{dom}(g)} (Te - T \circ f(\bar{x}) - (g(e) - g(f(\bar{x})))).$$

Let us set:

$$\varepsilon_1 := \sup_{e \in \mathrm{dom}(g)} (Te - T \circ f(\bar{x}) - (g(e) - g(f(\bar{x})))),$$
$$\varepsilon_2 := \sup_{x \in \mathrm{dom}(f)} (Sx - S\bar{x} - (T \circ f(x) - T \circ f(\bar{x}))).$$

Obviously, $S \in \partial_{\varepsilon_2}(T \circ f)(\bar{x})$, i.e., $S \in {}^{\circ}D(T \circ f)(\bar{x})$, and $T \in \partial_{\varepsilon_1} g(f(\bar{x}))$, i.e., $T \in {}^{\circ}Dg(f(\bar{x}))$, since $\varepsilon_1 \approx 0$ and $\varepsilon_2 \approx 0$. ▷

5.6.7. *Let $f_1, \ldots, f_n : X \to E$ be convex operators, and n be a standard natural number. If $f_1, \ldots, f_n$ are in general position, then for a point $\bar{x} \in \mathrm{dom}(f_1) \cap \ldots \cap \mathrm{dom}(f_n)$ we have*

$$D(f_1 + \ldots + f_n)(\bar{x}) = Df_1(\bar{x}) + \ldots Df_n(\bar{x}).$$

◁ The proof consists in applying **5.6.3** and the rule of ε-subdifferention for sum, with use made of the fact that the sum of a standard number of infinitely small addends is again infinitely small. ▷.

5.6.8. *Let $f_1,\dots,f_n\colon X \to E$ be convex operators, with n a standard number. Assume that $f_1,\dots,f_n$ are in a general position, E is a vector lattice, and $\bar{x} \in \operatorname{dom}(f_1 \vee \dots \vee f_n)$. If F is a standard K-space and $T \in L^+(E,F)$ is a positive linear operator, then an element $S \in L(X,F)$ is an infinitesimal subgradient of the operator $T \circ (f_1 \vee \dots \vee f_n)$ at a point $\bar{x}$ iff the following system of conditions is compatible*:

$$T = \sum_{k=1}^{n} T_k ; T_k \in L^+(E,F) \quad (k := 1,\dots,n);$$
$$\sum_{k=1}^{n} T_k \bar{x} \approx T(f_1(\bar{x}) \vee \dots \vee f_n(\bar{x})); \quad S \in \sum_{k=1}^{n} D(T_k \circ f_k)(\bar{x}).$$

◁ Let us determine the following operators:

$$(f_1,\dots,f_n)\colon X \to (E^n), \quad (f_1,\dots,f_n)(x) := (f_1(x),\dots,f_n(x)) ;$$
$$\aleph\colon E^n \to E, \ \aleph(e_1,\dots,e_n) := e_1 \vee \dots \vee e_n.$$

In this case the following presentation is valid:

$$T \circ f_1 \vee \dots \vee f_n = T \circ \aleph \circ (f_1,\dots,f_n).$$

From now on, allowing for **5.6.5** and recalling that $T \circ \aleph$ is a sublinear operator, we deduce the required result. ▷

5.6.9. Let X be a vector space, E be a certain K-space and $\mathfrak{A}$ be a pointwise order-bounded set in $L(X,E)$. Let us consider a convex operator $f := \varepsilon_{\mathfrak{A}} \circ \langle \mathfrak{A} \rangle^e$, where, as usual, $\varepsilon_{\mathfrak{A}}$ is the *canonical sublinear operator*

$$\varepsilon_{\mathfrak{A}} : l_\infty(\mathfrak{A}, E) \to E, \quad \varepsilon_{\mathfrak{A}}(f) := \sup f(\mathfrak{A}),$$

and the affine operator $\langle \mathfrak{A} \rangle^e$ for $e : l_\infty(\mathfrak{A}, E)$ acts by the rule $\langle \mathfrak{A} \rangle^e x := \langle \mathfrak{A} \rangle x + e$, $\langle \mathfrak{A} \rangle x\colon T \in \mathfrak{A} \to Tx$.

5.6.10. *If $g\colon E \to F$ is an increasing convex operator acting into a standard K-space F, and the image $f(X)$ contains an algebraic interior point* $\operatorname{dom}(g)$, *while an element $\bar{x}$ of X is such that $f(\bar{x}) \in \operatorname{dom}(g)$, then the following presentation is valid:*

$$\begin{aligned}&D(g \circ f)(\bar{x})\\ &= \{T \circ \langle \mathfrak{U} \rangle \colon T \circ \Delta_{\mathfrak{U}} \in Dg(f(\bar{x})),\quad T \geq 0,\ T \circ \Delta_{\mathfrak{U}} f(\bar{x}) \approx T \circ \langle \mathfrak{U} \rangle^{e^-}\ x\}.\end{aligned}$$

$\lhd$ If $S \in D(g \circ f)(\bar{x})$, then, by **5.6.3**, $S \in \partial_g (g \circ f)(\bar{x})$ for a certain $\varepsilon \approx 0$, and all we have to do is to apply the corresponding rule of ε-subdifferentiation. Whereas if $T \geq 0, T \circ \Delta_{\mathfrak{U}} \in Dg(f(\bar{x}))$ and $T \circ \Delta_{\mathfrak{U}} f(\bar{x}) \approx T \circ \langle \mathfrak{U} \rangle^e \bar{x}$, then for a certain $\varepsilon \approx 0$ we obviously get $T \circ \Delta_{\mathfrak{U}} \in \partial_\varepsilon g(f(\bar{x}))$. Let us, in addition, assign $\delta := T \circ \Delta_{\mathfrak{U}} f(\bar{x}) - T \circ \langle \mathfrak{U} \rangle^e \bar{x}$. Then, by hypothesis, $\delta \geq 0$ and $\delta \approx 0$. Therefore, $T \circ \langle \mathfrak{U} \rangle \in \partial_{\varepsilon + \delta} (g \circ f)(\bar{x})$. Now we have to remark that $\varepsilon + \delta \approx 0$. $\rhd$

5.6.11. *Under condition 5.6.5 let the mapping g be a sublinear Maharam operator. Then*

$$D(g \circ f)(\bar{x}) = \bigcup_{T \in Dg(f(\bar{x}))} \ \bigcup_{\delta \geq 0, T\delta \approx 0} T(\partial_\delta f(\bar{x})).$$

$\lhd$ By virtue of **5.6.5**, we can assume $g := T$. If for any $x \in X$ we have $Cx - C\bar{x} \leq f(x) - f(\bar{x}) + \delta$ and $T\delta \approx 0$, then, obviously, $TC \in \partial_{T\delta}(T \circ f)(\bar{x}) \subset \subset D(T \circ f)(\bar{x})$. To complete the proof, let us take $S \in D(T \circ f)(\bar{x})$. By virtue of **5.6.3**, there is an infinitely small ε, such that $S \in \partial_\varepsilon (T \circ f)(\bar{x})$. Applying the corresponding rule of ε-subdifferentiation, we find $\delta \geq 0$ and $C \in \partial_\delta f(\bar{x})$ such that $T\delta \leq \varepsilon$ and $S = TC$, which was required. $\rhd$

5.6.12. *Let Ξ be a set and $(f_\xi)_{\xi \in \Xi}$ be a uniformly regular family of convex operators. The following presentations are valid:*

$$D\Big(\sum_{\xi \in \Xi} f_\xi\Big)(\bar{x}) = \bigcup_{\substack{\delta \in l_\infty(\Xi E)\\ \delta \geq 0, \delta \approx 0}} \sum_{\xi \in \Xi} \partial_{\delta(\xi)} f_\xi (\bar{x});$$

$$\begin{aligned}&D(\sup_{\xi \in \Xi} f_\xi)(\bar{x})\\ &= \bigcup \Big\{ \sum_{\xi \in \Xi} \alpha_\xi \partial_{\delta(\xi)}(\bar{x}) \colon 0 \leq \alpha_\xi \leq 1_E,\ \sum_{\xi \in \Xi} \alpha_\xi = 1_E,\\ &\sum_{\xi \in \Xi} \alpha_\xi f_\xi(\bar{x}) \approx \sup_{\xi \in \Xi} f_\xi(\bar{x}),\quad \sum_{\xi \in \Xi} \alpha_\xi \delta(\xi) \approx 0 \Big\}.\end{aligned}$$

$\lhd$ The proof results immediately from **5.6.11** with the rule of disintegration taken into account (see [115]). $\rhd$

5.6.13. It is expedient to remark that the formulas **5.6.7** - **5.6.12** allow one to introduce refinements analogous to the case of standard entourage in **5.6.6** (which, possibly, does not include the point $\bar{x}$). It should be also emphasized that the given patterns enable one to deduce the whole spectrum of all possible formulas of subdifferential calculus (convolutions, Lebesque sets, etc.).

5.6.14. Let, as above, $f: X \to E^{\cdot}$ be a convex operator acting into a standard K-space E, and let $\mathfrak{X} := (\cdot)$ be *a generalized point in* $\operatorname{dom}(f)$, i.e., a net of elements of the $\operatorname{dom}(f)$. The operator $T \in L(X, E)$ is said to be *an infinitesimal subgradient of* f *at the generalized point* $\mathfrak{X}$, *if for a certain infinitely small* ε *we have*

$$f^*(T) \le \liminf (T\mathfrak{X} - f(\mathfrak{X})) + \varepsilon$$

(here, of course, the rule $T\mathfrak{X} := T \circ \mathfrak{X}$ is in action). Therefore, in standard entourage the infinitesimal subgradient is a common support operator at a generalized point (see [1, 115]). Let us denote by the symbol $Df(\mathfrak{X})$ the union of all infinitesimal subgradients f at $\mathfrak{X}$. This set is termed, by well-understandable reasons, the *infinitesimal subdifferential of* f at $\mathfrak{X}$. Let us deduce the two basic rules of subdifferetiation at a generalized point, which are of interest since no exact formulas for ε-subdifferentials are known.

5.6.15. *Let* $f_1, \dots, f_n$ *be a standard set of convex operators in general position, and let a generalized point* $\mathfrak{X}$ *lie in* $\operatorname{dom}(f_1) \cap \dots \cap \operatorname{dom}(f_n)$. *Then*

$$D(f_1 + \dots + f_n)(\mathfrak{X}) = Df_1(\mathfrak{X}) + \dots + Df_n(\mathfrak{X}).$$

◁ Let $T_k \in Df_k(\mathfrak{X})$ for $k := 1, \dots, n$, i.e.,

$$f_k^*(T_k) \le \liminf (T_k\mathfrak{X} - f_k(\mathfrak{X})) + \varepsilon_k$$

for a suitable infinitely small $\varepsilon_1, \dots, \varepsilon_n$. In this case

$$(f_1 + \dots + f_n)^*(T_1 + \dots + T_n) \le \sum_{k=1}^{n} f_k^*(T_k)$$
$$\le \sum_{k=1}^{n} (\liminf (T_k\mathfrak{X} - f_k(\mathfrak{X})) + \varepsilon_k)$$
$$\le \liminf \sum_{k=1}^{n} (T_k\mathfrak{X} - f_k(\mathfrak{X})) + \sum_{k=1}^{n} \varepsilon_k$$

by virtue of conventional properties of the Young-Fenchel transform and the lower limit. Now we have to remark that $\varepsilon_1+\ldots+\varepsilon_n \approx 0$ and conclude that the inclusion $\supset$ is valid for the sets considered in the equality we are interested in.

In order to check the reverse inclusion, let us, having reduced the case to $n = 2$, choose a $T \in D(f_1 + f_2)(\mathcal{X})$. Then, for some $\varepsilon \approx 0$ and T_1, T_2, such that $T_1 + T_2 = T$, we get

$$(f_1 + f_2)^*(T) = f_1^*(T_1) + f_2^*(T_2),$$
$$f_1^*(T_1) + f_2^*(T_2) - \liminf(T\mathcal{X} - (f_1 + f_2)(\mathcal{X})) \le \varepsilon.$$

Let us, by definition, assign

$$\delta_1 := f_1^*(T_1) - \liminf(T_1\mathcal{X} - f_1(\mathcal{X})),$$
$$\delta_2 := f_2^*(T_2) - \liminf(T_2\mathcal{X} - f_2(\mathcal{X})).$$

Obviously, for $k := 1,2$ we have

$$0 \le \sup_{x \in \operatorname{dom}(f_k)} (T_k x - f_k(x)) - \limsup(T_k\mathcal{X} - f_k(\mathcal{X})) \le \delta_k.$$

Therefore, we now have to make sure that δ_1 and δ_2 are infinitesimals. We consequitively derive

$$\begin{aligned}
&\delta_1 + \delta_2 \\
&\le \varepsilon + \liminf(T\mathcal{X} - (f_1 + f_2)(\mathcal{X})) - \sum_{k=1}^{2} \liminf(T_k\mathcal{X} - f_k(\mathcal{X})) \\
&\le (\varepsilon + \limsup(T_1\mathcal{X} - f_1(\mathcal{X})) - \liminf(T_1\mathcal{X} - f_1(\mathcal{X}))) \\
&\wedge (\varepsilon + \limsup(T_2\mathcal{X} - f_2(\mathcal{X})) - \liminf(T_2\mathcal{X} - f_2(\mathcal{X})) \\
&\le (\varepsilon + f_1^*(T_1) - \liminf(T_1\mathcal{X} - f_1(\mathcal{X}))) \\
&\wedge (\varepsilon + f_2^*(T_2) - \liminf(T_2\mathcal{X} - f_2(\mathcal{X}))) \\
&\le \varepsilon + \delta_1 \wedge \delta_2.
\end{aligned}$$

Hence, $0 \le \delta_1 \vee \delta_2 \le \varepsilon$, which completes the proof. ▷

5.6.16. *Let F be a standard K-space, and let $g: E \to F^{\cdot}$ be an increasing convex operator. If the sets $X \times \operatorname{epi}(g)$ and $\operatorname{epi}(f) \times F$ are in general position, then, for a generalized point $\mathcal{X}$ in $\operatorname{dom}(g \circ f)$, we have*

$$D(g \circ f)(\mathcal{X}) = \bigcup_{T \in Dg(f(\mathcal{X}))} D(T \circ f)(\mathcal{X}).$$

◃ If it is known that

$$(T \circ f)^*(S) \le \liminf (S\mathscr{X} - T \circ f(\mathscr{X})) + \varepsilon_1,$$
$$g^*(T) \le \liminf (T \circ f(\mathscr{X}) - g \circ f\mathscr{X})) + \varepsilon_2$$

for some infinitely small ε_1 and ε_2, then

$$(g \circ f)^*(S) \le (T \circ f)^*(S) + g^*(T)$$
$$\le \liminf (S\mathscr{X} - T \circ f(\mathscr{X})) + \varepsilon_1 + \liminf (T \circ f(\mathscr{X}) - g \circ f(\mathscr{X})) + \varepsilon_2$$
$$\le \liminf (S\mathscr{X} - g \circ f(\mathscr{X})) + \varepsilon_1 + \varepsilon_2 .$$

Therefore, $S \in D(g \circ f)(\mathscr{X})$ and the right-hand side of the formula under study symbolize the set incorporated in its left-hand side.

To complete the proof, let us take $S \in D(g \circ f)(\mathscr{X})$. Then there is an infinitely small ε and an operator T, such that

$$(g \circ f)^*(S) = (T \circ f)^*(S) + g^*(T) \le \liminf (S\mathscr{X} - g \circ f(\mathscr{X})) + \varepsilon .$$

Let us set

$$\delta_1 := (T \circ f)^*(S) - \liminf (S\mathscr{X} - T \circ f(\mathscr{X})),$$
$$\delta_2 := g^*(T) - \liminf (T \circ f(\mathscr{X}) - g \circ f(\mathscr{X})).$$

By the properties of the upper and lower limits, we deduce, first,

$$\delta_1 \ge (T \circ f)^*(S) - \limsup (S\mathscr{X} - T \circ f(\mathscr{X})) \ge 0,$$
$$\delta_2 \ge g^*(T) - \limsup (T \circ f(\mathscr{X}) - g \circ f(\mathscr{X})) \ge 0,$$

and, second,

$$\delta_1 + \delta_2 \le$$
$$\le \liminf (S\mathscr{X} - g \circ f(\mathscr{X})) + \varepsilon - \liminf (S\mathscr{X} - T \circ f(\mathscr{X}))$$
$$- \liminf (T \circ f(\mathscr{X}) - g \circ f(\mathscr{X})(\le (\limsup (S\mathscr{X} - T \circ f(\mathscr{X}))$$
$$- \liminf (S\mathscr{X} - T \circ f(\mathscr{X})) + \varepsilon) \wedge (\limsup (T \circ f(\mathscr{X}) - g \circ f(\mathscr{X}))$$
$$- \liminf (T \circ f(\mathscr{X}) - gf(\mathscr{X})) + \varepsilon) \le \delta_1 \wedge \delta_2 + \varepsilon ,$$

since the following obvious equalities are valid:

$$\limsup (T \circ f(\mathscr{X}) - g \circ f(\mathscr{X})) \le g^*(T),$$
$$\limsup (S\mathscr{X} - T \circ f(\mathscr{X})) \le (T \circ f)^*(S).$$

Therefore, $0 \le \delta_1 \vee \delta_2 \le \varepsilon$ and $\delta_1 \approx 0, \delta_2 \approx 0$, which implies $T \in Dg(f(\mathfrak{X}))$ and $S \in D(T \circ f)(\mathfrak{X})$. ▷

5.6.17. The point $\bar{x} \in \mathrm{dom}(f)$ is called an *infinitesimal solution* of the unconstrained program $f(x) \to \inf$, where $f\colon X \to E^{\cdot}$, provided $0 \in Df(\bar{x})$, i.e., if $\bar{x}$ is feasible and $f(\bar{x}) \approx \inf\{f(x)\colon x \in X\}$. The infinitesimal solution of an arbitrary program is understood in the natural way.

5.6.18. *In a standard unconstrained program* $f(x) \to \inf$ *there is an infinitesimal solution iff, first, the image* $f(X)$ *is bounded from below, and second, there is a standard generalized solution* $(x_\varepsilon)_{\varepsilon \in \mathcal{E}}$ *of the program under consideration, i.e.,* $x_\varepsilon \in \mathrm{dom}(f)$ *and* $e \le f(x_\varepsilon) \le e + \varepsilon$ *for all* $\varepsilon \in \mathcal{E}$, *where* $e := \inf f(X)$ *is the value of the program.*

◁ By virtue of the idealization and transfer principles, as well as of **5.6.3**, we deduce:

$$
\begin{aligned}
&(\exists \bar{x} \in X)\ 0 \in Df(\bar{x}) \leftrightarrow (\exists x \in X)(\forall^{\mathrm{st}} \varepsilon \in \mathcal{E})\ 0 \in \partial_\varepsilon f(x)\\
&\leftrightarrow (\forall^{\mathrm{st\,fin}} \mathcal{E}_0 \subset \mathcal{E})(\exists x \in X)(\forall \varepsilon \in \mathcal{E}_0)\ 0 \in \partial_\varepsilon f(x)\\
&\leftrightarrow (\forall^{\mathrm{st}} \varepsilon \subset \mathcal{E})(\exists x_\varepsilon \in X)\ 0 \in \partial_\varepsilon f(x_\varepsilon)\\
&\leftrightarrow (\forall \varepsilon \subset \mathcal{E})(\exists x_\varepsilon \in X)(\forall x \in X)\ f(x) \ge f(x_\varepsilon) - \varepsilon.\ \triangleright
\end{aligned}
$$

5.6.19. Let us consider a *regular convex program*

$$g(x) \le 0, \quad f(x) \to \inf.$$

Therefore, $g, f\colon X \to E^{\cdot}$ (for simplicity, $\mathrm{dom}(f) = \mathrm{dom}(g) = X$), for every $x \in X$ we have either $g(x) \le 0$ or $g(x) \ge 0$ and, moreover, for a certain $x_0 \in X$ the element $-g(x_0)$ is a unit in E.

5.6.20. *In standard entourage a feasible interior point* $\bar{x}$ *is an infinitesimal solution of the regular program under consideration iff the following system of conditions is compatible*:

$$
\begin{aligned}
&\alpha, \beta \in {}^{\circ}[0, 1_E],\ \alpha + \beta = 1_E,\ \ker(\alpha) = 0;\\
&\beta \circ g(\bar{x}) \approx 0,\ 0 \in D(\alpha \circ f)(\bar{x}) + D(\beta \circ g)(\bar{x}).
\end{aligned}
$$

◁ ← When the system of conditions is compatible, for a feasible x for some infinitely small ε_1 and ε_2 we have

$$\alpha f(\bar{x}) \le \alpha f(x) + \beta g(x) - \beta g(\bar{x}) + \varepsilon_1 + \varepsilon_2 \le \alpha f(x) + \varepsilon$$

for every standard $\varepsilon \in {}^\circ\mathcal{E}$. In particular, $\alpha(f(\bar{x}) - f(x)) \le \alpha\varepsilon$ for $\varepsilon \in {}^\circ\mathcal{E}$, since α is a standard mapping. By condition, $\ker(\alpha) = 0$ and the general properties of multipliers we see that $\bar{x}$ is an infinitesimal solution.

→ Let $e := \inf\{f(x) : x \in X, g(x) \le 0\}$ be the value of the program under discussion. By hypothesis and by virtue of the transfer principle, e is a standard element. Therefore, again applying the transfer principle, by the vector minimax theorem we can find standard multipliers $\alpha, \beta \in {}^\circ[0, 1_E]$ such that

$$\alpha + \beta = 1_E, \quad 0 = \inf_{x \in X} (\alpha(f(x) - e) + \beta \circ g(x)).$$

Using conventional considerations (see [1]), we can check if $\ker(\alpha) = 0$. Moreover, since $\bar{x}$ is an infinitesimal optimal solution, for a certain infinitely small ε we get $f(\bar{x}) - e = \varepsilon$. Hence, for any $x \in X$ the estimates $-\alpha\varepsilon \le \alpha f(x) - \alpha f(\bar{x}) + \beta g(x)$ are valid. In particular, $0 \ge \beta g(\bar{x}) \ge -\alpha\varepsilon \ge -\varepsilon$, i.e., $\beta g(\bar{x}) \approx 0$ and

$$0 \in \partial_{\alpha\varepsilon + \beta g(\bar{x})}(\alpha \circ f + \beta \circ g)(\bar{x}) \subset D(\alpha \circ f + \beta \circ g)(\bar{x}),$$

as $\alpha\varepsilon + \beta g(\bar{x}) \approx 0$. ▷

5.6.21. Let us consider a *program regular in the Slater sense*

$$Ax = A\bar{x}, \quad g(x) \le 0, \quad f(x) \to \inf,$$

i.e., first, $A \in L(X, \mathfrak{X})$ is a linear operator with values in a certain vector space $\mathfrak{X}$, the mappings $g : X \to E^{\cdot}$ and $f : X \to E^{\cdot}$ are convex operators (for convenience, $\mathrm{dom}(f) = \mathrm{dom}(g) = X$); second, F is an Archimedean ordered vector space, E is a standard K-space of bounded elements; and, finally, for a certain feasible point x_0 the element $-g(x_0)$ is a strong unit in F.

5.6.22. Criterion for infinitesimal optimality. *A feasible point $\bar{x}$ is the infinitesimal solution to a Slater-regular program iff the following system of conditions is compatible:*

$$\gamma \in L^+(F, E), \quad \mu \in L(\mathfrak{X}, E), \quad \gamma g(\bar{x}) \approx 0,$$
$$0 \in Df(\bar{x}) + D(\gamma \circ g)(\bar{x}) + \mu \circ A.$$

◁ ← When the system of conditions is compatible, for a feasible x and some infinitely small ε_1 and ε_2 we have

$$f(\bar{x}) \le f(x) + \varepsilon_1 + \gamma g(x) - \gamma g(\bar{x}) + \varepsilon_2 - \mu(Ax) + \mu(A\bar{x})$$
$$\le f(x) + \varepsilon_1 + \varepsilon_2 - \gamma g(\bar{x}) \le f(x) + \varepsilon$$

for any standard $\varepsilon \in {}^{\circ}\mathfrak{E}$.

→ If $\bar{x}$ is an infinitesimal solution, it is also an ε-solution for a suitable infinitely small ε. Now we have to apply the corresponding criterion of ε-optimum. ▷

5.6.23. A feasible point $\bar{x}$ is called *infinitesimal Pareto-optimal* in the program **5.6.21**, provided $\bar{x}$ is Pareto-ε-optimal for some infinitely small ε (relative to the (strong)order unit 1_E in the space E), i.e., if for a feasible x we have $f(x) - f(\bar{x}) \le \varepsilon 1_E$, then $f(x) - f(\bar{x}) = \varepsilon 1_E$ for $\varepsilon \in \mu(\mathbf{R}_+)$.

5.6.24. *Let a point $\bar{x}$ be infinitesimal Pareto-optimal in a Slater-regular program. Then for certain linear functionals α,β,γ on the spaces E,F and $\mathfrak{X}$, respectively, the following system of conditions is compatible*:

$$\alpha > 0, \quad \beta \ge 0, \quad \beta g(\bar{x}) \approx 0,$$
$$0 \in D(\alpha \circ f)(\bar{x}) + D(\beta \circ g)(\bar{x}) + \gamma \circ A.$$

If, in turn, the above relations are valid for a certain feasible point $\bar{x}$, and $\alpha(1_E) = 1$, $\ker(\alpha) \cap E^+ = 0$, *then $\bar{x}$ is an infinitesimal Pareto-optimal solution to the program under consideration.*

◁ The first part of the statement being proved results from the well-known characteristic of Pareto-ε-optimum with the properties of the infinitely small discussed earlier taken into account. If the hypothesis of the second part of the statement under discussion is fulfilled, then, making use of the definitions, for any feasible $x \in X$ we deduce:

$$0 \le \alpha(f(x) - f(\bar{x})) + \beta g(x) - \beta g(\bar{x}) + \varepsilon_1 + \varepsilon_2$$
$$\le \alpha(f(x) - f(\bar{x})) + \varepsilon_1 + \varepsilon_2 - \beta g(\bar{x})$$

for an appropriate infinitely small $\varepsilon_1, \varepsilon_2$. Let us set $\varepsilon := \varepsilon_1 + \varepsilon_2 - \beta g(\bar{x})$. Obviously, $\varepsilon \approx 0$ and, moreover, $\varepsilon \ge 0$. If now for a feasible x valid is $f(x) - f(\bar{x}) \le -\varepsilon 1_E$, then we get the equality $\alpha(f(x) - f(\bar{x})) = \varepsilon$. In other words, $\alpha(f(x) - f(\bar{x}) - \varepsilon 1_E) = 0$ and

$f(x) - f(\bar{x}) = \varepsilon 1_E$. This implies that $\bar{x}$ is a Pareto-ε-optimal solution. ▷

5.6.25. Using the pattern described above, one can obtain the properties of infinitesimal solutions for other basic forms of the problems of convex programming. One can, for instance, deduce nonstandard analogues of the theorem on characteristics of naturally defined infinitesimal optimal paths of finite-stage terminal dynamic problems.

PART 2

BOOLEAN-VALUED ANALYSIS

Boolean-valued analysis owes its origination to an outstanding achievement by P.J.Cohen who in the beginning of the sixties established that the addition of the negation of the continuum-hypothesis, CH, to the other axioms of Zermelo-Fraenkel set theory, ZFC, is consistent. Combined with an earlier result by K.Gödel on the CH-ZFC consistency, the fact proved by P.J.Cohen implies that CH is independent of the other ZFC axioms.

The discovery made by P.J.Cohen is associated with his overcoming a principal difficulty marked by G.Sheperdson and absent in the case considered by K.Gödel. The proof of (ZFC) + ($\neg$CH) consistency is impossible by means of standard models. To be more exact, having chosen a realization of the von Neumann universe, we cannot find a subclass in it which serves a $(ZFC) + (\neg CH)$ model, if we make use of THE available interpretation of the predicate of containment. P.J.Cohen managed to propose a new powerful method of constructing non-internal, i.e., nonstandard, ZHC models which he termed the forcing method. The techniques used by P.J.Cohen, such as the use of the axiom of the existence of a standard transitive model for ZFC and a forced transformation of the latter into a principally nonstandard model with the constraint method contradict the conventional mathematical intuition stemming, by the words of P.J.Cohen himself, "from our belief into a natural nearly physical model of the mathematical world" [28, p.202].

The difficulties hampering the understanding of the results by P.J.Cohen were, long before their origination, well expressed by N.N.Luzin in his famous report 'The present state of the theory of functions in a real variable' made by him at the All-Russian Congress of Mathematicians in 1927: "The first idea that occurs is that the determination of the cardinality of a continuum is a matter of a free axiom, as for parallel lines in geometry. At the same time, both when all other Eucledean axioms in geometry are presented, and we vary the axiom on parallel lines, the very sense of the uttered or written terms undergoes changes: 'a point', 'a straight line', etc., the sense of which words should change when we make the cardinality of the continuum moving along the alephic scale, proving all the time the consistency of this motion? The cardinality of the continuum, if only viewed as a set of points, is a unique reality, and it should have a specific location on the alephic scale, irrespective of the fact that the definition of the location is difficult or, as J.Hadamard would add, "even impossible for us, people" [163, pp. 11-12].

A typical viewpoint has been offered by P.S.Novikov: "...it is plausible (I share this opinion myself) that the Cohen result has purely negative value and exposes the end of the evolution of the 'naive' set theory in the Cantor sense" [200, p. 209].

The desire to eliminate the afore-mentioned difficulties in the understanding of the results and methods by P.J.Cohen led D.Scott and P.Solovay to the construction of Boolean-valued models for ZFC both possessing an appealing visualization from the standpoint of classical mathematicians and, at the same time, suitable for obtaining theorems on independence. Analogous models were also constructed by P.Vopenka at the same period, i.e., at the beginning of the sixties.

From all the above-said, the Boolean-valued models which achieve the same aims that those constructed by P.J.Cohen via forcing should be, in a sense, nonstandard, i.e., they should possess features not characteristic of conventional models.

In a qualitative sense, *the notion of Boolean-valued model includes a new concept of modelling*, which could be termed modelling by communication, or modelling by telephone. Let us clarify the essence of this concept by way of comparing it with traditional approaches. In a classical sense, when comparing two models of the same theory, we try to establish a one-to-one correspondence between the universes involved. Once such a bijection has been found by transferring the predicates and operators of one model into their analogs in the other one, we say that the models are isomorphic. Therefore, the presentation of isomorphism described implies a visual comparison of the models, i.e., the presentation of a bijection of the universes.

Let us imagine that we are deprived of the possibility of comparing models simultaneously physically element-by-element, but can exchange information with the owner of the counter-model by, for instance, telephone. In the process of communication we could easily establish that our counter-partner uses his model for studying objects which he calls by familiar terms when speaking about sets, their comparison and membership. As long as we are interested in ZFC, we ask him if the axioms of ZFC are true. Having worked with his model, he answers us in the affirmative. Having also checked that he is using the same deduction rules as adopted by us, we must admit that the model at his disposal is a model of the theory we are interested in. It would be expedient to emphasize that having come to this conclusion we have known nothing either about the objects comprising his model, not about the procedures he uses to distinguish true statements from false ones.

Therefore, *the new concept of modelling is associated with both the refusal to identify universes of discourse and the acceptance of new procedures for verifying statements.* In particular, when considering a Boolean-valued model $\mathbf{V}^{(B)}$ for ZFC, to each formula φ we assign an element $[\varphi]\in B$ lying in a given Boolean algebra B. The quantity $[\varphi]$ is called the truth-value of the formula φ. In this case a theorem of ZFC obtains the truth-value $\mathbf{1}_B$. In Boolean-valued models the construction starts with a fixed complete Boolean algebra B which serves as a base for constructing the Boolean-valued universe $\mathbf{V}^{(B)}$ and the domain of arrival to the truth-value by putting a certain element of the algebra B in correspondence to a formula of ZFC.

A detailed presentation of the afore-mentioned constructions can be found in Chapters **1** - **3** of this part of the book. The presented constructions and, first of all, the procedures of ascent and descent implementing the functorial relations between the von Neumann universe, $\mathbf{V}$, and the Boolean-valued universe, $\mathbf{V}^{(B)}$, comprise a technical foundation of applying Boolean-valued models to the problems of analysis. In the concluding chapters we we shall demonstrate the most important possibilities provided by Boolean-valued analysis, such as methods of transforming functional spaces into numerical sets, operators into functionals, vector-functions into conventional mappings, etc.. And again, as was the case in the first part of the present book, the choice of the circle of applications to the problems of functional analysis has been basically prompted by out personal scientific interests.

CHAPTER 1

UNIVERSES OF SETS

The credo of naive set theory includes, as is known, the dream about the 'Cantor paradise', i.e., about the universe, the world of sets, containing all possible formations that can be thought of as distinct. Realistic approximations to the unattainable ideal, i.e., adequate formal schemes making it possible to present a large spectrum of concrete sets while remaining within comfortable conditions of sufficient logical accuracy, are the subject of the modern set theory. In the first part of the book we have already encountered the ideas underlying a number of axiomatic set theories, such as Zermelo-Fraenkel set theory, the theories of external and internal sets.

The essence of these theories is the construction of universes ensuring 'approximations from below', to the world of naive sets, satisfactory for some specific purposes. It is within the frameworks of the corresponding axiomatics that we can exactly substantiate and realize in detail the qualitative phenomenological principles on which the standard and nonstandard mathematical models are based. In the present chapter we consider the formal apparatus of constructing universes of sets by transfinite processes of creating the so-called cumulative hierarchies. We are going to concentrate on a detailed description of the construction of the von Neumann universe, which often occurred in the first part of the present monograph. In this respect a thorough analysis will be carried out as regards the classes of sets within the formal system stemming from J.von Neumann, K.Gödel and P.Bernays and serving as a conservative extension of Zermelo-Fraenkel set theory.

1.1. Boolean Algebras

In this section we shall schematically present only those facts about Boolean algebras which are needed in the sequel. A more detailed presentation can be found in a number of monographs [74, 233, 265].

1.1.1. With the view of fixing terminology, let us recall some well-known notions partly used in the preceding sections.

An ordered set is a pair $(M,\leq)$, where $\leq$ is an order relation on M (see **I.3.1.10**). An *upper bound* of a subset X in the ordered set M is an element $a \in M$ such that $x \leq a$ for all $x \in X$. The least element of the set of upper bounds of the subset X is called its *least upper bound* or its *supremum* and is denoted by $\sup X$. In other words, $a = \sup X$ iff a is an upper bound of X and $a \leq b$ for any upper bound b of the set X. By reversal, i.e., by passing from $\leq$ to $\leq^{-1}$, we determine a *lower bound* or a *greatest lower bound*, $\inf X$, termed the *infimum* of the set X. If either the least upper or the greatest lower bound of a given set exists, it is unique and thus deserves the definite article Either of the bounds, $\inf X$ and $\sup X$, is referred to as *exact.*

A lattice is an ordered set L in which any two-element set $\{x, y\}$ has the supremum $x \vee y := \sup\{x,y\}$ and the infimum $x \wedge y := \inf\{x,y\}$. For lattices the following notation is used:

$$\vee X := \sup X, \quad \wedge X := \inf X,$$
$$\bigvee_{\alpha \in A} x_\alpha := \vee\{x_\alpha : \alpha \in A\}, \quad \bigwedge_{\alpha \in A} x_\alpha := \wedge\{x_\alpha : \alpha \in A\},$$
$$\bigvee_{k=1}^{n} x_k := x_1 \vee \ldots \vee x_n := \sup\{x_1, \ldots x_n\},$$
$$\bigwedge_{k=1}^{n} x_k := x_1 \wedge \ldots \wedge x_n := \inf\{x_1, \ldots x_n\}.$$

The binary operations $(x,y) \to x \vee y$ and $(x,y) \to x \wedge y$ arising in any lattice L satisfy the following laws:

(1) *commutativity:*

$$x \vee y = y \vee x, \quad x \wedge y = y \wedge x;$$

(2) *associativity:*

$$x \vee (y \vee z) = (x \vee y) \vee z, \quad x \wedge (y \wedge z) = (x \wedge y) \wedge z.$$

By induction, from (2) we deduce that *any nonempty finite set in a lattice has exact bounds.* If every subset of a lattice has such bounds, the lattice is called *complete.*

A lattice L is said to be *distributive*, provided the following relations hold in it:

(3) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$,

(4) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

If there exists the least (greatest) element of the lattice, it is called *zero* (*unity*, or *unit*), respectively. The zero and unity in the lattice L are denoted by the symbols $\mathbf{0}_L, \mathbf{1}_L$, respectively, or simply by $\mathbf{0}, \mathbf{1}$, provided it is clear what lattice L is discussed. It should be remarked that both $\mathbf{0}$ and $\mathbf{1}$ are neutral elements:

(5) $\mathbf{0} \vee x = x, \quad \mathbf{1} \wedge x = x$.

According to conventional definitions, $\vee\varnothing = \sup\varnothing := \mathbf{0}$, $\wedge\,\varnothing = \inf\varnothing := \mathbf{1}$. A *complement* x^* of the element x in the lattice L with zero and unity is determined as such an element of L that

(6) $x \wedge x^* = \mathbf{0}, \quad x \vee x^* = \mathbf{1}$.

Since the elements x and y in L are called *disjoint*, if $x \wedge y = \mathbf{0}$, we see that each element x is disjoint to each of its complements x^*. It should be, finally, remarked that if each element in L has at least one complement, then L is said to be a lattice with complements.

1.1.2. *A Boolean algebra* is a distributive lattice with complements. In particular, in a Boolean algebra B there is, by definition, zero, $\mathbf{0} := \mathbf{0}_B$, and unity, $\mathbf{1} := \mathbf{1}_B$.

At a first glance, the above definition might seem a bit strange, as it does not reveal why and what for a distributive lattice could be termed an algebra, since the term 'algebra' refers to conventional objects (cf.: a Lie algebra, a Banach algebra, a C^*-algebra, etc.). The arising ambiguity is easily eliminated, since in fact a Boolean algebra is an algebra over the two-element field. The principal importance of this peculiarity will be partially discussed in the section to follow. At the same time, it would be quite natural to view Boolean algebras in different contexts at different angles. Below we shall consider a Boolean algebra primarily as a distributive lattice with complements. It should be emphasized that concrete Boolean algebras important for functional analysis often arise as distributive lattices with complements.

It should be remarked that a formal example of a Boolean algebra is the one-element lattice, i.e., a singleton of the type $\{x\}$ with the obvious order $x \le x$. This algebra is termed *degenerate*. A simplest nondegenerate Boolean algebra is the two-element lattice $\{\mathbf{0}, \mathbf{1}\}$ with the order $\mathbf{0} \le \mathbf{1}, \mathbf{0} \le \mathbf{0}, \mathbf{1} \le \mathbf{1}$. A degenerate Boolean algebra is natural as an algebraic system but absurd in the context of Boolean-valued analysis we are interested in. At the same time, being the simplest algebraic system of little interest, the two-element lattice plays an important role in the chapters to follow. Therefore, let us make an agreement: speaking about a Boolean algebra B we shall always assume $\mathbf{0}_B \neq \mathbf{1}_B$, i.e., the degenerate algebra will be excluded from considerations.

In a Boolean algebra B each element $x \in B$ has the unique complement denoted by the symbol x^*. The arising mapping $x \to x^* (x \in B)$ is idempotent (i.e., $(\forall x \in B)\ (x^{**} := (x^*)^* = x)$) and it presents a *dual isomorphism* (=*antiisomorphism*) onto itself (i.e., it is an isomorphism of the ordered sets $(B,\le)$ and $(B,\le^{-1})$). In particular, the De Morgan formulas are valid:

$$\left(\bigvee_{\alpha\in A} x_\alpha\right)^* = \bigwedge_{\alpha\in A} x_\alpha^*, \quad \left(\bigwedge_{\alpha\in A} x_\alpha\right)^* = \bigvee_{\alpha\in A} x_\alpha^*,$$

where $x_\alpha \in B (\alpha \in A)$.

1.1.3. Thus, in an arbitrary algebra B the three operations $\vee, \wedge$ and $*$ are given; we call them *Boolean.* It is possible to give an equivalent definition to the Boolean algebra B, characterizing it as a universal algebra $(B, \vee, \wedge, *, \mathbf{0}, \mathbf{1})$ with two binary operations $\vee$ and $\wedge$, one unary operation $*$, and two chosen elements $\mathbf{0}$ and $\mathbf{1}$ obeying the conditions:

(1) the operations $\vee$ and $\wedge$ are commutative and associative (**1.1.1** (1,2));

(2) the operations $\vee$ and $\wedge$ are dually distributive relative to one another (**1.1.1** (3,4));

(3) the elements x and x^* are mutually complementary (**1.1.1** (6));

(4) $\mathbf{0}$ and $\mathbf{1}$ are neutral elements for the operations $\vee$ and $\wedge$, respectively (**1.1.1** (5)).

Having determined such a universal algebra B, we can introduce in it a relation of order, setting $x \le y$ if $x \wedge y = x$. In this case it appears that $(B, \le, \mathbf{0}, \mathbf{1})$ is a distributive lattice with complements, where $\vee$ and $\wedge$ coincide with the lattice operations, $*$ is complementation, and $\mathbf{0}$ and $\mathbf{1}$ are the least and the greatest elements, respectively. In literature one can find many equivalent systems of axioms which characterize Boolean algebras.

1.1.4. Using the basic operations $\vee$, $\wedge$ and $*$, other operations are introduced:

$$x - y := x \wedge y^*, \quad x \Rightarrow y := x^* \vee y,$$
$$x \Delta y := (x - y) \wedge (y - x) = (x \wedge y^*) \vee (y \wedge x^*),$$
$$x \Leftrightarrow y := (x \Rightarrow y) \wedge (y \Rightarrow x) = (x^* \vee y) \wedge (y^* \vee x).$$

Let us consider some easily checkable relations which will be repeatedly used below:

(1) $x \Rightarrow y = (x - y)^*, \quad x \Leftrightarrow y = (x \Delta y)^*$;

(2) $x \Rightarrow (y \Rightarrow z) = (x \wedge y) \Rightarrow z = (x \wedge y) \Rightarrow (x \wedge z)$;

(3) $x \leq y \Rightarrow z \leftrightarrow x \wedge y \leq z \leftrightarrow y - z \leq x^*$;

(4) $x \leq y \leftrightarrow x \Rightarrow y = \mathbf{1} \leftrightarrow x - y = 0$;

(5) $x = y \leftrightarrow x \Leftrightarrow y = \mathbf{1} \leftrightarrow x \Delta y = \mathbf{0}$.

It should be emphasized that the operation Δ, termed the *symmetric difference*, has the properties of a metric:

(6) $x \Delta y = \mathbf{0} \leftrightarrow x = y$;

(7) $x \Delta y = y \Delta x$;

(8) $x \Delta y \leq (x \Delta z) \vee (z \Delta y)$.

In this case, relative to such a metric the lattice operations become contractive, while complementation becomes an isometry:

$$(x \vee y)\Delta(u \vee v) \leq (x \Delta u) \vee (y \Delta v),$$
$$(x \wedge y)\Delta(u \wedge v) \leq (x \Delta u) \vee (y \Delta v),$$
$$x^* \Delta y^* = x \Delta y.$$

1.1.5. A Boolean algebra B is called *complete* (*σ-complete*), if any set (any countable set) in B has exact bounds. Instead of σ-complete algebras we often simply use the term *σ-algebras*. Associated with a Boolean algebra B are the mappings $\vee, \wedge : \mathcal{P}(B) \to B$, putting into correspondence to the set in B its supremum and infimum, respectively. These mappings are sometimes termed *infinite operations*. For these operations we have important relations such, for instance, as the distributive laws:

(1) $x \vee \bigwedge_{\alpha \in A} x_\alpha = \bigwedge_{\alpha \in A} x \vee x_\alpha$;

(2) $x \wedge \bigvee_{\alpha \in A} x_\alpha = \bigvee_{\alpha \in A} x \wedge x_\alpha$;.

From (1), (2) the following often used relations ensue:

(3) $\left(\bigvee_{\alpha\in A}\right)\Rightarrow x = \bigwedge_{\alpha\in A}(x_\alpha \Rightarrow x);$

(4) $\left(\bigwedge_{\alpha\in A}\right)\Rightarrow x = \bigvee_{\alpha\in A}(x_\alpha \Rightarrow x);$

(5) $x \Rightarrow \left(\bigvee_{\alpha\in A} x_\alpha\right) = \bigvee_{\alpha\in A}(x \Rightarrow x_\alpha);$

(6) $x \Rightarrow \left(\bigwedge_{\alpha\in A} x_\alpha\right) = \bigwedge_{\alpha\in A}(x \Rightarrow x_\alpha).$

Ensured are also the commutativity and associativity of the exact bounds, recalled earlier in some particular cases in **1.1.1** (1,2):

(7) $\bigvee_{\alpha\in A}\bigvee_{\beta\in B} x_{\alpha,\beta} = \bigvee_{\beta\in B}\bigvee_{\alpha\in A} x_{\alpha,\beta};$

(8) $\bigwedge_{\alpha\in A}\bigwedge_{\beta\in B} x_{\alpha,\beta} = \bigwedge_{\beta\in B}\bigwedge_{\alpha\in A} x_{\alpha,\beta};$

(9) $\bigvee\left(\bigcup_{\alpha\in A} X_\alpha\right) = \bigvee_{\alpha\in A}\bigvee X_\alpha;$

(10) $\bigwedge\left(\bigcup_{\alpha\in A} X_\alpha\right) = \bigwedge_{\alpha\in A}\bigwedge X_\alpha,$

where $X_\alpha \subset B\,(\alpha \in A)$. It should be emphasized that rules (1) - (6) are valid in an arbitrary Boolean algebra, while rules (7) - (10) hold in any ordered set under the obvious assumptions of existence of exact bounds.

1.1.6. Let us consider some ways of forming Boolean algebras.

(1) A nonempty subset B_0 of a Boolean algebra B is termed *a subalgebra* of B, if B_0 is closed relative to the Boolean operation $\vee,\wedge$ and $*$, i.e., $\{x \vee y, x \wedge y, x^*\} \subset B_0$, whatever $x, y \in B_0$. Concerning the order induced from B, the subalgebra B_0 is a Boolean algebra with the same zero and unit that B has. In particular, $B_0 := \{\mathbf{0}_B, \mathbf{1}_B\}$ is a subalgebra of B.

A subalgebra $B_0 \subset B$ is termed *regular* (*σ-regular*) iff for any set (any countable set) $A \subset B$ the exact bounds $\vee A$ and $\wedge A$ existing in B are in B_0. The intersection of an arbitrary family of subalgebras is a subalgebra as well. The same is also true for regular (σ-regular) subalgebras, which makes the definition to follow sound. The least subalgebra of the algebra

B containing a nonempty subset $M \subset B$ is called the *subalgebra generated by the set M. The regular (σ-regular) subalgebra generated by the set M* is introduced in an analogous way.

(2) *An ideal* of a Boolean algebra B is any nonempty set $J \subset B$ obeying the conditions:

$$x \in J \wedge y \in J \rightarrow x \vee y \in J,$$
$$x \in J \wedge y \leq x \rightarrow y \in J.$$

Examples of such ideals are sets $B_a := \{x \in B : x \leq a\}$, where $a \in B$. Those are called *principal*. If $\mathbf{0} \neq e \in B$, then the principal ideal B_e is an independent Boolean algebra relative to the order induced from B. In B_e the role of unit is played by the element e. The lattice operations are inherited from B, while the complementation of B_e has the form $x \rightarrow e - x \; (x \in B)$. The ideal J is termed *proper* provided $J \neq B$.

(3) Let us choose Boolean algebras B and B'. The mapping $h: B \rightarrow B'$ is called a *(Boolean) homomorphism*, if for any $x, y \in B$ the following equalities are fulfilled:

$$h(x \vee y) = h(x) \vee h(y),$$
$$h(x \wedge y) = h(x) \wedge h(y),$$
$$h(x^*) = h(x)^*.$$

The homomorphism h is *isotonic* ($x \leq y \rightarrow h(x) \leq h(y)$). If h is a homomorphism, then the image $h(B)$ of the algebra B is a subalgebra of B'. If h is bijective, then it is called an *isomorphism*, while the algebras B and B' themselves are called *isomorphic*. An injective homomorphism is usually called a *monomorphism*.

Let C be an arbitrary set, and let a bijection $h: B \rightarrow C$ be given. Then we can introduce an order in C by setting $h(x) \leq h(y)$ iff $x \leq y$. In this case C turns into a Boolean algebra, while h becomes an isomorphism of Boolean algebras.

(4) Let J be a proper ideal of a Boolean algebra B. Let us introduce an equivalence relation $\sim$ by using the rule

$$x \sim y \leftrightarrow x \Delta y \in J \quad (x, y \in B).$$

Let us denote by φ the canonical (factor-) mapping of the algebra B on the factor-set $B / J := B / \sim$. For the equivalence classes (cosets) $u, v \in B / J$ we denote $u \leq v$ if there are elements $x \in U$ and $y \in v$ such that $x \leq y$. An order relation in B / J has thus been determined. In this case B / J becomes a Boolean algebra which is termed the *factor-algebra of B by J*. The Boolean operations induced in B / J are such that φ becomes a homomorphism. If $h: B \to B'$ is a homomorphism, then $\ker h := \{x \in B: h(x) = \mathbf{0}\}$ is an ideal, and there is a unique monomorphism $g: B / \ker h \to B'$ for which $g \circ \varphi = h$, where $\varphi: B \to B' / \ker h$ is the factor-homomorphism. Therefore, any homomorphic image of a Boolean algebra is isomorphic to its factor-algebra by a suitable ideal.

(5) Let us choose a family of Boolean algebras $(B_\alpha)_{\alpha \in A}$. Let us supply the product $B := \Pi_{\alpha \in A} B_\alpha$ with coordinatewise order, setting $x \le y$ for $x, y \in B$ if $x(\alpha) \le y(\alpha)$ for all $\alpha \in A$. In this case B is a Boolean algebra. The Boolean operations in B coincide with the corresponding coordinatewise operations in the algebras B_α. The zero θ and unity e in B are determined by the equalities $\theta(\alpha) := \mathbf{0}_\alpha$ and $e(\alpha) := \mathbf{1}_\alpha$ $(\alpha \in A)$, where $\mathbf{0}_\alpha$ and $\mathbf{1}_\alpha$ are the zero and unity in B_α. The Boolean algebra B is termed the *Cartesian product* of the family of Boolean algebras $(B_\alpha)_{\alpha \in A}$.

(6) Let us again consider a family of Boolean algebras $(B_\alpha)_{\alpha \in A}$. There is a Boolean algebra B and a family of monomorphisms $\iota_\alpha: B_\alpha \to B$ $(\alpha \in A)$ obeying the following conditions:

(1) the family of subalgebras $(\iota_\alpha(B_\alpha))_{\alpha \in A}$ of the algebra B is independent, i.e., for any finite set of nonzero elements $x_k \in \iota_{\alpha_k}(B_{\alpha_k})$, where $\alpha_1, \dots, \alpha_n \in A$ and $\alpha_k \ne \alpha_l$ for $k \ne l$, fulfilled is $x_1 \wedge \dots \wedge x_n \ne \mathbf{0}$;

(2) the subalgebra in B generated by the union of all $\iota_\alpha(B_\alpha)$ coincides with B.

If a Boolean algebra B' and a family of monomorphisms $\iota'_\alpha: B_\alpha \to B'$ $(\alpha \in A)$ obey the same conditions, (1) and (2), then there is an isomorphism h of the algebra B on the algebra B' such that $\iota_\alpha \circ h = \iota'_\alpha$ $(\alpha \in A)$. The pair $(B, (\iota_\alpha)_{\alpha \in A})$ is called *the Boolean* (or *tensor*) *product of the family* $(B_\alpha)_{\alpha \in A}$ and is denoted by the symbol $\otimes_{\alpha \in A} B_\alpha$.

(7) *A completion* of a Boolean algebra B is a pair (ι, A) provided the following conditions are met:

(a) A is a complete Boolean algebra;

(b) ι is a monomorphism from B to A preserving the exact bounds of all sets;

(c) the proper algebra in A generated by the set $\iota(B)$ coincides with A.

The pairs (ι, A) and (ι', A') are said to *be isomorphic* if there is an isomorphism $h: A \to A'$ such that $h \circ \iota = \iota'$. For any Boolean algebra B there is a completion unique up to isomorphism (and thus referred to as the completion of B), which can be, for instance, obtained by the classical method of sections (stemming from Dedekind).

1.1.7. Examples

(1) For a nonempty set X, the set of subsets $\mathcal{P}(X)$ ordered by inclusion is a complete Boolean algebra, sometimes called *the Boolean* of X. In this case the Boolean operations coincide with the set-theoretic operations of union, intersection and complement.

(2) Let X be a topological space. The set of all clopen (i.e., open and closed simultaneously) subsets of the space X ordered by inclusion is a subalgebra of the Boolean $\mathcal{P}(X)$. Let us denote this subalgebra by the symbol $\mathbb{E}(X)$. The Boolean operations in $\mathbb{E}(X)$ are inherited from $\mathcal{P}(X)$ and, hence, coincide with the set-theoretic ones. However, $\mathbb{E}(X)$ is not a regular subalgebra, i.e., the infinite operations in $\mathcal{P}(X)$ and $\mathbb{E}(X)$ can differ essentially.

(3) A closed subset F of a topological space X is called *regular* if $F = \operatorname{cl}\operatorname{int} F$, i.e., if F coincides with the closure of the set of its interior points. Analogously, *a regular open set* G is defined by the relation $G = \operatorname{int}\operatorname{cl} G$. Let $\mathrm{RC}(X)$ and $\mathrm{RO}(X)$ be sets of regular closed subsets and regular open subsets, respectively, of the topological space X. The sets $\mathrm{RC}(X)$ and $\mathrm{RO}(X)$ ordered by inclusion are complete Boolean algebras. The mapping $F \to \operatorname{int} F$ $(F \in \mathrm{RC}(X))$ establishes an isomorphism between them. The algebras of $\mathrm{RC}(X)$ and $\mathrm{RO}(X)$ are contained in the Boolean $\mathcal{P}(X)$, not being, however, its subalgebras. Thus, for instance, in $\mathrm{RC}(X)$ the Boolean operations have the form

$$E \vee F = E \cup F, \quad E \wedge F = \operatorname{cl}\operatorname{int} E \cap F, \quad F^* = \operatorname{cl}(X - F).$$

(4) Let $\mathcal{B}(X)$ be the Borel σ-algebra of a topological space X (= σ-regular subalgebra of the Boolean $\mathcal{P}(X)$ generated by the topology). In $\mathcal{B}(X)$ let us consider an ideal $\mathcal{N}$ consisting of all meager sets (= first-category sets). Then the factor-algebra $\mathcal{B}(X)/\mathcal{N}$ is a complete Boolean algebra termed *the algebra of Borel sets modulo meager sets*. An isomorphic algebra is obtained if instead of $\mathcal{B}(X)$ we take an σ-algebra with the Baire property. (A set $M \subset X$ *has the Baire property* if for a certain open $G \subset X$ the symmetrical difference $M \Delta G$ is a meager set). If the space X is a Baire one, i.e., if in it there are no nonempty open meager sets, then the algebra in question is isomorphic to the algebra of regular closed sets $\mathrm{RC}(X)$.

(5) Let $\mathcal{B}$ be a σ-complete Boolean algebra, and let a positive countably additive function $\mu: \mathcal{B} \to \mathbf{R}$ be given. *Countable additivity*, as usual, means

$$\mu\left(\bigvee_{n=1}^{\infty} x_n\right) = \sum_{n=1}^{\infty}\mu(x_n)$$

for any sequence (x_n) of mutually disjoint elements of $\mathcal{B}$. Let us set $\mathcal{N} := \{x \in \mathcal{B}: \mu(x) = 0\}$, in which case $\mathcal{N}$ is a σ-complete ideal. On the factor-algebra $B := \mathcal{B} / \mathcal{N}$ there is a unique countably additive function $\bar{\mu}$, for which $\mu = \bar{\mu} \circ \varphi$, where $\varphi: \mathcal{B} \to B$ is a factor-homomorphism. The algebra B is complete, while the function $\bar{\mu}$ is strictly positive, i.e., $\bar{\mu}(x) = 0 \to x = \mathbf{0}$. If $\rho(x,y) := \bar{\mu}(x \Delta y)$, then ρ is a metric, and the metric space (B,ρ) is complete.

Let $(X,\mathcal{B},\mu)$ be a space with finite measure, i.e., X is a nonempty set, $\mathcal{B}$ is a σ-complete subalgebra in $\mathcal{P}(X)$, and μ is the same as above. Then the algebra B is termed *an algebra of measurable sets modulo sets of measure zero.*

(6) Let $(X,\mathcal{B},\mu)$ be the same as in (5), and let us denote by the symbol $M(\mu) := M(X,\mathcal{B},\mu)$ the space of equivalence classes of μ-measurable functions almost everywhere finite on X. The measurable functions *are equivalent* provided they can assume different values only on the set of measure zero. In the space $M(\mu)$ an order is introduced by setting $\bar{f} \le \bar{g}$ iff $f(x) \le g(x)$ for almost all $x \in X$. Here $\bar{f}$ is the class of equivalence of the function f. Then $M(\mu)$ is a lattice. Let $\mathbf{1}$ the equivalence class of the function identically equal to unity on X. Let us set $B := \{e \in M(\mu): e \wedge (\mathbf{1} - e) = \mathbf{0}\}$. In this case B is a complete Boolean algebra with respect to the order induced from $M(\mu)$

$$c \vee e = c + e - c \cdot e, \quad c \wedge e = c \cdot e, \quad e^* = \mathbf{1} - e \quad (c,e \in B),$$

where +, ·, - are the signs of addition, multiplication and subtraction, respectively, in the ring $M(\mu)$.

(7) Let H be a complex Hilbert space, and $B(H)$ be the algebra of all bounded linear operators in H. The *commutant* $\mathcal{U}'$ of a set $\mathcal{U} \subset B(H)$ is introduced by the formula $\mathcal{U}' := \{T \in B(H): (\forall S \in \mathcal{U})(TS = ST)\}$, while the *bicommutant* is introduced by the rule $\mathcal{U}'' := (\mathcal{U}')'$. *The von Neumann algebra* is any selfadjoint $(T \in \mathcal{U} \to T^* \in \mathcal{U})$ subalgebra $\mathcal{U} \subset B(H)$ which coincides with its bicommutant. Let us choose a commutative von Neumann algebra $\mathcal{U}$. The set of all orthoprojections contained in $\mathcal{U}$ will be denoted by the symbol $B_{\mathcal{U}}$. The order relation in $B_{\mathcal{U}}$ is introduced as follows:

$$\pi \le \rho \leftrightarrow \pi(H) \subset \rho(H) \quad (\pi,\rho \in B_{\mathcal{U}}).$$

In this case $B_{\mathcal{U}}$ is a complete Boolean algebra, the Boolean operations having the form:

$$\pi \vee \rho = \pi + \rho - \pi \circ \rho, \quad \pi \wedge \rho = \pi \circ \rho, \quad \pi^* = I_H - \pi.$$

1.1.8. Remarks

(1) The theory of Boolean algebras originated from the classical work by G.Boole 'An Investigation of the Laws of Thought, on Which Are Founded the Mathematical Theories of Logic and Probabilities' [17]. The purpose of this monograph was formulated by the author in the following way: "In the treatise offered to the attention of the reader we intend to investigate the fundamental laws of those operations which refine the reason in the process of consideration, in order to express them in a symbolic language of calculus and on this foundation to construct the science of logic and its method." Following this doctrine, G.Boole carries out, in fact, algebrization of the logical system that underlies classical mathematical considerations. As a result, he has become the author of the algebraic system termed a Boolean algebra.

(2) One of the most important examples considered in the afore-mentioned book is the propositional algebra. In modern terms, the *propositional algebra* is a Boolean algebra arising as a result of identifying the equivalent formulas in a set of all formulas of the propositional calculus.

The above-said can be in general general formalized as follows. Let $\mathfrak{T}$ be a first-order theory based on the classical (two-valued) logic. In the set of all formulas Φ in the theory $\mathfrak{T}$ let us introduce a relation of preorder, setting $\varphi \leq \psi$ iff the formula $\varphi \to \psi$ is a theorem of the theory $\mathfrak{T}$. Let us consider the equivalence relation $\sim$ in Φ:

$$\varphi \sim \psi \leftrightarrow \varphi \leq \psi \wedge \psi \leq \varphi \quad (\varphi, \psi \in \Phi).$$

Let $\mathfrak{A} = \Phi/\sim$ be the corresponding factor-set supplied with the induced order. In more detail, if $|\varphi|$ is the equivalence class of the formula $\varphi \in \Phi$, then $|\varphi| \leq |\psi|$ implies $\varphi \leq \psi$. The arising ordered set $\mathfrak{A} := \mathfrak{A}(\mathfrak{T})$ is a Boolean algebra, which is sometimes termed a Lindenbaum-Tarski algebra of the theory $\mathfrak{T}$. The Boolean operations in the algebra $\mathfrak{A}(\mathfrak{T})$ have the form

$$|\varphi| \vee |\psi| = |\varphi \vee \psi|,$$
$$|\varphi| \wedge |\psi| = |\varphi \wedge \psi|,$$
$$|\varphi|^* = |\neg \varphi|.$$

The translation of the logical problems of formal theories into the language of the corresponding Boolean algebras, the Lindenbaum-Tarski algebras, is called the *Boolean method*.

(3) The classical ways of making conclusions (syllogisms, the excluded middle,

modus ponens, generalization, etc.) are abstractions originated as a result of idealization of those real operations carried out in the brain in the process of reasoning. Inevitably making the reality rougher, the two-valued logic gives, strictly speaking, only a hardly approximate, incomplete description of the laws of reasoning, which explains the interest to non-classical logical systems. One of such systems has been elaborated within the framework of the intuitionistic approach. Without going into details, let us briefly describe the corresponding propositional algebra.

The pseudo-Boolean algebra is a lattice L with zero and unity, where for any $x, y \in L$ there is a pseudo-complement $x \Rightarrow y$ of the element x relative to y. By definition, *the pseudo-complement* $x \Rightarrow y$ is the greatest of the elements $z \in L$ obeying the inequality $z \wedge x \leq y$. Hence, the following equivalence is valid (cf.: **1.1.4** (3)):

$$z \leq x \Rightarrow y \leftrightarrow x \wedge z \leq y \quad (x, y, z \in L),$$

which may be also considered as the definition of $x \Rightarrow y$. A pseudo-Boolean algebra is a distributive lattice. A complete lattice is a pseudo-Boolean algebra iff the following distributive law is fulfilled in it:

$$x \wedge \bigvee_{\alpha \in A} x_\alpha = \bigvee_{\alpha \in A} x \wedge x_\alpha \quad (x, x_\alpha \in L).$$

An example of a complete pseudo-Boolean algebra is the set of all open subsets of a topological space ordered by inclusion. Pseudo-Boolean algebras are termed *Brouwer lattices* or, most often, *Heyting algebras.*

We can demonstrate that the Lindenbaum algebra of the intuitionistic logic is a Heyting algebra. Therefore, Heyting algebras characterize the intuitionistic logic is the same way as Boolean algebras characterize the classical logic (for details, see [13, 214, 215]).

(4) Investigation of certain types of nonclassical logics results, as was in the case of the intuitionistic logic, to various classes of algebraic systems which are distributive lattices. The most well-known varieties are as follows: an implicative lattice (= the lattice with relative pseudo-complements), a topological Boolean algebra (= a Boolean algebra B with the operation $\mathbf{I}: B \to B$, obeying the internal axioms: $\mathbf{I}:(x \wedge y) = \mathbf{I}x \wedge \mathbf{I}y$; $x \leq y \to \mathbf{I}x \leq \mathbf{I}y$, $\mathbf{I}^2 = \mathbf{I}$, $\mathbf{I0} = \mathbf{0}$, $\mathbf{I1} = \mathbf{1}$), a Post algebra, etc. (see [13, 72, 214, 215]). A general theory of lattices is an individual branch with its numerous and in-depth relations with various sections of mathematics.

(5) The origination of the above-mentioned logics (=lattices) is associated with 'the investigation of the laws of thought' in the sense of the the Boole's program we discussed above. The analysis of the laws of microworld has given rise to a principally different type of logic. The logic of quantum mechanics differs significantly from the classical, intuitionistic and modal logics.

The ortholattice is a lattice L with zero, unity and a unary operation (of orthocomplementation) $(\cdot)^{\perp}: L \to L$ obeying the following conditions:

$$x \wedge x^{\perp} = \mathbf{0}, \quad x \vee x^{\perp} = \mathbf{1};$$
$$x^{\perp\perp} := (x^{\perp})^{\perp} = x;$$
$$(x \vee y)^{\perp} = x^{\perp} \wedge y^{\perp}, \quad (x \wedge y)^{\perp} = x^{\perp} \vee y^{\perp}.$$

The distributive ortholattice is a Boolean algebra. The elements x and y are called *orthogonal*, and we write $x \perp y$ if $x \le y^{\perp}$ or, which is equivalent, $y \le x^{\perp}$. The ortholattice L is termed an *orthomodular lattice* or a *(quantum) logic*, provided for any $x, y \in L, x \le y$ there is an element $z \in L$ such that $x \perp z$ and $x \vee z = y$. The last peculiarity is equivalent to the fact that from $x \le y$ we get $y = x \vee (y \wedge x^{\perp})$. An example of the quantum logic is a lattice of all closed subspaces of a Hilbert space with the operation of orthogonal complementation.

1.2. Representation of Boolean Algebras

A principally important possibility of representing a Boolean algebra as an algebra of clopen subsets of a compact space is guaranteed by the Stone theorem. The basic goal of this section is to prove the theorem and to describe some opportunities that are granted by it.

1.2.1. Let $\mathbf{2} := \mathbb{Z}_2 := \mathcal{P}(\{\varnothing\}) := \{\mathbf{0}, \mathbf{1}\}$ be a two-element set with the field structure determined by the relations:

$$\mathbf{0} + \mathbf{0} := \mathbf{0}, \quad \mathbf{0} + \mathbf{1} = \mathbf{1} + \mathbf{0} := \mathbf{1}, \quad \mathbf{1} + \mathbf{1} := \mathbf{0},$$
$$\mathbf{0} \cdot \mathbf{1} = \mathbf{1} \cdot \mathbf{0} := \mathbf{0}, \quad \mathbf{0} \cdot \mathbf{0} := \mathbf{0}, \quad \mathbf{1} \cdot \mathbf{1} := \mathbf{1}.$$

It should be remarked that all the elements of the field $\mathbf{2}$ are idempotents. Let us now consider an arbitrary set B with the structure of an associative ring whose elements are idempotents: $b \in B \to b^2 = b$. In this case B is called a *Boolean ring*. Such a ring is commutative and obeys the identity $b = -b$ for $b \in B$. The Boolean ring is obvious to be a vector space over the field $\mathbf{2}$ and, moreover, a commutative algebra over this field.

It should be recalled that the unity of an algebra is by definition different from zero. The field $\mathbf{2}$ can be naturally identified with the subring of a Boolean ring composed of the zero and unity of the latter. This fact is reflected in the notation: for the zero of any ring, use is made of the symbol $\mathbf{0}$, whereas for the unity, of the symbol $\mathbf{1}$. Such an agreement, naturally, results in a quite common collision of notation (in the field $\mathbf{2}$ the addition and multiplication can be redefined, in which case $\mathbf{0}$ starts playing the role of $\mathbf{1}$, and vice versa).

A Boolean ring B is always considered with the order relation determined by the rule:

$$b_1 \geq b_2 \leftrightarrow b_1 b_2 = b_1 \quad (b_1, b_2 \in B).$$

The ordered set $(B,\leq)$ can be directly proved to be a distributive lattice with the least element **0**, and the greatest element **1**. In this case the lattice operations are related with the ring ones in the following way:

$$x \vee y = x + y + xy, \quad x \wedge y = xy.$$

Moreover, each element $b \in B$ has a unique complement, i.e. such an element b^*, that

$$b^* \vee b = \mathbf{1}, \quad b^* \wedge b = \mathbf{0}.$$

Obviously, $b^* = \mathbf{1} + b$. Hence, any Boolean ring is a Boolean algebra provided the order in it is determined in the way just discussed.

In turn, we can introduce the structure of a ring in the Boolean algebra B, setting

$$x + y := x \Delta y, \quad xy := x \wedge y \quad (x, y \in B).$$

In this case $(B, +, \cdot, \mathbf{0}, \mathbf{1})$ becomes a Boolean ring with unity, for which the newly-arisen order relation coincides with the one available.

Therefore, a Boolean algebra can be viewed as an algebra with unity over the ring **2**, whose every element is idempotent.

1.2.2. Let B be an arbitrary Boolean algebra.

(1) *A character* of the algebra B is a (Boolean, or, which is the same, ring) homomorphism $\chi: B \to \mathbf{2}$. By the symbol $X(B)$ we shall denote a set of all the characters of B with the topology of pointwise convergence. In more detail, the topology in $X(B)$ is induced by the product topology of $\mathbf{2}^B$, in which case $\mathbf{2}$ has a unique compact Hausdorff (discrete) topology. The arising space $\mathbf{2}^B$, a Cantor discontinuum, is compact and totally disconnected. The last fact implies that any two different points in this space have disjoint clopen neighbourhoods. It is also evident that $X(B)$ is a closed subset of $\mathbf{2}^B$. Therefore, $X(B)$ is compact and totally disconnected. The set $X(B)$ is termed *the character space* of the Boolean algebra B.

(2) As is known, a nonempty set $\mathfrak{F} \subset B$ is called *a filter* provided

$$x \in \mathfrak{F} \wedge y \in \mathfrak{F} \to x \wedge y \in \mathfrak{F},$$
$$x \in \mathfrak{F} \wedge x \leq y \to y \in \mathfrak{F}.$$

A filter different from B is termed *proper*. Elements of the set of all proper filters that are maximal by inclusion are called *ultrafilters*. Let $U(B)$ be the set of all ultrafilters in B, and let $U(b)$ be the set of ultrafilters containing b. Let us supply $U(B)$ with a topology, having chosen the system of sets $\{U(b): b \in B\}$ to be the basis of the topology. Such a definition of a topology is sound since, as can be easily checked, $U(x \wedge y) = U(x) \cap U(y) \quad (x, y \in B)$. The topological space of $U(B)$ is often referred to as *the Stone space of the Boolean algebra B* and is denoted by $\mathfrak{S}(B)$.

(3) Let $M(B)$ be a set of all maximal (proper) ideals of the algebra B. An ideal here can be understood both in accordance with **1.1.6** (2) and in the standard sense of the theory of rings. A set $J \subset B$ is an ideal iff $J^* := \{x^* : x \in J\}$ is a filter in B. Moreover, $J \in M(B) \leftrightarrow J^* \in U(B)$. Therefore, the mapping $J \to J^*$ presents a bijection between $M(B)$ and $U(B)$. The set $M(B)$ is usually called *the space of maximal ideals* and is assumed to have the induced topology, the one which turns the mapping $J \to J^*$ into a homeomorphism.

1.2.3.

(1) *A Boolean ring B is a field iff it contains exactly two elements* **0** *and* **1**. *Hence,* **2** *is a Boolean field unique up to isomorphism.*

◁ Indeed, a nonzero element $x \in B$ is invertible, and, hence, the following implications are valid:

$$xx^{-1} = \mathbf{1} \to xxx^{-1} = \mathbf{1} \to xx^{-1} = x \to x = \mathbf{1}. \ \triangleright$$

For $\chi \in X(B)$ by the symbol χ^* we shall denote the mapping $x \to \chi(x)^* \quad (x \in B)$. As is seen, $\ker \chi := \{x \in B : \chi(x) = \mathbf{0}\}$ is an ideal, while $\ker \chi^*$ is a filter.

(2) *The mappings* $\chi \mapsto \ker(\chi) \quad (\chi \in X(B))$ *and* $\chi \to \ker \chi^* \quad (\chi \in X(B))$ *are homeomorphisms of* $X(B)$ *on* $M(B)$ *and* $U(B)$, *respectively.*

◁ The mapping $\chi \mapsto \ker(\chi)$ is injective. If $J \in M(B)$, then B/J is a field and, according to (1), it is isomorphic to **2**. Let us set $\chi := \lambda \circ \varphi$, where $\varphi : B \to B/J$ is a factor-homomorphism, while $\lambda : B/J \to \mathbf{2}$ is an isomorphism. Obviously, $\ker \chi = J$ and, hence, the mapping under discussion is bijective. The remaining statements are obvious. ▷

(3) *The element* $x \in B$ *is equal to zero iff* $\chi(b) = \mathbf{0}$ *for all* $\chi \in X(B)$.

◁ Let us assume $x \neq \mathbf{0}$. Then the principal ideal $\{y \in B : y \le x^*\}$ is proper, and it can be extended to a maximal ideal $J \in M(B)$. This statement, the Krull theorem, is directly deduced from the Kuratowski-Zorn Lemma (see **I.2.3.9**). By virtue of **(2)**, $J = \ker\chi$ for a certain $\chi \in X(B)$. Since $x \notin J$, we must get $\chi(x) \neq \mathbf{0}$. ▷

1.2.4. Stone theorem. *Every Boolean algebra B is isomorphic to the Boolean algebra of clopen sets of a totally disconnected compact set unique up to homeomorphism, the Stone space of the algebra B.*

◁ Let $C(X(B),\mathbf{2})$ be the algebra of continuous $\mathbf{2}$-valued functions determined on a totally disconnected compact $X(B)$. The Gelfand transform $\mathscr{G}_B$ assigns to an element $x \in B$ the $\mathbf{2}$-valued function

$$\hat{x}: \chi \to \chi(x) \quad (\chi \in X(B)).$$

It is obvious that $\mathscr{G}_B: B \to C(X(B),\mathbf{2})$ is a homomorphism which is injective (by **1.2.3** (3)) Let us take an $f \in C(X(B),\mathbf{2})$ and set $V_f := \{\chi \in X(B): f(\chi) = \mathbf{1}\}$. The set V_f is clopen. By the definition of the topology, of $X(B)$ there are $b_1,\dots,b_k \in B$ and $c_1,\dots,c_l \in B$ such that

$$V_f := \{\chi \in X(B): \chi(b_n) = \mathbf{1}\ (n \leq k), \chi(c_m) = \mathbf{0}\ (m \leq l)\}.$$

Let us set $b_0 := b_1 \wedge \dots \wedge b_k$, $c_0 := c_1 \vee \dots \vee c_l$ and $b := b_0 \wedge c_0^*$. The set V_f can be presented as follows:

$$V_f = \{\chi \in X(B): \chi(b_0) = \mathbf{1} \wedge \chi(c_0) = \mathbf{0}\} =$$
$$= \{\chi \in X(B): \chi(b) = \mathbf{1}\} = \{\chi \in X(B): \hat{b}(\chi) = \mathbf{1}\}.$$

Therefore, $f = \hat{b}$, and, hence, $\mathscr{G}_B$ is an isomorphism.

Let us assume now that Q_1 and Q_2 are totally disconnected compacta, and that the mapping $h: C(Q_1,\mathbf{2}) \to C(Q_2,\mathbf{2})$ is an isomorphism of the algebras. If χ is a character of the algebra $C(Q_2,\mathbf{2})$, then $\chi \circ h$ is that of $C(Q_2,\mathbf{2})$. In this case the mapping $\chi \to \chi \circ h$ implements the homeomorphism of the character spaces. On the other hand, the character space of the algebra $C(Q_k,\mathbf{2})$ is homeomorphic to the compactum Q_k. Therefore, the compacta Q_1 and Q_2 are homeomorphic. Now we have to notice that the algebra $C(X(B),\mathbf{2})$ is isomorphic to the algebra of clopen sets of the space $X(B)$ and, hence, of the space $U(B)$ as well. ▷

1.2.5. Further on we, as a rule, shall consider complete Boolean algebras. The notion of a complete Boolean algebra is closely associated with *extremal compacta*, i.e., compacta which are totally disconnected spaces. It should be recalled that a topological space is termed *totally disconnected* or *extremally disconnected* or, to put it short, extremal, if the closure of its every open subset is open.

Ogasawara theorem. *A Boolean algebra is complete iff its Stone space is extremal.*

◁ Let B be a complete Boolean algebra, and h be an isomorphism of B on the algebra of clopen sets of the compactum Q: $U(B)$. Let us choose an open set $G \subset Q$. As Q is totally disconnected, then $G \cup \mathfrak{U}$, where $\mathfrak{U}$ is the union of the clopen sets contained in $\mathcal{C}$. Let $\mathfrak{U}' := \{h^{-1}(U) \colon U \in \mathfrak{U}\}$ and $b := \vee \mathfrak{U}'$. It is the clopen set $h(b)$ that is the closure of G. Indeed, $\mathrm{cl}\, G \subset h(b)$ and $h(b) \setminus \mathrm{cl}\, G$ is open. If the latter set is nonempty, then $h(c) \subset h(b) \setminus \mathrm{cl}\, G$ for a certain $\mathbf{0} \neq c \in B$. This, however, implies that $h(c) \vee h(u) \leq h(b)$ for all $u \in \mathfrak{U}'$, which contradicts the equality $b = \vee \mathfrak{U}$. Therefore, $\mathrm{cl}\, G = h(b)$ is an open set.

Let us now assume that the compactum Q is extremal. Let $\mathcal{C}$ be the collection of clopen subsets of Q, and let $G := \cup \mathcal{C}$. The set G is open and its closure $\mathrm{cl}\, G$ must also be open, since Q is extremal. Obviously, $\mathrm{cl}\, G$ is the least upper bound of the set $\mathcal{C}$ in the Boolean algebra of clopen sets $\mathfrak{B}(Q)$. ▷

1.2.6. Examples

(1) The Stone space of the algebra $\{\mathbf{0}, \mathbf{1}\}$ is a singleton. If a Boolean algebra is finite, it has $\mathbf{2}^n$ elements for a certain $n \in \mathbf{N}$, and its Stone space contains exactly n points.

(2) Let us choose a nonempty set X. The Stone space of the Boolean $\mathcal{P}(X)$ is a Stone-Cech compactification $\beta(X)$ of the set X which is viewed as a discrete topological space.

(3) If Q is a totally disconnected compact space, then the Stone space of the algebra $\mathfrak{B}(Q)$ is homeomorphic to Q.

(4) Let B, B' be Boolean algebras, and $h \colon B \to B'$ be a homomorphism. Let $\iota \colon B \to \mathfrak{B}(\mathfrak{S}(B))$ and $\iota' \colon B' \to \mathfrak{B}(\mathfrak{S}(B'))$ be the Stone presentations of the algebras B and B'. There is a unique continuous mapping $\theta \colon \mathfrak{S}(B') \to \mathfrak{S}(B)$ such that

$$h(x) = \iota^{-1}\theta^{-1}(\iota(x)) \quad (x \in B).$$

The mapping $h \to \mathfrak{S}(h) := \theta$ is a bijection between the sets of homomorphisms from B to B' and the continuous mappings from $\mathfrak{S}(B')$ to $\mathfrak{S}(B)$. If B'' is another Boolean algebra, and $g \colon B' \to B''$ is a homomorphism, then $\mathfrak{S}(g \circ h) = \mathfrak{S}(h) \circ \mathfrak{S}(g)$. Moreover,

$\mathfrak{S}(I_B) = I_{\mathfrak{S}(B)}$. Let Boole be the category of Boolean algebras and homomorphisms, while Comp be a category of compacta and continuous mappings. Then the above statements can be formulated in the following way.

Theorem. *The mapping* $\mathfrak{S}$ *is a contravariant functor from the category* Boole *to the category* Comp.

Two important particular cases of the situation under consideration deserve special attention.

(5) A Boolean algebra B_0 is isomorphic to the subalgebra of a Boolean algebra B iff the compact set $\mathfrak{S}(B_0)$ is a continuous image of the compact set $\mathfrak{S}(B)$.

(6) A Boolean algebra B' is a homomorphic image of the algebra B (or isomorphic to the factor-algebra B) (see **I.1.1** (4)) iff the compact set $\mathfrak{S}(B')$ is homeomorphic to a closed subset of the compact set $\mathfrak{S}(B)$.

(7) Let $B := \Pi_{\alpha \in A} B_\alpha$, where $(B_\alpha)_{\alpha \in A}$ is a nonempty family of Boolean algebras. The Stone space $\mathfrak{S}(B)$ of the algebra B coincides with the Stone-Cech compactification of the topological sum $\cup_{\alpha \in A} \mathfrak{S}(B_\alpha) \times \{\alpha\}$ of spaces $\mathfrak{S}(B_\alpha)$.

(8) Let $B := \otimes_{\alpha \in A} B_\alpha$ be the Boolean product of a nonempty family of Boolean algebras (**I.1.6** (6)). Then the Stone space $\mathfrak{S}(B)$ of the algebra B is homeomorphic to the product $\Pi_{\alpha \in A} \mathfrak{S}(B_\alpha)$.

(9) An *absolute* of a compact set X is such a compact set $\dot{X}$ that meets the following conditions: **(a)** X is a continuous irreducible preimage of $\dot{X}$ (i.e., there is a continuous surjection of $\dot{X}$ on X, and X is not a continuous image of any proper closed subset of $\dot{X}$); **(b)** any compact continuous irreducible preimage of the compactum X is homeomorphic to $\dot{X}$. If $\dot{B}$ is the completion of a Boolean algebra B, then $\mathfrak{S}(\dot{B}) = \dot{\mathfrak{S}}(B)$, i.e., an absolute of the Stone space of the algebra B is homeomorphic to the Stone space of its completion $\dot{B}$.

1.2.7. An *atom* of a Boolean algebra B is a nonzero element a of it such that $\{x \in B : \mathbf{0} \le x \le a\} = \{\mathbf{0}, a\}$, or, which is equivalent, $a \ne \mathbf{0}$ is an atom of the Boolean algebra B if for any $x \in B$ we have either $a \le x$ or $a \le x^*$. The algebra B is said to be *atomic* if for any nonzero element $x \in B$ there exists an atom $a \le x$. A Boolean algebra is termed *atomless* if it contains no atom.

Let us call the Boolean algebra B *fully distributive* if

$$\bigwedge_{m\in M}\bigvee_{n\in N} x_{m,n} = \bigvee_{f\in N^M}\bigwedge_{m\in M} x_{m,f(m)},$$

where $x_{m,n} \in B$ $(m \in M,\ n \in N)$, M and N are arbitrary sets, and $N^M := \{f : f : M \to N\}$.

Theorem. *Let B be a complete Boolean algebra. The following statements are equivalent:*

(1) *B is isomorphic to the Boolean $\mathcal{P}(A)$ for a nonempty A;*

(2) *B is fully distributive;*

(3) *B is atomic.*

◁ (1) → (2) It suffices to remark that both the set-theoretic union (join) and intersection (meet) obey full distributivity.

(2) → (3) Let us consider a double family $\{x_{b,t} \in B : b \in B, t \in \mathbf{2}\}$, where $\mathbf{2} := \{\mathbf{0},\mathbf{1}\}$, $x_{b,0} := b^*$ and $x_{b,1} := b$. In this case

$$\mathbf{1} = \bigwedge_{b\in B} x_{b,0} \vee x_{b,1} = \bigwedge_{b\in B}\bigvee_{t\in \mathbf{2}} x_{b,t}.$$

Since the Boolean algebra B is fully distributive, we have

$$\mathbf{1} = \vee\{c(f) : f : B \to \mathbf{2}\},$$

where $c(f) := \wedge\{x_{b,f(b)} : b \in B\}$. This implies that for $b \in B$ valid is $b = \vee\{b \wedge c(f) : f \in \mathbf{2}^B\}$. Hence, for a non-zero $b \in B$ there is a $g \in \mathbf{2}^B$ such that $b \wedge c(g) \neq 0$. On the other hand, for arbitrary $b \in B$ and $f \in \mathbf{2}^B$ only the two following cases are possible:

(a) $f(b) = \mathbf{0} \to x_{b,f(b)} = b^* \to c(f) \leq b^* \leftrightarrow b \wedge c(f) = 0$,

(b) $f(b) = \mathbf{1} \to x_{b,f(b)} = b \to c(f) \leq b$.

Therefore, if $b \neq \mathbf{0}$, then either $b \wedge c(g) = 0$ or $c(f) \leq b$, i.e., $c(f)$ is an atom of B provided $c(f) \neq \mathbf{0}$. However, since there are sufficiently many nonzero $c(f)$, then B is atomic.

(3) → (1) Let A be a set of all the atoms of the Boolean algebra B. For $x \in B$ let us denote by the symbol $h(x)$ the set of all atoms $a \in B$ such that $a \leq x$. The mapping $h : B \to \mathcal{P}(A)$ can be easily checked to be an isomorphism of the Boolean algebras. ▷

1.2.8. Remarks

(1) As is seen from theorem **1.2.4**, a Boolean algebra is completely determined by its Stone space. To be more exact, any property of the Boolean algebra B can be translated into the topological language, after which it becomes a property of the Stone space $\mathfrak{S}(B)$. This method of studying Boolean algebras is called *the realization method.*

(2) The basic idea underlying the Stone theorem **1.2.4** is also valid for the case of distributive lattices. For a distributive lattice L the role of the Stone space $\mathfrak{S}(L)$ is played by the set of all simple ideals (or filters) topologized in a special way. The proper ideal $J \subset L$ is called *simple* when

$$x \wedge y \in J \to x \in J \vee y \in J.$$

The Stone spaces of distributive lattices can be used either for constructing new lattices or for the topological description of lattice-theoretical properties (the realization method!) (see [13, 72, 214]).

1.3. Von Neumann-Gödel-Bernays Theory

As has been earlier remarked in **I.3.2.5**, the axiom schemata of replacement ZF_4^{φ} embraces an infinite number of axioms due to arbitrariness in the choice of a formula φ. One, however, can try to introduce new undefinable primitive objects which can be determined by formulas φ of ZF_4^{φ}. Then a great deal of the statements contained in the schemata ZF_4^{φ} could be formulated as a single axiom on such objects. In this case required are the axioms that could imply existence for an object corresponding to the formula. Since all the formulas are constructed by the same procedure within a finite number of sets, we cannot exclude the possibility of managing with a finite number of axioms. It is this basic idea stemming from von Neumann, that became a cornerstone of the axiomatics of set theory developed by Gödel and Bernays and designated by NBG.

The initial undefinable object (notion) of NBG is a class. A class which is an element of a class is called a *set*, the other classes termed *proper*. It is turning classes into objects that constitutes the basic difference between NBG and ZFC, the metalanguage of the latter treating ‘a class’ and ‘a property’ as synonyms.

When presenting the axiomatic theory NBG use, as a rule, is made of one of the two available modifications of the language of ZFC. The first modification consists in adding a new unary predicate symbol M to the language of ZFC, with $M(X)$ semantically implying that X is a set. The second modification uses different types of variables for sets and classes. It should be emphasized that the techniques mentioned are not obligatory for describing NBG, but are rather used for convenience.

1.3.1. The system NBG is a first-order theory with equality. Strictly speaking, the language of NBG does not differ from that of ZFC. However, capital Latin letters, $X, Y, Z, \ldots$ (with indices) are commonly used for variables, while lowercase Latin letters are left for the argo arising as a result of introducing abbreviated symbols which are not used in the language of NBG.

Let $M(X)$ be an abbreviation of the formula $(\exists Y)(X \in Y)$ (read as 'X is a set'). Let us introduce lowercase Latin letters $x, y, z \ldots$ (with indices) for the variables bounded by the sets. To be more exact, the formulas $(\forall x)\,\varphi(x)$ and $(\exists x)\,\varphi(x)$ are abbreviations of the formulas $(\forall X)\,(M(X) \to \varphi(X))$ and $(\exists X)\,(M(X) \wedge \varphi(X))$, respectively. Semantically these formulas imply: 'for any set φ is valid' and 'there is a set for which φ is valid', respectively. When using these abbreviated denotations the variable X must not enter the formula φ, as well as in the formulas comprising these abbreviations. The above rules of using uppercase and lowercase letters will, however, be observed only within the present section. After we have proved that the theory of classes is, in principle, formalizable, we will gradually return to the conventional and, hence, freer mathematical language.

Let us now get down to formulating special axioms of NBG.

1.3.2. The axiom of extensionality (for classes), NBG_1: *two classes coincide if (and only iff) they consist of the same elements*

$$(\forall X)\,(\forall Y)(X = Y \leftrightarrow (\forall Z)(Z \in X \leftrightarrow Z \in Y)).$$

1.3.3. The axioms for sets:

(1) the axiom of an (unordered) pair, NBG_2:

$$(\forall x)\,(\forall y)\,(\exists z)\,(\forall u)\,(u \in z \leftrightarrow u = x \vee u = y);$$

(2) the axiom of union, NBG_3:

$$(\forall x)\,(\exists y)\,(\forall z)\,(z \in y \leftrightarrow (\exists u)\,(u \in x \wedge z \in y));$$

(3) **the axiom of powersets**, NBG_4:

$$(\forall x)\ (\exists y)(\forall z)(z \in y \leftrightarrow z \subset x);$$

(4) **the axiom of infinity**, NBG_5:

$$(\exists x)\ (0 \in x \wedge (\forall y)\ (y \in x \leftrightarrow y \cup \{y\} \in x)).$$

These axioms are seen to coincide with their analogues of ZF formulated in **1.3.2.3**, **1.3.2.4**, **1.3.2.7** and **1.3.2.8**. One should, however, bear in mind that in verbal formulation the word 'set' can already mean here a class which is an element of the class, while in symbolic presentations of the axioms small Latin letters denote abbreviations (see **1.3.1**). Thus, for instance, a partially expanded axiom of powersets, NBG_4, has the form

$$(\forall X)(M(X) \to (\exists Y)\ (M(Y) \wedge (\forall Z)(M(Z) \to (Z \in Y \leftrightarrow Z \subset X)))).$$

In the presentation of the axiom of infinity use has been made of the following abbreviation:

$$0 \in x := (\exists y)\ (y \in x \wedge (\forall u)(u \notin y)).$$

The existence of an empty set is not assumed beforehand but results from the axiom of infinity. Nevertheless, this statement is sometimes included into the list of NBG as a separate axiom:

(5) $(\exists y)(\forall u)(u \notin y)$.

1.3.4. The axiom of replacement, NBG_6: *if the class X is single-valued, then for any set y the class of the second components of those pairs of X, whose first components belong to y, is a set:*

$$(\forall X)(\mathrm{Un}(X) \to (\forall y)(\exists z)(\forall u)(u \in z \leftrightarrow (\exists v)((v, z) \in X \wedge v \in y))).$$

As it has been assumed, the schema ZF_4^{φ} has turned into a single axiom. Let us here remark that the axiom schemata of replacement of ZF (see **I.3.2.5**) is also corresponded to by a single axiom, the axiom of replacement. This axiom states that for any set x and any class Y there is a set which consists of the elements common for both x and Y, i.e.,

$$(\forall x)(\forall Y)(\exists z)(\forall u)\ (u \in z \leftrightarrow u \in x \wedge u \in Y).$$

This axiom is weaker than the axiom of replacement (deduced from NBG_6 and theorem **1.3.14** to be proved below), but is in many cases more convenient for practical purposes.

The group of axioms to follow, NBG_6-NBG_{13}, is designated for the formation of classes. These axioms state that for certain properties expressed by formulas there exist classes of all sets possessing the required properties. In this case uniqueness results, as usual, from the axiom of extensionality, NBG_1.

1.3.5. The axiom of ∈-relation, NBG_7: *there is a class which consists exactly of those ordered pairs of sets whose first component is an element of the second one*:

$$(\exists X)(\forall y)(\forall z)\,((y,z)\in X \leftrightarrow y\in z).$$

1.3.6. The axiom of intersection, NBG_8: *for any two classes there is their intersection*:

$$(\forall X)(\forall Y)(\exists Z)(\forall u)\,(u\in Z \leftrightarrow u\in X \wedge u\in Y).$$

1.3.7. The axiom of complementation, NBG_9: *for any class there exists a class complementing it*:

$$(\forall X)(\exists Y)(\forall u)\,(u\in Y \leftrightarrow u\notin X).$$

This implies the existence of the universal class $\mathbf{U} := \overline{\varnothing}$ which is the complement of the empty class $\varnothing$.

1.3.8. The axiom of domain, NBG_{10}: *for every class X of ordered pairs there is a class $Y := \mathrm{dom}X$, whose elements are exactly the first elements of the class X*:

$$(\forall X)(\exists Y)(\forall u)\,(u\in Y \leftrightarrow (\exists v)\,((u,v)\in X)).$$

1.3.9. The axiom of the Cartesian product, NBG_{11}: *for every class X there is a class $Y := X\times \mathbf{U}$ consisting of all possible ordered pairs whose first components are elements of the class X*:

$$(\forall X)(\exists Y)(\forall u)(\forall v)\,((u,v)\in Y \leftrightarrow u\in X).$$

1.3.10. The axioms of permutation, NBG_{12} and NBG_{13}. Let $\sigma := (i_1, i_2, i_3)$ be permutations of the set $\{1,2,3\}$. The class Y will be called the *σ-conjugation* of the class X provided $(x_1, x_2, x_3)\in Y$ iff $(x_{i_1}, x_{i_2}, x_{i_3})\in X$. *For any class X there is its* $(2,3,1)$- *and* $(1,3,2)$ *-conjugations*:

$$(\forall X)(\exists Y)(\forall u)(\forall v)(\forall \omega)\,((u,v,\omega)\in Y \leftrightarrow (v,\omega,u)\in X);$$
$$(\forall X)(\exists Y)(\forall u)(\forall v)(\forall \omega)\,((u,v,\omega)\in Y \leftrightarrow (u,\omega,v)\in X).$$

1.3.11. The axiom of foundation, NBG_{14}: *in every nonempty class there is an element having no common elements with it*:

$$(\forall X)(X \neq \varnothing \rightarrow (\exists y)(y\in X \wedge y\cap X = \varnothing)).$$

1.3.12. The axiom of choice, NBG_{15}: *for every class X there is a selecting function, i.e., a single-valued class assigning an elemeny of X to each nonempty set of X*:

$$(\forall X)(\exists Y)(\forall u)\,(u \neq \varnothing \wedge u\in X \rightarrow (\exists! v)(v\in u \wedge (u,v)\in Y)).$$

This is a very strong form of the axiom of choice. It is equivalent to the existence of a simultaneous choice of one element from every nonempty set.

The above axiom makes the list of the axioms of NBG complete. The system NBG, unlike the theory ZFC, is seen to have a finite number of axioms. Another convenient feature of NBG is that in fact it treats sets and properties of sets as formal objects, thus implementing objectivization inaccessible to the expressive means of the language of ZFC.

1.3.13. From the group of the axioms on the formation of classes let us deduce some statements to be used to prove general theorems on existence of classes.

(1) *For any class there is its* $(2,1)$ *-conjugation*:

$$(\forall X)(\exists Z)(\forall u)(\forall v)\,((u,v)\in Z \leftrightarrow (v,u)\in X).$$

◁ The axiom of the Cartesian product guarantees existence for the class $X \times \mathbf{U}$. If we consecutively apply the axioms of the (2,3,1)-conjugation and (1,3,2)-conjugation to the class $X \times \mathbf{U}$, we get a class Y of all triplets (v,u,ω) such that $(v,u) \in X$. Now we make use of the axiom of domain to see that $Z := \mathrm{dom}(Y)$ is the sought class. ▷

(**2**) *For any two classes, there is their Cartesian product:*

$$(\forall X)(\forall Y)(\exists Z)(\forall \omega)\,(\omega \in Z \leftrightarrow (\exists u \in X)(\exists v \in Y)(\omega = (u,v))).$$

◁ To prove the above statement we shall consecutively use the axiom of the Cartesian product, statement (**1**), the axiom of intersection, and set $Z := (\mathbf{U} \times Y) \cap (X \times \mathbf{U})$. ▷

By virtue of **1.3.13** (2) for $n \geq 2$ determined is the class $\mathbf{U}^n$ of all ordered tuples.

(**3**) *For any class X there is a class $Z := (\mathbf{U}^n \times \mathbf{U}^m) \cap (X \times \mathbf{U}^m)$:*

$$(\forall X)(\exists Z)(\forall x_1)\dots(\forall x_n)(\forall y_1)\dots(\forall y_m)$$
$$((x_1,\dots,x_n,y_1,\dots,y_m) \in Z \leftrightarrow (x_1,\dots,x_n) \in X).$$

(**4**) *For any class X there is a class $Z := (\mathbf{U}^m \times \mathbf{U}^n) \cap (\mathbf{U}^m \times X)$:*

$$(\forall X)(\exists Z)(\forall x_1)\dots(\forall x_n)(\forall y_1)\dots(\forall y_m)$$
$$((y_1,\dots,y_m,x_1,\dots,x_n) \in Z \leftrightarrow (x_1,\dots,x_n) \in X).$$

◁ In order to prove (**3**) and (**4**) we should apply the axiom of the Cartesian product and the axiom of intersection. ▷

(**5**) *For any class X there is a class Z such that*

$$(\forall x_1)\dots(\forall x_n)(\forall y_1)\dots(\forall y_m)$$
$$((x_1,\dots,x_{n-1},y_1,\dots,y_m,x_n) \in Z \leftrightarrow (x_1,\dots,x_n) \in X).$$

◁ Use should be made of the axiom of permutation and of the axiom of the Cartesian product. ▷.

1.3.14. Theorem. *Let φ be a formula in the construction of which only variables of $X_1,\dots,X_n$, $Y_1,\dots,Y_m$ occur, and which is predicative, i.e., all bound variables of φ are restricted to sets. Then in* NBG *the following statement is provable:*

$$(\forall Y_1)\dots(\forall Y_m)(\exists Z)(\forall x_1)\dots(\forall x_n)$$
$$((x_1,\dots,x_n)\in Z \leftrightarrow \varphi(x_1,\dots,x_n, Y_1,\dots,Y_m)).$$

⊲ Let the formula φ be written, with the adopted abbreviations taken into account, in such a way that the only bound variables of φ are those for sets. It suffices to consider only φ containing no subformulas of the type $Y \in W$ and $X \in X$, since the latter are replaced with equivalent ones, $(\exists x)(x = Y \wedge x \in W)$ and $(\exists u)(u = X \wedge u \in X)$. Moreover, the symbol of equality can be eliminated from φ by substituting, in line with the axiom of extensionality, the expression $(\forall u)(u \in X \leftrightarrow u \in Y)$ for $X = Y$. The proof is carried out by induction on length k of the formula φ, i.e., by the number k of logical connectives and quantifiers belonging to φ.

For $k = 0$ the formula φ is atomic and has the form $x_i \in x_j$, or $x_j \in x_i$, or $x_i \in Y_l$ $(i < j \leq n, l < m)$. If $\varphi := x_i \in x_j$, then, according to the axiom of $\in$-relation, there is a class W_1 for which

$$(\forall x_i)(\forall x_j)((x_i,x_j)\in W_1 \leftrightarrow x_i \in x_j).$$

If $\varphi := x_j \in x_i$, then we first, using the same axiom, find a class W_2 with the property

$$(\forall x_i)(\forall x_j)((x_j,x_i)\in W_2 \leftrightarrow x_j \in x_i),$$

and then apply **1.3.13** (1). As a result, we choose a class W_3 for which

$$(\forall x_i)(\forall x_j)((x_i,x_j)\in W_3 \leftrightarrow x_j \in x_i).$$

Hence, in any of these two cases there is a class W such that the following formula is valid:

$$\Phi: (\forall x_i)(\forall x_j)((x_i,x_j)\in W \leftrightarrow \varphi(x_1,\dots,x_n, Y_1,\dots,Y_m)).$$

By virtue of **1.3.13** (4), in the formula Φ we can replace the subformula $(x_i,x_j)\in W$ with $(x_1,\dots,x_{i-1},x_i)\in Z_1$ for a certain other class Z_1, and add the quantifiers $(\forall x_1)\dots(\forall x_{i-1})$ in the beginning of the formula. Let Ψ be the formula obtained in this case. According to **1.3.13** (5), in the formula Ψ for a certain other class Z_2 it is possible to write $(x_1,\dots,x_i,x_{i+1},\dots,x_j)\in Z_2$ instead of the subformula $(x_1,\dots,x_{i-1},x_i,x_j)\in Z_1$ and to add the quantifiers $(\forall x_{i+1})\dots(\forall x_{j-1})$ at the beginning of the formula Ψ. And, finally, applying to Z_2 **1.3.13** (3), we find a class Z for which the following formula is true:

$$(\forall x_1)\ldots(\forall x_n)((x_1,\ldots,x_n)\in Z \leftrightarrow \varphi(x_1,\ldots,x_n, Y_1,\ldots,Y_m)).$$

For the remaining case, $x_i \in Y_l$ the sought requirement follows from the existence of the Cartesian products $W := \mathbf{U}^{i-1} \times Y_l$ and $Z := W \times \mathbf{U}^{n-i}$. Therefore, the theorem has been proved for $k = 0$.

Let us assume that for all $k < p$ the theorem has been proved and the formula φ has p logical connectives and quantifiers. It suffices to consider the cases when φ is obtainable from some formulas using negation, implications and the universal quantifier.

(a) $\varphi := \neg\psi$. By the induction hypothesis, there is a class V such that

$$(\forall x_1)\ldots(\forall x_n)((x_1,\ldots,x_n)\in V \leftrightarrow \psi(x_1,\ldots,x_n, Y_1,\ldots,Y_m)).$$

In accordance with the axiom of complement, there is a class $Z := \mathbf{U} - V := \mathbf{U} \setminus V$ obeying the required conditions.

(b) $\varphi := \psi \rightarrow \theta$. Again, by the induction hypothesis, there are classes V and W, such that for V and ψ all the statements of **(a)** are valid and, moreover,

$$(\forall x_1)\ldots(\forall x_n)((x_1,\ldots,x_n)\in W \leftrightarrow \theta(x_1,\ldots,x_n, Y_1,\ldots,Y_m)).$$

The sought class, $Z := \mathbf{U} - (V \cap (\mathbf{U} - W))$ does exist by virtue of the axiom of intersection and that of complementation.

(c) $\varphi := (\forall x)\psi$. Let V and ψ be the same as in **(a)**. If we apply the axiom of the domain to the class $X := \mathbf{U} - V$, then we obtain the class Z_1 for which

$$(\forall x_1)\ldots(\forall x_n)((x_1,\ldots,x_n)\in Z \leftrightarrow (\exists x)\ \neg\psi(x_1,\ldots,x_n, Y_1,\ldots,Y_m)).$$

The class $Z := \mathbf{U} - Z_1$ that exists by virtue of the axiom of complementation is the sought one, since $(\forall x)\psi$ is equivalent to $\neg(\exists x)(\neg\psi)$. ▷

1.3.15. Each of the axioms for the formation of classes, NBG_7-NBG_{13}, is a corollary to theorem **1.3.14**, provided the formula φ is chosen in an appropriate way. On the other hand, the theorem itself is seen from the proof to be deduced from the axioms of the formation of classes. It is remarkable that instead of an infinite number of statements in **1.3.14** we can manage with a finite number of the axioms NBG_7-NBG_{13}.

Theorem **1.3.14** allows one to prove the existence of various classes. Thus, for any class Y there exists class of all its subsets $\mathcal{P}(Y)$, as well as the union of all elements of the class $\cup(Y)$ determined by conventional formulas

$$(\forall u)(u \in \mathcal{P}(Y) \leftrightarrow u \subset Y),$$
$$(\forall u)(u \in \cup(Y) \leftrightarrow (\exists v)(v \in Y \wedge u \in v)).$$

The above statements can be easily proved if we choose $\varphi(X,Y) := X \subset Y$ and $\varphi(X,Y) := (\exists u)(x \in u \wedge u \in Y)$. By analogous considerations sound are the definitions of Z^{-1}, $\operatorname{im} Z$, $Z \upharpoonright Y$, $Z''Y$, $X \cup Y$, etc., where X, Y and Z are some classes.

1.3.16. Theorem. *Every theorem of* ZFC *is a theorem of* NBG.

◁ All the axioms of ZFC are theorems of NBG. Let us prove the only not obvious part of this statement which concerns the axiom of replacement ZF_4^{φ}. Let the formula φ contain no free occurrences of the variable y, and let $\{x,t,z_1,\dots,z_m\}$ be complete set of variables used in constructing φ. Let us then assume that, for all $x,u,v,z_1,\dots,z_m$, the following relation holds:

$$\varphi(x,u,z_1,\dots,z_m) \wedge \varphi(x,v,z_1,\dots,z_m) \rightarrow u = v.$$

The formula φ is predicative as all the variables in it are restricted to sets. By theorem **1.3.14**, there is a class Z such that

$$(\forall x)(\forall u)((x,u) \in Z \leftrightarrow \varphi(x,u,z_1,\dots,z_m)).$$

This property of φ shows that the class Z is single-valued, i.e., that $\operatorname{Un}(Z)$ is provable within NBG. By virtue of the axiom of replacement, NBG_6, there is a set y for which

$$(\forall v)(v \in y \leftrightarrow (\exists u)((u,v) \in \mathbf{Z} \wedge u \in x)).$$

Obviously, for y the required relation

$$(\forall z_1)\dots(\forall z_m)(\forall v)(v \in y \leftrightarrow (\exists u \in x)\,\varphi(x,v,z_1,\dots,z_m))$$

is fulfilled. ▷

1.3.17. Theorem. *Every theorem of* NBG *dealing with sets is a theorem of* ZFC.

⊲ The proof can be, for instance, found in [28]. It requires the use of some facts of model theory which go beyond the framework of the present book.

Statements **1.3.16** and **1.3.7** are often formulated in the following form.

1.3.18. Theorem. *Von Neumann-Gödel-Bernays set theory* NBG *is conservative over Zermelo-Fraenkel set theory* ZFC.

1.3.19. Among the other axiomatic set theories we should mention the Bernays-Morse theory that extends the theory NBG. The theory in question has special axioms, NBG_1-NBG_5, NBG_{14}, and the following schemata of the axiom of comprehension:

$$(\exists X)(\forall Y)\ (Y \in X \leftrightarrow M(Y) \wedge \varphi(Y, X_1, \dots, X_n)),$$

where φ is an arbitrary formula containing no occurrences of the variable X.

It is obvious from **1.3.14** that if in the formula φ all the domains of the definition of quantifiers is restricted to sets, then the axiom schema of comprehension is a theorem of NBG. The Bernays-Morse set theory allows quantification over arbitrary classes in the schemata of axioms of comprehension. This theory can be also modified by the axiom of choice, NBG_{15}.

1.4. Ordinals.

The concept of an ordinal is the key one in studying infinite sets. It is designated for transfinite iteration of various mathematical constructions, or considerations, as well as for measuring cardinality. The topic of the present section is to consider how to do it.

1.4.1. Let us consider classes X and Y. We say that X is an *order relation*, or simply an *order* on Y provided X is an antisymmetric, reflexive and transitive relation on Y. The antiasymmetry, reflexivity and transitivity of a relation are written in the same way as in the language of ZFC (see **I.3.1.10**). The order of X on Y is called *linear* or *total* if $Y \times Y \subset X \cup X^{-1}$. The relation X is said *to well-order* Y or *to be well-ordering*, or Y is said *to be well-ordered* (sometimes an abusive term *totally-ordered* is used, too), provided that X is an order on Y and any nonempty subclass of the class Y has a least element (relative to X). Classes X_1 and X_2 ordered by the relations R_1 and R_2, respectively, are termed *similar* or *equipotent* if there is exists a bijection h from X_1 on X_2 such that $(x,y) \in R_1 \leftrightarrow (h(x),h(y)) \in R_2$ for all $x, y \in X_1$.

1.4.2. Let us introduce the relation E by the formula

$$(x,y) \in E \leftrightarrow (x \in y) \vee x = y.$$

The class E does exist by virtue of the axiom of $\in$-relation, NBG_7, and theorem **1.3.14**. As is seen, E is an order relation on the universal class $\mathbf{U}$.

The class X is called *transitive* (not to be mixed up with a transitive relation!) if each of its elements is also its subset:

$$\mathrm{Tr}(X) := (\forall y)(y \in X \rightarrow y \subset X).$$

An ordinal class is the name of any transitive class well-ordered by the relation E. The record $\mathrm{Ord}(X)$ implies that X is ordinal. The ordinal class that is a set is termed *an ordinal* (or *an ordinal number*, or *a transfinite number*). The class of all ordinals is denoted by On. It should be recalled that ordinals are, as a rule, denoted by lowercase Greek letters, the following abbreviations used in this case:

$$\alpha < \beta := \alpha \in \beta, \quad \alpha \leq \beta := (\alpha \in \beta) \vee (\alpha = \beta), \quad \alpha + 1 := \alpha \cup \{\alpha\}.$$

If $\alpha < \beta$, then α is said to proceed β, while β is said to succeed α. Using the axiom of foundation, NBG_{14}, we can easily prove the following statement.

1.4.3. *A class is ordinal iff it is transitive and well-ordered by the relation E.*

◁ Let a transitive class X be totally ordered by the relation E. Choose a nonempty subclass $Y \subset X$ and show that Y has a least element. There is at least one element $y \in Y$. If $y = 0$, then y is the sought least element in Y. If $y \neq 0$, then, according to the axiom of foundation, we can find an element $x \in y$ such that $x \cap y = 0$. In this case x is the least element of the set y, since y is totally ordered. As the class Y is totally ordered by the relation E, the element x will be the least in the class Y as well. Hence, X is an ordinal class. The sufficiency of the given condition is thus proved, its necessity being obvious. ▷

Therefore, both in NBG and ZFC use can be made of a simpler definition of ordinal:

$$\mathrm{Ord}(X) \leftrightarrow \mathrm{Tr}(X) \wedge (\forall u \in X)(\forall v \in X)\,(u \in v \vee u = v \vee v \in u).$$

It would be expedient to emphasize that the equivalence of the definitions of ordinal considered can also be established without the axiom of choice.

The majority of the properties of ordinals given below can be deduced without employing the axiom of foundation, making use only of the initial definition of ordinal. This peculiarity which is important to, for instance, substantiate consistency of the axiom of foundation with the remaining axioms of ZF, is insignificant for our further purposes.

1.4.4. Below we shall require some additional facts. Let X and Y be arbitrary classes.

(1) *If X is ordinal, Y is transitive, and $X \neq Y$, then the relations $Y \subset X$ and $Y \in X$ are equivalent.*

⊲ For $Y \in X$ the class Y is a set, and $Y \subset X$ since X is transitive. Let us, in turn, assume that $Y \subset X$. Since $X \neq Y$, then $Z := X - Y \neq \varnothing$. The class Z has the least element, $x \in Z$ (in the sense of the order relation E). This implies that $x \cap Z = \varnothing$ or $x \subset Y$. Besides, $x \subset X$, since $x \in X$ and X is transitive. Let us choose an element $y \in Y$. Since X is linearly ordered, then $x \in y$ or $x = y$, or, finally, $y \in x$. With transitivity of Y taken into account, the first two relations yield $x \in Y$, which contradicts the membership $x \in Z$. Therefore, $y \in x$ and, hence, $Y \subset x$. Taking into consideration the inclusion $x \subset Y$ proved above, we get $x = Y$ and, finally, $x = Y \wedge x \in X \rightarrow Y \in X$. ⊳.

(2) *The intersection of any two ordinal classes is an ordinal class.*

⊲ This is obvious. ⊳

(3) *If X and Y are ordinal classes, then*

$$X \in Y \vee X = Y \vee Y \in X.$$

⊲ Let the intersection $X \cap Y = Z$ coincide with none of the classes X and Y. Then, according to **(1)** and **(2)**, $Z \in X$ and $Z \in Y$, i.e., $Z \in X \cap Y = Z$. For the set $Z \in X$, however, the relation $Z \in Z$ is impossible. Hence, either $Z = X$ and then $Y \subset X$, or $Z = Y$ and then $X \subset Y$. Now we have to refer to **(1)**. ⊳

1.4.5. Theorem. *The following statements are valid:*

(1) *only ordinals can be elements of any ordinal class;*

(2) *the class* On *is the only ordinal class that is not an ordinal;*

(3) *for any ordinal α, the set $\alpha+1$ is an ordinal, and the least one of all the ordinals succeeding α;*

(4) *whatever the nonempty class of ordinals $X \subset$ On might be, the union $\cup(X)$ is an upper bound of the set X in the ordered class* On.

◁ (1) Let us choose an ordinal class X and an element $x \in X$. Since X is transitive, we have $x \subset X$ and, hence, x is linearly ordered by the relation E. Let us prove $\mathrm{Tr}(x)$. If $z \in y \in x$, then $z \in X$ since X is transitive. Of three possible cases, $z = x$, $x \in z$ and $z \in x$, the first two result in closed cycles, $z \in y \in z$ and $z \in y \in x \in z$, respectively, which contradict the axiom of foundation. Therefore, $z \in x$ and, hence, $z \in y \rightarrow z \in x$, i.e., $y \subset x$, which proves $\mathrm{Tr}(x)$ and, at the same time, $\mathrm{Ord}(x)$.

(2) By **1.4.4** (3), the class On is linearly ordered, by **(1)**, it is transitive, and, hence, $\mathrm{Ord}(\mathrm{On})$. If On is a set, then On is an ordinal, which results in a contradiction, $\mathrm{On} \in \mathrm{On}$. Hence, On is an ordinal class but not an ordinal. For an arbitrary ordinal class X $X \notin \mathrm{On}$ yields $X = \mathrm{On}$. Indeed, statement **1.4.4** (3) opens only one possibility, $\mathrm{On} \in X$ which contradicts the fact that On is a proper class.

(3) If α is an ordinal, then, obviously, the set $\alpha + 1 := \alpha \cup \{\alpha\}$ is linearly ordered. For $x \in \alpha + 1$ we get either $x \in \alpha$ or $x = \alpha$, and in both cases $x \subset \alpha$. However, $\alpha \subset \alpha + 1$ and, hence, $x \subset \alpha + 1$, which proves that $\alpha + 1$ is transitive. And, finally, $\alpha + 1$ is an ordinal and $\alpha < \alpha + 1$. If $\alpha < \beta$ for a certain ordinal β, then $\alpha \in \beta$ and $\alpha \subset \beta$, i.e., $\alpha \cup \{\alpha\} \subset \beta$. By **1.4.4** (1), we have either $\alpha \cup \{\alpha\} \in \beta$ or $\alpha \cup \{\alpha\} = \beta$. Hence, $\alpha + 1 \leq \beta$.

(4) Let us assume $X \subset \mathrm{On}$ and $y \in Y := \cup X$, and choose such an element $x \in X$ that $y \in x$. Since x is an ordinal, then $y \subset x$ and, moreover, $y \subset Y$. As the class On is transitive (see **(2)**), $x \in X$ yields $x \subset \mathrm{On}$ and, hence $Y \subset \mathrm{On}$. Therefore, Y is a transitive subclass On and, hence, Y is an ordinal. If $\alpha \in X$, then $\alpha \subset Y$ and, in accord with **1.4.4** (1), $\alpha \leq Y$. While if β is an ordinal and $\beta \geq \alpha$ for all $\alpha \in X$, then $Y \subset \beta$ and, again by **1.4.4** (1), $Y \leq \beta$. Hence, $Y = \sup X$. ▷

1.4.6. The least upper bound of a set of ordinals x is usually denoted by $\lim(x)$. The ordinal α is called *limiting* if $\alpha \neq \varnothing$ and $\lim(\alpha) = \alpha$.The term *'limit ordinal'* is also employed. In equivalent words, α is a limiting ordinal if it is not presentable in the form $\alpha = \beta + 1$ with a $\beta \in \mathrm{On}$. The class of all limiting ordinals is designated by the symbol K_{II}. The ordinals not belonging to K_{II} form a class of nonlimiting ordinals $K_{\mathrm{I}} := \mathrm{On} - K_{\mathrm{II}} = \{\alpha \in \mathrm{On}\ (\exists \beta \in \mathrm{On})(\alpha = \beta + 1)\}$. Let us denote by the letter ω the least limiting ordinal (whose existence is ensured by theorem **1.4.5** and the axiom of infinity). We can show that ω coincides with the class of nonlimiting ordinals α such that each predecessor of α is also nonlimiting:

$$\omega = \{\alpha \in \mathrm{On}\colon \alpha \cup \{\alpha\} \in K_{\mathrm{I}}\}.$$

The ω elements are called *finite ordinals*, or *natural numbers*, or *positive integers*. The least ordinal, the zero set $0 := \varnothing$, is contained in ω. The successor $1 := 0 + 1 = 0 \cup \{0\} = \{\varnothing\}$ contains the only element 0. Then, $2 := 1 \cup \{1\} = \{0\} \cup \{1\} = \{0, 1\} = \{0, \{0\}\}$, $3 := 2 \cup \{2\} = \{0, \{0\}, \{\{0, \{0\}\}\}$, etc.. Therefore,

$$\omega := \{0, \{0\}, \{0, \{0\}\}, \ldots\} = \{0, 1, 2, \ldots\}.$$

The following notation is also used:

$$\mathbf{N} := \omega - \{0\} = \{1, 2, \ldots\}.$$

The next statement enumerates the basic properties of a set of natural numbers ω known as a whole as *the system of Peano axioms.*

1.4.7. Theorem. *The following statements are valid*:

(1) *zero is a natural number, i.e.,* $0 \in \omega$ *;*

(2) *for every natural number* $\alpha \in \omega$ *the successor* $\alpha + 1$ *is also a natural number*;

(3) $0 \neq \alpha + 1$ *for any natural number* α ;

(4) *for natural numbers* α *and* β $\alpha + 1 = \beta + 1$ *implies* $\alpha = \beta$;

(5) *if the class X contains an empty set and is such that for every ordinal, its successor is also in X, then* $\omega \subset X$ *.*

1.4.8. Theorem (the principle of transfinite induction). *Let* G *be a certain class with the following properties:* (1) $0 \in X$ *;* (2) *if* α *is an ordinal and* $\alpha \in X$, *then* $\alpha + 1 \in X$; (3) *if* x *is a set of ordinals contained in* X, *then* $\lim(x) \in X$. *Then* $\mathrm{On} \subset X$.

$\lhd$ Let us assume $\mathrm{On} \not\subset X$. Then the nonempty subclass $\mathrm{On} - X$ of the well-ordered class On has the least element $\alpha \in \mathrm{On} - X$, in which case this implies that $\alpha \cap (\mathrm{On} - X) = 0$ or $\alpha \subset X$ and $\alpha \neq 0$ (see **(1)**). If $\alpha \in K_{\mathrm{I}}$, i.e., $\alpha = \beta + 1$ for a certain $\beta \in \mathrm{On}$, then

$\beta \in \alpha \subset X \to \beta \in X$ and, by condition (2), $\alpha = \beta + 1 \in X$. While if $\alpha \in K_{\mathrm{II}}$, then, by condition (3), we deduce $\alpha = \lim(\alpha) \in X$. In both cases we have $\alpha \in X$, which contradicts the inclusion $\alpha \in \mathrm{On} - X$. ▷

1.4.9. Theorem (the principle of transfinite recursion). *Let G be a certain class which is a function. Then there is a unique function F for which*

(1) $\mathrm{dom}(F) = \mathrm{On}$;

(2) $F(\alpha) = (F|\alpha)$ *for any* $\alpha \in \mathrm{On}$, *where* $F|\alpha := F \cap (\alpha \times \mathrm{U})$ *is the restriction of* F *to* α.

◁ Let us determine the class Y by the relation

$$f \in Y \leftrightarrow \mathrm{Func}(f) \wedge \mathrm{dom}(f) \in \mathrm{On} \wedge (\forall \alpha \in \mathrm{dom}(f))(f(\alpha) = G(f|\alpha)).$$

If $f, g \in Y$, then either $f \subset g$ or $g \subset f$. Indeed, if $\beta := \mathrm{dom}(f)$ and $\gamma := \mathrm{dom}(g)$, then either $\beta \le \gamma$ or $\gamma \le \beta$. Assuming, for instance, $\gamma < \beta$, let us set $z := \{\alpha \in \mathrm{On}: \alpha < \gamma \wedge f(\alpha) \ne g(\alpha)\}$. If $z \ne 0$, then there is the least element $\delta \in z$. In this case for all $\alpha < \delta$ we get $f(\alpha) = g(\alpha)$, i.e., $f|\delta = g|\delta$. By the definition of the class Y, however, we also have $f(\delta) = G(f|\delta)$ and $g(\delta) = G(g|\delta)$, and, hence, $f(\delta) = g(\delta)$ and $\delta \notin z$. This contradicts the choice of δ and, thus, $z = 0$, i.e., $f(\alpha) = g(\alpha)$ for all $\alpha < \gamma$, which yields the required inclusion $g \subset f$. Let us set $F = \cup Y$. Obviously, F is a function, $\mathrm{dom}(F) \subset \mathrm{On}$ and $F(\alpha) = G(F|\alpha)$ for all $\alpha \in \mathrm{dom}(F)$. If $\alpha \in \mathrm{dom}(F)$, then $\alpha, G(F|\alpha) \in f$ for a certain $f \in Y$. Then $\alpha \in \beta := \mathrm{dom}(f) \subset \mathrm{dom}(F)$ and, since β is transitive, we get $\alpha \subset \mathrm{dom} F$. Therefore, the class $\mathrm{dom} F$ is transitive and, according to **1.4.4** (1), we have either $\mathrm{dom} F = \mathrm{On}$ or $\mathrm{dom} F \in \mathrm{On}$, the latter being impossible. Indeed, it follows from $\delta := \mathrm{dom} F \in \mathrm{On}$ that the function $f := F \cup \{(\delta, G(F))\}$ is included in Y, and, hence, $f \subset F$, which results in a contradiction, $f \subset F \to \mathrm{dom}(F) \subset \mathrm{dom}(F) \to \delta \in \mathrm{dom}(F) = \delta$. ▷

1.4.10. A binary relation R is termed *well-founded* if for any $x \in \mathrm{V}$ the class $R^{-1}(x)$ is a set and for any nonempty $x \in \mathrm{V}$ there is an element $y \in x$ such that $x \cap R^{-1}(y) = 0$. The last condition (assuming the axiom of choice) is equivalent to the fact that there is no infinite sequence (x_n) with the property $x_n \in R(x_{n+1})$ for all $n \in \omega$. An example of a well-founded relation is the relation $\in$. It is often more convenient to apply the principles of transfinite induction and recursion in the following form.

1.4.11. Theorem. *Let R be a well-founded relation. Then the following statements are valid:*

(1) (induction on R) *if the class X is such that for any $x \in \mathbf{V}$ the relation $R^{-1}(x) \subset X$ implies $x \in X$, then $X = \mathbf{V}$;*

(2) (recursion on R) *for any function $C: \mathbf{V} \to \mathbf{V}$ there is such a function F that* $\operatorname{dom}(F) = \mathbf{V}$ *and $F(x) = G(F|R^{-1}(x))$ for all $x \in \mathbf{V}$.*

1.4.12. Two sets are said to be *equipotent* (or *of the same cardinality*) if there is a one-to-one mapping of one onto the other. The ordinal which is equipotent to no preceding ordinal is termed a *cardinal*, any natural number being a cardinal. The cardinal that is not a natural number is called *infinite*. Therefore, ω is the least infinite cardinal. For any ordinal α by the symbol ω_α we shall denote the infinite cardinal for which the ordered set of all infinite cardinals less than ω_α is similar to α. If such a cardinal exists, then it is unique.

1.4.13. Theorem (the principle of measuring cardinality). *The following statements are valid:*

(1) *infinite cardinals form a well-ordered proper class;*

(2) *for any ordinal α there is a cardinal ω_α, in which case the mapping $\alpha \to \omega_\alpha$ is a similarity of the class of ordinals and that of infinite cardinals;*

(3) *there is a mapping $|\cdot|$ from the universal class $\mathbf{V}$ on the class of all cardinals such that the sets x and $|x|$ are equipotent for any $x \in \mathbf{V}$.*

$\lhd$ The proof can, for instance, be found in [180]. $\rhd$

The cardinal $|x|$ is called *the cardinality* or *cardinal number of the set x*. Hence, any set is equipotent to a unique cardinal, namely, to its cardinal number. The set x is *countable* provided $|x| = \omega_0 := \omega$, and it is *at most countable* if $|x| \leq \omega_0$.

1.4.14. For an arbitrary ordinal α by the symbol $\mathbf{2}^{\omega_\alpha}$ we shall denote the cardinality of the set $\mathcal{P}(\omega_\alpha)$, i.e., $\mathbf{2}^{\omega_\alpha} = |\mathcal{P}(\omega_\alpha)|$. Such a denotation is justified by the fact that $\mathbf{2}^x$ and $\mathcal{P}(x)$ are equipotent for any x where $\mathbf{2}^x$ is the class of all mappings from x to $\mathbf{2}$. A theorem proved by Cantor states that $|x| < |\mathbf{2}^x|$ whatever the set x is. In particular, $\omega_\alpha < 2^{\omega_\alpha}$ for any ordinal α. In this case, by theorem **1.4.13**, we get $\omega_{\alpha+1} \le 2^{\omega_\alpha}$. Whether there are intermediate powers between $\omega_{\alpha+1}$ and 2^{ω_α} or not, i.e., whether the equality $\omega_{\alpha+1} = 2^{\omega_\alpha}$ holds or not, that is the contents of the generalized problem of the continuum. For $\alpha = 0$ this is a classical *problem of the continuum.* The *continuum hypothesis,* CH (generalized continuum hypothesis, GCH) is the equality $\omega_1 = 2^{\omega}$ (or, respectively, the equality $\omega_{\alpha+1} = 2^{\omega_\alpha}$).

1.4.15. In the class $\mathrm{On} \times \mathrm{On}$ let us introduce an order which will be called *canonical.* Let us consider $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathrm{On}$. Let us also assume that for $(\alpha_1,\alpha_2) \le (\beta_1,\beta_2)$ one of the following conditions is fulfilled:

(1) $\alpha_1 = \beta_1$ and $\alpha_2 = \beta_2$;

(2) $\sup\{\alpha_2,\alpha_2\} < \sup\{\beta_1,\beta_2\}$;

(3) $\sup\{\alpha_2,\alpha_2\} = \sup\{\beta_1,\beta_2\}$ and $\alpha_1 < \beta_1$;

(4) $\sup\{\alpha_2,\alpha_2\} = \sup\{\beta_1,\beta_2\}$ and $\alpha_1 = \beta_1$ and $\alpha_2 < \beta_2$.

Therefore, the pairs (α,β) are compared relative to $\sup\{\alpha,\beta\}$, while the set of ordered pairs (α,β) with the same $\sup\{\alpha,\beta\}$ has the lexicographic order. We can easily prove that the class $\mathrm{On} \times \mathrm{On}$ with the canonical order is a well-ordered class. In an analogous way it can be checked that the class $\mathrm{On} \times \mathrm{On} \times \mathrm{On}$ is canonically well-ordered, etc..

1.4.16. Remarks

(1) The idea of transfinite iteration is one of the most fundamental and original discoveries made by G.Cantor. Based on it, he created a powerful method of the qualitative analysis of the notion of infinity and penetrated into the essence of the infinite. The notion of infinity can be traced in religious and philosophical doctrines since the ancient times. The whole totality of the ideas on the infinite was, however, preferably a humanitarian subject before G.Cantor, who made the very notion of the infinite a subject for mathematical investigation.

(2) The problem of the continuum stems from G.Cantor and was the first to be formulated in the famous report by D.Hilbert. Remaining unsolved for decades, this problem gave rise to in-depth studies of the fundamentals of set theory. In 1939 K.Gödel established consistency of the generalized continuum hypothesis with ZFC [60]. In 1963 P.J.Cohen proved that the negation of the generalized continuum hypothesis is also consistent with ZFC. Both these results brought about new ideas, methods and problems.

(3) Following G.Cantor, an ordinal is the order-type of a certain well-ordered set x; i.e., the class of all ordered sets similar to x. All the order-types, however, except for that of the empty set, are proper classes. This peculiarity makes it impossible to develop the theory of order-types (within NBG) since it is impossible to consider the class of order-types. The definition of ordinal **1.4.2**, by J.von Neumann, singles a canonical representative out of every order-type.

(4) In this section we have given only the basic facts on ordinals, a more detailed information can be found in [112, 180].

1.5. Hierarchies of Sets

The recursive definitions based on theorem **1.4.9** or its modifications give, in particular, decreasingly (or increasingly) nested transfinite sequences of sets called cumulative hierarchies. Of a special interest for us are the hierarchies resulting in the models of set theory.

1.5.1. Let us consider a certain set x_0 and two single-valued classes Q and R. Starting with them, let us construct a new single-valued class G. Let us first set $G(0) := x_0$. Then, if x is a function and $\mathrm{dom}(x) = \alpha + 1$ for a certain $\alpha \in \mathrm{On}$, then $G(x) := Q(x(\alpha))$, while if $\mathrm{dom}(\alpha) = \alpha$ is a limiting ordinal, then in order to obtain $G(x)$ let us first accumulate the set of the values of $x(\beta)$ for $\beta < \alpha$, and then apply R to the obtained set, i.e., $G(x) := R(\bigcup \mathrm{im}(x))$. In all the remaining cases we assume $G(x) = 0$. By theorem **1.4.9** on transfinite recursion, there exists a single-valued class F satisfying the conditions:

$$F(0) = x_0,$$
$$F(\alpha + 1) = Q(F(\alpha)),$$
$$F(\alpha) = R\left(\bigcup_{\beta<\alpha} F(\beta)\right) \quad (\alpha \in K_{\mathrm{II}}).$$

Such a function F is often called a cumulative hierarchy. The union of the elements of the class $\mathrm{im}(F)$, i.e., the class

$$\bigcup_{\alpha \in \mathrm{On}} F(\alpha) := \bigcup \mathrm{im}(F),$$

is often termed the *limit of the cumulative hierarchy* $(F(\alpha))_{\alpha \in \mathrm{On}}$.

1.5.2. Further on we shall be interested only in a special case when x_0 is the empty set, R is the identity mapping of the universal class $\mathbf{U}$, and Q is a class-function, $\mathrm{dom}(Q) = \mathbf{U}$. In this case cumulative hierarchies are constructed inductively, starting with the empty set, by successively applying the operation Q. Varying Q, we get different cumulative hierarchies.

The least ordinal $\mathbf{a}$ for which $x \in F(\mathbf{a}+1)$ is called the ***(ordinal) rank of the set x relative to the hierarchy*** $(F(\alpha))_{\alpha \in \mathrm{On}}$ and is denoted by $\mathrm{rank}x$. This definition is obviously determined by theorem **1.3.14**, according to which there we can find the class 'rank' obeying the condition

$$(\forall x)(\forall y)((x,y) \in \mathrm{rank} \leftrightarrow \varphi(x,y,F,\mathrm{On})),$$

where φ is a predicative formula

$$(\exists \alpha \in \mathrm{On})(y = \alpha \wedge x \in F(\alpha + 1) \wedge (\forall \beta \in \mathrm{On})(x \in F(\beta+1) \rightarrow \alpha \leq \beta)).$$

In this case $\mathrm{Un}(\mathrm{rank})$, $\mathrm{dom}(\mathrm{rank}) = \bigcup \mathrm{im} F$ and $\mathrm{im}(\mathrm{rank}) \subset \mathrm{On}$ hold, i.e., rank is a function from $\bigcup \mathrm{im}(F)$ to On. The notation of the rank will not include F since we know which hierarchy is considered.

1.5.3. As the simplest, let us consider the case when $(x_0 = 0, R = I_{\mathrm{U}}) Q := \mathcal{P}_{tr}$, where $\mathcal{P}_{tr}$ put into correspondence to any $x \in \mathbf{U}$ a class $\mathcal{P}_{tr}(x)$ of all transitive subsets of the set x. As long as a transitive subset of an ordinal is an ordinal, then $Q(\alpha) = \alpha \cup \{\alpha\} = \alpha + 1$ and $F(\alpha + 1) = \alpha + 1$ for every ordinal α. If α is determined, then

$$F(\alpha) = \bigcup_{\beta<\alpha} F(\beta) = \bigcup_{\beta+1<\alpha} F(\beta + 1) = \bigcup_{\beta+1<\alpha} \beta + 1 = \alpha.$$

Therefore, the limit of the increasinly nested cumulative hierarchy is the class of ordinals On.

1.5.4. If the role of Q is played by the operation of taking powersets $\mathcal{P}$, then we obtain a familiar (see **1.3.1**) cumulative hierarchy (we put $x_0 = 0, R = I_{\mathrm{U}}$):

$$V_0 := 0,$$
$$V_{\alpha+1} := \mathcal{P}(V_\alpha)\ (\alpha \in \mathrm{On}),$$
$$V_\alpha := \bigcup_{\beta<\alpha} V_\beta\ (\alpha \in K_{\mathrm{II}}).$$

The class $\mathbf{V} := \bigcup_{\alpha \in \mathrm{On}} V_\alpha$ is the von Neumann universe. It should be recalled that its lower levels have the form $V_1 = \mathcal{P}(0) = \{0\} = 1$, $V_2 = \mathcal{P}(1) = \{0, \{0\}\} = 2$, $V_3 = \mathcal{P}(V_2) = = \{0, \{0\}, \{\{0\}\}, \{0, \{0\}\}\} \neq 3$, etc..

1.5.5. *The following statements are valid:*

(1) V_α *is a transitive set for every* $\alpha \in \mathrm{On}$;

(2) $V_\beta \in V_\alpha$ *and* $V_\beta \subset V_\alpha$ *for any* $\alpha, \beta \in \mathrm{On}, \beta < \alpha$;

(3) *if* $x \in y \in \mathbf{N}$, *then* $\mathrm{rank}(x) < \mathrm{rank}(y)$;

(4) *the class of ordinals* On *is contained in the universe* $\mathbf{V}$;

(5) $\mathrm{rank}(\alpha) = \alpha$ *for* $\alpha \in \mathrm{On}$;

(6) *if* x *is a set and* $x \subset \mathbf{V}$, *then* $x \in \mathbf{V}$.

◁ (1) Let us proceed by transfinite induction. For $\alpha = 0$ the class $V_0 = 0$ is a transitive set. Assume that the set V_α is transitive. As $V_{\alpha+1} = \mathcal{P}(V_\alpha)$, we set that $V_{\alpha+1}$ is a set and for any x and y it follows from $x \in y \in V_{\alpha+1}$ that $y \subset V_\alpha$ and $x \in V_\alpha$. By the induction hypothesis, either $x \subset V_\alpha$ or $x \in V_{\alpha+1}$, and, hence, $y \in V_{\alpha+1}$. If $\alpha \in K_{\mathrm{II}}$ and V_β is a transitive set for all $\beta < \alpha$, then for any $x \in V_\alpha$ we get

$$(\exists \beta < \alpha)(x \in V_\beta) \to (\exists \beta < \alpha)(x \subset V_\beta) \to x \subset V_\alpha .$$

Besides, V_α is a set as the union of a set of sets.

(2) We have established in (1) that V_α is transitive. Therefore, it suffices to demonstrate that $V_\beta \in V_\alpha$ $(\beta < \alpha)$. Let us carry out transitive induction on α. For $\alpha = 1$ we have nothing to prove. Let $\alpha > 1$ and $V_\beta \in V_\alpha$ for all $\beta < \alpha$. The inequality $\beta < \alpha + 1$ holds only when $\alpha = \beta$ or $\beta < \alpha$. If $\alpha = \beta$, then

$$V_\beta = V_\alpha \in \mathcal{P}(V_\alpha) = V_{\alpha+1} .$$

If $\beta < \alpha$, then, by the induction hypothesis we have $V_\beta \in V_\alpha$, while by (1) we get $V_\alpha \subset V_{\alpha+1}$ and, hence, $V_\beta \in V_{\alpha+1}$. Now we have to remark that for $\beta < \alpha$ with the limiting ordinal $\alpha \in K_{\mathrm{II}}$ we always have $V_\beta \in V_\alpha$ since

$$V_\beta \in V_{\beta+1} \subset \bigcup_{\gamma<\alpha} V_\gamma = V_\alpha .$$

(3) It is obvious that $\alpha = \mathrm{rank}(x)$ iff $x \in V_{\alpha+1}$ and $x \notin V_\alpha$. Hence, if $x \in y$, then $y \not\subset V_\alpha$ and, therefore, $y \not\subset V_{\alpha+1}$. By definition, $\mathrm{rank}(y) > \alpha$.

(4), (5) Let us again make use of transitive induction. For $\alpha = 0$ we have $0 \in V_0 \subset \mathbf{V}$ and $\mathrm{rank}(0) = 0$, as $0 \notin V_0$. Put $\alpha \in \mathbf{V}$ and $\mathrm{rank}(\alpha) = \alpha$. Then $\alpha + 1 = \mathbf{a} \cup \{\alpha\} \subset V_{\alpha+1}$, or $\alpha + 1 \in \mathcal{P}(V_{\alpha+1}) = V_{\alpha+2}$. On the other hand, if $\alpha + 1 \in V_{\alpha+1}$, then $\alpha \cup \{\alpha\} \subset V_\alpha$ and we come to a contradiction $\alpha \in V_\alpha$. Therefore, $\alpha + 1 \notin V_{\alpha+1}$ and, hence, $(\alpha + 1) = \alpha + 1$. Let us assume that $\alpha \in K_{\mathrm{II}}$, and for all $\beta < \alpha$ we have $\beta \in \mathbf{V}$ and $\mathrm{rank}(\beta) = \beta$. In this case

$$\alpha = \{\beta \in \mathrm{On} : \beta < \alpha\} \subset \bigcup_{\beta<\alpha} V_{\beta+1} \subset V_\alpha ;$$

whence we deduce $\alpha \in V_{\alpha+1}$. Besides, the relation $\alpha \in V_\alpha$ implies that $\alpha \in V_\beta$ for a certain $\beta < \alpha$. Applying (3), and the induction hypothesis, we immediately arrive at a contradiction:

$$\beta = \mathrm{rank}(\beta) < \mathrm{rank}(\alpha) < \beta .$$

(6) Let us put $\alpha = \sup\{\mathrm{rank}(y) : y \in x\}$. Obviously, $x \subset V_{\alpha+1}$ and $x \subset V_{\alpha+2} \subset \mathbf{V}$. ▷

1.5.6. Theorem. *The axiom of foundation,* NBG_{14}, *is equivalent to the statement* $\mathbf{U} = \mathbf{V}$, *i.e., to the coincidence of the universal class with the von Neumann universe.*

◁ Let $\mathbf{U} = \mathbf{V}$ and let us choose a nonempty class X. There is an element $x \in X$ with the least rank α, i.e., $\mathrm{rank}(x) = \alpha$ and $\mathrm{rank}(x) \le \mathrm{rank}(y)$ for all $y \in X$. If $u \in x \cap X$, then, by virtue of **1.5.5** (3), $\mathrm{rank}(u) < \alpha = \mathrm{rank}(x)$, which contradicts the definition of α. Hence, $x \cap X = 0$.

Let us now prove that $\mathbf{U} \neq \mathbf{V}$ contradicts the axiom of foundation. Indeed, applying the axiom to a nonempty class $\mathbf{U} - \mathbf{V}$ we find a set $y \in \mathbf{U} - \mathbf{V}$ for which $y \cap (\mathbf{U} - \mathbf{V}) = 0$. The last relation yields $y \subset \mathbf{V}$, while from **1.5.5** (6) we deduce $y \in \mathbf{V}$, which contradicts the choice of y. ▷

1.5.7. Theorem. *The following statements are valid:*

(1) ($\in$-induction): *if the class X is such that for any set x it follows from $x \subset X$ that $x \in X$, then $X = \mathbf{V}$;*

(2) ($\in$-recursion): *if G is a single-valued class, then there is a unique function F*

determined over **V**, *for which* $F(x) = G(\mathrm{im}(F|x))$ *for* $x \in \mathbf{V}$;

(3) (induction on rank): *if for the class* X *and every set* x *it follows from* $\{y \in \mathbf{V}: \mathrm{rank}(y) < \mathrm{rank}(x)\} \subset X$ *that* $x \in \mathbf{V}$, *then* $\mathbf{V} = X$..

◁ As has been established in **1.5.6**, the universe **V** coincides with the class of all sets **U**. Therefore, the required statements result directly from **1.1.11** under the condition that the relation $\in = \{(x,y) \in \mathbf{V}^2: x \in y\}$ and $R := \{(x,y) \in \mathbf{V}^2: \mathrm{rank}(x) < \mathrm{rank}(y)\}$ are well-founded. For $\in$, the necessary property results from the axiom of foundation (see **1.4.10**). Let us now choose such a sequence $(x_n)_{n\in\omega}$ of the sets $x_n \in \mathbf{V}$, that $x_{n+1} \in R(x_n)$ $(n \in \omega)$. Then the sequence of the ordinals $\alpha_n := \mathrm{rank}(x_n)$ obeys the condition $\alpha_{n+1} < \alpha_n$ $(n \in \omega)$ (see **1.5.5** (3)). This contradicts the fact that the class On is well-ordered and, hence R is well-founded. ▷

1.5.8. Let $\sim$ be an equivalence on a class W. The union of all W elements which are equivalent to a given $x \in W$ forms, generally speaking, a proper class, which hampers the formation of a factor-class. This difficulty can be overcome by using the ordinal rank.

Frege-Russel-Scott theorem. *There is a function* $F: W \to \mathbf{V}$ *such that for all* $x, y \in W$ *we have*

$$F(x) = F(y) \leftrightarrow x \sim y.$$

◁ By theorem **1.3.14**, there is a class F such that for all $x, y \in W$, we obtain

$$(x,y) \in F \leftrightarrow \varphi(x,y,W,\sim,\mathrm{rank}),$$

where the predicative formula φ has the form

$$(\forall z)(z \in y \leftrightarrow x \in W \wedge x \sim z \wedge (\forall u)(x \sim u \to \mathrm{rank}(z) \le \mathrm{rank}(u))).$$

Therefore, F is a function, and $y = F(x)$ is the class of sets z equivalent to x and having the least ordinal rank among such sets. If $\alpha = \mathrm{rank}(x)$, then $F(x) \subset W \cap V_{\alpha+1}$ and, hence, $F(x)$ is a set. Besides, $\mathrm{dom}(F) = W$, and for any $x, y \in W$ we have $x \sim y \leftrightarrow F(x) = F(y)$. Indeed, if $F(x) = F(y)$, then there is a $\omega \in W$, for which $x \sim \omega$ and $y \sim \omega$, i.e., $x \sim y$. The reverse implication is obvious. ▷

If follows from the axiom of domain NBG_{10}, and **1.3.13** **(1)** that there is a class $\mathrm{im} F := \{F(x): x \in W\}$. Let us call this class the *factor-class of the class* W by the equivalence $\sim$, i.e., $W/\sim := \mathrm{im} F$. In this case we say that F *is the canonical factor-homomorphism* or *the canonical projection.*

1.5.9. Let B be a fixed set containing more than one element. Let us put $Q := \mathcal{P}^{(B)} : x \to B^x$ $(x \in \mathbf{V})$, where B^x is, as usual, the set of all mappings from x to B. The cumulative hierarchy arising in this case (see **1.5.1**, where $x_0 = 0, R = I_{\mathbf{V}}$) will be denoted by the symbol $(V_\alpha^{(B)})_{\alpha \in \mathrm{On}}$. It is obvious that the *B-valued universe*

$$\mathbf{V}^{(B)} := \bigcup_{\alpha \in \mathrm{On}} V_\alpha^{(B)}$$

is a subclass of the class $\mathbf{V}$ and consists of B-valued functions determined on the sets of B-valued functions. The standard interpretation of the symbol $\in$ in $\mathbf{V}^{(B)}$ yields nothing of interest, since for the B-valued functions u,v the relation $u \in v$ is valid only in trivial cases. The hierarchies (V_α) and $(V_\alpha^{(B)})$ are, however, essentially different and this peculiarity can give grounds for nonstandard interpretations of set theory in the universe $\mathbf{V}^{(B)}$, which will be discussed in more detail in Chapter **2** below.

1.5.10. For the sake of completeness let us consider one more cumulative hierarchy. The following operations with sets are called *Gödel operations* (they are eight all in all): the formation of an unordered pair, set-theoretic difference, Cartesian product; the (2,3,1)-, (3,2,1)- and (1,3,2)-conjugations (see **1.3.10**), as well as $X \to X^2 \cap \in$ and $X \to \mathrm{dom}(X)$. For any set (sets) X the closure $\mathrm{cl}_G(X)$ is the least set containing X and closed relative to the Gödel operations. Let us now set $Q(x) := \mathcal{P}(x) \cap \mathrm{cl}_G(x \cup \{x\})$. The hierarchy resulting in this case is termed the *constructible hierarchy* and denoted by $(L_\alpha)_{\alpha \in \mathrm{On}}$. The constructible universe is a class $\mathbf{L}$: $\bigcup_{\alpha \in \mathrm{On}} L_\alpha$; the elements of $\mathbf{L}$ being constructible sets (for details see [96, 184]).

1.5.11. Remarks

(**1**) The cumulative hierarchy $(V_\alpha)_{\alpha \in \mathrm{On}}$ was first considered by J.von Neumann. The relativization of the axiom of foundation to the class $\mathbf{V}$ is provable in the theory NBG-$\{\mathrm{NBG}_{14}\}$, which implies that NBG_{14} is consistent with the rest of the axioms of NBG. Other means can be employed to show that $\neg\, \mathrm{NBG}_{14}$ is also consistent with the other NBG axioms, i.e., that NBG_{14} is an independent axiom.

(**2**) If B is a complete Heyting lattice (see **1.1.8** (8)), then the universe $\mathbf{V}^{(B)}$ can be transformed into a model of intuitionistic set theory by using the structure of B and the hierarchy $(V_\alpha^{(B)})$. In particular, if B is a complete Boolean algebra, then there arises a

Boolean-valued model of set theory (for more details see **2.1.10** (3)).

(**3**) If $B := [0,1]$ is the interval of a real line from zero to unity, then the class $\mathbf{V}^{(B)}$ is naturally called the *universe of Zadeh-fuzzy sets* [280]. This universe can serve a model for a certain set theory with an appropriate many-valued logic, as well as constitute some basis for studying indistinct sets.

(**4**) The constructible universe $\mathbf{L}$ is the least transitive model of ZFC containing all ordinals. The class $\mathbf{L}$ satisfies the axiom of choice and the generalized continuum hypothesis. Therefore, both AC and GCH are consistent with ZF. The statement that all sets are constructible is termed the *axiom of constructibility* and is presented as $\mathbf{V} = \mathbf{L}$. The relativization of the formula $\mathbf{V} = \mathbf{L}$ to the class $\mathbf{L}$ is provable in ZF. Hence, $\mathbf{V} = \mathbf{L}$ is consistent with ZF. All these results, as well as the definition of constructible sets belong to K.Gödel [61] (see also [96, 184]). The corresponding statements on the consistency of the axiom of choice and GCH are also valid for NBG (see [96, 29, 180, 184]).

(**5**) It has been proved [255] that if B is a quantum logic (see **1.1.8** (5)), then the universe $\mathbf{V}^{(B)}$ serves as a model for a certain quantum set theory in the sense analogous to that discussed below in **2.4**. Traeting quantum theories as logic systems, constructing quantum set theory and developing a corresponding quantum mathematics, all these are interesting and actual problems, and a lot has been done in this direction. Adequate mathematical means and correct reference points can, possibly, be traced in the theory of von Neumann algebras and in various 'noncommutative' trends that has arisen from it (noncommutative probability theory, noncommutative integration, etc.).

CHAPTER 2

BOOLEAN-VALUED UNIVERSES

Various nonstandard methods of analysis are unified by studying special quite unconventional models of set theory. In particular, the apparatus of Boolean-valued analysis is based on the properties of a certain cumulative hierarchy $\mathbf{V}^{(B)}$, with its every succeeding layer composed of all possible functions departing from the preceding layers and arriving at a complete Boolean algebra B chosen beforehand. The principal topics of the present section is constructing such a hierarchy, i.e., the Boolean-valued universe $\mathbf{V}^{(B)}$, as well as studying the general properties of $\mathbf{V}^{(B)}$.

Special attention will be paid to introducing truth-values for formulas and exactly explaning the sense in which $\mathbf{V}^{(B)}$ can be viewed as a model of set theory. Presented in detail are the basic techniques that lay grounds for Boolean-valued analysis such as the principles of transfer, mixing and maximum. Considerations of logical rigour and independence made us pay special attention to constructing a separated universe and interpreting NBG in $\mathbf{V}^{(B)}$. The reader interested only in applications to analysis can, in the first reading, get acquainted with these more sophisticated fragments but quite briefly.

2.1 The Universe Over a Boolean Algebra

In this section a Boolean-valued universe is determined, Boolean truth-values are constructed for set-theoretical formulas and the simplest related facts are given.

2.1.1. Let us start with informal heuristic considerations which could facilitate acquaintance with some features of Boolean-valued universes and Boolean truth-values. Let $\mathbf{2} := \{\mathbf{0},\mathbf{1}\}$ be the two-element Boolean algebra (it is customary to identify all of them). Let us choose an arbitrary set $x \in \mathbf{V}$ and associate with it a certain (characteristic) function χ_x with the values in $\mathbf{2}$ determined (generally speaking, nonuniquely) by the conditions that $x \subset \mathrm{dom}(\chi_x)$ and $\chi_x(t) = \mathbf{1}$ whenever $t \in x$. Clearly, there are sound reasons to identify x with any such function χ_x. For these elements of the domain of definition $\mathrm{dom}(\chi_x)$ of the two-valued function χ_x to be interpretable as two-valued functions, we should, of course, have substituted all the elements on the level $V_\beta, \beta < \operatorname{rank}(x)$ where $\operatorname{dom}(\chi_x)$ is located with appropriate characteristic functions. If one wants to serve, in this sense, the whole world of sets, i.e., the universe $\mathbf{V}$, then one should start from the level zero $\varnothing$. By formalizing these observations, we come to the notion of the **2**-valued universe

$$\mathbf{V}^{(2)} := \{x : (\exists \alpha \in \mathrm{On})(x \in V_\alpha^{(2)})\},$$

where $\mathbf{V}^{(2)} := \varnothing, V_1^{(2)} := \{\varnothing\}, V_2^{(2)} := \{\{\varnothing\},(\{\varnothing\},\mathbf{1})\}$, etc.. In more detail, by analogy with $\mathbf{V}$, by $\in$-recursion we determine the cumulative hierarchy

$$V_\alpha^{(2)} := \{x : \mathrm{Fnc}(x) \wedge \mathrm{im}(x) \subset \mathbf{2} \wedge (\exists \beta < \alpha)(\mathrm{dom}(x) \in V_\alpha^{(2)})\}.$$

Obviously, $\mathbf{V}^{(2)}$ consists of two-valued functions, in which case we associate with every element $x \in \mathbf{V}^{(2)}$ a unique set $\bar{x} := \{y \in \mathbf{V}^{(2)} : x(y)\} = \mathbf{1}$. However, different elements of $\mathbf{V}^{(2)}$ can correspond to the same set. Therefore, let us identify the functions x and $y \in \mathbf{V}^{(2)}$, for which $\bar{x} = \bar{y}$, paying no attention to formal difficulties and restrictions which are to be met on this way. Let us choose arbitrary $x, y \in \mathbf{V}^{(2)}$. By virtue of the above identification, the equality $x = y$ is valid iff $\bar{x} = \bar{y}$. At the same time, it is natural to assume that the formula $x \in y$ is true only if $x \in \bar{y}$. Let us put $[x = y] := \mathbf{1}$, $[x \in y] := \mathbf{1}$ in the case when the formulas $x = y$, $x \in y$ are valid, and let $[x = y] := \mathbf{0}$, $[x \in y] := \mathbf{0}$ in the opposite case. In this case the following presentations are valid:

$$[x \in y] = \bigvee_{t \in \mathrm{dom}(y)} y(t) \wedge [t \in x],$$
$$[x = y] = \bigwedge_{t \in \mathrm{dom}(x)} x(t) \Rightarrow [t \in y] \wedge \bigwedge_{t \in \mathrm{dom}(y)} y(t) \Rightarrow [t \in x].$$

It would be expedient to compare these relations with the tautologies of set theory

$$u \in v \leftrightarrow (\exists \omega)(\omega \in v \wedge \omega = u),$$
$$u = v \leftrightarrow (\forall \omega)(\omega \in u \rightarrow \omega \in v) \wedge (\omega \in v \rightarrow \omega \in u).$$

2.1.2. Let B be a fixed complete algebra which is an element of the von Neumann universe $\mathbf{V}$. The Boolean-valued universe $\mathbf{V}^{(B)}$ arises as the limit of the cumulative hierarchy (**1.5.1**) provided $x_0 := 0$, $R := I_{\mathbf{V}}$, while Q is determined by the formula

$$y \in Q(x) \leftrightarrow \mathrm{Fnc}(y) \wedge \mathrm{dom}(y) \subset x \wedge \mathrm{im}(y) \subset B.$$

Therefore, the hierarchy $(V_\alpha^{(B)})_{\alpha \in \mathrm{On}}$ has the form

$$V_0^{(B)} := 0,$$
$$V_{\alpha+1}^{(B)} := \{y : \mathrm{Fnc}(y) \wedge \mathrm{dom}(y) \subset V_\alpha^{(B)} \wedge \mathrm{im}(y) \subset B\},$$
$$V_\alpha^{(B)} := \cup\{V_\alpha^{(B)} : \beta < \alpha\} \quad (\alpha \in K_{\mathrm{II}}).$$

Therefore, by definition we assign

$$\mathbf{V}^{(B)} := \bigcup_{\alpha \in \mathrm{On}} V_\alpha^{(B)}.$$

Since the empty set is a function with the empty domain of definition, let us write down the first and the second layers of the Boolean-valued universe: $V_1^{(B)} = \{0\}$, $V_2^{(B)} = \{0\} \cup \{(0,b) : b \in B\}$. The ordinal rank of the element $x \in \mathbf{V}^{(B)}$ will be denoted by $\rho(x)$.

2.1.3. Since the relation $y \in \mathrm{dom}(x)$ is well-founded, the following induction principle results from **1.4.11** (1) for $\mathbf{V}^{(B)}$:

$$(\forall x \in \mathbf{V}^{(B)}) \quad ((\forall y \in \mathrm{dom}(x))\ \varphi(y) \to \varphi(x)) \to (\forall x \in \mathbf{V}^{(B)})\ \varphi(x),$$

where φ is an arbitrary formula of ZFC.

2.1.4. Our nearest task is to ascribe a truth-value to every formula of ZFC, where free variables are replaced with elements of $\mathbf{V}^{(B)}$. Such a value must be an element of B and have the property that the theorems of ZFC become 'true' in $\mathbf{V}^{(B)}$, i.e., that that they get the highest truth-value, unity.

Let us, before all, introduce the truth-value for atomic formulas $x \in y$ and $x = y$. This is performed with two class-functions, $[\cdot \in \cdot]$ and $[\cdot = \cdot]$, from $\mathbf{V}^{(B)} \times \mathbf{V}^{(B)}$ to B. For arbitrary $x, y \in \mathbf{V}^{(B)}$ we will set

$$(1)\ [x \in y] := \bigvee_{z \in \mathrm{dom}(y)} y(z) \wedge [z = x],$$

$$(2)\ [x = y] := \bigwedge_{z \in \mathrm{dom}(y)} x(z) \Rightarrow [z \in x] \wedge \bigwedge_{z \in \mathrm{dom}(x)} x(z) \Rightarrow [z \in y].$$

By recursion on $(\rho(x), \rho(y))$, the above formulas determine the functions $[\cdot \in \cdot]$ and $[\cdot = \cdot]$, provided $\mathrm{On} \times \mathrm{On}$ is canonically well-ordered (see **1.4.15**). Indeed, at the level zero when $(\rho(x), \rho(y)) = (0,0)$ we get (see **1.1.1**):

$$[0 \in 0] = \vee\varnothing = \mathbf{0}_B, [0 = 0] = \wedge\varnothing = \mathbf{1}_B.$$

Besides, for $z \in \mathrm{dom}(y)$ (or $z \in \mathrm{dom}(x)$) we have $(\rho(x),\rho(z)) < (\rho(x),\rho(y))$ (and, respectively, $(\rho(z),\rho(y)) < (\rho(x),\rho(y))$).

One can choose another way and make use of transfinite recursion **1.4.9**. Namely, if for all $u,v \in V_\alpha^{(B)}$ the values of $[u \in v]$ and $[u = v]$ are determined, then for $x,y \in V_{\alpha+1}^{(B)}$ we can calculate

$$[x = y] = \bigwedge_{u \in \mathrm{dom}(x)} \left(x(u) \Rightarrow \bigvee_{v \in \mathrm{dom}(y)} y(v) \wedge [u = v] \right)$$
$$\wedge \bigwedge_{v \in \mathrm{dom}(y)} \left(y(v) \Rightarrow \bigvee_{u \in \mathrm{dom}(x)} x(u) \wedge [u = v] \right),$$

since $\mathrm{dom}(x) \subset V_\alpha^{(B)}$ and $\mathrm{dom}(y) \subset V_\alpha^{(B)}$. Now the values of $[x = z]$ for all $z \in \mathrm{dom}(y)$. Therefore, we can calculate

$$[x \in y] = \bigvee_{z \in \mathrm{dom}(y)} y(z) \wedge [z = x].$$

The case of a limiting ordinal causes no problem.

2.1.5. Let us consider the substantiation of the discussed recursive definition **2.1.4** in more detail. For $k := 1,2,3,4$ let us set

$$\pi_x^k(u,v) := \vee\{b \in B : (\exists c_1,c_2,c_3,c_4 \in B)((u,v,c_1,c_2,c_3,c_4) \in x \wedge c_k = b)\}.$$

Let π_1 and π_2 be functions putting into correspondence to every ordered hexad (u,v,c_1,c_2,c_3,c_4) the first and the second, respectively, components, u and v. Using this notation, let us describe a certain single-valued class Q. For an arbitrary $x \in \mathbf{V}$ the set $Q(x)$ consists of all possible hexads (u,v,c_1,c_2,c_3,c_4) obeying the following conditions:

$$\mathrm{Fnc}(u), \mathrm{Fnc}(v), \quad \mathrm{im}(u) \cup \mathrm{im}(v) \subset B,$$
$$\mathrm{dom}(u) \subset \pi_1``x, \quad \mathrm{dom}(v) \subset \pi_2``x;$$
$$b_1 = \bigvee_{z \in \mathrm{dom}(v)} v(z) \wedge \pi_x^3(u,z),$$
$$b_2 = \bigvee_{z \in \mathrm{dom}(u)} u(z) \wedge \pi_x^4(v,z),$$
$$b_3 = b_4 = \bigwedge_{z \in \mathrm{dom}(u)} u(z) \Rightarrow \pi_x^1(z,v) \wedge \bigwedge_{z \in \mathrm{dom}(v)} v(z) \Rightarrow \pi_x^2(u,z).$$

By **1.5.1**, there is a cumulative hierarchy $(F(\alpha))_{\alpha \in \mathrm{On}}$, for which

$$F(0) = (0,0,\mathbf{0}_B,\mathbf{0}_B,\mathbf{1}_B,\mathbf{1}_B),$$

$$F(\alpha+1)=Q(F(\alpha))\quad(\alpha\in\mathrm{On}),$$
$$F(\alpha)=\bigcup_{\beta<\alpha}F(\beta)\quad(\alpha\in K_{\mathrm{II}}).$$

The class $X:=\mathrm{im}(F)$ is obviously a function with $\mathrm{im}(X)\subset B^4$ and $\mathrm{dom}(X)\subset\mathbf{V}^{(B)}\times\mathbf{V}^{(B)}$. If $P_k\colon B^4\to B$ is the k-th projection then, according to the definition, we put

$$[\cdot\in\cdot]:=P_1\circ X,[\cdot=\cdot]:=P_3\circ X.$$

2.1.6. Let us now describe a way of considering any formula of set theory as a statement about the elements of the Boolean-valued universe. In other words, we are going to define the interpretation of set theory in $\mathbf{V}^{(B)}$ by using the functions $[\cdot\in\cdot]$, $[\cdot=\cdot]$ discussed in **2.1.4**. Let us, first of all, determine the *interpretation class* I as a class of all mappings from the set of symbols of the variables in the language of set theory into the universe $\mathbf{V}^{(B)}$. By the *interpretation of a variable* x we mean the evaluation that assigns to each $v\in I$ the element $\bar{x}(v):=v(x)$. As interpretations of the formulas $x\in y$ and $x=y$ let us choose the following functions:

$$v\to[\bar{x}(v)\in\bar{y}(v)],\quad v\to[\bar{x}(v)=\bar{y}(v)]\quad(v\in I).$$

For every formula $\varphi(x_1,\dots,x_n)$ with n free variables let us now determine the interpretation $Y\to[\varphi(\bar{x}_1(v),\dots,\bar{x}_n(v))]$by induction on the length of the formula φ, using the following rules:

$$[\varphi(x)\wedge\psi(y)]\colon v\to[\varphi(\bar{x}(v))]\wedge[\psi(\bar{y}(v))],$$
$$[\varphi(x)\vee\psi(y)]\colon v\to[\varphi(\bar{x}(v))]\vee[\psi(\bar{y}(v))],$$
$$[\neg\varphi(x)]\colon v\to[\varphi(\bar{x}(v))]^*,$$
$$[\varphi(x)\wedge\psi(y)]\colon v\to[\varphi(\bar{x}(v))]\Rightarrow[\psi(\bar{y}(v))],$$
$$[(\forall t)\,\varphi(t,x)]\colon v\to\wedge\{[\varphi(\bar{t},(v'),\bar{x}(v'))]\colon v'\in I_v(x)\},$$
$$[(\exists t)\,\varphi(t,x)]\colon v\to\vee\{[\varphi(\bar{t},(v'),\bar{x}(v'))]\colon v'\in I_v(x)\},$$

where $x:=(x_1,\dots,x_n)$, $y:=(y_1,\dots,y_m)$, $\bar{x}(v):=(\bar{x}_1(v),\dots,\bar{x}_n(v))$, $\bar{y}(v):=(\bar{y}_1(v),\dots,\bar{y}_m(v))$, $I_v(x):=\{v'\in I\colon v(x)=v'(x)\}$, and all free variables of the formulas φ and ψ are contained within $t,x_1,\dots,x_n$ and $t,y_1,\dots,y_m$, respectively. It should be remarked that $[\varphi(\bar{x}(v))]$ depends only on the values $\bar{x}_k(v)=v(x_k)$ $((k:=1,\dots,n)$; and, therefore, we shall write $[\varphi(u_1,\dots,u_n)]$ instead of $[\varphi(\bar{x}(v))]=[\varphi(\bar{x}_1(v),\dots,\varphi(\bar{x}_n(v))]$ provided $u_k:=\bar{x}_k(v)\in\mathbf{V}^{(B)}$ $(k:=1,\dots,n)$. If $\varphi:=\varphi(x_1,\dots,x_n)$ is a formula and $u_1,\dots,u_k\in\mathbf{V}^{(B)}$ then , by definition, we put

$$\mathbf{V}^{(B)}\models\varphi(u_1,\dots,u_n)\leftrightarrow[\varphi(u_1,\dots,u_n)]=1_B.$$

In this case we say that φ *is true inside* $\mathbf{V}^{(B)}$ *for the given values* $u_1,\dots,u_n$ *of the variables* $x_1,\dots,x_n$, or simply that $\varphi(u_1,\dots,u_n)$ *is valid in* $\mathbf{V}^{(B)}$.

2.1.7. The introduced notion of interpretation makes it possible to judge the elements in $\mathbf{V}^{(B)}$. More convenient for this purpose proves, however, to be a somewhat different language obtained by supplementing the alphabet of the language of set theory with one constant for every element in $\mathbf{V}^{(B)}$. In this case, as usual, the elements of $\mathbf{V}^{(B)}$ are identified with the corresponding symbols of the constants. The formulas and sentencess of the new language will be called *B-formulas* and *B-sentences*. Then every B-formula (B-expression) is obtained from a certain formula of set theory by inserting values of $\mathbf{V}^{(B)}$ in place of some (respectively, all) free variables. Let us now see in what way are the definitions of Boolean truth-values from **2.1.6** simplified. Namely, the Boolean estimate for any B-sentence can be obtained by letting

$$[\sigma \wedge \tau] := [\sigma] \wedge [\tau],$$
$$[\sigma \vee \tau] := [\sigma] \vee [\tau],$$
$$[\neg \sigma] := [\sigma]^*,$$
$$[\sigma \rightarrow \tau] := [\sigma] \Rightarrow [\tau],$$
$$[(\forall x)\varphi(x)] := \wedge\{[\varphi(u)] : u \in \mathbf{V}^{(B)}\},$$
$$[(\exists x)\varphi(x)] := \vee\{[\varphi(u)] : u \in \mathbf{V}^{(B)}\},$$

where σ and τ are B-sentences, while φ is a B-formula with one free variable x. The B-sentence σ is said to be *true in (inside)* $\mathbf{V}^{(B)}$, and we write $\mathbf{V}^{(B)} \models \sigma$ if $[\sigma] = \mathbf{1}_B$. Herefrom, unless otherwise specified, we shall use both the linguistic means of **2.1.6** and **2.1.7**. We shall also use the same letters when denoting both variables and elements of the universe $\mathbf{V}^{(B)}$. If several Boolean algebras, $B, C, \dots$ are considered simultaneously and there is a necessity to go into details, then, alongside with $[\varphi]$, we shall write $[\varphi]^B$, $[\varphi]^C$, etc..

2.1.8. Theorem. *If the formula* $\varphi(x_1,\dots,x_n)$ *is provable in predicate calculus with equality, then* $\mathbf{V}^{(B)} \models \varphi(x_1,\dots,x_n)$ *for any* $x_1,\dots,x_n \in \mathbf{V}^{(B)}$. *In particular, the following relations are valid:*

(1) $[x = x] = \mathbf{1}$;

(2) $x(y) \leq [y \in x]$ *for all* $y \in \mathrm{dom}(x)$;

(3) $[x = y] = [y = x]$;

(4) $[x = y] \wedge [y = z] \leq [x = z]$;

(5) $[x \in y] \wedge [x = z] \leq [z \in y]$;

(6) $[y \in x] \wedge [x = z] \leq [y \in z]$;

(7) $[x = y] \wedge \varphi(x) \leq \varphi(y)$ *for any formula* φ.

⊲ The axioms of predicate calculus are easily checked to be true inside $\mathbf{V}^{(B)}$, while the rules of inference preserve validity. To be more exact, if in predicate calculus the formula φ is deducible from the formulas $\varphi_1,\dots,\varphi_n$, then $[\varphi_1]\wedge\dots\wedge[\varphi_n] \leq [\varphi]$. Let us now prove validity for (1) -(7).

(1) This relation is established by induction on the well-founded relation $y \in \mathrm{dom}(x)$. Let us assume that $[y = y] = \mathbf{1}$ for all $y \in \mathrm{dom}(x)$. Then, by **2.1.4** (1), we get

$$[y \in x] = \bigvee_{t \in \mathrm{dom}(x)} x(t) \wedge [t = y] \geq x(y) \wedge [y = y] \geq x(y),$$

and, hence, according to **1.1.4** (4) we have

$$[x = x] = \bigwedge_{y \in \mathrm{dom}(x)} x(y) \Rightarrow [y \in x] = \mathbf{1}.$$

(2) Taking into account **2.1.4** (1) and what was proved in (1), for $y \in \mathrm{dom}(x)$ we get the following estimate:

$$[y \in x] \geq x(y) \wedge [y = y] = x(y).$$

(3) This relation results from the definition due to the symmetry of relation **2.1.4** (2) which gives the truth-value for equality.

Statements (4)-(6) are established by a simultaneous induction. Let $\rho(x,y,z) := (\alpha,\beta,\gamma) \in \mathrm{On}^3$ be such a permutation of the triple of ordinals $\rho(x)$, $\rho(y)$ and $\rho(z)$ that $\alpha \geq \beta \geq \gamma$. (The class On^3 is considered with the canonical well-ordering **1.4.15**.) Let us assume that $x, y, z \in \mathbf{V}^{(B)}$, and for all $u, v, \omega \in \mathbf{V}^{(B)}$ inequalities (4)-(6) hold if $\rho(u,v,\omega) < \rho(x,y,z)$. The induction step will be analysed in each case separately.

(4) Let $t \in \mathrm{dom}(x)$. Since $[x = y] \leq x(t) \Rightarrow [t \in y]$, by **1.1.4** (3), we have

$$x(t) \wedge [x = y] \leq [t \in y],$$
$$x(t) \wedge [x = y] \wedge [y = z] \leq [t \in y] \wedge [y = z].$$

Having remarked that $\rho(t,y,z) < \rho(x,y,z)$ and applying the induction hypothesis for (6), we get

$$[t \in y] \wedge [y = z] \leq [t = z],$$
$$x(t) \wedge [y = x] \wedge [y = z] \leq [t \in z].$$

Let us now again employ relation **1.1.4** (3). Then

$$[x = y] \wedge [y = z] \leq x(t) \Rightarrow [t = z],$$

and, hence,

$$[x = y] \wedge [y = z] \leq \bigwedge_{t \in \mathrm{dom}(x)} x(t) \Rightarrow [t \in z].$$

Analogously,

$$[x = y] \wedge [y = z] \leq \bigwedge_{t \in \mathrm{dom}(z)} z(t) \Rightarrow [t \in x].$$

By virtue of **2.1.4** (2) we conclude: $[x = y] \wedge [y = z] \leq [x = z]$.

(5) Let us consider $t \in \mathrm{dom}(y)$. Then $\rho(t,x,z) < \rho(x,y,z)$ and so, by the induction hypothesis for (6), we get

$$y(t) \wedge [t = x] \wedge [x = z] \leq y(t) \wedge [t = z] \leq [z \in y].$$

By **1.1.5** (2), this gives

$$[x = z] \wedge \bigvee_{t \in \mathrm{dom}(y)} y(t) \wedge [t = x] \leq [|z \in y],$$

or $[x = z] \wedge [x \in y] \leq [z \in y]$.

(6) Let again $t \in \mathrm{dom}(x)$. In this case

$$x(t) \wedge [x = z] \leq [t \in z],$$
$$[t = y] \wedge x(t) \wedge [x = z] \leq [t = y] \wedge [t \in z].$$

This time, once more, $\rho(t,y,z) < \rho(x,y,z)$ and, hence, by the induction hypothesis, for (5) and formula **1.1.5** (2), we derive

$$x(t) \wedge [x = z] \wedge [t = y] \leq [y \in z],$$
$$[x = z] \wedge \bigvee_{t \in \mathrm{dom}(x)} x(t) \wedge [t = y] \leq [y \in z].$$

Therefore, according to **2.1.4** (1), $[x = z] \wedge [y \in x] \leq [y \in z]$.

(7) This relation is proved by induction on the length of the formula with the relations established above taken into account. ▷

As a corollary to theorem **2.1.8**, let us recall the following rules for calculating the Boolean truth-values of bounded formulas.

2.1.9. *For any B-formula* φ *with a single free variable* x *and for every* $u \in \mathbf{V}^{(B)}$ *the following relations are valid:*

$$[(\exists x \in u)\ \varphi(x)] = \bigvee_{v \in \operatorname{dom}(u)} u(v) \wedge [\varphi(v)],$$
$$[(\forall x \in u)\ \varphi(x)] = \bigwedge_{v \in \operatorname{dom}(u)} u(v) \Rightarrow [\varphi(v)].$$

◁ These formulas are mutually dual and, hence, it is sufficient to prove one of them, for instance the first. By **2.1.8** (2) the following inequality holds

$$[(\exists x \in u)\ \varphi(x)] = \bigvee_{v \in \operatorname{dom}(u)} u(v) \wedge [\varphi(v)].$$

On the other hand, applying **2.1.4** (1) and **2.1.8** (7), we get

$$[(\exists x \in u)\ \varphi(x)] = \bigvee_{t \in \mathbf{V}^{(B)}} \bigvee_{v \in \operatorname{dom}(u)} u(v) \wedge [t = v] \wedge [\varphi(t)]$$
$$\leq \bigvee_{v \in \operatorname{dom}(u)} u(v) \wedge [\varphi(v)].\ ▷$$

Remarks

(1) For $u_1,\dots,u_n \in \mathbf{V}^{(B)}$ and $b \in B$ for every concrete formula φ of set theory, the expression $[u_1,\dots,u_n] = b$ is again a formula of set theory. In ZFC, however, the mapping $\varphi \to [\varphi]$ is not a definable class, allowing only a metalinguistic definition.

(2) The Boolean-valued universe $\mathbf{V}^{(B)}$ is used for proving relative consistency of set-theoretic propositions according to the following schema. Let $\mathfrak{T}$ and $\mathfrak{T}'$ be extensions of ZF such that consistency of ZF implies that of $\mathfrak{T}'$. Let us assume that B can be determined in such a way that $\mathfrak{T}' \vDash$ 'B is a complete Boolean algebra' and $\mathfrak{T}' \vDash [\varphi]^B = \mathbf{1}$ for every axiom φ of the theory $\mathfrak{T}$. In this case the consistency of ZF implies that of $\mathfrak{T}$ (see [10]).

(3) Let Ω be a complete Heyting lattice (see **1.1.8** (3)). The pseudo-complement b^* of an element $b \in \Omega$ is introduced by the formula $x^* := x \Rightarrow \mathbf{0}$, where $\Rightarrow$ is the operation of

relative pseudo-complementation. An insignificant modification of the formulas **2.1.4** determines the truth-values $[\cdot \in \cdot]^{\Omega}$ and $[\cdot = \cdot]^{\Omega}$ operating from $\mathbf{V}^{(\Omega)} \times \mathbf{V}^{(\Omega)}$ in Ω. The truth in $\mathbf{V}^{(\Omega)}$ is determined in the same way as in **2.1.6**. In this case in $\mathbf{V}^{(\Omega)}$ all the formulas of the intuitionistic predicate calculus prove to be true (see [56, 73, 246, 247].

2.2. Transformations of Boolean-valued universes

Any homomorphism of a Boolean algebra B induces a certain transformation of the universe $\mathbf{V}^{(B)}$. Studying such transformations and, in particular, elucidation of the problem how Boolean truth-values of formulas are in this case transformed is the topic to be discussed in this section.

2.2.1. Let π be a homomorphism of B in a complete Boolean algebra C. By recursion on a well-founded relation $y \in \mathrm{dom}(x)$ the mapping $\pi^*: \mathbf{V}^{(B)} \to \mathbf{V}^{(C)}$ is determined by the formulas

$$\mathrm{dom}(\pi * x): \{\pi * y: y \in \mathrm{dom}(x)\},$$
$$\pi * x: v \to \vee\{\pi(x(z)): z \in \mathrm{dom}(x) \wedge \pi * z = v\}.$$

If a homomorphism π is injective, then the mapping $\pi *$ is also injective. In this case

$$\pi * x: \pi * y \to \pi(x(y)) \quad (y \in \mathrm{dom}(x)).$$

◁ Indeed, it suffices to establish that for an arbitrary ordinal α, the restriction of $\pi *$ to $V_\alpha^{(B)}$ is injective. Let us assume that this statement holds for all $\beta < \alpha$. Let $x, y \in V_\alpha^{(B)}$ be such that $\pi * x: \pi * z \to \pi(x(z))$ $(z \in \mathrm{dom}(x))$ and $\pi * y: \pi * z \to \pi(y(z))$ $(z \in \mathrm{dom}(y))$. Therefore, we come to the inequality

$$\{(\pi * z, \pi(x(z))): z \in \mathrm{dom}(x)\} = \{(\pi * u, \pi(y(u))): u \in \mathrm{dom}(y)\}.$$

Since for a certain $\beta < \alpha$ the sets $\mathrm{dom}(x)$ and $\mathrm{dom}(y)$ are contained in $V_\beta^{(B)}$; therefore, $\pi *$ is injective on either of these sets. Since π is injective, we get

$$\{(z, x(z)): z \in \mathrm{dom}(x)\} = \{(u, (y(u)): u \in \mathrm{dom}(y)\},$$

or, which is the same, $x = y$. ▷

A homomorphism $\pi: B \to C$ is called *full* if $\pi(\vee M) = \vee \pi(M)$ for every set $M \subset B$. From now on π is a full homomorphism from B to a complete Boolean algebra C.

2.2.2. Theorem. *The following statements are valid:*

(1) *if ρ is a full homomorphism of an algebra C to a complete Boolean algebra D, then $(\rho \circ \pi)^* = \rho^* \circ \pi^*$;*

(2) *if a homomorphism π is injective (respectively, surjective), then the mapping π^* is injective (respectively, surjective);*

(3) *for all x and $y \in \mathbf{V}^{(B)}$ the following equalities hold:*

$$[\![\pi^* x = \pi^* y]\!]^C = \pi([\![x = y]\!]^B),$$
$$[\![\pi^* x \in \pi^* y]\!]^C = \pi([\![x \in y]\!]^B);$$

(4) *for any $x \in \mathbf{V}^{(B)}$ and $t \in \mathbf{V}^{(C)}$ the following equality holds:*

$$[\![t \in \pi^* x]\!]^C = \bigvee_{u \in \mathbf{V}^{(B)}} \pi([\![u \in x]\!]^B) \wedge [\![t = \pi^* u]\!]^C.$$

◁ (1) Let us assume that $(\rho \circ \pi)^* y = (\rho^* \circ \pi^*) y$ for all $y \in \mathrm{dom}(x)$. Then for $u := (\rho \circ \pi)^* y$, where $y \in \mathrm{dom}(x)$, we deduce (see **1.1.5** (9)):

$$((\rho \circ \pi)^* x) u$$
$$= \vee \{(\rho \circ \pi)(x(z)) : z \in \mathrm{dom}(x) \wedge (\rho^* \circ \pi^*) z = (\rho^* \circ \pi^*) y\}$$
$$= \vee \{\rho(\vee \{\pi(x(z)) : z \in \mathrm{dom}(x) \wedge \pi^* z = v\}) : v \in \mathrm{dom}(\pi^* x) \wedge \rho^* v$$
$$= (\rho^* \circ \pi^*) y\} = \vee \{\rho((\pi^* x)(v)) : v \in \mathrm{dom}(\pi^* x) \wedge \rho^* v = \rho^* (\pi^* y)\}$$
$$= (\rho^* (\pi^* x)) (\rho^* (\pi^* y)) = ((\rho^* \circ \pi^*) x) u.$$

Therefore, $(\rho \circ \pi)^* x = \rho^* (\pi^* x)$, and the required result follows from **2.1.3**.

(2) The case of an injective π has been discussed in **2.2.1**. Let us assume that π is a surjective mapping. In this case there is a principal ideal B_0 of a Boolean algebra B, and an isomorphism $\rho : C \xrightarrow{\text{on}} B_0$, for which ρ^{-1} coincides with the restriction π_0 of the homomorphism π to B_0. If $z \in \mathbf{V}^{(C)}$, then, according to (1), $x = I_C^* x = (\pi_0 \circ \rho)^* x = \pi_0^* (\rho^* x) \in \mathrm{im}(\pi_0^*)$. Hence, π_0^* maps $\mathbf{V}^{(B_0)}$ on $\mathbf{V}^{(C)}$. Now we have to remark that $\mathbf{V}^{(B_0)} \subset \mathbf{V}^{(B)}$, and the restriction of π^* to $\mathbf{V}^{(B_0)}$ coincides with π_0^*.

(3) The proof is carried out by induction on $(\rho(x),\rho(y))$ when the class $\mathrm{On} \times \mathrm{On}$ is canonically well-ordered (see **1.4.15**). Let us assume that the required formulas are fulfilled for any $u,v \in \mathbf{V}^{(B)}$ for $(\rho(u),\rho(v)) < (\rho(x),\rho(y))$. If $z \in \mathrm{dom}(x)$ or $z \in \mathrm{dom}(y)$, then, obviously, $\max\{(\rho(z),\rho(x)),(\rho(z),\rho(y))\} < (\rho(x),\rho(y))$. Hence, the following expressions are valid (see **1.1.5** (2,9)):

$$
\begin{aligned}
&[\pi * x \in \pi * y] \\
&= \bigvee_{t \in \mathrm{dom}(\pi * y)} (\pi * y)(t) \wedge [t = \pi * x] = \bigvee_{z \in \mathrm{dom}(y)} (\pi * y)(\pi * z) \wedge [\pi * z = \pi * x] \\
&= \bigvee_{z \in \mathrm{dom}(y)} (\vee \{\pi(y(u)) : u \in \mathrm{dom}(y) \wedge \pi * u = \pi * z\}) \wedge [\pi * z = \pi * x] \\
&= \bigvee_{z \in \mathrm{dom}(y)} \vee \{\pi(y(u)) \wedge [\pi * z = \pi * x] : u \in \mathrm{dom}(y) \wedge \pi * u = \pi * z\} \\
&= \bigvee_{u \in \mathrm{dom}(y)} \pi(y(u)) \wedge \pi([u = x]) = \pi\Big(\bigvee_{u \in \mathrm{dom}(y)} y(u) \wedge [u = x]\Big) \\
&= \pi([x \in y]).
\end{aligned}
$$

Analogous calculations are also used for the Boolean truth-values of equality (by successively applying **2.1.4** (2), **2.2.1**, **1.1.5** (10), and **2.1.4** (2):

$$
\begin{aligned}
&[\pi * x = \pi * y] \\
&= \bigvee_{t \in \mathrm{dom}(\pi * y)} (\pi * y)(t) \Rightarrow [t = \pi * x] \wedge \bigvee_{z \in \mathrm{dom}(\pi * x)} (\pi * x)(t) \Rightarrow [t \in \pi * y] \\
&= \bigwedge_{z \in \mathrm{dom}(y)} (\pi * y)(\pi * z) \Rightarrow [\pi * z \in \pi * x] \\
&\wedge \bigwedge_{z \in \mathrm{dom}(x)} (\pi * x)(\pi * z) \Rightarrow [\pi * z \in \pi * y] \\
&= \bigwedge_{z \in \mathrm{dom}(y)} \wedge \{\pi(y(u)) \Rightarrow \pi([u \in x]) : u \in \mathrm{dom}(y) \wedge \pi * u = \pi * z\} \\
&\wedge \bigwedge_{z \in \mathrm{dom}(x)} \wedge \{\pi(x(u)) \Rightarrow \pi([u \in y]) : u \in \mathrm{dom}(x) \wedge \pi * u = \pi * z\} \\
&= \bigwedge_{u \in \mathrm{dom}(x)} \pi(x(u) \Rightarrow [u \in y]) \wedge \bigwedge_{u \in \mathrm{dom}(y)} \pi(y(u) \Rightarrow [u \in x]) = \pi([x = y]).
\end{aligned}
$$

(4) By virtue of (3) and **2.1.8** (4), the following estimates are fulfilled for $x \in \mathbf{V}^{(B)}$ and $t \in \mathbf{V}^{(C)}$:

$$
\begin{aligned}
&[t \in \pi * x] \\
&= \bigvee_{s \in \mathrm{dom}(\pi * x)} (\pi * x)(s) \wedge [s = t] = \bigvee_{u \in \mathrm{dom}(x)} (\pi * x)(\pi * u) \wedge [\pi * u = t] \\
&\leq \bigvee_{u \in \mathbf{V}^{(B)}} \pi([u = x]) \wedge [\pi * u = t] \\
&= \bigvee_{u \in \mathbf{V}^{(B)}} [\pi * u = \pi * x] \wedge [\pi * u = t] \leq [t \in \pi * x]. \ \triangleright
\end{aligned}
$$

2.2.3. Theorem. *Let $\varphi(x_1,\dots,x_n)$ be a formula of* ZFC, $u_1,\dots,u_n \in \mathbf{V}^{(B)}$, *and π be a full homomorphism from B to C. Then the following statements are valid:*

(1) *if φ is a formula of class Σ_1 and the homomorphism π is arbitrary, then*

$$\pi\big([\varphi(u_1,\dots,u_n)]^B\big) \le [\varphi(\pi^* u_1,\dots,\pi^* u_n)]^C;$$

(2) *if φ is a bounded formula and π is arbitrary, or π is an epimorphism and φ is an arbitrary formula, then*

$$\pi\big([\varphi(u_1,\dots,u_n)]^B\big) = [\varphi(\pi^* u_1,\dots,\pi^* u_n)]^C.$$

$\lhd$ For atomic formulas this statement is ensured by **2.2.2**. The general case is established by induction on the complexity of the formula φ. In this case the nontrivial induction step arises only when φ has either the form $(\exists x)\varphi_0$ or $(\forall x)\varphi_0$. It is in this case than additional suppositions on φ and π are necessary.

(1) If in the induction step we have to do with a bounded universal quantifier, i.e., if φ has the form $(\forall x \in u)\varphi_0(x,u_1,\dots,u_n)$, then (see the definitions and **1.1.5** (3,10)), the following chain of equalities holds:

$$\begin{aligned}
&[\varphi(\pi^* u, \pi^* u_1,\dots,\pi^* u_2)] \\
&= \bigwedge_{x\in \operatorname{dom}(\pi^* u)} (\pi^* u)(x) \Rightarrow [\varphi_0(x,\pi^* u_1,\dots,\pi^* u_n)] \\
&= \bigwedge_{x\in \operatorname{dom}(u)} (\pi^* u)(\pi^* x) \Rightarrow [\varphi_0(\pi^* x,\pi^* u_1,\dots,\pi^* u_n)] \\
&= \bigwedge_{x\in \operatorname{dom}(u)} \wedge\{\pi(u(z)) \Rightarrow [\varphi_0(\pi^* x, \pi^* u_1,\dots,\pi^* u_n)] : z \in \operatorname{dom}(u) \wedge \pi^* z \\
&= \pi^* x\} = \bigwedge_{x\in \operatorname{dom}(u)} \pi(u(x) \Rightarrow [\varphi_0(x,u_1,\dots,u_n)]) \\
&= \pi[(\forall x \in u)\varphi_0(x,u_1,\dots,u_n)] = \pi[\varphi(u,u_1,\dots,u_n)].
\end{aligned}$$

Then, for an unbounded existential quantifier we immediately deduce from the definition

$$\begin{aligned}
&[(\exists x)\varphi_0(x,\pi^* u_1,\dots,\pi^* u_n)] \\
&\ge \vee\{[\varphi_0(x,\pi^* u_1,\dots,\pi^* u_n)] : x \in \operatorname{im}(\pi^*)\} \\
&= \vee\{[\varphi_0(\pi^* u,\pi^* u_1,\dots,\pi^* u_n)] : u \in \mathbf{V}^{(B)}\} \\
&= \vee\{\pi([\varphi_0(u,u_1,\dots,u_n)]) : u \in \mathbf{V}^{(B)}\} = \pi([(\exists x)\varphi_0(x,u_1,\dots,u_n)]).
\end{aligned}$$

(2) It should be, first of all, remarked that if π is a surjection, then π^* is also a surjection, i.e., $\operatorname{im}(\pi^*) = \mathbf{V}^C$ (see **2.2.2** (2)). Therefore, for the formula $\varphi := (\exists x)\varphi_0$ we get

$$
\begin{aligned}
&[\![\varphi(\pi * u_1,\ldots,\pi * u_n)]\!]\\
&= \vee\{[\![\varphi_0(x,\pi * u_1,\ldots,\pi * u_n)]\!]: x\in \mathbf{V}^{(C)} = \mathrm{im}(\pi^*)\}\\
&= \vee\{[\![\varphi_0(\pi * u,\pi * u_1,\ldots,\pi * u_n)]\!]: u\in \mathbf{V}^{(B)}\}\\
&= \vee\{\pi([\![\varphi_0(u,u_1,\ldots,u_n)]\!]): u\in \mathbf{V}^{(B)}\} = \pi([\![\varphi(u_1,\ldots,u_n)]\!]).
\end{aligned}
$$

The same considerations are also applicable to the formula φ of type $(\forall x)\varphi_0(x,u_1,\ldots,u_n)$.

If the domain of the existential quantifier under consideration is bounded, i.e., if $\varphi(u_1,\ldots,u_n)$ has the form $(\exists x\in u)\varphi_0(x,u_1,\ldots,u_n)$ and $u,u_1,\ldots,u_n\in \mathbf{V}^{(B)}$, then (see the definitions and **1.1.5** (2,9)) the following calculations hold:

$$
\begin{aligned}
&[\![\varphi(\pi * u,\pi * u_1,\ldots,\pi * u_n)]\!]\\
&= \bigvee_{x\in \mathrm{dom}(\pi^* u)} (\pi * u)(x) \wedge [\![\varphi_0(x,\pi * u_1,\ldots,\pi * u_n)]\!]\\
&= \bigvee_{x\in \mathrm{dom}(u)} (\pi * u)(\pi * x) \wedge [\![\varphi_0(\pi * x,\pi * u_1,\ldots,\pi * u_n)]\!]\\
&= \bigvee_{z\in \mathrm{dom}(u)} \pi(u(z) \wedge [\![\varphi_0(z,u_1,\ldots,u_n)]\!]) = \pi([\![\varphi(u,u_1,\ldots,u_n)]\!]).
\end{aligned}
$$

The case of a bounded universal quantifier has been considered earlier. ▷

2.2.4. Corollary. *Let π,φ and $u_1,\ldots,u_n$ be the same as in **2.2.3**, and let one of the following statements be fulfilled:*

(1) *$\varphi(x_1,\ldots,x_n)$ is a formula of class Σ_1, π is arbitrary;*

(2) *π is an epimorphism and $\varphi(x_1,\ldots,x_n)$ is an arbitrary formula.*

Then

$$
\mathbf{V}^{(B)} \models \varphi(u_1,\ldots,u_n) \to \mathbf{V}^{(C)} \models \varphi(\pi * u_1,\ldots,\pi * u_n).
$$

2.2.5. Corollary. *Let π,φ and $u_1,\ldots,u_n$ be the same as in **2.2.3**, and, moreover, let one of the following statements be fulfilled:*

(1) *φ is bounded and π is a monomorphism;*

(2) *π is an isomorphism and φ is arbitrary. In this case we have*

$$\mathbf{V}^{(B)} \models \varphi(u_1,\dots,u_n) \leftrightarrow \mathbf{V}^{(C)} \models \varphi(\pi^* u_1,\dots,\pi^* u_n).$$

2.2.6. Let us now consider a special case of the situation under study. Let B_0 be a regular subalgebra of a complete Boolean algebra B. This implies that B_0 is a complete subalgebra, and the exact bounds of any set in B_0 are independent of the fact whether they are calculated in B_0 or B. Under such circumstances $\mathbf{V}^{(B_0)} \subset \mathbf{V}^{(B)}$, in which case if $\imath$ is the identical embedding of B_0 in B, then $\imath^*$ is an embedding of $\mathbf{V}^{(B_0)}$ in $\mathbf{V}^{(B)}$. It follows from **2.2.5** (1) that if $\varphi(x_1,\dots,x_n)$ is a bounded formula, and if $u_1,\dots,u_n \in \mathbf{V}^{(B_0)}$, , then

$$\mathbf{V}^{(B_0)} \models \varphi(u_1,\dots,u_n) \leftrightarrow \mathbf{V}^{(B)} \models \varphi(u_1,\dots,u_n).$$

Since the two-element algebra $\mathbf{2} := \{0,1\}$ can be viewed as a regular subalgebra of the Boolean algebra B, then the above is also valid for the universe $\mathbf{V}^{(2)}$. Below we shall see that $\mathbf{V}^{(2)}$ is naturally isomorphic to the von Neumann universe $\mathbf{V}$.

2.2.7. For an arbitrary set $x \in \mathbf{V}$ let us determine an element $x^\wedge \in \mathbf{V}^{(2)} \subset \mathbf{V}^{(B)}$ by recursion on the well-founded relation $y \in x$. To this end let us put

$$\operatorname{dom}(x^\wedge) := \{y^\wedge : y \in x\}, \quad \operatorname{im}(x^\wedge) := \{\mathbf{1}_B\}.$$

From **2.2.2** (3) for any $x, y \in V$ it follows:

$$[x^\wedge \in y^\wedge]^B \in \mathbf{2}, \quad [x^\wedge = y^\wedge]^B \in \mathbf{2}.$$

The mapping $x \to x^\wedge$ $(x \in \mathbf{V})$ is called the *canonical embedding* of the class of all sets $\mathbf{V}$ in the Boolean-valued universe $\mathbf{V}^{(B)}$. The elements of $\mathbf{V}^{(B)}$ which have the form $x^\wedge$ for a certain $x \in \mathbf{V}$ are termed *standard.* Sometimes $x^\wedge$ is called the *standard name of the set x in* $\mathbf{V}^{(B)}$.

2.2.8. Theorem. *The following statements are valid*:

(1) *if $x \in \mathbf{V}$ and $y \in \mathbf{V}^{(B)}$, then*

$$[y \in x^\wedge] = \vee\{[y = u^\wedge] : u \in x\};$$

(2) *if $x, y \in \mathbf{V}$, then*

$$x \in y \leftrightarrow \mathbf{V}^{(B)} \models x^{\wedge} \in y^{\wedge}, \quad x = y \leftrightarrow \mathbf{V}^{(B)} \models x^{\wedge} = y^{\wedge};$$

(**3**) *the mapping* $x \to x^{\wedge}$ *is injective*;

(**4**) *for any* $y \in \mathbf{V}^{(2)}$ *there is a unique element* $x \in \mathbf{V}$ *such that* $\mathbf{V}^{(B)} \models x^{\wedge} = y$;

(**5**) *if* π *is a full homomorphism from* B *to* C, *then for every* $x \in \mathbf{V}$ *we have* $\pi * x^{\wedge} = \hat{x}{}^{\wedge}$, *where* $\overset{\wedge}{(\ )}{}^{\wedge}$ *is a canonical embedding of* $\mathbf{V}$ *to* $\mathbf{V}^{(C)}$.

$\lhd$ Direct calculations with account taken of definitions **2.1.4** and **2.2.7** afford

$$[y \in x^{\wedge}] = \bigvee_{t \in \mathrm{dom}(x^{\wedge})} x^{\wedge}(t) \wedge [t = y]$$
$$= \bigvee_{t \in x} x^{\wedge}(t^{\wedge}) \wedge [t^{\wedge} = y] = \bigvee_{t \in x} [t^{\wedge} = y].$$

(2) Let us assume that, for all $z \in \mathbf{V}$ such that $\mathrm{rank}(z) < \mathrm{rank}(y)$, the following relations hold:

$$(\forall x)\big(x \in z \leftrightarrow [x^{\wedge} \in z^{\wedge}] = \mathbf{1}\big),$$
$$(\forall x)\big(x = z \leftrightarrow [x^{\wedge} = z^{\wedge}] = \mathbf{1}\big),$$
$$(\forall x)\big(z \in x \leftrightarrow [z^{\wedge} \in x^{\wedge}] = \mathbf{1}\big).$$

According to (1), $[x^{\wedge} \in y^{\wedge}] = \vee\big\{[t^{\wedge} = x^{\wedge}] : t \in y\big\}$. Since $\mathrm{rank}(t) < \mathrm{rank}(y)$ for $t \in y$, by the inductive hypothesis we conclude that $[x^{\wedge} \in y^{\wedge}] = \mathbf{1}$ iff $[t^{\wedge} = x^{\wedge}] = \mathbf{1}$ or $t = x$ for some $t \in y$. Then, by definition,

$$[x^{\wedge} = y^{\wedge}] = \bigwedge_{t \in x} [t^{\wedge} \in y^{\wedge}] \wedge \bigwedge_{s \in y} [s^{\wedge} = x^{\wedge}]$$

and $\mathrm{rank}(s) < \mathrm{rank}(y)$ for $s \in y$. Therefore, taking into consideration the above and the inductive hypothesis, we deduce that the right-hand side of the last equality is equal to unity iff $t \in y$ for all $t \in x$, and $s \in x$ for all $s \in y$, i.e., if $x = y$. If we again use (1), we get

$$[y^{\wedge} \in x^{\wedge}] = \vee\{[y^{\wedge} = t^{\wedge}] : t \in x\}.$$

Hence, $[y^{\wedge} \in x^{\wedge}] = \mathbf{1}$ is valid only if $[y^{\wedge} \in t^{\wedge}] = \mathbf{1}$ for some $t \in x$. The last statement is, by virtue of the above, equivalent to the relation $(\exists t \in x)(t \in y)$, i.e., $y \in x$.

(3) This statement ensues from (2).

(4) Let us assume that $y \in \mathbf{V}^{(2)}$ and for any $t \in \operatorname{dom}(y)$ there are such $u \in \mathbf{V}$ that $[t = u^\wedge] = \mathbf{1}$. Let us determine $x \in \mathbf{V}$ by the identity

$$x := \{u \in \mathbf{V} : (\exists t \in \operatorname{dom}(y))(y(t) = \mathbf{1} \wedge [u^\wedge = t] = \mathbf{1})\}.$$

In this case for $u \in x$ we get

$$[u^\wedge = y] = \bigvee_{t \in \operatorname{dom}(y)} y(t) \wedge [t = u^\wedge] = 1.$$

Besides, using the inductive hypothesis, we deduce for $t \in \operatorname{dom}(y)$:

$$y(t) \le [t \in x^\wedge] = \bigvee_{u \in x} [t = u^\wedge].$$

Summing up the above, we can conclude

$$[x^\wedge = y] = \bigwedge_{t \in \operatorname{dom}(y)} y(t) \Rightarrow [t \in x^\wedge] \wedge \bigwedge_{u \in x} [u^\wedge \in y] = \mathbf{1}.$$

(5) Let us carry out induction on the well-founded relation $y \in x$. Let us assume $(\forall y \in x)\left(\pi * y^\wedge = \hat{y^\wedge}\right)$. In this case

$$\operatorname{dom}(\pi * x^\wedge) = \{\hat{y^\wedge} : y \in x\} = \operatorname{dom}(\hat{x^\wedge}).$$

Therefore, for $y \in x$ we have

$$\begin{aligned}
(\pi * x^\wedge)\left(\hat{y^\wedge}\right) &= (\pi * x^\wedge)(\pi * y^\wedge) = \\
&= \vee\{\pi(x^\wedge(y^\wedge)) : z \in \operatorname{dom}(x) \wedge \pi * z = \pi * y^\wedge\} \ge \\
&\ge \pi\left(x^\wedge(y^\wedge)\right) = \mathbf{1}_B = \hat{x^\wedge}(\hat{y^\wedge}).
\end{aligned}$$

Therefore, $\pi * x^\wedge = \hat{x^\wedge}$, which ensures the inductive step. ▷

2.2.9. *Let* $u_1, \dots, u_n \in \mathbf{V}$, *and* $\varphi(x_1, \dots, x_n)$ *be a formula of* ZFC. *Then the following statements hold:*

(1) $\varphi(u_1,\dots,u_n) \in \mathbf{V}^{(2)} \models \varphi(u_1^\wedge,\dots,u_n^\wedge)$;

(2) *if* φ *is a bounded formula, then*

$$\varphi(u_1,\dots,u_n) \leftrightarrow \mathbf{V}^{(B)} \models \varphi(u_1^\wedge,\dots,u_n^\wedge);$$

(3) *if* φ *is a formula of class* Σ_1, *then*

$$\varphi(u_1,\dots,u_n) \to \mathbf{V}^{(B)} \models \varphi(u_1^\wedge,\dots,u_n^\wedge).$$

◁ Let us remark that only statement (1) has to be proved, as both (2) and (3) result from (1), **2.2.4** (1) and **2.2.5** (1). For the atomic formulas (1) is ensured by **2.2.8** (2). Then, due to complexity of the formula φ, a nontrivial step arises in induction only when the existential quantifier appears. Let us assume that φ has the form $(\exists x)\psi(x,u_1,\dots,u_n)$ and $\varphi[u_1^\wedge,\dots,u_n^\wedge] = \mathbf{1}$, and let for ψ the statement (1) be fulfilled. In this case

$$\mathbf{1} = \vee\{[\psi(u,u_1^\wedge,\dots,u_n^\wedge)]^2 : u \in \mathbf{V}^{(2)}\}.$$

Therefore, $[\psi(v,u_1^\wedge,\dots,u_n^\wedge)] = \mathbf{1}$ for a certain $v \in \mathbf{V}^{(2)}$. By **2.2.8** (4), there is such an $u_0 \in \mathbf{V}$ that $[u_0^\wedge = u] = 1$. Hence, according to **2.1.8** (7), we get

$$\mathbf{1} = [\psi(v,u_1^\wedge,\dots,u_n^\wedge)] \wedge [v = u_0^\wedge] < [\psi(u_0^\wedge,\dots,u_n^\wedge)].$$

By the induction hypothesis, we have $\psi(u_0,\dots,u_n)$. Hence, $\varphi(u_1,\dots,u_n)$ is also valid. Vice versa, if $\varphi(u_1,\dots,u_n)$, then for some $u_0 \in \mathbf{V}$ we get $\psi(u_0,u_1,\dots,u_n)$. And again, by the induction hypothesis, $[\psi(u_0^\wedge,u_1^\wedge,\dots,u_n^\wedge)] = 1$. Since, however, $[(\exists x)\psi(x,u_1^\wedge,\dots,u_n^\wedge)] \geq$ $\geq [\psi(u_0^\wedge,u_1^\wedge,\dots,u_n^\wedge)]$; therefore, $[\varphi(u_1^\wedge,\dots,u_n^\wedge)] = \mathbf{1}$. ▷

2.2.10. Remarks

(1) Let $\mathcal{U}$ be an ultrafilter in a Boolean algebra B, while $\mathcal{U}'$ be the ideal dual to it, i.e., $\mathcal{U}' := \{b^* : b \in \mathcal{U}\}$. Then the factor-algebra $B/\mathcal{U}'$ has two elements and can be identified with the Boolean algebra $\mathbf{2} := \{\mathbf{0},\mathbf{1}\}$. The factor-homomorphism $\pi: B \to \mathbf{2}$ is not, generally speaking, full, and hence **2.2.4** and **2.2.5** cannot be applied to establish a relationship between the truth-values in $\mathbf{V}^{(B)}$ and $\mathbf{V}^{(2)}$. If, however, π is full then, by virtue of **2.2.5**, it is evident that for any formula $\varphi(x_1,\dots,x_n)$ and any set $u_1,\dots,u_n \in \mathbf{V}^{(B)}$ we get

$$\mathbf{V}^{(2)} \models \varphi(\pi * u_1,\dots,\pi * u_n) \leftrightarrow [\varphi(u_1,\dots,u_n)] \in \mathcal{U},$$

since for $b \in B$ the relations $\pi(b) = \mathbf{1}$ and $b \in \mathcal{U}$ are equivalent.

(2) By way of factorizing, a model other than $\mathbf{V}^{(2)}$ can be constructed from the universe $\mathbf{V}^{(B)}$ and the ultrafilter $\mathcal{U}$.

Let us introduce in $\mathbf{V}^{(B)}$ the relation $\sim_{\mathcal{U}}$ by the formula

$$\sim_{\mathcal{U}} := \{(x,y) \in \mathbf{V}^{(B)} \times \mathbf{V}^{(B)} : [x = y] \in \mathcal{U}\}.$$

It is obvious that $\sim_{\mathcal{U}}$ is an equivalence on $\mathbf{V}^{(B)}$. By the symbol $\mathbf{V}^{(B)}/\mathcal{U}$ let us denote the factor-class (see **1.5.8**) of the universe $\mathbf{V}^{(B)}$ by $\sim_{\mathcal{U}}$, considered with the binary relation

$$\in_{\sim\mathcal{U}} := \{(\tilde{x}, \tilde{y}) : x,y \in \mathbf{V}^{(B)} \wedge [x \in y] \in \mathcal{U}\},$$

where $x \rightarrow \tilde{x}$ is the canonical factor-mapping from $\mathbf{V}^{(B)}$ to $\mathbf{V}^{(B)}/\mathcal{U}$. We can also show that

$$\mathbf{V}^{(B)}/\mathcal{U} \models \varphi(\tilde{x}_1,\dots,\tilde{x}_n) \leftrightarrow [\varphi(x_1,\dots,x_n)] \in \mathcal{U}$$

for $x_1,\dots,x_n \in \mathbf{V}^{(B)}$ and the formula φ.

The reader familiar with the theory of ultraproducts will recognize in (2) a known Łoš theorem (see [26,46, 96]). One can easily make sure in the in-depth relations between these phenomena. In (3) and (4) ultraproducts are obtained by the factorization of an appropriate Boolean-valued universe.

(3) Let T be a nonempty set of (not necessarily all) principal ultrafilters on a Boolean algebra B, and $\mathbf{V}^T$ be, as usual, the class of all mappings from T to $\mathbf{V}$. By virtue of **2.2.8** (4), for every $x \in \mathbf{V}^{(2)}$ there is a unique element $x^\wedge \in \mathbf{V}$ such that $[(x^\wedge)^\wedge = x] = \mathbf{1}$. Let us now determine a mapping $h: \mathbf{V}^{(B)} \rightarrow \mathbf{V}^T$, assuming

$$h(x) := \{(t, \pi_t * x) : t \in T\} \quad (x \in \mathbf{V}^{(B)}),$$

where π_t is the full homomorphism from B to $\mathbf{2}$ determined by the ultrafilter t, i.e., $\pi_t(b) = \mathbf{1}$ if $b \in t$, and $\pi_t(b) = \mathbf{0}$ if $b \in t'$. It is also possible to demonstrate that h is a surjective mapping. On the other hand, h is injective iff any element $b \in B$ belongs to an ultrafilter $t \in T$, i.e., $(\forall b \in B)(\exists t \in T)\ (b \in t)$ (which asserts that T defines a dense set of points in the Stone space of the algebra B, or that B is atomic, or that B is isomorphic to the Boolean $\mathcal{P}(T)$). It is the last statement that is the Łos theorem mentioned above. In this case for any $u_1,\dots,u_n \in \mathbf{V}^{(B)}$ and the formula $\varphi(x_1,\dots,x_n)$ we have

$$[\varphi(u_1,\dots,u_n)] \leq b \leftrightarrow (\forall t \in T) \quad \big([\varphi(\pi_t * u_1,\dots,\pi_t * u_n)] = \mathbf{1} \rightarrow b \in t\big).$$

(4) Let T be a set and $\mathcal{U}$ be an ultrafilter in the Boolean $\mathcal{P}(T)$. Let $\mathbf{V}^T/\mathcal{U}$ be the usual ultrapower of the class $\mathbf{V}$ over $\mathcal{U}$ with the canonical factor-mapping $g\colon \mathbf{V}^T \to \mathbf{V}^T/\mathcal{U}$ (see **1.5.7**). Let us put $\lambda(\tilde{x}) := g \circ h(x)$, where h is determined in (3), while $x \to \tilde{x}$ is the same as in (3). Therefore, a bijection λ is determined between $\mathbf{V}^{(\mathcal{P}(T))}/\mathcal{U}$ and $\mathbf{V}^T/\mathcal{U}$. In this case for any formula $\varphi(x_1,\dots,x_n)$ and functions $u_1,\dots,u_n \in \mathbf{V}^T$ we get

$$\mathbf{V}^T/\mathcal{U} \models \varphi(\tilde{u}_1,\dots,\tilde{u}_n) \leftrightarrow \{t \in T\colon \varphi(u_1(t),\dots,u_n(t))\} \in \mathcal{U}\,.$$

(5) It is expedient to compare **2.2.4** and **2.2.5** with the following statement. If M is a transitive model of ZFC (i.e., if M is a transitive class which is a model of ZFC), $u_1,\dots,u_n \in M$, $\varphi(x_1,\dots,x_n)$ is a bounded formula and $\psi(x_1,\dots,x_n)$ is a formula of class Σ_1, then

$$M \models \varphi(u_1,\dots,u_n) \leftrightarrow \varphi(u_1,\dots,u_n),$$
$$M \models \psi(u_1,\dots,u_n) \to \psi(u_1,\dots,u_n).$$

2.3. Mixings and the Maximum Principle

Let us consider a family of functions $(f_\xi)_{\xi\in\Xi}$ with domain A. If $(A_\xi)_{\xi\in\Xi}$ is a family of pairwise disjoint subsets of A, then we can define on A the function f whose restriction to A_ξ coincides with the restriction of f_ξ to A_ξ for all $\xi \in \Xi$. This function can be naturally called a disjunctive mixing of the family $(f_\xi)_{\xi\in\Xi}$. The Boolean-valued universe is complete in the sense that it contains all disjoint mixings of families of its elements. This peculiarity allows one to construct various special elements inside $\mathbf{V}^{(B)}$. Now let us go over to a more exact presentation.

2.3.1. A set consisting of mutually disjoint elements of a Boolean algebra is called an *antichain*. To be more exact, the set $A \subset B$ is called an antichain provided $a_1 \wedge a_2 = \mathbf{0}$ for any distinct $a_1, a_2 \in A$. If an antichain has the form $A := \{a_\xi\colon \xi \in \Xi\}$, then it is always assumed that $a_\xi \wedge a_\eta = \mathbf{0}$ as soon as $\xi \neq \eta$. The antichain $A \subset B$ is termed a *partition of the element* $b \in B$ (or a *partition of unity* when b is the unity of the algebra B), provided $b = \vee A$.

Let us consider an antichain $(b_\xi)_{\xi\in\Xi}$ in the Boolean algebra B and a family $(x_\xi)_{\xi\in\Xi}$ of elements of the universe $\mathbf{V}^{(B)}$. The *disjoint mixing* or simply the *mixing of the family*

(x_ξ) *relative to the antichain* (b_ξ) (sometimes they say *with probabilities* (b_ξ) or simply *by* (b_ξ)) is an element $x \in \mathbf{V}^{(B)}$ meeting the conditions

$$\operatorname{dom}(x) := \cup\{\operatorname{dom}(x_\xi) : \xi \in \Xi\},$$
$$x(t) := \vee\{b_\xi \wedge x_\xi(t) : \xi \in \Xi\} \quad (t \in \operatorname{dom}(x)).$$

The last equality implies that $x_\xi(t) = \mathbf{0}$ for $t \in \operatorname{dom}(x) - \operatorname{dom}(x_\xi)$. As long as $\alpha := \sup_{\xi \in \Xi} \rho(x_\xi) \in \mathrm{On}$, we have $\operatorname{dom}(x) \in \mathbf{V}^{(B)}_{\alpha+1}$. Hence, the given relation indeed determines a certain element $x \in \mathbf{V}^{(B)}$. The following conventional symbolic notation is used: $\operatorname{mix}_{\xi \in \Xi}(b_\xi x_\xi) := \operatorname{mix}\{b_\xi x_\xi : \xi \in \Xi\} : x$. In order to study the basic properties of mixings let us prove an auxiliary fact.

2.3.2. *Take* $x \in \mathbf{V}^{(B)}$ *and* $b \in B$. *Determine the function* bx *by the following relations*:

$$\operatorname{dom}(bx) := \operatorname{dom}(x), \quad bx : t \to b \wedge x(t) \quad (t \in \operatorname{dom}(x)).$$

Then $bx \in \mathbf{V}^{(B)}$, *and for any* x *and* $y \in \mathbf{V}^{(B)}$, *the following equalities are valid*:

$$[x \in by] = b \wedge [x \in y], \quad [bx = by] = b \Rightarrow [x = y].$$

⊲ The first relation can be checked by a direct calculation of Boolean truth-values using the infinite distributive law **1.1.5** (2). Indeed,

$$\begin{aligned}[x \in by] &= \bigvee_{t \in \operatorname{dom}(by)} (by)(t) \wedge [t = x] \\ &= b \wedge \bigvee_{t \in \operatorname{dom}(y)} y(t) \wedge [t = x] = b \wedge [x \in y].\end{aligned}$$

Then, using the preceding equality and successively applying **1.1.4** (2), **1.1.5** (6), **1.1.4** (4), **1.1.4** (2) and (**1.1.5** (6), we deduce

$$\begin{aligned}
&[bx = by] \\
&= \bigwedge_{t \in \operatorname{dom}(by)} (by)(t) \Rightarrow [t \in bx] \wedge \bigwedge_{t \in \operatorname{dom}(bx)} (bx)(t) \Rightarrow [t \in by] \\
&= \bigwedge_{t \in \operatorname{dom}(y)} (b \wedge y(t)) \Rightarrow (b \wedge [t \in x]) \wedge \bigwedge_{t \in \operatorname{dom}(x)} (b \wedge x(t)) \Rightarrow (b \wedge [t \in y]) \\
&= \bigwedge_{t \in \operatorname{dom}(y)} ((b \wedge y(t)) \Rightarrow b) \wedge ((b \wedge y(t)) \Rightarrow [t \in x]) \\
&\wedge \bigwedge_{t \in \operatorname{dom}(x)} ((b \wedge x(t)) \Rightarrow b) \wedge ((b \wedge x(t)) \Rightarrow [t \in y]) \\
&= \bigwedge_{t \in \operatorname{dom}(y)} b \Rightarrow (y(t) \Rightarrow [t \in x]) \wedge \bigwedge_{t \in \operatorname{dom}(x)} b \Rightarrow (x(t) \Rightarrow [t \in y]) \\
&= b \Rightarrow [x = y]. \ \triangleright
\end{aligned}$$

2.3.3. Theorem (the mixing principle). *Let* $(b_\xi)_{\xi\in\Xi}$ *be an antichain in* B *and* $(x_\xi)_{\xi\in\Xi}$ *be a family of elements of* $\mathbf{V}^{(B)}$. *Put* $x := \operatorname{mix}_{\xi\in\Xi}(b_\xi x_\xi)$. *Then*

$$[x = x_\xi] \geq b_\xi \quad (\xi\in\Xi).$$

If, moreover, $(b_\xi)_{\xi\in\Xi}$ *is a partition of unity and an element* $y\in\mathbf{V}^{(B)}$ *obeys the relation* $[y = x_\xi] \geq b_\xi$ *for all* $\xi\in\Xi$, *then* $[x = y] = \mathbf{1}$.

◁ By the definition of mixing, for any $\xi\in\Xi$ we have $b_\xi x = b_\xi x_\xi$. Applying **2.3.2**, we deduce

$$\mathbf{1} = [b_\xi x = b_\xi x_\xi] = b_\xi \Rightarrow [x_\xi = x].$$

Therefore, $[x_\xi = x] \geq b_\xi$ $(\xi\in\Xi)$ (according to **1.1.4** (4)).

Let us now assume that (b_ξ) is a partition of unity and $[y = x_\xi] \geq b_\xi$ $(\xi\in\Xi)$. Then by **2.1.8** (4), we get

$$b_\xi \leq [x = x_\xi] \wedge [x_\xi = y] \leq [x = y] \, (\xi\in\Xi).$$

Hence,

$$\mathbf{1} = \vee\{b_\xi : \xi\in\Xi\} \leq [x = y] \leq \mathbf{1},$$

which completes proof. ▷

2.3.4. *Let* $x\in\mathbf{V}^{(B)}$, *and let us determine* $\bar{x}\in\mathbf{V}^{(B)}$ *by the relations*

$$\operatorname{dom}(\bar{x}) := \operatorname{dom}(x), \quad \bar{x}(t) := [t\in x] \quad (t\in\operatorname{dom}(x)).$$

In this case

$$\mathbf{V}^{(B)} \models x = \bar{x}.$$

◁ The aim can be achieved by performing the following simple calculations, making use of the definition of **2.1.4**, as well as of **1.1.4** (4) and **2.1.8** (2):

$$[x=\bar{x}]$$
$$=\bigwedge_{t\in\operatorname{dom}(x)} x(t)\Rightarrow[t\in\bar{x}]\wedge\bigwedge_{t\in\operatorname{dom}(\bar{x})}[t\in x]\Rightarrow[t\in x]$$
$$=\bigwedge_{t\in\operatorname{dom}(x)} x(t)\Rightarrow\left(\bigvee_{u\in\operatorname{dom}(\bar{x})}\bar{x}(u)\wedge[u=t]\right)$$
$$\geq\bigwedge_{t\in\operatorname{dom}(x)} x(t)\Rightarrow x(t)\Rightarrow[t\in x]=\mathbf{1}.\ \triangleright$$

2.3.5. *Let us choose a partition of unity* $(b_\xi)_{\xi\in\Xi}\subset B$ *and a family* $(x_\xi)_{\xi\in\Xi}\subset\mathbf{V}^{(B)}$. *Let us set* $x:=\operatorname{mix}_{\xi\in\Xi}(b_\xi x_\xi)$. *Then the following statements are valid:*

(1) *if* $(x'_\xi)_{\xi\in\Xi}\subset\mathbf{V}^{(B)}$ *and* $\mathbf{V}^{(B)}\models x_\xi=x'_\xi$ $(\xi\in\Xi)$, *then*

$$\mathbf{V}^{(B)}\models x=\operatorname*{mix}_{\xi\in\Xi}(b_\xi x'_\xi);$$

(2) *if an element* $y\in\mathbf{V}^{(B)}$ *is such that* $\operatorname{dom}(y)=\operatorname{dom}(x)$ *and*

$$y(t):=\bigvee_{\xi\in\Xi} b_\xi\wedge[t\in x_\xi]\quad(t\in\operatorname{dom}(y)),$$

then $\mathbf{V}^{(B)}\models x=y$.

◁ Let $x':=\operatorname{mix}_{\xi\in\Xi}(b_\xi x'_\xi)$. From the conditions we deduce

$$b_\xi\leq[x_\xi=x'_\xi]\wedge[x_\xi=x']\wedge[x'_\xi=x']\leq[x=x'],$$

and, hence, $[x=x']=\mathbf{1}$. The last claim (2) follows from claim (1) and **2.3.4.** ▷.

2.3.6. *For any* $b\in B$ *and* $x\in\mathbf{V}^{(B)}$ *the following formulas are valid:*

$$[bx=x]=b\vee[x=\varnothing],\quad[bx=\varnothing]=b^*\vee[x=\varnothing].$$

In particular,

$$\mathbf{V}^{(B)}\models bx=\operatorname{mix}\{bx,b^*,\varnothing\}.$$

◁ It should be remarked that $[t\in bx\rightarrow t\in x]=\mathbf{1}$, since, by virtue of **2.3.2**, $[t\in bx]=b\wedge[t\in x]\leq[t\in x]$. Therefore, $[bx=x\leftrightarrow(\forall t)(t\in x\rightarrow t\in bx)]=\mathbf{1}$. With this equality taken into account, we calculate

$$
\begin{aligned}
[bx = x] &= \bigwedge_{t\in V^{(B)}} [t\in x] \Rightarrow [t\in bx]\\
&= \bigwedge_{t\in V^{(B)}} [t\in x]^* \vee (b\wedge [t\in x])\\
&= \bigwedge_{t\in V^{(B)}} (b\vee [t\in x]^*)\wedge ([t\in x]^*\vee [t\in x])\\
&= \bigwedge_{t\in V^{(B)}} b\vee [t\in x]^* = b\vee \bigwedge_{t\in V^{(B)}} b\vee [t\in x]^*\\
&= b\vee [(\forall t)(t\notin x)] = b\vee [x=\varnothing].
\end{aligned}
$$

On the other hand, appealing to **2.3.2** again, and making use of the fact that $b\varnothing = \varnothing$, we can write

$$b^*\vee [x=\varnothing] = b\Rightarrow [x=\varnothing] = [bx = b\varnothing] = [bx=\varnothing]. \ \triangleright$$

2.3.7. *Let us assume that* (b_ξ) *is a partition of unity in* B, *and let a family* $(x_\xi)\subset \mathbf{V}^{(B)}$ *be such that* $\mathbf{V}^{(B)}\models x_\xi \neq x_\eta$ *for any* $\xi\neq\eta$. *Then there is an element* $x\in\mathbf{V}^{(B)}$, *for which* $[x = x_\xi] = b_\xi$ *for all* ξ.

$\triangleleft$ Let us put $x := \mathrm{mix}(b_\xi x_\xi)$ and $a_\xi := [x = x_\xi]$. By hypothesis, we have

$$a_\xi \wedge a_\eta = [x = x_\xi]\wedge [x_\eta = x] \le [x_\xi \neq x_\eta]^* = \mathbf{0}$$

for $\xi\neq\eta$. Moreover, due to the properties of mixing, $b_\xi \le a_\xi$ for all ξ. Therefore, (a_ξ) is also a partition of unity in B. On the other hand,

$$b_\xi^* = \bigvee_{\eta\neq\xi} b_\eta \le \bigvee_{\eta\neq\xi} a_\eta = a_\xi^*,$$

and, hence, $b_\xi^* \le a_\xi^* \to b_\xi \ge a_\xi$. Therefore, the partitions of unity (b_ξ) and (a_ξ) coincide. $\triangleright$

The following fact, whose proof is based on a two-point mixing, often makes it possible to reduce the amount of bulky calculations.

2.3.8. *Let us consider* B*-formulas* $\varphi(x)$ *and* $\psi(x)$. *Assume that for a certain* $u_0\in\mathbf{V}^{(B)}$ *we have* $[\varphi(u_0)] := \mathbf{1}$. *Then*

$$
\begin{aligned}
&[(\forall x)(\varphi(x)\to\psi(x))] = \wedge\{[\psi(u)] : u\in\mathbf{V}^{(B)} \wedge [\varphi(u)] = \mathbf{1}\},\\
&[(\exists x)(\varphi(x)\to\psi(x))] = \vee\{[\psi(u)] : u\in\mathbf{V}^{(B)} \wedge [\varphi(u)] = \mathbf{1}\}.
\end{aligned}
$$

◁ Prove the first equality. To begin with, it is evident (see **2.1.7**) that

$$c := [(\forall x)(\varphi(x) \to \psi(x))] = \bigwedge_{t \in \mathbf{V}^{(B)}} [\varphi(t)] \Rightarrow [\psi(t)]$$
$$\le \bigwedge_{t \in \mathbf{V}^{(B)},\, [\varphi(t)] = 1} [\varphi(t)]^* \vee [\psi(t)] = \bigvee_{t \in \mathbf{V}^{(B)},\, [\varphi(t)] = 1} [\psi(t)] =: d.$$

In order to substantiate the reverse inequality $d \le c$, let us choose an arbitrary element $t \in \mathbf{V}^{(B)}$ and put $u := \operatorname{mix}\{bt, b^* u_0\}$, where $b := [\varphi(t)]$. Then, by virtue of **2.1.8** (7) and **2.3.3**, we can estimate

$$b \le [\varphi(t)] \wedge [t = u] \le [\varphi(u)],$$
$$b^* \le [\varphi(u_0)] \wedge [u = u_0] \le [\varphi(u)].$$

Therefore, $[\varphi(u)] = 1$. In this case, by the same considerations,

$$b \wedge [\psi(u)] \le [u = t] \wedge [\psi(u)] \le [\psi(t)].$$

Hence, the following estimates are valid:

$$[\psi(u)] \le b^* \vee (b \wedge [\psi(u)]) \le b^* [\psi(t)]$$
$$= b \Rightarrow [\psi(t)] = [\varphi(t)] \Rightarrow [\psi(t)].$$

As long as $d \le [\psi(u)]$, we have $d \le [\varphi(t)] \Rightarrow [\psi(t)]\ (t \in \mathbf{V}^{(B)})$. Now, passing to the infimum over t on the right-hand side of the last inequality, we get $d \le c$.

The last equality is dual to the previous one, since it is deduced from it according to the De Morgan formulas (see **1.1.2**). ▷

2.3.9. Let us now establish the central result of the present section, the maximum principle, stating that in the formula

$$[(\exists x)\varphi(x)] = \vee\{[\varphi(u)] : u \in \mathbf{V}^{(B)}\}$$

the least exact upper bound is attained at a certain element $u_0 \in \mathbf{V}^{(B)}$.

Let us first recall a fundamental property of complete Boolean algebras. Let B be a complete Boolean algebra. A set $E \subset B$ is called *minorizing*, or *minorant*, or *coinitial* in a subset $B_0 \subset B$ if for any $0 < b \in B_0$ there is such an $x \in E$ that $0 < x \le b$.

(1) Theorem (the exhaustion principle). *Let M be a nonempty set of elements of a complete Boolean algebra B, and let E be a set minorizing in the component $B_0 \subset B$*

generated by the set M. Then there is an antichain $E_0 \subset E$ such that $\vee E_0 = \vee M$, and for each $x \in E_0$ there is a $y \in M$ for which $x \le y$.

◁ Let us consider a set $\mathfrak{U}$ of all antichains A obeying the following conditions: (1) $A \subset E$; (2) for any $x \in A$ there is a $y \in M$ for which $x \le y$. If $\mathbf{0} \ne y \in M$, then, by the condition of minorance, $y \ge x$ for some $\mathbf{0} \ne x \in E$. Hence, $\{x\} \in \mathfrak{U}$ and $\mathfrak{U}$ is non-empty. The set $\mathfrak{U}$ ordered by inclusion is easily checked to obey the conditions of the Kuratowski-Zorn lemma. Therefore, there is a maximal element $E_0 \in \mathfrak{U}$. The task is to show that the elements $b_0 := \vee E_0$ and $b := \vee M$ coincide. It follows from the definition of $\mathfrak{U}$ that $b_0 \le b$. If $b_0 \ne b$, then there are such elements $\mathbf{0} \ne x_0 \in B$ and $x \in M$, that $x_0 \wedge b_0 = \mathbf{0}$ and $x_0 \le x$. By the condition of minorance, $\mathbf{0} < y \le x$ for some $y \in E$. The set $E_0 \cup \{y\}$ is incorporated in $\mathfrak{U}$ and is essentially wider than E_0. This contradicts the fact that E_0 is minimal, and, hence, $b_0 = b$. ▷

(2) Corollary. *For any non-empty set $M \subset B$ there is an antichain $A \subset B$ with the following properties: $\vee A = \vee B$, and for any $x \in A$ there is a $y \in M$ such that $x \le y$.*

◁ We should choose a minorant set $E := \bigcup_{y \in M} [\mathbf{0}, y]$ and make use of **(1)**. ▷

2.3.10. Theorem (the maximum principle). *Let $\varphi(x, x_1, \dots, x_n)$ be a certain formula, and $u_1, \dots, u_n$ be arbitrary elements of $\mathbf{V}^{(B)}$. Then there is a $u_0 \in \mathbf{V}^{(B)}$ such that*

$$[(\exists x)\varphi(x, u_1, \dots, u_n)] = [\varphi(u_0, u_1, \dots, u_n)].$$

In particular, if $\mathbf{V}^{(B)} \models (\exists x)\varphi(x, u_1, \dots, u_n)$, then $\mathbf{V}^{(B)} \models \varphi(u_0, u_1, \dots, u_n)$ for some $u_0 \in \mathbf{V}^{(B)}$.

◁ By definition, we have

$$\begin{aligned} b &:= [(\exists x)\varphi(x, u_1, \dots, u_n)] \\ &= \bigvee_{u \in \mathbf{V}^{(B)}} [\varphi(u, u_1, \dots, u_n)]. \end{aligned}$$

The class $A := \{[\varphi(u, u_1, \dots, u_n)] : u \in \mathbf{V}^{(B)}\}$ is a subset of the algebra B. According to **2.3.9** (2), there is a partition $(b_\xi)_{\xi \in \Xi}$ of the element b and a family $(u_\xi)_{\xi \in \Xi}$ of the elements of $\mathbf{V}^{(B)}$, for which the following relations hold:

$$\begin{aligned} &b_\xi \le [\varphi(u_\xi, u_1, \dots, u_n)] \quad (\xi \in \Xi), \\ &b = \vee\{[\varphi(u_\xi, u_1, \dots, u_n)] : \ (\xi \in \Xi)\}. \end{aligned}$$

Let us set u_0: $\operatorname{mix}_{\xi\in\Xi}(b_\xi u_\xi)$ and recall that by **2.3.3** we have $b_\xi \le [u_0 = u_\xi]$ $(\xi \in \Xi)$. Obviously,

$$[\varphi(u_0,u_1,\dots,u_n)] \le b.$$

On the other hand, applying **2.1.8** (7), we get

$$b_\xi \le [u_0 = u_\xi] \wedge [\varphi(u_\xi,u_1,\dots,u_n)] \le [\varphi(u_0,\dots,u_n)].$$

Therefore,

$$[\varphi(u_0,\dots,u_n)] \ge \bigvee_{\xi\in\Xi} b_\xi = b.$$

The second part of the theorem is a direct corollary to the first one. ▷

2.4. The Transfer Principle

In this section we shall check if the universe $\mathbf{V}^{(B)}$ constructed over a complete Boolean algebra B can serve, together with the Boolean truth-values $[\cdot \in \cdot]$ and $[\cdot = \cdot]$, a Boolean model of ZFC. Or, more exactly, if the following fact is valid.

2.4.1. Theorem (the transfer principle). *Any theorem of* ZFC *is valid in* $\mathbf{V}^{(B)}$, *or, symbolically,* $\mathbf{V}^{(B)} \models ZFC$.

The proof of this theorem consists in checking the relations $\mathbf{V}^{(B)} \models ZF_k$ for $k = 1,2,\dots,6$ in $\mathbf{V}^{(B)} \models AC$. In this case the greater part of the effort is to be spent on routine calculations given below for the completeness of presentation.

2.4.2. *The axiom of extensionality* ZF_1 *is true in* $\mathbf{V}^{(B)}$:

$$\mathbf{V}^{(B)} \models (\forall x)(\forall y)\,(x = y \leftrightarrow (\forall z)(z \in x \leftrightarrow z \in y)).$$

◁ The proof results immediately from the definition of the Boolean truth-value of equality **2.1.4** (2) and from **2.1.9**. Indeed, for any x and $y \in \mathbf{V}^{(B)}$ we put

$$c := c(x,y) := [(\forall z \in x)(z \in y)] = \bigwedge_{z \in \mathrm{dom}(x)} x(z) \Rightarrow [z \in y].$$

Obviously, $c(x,y) \wedge c(y,x) = [x = y]$ but, on the other hand,

$$c(x,y) \wedge c(y,x) = [(\forall z)(z \in x \leftrightarrow z \in y)].$$

Hence, by **1.1.4** (5), we conclude

$$[x = y \leftrightarrow (\forall z)(z \in x \leftrightarrow z \in y)] = \mathbf{1} \quad (x, y \in \mathbf{V}^{(B)}).$$

Now, taking infima over x and y, we complete the proof. ▷

2.4.3. *The axiom of union* ZF_2 *is true in* $\mathbf{V}^{(B)}$:

$$\mathbf{V}^{(B)} \models (\forall x)(\exists y)(z \in y \leftrightarrow (\exists u \in x)(z \in u)).$$

◁ Let us choose an arbitrary element $x \in \mathbf{V}^{(B)}$ and determine $y \in \mathbf{V}^{(B)}$ by the relations

$$\mathrm{dom}(y) := \cup\{\mathrm{dom}(u) : u \in \mathrm{dom}(x)\},$$
$$y(t) := [(\exists u \in x)(t \in u)] \quad (t \in \mathrm{dom}(y)).$$

It suffices to show that $[y = \cup x] = \mathbf{1}$.

In line with **2.1.9**, it holds that

$$[y \subset \cup x] = [(\forall t \in y)(\exists u \in x)(t \in u)]$$
$$= \bigwedge_{t \in \mathrm{dom}(y)} [(\exists u \in x)(t \in u)] \Rightarrow [(\exists u \ni x)(t \in u)] = \mathbf{1}.$$

Let us, then, remark that for $u \in \mathrm{dom}(x)$ and $z \in \mathrm{dom}(u)$ we have (see **2.1.8** (2) and **2.1.9**):

$$x(u) \wedge u(z) \le x(u) \wedge [z \in u] \le \bigvee_{u \in \mathrm{dom}(x)} x(u) \wedge [z \in u]$$
$$= [(\exists u \in x)(z \in u)] = y(z) \le [(z \in y)].$$

The above relation yields $x(u) \Rightarrow (u(z) \Rightarrow [z \in y]) = \mathbf{1}$ (see **1.1.4** (2-4)). Taking into account this equality as well as $x(u) \Rightarrow (u(z) \Rightarrow [z \in y]) = \mathbf{1}$ **2.1.9** and **1.1.5** (6), we can calculate

$$[\cup x \subset y] = [(\forall u \in x)(\forall z \in u)(z \in y)]$$
$$= \bigwedge_{u \in \mathrm{dom}(x)} x(u) \Rightarrow \left(\bigwedge_{z \in \mathrm{dom}(u)} u(z) \Rightarrow [z \in y] \right)$$

$$= \bigwedge_{u \in \mathrm{dom}(x)} \bigwedge_{z \in \mathrm{dom}(u)} x(u) \Rightarrow \big(u(z) \Rightarrow [z \in y]\big) = \mathbf{1}.$$

Therefore, $[y = \cup x] = \mathbf{1}$, and, hence,

$$[(\exists u)(u = \cup x)] = \bigvee_{u \in \mathbf{V}^{(B)}} [u = \cup x] \geq [y = \cup x] = \mathbf{1}.$$

If we take the infimum over $x \in \mathbf{V}^{(B)}$, we get the required result

$$[(\forall x)(\exists y)(y = \cup x)] = \bigwedge_{x \in \mathbf{V}^{(B)}} [(\exists y)(y = \cup x)] = \mathbf{1}. \ \triangleright$$

2.4.4. *The axiom of powersets* ZF_3 *is true in* $\mathbf{V}^{(B)}$:

$$\mathbf{V}^{(B)} \models (\forall x)(\exists y)(\forall z)\,(z \in y \leftrightarrow z \subset x).$$

$\triangleleft$ Let us consider an arbitrary element $x \in \mathbf{V}^{(B)}$ and determine $y \in \mathbf{V}^{(B)}$ in such a way that

$$\mathrm{dom}(y) := B^{\mathrm{dom}(x)},$$
$$y(z) := [z \subset x] \quad (z \in \mathrm{dom}(y)).$$

It suffices to demonstrate that $[z \in y \leftrightarrow z \subset x] = \mathbf{1}$ for every $z \in \mathbf{V}^{(B)}$. It is obvious that

$$[z \in y] = \bigvee_{t \in \mathrm{dom}(y)} y(t) \wedge [t = z]$$
$$= \bigvee_{t \in \mathrm{dom}(y)} [t \subset x] \wedge [t = z] \leq [z \subset x].$$

Hence, $[z \in y \leftrightarrow z \subset x] = \mathbf{1}$ (see **1.1.4** (4)). Now we have to substantiate the equality $[z \subset x \to z \in y] = \mathbf{1}$. To this end, let us somewhat modify z, i.e., let us consider an element $z' \in \mathrm{dom}(y)$ such that $\mathrm{dom}(z') := \mathrm{dom}(x)$ and $z'(t) := [t \in z] \quad (t \in \mathrm{dom}(z'))$. Then for every $t \in \mathbf{V}^{(B)}$ we obtain

$$[t \in z'] = \bigvee_{u \in \mathrm{dom}(z')} z'(u) \wedge [t = u]$$
$$= \bigvee_{u \in \mathrm{dom}(z')} [u \in z] \wedge [u = t] \leq [t \in z],$$

and, hence, $[z' \subset z] = \mathbf{1}$. On the other hand, by virtue of **2.1.8** (5) and **2.1.9**, we obtain

$$[t \in z \cap x] = \bigvee_{u \in \mathrm{dom}(x)} x(u) \wedge [t = u] \wedge [t \in z]$$
$$\leq \bigvee_{u \in \mathrm{dom}(x)} z'(u) \wedge [t = u] \wedge [t \in z'],$$

and, thus, $[z \cap x \subset z'] = \mathbf{1}$ (and again **1.1.4** (4)!). Moreover,

$$[z \subset x] = \bigwedge_{t \in \mathbf{V}^{(B)}} [t \in z] \Rightarrow [t \in x] \leq \bigwedge_{t \in \mathrm{dom}(z')} z'(t) \Rightarrow [t \in x]$$
$$= [(\forall t \in z')(t \in x)] = [z' \subset x] = y(z') \leq [z' \in y].$$

Summing up all that has been said about z and z', we derive

$$[z \subset x] \leq [x \cap z \subset z'] \wedge [z' \subset z] \wedge [z \subset x] \leq [z = z'],$$
$$[z \subset x] \leq [z' \in y].$$

The last two relations immediately yield

$$[z \subset x] = [z \subset x] \wedge [z = z'] \leq [z' \in y] \wedge [z = z'] \leq [z \in y],$$

i.e., $[z \subset x] \leq [z \in y]$, which is equivalent to the sought result by virtue of **1.1.4** (4).

2.4.5. *The axiom of replacement* ZF_4^{φ} *is true in* $\mathbf{V}^{(B)}$*:*

$$\mathbf{V}^{(B)} \models (\forall u)(\forall v_1)(\forall v_2)(\varphi(u,v_1) \wedge \varphi(u,v_2) \to v_1 = v_2)$$
$$\to ((\forall x)(\exists y)(\exists s \in x)(\exists t)(\varphi(s,t) \to t \in y)).$$

$\triangleleft$ In predicate calculus with equality the axiom of replacement can be deduced from that of choice (see **1.2.5**) and the formula

$$\Phi := (\forall x)((\forall t \in x)(\exists u)\varphi(t,u) \to (\exists y)(\forall t \in x)(\exists u \in y)\varphi(t,u))$$

(y does not occur freely in φ), i.e., $\Phi \wedge \Psi \to ZF_4^{\varphi}$, where Ψ is the axiom of choice. Therefore, it suffices to show that $\mathbf{V}^{(B)} \models \Phi$ and $\mathbf{V}^{(B)} \models \Psi$.

(a) $\mathbf{V}^{(B)} \models \Psi := (\forall x)(\exists y)(\forall t)\,(t \in y \leftrightarrow t \in x \wedge \psi(t)).$

Let us choose an arbitrary element $x \in \mathbf{V}^{(B)}$ and consider the function $y \in \mathbf{V}^{(B)}$ determined by the formulas

$$\mathrm{dom}(y) := \mathrm{dom}(x),$$
$$y(t) := x(t) \wedge [\psi(t)] \quad (t \in \mathrm{dom}(y)).$$

Then $[\![(\forall t)(t\in y \leftrightarrow t\in x \wedge \psi(t))]\!] = a\wedge b$, where

$$a := [\![(\forall t\in y)(t\in x \wedge \psi(t))]\!],\ b := [\![(\forall t\in x)(\psi(t)\to t\in y)]\!].$$

From **2.1.8** (2) and **2.1.9**, however, it is easily deduced that $a = b = 1$. Indeed,

$$a = \bigwedge_{t\in \mathrm{dom}(y)} y(t) \Rightarrow [\![t\in x\wedge \psi(t)]\!]$$
$$= \bigwedge_{t\in \mathrm{dom}(y)} x(t)\wedge [\![\psi(t)]\!] \Rightarrow [\![t\in x]\!]\wedge[\![\psi(t)]\!] = 1.$$

Analogously,

$$b = \bigwedge_{t\in \mathrm{dom}(x)} x(t) \Rightarrow ([\![\psi(t)]\!]\Rightarrow [\![t\in y]\!])$$
$$= \bigwedge_{t\in \mathrm{dom}(x)} x(t)\wedge [\![\psi(t)]\!] \Rightarrow [\![t\in x]\!]\wedge[\![\psi(t)]\!] = 1.$$

(**b**) $\mathbf{V}^{(B)}\models \Phi$. Let x be an arbitrary element of $\mathbf{V}^{(B)}$. Since B is a set, for every fixed $t\in \mathrm{dom}(x)$ the class

$$K := \{[\![\varphi(t,u)]\!] : u\in \mathbf{V}^{(B)}\}\subset B$$

is a set. The axiom of replacement for sets (i.e., in $\mathbf{V}$) states that there is such an ordinal $\alpha(t)$ that

$$\{[\![\varphi(t,u)]\!] : u\in \mathbf{V}^{(B)}_{\alpha(t)}\} = K.$$

Let us put $\alpha := \sup\{\alpha(t) : t\in \mathrm{dom}(x)\}$ and determine $y\in \mathbf{V}^{(B)}$ by the formulas

$$\mathrm{dom}(y) := \mathbf{V}^{(B)}_{\alpha},\ \mathrm{im}(y) = \{1\}.$$

In this case y is the sought element, as is shown by the following calculations:

$$[\![(\forall t\in x)(\exists u)\varphi(t,u)]\!] = \bigwedge_{t\in \mathrm{dom}(x)} x(t)\Rightarrow \Big(\bigvee_{u\in \mathbf{V}^{(B)}} [\![\varphi(t,u)]\!]\Big)$$
$$= \bigwedge_{t\in \mathrm{dom}(x)} x(t)\Rightarrow \Big(\bigvee_{u\in \mathbf{V}^{(B)}_{\alpha(t)}} [\![\varphi(t,u)]\!]\Big)$$
$$= \bigwedge_{t\in \mathrm{dom}(x)} x(t)\Rightarrow \Big(\bigvee_{u\in \mathbf{V}^{(B)}_{\alpha}} [\![\varphi(t,u)]\!]\Big)$$
$$= \bigwedge_{t\in \mathrm{dom}(x)} x(t)\Rightarrow [\![(\exists u\in y)\varphi(t,u)]\!] = [\![(\forall t\in x)(\exists u\in y)\varphi(t,u)]\!]. \ \triangleright$$

2.4.6. *The axiom of infinity* ZF_5 *is true in* $\mathbf{V}^{(B)}$:

$$\mathbf{V}^{(B)} \models (\exists x)(0 \in x \wedge (\forall t)(t \in x \to t \cup \{t\} \in x)).$$

◁ This axiom is fulfilled if we put $x := \omega^{\wedge}$ (see **2.2.7**). It is, first of all obvious that $[0^{\wedge} \in \omega^{\wedge}] = \mathbf{1}$, since $0^{\wedge} \in \mathrm{dom}(\omega^{\wedge})$. It should be remarked that for $t \in \mathbf{V}$ and $u := t \cup \{t\}$ we have $[u^{\wedge} = t^{\wedge} \cup \{t^{\wedge}\}] = \mathbf{1}$. Indeed, in line with **2.2.8** (1) we have

$$[v \in u^{\wedge}] = \bigvee_{s \in u} [s^{\wedge} = v] = [t^{\wedge} = v] \vee \bigvee_{s \in t} [s^{\wedge} = v]$$
$$[t^{\wedge} = v] \vee [v \in t^{\wedge}] = [t^{\wedge} = v \vee v \in t^{\wedge}] = [v \in t^{\wedge} \cup \{t^{\wedge}\}].$$

Taking into account this relation as well as **2.1.9** and **2.2.8** (2), we can easily calculate

$$[(\forall t \in \omega^{\wedge})(t \cup \{t\}) \in \omega^{\wedge}] = \bigwedge_{t \in \omega} [t^{\wedge} \cup \{t^{\wedge}\} \in \omega^{\wedge}]$$
$$= \bigwedge_{t \in \omega} [(t \cup \{t^{\wedge}\})^{\wedge} \in \omega^{\wedge}] = \mathbf{1}. \ ▷$$

2.4.7. *The axiom of foundation* ZF_6 *is true in* $\mathbf{V}^{(B)}$:

$$\mathbf{V}^{(B)} \models (\forall x)(\exists y)(x = 0 \vee (y \cap x = 0)).$$

◁ Let us choose an arbitrary element $x \in \mathbf{V}^{(B)}$. Let us show that

$$b := [x \neq 0 \wedge (\forall y \in x)(y \cap x \neq 0)] = \mathbf{0}_B.$$

Let us assume $b \neq \mathbf{0}_B$. Since $b \leq [(\exists u)(u \in x)]$, there is an element $y_0 \in \mathbf{V}^{(B)}$, for which $[y_0 \in x] \wedge b \neq \mathbf{0}$ and $\rho(y_0) \leq \rho(y)$ for $[y \in x] \wedge b \neq \mathbf{0}(y \in \mathbf{V}^{(B)})$. Since, besides, for any $y \in \mathbf{V}^{(B)}$ the estimate

$$[y \in x] \wedge b \leq [y \cap x \neq 0] = \bigvee_{z \in \mathrm{dom}(y)} y(z) \wedge [z \in x]$$

is valid, $[z \in x] \wedge [y_0 \in x] \wedge b \neq \mathbf{0}$ for some $z \in \mathrm{dom}(y_0)$. However, $\rho(z) < \rho(y_0)$, which contradicts the choice of y_0. Therefore, $b = \mathbf{0}_B$ and, hence,

$$\mathbf{1}_B = b^* = [\neg(x \neq 0 \wedge (\forall y \in x)(y \cap x \neq 0))]$$
$$= [(\exists y)(x = 0 \vee (y \in x \wedge y \cap x = 0))].$$

The proof is completed by passing to the infimum over $x \in \mathbf{V}^{(B)}$. ▷

2.4.8 Now we have to check validity for the axiom of choice inside $\mathbf{V}^{(B)}$. To this end, some additional constructions will be necessary. Let us consider arbitrary elements $x, y \in \mathbf{V}^{(B)}$. Let us determine a singleton $\{x\}^B$, an unordered pair $\{x, y\}^B$ and an ordered pair $(x,y)^B$ inside $\mathbf{V}^{(B)}$ with the relations

$$\operatorname{dom}(\{x\}^B) := \{x\}, \quad \operatorname{im}(\{x\}^B) := \{\mathbf{1}\};$$
$$\operatorname{dom}(\{x,y\}^B) := \{x,y\}, \quad \operatorname{im}(\{x,y\}^B) := \{\mathbf{1}\};$$
$$(x,y)^B := \{\{x\}^B, \ \{x,y\}^B\}^B.$$

The elements $\{x\}^B$, $\{x,y\}^B$ and $(x,y)^B \in \mathbf{V}^{(B)}$ correspond to their names.

The following statements are valid:

$$\mathbf{V}^{(B)} \models (\forall t)(t \in \{x\}^B \leftrightarrow t = x),$$
$$\mathbf{V}^{(B)} \models (\forall t)(t \in \{x,y\}^B \leftrightarrow t = x \vee t = y),$$
$$\mathbf{V}^{(B)} \models \text{'}(x,y)^B \textit{ is the ordered pair of elements } x \textit{ and } y\text{'},$$

or, in abbreviated form,

$$[\{x\}^B = \{x\}] = [\{x,y\}^B = \{x,y\}] = [(x,y)^B = (x,y)] = \mathbf{1}.$$

◁ Let us check, for instance, the statement on an unordered pair. For any $t \in \mathbf{V}^{(B)}$ we have

$$[t \in \{x,y\}^B] = \vee\{[t = s] : s \in \operatorname{dom}(\{x,y\}^B)\}$$
$$= [t = x] \vee [t = y] = [\![t = x \vee t = y]\!].$$

Hence,

$$[(\forall t)(t \in \{x,y\}^B \leftrightarrow t = x \vee t = y)] = \mathbf{1}. \ ▷$$

2.4.9. The notions introduced in the preceding item can be easily generalized for the case of an arbitrary $n > 2$. Let $x: n \to \mathbf{V}^{(B)}$. Then, by definition, $s := (x(0), \dots x(n-1))^B \in \mathbf{V}^{(B)}$, provided there is a mapping $y: n \to \mathbf{V}^{(B)}$ such that

$$y(0) = x(0), \quad y(n-1) = s,$$
$$y(k) = (y(k-1), x(k))^B \quad (0 < k \leq n-1).$$

Obviously, the function from $(\mathbf{V}^{(B)})^n$ to $\mathbf{V}^{(B)}$ is thus defined:

$$(x_0,\ldots,x_{n-1}) \to (x_0,\ldots,x_{n-1})^B \quad (x_0,\ldots,x_{n-1} \in \mathbf{V}^{(B)}).$$

Let us note an important property of this function, confining ourselves for simplicity with the case in which $n = 2$. It should be recalled that for any $x, y, x', y' \in \mathbf{V}$ the equivalence

$$(x,y) = (x',y') \leftrightarrow x = x' \wedge y = y'$$

is valid. This statement is a theorem of ZF and, hence, it is also valid in the model $\mathbf{V}^{(B)}$ (by **2.4.2** - **2.4.7**). Therefore, for any $x, y, x', y' \in \mathbf{V}$ we have

$$[(x,y) = (x',y')] = [x = x'] \wedge [y = y'].$$

As long as $(x,y)^B$ is an ordered pair inside $\mathbf{V}^{(B)}$, we must have

$$[(x,y)^B = (x',y')^B] = [x = x'] \wedge [y = y'].$$

In particular,

$$\mathbf{V}^{(B)} \models (x,y)^B = (x',y')^B \leftrightarrow \mathbf{V}^{(B)} \models x = x' \wedge y = y',$$

i.e., the function $(\,;\,)^B$ is 'injective in the internal sense'. It goes without saying that it is also injective in the sense of $\mathbf{V}$, i.e., if $(x,y)^B$ and $(x',y')^B$ coincide as elements of $\mathbf{V}$, then $x = x'$ and $y = y'$. But still these are two different properties.

2.4.10. Let us recall that by theorem **1.4.3** an ordinal can be defined as a transitive set linearly ordered by the relation of membership E. In a symbolic presentation it is as follows:

$$\begin{aligned}&\mathrm{Ord}(x) \leftrightarrow ((\forall u \in x)(\forall v \in u)(v \in x) \wedge \\ &\wedge (\forall u \in x)(\forall v \in x)(u \in v \vee u = v \vee v \in u)).\end{aligned}$$

We can see here that $\mathrm{Ord}(x)$ is a bounded formula, and, hence, according to **2.2.9** (2) valid is

$$\alpha \in \mathrm{On} \leftrightarrow \mathbf{V}^{(B)} \models \mathrm{Ord}(\alpha^\wedge).$$

Besides, in **2.2.8 (2)** we have established that

$$[\alpha^\wedge = \beta^\wedge] = \mathbf{1} \leftrightarrow \alpha = \beta \; (\alpha, \beta \in \mathrm{On}).$$

2.4.11. *The axiom of choice* AC *is true in* $\mathbf{V}^{(B)}$:

$$\mathbf{V}^{(B)} \models (\forall x)(\exists y)\ (y \text{ is a function of choice on } x).$$

◁ In the theory of ZF it is provable that on the set x there is a function of choice provided there is an ordinal α and a function f such that $\alpha = \mathrm{dom}(f)$ and $\mathrm{im}(f) \supset u := \cup x$. Indeed, y can be determined by the formula

$$(t,s) \in y \leftrightarrow s \in t \wedge t \in x \wedge (\exists \alpha_0 \in \alpha)(f(\alpha_0) = s)$$
$$\wedge (\forall \beta \in \alpha)(f(\beta) \in t \to \alpha_0 \leq \beta).$$

By virtue of **2.4.2 - 2.4.7**, this statement is also true inside $\mathbf{V}^{(B)}$, so it suffices to show that

$$\mathbf{V}^{(B)} \models (\forall u)(\exists \alpha)(\exists f)(\mathrm{Ord}(\alpha) \wedge \mathrm{Fnc}(f) \wedge \mathrm{dom}(f) = \alpha \wedge \mathrm{im}(f) \supset u).$$

Let us choose an arbitrary element $u \in \mathbf{V}^{(B)}$ and, using the axiom of choice for sets, find an ordinal α and a function g in such a way that $\mathrm{dom}(g) = \alpha$ and $\mathrm{dom}(u) \subset \mathrm{im}(g) \subset \mathbf{V}^{(B)}$. Let us determine $f \in \mathbf{V}^{(B)}$ via the relation

$$f := \left\{(\beta^{\wedge}, g(\beta))^{B} : \beta < \alpha\right\} \times \{\mathbf{1}_B\}.$$

Let us show that f obeys all the required conditions.

(a) $\mathbf{V}^{(B)} \models$ 'f is a binary relation'. Indeed, for an arbitrary $f \in \mathbf{V}^{(B)}$ we have

$$[t \in f] = \bigvee_{\beta < \alpha} [t = (\beta^{\wedge}, g(\beta))^{B}]$$
$$\leq \vee\{[t = (x,y)^{B}] : x, y \in \mathbf{V}^{(B)}\} = [(\exists x)(\exists y)(t = (x,y))].$$

(b) $\mathbf{V}^{(B)} \models \mathrm{Fnc}(f)$. Taking into account **(a)**, we have only to show that f is unique inside $\mathbf{V}^{(B)}$. Let us choose arbitrary $t, s_1, s_2 \in \mathbf{V}^{(B)}$ and calculate, applying in succession **2.1.4** (1), **2.4.9**, **2.1.8** (4) and **2.2.8** (2):

$$[(t,s_1) \in f \wedge (t,s_2) \in f] = [(t,s_1)^{B} \in f] \wedge [(t,s_2)^{B} \in f]$$
$$= \bigvee_{\beta < \alpha} \bigvee_{\gamma < \alpha} [(t,s_1)^{B} = (\beta^{\wedge}, g(\beta))^{B}] \wedge [(t,s_2)^{B} = (\gamma^{\wedge}, g(\gamma))^{B}]$$
$$= \bigvee_{\beta < \alpha} \bigvee_{\gamma < \alpha} [t = \beta^{\wedge}] \wedge [t = \gamma^{\wedge}] \wedge [s_1 = g(\beta)] \wedge [s_2 = g(\gamma)]$$
$$\leq \bigvee_{\beta < \alpha} \bigvee_{\gamma < \alpha} [\beta^{\wedge} = \gamma^{\wedge}] \wedge [s_1 = g(\beta)] \wedge [s_2 = g(\gamma)]$$

$$= \bigwedge_{\beta<\beta} [s_1 = g(\beta)] \wedge [s_2 = g(\beta)] \leq [s_1 = s_2].$$

(c) $\mathbf{V}^{(B)} \models \mathrm{Ord}(\alpha^\wedge) \wedge \mathrm{dom}(f) = \alpha^\wedge$. The relation $\mathbf{V}^{(B)} \models \mathrm{Ord}(\alpha^\wedge)$ has been discussed in **2.4.10**. Then for $t \in \mathbf{V}^{(B)}$ we have

$$[t \in \mathrm{dom}(f)] = [(\exists s)(t,s) \in f] = \bigvee_{s\in\mathbf{V}^{(B)}} [(t,s) \in f]$$

$$= \bigvee_{s\in\mathbf{V}^{(B)}} \bigvee_{\beta<\alpha} [(t,s) = (\beta^\wedge, g(\beta))]$$

$$= \bigvee_{\beta<\alpha} \bigwedge_{s\in\mathbf{V}^{(B)}} [t = \beta^\wedge] \wedge [s = g(\beta)]$$

$$= \bigvee_{\beta<\alpha} [t = \beta^\wedge] = \bigvee_{\beta\in\mathrm{dom}(\alpha^\wedge)} [t = \beta] = [t \in \alpha^\wedge].$$

(d) $\mathbf{V}^{(B)} \models \mathrm{im}(f) \supset u$. Let us choose an $s \in \mathbf{V}^{(B)}$ and carry out the following calculations:

$$[s \in u] = \bigvee_{v\in\mathrm{dom}(u)} u(v) \wedge [s = v] \leq \bigvee_{\beta<\alpha} [s = g(\beta)]$$

$$= \bigvee_{\beta<\alpha} \left([s = g(\beta)] \wedge \bigvee_{t\in\mathbf{V}^{(B)}} [\beta^\wedge = t] \right)$$

$$= \bigvee_{\beta<\alpha} \bigvee_{t\in\mathbf{V}^{(B)}} [(t,s) = (\beta^\wedge, g(\beta))]$$

$$= \bigvee_{t\in\mathbf{V}^{(B)}} [(t,s) \in f] = [(\exists t)(t,s) \in f] = [s \in \mathrm{im}(f)]. \ \triangleright$$

The proof of theorem **2.4.1** is complete.

2.4.12. Remarks

(1) Substituting for the logical part of the language of ZF the laws of the intuitionistic logic (see **2.1.10** (3)), we come to the intuitionistic set theory ZF_1. The ZF_1 models can be constructed using the scheme presented. Namely, if Ω is a complete Heyting lattice, then the universe $\mathbf{V}^{(\Omega)}$ becomes a Heyting-valued model of the theory ZF_1 provided the corresponding truth-values $[\cdot \in \cdot]$ and $[\cdot = \cdot]$ acting from $\mathbf{V}^{(\Omega)} \times \mathbf{V}^{(\Omega)}$ to $\mathbf{V}^{(\Omega)}$ are determined. For details see [56, 73, 247].

(2) Let B be a (quantum) logic (see **1.5.11** (5)). If the truth-values $[\cdot \in \cdot]$ and $[\cdot = \cdot]$ are determined by **2.1.4** and the formulas are evaluated as in **2.1.7**, then in the universe $\mathbf{V}^{(B)}$ the axioms ZF_2-ZF_6 and AC are valid. Therefore, set theory can be developed in $\mathbf{V}^{(B)}$. In particular, the real numbers inside $\mathbf{V}^{(B)}$ will correspond to those observed in the mathematical model of a quantum-mechanical system (see [255]).

2.5.5. Separated Boolean-Valued Universe

In this section a separated Boolean-valued universe is built and the interpretation of NBG in it is given.

2.5.1. For elements x and y of the universe $\mathbf{V}^{(B)}$ the relation $\mathbf{V}^{(B)} \models x = y$ does not at all imply that x and y coincide as sets, i.e., as elements of $\mathbf{V}$. Indeed, if for every ordinal α we determine $x_\alpha \in \mathbf{V}^{(B)}$ by the formulas $\mathrm{dom}(x_\alpha) \in V_\alpha^{(B)}$, $\mathrm{im}(x_\alpha) := \{0\}$, then, as can be easily checked, $[x_\alpha = 0] = \mathbf{1}$ for all α. Therefore, each element of the class $\{x_\alpha : \alpha \in \mathrm{On}\}$ is presented as an empty set inside $\mathbf{V}^{(B)}$. We can make sure that for any $x \in \mathbf{V}^{(B)}$ there is a proper class $y \in \mathbf{V}^{(B)}$ such that $[x = y] = \mathbf{1}$. This peculiarity results in certain technical difficulties and, in particular, hampers translations from the language of $\mathbf{V}^{(B)}$ into that of $\mathbf{V}$. This defect of the $\mathbf{V}^{(B)}$ model is eliminated by a proper factorization (see **1.5.8**).

2.5.2. In the universe $\mathbf{V}^{(B)}$ let us introduce the equivalence $\sim$:

$$\sim := \{(x, y) \in \mathbf{V}^{(B)} \times \mathbf{V}^{(B)} : [x = y] = \mathbf{1}_B\}.$$

Let us consider a factor-class $\tilde{\mathbf{V}}^{(B)} := \mathbf{V}^{(B)} / \sim$, and let $\mathbf{V}^{(B)} \to \tilde{\mathbf{V}}^{(B)}$ be the canonical mapping. The class $\tilde{\mathbf{V}}^{(B)}$ is called a *separated Boolean-valued universe.* Let us introduce the Boolean truth-values for equality $[\cdot = \cdot]_s$, and of membership $[\cdot \in \cdot]_s$ for the class $\tilde{\mathbf{V}}^{(B)}$ by way of descending the corresponding functions $[\cdot = \cdot]_s$ and $[\cdot \in \cdot]_s$ on a factor-class:

$$[\cdot = \cdot]_s := [\cdot = \cdot] \circ (\pi^{-1} \times \pi^{-1}),$$
$$[\cdot \in \cdot]_s := [\cdot \in \cdot] \circ (\pi^{-1} \times \pi^{-1}).$$

Now for any formula $\varphi(u_1, \dots, u_n)$ and for arbitrary $\tilde{x}_1, \dots, \tilde{x}_n \in \tilde{\mathbf{V}}^{(B)}$ let us determine $[\varphi(\tilde{x}_1, \dots, \tilde{x}_n)] \in B$ in the same way as in **2.1.7**. Then we must get

$$[\varphi(x_1, \dots, x_n)] = [\varphi(\pi x_1, \dots, \pi x_n)]_s \quad (x_1, \dots, x_n \in \mathbf{V}^{(B)}).$$

The validity of formulas in $\tilde{\mathbf{V}}^{(B)}$ is set in the same way as in **2.1.6**:

$$\tilde{\mathbf{V}}^{(B)} \models \varphi(\tilde{x}_1, \dots, \tilde{x}_n) \leftrightarrow [\varphi(\tilde{x}_1, \dots, \tilde{x}_n)]_s = \mathbf{1}_B.$$

The soundness of the given definitions is obvious since by virtue of **2.1.8** (7) for any formula φ of ZFC we have

$$\mathbf{1} = [x = y] \to [\varphi(x)] = [\varphi(y)] \quad (x, y \in \mathbf{V}^{(B)}).$$

Therefore, when calculating Boolean values in a separated universe use can be made of arbitrary representatives of the required residue classes. This remark, in particular, yields that theorem **2.1.8** remains valid when $\tilde{\mathbf{V}}^{(B)}$ is substituted for $\mathbf{V}^{(B)}$ and when Boolean truth-values are supplied with the index s.

As a somewhat unexpected example, let us consider the following definition: for $\tilde{x} \in \tilde{\mathbf{V}}^{(B)}$ the symbol $\vee\tilde{x}$ denotes the *level of* $\tilde{x}$, i.e., the element

$$\vee\tilde{x} := \bigvee_{t \in \mathrm{dom}(x)} x(t),$$

where $\tilde{x}$ is the equivalence class of the element $x \in \mathbf{V}^{(B)}$. At first sight these definitions seem to be not quite legitimate, since the domain of the definition of the elements equal inside must not obligatorily coincide. At the same time,

$$[(\exists y \in \tilde{x})]_s = [(\exists y \in \tilde{x}) y = y]_s$$
$$= \bigvee_{t \in \mathrm{dom}(x)} x(t) \wedge [t = t] = \bigvee_{t \in \mathrm{dom}(x)} x(t) = \vee\tilde{x}.$$

Obviously, $\vee\tilde{x} = [x \neq \varnothing]_s$, and thus the notion of level is correct. In an analogous way, for any element $\tilde{x}$ of $\tilde{\mathbf{V}}^{(B)}$ and an element b of the Boolean algebra B, the element $\tilde{bx}: t \to b \wedge x(t)$ $((t \in \mathrm{dom}(x))$ is determined. Indeed, if $[x_1 = x_2] = \mathbf{1}$, then, by virtue of the earlier established **2.3.2**, $[bx_1 = bx_2] = b \Rightarrow [x_1 = x_2] = \mathbf{1}$. In this respect often is used the designation $0 = \varnothing$, which, in particular, implies that $0\varnothing = \varnothing = 0\tilde{x}$ for any $x \in \tilde{\mathbf{V}}^{(B)}$.

2.5.3. It should be remarked that the facts presented in **2.2** - **2.4** are, with some obvious specifications and refinements, also true in $\tilde{\mathbf{V}}^{(B)}$. Thus, in the sense of **2.4** $\tilde{\mathbf{V}}^{(B)}$ is a model of the theory of ZFC. Analogously, if ρ is a complete homomorphism of Boolean algebras, then ρ^* has any equivalence class invariant, and, hence ρ^* induces the only mapping of the corresponding separated universes which is also denoted by ρ^*, i.e., it is analogous with **2.2.2**, etc.. If $(x_\xi) \subset \mathbf{V}^{(B)}$, (b_ξ) is a disjoint family in B and $x = \mathrm{mix}(b_\xi x_\xi)$, then the element $\tilde{x} := \pi x$ will preserve the name 'a mixing' and the notation $\tilde{x} = \mathrm{mix}(b_\xi \tilde{x}_\xi)$ $(\tilde{x}_\xi = \pi x_\xi)$. Such a definition of mixing in $\tilde{\mathbf{V}}^{(B)}$ is correct (see **2.3.5** (1)). Therefore, if $\tilde{x} \in \tilde{\mathbf{V}}^{(B)}$ and $(\tilde{x}) \subset \tilde{\mathbf{V}}^{(B)}$, then the presentation $\tilde{x} = \mathrm{mix}(b_\xi \tilde{x}_\xi)$ implies that

$$b_\xi \leq [\tilde{x} = \tilde{x}_\xi]_s \ (\xi \in \Xi).$$

It should be remarked that if (b_ξ) is a partition of unity, then the mixing $\operatorname{mix}(b_\xi x_\xi)$ is unique (due to separation!) (see **2.3.3**).

The equality (see **2.4.9**)

$$[(x,y)^B = (x',y')^B] = [x = x'] \wedge [y = y']$$

shows the mapping $(\cdot,\cdot)^B$ to be stable relative to the equivalence relation in **2.5.2**. Hence, there is an injective embedding $\tilde{\mathbf{V}}^{(B)} \times \tilde{\mathbf{V}}^{(B)} \to \tilde{\mathbf{V}}^{(B)}$ denoted by the same symbol $(\cdot,\cdot)^B$, for which $(\pi x, \pi y)^B = \pi((x,y)^B)$. In this case

$$[(\tilde{x},\tilde{y})^B = (\tilde{x},\tilde{y})]_s = 1 \ (\tilde{x},\tilde{y} \in \mathbf{V}^{(B)}).$$

The maximum principle is also preserved and has the following refinement.

2.5.4. *Let $\varphi(u,u_1,\dots,u_n)$ be a formula, $\tilde{x}_1,\dots,\tilde{x}_n \in \tilde{\mathbf{V}}^{(B)}$ and $\tilde{\mathbf{V}}^{(B)} \models (\exists! u)\varphi(u,\tilde{x}_1,\dots,\tilde{x}_n)$. Then there is a unique element $\tilde{x}_0 \in \tilde{\mathbf{V}}^{(B)}$ such that $\tilde{\mathbf{V}}^{(B)} \models \varphi(\tilde{x}_0,\tilde{x}_1,\dots,\tilde{x}_n)$.*

◁ Let $\tilde{x}_k := \pi(x_k)$, where $x_k \in \mathbf{V}^{(B)}$ $(k := 1,\dots,n)$. In this case $\mathbf{V}^{(B)} \models (\exists! u)\varphi(u,x_1,\dots,x_n)$. By transfer, there is an element $x_0 \in \mathbf{V}^{(B)}$, for which $\mathbf{V}^{(B)} \models \varphi(x_0,x_1,\dots,x_n)$. Let us put $\tilde{x}_0 := \pi(x_0)$. Obviously, $\tilde{\mathbf{V}}^{(B)} \models \varphi(\tilde{x}_0,\tilde{x}_1,\dots,\tilde{x}_n)$. If for an element $z \in \tilde{\mathbf{V}}^{(B)}$ we have $\tilde{\mathbf{V}}^{(B)} \models \varphi(z,\tilde{x}_1,\dots,\tilde{x}_n)$, then we get $\tilde{\mathbf{V}}^{(B)} \models \varphi(\tilde{x}_0,\dots,\tilde{x}_n) \wedge \varphi(z,\tilde{x}_1,\dots,\tilde{x}_n)$. By hypothesis, $\tilde{\mathbf{V}}^{(B)} \models z = \tilde{x}_0$, which implies, since $\tilde{\mathbf{V}}^{(B)}$ is separated, that $z = \tilde{x}_0$. ▷

2.5.5. For arbitrary b and $c \in B$ let us put (see **1.1.4**)

$$[b = c] := b \Leftrightarrow c = (b \Delta c)^* = (b \wedge c) \vee (b^* \wedge c^*).$$

It should be remarked that in line with **1.1.4** (3) $a \leq [b = c]$ iff $a \wedge b = a \wedge c$. Let us consider a function $f\colon \operatorname{dom}(f) \to B$, whose domain of definition $\operatorname{dom}(f)$ is contained in $\tilde{\mathbf{V}}^{(B)}$. They say that f is *extensional* if

$$[x = y]_s \leq [f(x) = f(y)] \ (x,y) \in \operatorname{dom}(f)).$$

The existence of f can be easily seen to be equivalent to the relation

$$f(x) \wedge [x = y]_s \leq f(y) \ (x,y) \in \operatorname{dom}(f)).$$

If $u\colon \operatorname{dom}(u) \to B$ is an arbitrary function and $\operatorname{dom}(u) \subset \tilde{\mathbf{V}}^{(B)}$, then u can be related with the extensional function $\bar{u}\colon \tilde{\mathbf{V}}^{(B)} \to B$ using the formula

$$\bar{u}\colon x \to \bigvee_{t\in\operatorname{dom}(u)} u(t) \wedge [t = x]_s \quad (x \in \tilde{\mathbf{V}}^{(B)}).$$

Another class of extensional functions arises in the following way. Let φ be a B formula. Then the following function is extensional

$$\bar{\varphi}\colon x \to [\varphi(x)]_s \quad (x \in \tilde{\mathbf{V}}^{(B)}).$$

2.5.6. Theorem. *If $u\colon \operatorname{dom}(u) \to B$ is a function, in which case $\operatorname{dom}(u) \subset \tilde{\mathbf{V}}^{(B)}$ and $\operatorname{dom}(u) \in \mathbf{V}$, then there exists a unique element $x \in \tilde{\mathbf{V}}^{(B)}$ such that $\bar{u}(t) = [t \in x]_s$ at all $t \in \tilde{\mathbf{V}}^{(B)}$. Vice versa, if $x \in \tilde{\mathbf{V}}^{(B)}$, then there exists a function $u\colon \operatorname{dom}(u) \to B$, for which $\operatorname{dom}(u) \subset \tilde{\mathbf{V}}^{(B)}$, $\operatorname{dom}(u) \in \mathbf{V}$ and $\bar{u}(t) = [t \in x]_s$ $(t \in \tilde{\mathbf{V}}^{(B)})$.*

$\triangleleft$ Let D be a subset of an unseparated universe whose image under the canonical factor-mapping π coincides with $\operatorname{dom}(u)$. Let us determine an element $x' \in \mathbf{V}^{(B)}$ by the formula

$$\operatorname{dom}(x') := D, \quad x'(t) := u(\pi t) \ (t \in D).$$

Let us, finally, put $x := \pi(x')$. Then for $t \in \tilde{\mathbf{V}}^{(B)}$ we get

$$[t \in x]_s = \bigvee_{y\in D} x'(y) \wedge [t \in \pi y]_s = \bigvee_{y\in\operatorname{dom}(u)} x(y) \wedge [y = t] = \bar{u}(t).$$

If some other element $z \in \tilde{\mathbf{V}}^{(B)}$ has this property, then $[t \in x]_s = [t \in z]_s$ for all $t \in \tilde{\mathbf{V}}^{(B)}$. Hence,

$$\tilde{\mathbf{V}}^{(B)} \models (\forall t)\,(t \in x \leftrightarrow t \in z).$$

By virtue of the axiom of extensionality, inside $\tilde{\mathbf{V}}^{(B)}$ we get $[x = z]_s = \mathbf{1}$. As $\tilde{\mathbf{V}}^{(B)}$ is separated, then $x = z$.

Now, conversely, let $x \in \tilde{\mathbf{V}}^{(B)}$, and let x' be such an element of the separated universe that $x = \pi(x')$. Let us put $\operatorname{dom}(u) := \pi``(\operatorname{dom}(x'))$ and determine $u\colon \operatorname{dom}(u) \to B$ in such way that $u(\pi t) = x'(t)$ $(t \in \operatorname{dom}(x'))$. In this case for any $t \in \tilde{\mathbf{V}}^{(B)}$ we have

$$[t \in x]_s = \bigvee_{y \in \mathrm{dom}(x')} x'(y) \wedge [t = \pi y]_s$$
$$= \bigvee_{y \in \mathrm{dom}(u)} u(y) \wedge [y = t]_s = \bar{u}(t). \rhd$$

2.5.7. Below we shall basically operate with a separated Boolean-valued universe $\tilde{\mathbf{V}}^{(B)}$. In this case when calculating Boolean truth-values we often, without further specifications, replace elements of $\tilde{\mathbf{V}}^{(B)}$ with their representatives in $\mathbf{V}^{(B)}$ (it is, for instance, common practice when working with spaces of the equivalence classes of measurable functions). Besides, starting from the sentence to follow, we shall omit the sign ~ and index s and write simply $\mathbf{V}^{(B)}$ $[\cdot = \cdot]$ and $[\cdot \in \cdot]$ instead of $\tilde{\mathbf{V}}^{(B)}$, $[\cdot = \cdot]_s$ and $[\cdot \in \cdot]_s$.

As is seen from **2.5.6**, any element of $\mathbf{V}^{(B)}$ determines a certain extensional mapping on $\mathbf{V}^{(B)}$ with values in B, but only some special-type extensional mappings from $\mathbf{V}^{(B)}$ in B are determined by elements from $\mathbf{V}^{(B)}$. This peculiarity serves as a motivation for the following definition.

2.5.8. *A class inside* $\mathbf{V}^{(B)}$ or the $\mathbf{V}^{(B)}$*-class* is any extensional mapping $X: \mathbf{V}^{(B)} \to B$, which is a class in the conventional sense, i.e., in the sense of $\mathbf{V}$. Therefore, we assign to each element $x \in \mathbf{V}^{(B)}$ the $\mathbf{V}^{(B)}$-class

$$< x > := [\cdot \in x] : t \to [t \in x] \quad (t \in \mathbf{V}^{(B)}).$$

Such a correspondence is obviously injective. Let us now introduce Boolean truth-values, setting for $\mathbf{V}^{(B)}$-classes X and Y and an element $z \in \mathbf{V}^{(B)}$:

$$[< z > \in X] := X(z),$$
$$[X = Y] := \bigwedge_{u \in \mathbf{V}^{(B)}} [< u > \in X] \Leftrightarrow [< u > \in Y],$$
$$[X \in Y] := \bigvee_{u \in \mathbf{V}^{(B)}} [< u > = X] \wedge [< u > \in Y].$$

The first and third formulas are consistent, since, as X is extensional, we have

$$[< z > \in X] = \bigvee_{u \in \mathbf{V}^{(B)}} X(u) \wedge [u = z]$$

and, moreover, $[< z > = < u >] = [z = u]$ at all $z \in \mathbf{V}^{(B)}$. It follows from the definitions that $[X = Y] = \mathbf{1}$ implies $X = Y$. The function $\mathbf{U}_B: x \to \mathbf{1}_B$ $(x \in \mathbf{V}_B)$ is a universal class inside $\mathbf{V}_B$. An empty $\mathbf{V}_B$-class is the function identical to zero on $\mathbf{V}_B$.

2.5.9. It should be recalled that a formula is termed *predicative* if only the variables for sets appear to be bounded in it (see **1.3.14**).

(1) Let us define the Boolean truth-value for a predicative formula. For propositional connectives it is carried out in the same way as in **2.1.7**, and we thus have to detalize only the case for the quantifiers, whose action is restricted to the class of sets. In this case we can consider only formulas containing no subformulas of the type $X_1 \in X_2$, since the latter is equivalent to the formula $(\exists x)(x = X_1 \wedge x \in X_2)$.

Thus, let φ be a predicative formula with free variables $X, X_1, \dots, X_n$, while $Y_1, \dots, Y_n$ be some $\mathbf{V}_B$-classes. Let us, by definition, put

$$[(\forall x)\varphi(x, Y_1, \dots, Y_n)] = \bigwedge_{y \in \mathbf{V}^{(B)}} [\varphi(y, Y_1, \dots, Y_n)],$$
$$[(\exists x)\varphi(x, Y_1, \dots, Y_n)] = \bigwedge_{y \in \mathbf{V}^{(B)}} [\varphi(y, Y_1, \dots, Y_n)].$$

Let us say that the predicative formula $\varphi(X_1, \dots, X_n)$ *is valid inside* $\mathbf{V}^{(B)}$ *for the given values* $Y_1, \dots, Y_n$ *of the variables* $X_1, \dots, X_n$ provided $[\varphi(Y_1, \dots, Y_n)] = \mathbf{1}$. As in **2.1.6**, we shall write

$$\mathbf{V}^{(B)} \models \varphi(Y_1, \dots, Y_n) \leftrightarrow [\varphi(Y_1, \dots, Y_n)] = \mathbf{1}.$$

(2) The notion of validity in the model $\mathbf{V}^{(B)}$ is spread on nonpredicative formulas in the following way. If $\varphi(X, X_1, \dots, X_n)$ is a predicative formula, then we set

$$\mathbf{V}^{(B)} \models (\forall X)\varphi(X, Y_1, \dots, Y_n) \quad (\mathbf{V}^{(B)} \models (\exists X)\varphi(X, Y_1, \dots, Y_n))$$

iff $[\varphi(Y, Y_1, \dots, Y_n)] = \mathbf{1}$ for any $\mathbf{V}^{(B)}$-class Y (respectively, there is such a $\mathbf{V}^{(B)}$-class Y that $[\varphi(Y, Y_1, \dots, Y_n)] = \mathbf{1}$).

The $\mathbf{V}^{(B)}$-class Y is called a $\mathbf{V}^{(B)}$*-set*, if $\mathbf{V}^{(B)} \models M(Y)$, where $M(X) := (\exists Z)(X \in Z)$ (see **1.3.1**). It would be simpler to say 'a B-set' instead of 'a $\mathbf{V}^{(B)}$-set'; this term, however, will be preserved to be used somewhere else (see **3.4**).

2.5.10. *For every* $x \in \mathbf{V}^{(B)}$ *a* $\mathbf{V}^{(B)}$*-class* $< x >$ *is a* $\mathbf{V}^{(B)}$*-set. Conversely, if a* $\mathbf{V}^{(B)}$*-class* X *is a* $\mathbf{V}^{(B)}$*-set, then x=X for a certain* $x \in \mathbf{V}^{(B)}$.

$\lhd$ For an arbitrary element $x \in \mathbf{V}^{(B)}$ we have

$$[< x > \in < \{x\}^B >] = [< x > \in \{x\}^B] = \mathbf{1},$$

and, hence, $\mathbf{V}^{(B)} \models M(< x >)$. Let us assume that for a $\mathbf{V}^{(B)}$-class X $\mathbf{V}^{(B)} \models M(< X >)$ holds. Then, by definition (see **2.5.9** (2)), there is a $\mathbf{V}^{(B)}$-class Z, for which

$$\bigvee_{t \in \mathbf{V}^{(B)}} Z(t) \wedge [< t > = X] = 1.$$

Hence, by virtue of the exhaustion principle, we can choose such a partition of the unity, $(b_\xi)_{\xi \in \Xi}$ and such a family $(x_\xi)_{\xi \in \Xi} \subset \mathbf{V}^{(B)}$ that

$$[< x_\xi > = X] \geq b_\xi .$$

If $x := \operatorname{mix}(b_\xi x_\xi)$, then

$$[< x > = X] \geq [< x > = < x_\xi >] \wedge [< x_\xi > = X] \geq b_\xi ,$$

and, hence, $[< x > = X] = 1$ or $< x > = X$. ⊳

On the basis of the fact just established we shall henceforth identify an element $x \in \mathbf{V}^{(B)}$ and the corresponding $\mathbf{V}^{(B)}$-set $< x >$.

2.5.11. Let C be a complete Boolean algebra and $\pi : B \to C$ be a full homomorphism. Let us consider a $\mathbf{V}^{(B)}$-class X, and put by definition,

$$(x,b) \in \pi^* X \leftrightarrow b = \bigvee_{t \in \mathbf{V}^{(B)}} (\pi \circ X)(t) \wedge [x = \pi^* t]^C .$$

In this case $\pi * X$ is a class inside $\mathbf{V}^{(B)}$. Indeed, $\pi * X$ is a $\mathbf{V}$ subclass by virtue of theorem **1.3.14**, since

$$\pi * X = \{(x,b) : \varphi(x, b, B, C, X, \pi^* , [\cdot = \cdot], \mathbf{V}^{(B)})\}$$

for the predicative formula

$$\varphi(Y, Z, B, \ldots) := Z = \bigvee_{t \in \mathbf{V}^{(B)}} (\pi \circ X)(t) \wedge [Y = \pi * t] .$$

Besides, $\pi * X$ is an extensional function:

$$(\pi * X)(x) \wedge [x = y] = \bigvee_{t \in \mathbf{V}^{(B)}} (\pi \circ X)(t) \wedge [x = \pi * t]$$
$$\wedge [x = y] \leq \bigvee_{t \in \mathbf{V}^{(B)}} (\pi \circ X)(t) \wedge [y = \pi * t] = (\pi * X)(y) .$$

One can easily notice that statement **2.2.2** (1) remains valid for classes, i.e., if ρ is a full homomorphism, then

$$(\rho \circ \pi) * X = (\rho * \circ \pi^*) X.$$

Then, if $\mathbf{V}^{(B)} \models M(X)$, then $\mathbf{V}^{(C)} \models M(\pi * X)$. Indeed, if $X = < x >$, $x \in \mathbf{V}^{(B)}$, then, by **2.2.4** (4), we have

$$\begin{aligned}(\pi * x)(t) &= \bigvee_{u \in \mathbf{V}^{(B)}} \pi([u = x]) \wedge [t = \pi * u] \\ &= \bigvee_{u \in \mathbf{V}^{(B)}} (\pi \circ < x >)(u) \wedge [t = \pi * u] = (\pi^* < x >)(t).\end{aligned}$$

Therefore, $< \pi * x > = \pi^* < x > = \pi * X$. The converse statement is also valid if π is injective.

It should be also remarked that due to **2.2.2** (4) the above statement agrees with **2.2.1**.

2.5.12. For every $\mathbf{V}^{(B)}$-class X and for any predicative B-formula φ with a single free variable the following presentations are valid:

$$[(\forall x \in \pi * X)\ \varphi(x)]^C = \bigwedge_{t \in \mathbf{V}^{(B)}} \pi \circ X(t) \Rightarrow [\varphi(\pi * t)]^C,$$

$$[(\exists x \in \pi * X)\ \varphi(x)]^C = \bigwedge_{t \in \mathbf{V}^{(B)}} \pi \circ X(t) \wedge [\varphi(\pi * t)]^C.$$

$\lhd$ It suffices to substantiate one of these relations, for instance, the first one. Here are the corresponding calculations (where use has been made of **1.1.5** (3), **2.1.8** (7) and $(a \wedge b) \Rightarrow (c \wedge b) = (a \wedge b) \Rightarrow c$):

$$\begin{aligned}[(\forall x \in \pi * X)\ \varphi(x)] &= \bigwedge_{x \in \mathbf{V}^{(C)}} [x \in \pi * X)] \Rightarrow [\varphi(x)] \\ &= \bigwedge_{x \in \mathbf{V}^{(C)}} \left(\bigvee_{t \in \mathbf{V}^{(B)}} \pi \circ X(t) \wedge [x = \pi * t] \right) \Rightarrow [\varphi(x)] \\ &= \bigwedge_{t \in \mathbf{V}^{(B)}} \bigwedge_{x \in \mathbf{V}^{(C)}} (\pi \circ X(t) \wedge [x = \pi * t]) \Rightarrow [\varphi(x)] \\ &\leq \bigwedge_{t \in \mathbf{V}^{(B)}} \pi \circ X(t) \Rightarrow [\varphi(\pi * t)] \\ &= \bigwedge_{t \in \mathbf{V}^{(B)}} \bigwedge_{x \in \mathbf{V}^{(C)}} (\pi \circ X(t) \wedge [x = \pi * t]) \Rightarrow ([\varphi(\pi * t)] \wedge [x = \pi * t]) \\ &\leq \bigwedge_{t \in \mathbf{V}^{(B)}} \bigwedge_{x \in \mathbf{V}^{(C)}} (\pi \circ X(t) \wedge [x = \pi * t]) \Rightarrow [\varphi(x)] \\ &= \bigwedge_{t \in \mathbf{V}^{(C)}} \left(\bigvee_{t \in \mathbf{V}^{(B)}} \pi \circ X(t) \wedge [x = \pi * t] \right) \Rightarrow [\varphi(x)]\end{aligned}$$

$$= \bigwedge_{x \in V^{(C)}} [x \in \pi * X] \Rightarrow [\varphi(x)] = [(\forall x \in \pi * X)\ \varphi(x)]. \rhd$$

2.1.13. For any $V^{(B)}$-classes X *and* Y *the following relations hold*:

$$[\pi * X = \pi * Y] = \pi[X = Y],\ [\pi * X \in \pi * Y] = \pi[X \in Y].$$

$\lhd$ Let us first of all remark that $\pi \circ Y(t) = (\pi * Y)(\pi * t)$ or $\pi[t \in Y] = [\pi * t \in \pi * Y]$ for $t \in V^{(B)}$ (this result follows from **2.5.8** and **2.5.11** by virtue of **2.2.2** (3)). Then, using the first formula of **2.5.12**, we easily deduce

$$[\pi * X \subset \pi * Y]^C = [(\forall x \in \pi * X)\ (x \in \pi * Y)]^C$$
$$= \bigwedge_{t \in V^{(B)}} \pi \circ X(t) \Rightarrow [\pi * t \in \pi * Y]^C$$
$$= \bigwedge_{t \in V^{(B)}} \pi\left([t \in X]^B \Rightarrow [t \in Y]^B\right) = \pi[X \subset Y].$$

These relations give

$$[\pi * X = \pi * Y] = [\pi * X \subset \pi * Y] \wedge [\pi * Y \subset \pi * X] = \pi[X = Y].$$

Finally, taking into account the above, according to the second formula of **2.5.12**, we get

$$[\pi * X \in \pi * Y]^C = [(\exists t \in \pi * Y)\ (t = \pi * X)]^C$$
$$= \bigvee_{t \in V^{(B)}} \pi \circ Y(t) \wedge [\pi * t \in \pi * X]^C$$
$$= \bigvee_{t \in V^{(B)}} \pi\left(Y(t) \wedge [t = X]^B\right) = \pi[X \in Y]^B. \rhd$$

2.5.14. As in **2.2**, we can derive various corollaries from the results obtained above. Let us note only the following.

(1) *If* $\varphi(Y_1,\dots,Y_n)$ *is a bounded predicative formula, then for any* $V^{(B)}$*-classes* $X_1,\dots,X_n$ *we have*

$$\pi[\varphi(X_1,\dots,X_n)] = [\varphi(\pi * X_1,\dots,\pi * X_n)].$$

This, in particular, implies that π *is a monomorphism, then*

$$V^{(B)} \models \varphi(X_1,\dots,X_n) \leftrightarrow V^{(C)} \models \varphi(\pi * X_1,\dots,\pi * X_n).$$

(2) *If φ is a predicative formula of class Σ_1, then for the same $X_1,\dots,X_n$ we get*

$$\pi[\varphi(X_1,\dots,X_n)] \le [\varphi(\pi * X_1,\dots,\pi * X_n)].$$

In particular, the following implication is valid:

$$\mathbf{V}^{(B)} \models \varphi(X_1,\dots,X_n) \to \mathbf{V}^{(C)} \models \varphi(\pi * X_1,\dots,\pi * X_n).$$

⊲ The proof is carried out by the scheme of **2.2.3**. As an example, let us take the case of a bounded universal quantifier: $\varphi := (\forall x \in Y)\psi$. By **2.5.12** and **2.5.13**, for the $\mathbf{V}^{(B)}$-classes $X_1,\dots,X_n$ we have

$$\begin{aligned}
&[\varphi(\pi * Y, \pi * X_1,\dots,\pi * X_n)] \\
&= \bigwedge_{x \in \mathbf{V}^{(B)}} [\pi * x \in \pi * Y] \Rightarrow [\psi(\pi * x, \pi * X_1,\dots,\pi * X_n)] \\
&= \bigwedge_{x \in \mathbf{V}^{(B)}} \pi[X \in Y] \Rightarrow \pi[\psi(x, X_1,\dots,X_n)] \\
&= \pi\Big(\bigwedge_{x \in \mathbf{V}^{(B)}} [X \in Y] \Rightarrow [\psi(x, X_1,\dots,X_n)]\Big) \\
&= \pi[(\forall x \in Y\ \psi(x, X_1,\dots,X_n)] = \pi[\varphi(Y, X_1,\dots,X_n)]. \ ⊳
\end{aligned}$$

2.5.15. Making use of the canonical embedding $(\cdot)^\wedge : \mathbf{V} \to \mathbf{V}^{(B)}$, to each class $x \subset \mathbf{V}$ we assign a $\mathbf{V}^2$-class X' by the formula:

$$X'(t) := \mathbf{1}_2 \text{ if } (\exists x \in X)(t = x^\wedge),$$
$$X'(t) := \mathbf{0}_2 \text{ in the opposite case.}$$

It trivially follows from **2.1.8** (4) that X' is extensional. Then, let us put $X^\wedge := \iota * " X'$, where ι is the identical embedding of $\mathbf{2}$ in B. Hence, $X^\wedge$ is a $\mathbf{V}^{(B)}$-class, for which

$$X^\wedge(t) = \vee\{[t = x^\wedge] : x \in X\} \quad (t \in \mathbf{V}^{(B)}).$$

It should be remarked that since $\mathrm{Ord}(X)$ is a bounded predicative function, by **2.2.8** (4), **2.2.9** (1) and **2.5.14**, $\mathrm{On}^\wedge$ is an ordinal class inside $\mathbf{V}^{(B)}$, i.e., $\mathbf{V}^{(B)} \models \mathrm{Ord}(\mathrm{On}^\wedge)$.

Formulas **2.5.12** can obviously be specialized:

$$[(\forall x \in Y^\wedge)\ \varphi(x)] = \wedge\{[\varphi(x^\wedge)] : x \in Y\},$$
$$[(\exists x \in Y^\wedge)\ \varphi(x)] = \vee\{[\varphi(x^\wedge)] : x \in Y\}.$$

2.5.16. Let φ and ψ be predicative functions with free variables $X, X_1, \dots, X_n$, and let $Y_1, \dots, Y_n$ be some $\mathbf{V}^{(B)}$-classes. Then, if $[\varphi(x_0, Y_1, \dots, Y_n)] = \mathbf{1}$ for a certain $x_0 \in \mathbf{V}^{(B)}$, we have

$$[(\exists x)(\varphi(x, Y_1, \dots, Y_n) \to \psi(x, Y_1, \dots, Y_n))]$$
$$= \vee\{[\psi(x, Y_1, \dots, Y_n)]: x \in \mathbf{V}^{(B)} \wedge [\varphi(x, Y_1, \dots, Y_n)] = \mathbf{1}\},$$
$$[(\forall x)(\varphi(x, Y_1, \dots, Y_n) \to \psi(x, Y_1, \dots, Y_n))]$$
$$= \wedge\{[\psi(x, Y_1, \dots, Y_n)]: x \in \mathbf{V}^{(B)} \wedge [\varphi(x, Y_1, \dots, Y_n)] = \mathbf{1}\}.$$

◁ The proof is carried out by the same scheme as in **2.3.8.** ▷

2.5.17. Theorem (the maximum principle). *Let $\varphi(x)$ be a predicative B-formula with one free variable (which implies that φ can contain constants which are $\mathbf{V}^{(B)}$-classes or $\mathbf{V}^{(B)}$-sets). Then the following statements are valid:*

(1) *there is an element $x_0 \in \mathbf{V}^{(B)}$ for which*

$$[(\exists x)\ \varphi(x)] = [\varphi(x_0)];$$

(2) *if $\mathbf{V}^{(B)} \models (\exists x)\ \varphi(x)$, then there is an element $x_0 \in \mathbf{V}^{(B)}$ for which $\mathbf{V}^{(B)} \models \varphi(x_0)$;*

(3) *if $\mathbf{V}^{(B)} \models (\exists! x)\ \varphi(x)$, then there is a unique element $x_0 \in \mathbf{V}^{(B)}$ for which $\mathbf{V}^{(B)} \models \varphi(x_0)$.*

◁ The proof based on the mixing principle (see **2.5.3**) does not differ from the considerations given in **2.3.10** and **2.5.4.** ▷

2.5.18. Theorem (the transfer principle). *All the theorems of* NBG *are true within* $\mathbf{V}^{(B)}$.

◁ It suffices to make sure that the axioms of NBG are true inside $\mathbf{V}^{(B)}$.

(a) The axiom of extensionality for classes inside $\mathbf{V}^{(B)}$ is true, which follows directly from definitions **2.5.8** and **2.5.9**. The statement $\mathbf{V}^{(B)} \models \mathrm{NBG}_2$-$\mathrm{NBG}_5$ was proved in **2.4**.

(b) $\mathbf{V}^{(B)} \models \mathrm{NBG}_6$. The proof is carried out as in **2.4.5**, with the expressions $\varphi(t,u)$ substituted for by $(t,u) \in X$ throughout (see **2.4.5** and **1.3.4**).

(c) $\mathbf{V}^{(B)} \models \wedge_{k=7}^{13} \mathrm{NBG}_k$. It suffices to establish that inside $\mathbf{V}^{(B)}$ true is the statement **1.3.14**, the axioms NBG_7-NBG_{13} being its partial cases. Let the formula $\varphi(X_1,\dots,X_n,Y_1,\dots,Y_m)$ obey all the conditions of **1.3.14**. Let us consider arbitrary $\mathbf{V}^{(B)}$-classes $Y_1,\dots,Y_m$ and determine a $\mathbf{V}^{(B)}$-class Z by the formula

$$Z(t) := [(\exists x_1,\dots,x_n)(t = (x_1,\dots,x_n) \\ \wedge \varphi(x_1,\dots,x_n, Y_1,\dots,Y_m))].$$

It is easily checked that in this case

$$\mathbf{V}^{(B)} \models (\forall x_1,\dots,x_n)(\exists t)((t = (x_1,\dots,x_n) \\ \wedge t \in Z \leftrightarrow \varphi(x_1,\dots,x_n,Y_1,\dots,Y_N))).$$

(d) $\mathbf{V}^{(B)} \models \mathrm{NBG}_{14}$. We get the required result if we substitute the lowercase Latin letter x with the capital one X in **2.4.7**.

(e) $\mathbf{V}^{(B)} \models \mathrm{NBG}_{15}$. Let G be a function from On to $\mathbf{V}^{(B)}$. Let us put

$$F(t) := \vee\{[t = (\alpha^\wedge, G(\alpha))^B] : \alpha \in \mathrm{On}\}.$$

Then F is a $\mathbf{V}^{(B)}$-class and by analogy with **2.4.10**, we can successively calculate: $[\mathrm{Fnc}(F)] = \mathbf{1}$, $[\mathrm{Ord}(\mathrm{On}^\wedge) \wedge \mathrm{dom}(F) = \mathrm{On}^\wedge] = \mathbf{1}$ and $[\mathrm{im}(F) \supset \mathbf{U}_B] = \mathbf{1}$. Therefore, inside $\mathbf{V}^{(B)}$ the universal class $\mathbf{U}_B$ can be well-ordered. Hence, $\mathbf{V}^{(B)} \models$ 'there exists a selecting function for the class $\mathbf{U}_B$'. ▷

2.5.19. On the basis of theorem **2.1.18** there arises a possibility of operating with classes inside $\mathbf{V}^{(B)}$. As an example, let us consider the following definition of category in a Boolean-valued model.

A category $\mathfrak{K}$ inside $\mathbf{V}^{(B)}$ consists of classes Ob $\mathfrak{K}$, Mor $\mathfrak{K}$ and Com inside $\mathbf{V}^{(B)}$, which are called *the class of objects, the class of morphisms and the composition of the category* $\mathfrak{K}$,respectively, such that $\mathbf{V}^{(B)} \models (\mathfrak{K}1) - (\mathfrak{K}3)$:

($\mathfrak{K}$ 1) there are mappings D and R from Mor $\mathfrak{K}$ to Ob $\mathfrak{K}$ such that for any objects a and b the class $\mathfrak{K}(a,b) := H_{\mathfrak{K}}(a,b) := \{\alpha \in \mathrm{Mor}\,\mathfrak{K} : D(\alpha) = a, R(\alpha) = b\}$ is a set (termed the *set of morphisms from* a *to* b);

($\mathfrak{K}$ 2) Com is an associative partial binary operation on Mor $\mathfrak{K}$ and

$$\operatorname{dom}(\mathrm{Com}) := \{(\alpha,\beta) \in (\mathrm{Mor}\,\mathfrak{K})^2 : D(\beta) = R(\alpha)\};$$

($\mathfrak{K}$ 3) for every object $a \in \mathrm{Ob}\,\mathfrak{K}$ there is a morphism 1_a which is called *the identity morphism of the object a* such that $D(1_a) = R(1_a) = a$, $\mathrm{Com}(1_a,\alpha) = \alpha$ for $R(\alpha) = a$ and $\mathrm{Com}(\beta,1_a) = \beta$ for $D(\beta) = a$.

We usually write $\beta\alpha$ or $\beta \circ \alpha$ instead of $\mathrm{Com}(\alpha,\beta)$.

2.5.20. Remarks

(1) The model $\mathbf{V}^{(B)}$ can be characterized axiomatically. Namely, there is a class $\mathbf{V}^{(B)}$ unique up to a bijection preserving all the Boolean truth-values and obeying the following conditions:

(a) there are two mappings, $[\cdot \in \cdot], [\cdot = \cdot]: \mathbf{V}^{(B)} \times \mathbf{V}^{(B)} \to B$ such that the conventional axioms of equality hold inside $\mathbf{V}^{(B)}$ (see **2.1.7**, **2.1.8**);

(b) $\mathbf{V}^{(B)}$ is separated, i.e., $[x = y] = \mathbf{1}_B$ implies that $x = y$ for $x, y \in \mathbf{V}^{(B)}$;

(c) the axioms of extensionality and foundation are true inside $\mathbf{V}^{(B)}$;

(d) for $\mathbf{V}^{(B)}$, statement **2.5.6** is valid.

(2) Let π be a full homomorphism from B to a complete Boolean algebra C. Then π^* is a unique mapping from $\mathbf{V}^{(B)}$ to $\mathbf{V}^{(C)}$, for which, first,

$$[\pi^* x = \pi^* y]^C = \pi[x = y]^B \quad (x,y \in \mathbf{V}^{(B)}),$$

and, second, for $y \in \mathbf{V}^{(B)}$ and $z \in \mathbf{V}^{(C)}$,

$$[z \in \pi^* y]^C \le \bigvee_{x \in \mathbf{V}^{(B)}} [z = \pi^* x].$$

CHAPTER 3

FUNCTORS OF BOOLEAN-VALUED ANALYSIS

Owing to the transfer and maximum principles, various constructions of conventional mathematical practice are possible within a Boolean-valued universe. In particular, in such a model there are fields of real numbers, Banach spaces, differential operators, etc.. The objects presenting them serve, in a certain sense, as a nonstandard realizations of the initial mathematical constructions. Therefore, assuming that the model $\mathbf{V}^{(B)}$ is a nonstandard presentation of the mathematical world and taking into account the fact that $\mathbf{V}^{(B)}$ is constructed within the von Neumann universe, we are in a position as if we look inside the Boolean-valued world and see a standard presentation of nonstandard objects. While examining algebras B item-by-item, an observer sees a great number of modifications of the same idea encoded in a set-theoretic formula. It is comparing them with one another that comprises the method of studying the mathematical idea they are based on. Besides, one discovers that essentially different analytical objects are nothing but presentations of the same concept. This fact enables one to clarify the internal reasons accounting for many analogies, as well as facilitates the appearance of new opportunities to study old objects.

(The situation presented above resembles the famous image of the Platon cave. If someone from this cave has managed to flee into the sunlight, then he, wishing to tell the others what he has seen, could light several fires outside this cave. In that event the things would throw not one but many distinctive shadows on the cave walls and the prisoners of the cave could penetrate into the essence of the things in-depth by way of comparing their shadows.)

The comparative analysis with the help of Boolean-valued models is carried out in two stages which can be conditionally called syntactic and semantic. At the syntactic stage the mathematical statement under investigation (a definition, a construction, a property, etc.) is transformed into a formal text of the symbolic language of set theory, or, to be more precise, into a text in a suitable jargon. In this stage we often have to investigate the complexity of the text under study and, in particular, to see if the text or some of its fragments is bounded formulas (Σ_1-formulas). The semantic stage is interpretation of a formal text in a Boolean-valued universe. In this stage we use the terms of the conventional set theory, i.e., the universe $\mathbf{V}$, to interpret (decode, translate) meaningful texts for objects of the Boolean-valued universe $\mathbf{V}^{(B)}$. This is carried out with elements and subsets of the Boolean-valued universe and the von Neumann universe by using exactly defined operations. In the present chapter we shall consider the basic operations of Boolean-valued analysis, i.e., the canonical embedding, descent, ascent and immersion. The most important properties of these operations can be conveniently explored by using the notions of category and functor, the latter being not obligatory. Therefore, assuming that the reader is not a specialist in the theory of categories, we shall make use of the initial notions of the aforementioned theory.

3.1. Canonical Embedding

This section is devoted to a detailed investigation of the method of embedding the class of all sets into a Boolean-valued universe.

3.1.1. Theorem. *The following statements are valid:*

(1) *if a class* $X \subset \mathbf{V}$ *and an element* $z \in \mathbf{V}^{(B)}$ *are such that* $\mathbf{V}^{(B)} \models z \in X^\wedge$, *then* $z = \operatorname{mix}_{x \in X}(b_x x^\wedge)$ *for a certain partition of unity* $(b_x)_{x \in X}$ *in* B;

(2) *for a* $\mathbf{V}^{(2)}$*-class* Y *there is a unique class* $X \subset \mathbf{V}$ *such that* $\mathbf{V}^{(2)} \models X^\wedge = Y$;

(3) *for* $X \subset \mathbf{V}$ *and* $Y \subset \mathbf{V}$ *we have:*

$$X \in Y \leftrightarrow \mathbf{V}^{(B)} \models X^\wedge \in Y^\wedge, \quad X = Y \leftrightarrow \mathbf{V}^{(B)} \models X^\wedge = Y^\wedge;$$

(4) *if* $\pi : B \to C$ *is a full homomorphism, then* $\pi^* : X^\wedge = X^{\overset{\wedge}{\wedge}}$ *for every class* $X \subset \mathbf{V}$, *where* $X^{\overset{\wedge}{\wedge}}$ *is the canonical embedding of* X *in* $\mathbf{V}^{(C)}$.

$\triangleleft$ (1) For $x \in X$ let us put $b_x := [x^\wedge = z]$. Then, by **2.2.8** (2), for $x, y \in X, x \neq y$,

$$b_x \wedge b_y \leq [x^\wedge = y^\wedge] = \mathbf{0}.$$

On the other hand,

$$\vee\{b_x : x \in X\} = X^\wedge(z) = [z \in X^\wedge] = \mathbf{1},$$

so that $(b_x)_{x \in X}$ is a partition of unity, and $z = \operatorname{mix}_{x \in X}(b_x x^\wedge)$.

(2) The proof follows from **2.2.8**. Indeed, if $X' := \{y \in \mathbf{V}^{(2)} : [y \in Y] = \mathbf{1}_2\}$ and $X := \{x \in \mathbf{V} : x^\wedge \in X'\}$, then, according to **2.2.8** (3,4), for $t \in \mathbf{V}^{(2)}$ we get

$$X \wedge (t) = \vee\{[t = x^\wedge]^2 : x \in X\} = \vee\{[t = x^\wedge]^2 : Y(x) = \mathbf{1}_2\}$$

$$= \vee\{Y(x) \wedge [t = x^{\wedge}]^2 : x \in \mathbf{V}^{(2)}\} = Y(t).$$

The uniqueness follows from **2.2.8** (4) and **2.5.15**.

(3) We have to compare **2.5.15** and (2).

(4) If ι_1 and ι_2 are embeddings of the algebra **2** in B or C, respectively, then $\pi \circ \iota_1 = \iota_2$ and, in line with **2.5.11**,

$$\pi * X^{\wedge} = \pi * \circ \iota *_1 (X^{\wedge}) = \iota *_2 X^{\wedge} = X^{\overset{\wedge}{\wedge}}. \triangleright$$

3.1.2. *If x and y are sets then*

$$\{x\}^{\wedge} = \{x^{\wedge}\}^B, \quad \{x, y\}^{\wedge} = \{x^{\wedge}, y^{\wedge}\}^B, \quad (x,y)^{\wedge} = (x^{\wedge}, y^{\wedge})^B.$$

$\triangleleft$ All these three formulas are bounded and, hence, from **2.2.9** we deduce

$$\mathbf{V}^{(B)} \models \{x\}^{\wedge} = \{x^{\wedge}\} \wedge \{x,y\}^{\wedge} = \{x^{\wedge}, y^{\wedge}\} \wedge (x, y)^{\wedge} = (x^{\wedge}, y^{\wedge}).$$

All we have to do now is to use the necessary relations of **2.4.8.** $\triangleright$

3.1.3. *Let a formula φ of class Σ_1 obey all the conditions of theorem* **1.3.14.** *Let us choose classes $Z_1,\dots,Z_n$, $Y_1,\dots,Y_m$, and let the class Y be determined by the formula*

$$Y := \{(x_1,\dots,x_n) : x_1 \in Z_1 \wedge \dots \wedge x_n \in Z_n \\ \wedge \varphi(x_1,\dots,x_n, Y_1,\dots,Y_m)\}.$$

In this case inside $\mathbf{V}^{(B)}$ the following relation holds:

$$Y^{\wedge} = \{(x_1,\dots,x_n) : x_1 \in Z_1^{\wedge} \wedge \dots \wedge x_n \in Z_n^{\wedge} \wedge \varphi(x_1,\dots,x_n, Y_1^{\wedge},\dots,Y_m^{\wedge})\}.$$

$\triangleleft$ According to theorem **1.3.14**, Y is the only class obeying the conditions $\Phi(Z_1,\dots,Z_n, Y_1,\dots,Y_m)$ and $\Psi(Z_1,\dots,Z_n, Y_1,\dots,Y_m)$, where Φ and Ψ have the form

$$\Phi := (\forall u \in Y)(\exists x_1 \in Z_1)\dots(\exists x_n \in Z_n)(u = (x_1,\dots,x_n) \wedge \varphi(x_1,\dots,Y_m)),$$
$$\Psi := (\forall x_1 \in Z_1)\dots(\forall x_n \in Z_n)(\exists u) \\ (u = (x_1,\dots,x_n) \wedge \varphi(x_1,\dots,Y_m) \to u \in Y).$$

Obviously, Φ and Ψ are formulas of class Σ_1 and, hence, by **2.5.14**, we have

$$\mathbf{V}^{(B)} \models \Phi(Z_1^{\wedge}, \dots, Y_m^{\wedge}) \wedge \Psi(Z_1^{\wedge}, \dots, Y_m^{\wedge}).$$

The relation obtained is equivalent to the sought one. ▷

3.1.4. *For any classes* $X \subset \mathbf{V}$ *and* $Y \subset \mathbf{V}$ *the following statements are valid:*

(1) $\mathbf{V}^{(B)} \models (X \cup Y)^{\wedge} = X^{\wedge} \cup Y^{\wedge}$;

(2) $\mathbf{V}^{(B)} \models (X \times Y)^{\wedge} = X^{\wedge} \times Y^{\wedge}$;

(3) $\mathbf{V}^{(B)} \models (\cup X)^{\wedge} = \cup(X^{\wedge})$;

(4) $\mathrm{Rel}(X) \to \mathbf{V}^{(B)} \models \mathrm{Rel}(X^{\wedge})$;

(5) $(F: X \to Y) \to \mathbf{V}^{(B)} \models F^{\wedge}: X^{\wedge} \to Y^{\wedge}$;

(6) $\mathrm{Rel}(X) \to \mathbf{V}^{(B)} \models (X``Y)^{\wedge} = (X^{\wedge})``(Y^{\wedge})$;

(7) $\mathrm{Rel}(X) \to \mathbf{V}^{(B)} \models \mathrm{dom}(X^{\wedge}) = \mathrm{dom}(X)^{\wedge} \wedge \mathrm{im}(X^{\wedge}) = \mathrm{im}(X)^{\wedge}$.

◁ Formulas (1) - (5) result from **3.1.3** (see **I.3.1.11**, **I.3.1.12**). Statements (6) and (7) are beyond the scope of applicability of **3.1.3** and they are therefore deduced by direct calculations (with use made of **2.4.9**, **3.1.1** and **3.1.2**):

(6)
$$\begin{aligned}
&[t \in (X^{\wedge})``(Y^{\wedge})] = [(\exists u \in X^{\wedge})(\exists v \in Y)(u = (v,t))] \\
&= \bigvee_{u \in X} \bigvee_{v \in Y} [u^{\wedge} = (v^{\wedge}, t)] = \bigvee_{v \in Y} \bigvee_{(z,\omega) \in X} [z^{\wedge} = v^{\wedge}] \wedge [\omega^{\wedge} = t] \\
&= \wedge\{[\omega^{\wedge} = t] : v \in Y \wedge (v,\omega) \in X\} \\
&= [(\exists \omega \in (X^{\wedge})``(Y^{\wedge}))(t = \omega)] = [t \in (X``Y)^{\wedge}].
\end{aligned}$$

(7)
$$\begin{aligned}
&[t \in \mathrm{dom}(X^{\wedge})] = [(\exists u \in X^{\wedge})(\exists v)(u = (t,v))] \\
&= \bigvee_{(z,\omega) \in X} \bigvee_{v \in \mathbf{V}^{(B)}} [z^{\wedge} = t] \wedge [\omega^{\wedge} = v] \\
&= \vee\{[z^{\wedge} = t] : z \in \mathrm{dom}(X)\} = [t \in \mathrm{dom}(X)^{\wedge}]. \ \triangleright
\end{aligned}$$

3.1.5. Theorem. *Let X and Y be nonempty sets, $F \subset X \times Y$ and consider a correspondence $\Phi := (F, X, Y)$. Then an element $\Phi^{\wedge} \in \mathbf{V}^{(B)}$ obeys the following conditions:*

(1) $\mathbf{V}^{(B)} \models \Phi^{\wedge}$ *is a correspondence from* $X^{\wedge}$ *to* $Y^{\wedge}$, *and* $\mathrm{Gr}\Phi^{\wedge} = F^{\wedge}$;

(2) $\mathbf{V}^{(B)} \models \Phi^{\wedge}(A^{\wedge}) = \Phi(A)^{\wedge}$ *at all* $A \in \mathcal{P}(X)$;

(3) $\mathbf{V}^{(B)} \models (\Psi \circ \Phi)^{\wedge} = \Psi^{\wedge} \circ \Phi^{\wedge}$ *for any correspondence* Ψ;

(4) $\mathbf{V}^{(B)} \models (I_X)^{\wedge} = I_{X^{\wedge}}$.

◁ (1) If the formula $\varphi(X, Y, F, \Phi)$ states that Φ is a correspondence from $\varphi(X, Y, F, X)$ to Y and $F = \mathrm{Gr}\Phi$, then φ is a bounded formula and the required result follows from **2.2.9**.

(2) The result follows from **3.1.4** (6).

(3), (4). Here we again have to do with bounded formulas, and, hence, it suffices to refer to **2.2.9**. ▷

3.1.6. Corollary. *For any mapping $f: X \to Y$ the element $f^{\wedge}$ obeys the conditions:*

(1) $\mathbf{V}^{(B)} \models f^{\wedge}: X^{\wedge}) \to Y^{\wedge}$;

(2) $\mathbf{V}^{(B)} \models f^{\wedge}(x^{\wedge}) = (\mathrm{x})^{\wedge}$ *for all* $x \in X$;

(3) $\mathbf{V}^{(B)} \models (g \circ f)^{\wedge} = g^{\wedge} \circ f^{\wedge}$ *for any* $g: Y \to Z$.

3.1.7. Let us introduce the categories $\mathcal{V}_*$ and $\mathcal{V}_*^{(B)}$ that are associated with the universes $\mathbf{V}$ and $\mathbf{V}^{(B)}$. Without specifying it every time, let us assume that the classes of objects and morphisms of a category do not intersect (this can be achieved by using different indices). Let $\mathcal{V}_*$ be the category of nonempty sets and correspondences, so that $\mathrm{Ob}\,\mathcal{V}_* = \mathbf{V} \setminus \{\varnothing\}$ and $\mathcal{V}_*(x, y)$ is a set of all nonempty correspondences from x to y, the composition being the conventional superposition of correspondences.

The class of objects of the category $\mathcal{V}_*^{(B)}$ is formed by nonempty $\mathbf{V}^{(B)}$-sets:

$$\mathrm{Ob}\,\mathcal{V}_*^{(B)} := \{x \in \mathbf{V}^{(B)} : [x \neq \varnothing] = 1\}.$$

The set of morphisms from an object $x \in \operatorname{Ob} \mathfrak{V}_*^{(B)}$ into an object $y \in \operatorname{Ob} \mathfrak{V}_*^{(B)}$ is determined by the formula

$$\mathfrak{V}_*^{(B)}(x,y) := \{\alpha \in \mathbf{V}^{(B)} : [\alpha \text{ is a correspondence from } x \text{ to } y \wedge \operatorname{Gr}\alpha \neq \varnothing] = 1\}.$$

If α and β are morphisms of the category $\mathfrak{V}_*^{(B)}$, in which case $[D(\beta) = R(\alpha)] = 1$, then there is a unique element $\gamma \in \mathbf{V}^{(B)}$ (the principle of maximum!) such that $[\gamma = \beta \circ \alpha] = 1$. It is this element γ that is assumed to be the composition of the morphisms α and β in the category $\mathbf{V}^{(B)}$.

The subcategories of $\mathfrak{V}_*$ and $\mathfrak{V}_*^{(B)}$ consisting of the same objects and of the mappings as morphisms will be denoted by $\mathfrak{V}$ and $\mathfrak{V}^{(B)}$, respectively. By the symbol $\mathfrak{F}^\wedge$ we shall denote the mapping from $\mathfrak{V}_*$ to $\mathfrak{V}_*^{(B)}$, putting into correspondence to the set $x \in \mathbf{V} \setminus \{0\}$ and to the correspondence α the elements $x^\wedge \in \mathbf{V}^{(B)}$ and $\alpha^\wedge \in \mathbf{V}^{(B)}$. The following theorem results from **3.1.5** and **3.1.6**.

3.1.8. Theorem. *The mapping $\mathfrak{F}^\wedge$ is a covariant functor from the category $\mathfrak{V}_*$ to the category $\mathfrak{V}_*^{(B)}$ (as well as from the category $\mathfrak{V}$ to the category $\mathfrak{V}^{(B)}$).*

The functor $\mathfrak{F}^\wedge$ (as well as its restriction on the subcategory $\mathfrak{V}$) is termed *the canonical embedding functor* or *the standard name functor.*

3.1.9. Let us dwell on the properties of ordinals inside $\mathbf{V}^{(B)}$.

(1) It is known (see **2.4.10**) that $\operatorname{Ord}(X)$ is a bounded formula. Since $\lim(\alpha) \leq \alpha$ for any ordinal α, then the formula $\operatorname{Ord}(x) \wedge x = \lim(x)$ can be written as

$$\operatorname{Ord}(x) \wedge (\forall t \in x)(\exists s \in x)(t \in s),$$

and, hence, it is also bounded. Finally, the presentation

$$\operatorname{Ord}(x) \wedge x = \lim(x) \wedge (\forall t \in x)(t = \lim(t) \rightarrow t = 0)$$

states that 'the least limiting ordinal' is also a bounded formula. Therefore, by **2.2.9**, α is the (least) limiting ordinal iff $\mathbf{V}^{(B)} \models$ '$\alpha^\wedge$ is the (least) limiting ordinal'. Since ω is the least limiting ordinal (see **1.4.6**); therefore, $\mathbf{V}^{(B)} \models$ '$\omega^\wedge$ is the least limiting ordinal'.

(2) It follows from **1.4.5** (2), **2.5.15** and **2.5.16** that $\mathbf{V}^{(B)} \models$ '$\mathrm{On}^\wedge$ is the only ordinal class which is not an ordinal'. Therefore, for any $x \in \mathbf{V}^{(B)}$ the following relation is valid:

$$[\![\mathrm{Ord}(x)]\!] = \vee\{[\![x = \alpha^{\wedge}]\!] : \alpha \in \mathrm{On}\}.$$

(3) *For an arbitrary* $x \in \mathbf{V}^{(B)}$, *the equality* $\mathbf{V}^{(B)} \models \mathrm{Ord}(x)$ *holds iff there is an ordinal* $\beta < \mathrm{On}$ *and a partition of unity* $(b_\alpha)_{\alpha \in \beta} \subset B$ *such that* $x = mix_{\alpha\in\beta}(b_\alpha \alpha^{\wedge})$. *In other words, any ordinal inside* $\mathbf{V}^{(B)}$ *is a mixing of a certain set of standard ordinals.*

◁ This result follows from (2) and **3.1.1** (1). ▷

(4) From **2.5.15** we get the quantification formulas over ordinals:

$$[\![(\forall x)(\mathrm{Ord}(x) \to \psi(x)]\!] = \bigwedge_{\alpha \in \mathrm{On}} [\![\psi(\alpha^{\wedge})]\!],$$
$$[\![(\exists x)(\mathrm{Ord}(x) \wedge \psi(x)]\!] = \bigwedge_{\alpha \in \mathrm{On}} [\![\psi(\alpha^{\wedge})]\!].$$

3.1.10. A class X is called *finite* if it coincides with the image of a function determined on a finite ordinal. Symbolically this is expressed as $\mathrm{fin}(X)$, so that

$$\mathrm{fin}(X) := (\exists n)(\exists f)(n \in \omega \wedge \mathrm{Fnc}(f) \wedge \mathrm{dom}(f) = n \wedge \mathrm{im}(f) = X).$$

The above formula is obviously not bounded. By virtue of the axiom of replacement NBG_6, it is clear that $\mathrm{fin}(X) \to M(X)$, and, hence, we shall speak about finite sets instead of finite classes. The class of all finite subsets of the class X is denoted by $\mathcal{P}_{\mathrm{fin}}(X)$:

$$\mathcal{P}_{\mathrm{fin}}(X) := \{Y \in \mathcal{P}(X) : \mathrm{fin}(Y)\}.$$

Let us clarify what happens with finite sets under the canonical embedding of $\mathbf{V}$ in $\mathbf{V}^{(B)}$, i.e., what the class $\mathcal{P}_{\mathrm{fin}}(X)^{\wedge}$ is. Let us first show that

$$\mathbf{V}^{(B)} \models \mathcal{P}_{\mathrm{fin}}(X)^{\wedge} \subset \mathcal{P}_{\mathrm{fin}}(X^{\wedge}).$$

◁ It should be remarked that if f is a mapping of a certain $n \in \omega$ in X, then $[\![(f^{\wedge}) \in \mathcal{P}_{\mathrm{fin}}(X^{\wedge})]\!] = \mathbb{1}$. Indeed, according to **3.1.6**, $[\![f^{\wedge} : n^{\wedge} \to X^{\wedge}]\!] = [\![n^{\wedge} \in \omega^{\wedge}]\!] = \mathbb{1}$, and, hence,

$$[\![\mathrm{im}(f^{\wedge}) \in \mathcal{P}(X^{\wedge}) \wedge \mathrm{fin}(\mathrm{im}(f^{\wedge}))]\!] = \mathbb{1}.$$

For an arbitrary $t \in \mathbf{V}^{(B)}$ we can now easily calculate (see **2.2.8** (1), **3.1.4** (7), **3.1.6**):

$$
\begin{aligned}
&[t \in \mathcal{P}_{\text{fin}}(X)^\wedge] \\
&= \bigvee_{u \in \mathcal{P}_{\text{fin}}(X)} [t = u^\wedge] = \bigvee_{n \in \omega} \bigvee_{f:n \to X} [t = \operatorname{im}(f)^\wedge] \\
&= \bigvee_{n \in \omega} \bigvee_{f:n \to X} [t = \operatorname{im}(f^\wedge)] \wedge [n^\wedge \in \omega^\wedge] \wedge [f^\wedge : n^\wedge \to X^\wedge] \\
&\le [t \in \mathcal{P}_{\text{fin}}(X^\wedge)].
\end{aligned}
$$

3.1.11. *For any class X we have*

$$\mathbf{V}^{(B)} \models \mathcal{P}_{\text{fin}}(X)^\wedge = \mathcal{P}_{\text{fin}}(X^\wedge).$$

$\lhd$ Let us assume that for a $t \in \mathbf{V}^{(B)}$ the following relation holds:

$$[t \in \mathcal{P}_{\text{fin}}(X^\wedge)] = [(\exists n \in \omega^\wedge)(\exists f)(f : n \leftrightarrow X^\wedge \wedge t = \operatorname{im}(f)] = \mathbf{1}.$$

Then there is such a countable partition of unity $(b^{(n)})_{n \in \omega} \subset B$ that

$$[(\exists f)(f : n \to X^\wedge \wedge t = \operatorname{im}(f)] \ge b^{(n)} \quad (n \in \omega).$$

For every $n \in \omega$ we can, by virtue of the maximum principle, find such an $f_n' \in \mathbf{V}^{(B)}$ which obeys the inequality

$$[f_n' : n^\wedge \to X^\wedge] \wedge [t = \operatorname{im}(f_n')] \ge b^{(n)}.$$

Let us make use of statement **3.1.6**, and choose an $f_n'' \in \mathbf{V}^{(B)}$ in such a way that $[f_n'' : n^\wedge \to X^\wedge] \ge (b^{(n)})^*$, and let us set $f_n : \operatorname{mix}\{b^{(n)} f_n', \ (b^{(n)})^* f_n''\}$. Then $[f_n : n^\wedge \to X^\wedge] = \mathbf{1}$ and $[t = \operatorname{im}(f_n)] \ge b^{(n)}$. Then, for every $k \in n$ we have $[f_n(k^\wedge) \in X^\wedge] = \mathbf{1}$. Hence, $f_n(k) = \operatorname{mix}(b_x^{(k)} x^\wedge)$ for a certain partition of unity $(b_x^{(k)})_{x \in X}$ (see **3.1.1** (1)). Therefore,

$$[f_n(k^\wedge) = x^\wedge] \ge b_x^{(k)} \quad (x \in X, k \in n).$$

Let X^n be, as usual, a class of all mappings from n to X. It should be remarked that for $g \in X^n$ and $k \in n$ we get

$$[f_n(k^\wedge) = g^\wedge(k^\wedge)] = [f_n(k^\wedge) = g^\wedge(k)^\wedge] \ge b_{g(k)}^{(k)},$$

and, hence, $[f_n = g^\wedge] \ge b_{g,n}$, where $b_{g,n} := \wedge\{b_{g(k)}^{(k)} : k \in n\}$. In this case, however, the following relation also holds:

$$[\![\operatorname{im}(f) = \operatorname{im}(g^{\wedge})]\!] \geq b_{g,n} \quad (g \in X^n).$$

By definition, $\operatorname{im}(g) \in \mathcal{P}_{\text{fin}}(X)$, while by virtue of **3.1.4** (7) we have

$$[\![\operatorname{im}(g^{\wedge}) \in \mathcal{P}_{\text{fin}}(X)^{\wedge}]\!] = \mathbf{1}.$$

Hence, we get

$$[\![t \in \mathcal{P}_{\text{fin}}(X)^{\wedge}]\!] \geq [\![t = \operatorname{im}(f_n)]\!]$$
$$\wedge [\![\operatorname{im}(f_n) = \operatorname{im}(g^{\wedge})]\!] \wedge [\![\operatorname{im}(g^{\wedge}) \in \mathcal{P}_{\text{fin}}(X)^{\wedge}]\!] \geq b^{(n)} \wedge b_{g,n}.$$

Making use of the definition of the element $b_{g,n}$ and distributive laws **1.1.5** (1,2), we can calculate

$$\bigvee \{b^{(n)} \wedge b_{g,n} : n \in \omega \wedge g \in X^n\} = \bigvee_{n \in \omega} b^{(n)} \wedge \left(\bigvee_{g \in X^n} \bigwedge_{k \in n} b^{(k)}_{g(k)} \right)$$
$$= \bigvee_{n \in \omega} b^{(n)} \wedge \left(\bigwedge_{k \in n} \bigvee_{g \in X^n} b^{(k)}_{g(k)} \right) = \bigvee_{n \in \omega} b^{(n)} \wedge \left(\bigwedge_{k \in n} \bigvee_{g \in X} b^{(k)}_{x} \right) = \bigvee_{n \in \omega} b^{(n)} = \mathbf{1}.$$

As is seen, $[\![t \in \mathcal{P}_{\text{fin}}(X)^{\wedge}]\!] = \mathbf{1}$, and, applying **2.5.16**, we deduce $[\![\mathcal{P}_{\text{fin}}(X^{\wedge}) \subset \mathcal{P}_{\text{fin}}(X)^{\wedge}]\!] = \mathbf{1}$. The reverse inclusion is established in **3.1.10**. $\triangleright$

3.1.12. *For any class X and every $n \in \omega$ the following relations are valid:*

(1) $\mathbf{V}^{(B)} \models (X^n)^{\wedge} = (X^{\wedge})^{n^{\wedge}}$;

(2) $\mathbf{V}^{(B)} \models \mathcal{P}(X)^{\wedge} \subset \mathcal{P}(X^{\wedge})$.

$\triangleleft$ (1) Taking into account **3.1.6**, we can write for an arbitrary $t \in \mathbf{V}^{(B)}$:

$$[\![t \in (X^n)^{\wedge}]\!] = \bigvee \{[\![t = u^{\wedge}]\!] : u \in X^n\}$$
$$= \bigvee \{[\![t = u^{\wedge}]\!] \wedge [\![u^{\wedge} : n^{\wedge} \to X^{\wedge}]\!] : u \in X^n\}$$
$$\leq \bigvee \{[\![t = u]\!] \wedge [\![u : n^{\wedge} \to X^n]\!] : u \in \mathbf{V}^{(B)}\}$$
$$= [\![(\exists u)(u : n^{\wedge} \to X^{\wedge} \wedge t = u)]\!] = [\![t \in (X^{\wedge})^{n^{\wedge}}]\!].$$

Therefore, we have established

$$[(X^n)^\wedge \subset (X^\wedge)^{n^\wedge}] = \mathbf{1}.$$

To prove the reverse inclusion, let us consider such an element $u \in \mathbf{V}^{(B)}$ that $[u{:}n^\wedge \to (X^\wedge] = \mathbf{1}$. In this case $[u(k^\wedge) \in X^\wedge] = \mathbf{1}$ $(k \in n)$ and, hence, $[u(k^\wedge) = \operatorname{mix}(b_x^{(k)} x^\wedge)] = \mathbf{1}$ for a certain partition of unity $(b_x^{(k)})_{x \in X}$. Going over to refined partitions of unity, we can, if necessary, choose such a partition of unity (b_ξ) and such families $(x_{k,\xi}) \subset X$ $(k \in n)$ that $[u(k^\wedge) = \operatorname{mix}(b_\xi x_{k,\xi}^\wedge)] = \mathbf{1}$ for all $k \in n$. Let us determine the functions $u_\xi : n \to X$ by the relations $u_\xi(k) := x_{k,\xi}$. Then $[u = u_\xi^\wedge] \geq b_\xi$ and $[u_\xi^\wedge \in (X^n)^\wedge] = \mathbf{1}$ and, hence, $[u \in (X^n)^\wedge] = \mathbf{1}$. By virtue of **2.5.16**, we get $[(X^\wedge)^{n\wedge} \subset (X^n)^\wedge] = \mathbf{1}$.

(2) This result is obtained by direct calculations. ▷

3.1.13. Remarks

(1) Cardinals inside the model $\mathbf{V}^{(B)}$ are a greater problem than ordinals (see **3.1.9**). One can easily notice that $\neg\operatorname{Card}(x)$ is a Σ_1-formula and, hence, $[\operatorname{Card}(\alpha^\wedge)] = \mathbf{1} \to \operatorname{Card}(\alpha)$. The formula $\neg\operatorname{Card}(x)$ is not, however, of class Σ_1 and, hence, the reverse implication might be violated and an ordinal might lose the property to be a cardinal under the canonical embedding in $\mathbf{V}^{(B)}$. In fact for infinite cardinals $\lambda < \aleph$ one can choose such a complete Boolean algebra B that $\mathbf{V}^{(B)} \models |\lambda^\wedge| = |\aleph^\wedge|$, this peculiarity called the *displacement of cardinal numbers.* One can also choose such a B that $\mathbf{V}^{(B)} \models 2^{\omega_\alpha} = \omega_{\beta+1}$ for some $\alpha < \beta$. In this way one can establish the consistency of $\neg$ GCH with ZFC [10, 96, 248].

(2) In spite of what has been said in (1), cardinals in $\mathbf{V}^{(B)}$ behave themselves provided B is made to obey the countability of antichains, i.e., if any antichain in B is no more than countable (B is also said to be of *countable type*). For such a B we have

$$\mathbf{V}^{(B)} \models \operatorname{Card}(\alpha^\wedge) \leftrightarrow \operatorname{Card}(\alpha),$$
$$\mathbf{V}^{(B)} \models (\omega_\alpha)^\wedge = \omega_\alpha{}^\wedge.$$

(3) The properties of constructible sets (see **1.5.10**) inside $\mathbf{V}^{(B)}$ are like those of ordinals. Namely, if $L(x)$ is a formula stating that x is a constructible set, then

$$[L(u)] = \vee\{[u = v] : v \in \mathbf{L}\} \quad (u \in \mathbf{V}^{(B)})$$

and statements **3.1.9** (2) - **3.1.9** (4) are preserved, provided Ord is replaced with L in them (see [10, 96, 248]).

(4) By virtue of **3.1.11**, it might seem that in **3.1.12** (2) there is equality, i.e., $[\mathcal{P}(X^\wedge) = \mathcal{P}(X)^\wedge] = \mathbf{1}$. This, however, is not the case since if B is an algebra of regular closed subsets of the Cantorian ω-discontinuum, then $[\mathcal{P}(\omega^\wedge) \neq \mathcal{P}(\omega)^\wedge] = \mathbf{1}$.

3.2. The Descent Functor

Presented here are the basic techniques of translating propositions about elements of the universe $\mathbf{V}^{(B)}$ in statements on sets, the role of the translator performed by the operation of descent. The word 'descent' is used for denoting both the result and the method of presenting elements of $\mathbf{V}^{(B)}$ in the universe $\mathbf{V}$. Therefore, speaking not formally, the descent is acting from $\mathbf{V}^{(B)}$ to $\mathbf{V}$.

3.2.1. Let us consider an arbitrary $\mathbf{V}^{(B)}$-class $X: \mathbf{V}^{(B)} \to B$ and put

$$X{\downarrow} := \{x \in \mathbf{V}^{(B)} : [x \in X] = \mathbf{1}_B\}.$$

This equality defines a certain subclass $X{\downarrow}$ of the universal class $\mathbf{V}$ which is called the *descent* of the $\mathbf{V}^{(B)}$-class X. Let $X_\varphi := \overline{\varphi}$ be a class inside $\mathbf{V}^{(B)}$ which is definable by the B-formula φ (see **2.5.5**). Then the descent of the class X_φ has the form

$$X_\varphi{\downarrow} = \{x \in \mathbf{V}^{(B)} : [\varphi(x)] = \mathbf{1}\}.$$

In this case the formula $x \in X_\varphi{\downarrow}$ is expressed by the words 'x satisfies φ inside $\mathbf{V}^{(B)}$'. Thus, for instance, if $f \in \mathbf{V}^{(B)}$ and $[\mathrm{Fnc}(f)] = \mathbf{1}$, then f is said to be a *function inside* $\mathbf{V}^{(B)}$ or *in the model* $\mathbf{V}^{(B)}$. It is obvious that the descent of the universal $\mathbf{V}^{(B)}$-class $\mathbf{U}_B$ coincides with $\mathbf{V}^{(B)}$. Let us immediately remark two expedient formulas resulting immediately from **2.5**:

$$[X_\varphi \subset X_\psi] = \wedge\{[\psi(x)] : x \in X_\varphi{\downarrow}\},$$
$$[X_\varphi \cap X_\psi \neq \varnothing] = \vee\{[\psi(x)] : x \in X_\varphi{\downarrow}\},$$

where φ and ψ are arbitrary B-formulas.

Below we shall systematically use the following technique of abbreviation. Let the symbol f be a (conventional) notation for a certain n-ary function, for instance, $\{\cdot,\cdot\}$, $(\cdot,\cdot)$,

$\Phi(\cdot)$, $\pi_\Phi(\cdot)$, etc.. Then for any $x_1,\dots,x_n \in \mathbf{V}^{(B)}$ there exists a unique element $x_f \in \mathbf{V}^{(B)}$ such that

$$[x_f = f(x_1,\dots,x_n)] = [(\exists x)\,(x_1,\dots,x_n,x) \in f].$$

Under these circumstances instead of $x_f \downarrow$ we simply write $f(x_1,\dots,x_n)\downarrow$. For instance, $\Phi(A)\downarrow$ is the class determined by the relation

$$y \in \Phi(A) \leftrightarrow ([(\exists x \in A)(y \in \Phi(x))] = \mathbf{1}).$$

3.2.2. Let X be a subclass of the class $\mathbf{V}^{(B)}$ (i.e., $X \in \mathbf{V}^{(B)}$ in the sense of $\mathbf{V}$). They say that X is *cyclic* (or *extended*, or *disjointly complete*) and write Cyc$X$ provided X is closed relative to mixings of its any subfamilies over arbitrary partitions of unity. In other words, our class X is cyclic when for any partition of unity, $(b_\xi)_{\xi\in\Xi} \subset B$, and any family $(x_\xi)_{\xi\in\Xi} \subset X$ we have $\mathrm{mix}_{\xi\in\Xi}(b_\xi x_\xi) \in X$. The intersection of any set of cyclic sets is obviously a cyclic set, too.

The least cyclic set containing the given set $M \in \mathbf{V}^{(B)}$ is called the *cyclic envelope*, the *cyclic hull*, or the *cyclic extension* of M and is denoted by cyc(M). Obviously, the set $M \in \mathbf{V}^{(B)}$ is cyclic iff $M = \mathrm{cyc}(M)$.

3.2.3. *Let X and Y are classes inside $\mathbf{V}^{(B)}$. The following statements are valid:*

(1) $[X \neq \varnothing] = 1 \to X\downarrow \neq \varnothing \wedge \mathrm{Cyc}(X\downarrow)$;

(2) $X \in \mathbf{V}^{(B)} \to X\downarrow \in \mathbf{V}$;

(3) $X = Y \leftrightarrow X\downarrow = Y\downarrow$.

$\triangleleft$ (1) By the maximum principle, the class $X\downarrow$ is nonempty. If $(x_\xi)_{\xi\in\Xi} \subset X\downarrow$ and $(b_\xi)_{\xi\in\Xi}$ is a partition of unity, then for $x := \mathrm{mix}_{\xi\in\Xi}(b_\xi x_\xi)$ we have

$$[x \in X] \geq [x = x_\xi] \wedge [x_\xi \in X] \geq b_\xi \quad (\xi \in \Xi).$$

Therefore, $[x \in X] \geq \vee_{\xi\in\Xi} b_\xi = \mathbf{1}$ and $x \in X\downarrow$.

(2) Let us assume that $X \in \mathbf{V}^{(B)}$ and $x \in X\downarrow$. Let $u{:}\,\mathrm{dom}(u) \to B$ be such a function that $\mathrm{dom}(u) \subset \mathbf{V}^{(B)}$, $\mathrm{dom}(u) \in \mathbf{V}$ and $\bar{u}(\cdot) = [\cdot \in X]$ (see **2.5.6**). Then

$$\vee\{u(t) \wedge [t = x] : t \in \mathrm{dom}(u)\} = \mathbf{1}.$$

Using the exhaustion principle **2.3.9**, we find a partition of unity $(b_\xi) \subset B$ and a family $(t_\xi) \subset \mathrm{dom}(u)$ such that $u(t_\xi) \wedge [x = t_\xi] \geq b_\xi$, which implies the equality $x = \mathrm{mix}(b_\xi t_\xi)$. By $\mathrm{Part}(B)$ we denote a set of all partitions of unity in B and put

$$Y := \cup\{(\mathrm{dom}(u))^\theta : \theta \in \mathrm{Part}(B)\}.$$

Let us consider a function F which assigns to each x the set of ordered pairs (θ, v) for which $\theta \in \mathrm{Part}(B)$, $v : \theta \to \mathrm{dom}(u)$ and if $\theta := (b_\xi)$. Then $x = \mathrm{mix}(b_\xi x_\xi)$, where $x_\xi := v(b_\xi)$. Obviously, $\mathrm{dom}(F) \supset X\downarrow$, $\mathrm{im}(F) \subset \mathcal{P}(\mathrm{Part}(B) \times Y)$ and $F(x) \cap F(y) = \varnothing$ for $x \neq y$. Therefore, $|X\downarrow| \leq |\mathcal{P}(\mathrm{Part}(B) \times Y)|$ and $X\downarrow \in \mathbf{V}$.

(3) If $X\downarrow = Y\downarrow$, then, by **2.5.16**,

$$[X \subset Y] = \bigwedge_{t \in X\downarrow} [t \in Y] = \bigwedge_{t \in Y\downarrow} [t \in Y] = \mathbf{1}.$$

Analogously, $[Y \subset Y] = \mathbf{1}$ and, hence, $[X = Y] = \mathbf{1}$. ▷

3.2.4. Let X and Y be two $\mathbf{V}^{(B)}$-classes, while $X \times_B Y$ be their Cartesian product, inside $\mathbf{V}^{(B)}$, which exists by virtue of **1.3.13** (2) and **2.5.18**. The mapping

$$(\cdot;\cdot)^B : (x, y) \to (x, y)^B \quad (x \in X\downarrow, y \in Y\downarrow)$$

implements a bijection from the class $X\downarrow \times Y\downarrow$ onto the class $(X \times_B Y)\downarrow$. In this case

$$[\mathrm{Pr}_{X\downarrow}(x, y) = \mathrm{Pr}_X(x, y)] = [\mathrm{Pr}_{Y\downarrow}(x, y) = \mathrm{Pr}_Y(x, y)] = \mathbf{1}$$
$$(x \in X\downarrow, y \in Y\downarrow),$$

where $\mathrm{Pr}_{X\downarrow}, \mathrm{Pr}_{Y\downarrow}$ are the canonical projections on the factors $X\downarrow$ and $Y\downarrow$, respectively, while $\mathrm{Pr}_X, \mathrm{Pr}_Y$ are canonical projections inside $\mathbf{V}^{(B)}$ on X and Y, respectively. (One should bear in mind that Pr_X and Pr_Y are classes inside $\mathbf{V}^{(B)}$, while $\mathrm{Pr}_{X\downarrow}$ and $\mathrm{Pr}_{Y\downarrow}$ are classes in the sense of $\mathbf{V}$.)

◁ As has been noted earlier (see **2.4.9** and **2.5.3**), the function $(\cdot,\cdot)^B$ is an injective embedding of the class $\mathbf{V}^{(B)} \times \mathbf{V}^{(B)}$ in the class $\mathbf{V}^{(B)}$. It is hence sufficient to establish that $(\cdot,\cdot)^B$ maps $X\downarrow \times Y\downarrow \subset \mathbf{V}^{(B)} \times \mathbf{V}^{(B)}$ on $(X \times_B Y)\downarrow$. For any $x \in X\downarrow$ and $y \in Y\downarrow$ we have

$$[\![(x,y)^B \in X \times Y]\!]$$
$$= [\![(\exists u)(\exists v)(u \in X \wedge v \in Y \wedge (u,v) = (x,y)^B)]\!]$$
$$= \bigvee_{u \in \mathbf{V}^{(B)}} \bigvee_{v \in \mathbf{V}^{(B)}} [\![u \in X]\!] \wedge [\![v \in Y]\!] \wedge [\![(u,v) = (x,y)^B]\!]$$
$$\geq [\![x \in X]\!] \wedge [\![y \in Y]\!] \wedge [\![(x,y) = (x,y)^B]\!] = 1.$$

Therefore, $(x,y)^B \in (X \times_B Y)\downarrow$. Let us now consider an arbitrary element $z \in (X \times_B Y)\downarrow$ and remark that according to the maximum principle we find elements x and $y \in \mathbf{V}^{(B)}$ such that

$$1 = [\![z \in X \times Y]\!] = [\![(\exists u \in X)(\exists v \in Y)(z = (u,v))]\!]$$
$$= [\![x \in X]\!] \wedge [\![y \in Y]\!] \wedge [\![z = (x,y)]\!].$$

Hence, $x \in X\downarrow$, $y \in Y\downarrow$ and $z = (x,y)^B$. Finally, for $x \in X\downarrow$, $y \in Y\downarrow$ and $z \in \mathbf{V}^{(B)}$ we get

$$[\![z = \mathrm{Pr}_X(x,y)]\!] = [\![((x,y),z) \in \mathrm{Pr}_X]\!] = [\![z = x]\!] = [\![z = \mathrm{Pr}_{X\downarrow}(x,y)]\!],$$

which ensures validity for the required relation on the canonical projection on X. The situation is analogous for the projection on the second factor. ▷

3.2.5. Let us consider a binary relation X inside $\mathbf{V}^{(B)}$. This implies that the class X is inside $\mathbf{V}^{(B)}$ and $[\![X$ is a binary relation $]\!] = 1$. In accordance with **3.2.4** and the axiom of domain NBG, there is a class Y such that

$$(x,y) \in Y \leftrightarrow (x,y)^B \in X\downarrow.$$

Indeed, we must put

$$Y := \mathrm{dom}((\cdot,\cdot)^B \cap (\mathbf{V}^{(B)} \times \mathbf{V}^{(B)} \times X\downarrow)).$$

It is obvious that Y is a binary relation and that $(\cdot,\cdot)^B$ carries out a bijection between Y and $X\downarrow$. The class Y is called the *descent of the binary relation* X, and we shall preserve the symbol $X\downarrow$ for its notation. In an absolutely analogous way we determine the descent of an $n^\wedge$-ary relation X, namely:

$$X\downarrow := \{(x_1,\ldots,x_n) \in (\mathbf{V}^{(B)})^n : (x_1,\ldots,x_n)^B \in X\downarrow\}.$$

Therefore, the descent of the class X and that of the binary relation X are not the same, while the common notation $X\downarrow$ is a conveniently free choice, which should be taken into consideration to avoid ambiguity. For instance, the equality $(X \times_B Y)\downarrow = X\downarrow \times Y\downarrow$ is to be viewed only as another presentation of the first part of **3.2.4**. The same remarks are also true for the descents of correspondences, categories, etc., to be defined below.

3.2.6. Theorem. *For classes X and Y, the following formulas are valid:*

(1) $\operatorname{dom}(X)\downarrow = \operatorname{dom}(X\downarrow)$, $\operatorname{im}(X)\downarrow = \operatorname{im}(X\downarrow)$;

(2) $(X \cap Y)\downarrow = X\downarrow \cap Y\downarrow$;

(3) $(X|Y)\downarrow = (X\downarrow)|(Y\downarrow)$;

(4) $(X^{-1})\downarrow = (X\downarrow)^{-1}$;

(5) $(X \circ Y)\downarrow = (X\downarrow)\circ(Y\downarrow)$;

(6) $(X``Y)\downarrow = (X\downarrow)``(Y\downarrow)$;

(7) $(\mathbf{V}^{(B)} \models \operatorname{Fnc}(X)) \leftrightarrow \operatorname{Fnc}(X\downarrow)$;

(8) $(\mathbf{V}^{(B)} \models X \subset Y) \leftrightarrow X\downarrow \subset Y\downarrow$;

(9) $[x = y] \le [X(x) = X(y)]$ $(x, y \in \mathbf{V}^{(B)})$;

(10) $(X\downarrow)^n = (X^{n^\wedge})\downarrow$ $(n \in \omega)$.

(The arrow $\downarrow$ denotes the descents of the classes in (2), (8) that are on the left-hand sides of equalities (1), (6), (10), as well as those of the relations in the remaining places.)

$\lhd$ (1) By virtue of the maximum principle, for any $x \in \mathbf{V}^{(B)}$ there is such a $y \in \mathbf{V}^{(B)}$ that

$$[x \in \operatorname{dom}(X)] = [(\exists u)((x,u) \in X)] = [(x,y)^B \in X].$$

It follows from the above relation that $x \in \operatorname{dom}(X)\downarrow$ results in $x \in \operatorname{dom}(X\downarrow)$. Conversely, if $x \in \operatorname{dom}(X\downarrow)$ then $[(x,y) \in X] = 1$ for some $y \in \mathbf{V}^{(B)}$. Hence,

$$[\![x \in \mathrm{dom}(X)]\!] = \vee\{[\![(x,u) \in X]\!] : u \in \mathbf{V}^{(B)}\} \geq [\![(x,y) \in X]\!],$$

which affords $x \in \mathrm{dom}(X){\downarrow}$. The second relation is proved analogously.

(2) By definition, for an arbitrary $x \in \mathbf{V}^{(B)}$ we have

$$[\![x \in X \cap Y]\!] = [\![x \in X \wedge x \in Y]\!] = [\![x \in X]\!] \wedge [\![x \in Y]\!].$$

Therefore, $x \in (X \cap Y){\downarrow}$ iff we have $x \in X{\downarrow}$ and $x \in Y{\downarrow}$ simultaneously.

(3) Applying (2) and the definition of the restriction $X|Y$, we deduce

$$(X|Y){\downarrow} = (X \cap (Y \times U_B)){\downarrow} = X{\downarrow} \cap (Y{\downarrow} \times \mathbf{V}^{(B)}) = (X{\downarrow})|(Y{\downarrow}).$$

(4) The proof results from the definition of X^{-1}.

(5) For any class Z lets us denote by σX the class obtained from Z by a σ-conjugation, where $\sigma := (\iota_1, \iota_2, \iota_3)$ is a permutation of the set $\{1,2,3\}$ (see **1.3.10**). We then can easily make sure that $(\sigma Z){\downarrow} = \sigma(Z{\downarrow})$. If $Z \in \mathbf{V}^{(B)}$ is such that $\mathbf{V}^{(B)} \models Z = (Y \times \mathbf{U}_B) \cap (\mathbf{U}_B \times X)$ and $\sigma := \{1,2,3\}$, then

$$\mathbf{V}^{(B)} \models X \circ Y = \mathrm{dom}(\sigma Z).$$

Now, taking into account (1), (2) and **3.2.4**, we can write the following chain of equalities

$$(X \circ Y){\downarrow} = \mathrm{dom}(\sigma Z){\downarrow} = \mathrm{dom}(\sigma(Z{\downarrow}))$$
$$= \mathrm{dom}(\sigma((Y{\downarrow} \times \mathbf{V}^{(B)}) \cap (\mathbf{V}^{(B)} \times X{\downarrow})) = (X{\downarrow}) \circ (Y{\downarrow}).$$

(6) If we successively apply (1) and (3), we get

$$(X``Y){\downarrow} = (\mathrm{im}(X|Y)){\downarrow} = \mathrm{im}((X|Y){\downarrow})$$
$$= \mathrm{im}((X{\downarrow})|(Y{\downarrow})) = (X{\downarrow})``(Y{\downarrow}).$$

(7) Let us assume that $[\![\mathrm{Fnc}(X)]\!] = \mathbf{1}$. Then $X{\downarrow}$ is a binary relation and, besides,

$$[\![(x,y) \in X]\!] \wedge [\![(x,z) \in X]\!] \leq [\![y = z]\!]$$

for any $x, y, z \in \mathbf{V}^{(B)}$. Hence, for $(x,y) \in X{\downarrow}$ and $(x,z) \in X{\downarrow}$ we get $[\![y = z]\!] = \mathbf{1}$, i.e.,

$y = z$. In other words, Fnc(X)$\downarrow$ is fulfilled. In turn, if $X\downarrow$ is a single-valued binary relation, then by **2.5.16**, we deduce

$$[\mathrm{Fnc}(X)] = \bigwedge_{x\in \mathbf{V}^{(B)}} \wedge \{[y = z]: (x,y)\in X\downarrow \cap Y\downarrow, (x,z)\in X\downarrow\} = \mathbf{1}.$$

(8) Making use of (2) and **3.2.3** (3), we can write

$$\mathbf{1} = [X\subset Y] \leftrightarrow \mathbf{1} = [X\cap Y = X] \leftrightarrow X\downarrow \cap Y\downarrow = X\downarrow \leftrightarrow X\downarrow \subset Y\downarrow.$$

(9) The formula $(\forall x)(\forall y)(x = y \to X``\{x\} = X``\{y\})$ is a theorem of ZF, and has, therefore, the truth-value unity. Writing the truth values first for quantifiers and then for implication, we will get the required result.

(10) If $[t{:}n^\wedge \to X] = \mathbf{1}$, then for every $k\in n$ there is a unique element $x\in X\downarrow$, for which $[t(k^\wedge) = x] = \mathbf{1}$. Setting $s(k){:}= x$ for $k\in n$, we will obtain a mapping $s{:}n\to X\downarrow$, which will be denoted by $t\downarrow := s$. Hence,

$$[t\downarrow(k) = t(k^\wedge)] = \mathbf{1} \quad (k\in n).$$

Conversely, if $s{:}\, n\to X\downarrow$, then $t\in \mathbf{V}^{(B)}$ will be determined by the relation

$$t{:}= \{(k^\wedge, s(k))^B{:}\, k\in n\}\times \mathbf{1}_B.$$

In this case $[t{:}n^\wedge \to X] = 1$, $[t(k^\wedge) = s(k)] = \mathbf{1}$ for $k\in n$, and $t\downarrow = s$. From all what was said above we can conclude that the mapping $t\to t\downarrow$ is a bijection between $\{x\in \mathbf{V}^{(B)}{:}[x\in X^{n^\wedge}] = \mathbf{1}\}$ and $(X\downarrow)^n$.

Let us now recall the definition $s{:}= (x(0),\dots,x(n-1))^B$ (see **2.4.9**). Let $x{:}n\to X\downarrow$ and $y{:}\, n\to X\downarrow$ be such that $y(0) = x(0)$, $y(k) = (y(k-1), x(k))^B$ for $0\neq k\in n$ and $y(n-1) = s$. According to what has been earlier proved, there are such $p,q\in \mathbf{V}^{(B)}$ that $[p,q : n^\wedge \to X] = \mathbf{1}$, in which case $p\downarrow = x$ and $q\downarrow = y$. We then can easily check that

$$[p(0) = q(0) \wedge (\forall k\in n^\wedge)(k\neq 0 \to q(k) = (q(k-1), p(k)))] = \mathbf{1}.$$

Therefore, $[q(n^\wedge - 1) = (p(0^\wedge),\dots,p(n^\wedge - 1))\in X^{n^\wedge}] = \mathbf{1}$. On the other hand, $[s = q(n^\wedge - 1)] = \mathbf{1}$, and, hence, $s\in (X^{n^\wedge})\downarrow$. Thus, the mapping

$$(x(0),\dots,x(n-1)) \to (x(0),\dots,x(n-1))^B$$

is an injection of $(X\downarrow)^n$ in $(X^{n^\wedge})\downarrow$. Analogous considerations show that in this case the image of $(X\downarrow)^n$ is all $(X^{n^\wedge})\downarrow$. ▷

3.2.7. Somewhat different from **3.2.6** is the matter with the descents of the complement of a class and the families of classes. Let us consider an arbitrary class $Y \subset \mathbf{V}^{(B)}$. Since the formula $x \in \mathbf{V}^{(B)} \wedge (\forall y \in Y)\,([x = y] = 0)$ is predicative, there is a class Y^c determined by the relation

$$x \in Y^c \leftrightarrow x \in \mathbf{V}^{(B)} \wedge (\forall y \in Y)\,([x = y] = \mathbf{0}).$$

Let now X be a class inside $\mathbf{V}^{(B)}$. By the symbol X^c let us denote the $\mathbf{V}^{(B)}$-class that is the complement of the class X inside $\mathbf{V}^{(B)}$, i.e.,

$$\mathbf{V}^{(B)} \models (\forall x)(x \in X^c \leftrightarrow x \notin X).$$

The existence of a $\mathbf{V}^{(B)}$-class X^c results from **2.5.18**. Let us consider a formula

$$\varphi\big(y, B, Y, \mathbf{V}^{(B)}, [\cdot = \cdot]\big) := (\forall a)(\forall b)(\forall x)(b : a \to Y$$
$$\text{'}b \text{ is a partition of unity'} \wedge x\colon a \to Y \wedge y$$
$$= \operatorname*{mix}_{\alpha \in a}(b(\alpha) \cdot x(\alpha))),$$

stating that y is a mixing of a certain family of elements of the class Y. We can make sure that the formula is predicative and, hence, there is a class $\operatorname{mix}(Y)$ such that

$$(\forall y)(y \in \operatorname{mix}(Y) \leftrightarrow \varphi\big(y, B, Y, \mathbf{V}^{(B)}, [\cdot = \cdot]\big).$$

As an example, let us indicate the fact that for an arbitrary class $X \subset \mathbf{V}$ we have $X^\wedge\downarrow = \operatorname{mix}(X_1)$, where $X_1 := \{x^\wedge : x \in X\}$, while the injection of X to $\operatorname{mix}(X_1)$ is realized by the canonical embedding (see **3.1.1** (1)).

3.2.8. *If the class Y is a set then*

$$\operatorname{mix}(Y) = \operatorname{cyc}(Y).$$

◁ We only have to demonstrate that the set $\operatorname{mix}(Y)$ of all possible mixings $\operatorname{mix}_{y \in Y}(b_y y)$ of the elements of the set Y is cyclic. Let us consider a partition of unity $(b_\xi)_{\xi \in \Xi}$ and a family

$$y_\xi := \underset{y \in Y}{\mathrm{mix}}(b_{\xi,y} y) \quad (\xi \in \Xi).$$

Let us put $y_0 := \mathrm{mix}_{\xi \in \Xi}(b_\xi y_\xi)$ and $b_{(\xi,y)} := b_\xi \wedge b_{\xi,y}$ $(\xi \in \Xi, y \in Y)$. If $(\xi, y) \neq (\eta, z)$, then

$$b_{(\xi,y)} \wedge b_{(\eta,z)} = b_\xi \wedge b_\eta \wedge b_{\xi,y} \wedge b_{\eta,z} = \mathbf{0}.$$

Besides, we can easily calculate (see **1.1.5** (2)):

$$\bigvee_{(\xi,y) \in \Xi \times Y} b_{(\xi,y)} = \bigvee_{\xi \in \Xi} \left(b_\xi \wedge \bigvee_{y \in Y} b_{\xi,y} \right) = \mathbf{1}.$$

Therefore, $\left(b_{(\xi,y)}\right)$ is a partition of unity. Then, for any $y \in Y$ we get

$$[y_0 = y] \geq [y_0 = y_\xi] \wedge [y_\xi = y] \geq b_\xi \wedge b_{\xi,y}.$$

We see here that $y_0 = \mathrm{mix}(Y)$, i.e., $\mathrm{mix}(Y)$ is a cyclic set. ▷

3.2.9. *For any nonempty classes X and Y inside $\mathbf{V}^{(B)}$ we have*

(1) $X^c \downarrow = X \downarrow^c$;

$(X \cup Y)\downarrow = \mathrm{mix}(X \downarrow \cup Y\downarrow)$.

◁ (1) By virtue of definitions and proposition **2.5.16**, the following equivalences hold:

$$\begin{aligned} &x \in X^c \downarrow \leftrightarrow [x \in X^c] = \mathbf{1} \leftrightarrow [x \notin X] = \mathbf{1} \\ &\leftrightarrow [x \in X] = \mathbf{0} \leftrightarrow \vee\{[x = s] : s \in X\downarrow\} = \mathbf{0} \\ &\leftrightarrow (\forall s \in X\downarrow)([s = x] = \mathbf{0}) \leftrightarrow x \in (X\downarrow)^c. \end{aligned}$$

(2) It is seen from proposition **3.2.6** (8) that $X \downarrow \cup Y\downarrow \subset (X \cup Y) \downarrow$. Conversely, if $z \in (X \cup Y) \downarrow$, then

$$(\exists x \in X)(\exists y \in Y)(x = z \vee y = z).$$

Using the maximum principle, we choose $x_0, y_0 \in \mathbf{V}^{(B)}$ in such a way that $b \vee c = 1$, where $b := [x_0 \in X] \wedge [x_0 \in z]$ and $c := [y_0 \in Y] \wedge [y_0 \in z]$. Pick arbitrary $x_1 \in X \downarrow$ and $y_1 \in Y \downarrow$, and put $x = \mathrm{mix}\{b x_0, b^* x_1\}$, $y = \mathrm{mix}\{c y_0, c^* y_1\}$. Then $x \in X \downarrow$, since

$$b \le [x = x_0] \wedge [x_0 \in X] \le [x \in X],$$
$$b^* \le [x_1 = x] \wedge [x_1 \in X] \le [x \in X].$$

By an analogous reason, $y \in Y\downarrow$. Besides,

$$b \le [x = x_0] \wedge [x_0 = z] \le [x = z],$$
$$b^* \le c \le [y = y_0] \wedge [y_0 = z] \le [y = z],$$

i.e., $z = \mathrm{mix}\{bx, b^*y\}$ and $z \in \mathrm{mix}(X\downarrow \cup Y\downarrow)$. ▷

Here we should make an additional remark:

$$(3)\ (X \cup Y)\downarrow = \bigcup_{b \in B} bX\downarrow \oplus b^* Y\downarrow,$$

where $bX\downarrow \oplus b^*Y\downarrow$ is a set of elements of the type $\mathrm{mix}\{bx, b^*y\}$ $(x \in X\downarrow, y \in Y\downarrow)$.

3.2.10. Descending has to be sometimes used repeatedly. Let us see how this happens. Let X be a class. Organize a class-function Y by the formula

$$Y := \{(x,y) : x \in \mathbf{V}^{(B)} \wedge y = x\downarrow\}.$$

The double or repeated descent of the class X is the class $\cup \mathrm{im}(Y|(X\downarrow))$ denoted by $X\downarrow\downarrow$. Therefore,

$$X\downarrow\downarrow = \cup\{x\downarrow : x \in X\downarrow\}.$$

It goes without saying that if $X \in \mathbf{V}^{(B)}$, then $X\downarrow\downarrow \in \mathbf{V}$ (see **3.2.3** (2)).

3.2.11. *For any nonempty* $\mathbf{V}^{(B)}$*-class* X *the following relations are valid:*

(1) $(\cup X)\downarrow = \cup(X\downarrow\downarrow)$;

(2) $(\cap Y)\downarrow = \cap(X\downarrow\downarrow)$;

(3) $\mathcal{P}(X)\downarrow\downarrow \subset \mathcal{P}(X\downarrow)$.

◁ The proof rests on **2.5.16**. Here are the necessary calculations:

(1) $u \in \bigcup(X\downarrow\downarrow) \leftrightarrow (\exists v \in X\downarrow\downarrow)(u \in v)$
$\leftrightarrow (\exists z \in X\downarrow)(u \in z\downarrow) \leftrightarrow (\exists z \in X\downarrow)([u \in z] = \mathbf{1})$
$\leftrightarrow [(\exists z \in X)(u \in z)] = \mathbf{1} \leftrightarrow [u \in \bigcup X] = \mathbf{1} \leftrightarrow u \in (\bigcup X)\downarrow.$

(2) $u \in \bigcap(X\downarrow\downarrow) \leftrightarrow (\forall v \in X\downarrow\downarrow)(u \in v)$
$\leftrightarrow (\forall z \in X\downarrow)(u \in z\downarrow) \leftrightarrow (\forall z \in X\downarrow)([u \in z] = \mathbf{1})$
$\leftrightarrow [(\forall z \in X)(u \in z)] = \mathbf{1} \leftrightarrow [u \in \bigcap X] = \mathbf{1} \leftrightarrow u \in (\bigcap X)\downarrow.$

(3) $u \in \mathcal{P}(X)\downarrow\downarrow \leftrightarrow (\exists z \in \mathcal{P}(X)\downarrow)(u \in z\downarrow)$
$\leftrightarrow (\exists z)([z \subset X] = \mathbf{1} \wedge u = z\downarrow) \leftrightarrow (\exists z)(z\downarrow \subset X\downarrow \wedge u = z\downarrow)$
$\rightarrow u \subset X\downarrow \leftrightarrow u \in \mathcal{P}(X\downarrow)$. ▷

3.2.12. Theorem. *Let* $X, Y, f \in \mathbf{V}^{(B)}$ *be such that* $[X \neq \varnothing] = [Y \neq \varnothing] = [f : X \rightarrow Y] = \mathbf{1}$. *Then there is a unique mapping* $f\downarrow : X\downarrow \rightarrow Y\downarrow$, *the descent of* f *such that*

$$[f(x) = f\downarrow(x)] = \mathbf{1} \quad (x \in X\downarrow).$$

The descent of a mapping has the following properties:

(1) f *is extensional, i.e.,*

$$[x = x'] \leq [f\downarrow(x) = f\downarrow(x')] \quad (x \in X\downarrow);$$

(2) *if* $Z, g \in \mathbf{V}^{(B)}$ *are such that* $[\![Z \neq \varnothing]\!] = [\![g : Y \rightarrow Z]\!] = \mathbf{1}$, *then*

$$(g \circ f)\downarrow = g\downarrow \circ f\downarrow;$$

(3) $f\downarrow$ *is surjective (injective* or *bijective) iff* $[f$ *is surjective (injective* or *bijective)* $] = \mathbf{1}$.

◁ Let h be the descent of the relation f in the sense of **3.2.5**. It follows from **3.2.6** (1,7) that $h : X\downarrow \rightarrow Y\downarrow$. Then, since $(x, h(x))^B \in f\downarrow$ for any $x \in X\downarrow$, then

$$[h(x) = f(x)] = [(x, h(x)) \in f] = [(x, h(x))^B \in f] = \mathbf{1}.$$

The mapping h is uniquely determined by this property, since if $g : X\downarrow \rightarrow Y\downarrow$ has the same property, then

$$[h(x) = g(x)] \geq [g(x) = f(x)] \wedge [h(x) = f(x)] = \mathbf{1}$$

and $h(x) = g(x)$ for every $x \in X\downarrow$ (since $\mathbf{V}^{(B)}$ is separated). Let us use the determining property of the mapping h and **3.2.6** (9) and estimate

$$[x = x'] \le [f(x) = f(x')] \wedge [f(x) = h(x)] \\ \wedge [f(x') = h(x')] \le [h(x) = h(x')].$$

Therefore, we have established (1), while (2) follows from **3.2.6** (5). Now we have to establish (3). The statement on the surjectivity is easily deduced from **3.2.6** (6), while the bijectivity is the conjunction of surjectivity and bijectivity. The injectivity of f inside $\mathbf{V}^{(B)}$ is equivalent to the relation

$$[x = x'] = [f(x) = f(x')] = [h(x) = h(x')] \quad (x, x' \in X\downarrow).$$

Hence, $x = x'$ iff $h(x) = h(x')$, which fact implies that the mapping h is injective. ▷

3.2.13. Theorem. *Let $X, Y, F \in \mathbf{V}^{(B)}$ are such that $[X \ne \varnothing] = [Y \ne \varnothing] = [\varnothing \ne F \subset X \times Y] = 1$. Let $\Phi \in \mathbf{V}^{(B)}$ be a correspondence from X to Y with the graph F inside $\mathbf{V}^{(B)}$, i.e., $\mathbf{V}^{(B)} \models \Phi = (F, X, Y)$. Then the triplet $\Phi\downarrow := (F\downarrow, X\downarrow, Y\downarrow)$ which is the descent of Φ, is the unique correspondence obeying the equality*

$$\Phi\downarrow(x) = \Phi(x)\downarrow \quad (x \in X\downarrow).$$

The descent of a correspondence has the following properties:

(1) $\Phi(A)\downarrow \in \Phi\downarrow(A)\downarrow$ *for any* $A \in \mathbf{V}^{(B)}$ *such that* $[A \subset X] = 1$;

(2) $\pi_{\Phi}(A)\downarrow = \pi_{\Phi\downarrow}(A\downarrow)$ *for all* $A \in \mathbf{V}^{(B)}$ *for which* $[A \subset X] = 1$;

(3) $(\Psi \circ \Phi)\downarrow = \Psi\downarrow \circ \Phi\downarrow$;

(4) $(I_X)\downarrow = I_{X\downarrow}$.

◁ All these statements but (3) are elementarily deduced from **3.2.6**. It should be only remarked that the determining equality $\Phi\downarrow(x) = \Phi(x)\downarrow \quad (x \in X\downarrow)$ must be understood with use made of the remark on **3.2.1**. It is by virtue of the maximum principle that there exists a $\overline{\Phi} \in \mathbf{V}^{(B)}$ such that $[\overline{\Phi}: X \to \mathcal{P}(Y)] = 1$ and $[\Phi(x) = \overline{\Phi}(x)] = 1$ for all $x \in X\downarrow$. According to **3.2.12**, $\overline{\Phi}\downarrow: X\downarrow \to \mathcal{P}(Y)\downarrow$ and $[\overline{\Phi}(x) = \overline{\Phi}\downarrow(x)] = 1$ for $x \in X\downarrow$. In this case, however, $\overline{\Phi}\downarrow$ is determined by the relation

$$\Phi\downarrow(x) = (\overline{\Phi}\downarrow(x))\downarrow = \overline{\Phi}(x)\downarrow\downarrow \quad (x\in X\downarrow).$$

This, in particular, affords $\Phi\downarrow(A\downarrow) = \overline{\Phi}(A)\downarrow\downarrow$. Taking these remarks into account, let us now prove (2). Observe that we have

$$[\pi_{\Phi}(A) = \cap\overline{\Phi}(A)] = 1;$$

i.e., $\pi_{\Phi}(A) = \cap\{\overline{\Phi}(a): a\in A\}$ is fulfilled inside $\mathbf{V}^{(B)}$. From this relation, making use of the rule **3.2.11** (2), we deduce:

$$\begin{aligned}\pi_{\Phi}(A)\downarrow &= (\cap\overline{\Phi}(A))\downarrow = \cap\overline{\Phi}(A)\downarrow\downarrow)\\ &= \cap\{\Phi\downarrow(a): a\in A\downarrow\} = \pi_{\Phi\downarrow}(A\downarrow).\ \triangleright\end{aligned}$$

3.2.14. (1) *Let X and Y be nonempty sets inside $\mathbf{V}^{(B)}$, while a family $(f_\xi)_{\xi\in\Xi}\subset\mathbf{V}^{(B)}$ be such that*

$$[f_\xi : X\to Y] = 1 \quad (\xi\in\Xi).$$

Then for a partition of unity $(b_\xi)_{\xi\in\Xi}\subset B$, the mixing $\operatorname{mix}_{\xi\in\Xi}(b_\xi f_\xi)$ is a function from X to Y inside $\mathbf{V}^{(B)}$ and

$$\operatorname*{mix}_{\xi\in\Xi}(b_\xi f_\xi)\downarrow(x) = \operatorname*{mix}_{\xi\in\Xi}(b_\xi f_\xi\downarrow(x)) \quad (x\in X\downarrow).$$

$\triangleleft$ Let us set $g := \operatorname{mix}_{\xi\in\Xi}(b_\xi f_\xi)$. Since

$$b_\xi \le [g = f_\xi]\wedge[f_\xi : X\to Y] < [g: X\to Y],$$

then $[g: X\to Y] = 1$, i.e., g is a function from X to Y. Besides, by virtue of **3.2.12**, for every $x\in X\downarrow$ we have

$$\begin{aligned}b_\xi &\le [g\downarrow(x) = g(x)]\wedge[g(x) = f_\xi(x)]\\ &\wedge[f_\xi\downarrow(x) = f_\xi(x)] \le [g\downarrow(x) = f_\xi\downarrow(x)].\end{aligned}$$

Hence, we get $g\downarrow(x) = \operatorname{mix}_{\xi\in\Xi}(b_\xi f_\xi\downarrow(x))$. $\triangleright$

(2) *Let X, Y and (b_ξ) be the same as above, while $(\Phi_\xi)_{\xi\in\Xi}$ be a family of elements*

of $\mathbf{V}^{(B)}$ *which are correspondences from* X *to* Y *inside* $\mathbf{V}^{(B)}$. *Then the mixing* $\operatorname{mix}_{\xi\in\Xi}(b_\xi\Phi_\xi)$ *is a correspondence from* X *to* Y, *in which case*

$$\operatorname{mix}_{\xi\in\Xi}(b_\xi\Phi_\xi)\downarrow(x)=\operatorname{mix}_{\xi\in\Xi}(b_\xi\Phi_\xi\downarrow(x))\quad(x\in X\downarrow).$$

◁ The proof is analogous to **3.2.14** (1). ▷

3.2.15. By the symbol $\mathfrak{F}^{\downarrow}$ let us denote the mapping that assigns to each nonempty $\mathbf{V}^{(B)}$-set X its descent $X\downarrow$, and to every correspondence Φ inside $\mathbf{V}^{(B)}$, the correspondence $\Phi\downarrow$.

Theorem. *The mapping* $\mathfrak{F}^{\downarrow}$ *is a covariant functor from the category* $\mathfrak{V}_*^{(B)}$ *to the category* $\mathfrak{V}_*$ *(from the category* $\mathfrak{V}^{(B)}$ *to the category* $\mathfrak{V}$, *respectively).*

3.2.16. Theorem. *Let* $\mathfrak{K}$ *be a category inside* $\mathbf{V}^{(B)}$. *Then there is a unique category* $\mathfrak{K}'$ *(in the sense of* $\mathbf{V}$*) such that* $\operatorname{Ob}\mathfrak{K}'=(\operatorname{Ob}\mathfrak{K})\downarrow$, $\operatorname{Mor}\mathfrak{K}'=(\operatorname{Mor}\mathfrak{K})\downarrow$ *and* $\mathrm{Com}'=\mathrm{Com}\downarrow$, *where* Com' *is a composition of the category* $\mathfrak{K}'$ *and* $\mathbf{V}^{(B)}\models\mathrm{Com}$ *is the composition of the category* $\mathfrak{K}$.

◁ It follows from **3.2.6** (7) that Com' is a partial binary operation on the class $(\operatorname{Mor}\mathfrak{K})\downarrow$. As long as $[\mathrm{Com}(\alpha,\beta)=\mathrm{Com}'(\alpha,\beta)]=\mathbf{1}$ for any $(\alpha,\beta)\in\operatorname{Mor}\mathfrak{K}'$ and since Com is associative inside $\mathbf{V}^{(B)}$, then Com' is also associative. Let D and R be $\mathbf{V}^{(B)}$-classes taking part in the definition of the category $\mathfrak{K}$ (see **2.5.19**). Let us set $D':=D\downarrow$ and $R':=R\downarrow$. By virtue of **3.2.6** (1) and **3.2.6** (7), D' and R' are mappings from $\operatorname{Mor}\mathfrak{K}'$ to $\operatorname{Ob}\mathfrak{K}'$. Employing **3.2.6** (1) again, we conclude that for $(\alpha,\beta)\in\operatorname{Mor}\mathfrak{K}'$ the relations $(\alpha,\beta)\in\operatorname{dom}(\mathrm{Com}')$ and $[(\alpha,\beta)\in\operatorname{dom}(\mathrm{Com})]=\mathbf{1}$ are equivalent. On the other hand, the equality $R'(\alpha)=D'(\beta)$ is fulfilled only if $[R(\alpha)=D(\beta)]=\mathbf{1}$. The existence of identity morphisms in $\mathfrak{K}'$ is obvious. Hence, $\mathfrak{K}$ satisfies all the conditions of definition **2.5.19**. ▷

3.2.17. The category $\mathfrak{K}'$ of **3.2.16** is called the *descent of the category* $\mathfrak{K}$ and is denoted by $\mathfrak{K}\downarrow$. Let Set_*^B be the category of nonempty sets and correspondences inside $\mathbf{V}^{(B)}$. In more detail, $\mathrm{MorSet}_*^B,\mathrm{ObSet}_*^B,\mathrm{Com}:\mathbf{V}^{(B)}\to B$ have the form:

$$\mathrm{ObSet}_*^B: x\to[x\neq\varnothing],$$
$$\mathrm{MorSet}_*^B:\alpha\to[(\exists x)(\exists y)(\exists f)(x\neq\varnothing\wedge y\neq\varnothing$$
$$\wedge f\neq\varnothing\wedge f\subset x\times y\wedge\alpha=(f,x,y)],$$

$$\mathrm{Com}: u \to [(\exists\alpha)(\exists\beta)(\exists\gamma)(\alpha,\beta,\gamma \text{ are correspondences}) \wedge \gamma = \alpha \circ \beta \wedge u = (\alpha,\beta,\gamma)].$$

The descent of the category Set_*^B is easily seen to coincide with the category $\mathcal{V}_*^{(B)}$ introduced in **3.1.7**. The category of nonempty sets and mappings inside $\mathbf{V}^{(B)}$ is determined in an analogous way, and it is also obvious that $\mathcal{V}^{(B)} = \mathrm{Set}^B \downarrow$.

3.2.18. Remarks

(1) As has been noted in **3.2.5**, the general symbol $\downarrow$ is used for denoting different operations of the same nature, so that the designation $X\downarrow$ is uniquely understood only if some additional information is available on which object X is descending. This analogous to using the sign '+' for denoting different group operations: the addition of numbers, vectors, linear operators, etc.. The exact interpretation is easily restored contextually.

(2) The double descent **3.2.10** also arises with respect to other set-theoretic operations. Thus, for instance, if $\prod X$ is a class of all mappings f from X to $\cup X$ such that $f(x)\in X$ for any $x\in X$, while $\sum X := \cup\{x \times \{x\} : x\in X\}$, then for every $X\in\mathbf{V}^{(B)}$ there are natural bijections

$$(\textstyle\prod X)\downarrow\downarrow = \prod(X\downarrow\downarrow), \quad (\sum X)\downarrow = \sum(X\downarrow\downarrow).$$

(In the expression $(\prod X)\downarrow\downarrow$ the repeated descent is that of mappings!).

(3) It is obvious that in **3.2.11** (3) the inclusion is strict (provided $B \neq 2$). It should be also remarked that $\mathcal{P}(X)\downarrow$ is an algebraic system of signature $(\vee, \wedge, {}^*, 0, 1)$. We can show that this is a complete Boolean algebra which is a completion of the set $\mathcal{P}(X)\downarrow\downarrow$ ordered by inclusion in the following sense. There is an order-preserving injection $\iota: \mathcal{P}(X)\downarrow\downarrow \to \mathcal{P}(X)\downarrow$, in which for $a \in \mathcal{P}(X)\downarrow$, $a < \mathbf{1}$ there is a $b \in \mathcal{P}(X)\downarrow\downarrow$, so that we have $a \le \iota b < \mathbf{1}$, which is quite analogous to the notion of the completion of a Boolean algebra (see [96, 233]).

(4) When proving **3.2.6** (10) we have established that, in particular, for any $X\in\mathbf{V}^{(B)}$ the mapping $\downarrow$ implements a bijection between the sets $\mathcal{V}(n, X\downarrow)$ and $\mathcal{V}^{(B)}(n^\wedge, X)$. In reality this fact is of quite general character and reflects an in-depth interrelation between the functors $\mathfrak{F}^\wedge$ and $\mathfrak{F}^\downarrow$ (for details see **3.5**).

3.3. The Ascent Functor

In this section ascent is introduced as procedure reverse in relation to descent, the corresponding functor is determined and its properties are studied.

3.3.1. Let us consider an arbitrary subclass X of the class $\mathbf{V}^{(B)}$.

(1) *There is a* $\mathbf{V}^{(B)}$*-class* Y *given by the formula*

$$Y(t) := \vee\{[\![t = x]\!]: x \in X\} \quad (t \in \mathbf{V}^{(B)}).$$

$\lhd$ Indeed, by theorem **1.3.14**, there is a class Y (in the sense of $\mathbf{V}$) such that

$$(y,b) \in Y \leftrightarrow y \in \mathbf{V}^{(B)} \wedge b \in B \wedge \left(b = \bigvee_{x \in X} [\![x = y]\!]\right).$$

As is seen, the class Y is single-valued and $Y = \mathbf{V}^{(B)}$, i.e., Y is a mapping from $\mathbf{V}^{(B)}$ to B. Besides, the mapping is extensional since, by virtue of **2.1.8** (4),

$$\begin{aligned} Y(t) \wedge [\![t = s]\!] &= \vee\{[\![t = x]\!] \wedge [\![t = s]\!]: x \in X\} \\ &\leq \vee\{[\![s = x]\!]: x \in X\} = Y(s). \end{aligned}$$

Hence, Y is a class inside $\mathbf{V}^{(B)}$. $\rhd$

Thus, we assign to every class $X \subset \mathbf{V}^{(B)}$ a class Y inside $\mathbf{V}^{(B)}$, which is called the *ascent of the class* X and is denoted by $X\uparrow$. In the case when X is a set, there is a unique element $y \in \mathbf{V}^{(B)}$ such that $X\uparrow(t) = [\![t \in y]\!]$ for all $t \in \mathbf{V}^{(B)}$ (see **2.5.15**). It is this y that is hereafter assumed to be the ascent of the set X, according to the remarks made in **2.5.10**. As an example, let us note that for a class $X \subset \mathbf{V}$ the class $X^\wedge$ is the ascent of the class $X_1 := \{x^\wedge : x \in X\}$ (see **2.5.15**).

(2) Let us now assume that X is a binary relation such that $X \subset \mathbf{V}^{(B)} \times \mathbf{V}^{(B)}$. In order to ascend the relation X, it should first be embedded into $\mathbf{V}^{(B)}$, and then the above procedure should be used. To achieve our purpose, let us make use of the function $(x,y) \to (x,y)^B$ (see **3.2.4**). Therefore, we give the following definition of the ascent of a binary relation:

$$X\uparrow : t \to \vee\{[\![t = (x,y)^B]\!]: (x,y) \in X\}.$$

In particular, if X is the product of classes $Y \subset \mathbf{V}^{(B)}$ and $Z \subset \mathbf{V}^{(B)}$, then we can ascend the product

$$(Y \times Z)\uparrow : t \to \vee\{[t = (x,y)^B]: y \in Y \wedge z \in Z\}.$$

3.3.2. *Let $X \subset \mathbf{V}^{(B)}$ be a nonempty class and φ be a B-formula. In this case*

$$[(\forall u \in X\uparrow)\, \varphi(u)] = \wedge\{[\varphi(u)]: u: X\},$$
$$[(\exists u \in X\uparrow)\, \varphi(u)] = \vee\{[\varphi(u)]: u: X\}.$$

◁ Let us present the deduction of the latter formula (see **1.1.5** (2,7)):

$$[(\exists u \in X\uparrow)\, \varphi(u)] = [(\exists u)(u \in X\uparrow \wedge\, \varphi(u))]$$
$$= \bigvee_{v \in \mathbf{V}^{(B)}} \bigvee_{u \in X} [u = v] \wedge [\varphi(v)]$$
$$= \bigvee_{u \in X} \left(\bigvee_{v \in \mathbf{V}^{(B)}} [v = u] \wedge [\varphi(v)] \right) = \vee\{[\varphi(u)]: u \in X\}.$$

The case of a universal quantifier is considered in an analogous way. ▷

3.3.3. *Whatever a class $X \subset \mathbf{V}^{(B)}$ and a nonempty $\mathbf{V}^{(B)}$-class Y: $\mathbf{V}^{(B)} \to B$ might be, the following rules for cancelling arrows are valid:*

(1) $X\uparrow\downarrow = \mathrm{mix}(X)$;

(2) $Y\uparrow\downarrow = Y$.

◁ (1) The case of an empty X is trivial. If $x \in X$, then $[x \in X\uparrow] = \mathbf{1}$ and, hence, $x \in X\uparrow\downarrow$. It follows from this fact and from **3.2.3** that $\mathrm{mix}(X) \subset X\uparrow\downarrow$. The reverse inclusion is deduced from **3.3.2** and from the mixing principle.

(2) By virtue of **2.5.16**, for an arbitrary $y \in \mathbf{V}^{(B)}$ we have

$$[y \in Y\downarrow\uparrow] = \vee\{[y = t]: t \in Y\downarrow\} = [(\exists t \in Y)(t = y)] = [t \in Y]. \ \triangleright$$

(3) Further on, when speaking about the mixing of a family of ordered pairs, it would be expedient to make use of the following proposition.

Let $(b_\xi)_{\xi \in \Xi}$ be a partition of unity, while $(x_\xi)_{\xi \in \Xi}$ and $(y_\xi)_{\xi \in \Xi}$ be families of elements of $\mathbf{V}^{(B)}$. Then, for mixings, we have

$$\operatorname*{mix}_{\xi\in\Xi} b_\xi (x_\xi, y_\xi)^B = \left(\operatorname*{mix}_{\xi\in\Xi} b_\xi x_\xi, \operatorname*{mix}_{\xi\in\Xi} b_\xi y_\xi\right)^B.$$

◁ Let us first show that $b(x,y)^B = b(bx,by)^B$ for any $x, y \in \mathbf{V}^{(B)}$ and $b \in B$. Indeed, we only have to successively apply **2.3.2**, **2.4.9** and **2.3.6**:

$$[b(x,y)^B = b(bx,by)^B] = b \Rightarrow [(x,y)^B = (bx,by)^B]$$
$$= b \Rightarrow ([x = bx] \wedge [y = by]) = b \Rightarrow ((b^* \Rightarrow [x = \varnothing])$$
$$\wedge (b^* \Rightarrow [y = \varnothing])) = b^* \vee ((b \vee [x = \varnothing]) \wedge (b \vee [y = \varnothing]))$$
$$= (b^* \vee b \vee [x = \varnothing]) \wedge (b^* \vee b \vee [y = \varnothing]) = 1.$$

Let us now put

$$x := \operatorname*{mix}_{\xi\in\Xi} b_\xi x_\xi, \ y := \operatorname*{mix}_{\xi\in\Xi} b_\xi y_\xi.$$

In this case, taking account of what has been proved above, we get

$$b_\xi(x_\xi, y_\xi)^B = b_\xi(b_\xi x_\xi, b_\xi y_\xi)^B = b_\xi(b_\xi x, b_\xi y)^B = b_\xi(x,y)^B.$$

Now we have to refer to the mixing principle. ▷

The fact just established allows one to consider mixings in the class $\mathbf{V}^{(B)} \times \mathbf{V}^{(B)}$. Namely, we will assume that, by definition,

$$\operatorname*{mix}_{\xi\in\Xi} b_\xi (x_\xi y_\xi) := \left(\operatorname*{mix}_{\xi\in\Xi} b_\xi x_\xi, \operatorname*{mix}_{\xi\in\Xi} b_\xi y_\xi\right).$$

Then the mapping $(x,y) \to (x,y)^B$ preserves mixings.

3.3.4. Theorem. *For any classes $X \subset \mathbf{V}^{(B)}$ and $Y \subset \mathbf{V}^{(B)}$ the following statements are valid:*

(1) $\mathbf{V}^{(B)} \models X\!\uparrow\, \subset Y\!\uparrow$ *if* $X \subset Y$;

(2) $\mathbf{V}^{(B)} \models (X \cup Y)\!\uparrow\, = X\!\uparrow\, \cup Y\!\uparrow$;

(3) $\mathbf{V}^{(B)} \models (\operatorname{mix}(X) \cap \operatorname{mix}(Y))\!\uparrow\, = X\!\uparrow\, \cap Y\!\uparrow$;

(4) $\mathbf{V}^{(B)} \models (X \times Y)\!\uparrow\, = X\!\uparrow\, \times Y\!\uparrow$.

If, however, X and Y are relations and Z is a class, then the following relations are also fulfilled:

(5) $\mathbf{V}^{(B)} \models \mathrm{dom}(X)\uparrow = \mathrm{dom}(X\uparrow) \wedge \mathrm{im}(X)\uparrow = \mathrm{im}(X\uparrow)$;

(6) $\mathbf{V}^{(B)} \models (X^{-1})\uparrow = (X\uparrow)^{-1}$;

(7) $\mathbf{V}^{(B)} \models (\mathrm{mix}(X)``\mathrm{mix}(Z))\uparrow = (X\uparrow)``(Z\uparrow)$;

(8) $\mathbf{V}^{(B)} \models (\mathrm{mix}(X) \circ \mathrm{mix}(Y))\uparrow = (X\uparrow) \circ (Y\uparrow)$;

(9) $\mathbf{V}^{(B)} \models (Z^n)\uparrow = (Z\uparrow)^{n^\wedge}$.

⊲ (1) The result follows from the definition of descent.

(2) This fact is substantiated by the following obvious relations:

$$[t \in (X \cup Y)\uparrow] = \vee\{[t = u] : u \in X \cup Y\}$$
$$= \bigvee_{u \in X} [t = u] \vee \bigvee_{u \in Y} [t = u] = [t \in X\uparrow \vee t \in Y\uparrow].$$

(3) Let us assume that inside $\mathbf{V}^{(B)}$ the ascent of the intersection of the classes X and Y coincides with the intersection of their ascents. Then, according to **3.2.6** (2) and **3.3.3**, we get

$$\mathrm{mix}(X \cap Y) = (X \cap Y)\uparrow\downarrow = (X\uparrow \cap Y\uparrow)\downarrow$$
$$= X\uparrow\downarrow \cap Y\uparrow\downarrow = \mathrm{mix}(X) \cap \mathrm{mix}(Y).$$

Conversely, let the cyclic hull of the intersection of the classes X and Y be equal to the intersection of their cyclic hulls. Then, applying **3.2.6** (2) and **3.3.3** again, we get

$$(X \cap Y)\uparrow\downarrow = X\uparrow\downarrow \cap Y\uparrow\downarrow = (X\uparrow \cap Y\uparrow)\downarrow,$$

and, hence, $[(X \cap Y)\uparrow = X\uparrow \cap Y\uparrow] = \mathbf{1}$ (according to **3.2.3** (3)). To complete the proof, we have to apply the above-established facts to the classes $\mathrm{mix}(X)$ and $\mathrm{mix}(Y)$ and to use the rules for cancelling arrows **3.3.3**.

(4) In conformity with rules **3.3.2**, we calculate

$$[z \in X\uparrow \times Y\uparrow] = [(\exists u \in X\uparrow)(\exists v \in Y\uparrow) z = (u,v)]$$
$$= \bigvee_{u \in X} \bigvee_{v \in Y} [z = (u,v)] = \bigvee_{(u,v) \in X \times Y} [z = (u,v)^B] = [z \in (X \times Y)\uparrow].$$

(5) Assuming X to be a binary relation, we can easily check the validity of the following chain of equalities (see **1.1.5** (2,7)):

$$[x \in \operatorname{dom}(X\uparrow)] = [(\exists y)((x,y \in X\uparrow)] = \bigvee_{y \in \mathbf{V}^{(B)}} \bigvee_{(s,t) \in X} [(x,y)^B = (s,t)^B]$$
$$= \bigvee_{(s,t) \in X} \bigvee_{y \in \mathbf{V}^{(B)}} [x = s] \wedge [y = t] = \bigvee_{s \in \operatorname{dom} X} [x = s] = [x \in \operatorname{dom}(X)\uparrow].$$

The statement on $\operatorname{im}(X)$ is established analogously.

(6) $[(x,y) \in (X\uparrow)^{-1}] = [(y,x) \in X\uparrow]$
$$= \bigvee_{(s,t) \in X} [(s,t) = (y,x)] = \bigvee_{(t,s) \in X^{-1}} [(t,s) = (x,y)]$$
$$= [(x,y) \in (X^{-1})\uparrow] = \bigvee_{(t,s) \in X^{-1}} [(t,s) = (x,y)].$$

(7), (8) It is obvious that

$$\operatorname{mix}(X) \cap (\operatorname{mix}(Z) \times \mathbf{V}^{(B)}) = \operatorname{mix}(X) \cap \operatorname{mix}(Z \times \mathbf{V}^{(B)});$$
$$(\operatorname{mix}(Y) \times \mathbf{V}^{(B)}) \cap (\mathbf{V}^{(B)} \times \operatorname{mix}(X))$$
$$= \operatorname{mix}(Y \times \mathbf{V}^{(B)}) \cap \operatorname{mix}(\mathbf{V}^{(B)} \times X).$$

Then we have to follow the scheme of **3.2.6** (5,6), making use of (3), (4) and the fact that $[\mathbf{V}^{(B)}\uparrow = \mathbf{U}_B] = \mathbf{1}$.

(9) It should be observed that if remarks **3.3.3** (3) are taken into account, then we conclude that $\operatorname{mix}(Z^n) = \operatorname{mix}(Z)^n$. Thus, by virtue of **3.2.6** (10) and **3.3.3** (1), we infer

$$((Z\uparrow)^{n^\wedge})\downarrow = (Z\uparrow\downarrow)^n = (Z^n)\uparrow\downarrow,$$

which yields, according to **3.2.3** (3), the sought equality. ▷

3.3.5. Let us consider a class X whose elements are subsets of $\mathbf{V}^{(B)}$, i.e., $X \subset \mathcal{P}(\mathbf{V}^{(B)})$. The *double* or *repeated ascent* of the class X, denoted by $X\uparrow\uparrow$, is the ascent of the class $\{x\uparrow : x \in X\}$. Hence,

$$[t \in X\uparrow\uparrow] = \bigvee\{[t = x\uparrow] : x \in X\} \quad (t \in \mathbf{V}^{(B)}).$$

Let us introduce one more notation:

$$\mathrm{mix}''X: = \{\mathrm{mix}(u): u \in X\}.$$

It is obvious that $[X\uparrow\uparrow = (\mathrm{mix}''X)\uparrow\uparrow] = \mathbf{1}$. By $\mathcal{P}_0(X)$ we shall denote a class of nonempty elements $\mathcal{P}(X)$, i.e.,

$$\mathcal{P}_0(X): = \{z: z \subset X \wedge z \neq \varnothing\}.$$

3.3.6. *Let X be a nonempty $\mathbf{V}^{(B)}$-class and $Y \subset \mathcal{P}(\mathbf{V}^{(B)})$. Then*

(1) $\mathbf{V}^{(B)} \models \cup(Y\uparrow\uparrow) = (\cup Y)\uparrow$;

(2) $\mathbf{V}^{(B)} \models \cap(Y\uparrow\uparrow) = \cap(\mathrm{mix}''(Y\uparrow))$;

(3) $\mathbf{V}^{(B)} \models \cup X = (\cup(X\downarrow\downarrow))\uparrow$;

(4) $\mathbf{V}^{(B)} \models \mathcal{P}_0(X\downarrow)\uparrow\uparrow = \mathcal{P}_0(X)$.

$\lhd$ The proof is suggested as an exercise. $\rhd$

3.3.7. Let us now return to theorem **3.3.4** and remark that according to items (1) and (4) of this theorem, the ascent of a relation is again a relation. From the functional viewpoint, however, it is important that 'the images of points and sets', $X(t)$ and $X''A$, are also preserved in ascending, which is not always the case (see **3.3.4** (7)). Moreover, in ascending, the function can lose its property of being single-valued. This can be easily understood if we take it into account that the procedure 'ascent - descent' results in taking the cyclic hull (**3.3.3** (1)), while the functions obtained by way of descending extensional (**3.2.6** (9)). Here is an example. Let $X \subset \mathbf{V}^{(B)}$ be a cyclic set and $f: X \to \{0^{\wedge}, 1^{\wedge}\}$ be a two-valued function. Let us assume that $f(x) = 0^{\wedge}$ and $f(y) = 1^{\wedge}$ for some $x, y \in X$, $x \neq y$, while an element $b \in B$ is other than $\mathbf{0}$ and $\mathbf{1}$. In on an element $z := \mathrm{mix}\{bx, b^* y\} \in X$ the function f assumes the value $0^{\wedge}$, then $0 < b^* \leq [z = y] \not\leq [f(z) = f(y)] = \mathbf{0}$. Analogously, for $f(z) = 1^{\wedge}$ we get $0 < b \leq [z = y] \not\leq [f(z) = f(y)] = \mathbf{0}$.

On the other hand, $[z = y] \leq [f\uparrow(z) = f\uparrow(y)]$ (see **3.2.6** (9)). Therefore, either $[f\uparrow(y) = f(y)] \neq 1$ or $[f\uparrow(x) = f(x)] \neq \mathbf{1}$, i.e., $[f\uparrow(x) = f(x)] = \mathbf{1}$ is fulfilled not for any $x \in X$. Hence, preservance for the functional dependence in ascending should be specially considered.

3.3.8. *For an arbitrary relation* $X \subset \mathbf{V}^{(B)} \times \mathbf{V}^{(B)}$ *the following conditions are equivalent*:

(1) *if* $b \leq [x_1 = x_2]$ *for* $x_1, x_2 \in \mathrm{dom}(X), b \in B$, *then for any* $u \in \mathbf{V}^{(B)}$ *we get*

$$\vee\{b \wedge [y_1 = u]: y_1 \in X(x_1)\} = \vee\{b \wedge [y_2 = u]: y_2 \in X(x_2)\};$$

(2) *if* $x_1, x_2 \in \mathrm{dom} X$ *and* $y_1 \in X(x_1)$, *then*

$$[x_1 = x_2] \leq \vee\{[y_1 = y_2]: y_2 \in X(x_2)\};$$

(3) $\mathrm{mix}(X(x)) = (\mathrm{mix} X)(x) \quad (x \in \mathrm{dom}(X))$;

(4) $[X\uparrow(x) = X(x)\uparrow] = \mathbf{1} \quad (x \in \mathrm{dom}(X))$;

(5) $[x_1 = x_2] \leq [X(x_1)\uparrow = X(x_2)\uparrow] = \mathbf{1} \quad (x_1, x_2 \in \mathrm{dom}(X))$.

$\lhd$ (1) → (2). Let us set in (1) $b := \lfloor |x_1 = x_2| \rfloor$ and $u := y_1$.

(2) → (3). The inclusion $\subset$ is obvious. In order to prove the reverse inclusion, let us choose a partition of unity $(b_\xi) \subset B$ and a family $((x_\xi, y_\xi)) \subset X$ and denote $(x,y) = \mathrm{mix}(b_\xi(x_\xi y_\xi))$. The task is to establish that $y \in \mathrm{mix}(X(x))$. It follows from (2) that

$$b_\xi \leq [x = x_\xi] \leq \vee\{[y' = y_\xi]: y' \in X(x)\} = [y_\xi \in X(x)\uparrow].$$

Therefore, $b_\xi \leq [y = y_\xi] \wedge [y_\xi \in X(x)\uparrow] \leq [y \in X(x)\uparrow]$, so that $[y \in X(x)\uparrow] = \mathbf{1}$. But in this case $y \in X(x)\uparrow\downarrow = \mathrm{mix}(X(x))$, which completes the proof.

(3) → (4). In view of **3.3.3** (1) and **3.2.6** (6), we have

$$X(x)\uparrow\downarrow = \mathrm{mix}(X(x)) = (\mathrm{mix} X)(x) = (X\uparrow\downarrow)(x) = (X\uparrow(x))\downarrow.$$

Using **3.3.3** (2) again, we come the the required relation.

(4) → (5). It suffices to apply **3.2.6** (9).

(5) → (1). By **2.3.2**, if $b \leq [x_1 = x_2]$ and $x_1, x_2 \in \mathrm{dom}(X)$, then $b(X(x_1)\uparrow) = b(X(x_2)\uparrow)$. On the other hand, according to the definition of descent,

$$[u \in b(X(x_k)\uparrow)] = \vee\{\wedge [u = y]: y \in X(x_k)\},$$

which leads us to the required result. ▷

3.3.9. Let us now return to the notion of extensionality which we had to do with in **3.2.6** (9) and **3.2.12** (1) and which is now considered under more general conditions. The binary relation $R \subset \mathbf{V}^{(B)} \times \mathbf{V}^{(B)}$ is called *extensional in second coordinate*, provided it obeys one (and, hence, all) of the equivalent conditions **3.3.8** (1) - (5). Let us remark that if R is a function, then each of conditions (2) and (5) of **3.3.8** turns into the following relation (cf. **2.5.5**)

$$[x_1 = x_2] \leq [R(x_1) = R(x_2)] \quad (x_1, x_2 \in \mathrm{dom}(R)).$$

Let $X \subset \mathbf{V}^{(B)}$ and $Y \subset \mathbf{V}^{(B)}$ be sets. A correspondence $\Phi := (F, X, Y)$ is called extensional if its graph F is a relation extensional in second coordinate. If, moreover, $\mathrm{dom}\,\Phi = \mathrm{mix}\ \mathrm{dom}\Phi$ and $\Phi(x) = \mathrm{mix}\,\Phi(x)$ for every $x \in \mathrm{dom}\Phi$, then Φ is said to be *fully extensional.* It is evident that the fact that Φ is fully extensional implies $F = (X \times Y) \cap \mathrm{mix}(F)$.

The sets A and $C \subset \mathbf{V}^{(B)}$ are said to *be in general position* provided

$$[a = c] \leq \vee\{[a = b] \wedge [b = c] : b \in A \cap C\}$$

for any $a \in A$ and $c \in C$. When this condition is fulfilled, the last relation is in fact an equality, since $[a = b] \wedge [b = c] \leq [a = c]$.

The following statements are equivalent:

(1) $\mathbf{V}^{(B)} \models (A \cap C)\uparrow = A\uparrow \cap C\uparrow$;

(2) $\mathrm{mix}(A \cap C) = \mathrm{mix}(A) \cap \mathrm{mix}(C)$;

(3) *A and C are in general position.*

◁ The equivalence of (1) and (2) results from **3.2.6** (1), **3.3.3** (1) and **3.3.4** (3). Let us prove (1) → (3). It should be remarked that the inclusion $A\uparrow \cap C\uparrow \subset (A \cap C)\uparrow$ is equivalent to the formula

$$(\forall a \in A\uparrow)(\forall c \in C\uparrow)(a = c \rightarrow (\exists b \in A \cap C) \quad (a = b \wedge b = c)).$$

The Boolean truth-value of this formula is as follows:

$$\bigwedge_{a \in A, c \in C} [a = c] \Rightarrow \bigvee_{b \in A \cap C} [a = b] \wedge [b = c].$$

This implies that (3) is equivalent to the inclusion $A\uparrow \cap C\uparrow \subset (A\cap C)\uparrow$ inside $\mathbf{V}^{(B)}$. The reverse inclusion is always valid. ▷

Therefore, if $A\subset C$, then A and C are in general position by a trivial reason. In a general position there are any two sets of the type $A:=\{a^{\wedge}:a\in A'\}$, where $A'\in\mathbf{V}$.

The *ascent of the correspondence* $\Phi:=(F,X,Y)$ is by definition the element $\Phi\uparrow:=(F\uparrow,X\uparrow,Y\uparrow)^B\in\mathbf{V}^{(B)}$, where $F\uparrow$ is the ascent of the relation F (see **3.3.1** (2)).

3.3.10. Theorem. *Let X and Y be subsets of the class $\mathbf{V}^{(B)}$ and Φ is an extensional correspondence from X to Y. The ascent $\Phi\uparrow$ is a unique correspondence from $X\uparrow$ to $Y\uparrow$ inside $\mathbf{V}^{(B)}$ such that*

$$[\![\operatorname{dom}(\Phi\uparrow)=(\operatorname{dom}\Phi)\uparrow]\!]=\mathbf{1},$$
$$[\![\Phi\uparrow(x)=\Phi(x)\uparrow]\!]=\mathbf{1}\quad(x\in\operatorname{dom}\Phi).$$

The ascent of a correspondence has the following properties:

(1) *if* $\operatorname{dom}\Phi$ *and a set* $A\subset X$ *are in general position, then*

$$\mathbf{V}^{(B)}\models\Phi(A)\uparrow=\Phi\uparrow(A\uparrow);$$

(2) *the composition* $\Psi\circ\Phi$ *of extensional correspondences* Φ *and* Ψ *is an extensional correspondence, and if, besides,* $\operatorname{dom}\Psi\circ\Phi=\operatorname{dom}\Phi$ *and the sets* $\operatorname{dom}\Psi$ *and* $\Phi(x)$ *are in general position for all* $x\in\operatorname{dom}\Phi$, *then*

$$\mathbf{V}^{(B)}\models(\Psi\circ\Phi)\uparrow=\Psi\uparrow\circ\Phi\uparrow;$$

(3) $\mathbf{V}^{(B)}\models(I_X)\uparrow=I_{X\uparrow}$.

◁ By virtue of **3.3.4** and **3.3.8** it suffices to verify uniqueness for $\Phi\uparrow$ and properties (1) - (3). The case of the empty correspondence is omitted as obvious. Let Ψ be a correspondence inside $\mathbf{V}^{(B)}$ obeying the same relations that $\Phi\uparrow$, i.e., $[\![\operatorname{dom}\Psi=(\operatorname{dom}\Phi)\uparrow]\!]=\mathbf{1}$ and $[\![\Psi(x)=\Phi(x)\uparrow]\!]=\mathbf{1}$ $(x\in\operatorname{dom}\Phi)$. In this case $\mathbf{V}^{(B)}\models\operatorname{dom}\Psi=\operatorname{dom}(\Phi\uparrow)$ and

$$[\![(\forall x\in\operatorname{dom}\Psi)\ \Psi(x)=\Phi\uparrow(x)]\!]$$
$$=\bigwedge_{x\in\operatorname{dom}\Phi}[\![\Psi(x)=\Phi\uparrow(x)]\!]=\bigwedge_{x\in\operatorname{dom}\Phi}[\![\Psi(x)=\Phi(x)\uparrow]\!]=\mathbf{1}.$$

(1) Making use of **3.3.9** (1) and the properties of $\Phi\uparrow$ established above, the following equivalences can be written for an arbitrary $y \in \mathbf{V}^{(B)}$:

$$y \in \Phi\uparrow(A\uparrow) \leftrightarrow (\exists x)(x \in (\mathrm{dom}\Phi))\uparrow \wedge x \in A\uparrow \wedge y \in \Phi\uparrow(x) \leftrightarrow$$
$$\leftrightarrow (\exists x)(x \in (A \cap \mathrm{dom}\,\Phi)\uparrow \wedge y \in \Phi\uparrow(x)) \leftrightarrow$$
$$\leftrightarrow (\exists x \in (A \cap \mathrm{dom}\Phi)\uparrow) y \in \Phi(x).$$

Hence, the next equalities hold:

$$[y \in \Phi\uparrow(A\uparrow)] = \bigvee_{x \in A \cap \mathrm{dom}\Phi} [y \in \Phi(x)\uparrow]$$
$$= \bigvee_{x \in A \cap \mathrm{dom}\Phi} \bigvee_{v \in \Phi(x)} [y = v] = \bigvee_{v \in \Phi(A)} [y = v] = [y \in \Phi(A)\uparrow].$$

(2) Let us show that the correspondence $\Theta := \Psi \circ \Phi$ is extensional. Take $x_1, x_2 \in \mathrm{dom}\Theta$, $y_1 \in \Phi(x_1)$ and $z_1 \in \Psi(y_1)$. According to **3.3.8** (2), the following estimates are valid:

$$\bigvee_{z_2 \in \Theta(x_2)} [z_1 = z_2] = \bigvee_{y_2 \in \Phi(x_2)} \left(\bigvee_{z_2 \in \Psi(y_2)} [z_1 = z_2] \right)$$
$$\geq \bigvee_{y_2 \in \Phi(x_2)} \lfloor | y_1 = y_2 | \rfloor \geq \lfloor | x_1 = x_2 | \rfloor.$$

Using **3.3.8** (2) again, we remark that Θ is extensional. Therefore, by what has already been proved, for Θ we infer:

$$[\Theta\uparrow(x) = \Theta(x)\uparrow] = \mathbf{1} \quad (x \in \mathrm{dom}(\Theta)).$$

Taking now into account the facts established in (1), we can write inside $\mathbf{V}^{(B)}$ the following:

$$\Theta\uparrow(x) = \Theta(x)\uparrow = \Psi(\Phi(x))\uparrow = \Psi\uparrow(\Phi(x)\uparrow) =$$
$$= \Psi\uparrow(\Phi\uparrow(x)) = (\Psi\uparrow \circ \Phi\uparrow)(x) \quad (x \in \mathrm{dom}(\Theta)).$$

Hence, **3.3.2** yields the next relation

$$\mathbf{V}^{(B)} \models (\forall x \in \mathrm{dom}(\Theta\uparrow) = \mathrm{dom}(\Phi\uparrow))(\Theta\uparrow(x) = (\Psi\uparrow \circ \Phi\uparrow)(x)),$$

which is equivalent to the sought result, since $(\Psi\uparrow \circ \Phi\uparrow) = \mathrm{dom}(\Theta\uparrow)$.

(3) This is obvious. ▷

3.3.11. Theorem. *Let X and Y be subsets of the class $\mathbf{V}^{(B)}$, while f be an extensional mapping from X to Y. Then $f\uparrow$ is a unique element of $\mathbf{V}^{(B)}$ for which*

$$[f\uparrow : X\uparrow \to Y\uparrow] = [f\uparrow(x) = f(x)] = \mathbf{1}\ (x \in X).$$

The ascent of a mapping has the following properties:

(1) *if Z is a subset of $\mathbf{V}^{(B)}$ and $g: Y \to Z$ is an extensional mapping, then the mapping $g \circ f$ is also extensional and*

$$\mathbf{V}^{(B)} \models (g \circ f)\uparrow = g\uparrow \circ f\uparrow;$$

(2) $\mathbf{V}^{(B)} \models f(A)\uparrow = f\uparrow(A\uparrow) \quad (A \subset X)$;

(3) $\mathbf{V}^{(B)} \models$ *'the mapping $f\uparrow$ is injective' iff f is injective;*

(4) $\mathbf{V}^{(B)} \models$ *'the mapping $f\uparrow$ is surjective' iff* $\operatorname{mix}(\operatorname{im} f) = \operatorname{mix}(Y)$.

3.3.12. Proposition **3.3.3** directly yields rules for cancelling arrows for correspondences and mappings.

Let Φ and f be extensional correspondences from X to Y, in which case let Ψ be a correspondence inside $\mathbf{V}^{(B)}$. Then the following equalities are valid:

(1) $\Phi\uparrow\downarrow(x) = \operatorname{mix}(\Phi(x)) \quad (x \in \operatorname{dom}(\Phi))$;

(2) $f\uparrow\downarrow(x) = f(x) \quad (x \in \operatorname{dom}(f))$;

(3) $\Psi\uparrow\downarrow = \Psi$;

(4) $\pi_{\Phi\uparrow\downarrow}(A) = \pi_{\Phi\uparrow}(A\uparrow)\downarrow \quad (A \subset X)$;

(5) $\pi_{\Phi\uparrow\downarrow}(A)\uparrow = \pi_{\Phi\uparrow}(A\uparrow) \quad (A \subset X)$.

If, moreover, Φ is fully extensional and $A \subset \operatorname{dom}\Phi$ then

(6) $\pi_{\Phi}(A)\uparrow = \pi_{\Phi\uparrow}(A\uparrow)$.

$\lhd$ (1) From **3.2.13**, **3.3.10** and **3.3.3** (1) we directly deduce $(x \in \operatorname{dom}(\Phi))$:

$$\Phi\uparrow\downarrow(x) = \Phi\uparrow(x)\downarrow = \Phi(x)\uparrow\downarrow = \mathrm{mix}(\Phi(x)).$$

(2), (3) These are obvious.

(4) For an arbitrary $A \subset X$ we get

$$z \in \pi_{\Phi\uparrow}(A\uparrow)\downarrow \leftrightarrow [(\forall a \in A\uparrow)\, z \in \Phi\uparrow(a)] = \mathbf{1}$$
$$\leftrightarrow \bigwedge_{a\in A} [z \in \Phi\uparrow(a)] = \mathbf{1} \leftrightarrow (\forall a \in A)(z \in \Phi\uparrow(a)\downarrow)$$
$$\leftrightarrow (\forall a \in A)\, z \in \Phi\uparrow\downarrow(a) \leftrightarrow z \in \pi_{\Phi\uparrow\downarrow}(A).$$

(5) The sought equality results from the above-proved by virtue of **3.3.3** (2).

(6) According to (1), for a fully extensional Φ we have

$$\pi_{\Phi\uparrow\downarrow}(A) = \bigcap_{a\in A} \Phi\uparrow\downarrow(a) = \bigcap_{a\in A} \Phi(a) = \pi_{\Phi}(A).$$

The required result now ensues from (5). ▷

3.3.13. Let us now consider the category $\mathcal{PV}_*^{(B)}$ that consists of nonempty subsets of the class $\mathbf{V}^{(B)}$ and extensional correspondences having a nonempty graph with the conventional superposition as the composition:

$$\mathrm{Ob}\,\mathcal{PV}_*^{(B)} := \mathcal{P}(\mathbf{V}^{(B)}) - \{\varnothing\};$$

$$\mathcal{PV}_*^{(B)}(X, Y) := \{\Phi : \Phi \text{ is an extensional correspondence from } X \text{ to } Y \text{ and } \mathrm{Gr}\Phi \neq \varnothing\},$$

$$\mathrm{Com}(\Phi, \Psi) := \Psi \circ \Phi\ (\Phi, \Psi \in \mathrm{Mor}\,\mathcal{PV}_*^{(B)}).$$

The subcategory of the category $\mathcal{PV}_*^{(B)}$ which consists of cyclic sets and fully extensional correspondences will be denoted by $\mathcal{CPV}_*^{(B)}$. Let $\mathcal{PV}^{(B)}$ and $\mathcal{CPV}^{(B)}$ be subcategories of the categories $\mathcal{PV}_*^{(B)}$ and $\mathcal{CPV}_*^{(B)}$, respectively, with with the same classes of objects but with classes of extensional mappings as morphisms. The soundness of definition is ensured by **3.3.10** and **3.3.11**. Let us now consider a mapping $\mathfrak{F}\uparrow$ assigning to every object X and every morphism Φ of the category $\mathcal{PV}_*^{(B)}$ their ascents $X\uparrow$ and $\Phi\uparrow$, respectively. By virtue of theorem **3.3.10**, $\mathfrak{F}\uparrow$ acts into the category $\mathcal{V}_*^{(B)}$ (see **3.1.7**).

3.3.14. Theorem. *The mapping $\mathfrak{F}\uparrow$ is a covariant functor from the category $\mathcal{PV}^{(B)}$ to the category $\mathcal{V}^{(B)}$.*

3.3.15. Remarks

(1) The use of the symbol $\uparrow$ for denoting various types of ascents is analogous to the situation of the notation of descents. Therefore, the warnings and agreements made in **3.2.5** and **3.2.18** (1) should be taken into account.

(2) The functors $\mathfrak{F}^{\wedge}$ and $\mathfrak{F}^{\uparrow}$ operate in the same category and in many respects resemble one another (compare, for instance, definitions **2.5.15** and **3.3.1** (1), formulas **3.3.2** with analogous formulas of **2.5.15**, **3.3.3** and **3.1.1** (1), **3.3.4** and **3.1.4**, **3.1.10** and **3.1.5**). A more detailed analogy will be discussed in the section to follow.

(3) Formulas **3.3.2** and their counterparts of **2.5.15** are the particular cases of the following rules. If φ and ψ are predicative formulas in $n+1$ and $m+1$ free variables, respectively, while $X_1,\dots,X_n$ and $Y_1,\dots,Y_m$ are certain $\mathbf{V}^{(B)}$-classes, then

$$[(\forall u)(\varphi(u,\bar{X}) \to \psi(u,\bar{Y})] = \wedge\{[\psi(u,\bar{X})]: x \in A\},$$
$$[(\exists u)(\varphi(u,\bar{X}) \to \psi(u,\bar{Y})] = \vee\{[\psi(u,\bar{X})]: x \in A\},$$

where A is any subclass of the class $\mathbf{V}^{(B)}$ obeying the condition

$$\operatorname{mix}(A) = \{x \in \mathbf{V}^{(B)} : [\varphi(x,\bar{X})] = \mathbb{1}\} \quad (\bar{X} = (X_1,\dots,X_n)).$$

(4) Ascending has already been implicitly used in **2.4**. Let us dwell on this point. Let x be a subset of an unseparated universe, while $x' \subset \mathbf{V}^{(B)}$ be its image under factorization (see **2.5.2**, **2.5.7**): $x' := \pi``x := \{\pi t : t \in x\}$. Let us determine an element y of the unseparated universe by the formulas: $\operatorname{dom}(y) := x$, $\operatorname{im}(y) := \{\mathbb{1}\}$. Then $[\pi y = x'\uparrow] = \mathbb{1}$. Indeed,

$$[\pi t \in x'\uparrow] = \bigvee_{u \in x'} [\pi t = u] = \bigvee_{u \in x} [\pi t = \pi u]$$
$$= \bigvee_{u \in \operatorname{dom}(y)} y(u) \wedge [t = u] = [t \in y] = [\pi t = \pi u].$$

Therefore, the element y of **2.4.5** (b), $\{x\}^B$ and $\{x,y\}^B$ of **2.4.8**, f of **2.4.11** (1-3), are all ascents in the unseparated universe. Besides, $X^{\wedge}$ is the ascent of the class $\{x^{\wedge} : x \in X\}$ (see **3.3.1** (1)).

(5) In statements **3.3.10** (1,2) the condition of general position cannot be omitted.

The corresponding counter-examples can be easily constructed on using the following considerations. Let us assume that $A \subset X$ and Φ is a correspondence from X to X with the graph $\{(x,x): x \in M\}$. If $A \subset X$, in which case $A \cap M = \varnothing$ but $A \cap \operatorname{mix} M \neq \varnothing$, then $\Phi(A) = \varnothing$ and $[\Phi(A)\uparrow = \varnothing] = \mathbf{1}$. On the other hand, $[\Phi\uparrow(A\uparrow) \neq \varnothing] = \mathbf{1}$, since for $z \in A \cap \operatorname{mix} M$ we have $[z \in \Phi\uparrow(A\uparrow)] = \mathbf{1}$.

It should be also remarked that in similar considerations ([114, 120, 121, 134]) the condition of general position is absent and, hence, it is always implicitly assumed that $A \subset \operatorname{dom}\Phi$ or $\operatorname{im}\Phi \subset \operatorname{dom}\Psi$. This might result in ambiguity when working with general correspondences. This is, however, not dangerous for correspondences defined everywhere and, in particular, for mappings. The remarks made refer to the rules for calculating polars as well (see **3.3.12** (6)).

3.4. The Immersion Functor

In applications of Boolean-valued models to analysis the following technique proves to be greatly expedient. The analytical object under investigation is immersed in a Boolean-valued universe in such a way that inside the model it becomes a simpler and (or) well-studied object. This procedure proves functional, i.e., it enables one to study not only the internal structure of certain objects, but also their interrelations.

3.4.1. The sets appearing as descents have a supplementary algebraic structure and, therefore, only objects associated in a necessary way with a complete Boolean algebra can be immersed in $\mathbf{V}^{(B)}$. Let us introduce the necessary terminology. Consider an arbitrary set X. The mapping $d: X \times X \rightarrow B$ is called a *B-semimetric* provided that for any $x, y, z \in X$ the following conditions are fulfilled:

(1) $d(x,y) = 0$;

(2) $d(x,y) = d(x,y)$;

(3) $d(x,y) \leq d(x,z) \vee d(z,y)$.

If, moreover, $d(x,y) = 0$ yields $x = y$, then d is called a *B-metric* or a *Boolean metric* on X. The pair (X,d) is termed a *B-set* or a *Boolean set*, provided X is a set, while d is a Boolean metric on X.

When the set X is contained in the class $\mathbf{V}^{(B)}$, then X is furnished with the canonical B-metric:

$$d(x,y) := [\![x \neq y]\!] = [\![x = y]\!]^* \quad (x,y \in X).$$

The fact that d is a B-metric follows from **2.1.8** (1,3,4) and the separation of $\mathbf{V}^{(B)}$. While considering the subsets of the class $\mathbf{V}^{(B)}$ as B-sets, we will always mean the Boolean metric mentioned above. Many notions of Chapter 2 are naturally transferred to B-sets by way of dualizing with the help of complementation in the algebra B. Thus, we sometimes omit some small details in introducing new notions.

3.4.2. Let (b_ξ) be a partition of unity in B, and let (x_ξ) be a family of elements of a B-set X. *The mixing of the family* (x_ξ) *by* (b_ξ) is an element $x \in X$ such that $b_\xi \wedge d(x,x_\xi) = 0$ for all ξ. As before, the mixing will be denoted by the symbol $x = \operatorname{mix} b_\xi x_\xi$. The mixing (if it exists) is unique. Indeed, if $y \in X$ and $(\forall \xi)(b_\xi \wedge d(y,x_\xi) = 0)$, then

$$b_\xi \wedge d(x,y) \leq b_\xi \wedge (d(x,x_\xi) \vee d(x_\xi,y)) = 0.$$

The infinite distributive law **1.1.5** (2) in B implies

$$d(x,y) = \vee\{b_\xi \wedge d(x,y)\} = 0,$$

and, hence, $x = y$.

It should be emphasized that unlike in the universe $\mathbf{V}^{(B)}$ (see **2.3**), not all mixings in a B-set exist.

3.4.3. Let us consider a B-set (X,d). For a subset $A \subset X$ by the symbol $\operatorname{mix} A$ we shall denote a set of all mixings of elements of A. If $\operatorname{mix} A = A$, then A is said to be a *cyclic subset* in X. The intersection of all cyclic sets containing A is denoted by $\operatorname{cyc}(A)$. A Boolean set X is called *extended* (or *fully cyclic*) if it contains mixings $\operatorname{mix}(b_\xi x_\xi)$ of any families $(x_\xi) \subset X$ relative to any partitions of unity $(b_\xi) \subset B$. In the case when such mixings exist only for finite sets of elements, the X itself is called *decomposable*. In the same way as in **3.2.8** it is shown that if X is an extended B-set, then $\operatorname{mix}(A) = \operatorname{cyc}(A)$ for any $A \subset X$. The cyclic subset of a B-set is not always an extended B-set. Each cyclic subset of $\mathbf{V}^{(B)}$ with its canonical B-metric is an extended B-set.

3.4.4. *Let A be a set, and let for any $\alpha \in A$ be given a B-set (X_α, d_α). Put $X := \prod_{\alpha \in A} X_\alpha$ and define the mapping $d: X \times X \to B$ in the following way:*

$$d(x,y) := \vee\{d_\alpha(x(\alpha), y(\alpha)) : \alpha \in A\}.$$

Then d is a Boolean metric on X, and, moreover, (X,d) is extended iff X_α is extended for any $\alpha \in A$.

$\lhd$ We can easily prove that the above mapping is a B-metric. Besides, if (b_ξ) is a partition of unity, while (x_ξ) is a family of elements of the product X, then $x = \text{mix}(b_\xi x_\xi(\alpha))$ iff $x(\alpha) = \text{mix}(b_\xi x_\xi(\alpha))$ for all $\alpha \in A$. Therefore, X can be claimed to be extended. $\rhd$

Below the product of B-sets will be always viewed as a B-set with the Boolean metric given in **3.4.4.**

3.4.5. *Let A be a subset of an extended B-set (X,d). Then for any $x \in X$ the distance*

$$\text{dist}(x, A) := \wedge\{d(x,a) : a \in A\}$$

is attained for a certain $a \in \text{mix} A$. In other words, for every $x \in X$ there is an $a \in \text{mix} A$ such that $\text{dist}(x, A) = d(x,a)$.

$\lhd$ If $b_0 := \text{dist}(x, A)$, then there is a partition (b_ξ) of an element b_0^* and a family $(a_\xi) \subset A$ such that $b_\xi \wedge d(x, a_\xi) = 0$ for all ξ. Let us put $a := \text{mix}\{b_0 a_0, b_\xi a_\xi\}$, where a_0 is an arbitrary element of A. Since $(b_\xi) \cup \{b_0\}$, is a partition of unity then $a \in \text{mix}(A)$. Besides, for any ξ we get

$$b_\xi \wedge d(x,a) \leq (b_\xi \wedge d(x, a_\xi)) \vee (b_\xi \wedge d(a_\xi, a)) = 0.$$

Hence, $b_0^* \wedge d(x,a) = \vee\{b_\xi \wedge d(x,a)\} = 0$ or $d(x,a) \leq b_0$. The converse statement is immediate. $\rhd$

3.4.6. Let us note three expedient corollaries to **3.4.5**.

(1) *The distance from a point $x \in X$ to the subset A of an extended B-set X is equal to zero iff $x \in \text{mix}(A)$.*

(2) The distance between two sets $A_1 \subset X$ and $A_2 \subset X$ is defined by the formula

$$\bar{d}(A_1, A_2) := \bigvee_{a \in A_1} \text{dist}(a, A_2) \vee \bigvee_{a \in A_2} \text{dist}(A_1, a).$$

We can easily check that $\bar{d}$ is a Boolean semimetric on $\mathcal{P}(X)$ but is not, generally speaking, a metric. It would be natural to call $\bar{d}$ a *Hausdorff B-semimetric associated with* $\bar{d}$. If X is extended, then $\bar{d}(A_1, A_2) = 0$ iff $\operatorname{mix} A_1 = \operatorname{mix} A_2$.

(3) *Let $\mathcal{P}_{\mathrm{cyc}}(X)$ be the set of all cyclic subsets of a B-set (X,d). Then (X,d) is extended iff $(\mathcal{P}_{\mathrm{cyc}}(X), \bar{d})$ is an extended B-set.*

◁ Indeed, let X be extended. Then, by virtue of (2), $\bar{d}$ is a metric on $\mathcal{P}_{\mathrm{cyc}}(X)$ and we have only to prove that $(\mathcal{P}_{\mathrm{cyc}}(X), \bar{d})$ is extended. To this end, let us consider a partition of unity (b_ξ) and a family (A_ξ) in $\mathcal{P}_{\mathrm{cyc}}(X)$. Define $A \subset X$ as the union of all mixings from $\operatorname{mix}(b_\xi x_\xi)$, where $x_\xi \in A_\xi$ for all ξ. In this case by virtue of the distributive laws **1.1.5** (1,2) for any $x \in A$ and $x' \in A_\xi$, we get

$$b_\xi \wedge \operatorname{dist}(x', A) = \wedge\{b_\xi \wedge d(x', a) : a \in A\} = 0,$$
$$b_\xi \wedge \operatorname{dist}(x, A_\xi) = \wedge\{b_\xi \wedge d(x, a) : a \in A_\xi\} = 0$$

and, finally, $b_\xi \wedge \bar{d}(A, A_\xi) = 0$. The last equality is valid for all ξ and, hence, $A = \operatorname{mix}(b_\xi A_\xi)$. Using the scheme of **3.2.8**, we can prove that A is cyclic. The converse statement results from the fact that the mapping $x \to \{x\}$ is an injection of X to $\mathcal{P}_{\mathrm{cyc}}(X)$, in which case $\bar{d}(\{x\},\{y\}) = d(x,y)$ for any $x, y \in X$. ▷

3.4.7. Let us consider B-sets (X,d_X) and (Y,d_Y). The correspondence Φ from X to Y is called *contractive*, a *contraction correspondence*, or simply a *contraction* provided

$$\bar{d}_Y(\Phi(x),\Phi(y)) \le d_X(x,y) \quad (x,y \in \operatorname{dom}\Phi),$$

where $\bar{d}_Y$ is a Hausdorff B-semimetric associated with d_Y.

(1) *The property of being a contraction for a correspondence Φ is equivalent to each of the conditions* (cf. **3.3.8** (1,2)):

(a) *if $d_X(x_1, x_2) \le b$ $(x_1, x_2 \in \operatorname{dom}\Phi)$, then for every $y \in Y$ we have*

$$b \vee \operatorname{dist}(y, \Phi(x_1)) = b \vee \operatorname{dist}(y, \Phi(x_2));$$

(b) $\operatorname{dist}(y_1, \Phi(x_2)) \le d_X(x_1, x_2)$ *for all $x_1, x_2 \in \operatorname{dom}\Phi$ and $y_1 \in \Phi(x_1)$.*

If X and Y are subsets in $\mathbf{V}^{(B)}$, then to denote the same property of a correspondence after the introduced definition we have to use two (contrasting in the conventional sense) terms, i.e., contraction and extensionality. To avoid ambiguity one has to remember that extensionality is interpreted with the Boolean truth-value of equality $[\cdot = \cdot]$, while contraction pertains to the B-*metric* under study.

A correspondence Φ will be termed *fully contractive* if it is contractive and

$$\Phi(x) = \operatorname{mix}(\Phi(x)) \quad (x \in \operatorname{dom}\Phi).$$

(2) *The descent of any correspondence is a fully contractive (or, which is the same, fully extensional) correspondence.*

◁ The result sought implies that if Ψ is a correspondence inside $\mathbf{V}^{(B)}$ and $\Phi := \Psi\downarrow$, then Φ is an extensional correspondence and $\Phi(x)$ is a cyclic set for every $x \in \operatorname{dom}\Phi$. By virtue of **3.2.6** (9), **3.2.13** and **3.3.8** (5), Φ is extensional, while by **3.2.3** (1) and **3.2.13** (1) $\Phi(x)$ is cyclic. ▷

The mapping $f: X \to Y$ is contractive (a contraction mapping) if

$$d_Y(f(x), f(x')) \leq d_X(x, x') \quad (x, x' \in X).$$

If in the last relation the equality is fulfilled, then f is said to be a *B-isometry*. A bijective B-isometry is termed the *isomorphism* of B-sets.

3.4.8. Any set $X \in \mathbf{V}$ can be turned into a B-set if we define on it the discrete B – metric:

$$d(x,y) := \begin{cases} \mathbf{1}_B, & \text{if } x \neq y, \\ \mathbf{0}_B, & \text{if } x \neq y. \end{cases}$$

In this case the pair (X,d) is called a *discrete B-set*. In a discrete B-set there is no mixing $\operatorname{mix}(b_\xi x_\xi)$ only if the set of elements (x_ξ) contains more than one element, while the partition of unity (b_ξ) is different from the trivial partition $\{\mathbf{0}_B, \mathbf{1}_B\}$. Any correspondence given on a discrete B-set is contractive.

Discrete and extended B-sets are two extreme examples of 'B-qualification' presented by the elements of the universes $\mathbf{V}$ and $\mathbf{V}^{(B)}$ (see **3.2.3**), compromise variations presented by the class $\mathcal{P}(\mathbf{V}^{(B)})$. In analysis, B-sets of other origin are often encountered.

3.4.9. *Let π be a full monomorphism of B to a Boolean algebra C. Let us put*

$$d_\pi(x,y) := \wedge\{b^* : \pi(b) \wedge x = \pi(b) \wedge y\} \quad (x,y \in C).$$

Then d_π is a B metric on C and Boolean operations on C are contractive mappings.

◁ If $\pi = I_B$, then $d_\pi(b, b') = (b \Leftrightarrow b')^* = b\Delta b'$. Let us consider one more complete Boolean algebra C' and a full monomorphism $\pi' : B \to C'$. Then the homomorphism $h: C \to C'$ will be a contractive mapping from B-sets (C, d_π) to $(C', d_{\pi'})$ iff $h \circ \pi = \pi'$. Indeed, the fact that h is contractive in the metrics d_π and $d_{\pi'}$ implies that $\pi(b) \wedge x = \pi(b) \wedge y$ implies $\pi'(b) \wedge h(x) = \pi'(b) \wedge h(y)$ for any $x, y \in C$ and $b \in B$. If $\pi' = h \circ \pi$, then, applying h to the equality $\pi(b) \wedge x = \pi(b) \wedge y$, we obtain $\pi'(b) \wedge h(x) = \pi'(b) \wedge h(y)$. Conversely, if in the last equality we choose $x = \mathbf{1}_C$ and $y := \pi(b)$, then we get either $\pi'(b) = \pi'(b) \wedge h\pi(b)$ or $\pi'(b) \leq h \circ \pi(b)$. Since $b \in B$, we deduce $\pi' = h \circ \pi$. ▷

3.4.10. Let us consider another construction with B-sets analogous to **2.2.10**. Let ψ be an ultrafilter on a Boolean algebra D. Let us consider a Boolean set (X, d_X) with a D-valued B-metric d_X. Introduce a binary relation $\sim_\psi$ in X by the formula

$$(x,y) \in \sim_\psi \leftrightarrow d_X(x,y)^* \in \psi.$$

The definition of a Boolean metric implies that $\sim_\psi$ is an equivalence. Let $X/\sim_\psi$ be the factor-set of the set X by the relation $\sim_\psi$, while $\pi_X : X \to X/\sim_\psi$ be the canonical mapping. If the same is carried out with the Boolean set (D, Δ), then as a $D/\sim_\psi$ we have the two-element Boolean algebra, so that $D/\sim_\psi \cong \{\mathbf{0}_D, \mathbf{1}_D\}$. As is seen, there is a unique mapping $\tilde{d} : X/\sim_\psi \to D/\sim_\psi$ such that $\tilde{d}(\pi_X x, \pi_X y) = \pi_D(d(x,y))$ $(x, y \in X)$. Besides, $\tilde{d}$ is a discrete Boolean metric on $X/\sim_\psi$. If d_X is a discrete metric, then $\sim_\psi = I_X$ and $X/\sim_\psi = X$. Some set-theoretical operations in X and $X/\sim_\psi$ are interrelated by simple relations. If (X_α) is a family of subsets of the set X, then $(\cup X_\alpha)/\sim_\psi = \cup(X_\alpha/\sim_\psi)$. Between $X^n/\sim_\psi$ and $(X/\sim_\psi)^n$ there is a natural bijection given by the formula

$$\pi_{X^n}(x_1, \dots, x_n) \to (\pi_X x_1, \dots, \pi_X x_n) \quad (x_1, \dots, x_n \in X).$$

It should be also remarked that if $A \subset X$, then $A/\sim_\psi = \pi_X(A)$ and $\pi_A = \pi_X | A$.

Let us choose one more B-set (Y, d_Y), and let $F \subset X \times Y$. It can be easily checked that in this case

$$\mathrm{dom}(F/\sim_\psi) = \mathrm{dom}(F)/\sim_\psi, \ \mathrm{im}(F/\sim_\psi) = \mathrm{im}(F)/\sim_\psi.$$

3.4.11. *Let ρ be an arbitrary automorphism (=homomorphism into itself) of a Boolean algebra B, and ψ_ρ be an element of $\mathbf{V}^{(B)}$ determined by the function $\{(b^\wedge, \rho(b)) : b \in B\}$ in line with* **2.5.6.** *Then the following statements are valid:*

(1) $\rho(b) = [b^\wedge \in \psi_\rho]$ *for any* $b \in B$;

(2) *for a set* $A \subset B$ *we have* $[A^\wedge \subset \psi_\rho \to (\wedge A)^\wedge \in \psi_\rho] = \mathbf{1}$ *iff* $\rho(\wedge A) = \wedge \rho(A)$;

(3) $[\psi_\rho$ *is an ultrafilter on* $B^\wedge] = \mathbf{1}$.

⊲ (1) This result is checked by applying **2.2.8** (1,2).

(2) Using (1), we deduce for $A \subset B$:

$$[A^\wedge \subset \psi_\rho] = \bigwedge_{a \in A} [a \in \psi_\rho] = \bigwedge_{a \in A} \rho(a) = \wedge \rho(A).$$

Since $\rho(\wedge A) \leq \wedge \rho(A)$ because ρ is isotonic, the inequality $[A^\wedge \subset \psi_\rho] \leq [(\wedge A)^\wedge \in \psi_\rho]$ is equivalent to the equality $\rho(\wedge A) = \wedge \rho(A)$.

(3) Let us first of all observe that $V^{(B)} \models \psi_\rho \subset B^\wedge$. Indeed, for every $t \in \mathbf{V}^{(B)}$ we have

$$[t \in \psi_\rho] = \bigvee_{b \in B} \rho(b) \wedge [t = b^\wedge] \leq \bigvee_{b \in B} [t = b^\wedge] = [t \in B^\wedge].$$

Then, it follows from (1) that $[0^\wedge \notin \psi_\rho] = \mathbf{1}$, while (2) yields that $[\psi_\rho$ is a filter base $] = \mathbf{1}$. Besides, if $b \in B$, then

$$[(\exists a \in \psi_\rho)(a \leq b^\wedge)] = \bigvee_{a \in B} \rho(a) \wedge [a^\wedge \leq b^\wedge] = \bigvee_{a \leq b} \rho(a)$$
$$= \rho(b) = [b^\wedge \in \psi_\rho],$$

so that

$$[(\forall b \in B^\wedge)((\exists a \in \psi_\rho)\, a \leq b) \to b \in \psi_\rho] = \mathbf{1}.$$

Therefore, ψ_ρ is a filter in $B^\wedge$ inside $\mathbf{V}^{(B)}$ and we have to show that $\mathbf{V}^{(B)} \models$ 'for any $b \in B^\wedge$, or $b \in \psi_\rho$, or $b^* \in \psi_\rho$'. This statement is validated by the following formulas:

$$[(\forall b\in B^{\wedge})(b\in\psi_{\rho}\vee b^{*}\in\psi_{\rho})]$$
$$=\bigwedge_{b\in B}[b^{\wedge}\in\psi_{\rho}]\vee[(b^{*})^{\wedge}\in\psi_{\rho}]=\bigwedge_{b\in B}\rho(b)\vee\rho(b^{*})$$
$$=\wedge\{\rho(b\vee b^{*}): b\in B\}=\rho(1)=1.\ \triangleright$$

3.4.12. Let $\psi := \psi_{\iota}$, where ι is the identity homomorphism on B. According to **3.4.11**, $\mathbf{V}^{(B)}\models$ 'ψ is an ultrafilter on $B^{\wedge}$, and $A^{\wedge}\subset\psi$ implies $\wedge(A)^{\wedge}\in\psi$', whatever a set $A\subset B$.

Let us choose an arbitrary B-set (X,d). It is obvious from **3.1.16** that $(X^{\wedge},d^{\wedge})$ is a B-set inside $\mathbf{V}^{(B)}$. By virtue of **3.4.10, 3.4.11** and by the maximum principle, there are such $\tilde{X}, \sim := \sim_{\psi}$ and $\pi_X\in\mathbf{V}^{(B)}$ that

(1) $\mathbf{V}^{(B)}\models$ "$\sim$ is an equivalence relation on $X^{\wedge}$";

(2) $\mathbf{V}^{(B)}\models \tilde{X} := X^{\wedge}/\sim$;

(3) $\mathbf{V}^{(B)}\models$ "$\pi_X : X\to\tilde{X}$ is the factor-mapping";

(4) $[(x^{\wedge},y^{\wedge})^{B}\in\sim] = d(x,y)^{*}\quad (x,y\in X)$.

If we apply the described procedure to a B-set (B,Δ) (see **3.4.9**), then as $\tilde{B}$ we get the two-element Boolean algebra, so that $\mathbf{V}^{(B)}\models \tilde{B}\{\mathbf{0}_B^{\wedge},\mathbf{1}_B^{\wedge}\}^{B}$. Therefore, inside $\mathbf{V}^{(B)}$ there is a unique $\{\mathbf{0}_B^{\wedge},\mathbf{1}_B^{\wedge}\}$-valued Boolean metric $\tilde{d}$ on $\tilde{B}$, for which

$$\mathbf{V}^{(B)}\models(\forall x,y\in X^{\wedge})\ d(\pi_X(x),\pi_X(y))=\pi_B(d^{\wedge}(x,y)).$$

As is seen from **3.4.10**, for the discrete B-set (X,d) we get $\sim = I_{X^{\wedge}}$ and $X^{\sim} = X^{\wedge}$.

We shall say that subsets A and C of a certain B-set (X,d) *are in general position* provided

$$d(a,c)\geq\wedge\{d(a,b)\vee d(b,c): b\in A\cap C\}$$

for any $a\in A$ and $c\in C$. As was the case in **3.3.9**, the above relation is in fact fulfilled with equality, since $d(a,c)\leq d(a,b)\vee d(b,c)$.

(5) *The sets A and C are in general position iff*

$$\mathbf{V}^{(B)}\models(A\cap C)^{\sim}=A^{\sim}\cap C^{\sim}.$$

◁ It should be remarked that $(A \cap C)^{\sim} = \pi_X((A \cap C)^{\wedge}) = \pi_X(A^{\wedge} \cap C^{\wedge})$ and $A^{\sim} \cap C^{\sim} = \pi_X(A^{\wedge}) \cap \pi_X(C^{\wedge})$. Hence, the inclusion $(A \cap C)^{\sim} \subset A^{\sim} \cap C^{\sim}$ is always valid, while $A^{\sim} \cap C^{\sim} \subset (A \cap C)^{\sim}$ is equivalent to the formula

$$(\forall a \in A^{\wedge})(\forall c \in C^{\wedge})(a \sim c \to (\exists b \in (A \cap C)^{\wedge})(b \sim a \wedge b \sim c)).$$

Writing out the Boolean truth-value of the last formula and making use of the equality $[a^{\wedge} \sim c^{\wedge}] = d(a,c)^*$, we get

$$\bigwedge_{a \in A, c \in C} d(a,c)^* \Rightarrow \left(\bigvee_{b \in A \cap C} d(a,b)^* \wedge d(b,c)^* \right) = 1.$$

It is now evident that $[A^{\sim} \cap C^{\sim} \subset (A \cap C)^{\sim}] = \mathbf{1}$ iff for any $a \in A$ and $c \in C$ we obtain

$$d(a,c) \vee \left(\bigwedge_{b \in A \cap C} d(a,b) \vee d(b,c) \right)^* = \mathbf{1}.$$

It is this result that implies that A and C are in general position. ▷

3.4.13. Theorem. *Let (X, d_X) and (Y, d_Y) be certain B-sets and Φ be a contractive correspondence from X to Y. Then inside $\mathbf{V}^{(B)}$ there is a unique correspondence $\Phi^{\sim}$ from $X^{\sim}$ to $Y^{\sim}$ such that*

$$\mathrm{dom}(\Phi^{\sim}) = \mathrm{dom}(\Phi)^{\sim},$$
$$[\Phi^{\sim}(\pi_X x^{\wedge}) = \pi_Y(\Phi(x)^{\wedge})] = \mathbf{1} \quad (x \in \mathrm{dom}\Phi).$$

In this case the following relations are valid:

(1) *if the sets $A \subset X$ and* $\mathrm{dom}\,\Phi$ *are in general position, then*

$$\mathbf{V}^{(B)} \models \Phi(A)^{\sim} = \Phi^{\sim}(A^{\sim});$$

(2) *the superposition $\Psi \circ \Phi$ of contractive correspondences Φ and Ψ is contractive, and if $\Psi \circ \Phi = \mathrm{dom}\Phi$ and the sets $\mathrm{dom}\Psi$ and $\Phi(x)$ are in a general position for all $x \in \mathrm{dom}\Phi$, then*

$$\mathbf{V}^{(B)} \models (\Psi \circ \Phi)^{\sim} = \Psi^{\sim} \circ \Phi^{\sim};$$

(3) $\mathbf{V}^{(B)} \models (I_X)^{\sim} = I_{X^{\sim}}$.

$\lhd$ As is known from **3.1.5**, $\mathbf{V}^{(B)} \models$ '$\Phi^{\wedge}$ is a correspondence from $X^{\wedge}$ to $Y^{\wedge}$'. Let us put $\Phi^{\sim} := \pi_Y \circ \Phi^{\wedge} \circ \pi_X^{-1}$. It is obvious that $\mathbf{V}^{(B)} \models$ '$\Phi^{\sim}$ is a correspondence from $X^{\sim}$ to $Y^{\sim}$ and $\operatorname{dom} \Phi^{\sim} = \pi_X(\operatorname{dom}(\Phi^{\wedge})) = \pi_X((\operatorname{dom}(\Phi)^{\wedge}) = (\operatorname{dom}\Phi)^{\sim}$'. Let us now show that for any $x \in Z := \operatorname{dom}\Phi$ and $y \in \mathbf{V}^{(B)}$ the Boolean truth-values $b_1 := [y \in \Phi^{\sim} \circ \pi_X(x^{\wedge})]$ and $b_2 := [y \in \pi_Y \circ \Phi^{\wedge}(x^{\wedge})]$ coincide. Indeed,

$$
\begin{aligned}
b_1 :&= [(\exists s \in Z^{\wedge})(\exists t \in Y^{\wedge})(y = \pi_Y(t) \wedge t \in \Phi^{\wedge}(s) \wedge \pi_X(s) = \pi_X(x^{\wedge}))] \\
&= \bigvee_{s \in Z} \bigvee_{t \in Y} [t^{\wedge} \in \Phi(s)^{\wedge}] \wedge [y = \pi_Y(t^{\wedge})] \wedge [\pi_X(s) = \pi_X(x^{\wedge})] \\
&\geq \bigvee_{t \in Y} [y = \pi_Y(t^{\wedge})] \wedge [t^{\wedge} \in \Phi(x)^{\wedge}] \\
&= [(\exists t \in Y^{\wedge})(y = \pi_Y(t) \wedge t \in \Phi^{\wedge}(x^{\wedge}))] = b_2.
\end{aligned}
$$

On the other hand, making use of the equalities

$$
\begin{aligned}
&d_X(s,x)^* = [\pi_X(s^{\wedge}) = \pi_X(x^{\wedge})], \\
&\bar{d}_Y(\Phi(x), \Phi(s))^* = [\pi_Y(\Phi(x)^{\wedge}) = \pi_Y(\Phi(s)^{\wedge})]
\end{aligned}
$$

and taking it into account that the correspondence Φ is contractive, we infer

$$
\begin{aligned}
b_1 &\leq \bigvee_{s \in Z} \bigvee_{t \in Y} [\pi_Y(\Phi(s)^{\wedge}) = \pi_Y(\Phi(x)^{\wedge})] \wedge = [t^{\wedge} \in \Phi(s)^{\wedge}] \\
&\wedge [y = \pi_Y(t^{\wedge})] \leq \bigvee_{s \in Z} [y \in \pi_Y(\Phi^{\wedge}(x^{\wedge}))] = b_2.
\end{aligned}
$$

Therefore, $b_1 = b_2$, which immediately implies the validity of the defining relation $[\pi_Y(\Phi(x)^{\wedge}) = \Phi^{\sim}(\pi_X(x^{\wedge}))] = \mathbf{1}$ $(x \in Z)$. Hence, the relation

$$
\mathbf{V}^{(B)} \models (\forall x \in (\operatorname{dom}\Phi)^{\wedge})\, \Phi^{\sim}(\pi_X x) = \pi_Y \Phi^{\wedge}(x)
$$

holds, which fact, in turn, implies that $\Phi^{\sim}$ is unique, since $\operatorname{dom}(\Phi^{\sim}) = (\operatorname{dom}\Phi)^{\sim} = \pi_X((\operatorname{dom}(\Phi)^{\wedge})$.

(1) Using **3.4.12** (5), we can easily note that

$$
\Phi^{\sim}(A^{\sim}) = \Phi^{\sim}(A^{\sim} \cap \operatorname{dom}(\Phi^{\sim})) = \Phi^{\sim}((A \cap \operatorname{dom}(\Phi)^{\sim}).
$$

On the other hand, $\Phi(A)^{\sim} = \Phi(A \cap \operatorname{dom}\Phi)^{\sim}$ and, hence, without loss of generality, it can be assumed that $A \subset \operatorname{dom}\Phi$. In this case, however, by the defining property of $\Phi^{\sim}$, we can write inside $\mathbf{V}^{(B)}$ the chain of equalities:

$$\Phi^{\sim}(A^{\sim}) = \bigcup_{a\in A^{\sim}} \Phi^{\sim}(a) = \bigcup_{a\in A^{\wedge}} \Phi^{\sim}(\pi_X a)$$
$$= \bigcup_{a\in A^{\wedge}} \pi_Y(\Phi^{\wedge}(a)) = \pi_Y(\Phi^{\wedge}(A^{\wedge})) = \pi_Y(\Phi(A)^{\wedge}) = \Phi(A^{\sim}).$$

(2) Let Ψ be a contraction correspondence from Y to U. Let us choose $x_1, x_2 \in Z$, $y_1 \in \Phi(x_1)$ and $u_1 \in \Psi(y_1)$. Then, according to **3.4.7** (1),

$$\operatorname{dist}(u_1, \Psi\circ\Phi(x_2)) \le \wedge\{\operatorname{dist}(u_1, \Psi(y)) : y\in\Phi(x_2)\}$$
$$\le \wedge\{d(y_1, y) : y\in\Phi(x_2)\} = \operatorname{dist}(y_1, \Phi(x_2)) \le d(x_1, x_2).$$

Therefore, since x_1, x_2, y_1 and u_1 are arbitrary, we conclude that the correspondence $\Psi\circ\Phi$ is contractive. Then, making use of (1), **3.1.5** (2) and the defining relations for $(\Psi\circ\Phi)^{\sim}$, $\Psi^{\sim}$ and $\Phi^{\sim}$, we can write ($x\in Z$):

$$(\Psi^{\sim}\circ\Phi^{\sim})(\pi_X x^{\wedge}) = \Psi^{\sim}(\Phi(x)^{\sim}) = \Psi(\Phi(x))^{\sim}$$
$$= \pi_Y((\Psi\circ\Phi)(x)^{\wedge}) = \pi_Y((\Psi\circ\Phi)^{\wedge}(x^{\wedge})) = (\Psi\circ\Phi)^{\sim}(\pi_X x^{\wedge}).$$

Hence, $[(\Psi\circ\Phi)^{\sim} = \Phi^{\sim}\circ\Phi^{\sim}] = \mathbf{1}$, since $Z^{\sim} = \operatorname{dom}(\Psi^{\sim}\circ\Phi^{\sim})$.

(3) The result follows obviously from **3.1.5** (4). ▷

3.4.14. Theorem *For any contraction mapping $f: X\to Y$ there is a unique element $f^{\sim}\in\mathbf{V}^{(B)}$ such that*

$$[f^{\sim}: X^{\sim}\to Y^{\sim}] = [f^{\sim}\circ\pi_X = \pi_Y\circ f^{\wedge}] = \mathbf{1}.$$

In this case the following statements are valid:

(1) $\mathbf{V}^{(B)} \models f(A)^{\sim} = f^{\sim}(A^{\sim})$ *for every* $A\subset X$;

(2) *if $g: Y\to Z$ is a contraction mapping, then $g\circ f$ is a contraction mapping and* $\mathbf{V}^{(B)} \models (g\circ f)^{\sim} = g^{\sim}\circ f^{\sim}$;

(3) $\mathbf{V}^{(B)} \models$ '$f^{\sim}$ *is injective' iff f is a B-isometry;*

(4) $\mathbf{V}^{(B)} \models$ '$f^{\sim}$ *is surjective iff* $\vee\{d(f(x), y) : x\in X\} = \mathbf{1}$ *for every* $y\in Y$.

3.4.15. Let us consider categories BSet$_*$ and CBSet$_*$. The objects of these categories are nonempty B-sets and nonempty extended B-sets, respectively, The composition of morphisms is the superposition of correspondences. The subcategories of the categories BSet$_*$ and CBSet$_*$ consisting of the same objects and of contraction mappings will be denoted by BSet and CBSet, respectively. Let $\mathfrak{F}^{\sim}$ be the function assigning to the object X and morphism Φ of the category BSet the elements $\mathfrak{F}^{\sim}(X) := X^{\sim}$ and $\mathfrak{F}^{\sim}(\Phi) := \Phi^{\sim}$.

3.4.16. The mapping $\mathfrak{F}^{\sim}$ is a covariant functor from the category BSet to the category $\mathfrak{V}^{(B)}$.

3.4.17. Remarks

(1) The concept of a Boolean metric appeared at the beginning of the 1950s as a result of studying various 'distances' on abstract sets with the values in ordered systems (see [15, 43, 216]. There has, however, been discovered no particularly rich geometry associated with this concept, which fact accounts for B-metrics being unpopular in the years to follow. The reason of this curiosity can be understood through theorems **3.4.13** and **3.5.4**.

The geometry of Boolean metrics is much more meaningful and interesting when combined with topological and functional-analytic structures. In this case the presence of a duly agreed B-metric shows it to be expedient to study the considered structure by the method of Boolean-valued models.

(2) The mapping $[\cdot = \cdot]: X^2 \to B$ is called a *Boolean-valued equality,* provided it obeys conditions **2.2.8** (1,3,4). Such mappings are widely used for Boolean-valued interpretations of first-order theories (see [56]). The notion of a Boolean-valued equality is easily seen to be just a 'mirror mapping' of the idea of a Boolean metric, since conditions **2.2.8** (1,3,4) are fulfilled only iff the mapping $(x,y) \to [x = y]^*$ is a Boolean metric. In this context the idea of a Boolean metric proves to be quite fruitful.

(3) Definitions **3.4.1** adopted in this section are motivated by the fact that in analysis the structures under study are often related to a certain B-(semi)metric, while a B-valued equality is to be introduced artificially.

(4) We can demonstrate that the statement converse to **3.4.6** is also true. Namely, if ψ is an ultrafilter on $B^{\wedge}$ inside $\mathbf{V}^{(B)}$, then the mapping $\rho_\psi : B \to B$ determined by the formula $\rho_\psi(b) := [b^{\wedge} \in \psi]$ is an automorphism of $B^{\wedge}$. In this case $\rho_{\psi_\rho} = \rho$ and $[\psi_{\rho_\psi} = \psi] = 1$.

(5) As to statements **3.4.13** (1,2), one can apply to them the same remarks as in **3.3.15** (5).

3.5. Interrelations of Basic Functors

Among the basic functors described in the preceding four paragraphs, there are interesting and quite expedient for applications interrelations. Their study is the contents of the present section.

3.5.1. It should be recalled that for an arbitrary $X \in \mathcal{P}(\mathbf{V}^{(B)})$ the set $(x_\xi) \subset X$ consists of all possible mixings $\mathrm{mix}(b_\xi x_\xi)$ of $(x_\xi) \subset X$ families relative to any partitions of unity $(b_\xi) \subset B$ (see **3.2.8**). Let us expand mix onto extensional correspondences.

Let X *and* Y *be subsets of the class* $\mathbf{V}^{(B)}$, *while* Φ *be an extensional correspondence from* X *to* Y. *There is a unique fully extensional correspondence* Ψ *from* $\mathrm{mix}(X)$ *to* $\mathrm{mix}(Y)$, *for which*

$$\Psi(x) = \mathrm{mix}\,\Phi(x) \quad (x \in \mathrm{dom}\Phi).$$

$\lhd$ Indeed, we should set $\Psi := \Phi\uparrow\downarrow$ and make use of statements **3.3.12** (1) and **3.4.7** (2). From **3.2.13** and **3.3.3** (1) we see that $\mathrm{Gr}(\Psi) = \mathrm{mix}\,\mathrm{Cr}(\Phi)$. $\rhd$

By definition, we put $\mathrm{mix}\Phi := \Psi$. If Θ is another extensional correspondence and $\mathrm{dom}\Theta \subset Y$, then, by **3.2.13** (3) and **3.3.4** (8) we have $\mathrm{mix}(\Theta \circ \Phi) = \mathrm{mix}(\Theta) \circ \mathrm{mix}(\Phi)$ iff $(\Theta \circ \Phi)\uparrow = \Theta\uparrow \circ \Phi\uparrow$. Moreover, it is obvious that $(I_X) = I_{\mathrm{mix}(X)}$.

3.5.2. Let us choose a nonempty set X. By the symbol $B_0(X)$ we shall denote a set of all partitions of unity in B of the type $(b_x = b(x))_{x \in X}$:

$$b \in B_0(X) \leftrightarrow (b \in B^X \wedge (\forall x \in X)(\forall y \in X)(x \neq y) \to b(x) \wedge b(y) = 0)).$$

Assign to an element $y \in X$ the partition of unity $\iota y := \iota_X y := (b_x)_{x \in X}$, where $b_x = \mathbf{1}$ for $x = y$ and $b_x = 0$ for $x \neq y$. It is obvious that ι_X is an injection from X to $B_0(X)$. For elements $u, v \in B_0(X)$ let us, according to the definition, put

$$d(u,v) := \wedge\{u(x)^* \vee v(x)^* : x \in X\}.$$

It can easily checked that d is a B-metric on $B_0(X)$. Moreover, $(B_0(X),d)$ is an extended B-set. The last fact is, in essence, established by the same considerations as in **3.2.8**. Hence, $B_0(\cdot)$ is a mapping from $\mathbf{V}$ to CBSet.

Let us extend this mapping onto correspondences. Choose a correspondence $\Phi := (F,Y,Y)$ and determine $B_0(\Phi) := (G, B_0(X), B_0(Y))$, where

$$G := \{(u,v) \in B_0(X) \times B_0(Y) \leftrightarrow$$
$$\leftrightarrow (\forall x \in X)(\forall y \in Y)(u(x) \wedge v(y) \neq 0 \rightarrow (x,y) \in F)\}.$$

If Φ is unique, then $B_0(\Phi)$ is unique. Directly from definitions we can deduce:

$$B_0(I_X) = I_{B_0(X)},$$
$$B_0(\Psi \circ \Phi) = B_0(\Psi) \circ B_0(\Phi),$$
$$\Phi = \iota_Y^{-1} \circ B_0(\Phi) \circ \iota_X.$$

Hence, the mapping $B_0(\cdot)$ is a covariant functor from $\mathfrak{V}_*$ to CBSet$_*$.

3.5.3. Some interrelations between basic operations of Boolean-valued analysis have been earlier presented in the form of the rules of arrow cancellation. Let us now supply these rules with functorial formulations.

(1) *The descent functor $\mathfrak{F}^{\downarrow}$ and the ascent functor $\mathfrak{F}^{\uparrow}$ establish a homomorphism between the categories $\mathfrak{V}^{(B)}$ and $\mathfrak{CPV}^{(B)}$. This implies that $\mathfrak{F}^{\uparrow} \circ \mathfrak{F}^{\downarrow}$ and $\mathfrak{F}^{\downarrow} \circ \mathfrak{F}^{\uparrow}$ coincide with identity functors on $\mathfrak{V}^{(B)}$ and $\mathfrak{CPV}^{(B)}$, respectively.*

◁ The identity for the functor $\mathfrak{F}^{\uparrow} \circ \mathfrak{F}^{\downarrow}$ is prompted by the rules for 'descent-ascent' **3.3.3** (2) and **3.3.12** (3), while that of the functor $\mathfrak{F}^{\downarrow} \circ \mathfrak{F}^{\uparrow}$ from the rule of 'ascent-descent' **3.3.3** (1) and **3.3.12** (1). ▷

Let us recall some notions of the theory of categories. Let us choose categories $\mathfrak{H}, \mathfrak{K}$ and covariant functors $\mathfrak{F} : \mathfrak{H} \rightarrow \mathfrak{K}$, $\mathfrak{G} : \mathfrak{K} \rightarrow \mathfrak{H}$. *The natural transform* $\Theta : \mathfrak{F} \rightarrow \mathfrak{G}$ of the functor $\mathfrak{F}$ to the functor $\mathfrak{G}$ is a mapping $\Theta : \mathrm{Ob}\,\mathfrak{H} \rightarrow \mathrm{Mor}\,\mathfrak{K}$ such that

(a) $\Theta_a := \Theta(a) \in \mathfrak{K}(\mathfrak{F}(a), \mathfrak{G}(a))$ for every $a \in \mathrm{Ob}\,\mathfrak{H}$, and

(b) $\mathfrak{G}(\alpha)\Theta_a = \Theta_b \mathfrak{F}(a)$ for any morphism $\alpha : a \rightarrow b$.

The functors $\mathfrak{F}$ and $\mathfrak{G}$ are termed *isomorphic* and are marked as $\mathfrak{F} \sim \mathfrak{G}$, provided there is a natural transform $\Theta : \mathfrak{F} \rightarrow \mathfrak{G}$ such that Θ_a is an isomorphism for every $a \in \mathrm{Ob}\,\mathfrak{H}$. The categories $\mathfrak{H}$ and $\mathfrak{K}$ are said to be *equivalent* provided there is such a pair of functors

$\mathfrak{F}:\mathfrak{K}\to\mathfrak{L}$ and $\mathfrak{G}:\mathfrak{L}\to\mathfrak{K}$ that $\mathfrak{F}\circ\mathfrak{G}\sim I_{\mathfrak{L}}$ and $\mathfrak{G}\circ\mathfrak{F}\sim I_{\mathfrak{K}}$. In this case one often uses the phrase 'the functors $\mathfrak{F}$ and $\mathfrak{G}$ establish equivalence for the categories $\mathfrak{K}$ and $\mathfrak{L}$'.

The category $\mathfrak{K}^0$ *dual to* $\mathfrak{K}$ consists, by definition, of the same objects and morphisms as $\mathfrak{K}$, but in the definition of $\mathfrak{K}^0$ the mappings D and R swap their places and the order of morphisms in the composition is reversed (see **2.5.19**). The *product* of the categories $\mathfrak{K}$ and $\mathfrak{L}$ is determined by the relations

$$\begin{aligned}&\mathrm{Ob}\,\mathfrak{K}\times\mathfrak{L}:=\mathrm{Ob}\,\mathfrak{K}\times\mathrm{Ob}\,\mathfrak{L};\\&\mathfrak{K}\times\mathfrak{L}((a,b),(a',b')):=\mathfrak{K}(a,a')\times\mathfrak{L}(b,b');\\&(\alpha',\beta')\circ(\alpha,\beta):=(\alpha'\alpha,\beta',\beta),\end{aligned}$$

where $a,a'\in\mathrm{Ob}\,\mathfrak{L}$; $b,b'\in\mathrm{Ob}\,\mathfrak{L}$; $\alpha,\alpha'\in\mathrm{Mor}\,\mathfrak{K}$; $\beta,\beta'\in\mathrm{Mor}\,\mathfrak{L}$.

Let us now introduce the notion of a conjugate functor. Again consider the functors $\mathfrak{F}:\mathfrak{K}\to\mathfrak{L}$ and $\mathfrak{G}:\mathfrak{L}\to\mathfrak{K}$. Then determine two new functors $\mathfrak{F}^h$ and $\mathfrak{G}_h$ from the category $\mathfrak{K}^0\times\mathfrak{L}$ to the category Set of sets and mappings. For any $(a,b)\in\mathrm{Ob}\,\mathfrak{K}\times\mathrm{Ob}\,\mathfrak{L}$, $\alpha\in\mathfrak{K}(a,a')$, $\beta\in\mathfrak{L}(b,b')$, set

$$\begin{aligned}&\mathfrak{F}^h(a,b):=\mathfrak{L}(\mathfrak{F}(a),b),\ \mathfrak{F}_h(a,b):=\mathfrak{K}(a,\mathfrak{G}(b)),\\&\mathfrak{F}^h(\alpha,\beta):=f\to\beta f\mathfrak{F}(\alpha),\ \mathfrak{F}_h(\alpha,\beta):g\to\mathfrak{G}(\beta)g\alpha,\end{aligned}$$

where $f\in\mathfrak{L}(\mathfrak{F}(a,b))$, $g\in\mathfrak{K}(a,\mathfrak{G}(b))$. If the functors $\mathfrak{F}^h$ and $\mathfrak{G}_h$ are isomorphic, then we say that $\mathfrak{G}$ is *right-conjugate* to the functor $\mathfrak{F}$, while $\mathfrak{F}$ is *left-conjugate* to the functor $\mathfrak{G}$, or, to put it short, that $\mathfrak{F}$ and $\mathfrak{G}$ is a *conjugate pair of functors.* The isomorphism $\Theta:\mathfrak{F}^h\to\mathfrak{G}_h$ is referred to as *conjugation*, while the inverse isomorphism Θ^{-1}, as *co-conjugation.*

Let $\mathfrak{L}$ be a subcategory of the category $\mathfrak{K}$. An object $b\in\mathrm{Ob}\,\mathfrak{L}$ is called an $\mathfrak{L}$-reflector of the object $a\in\mathrm{Ob}\,\mathfrak{K}$ provided there is such a morphism $\delta:a\to b$ that any morphism $(\alpha:a\to c)\in\mathfrak{K}$, $c\in\mathrm{Ob}\,\mathfrak{L}$ is presentable as $\alpha=\beta\delta$ for a uniquely determined morphism $\beta:b\to c$. If for any object of the category $\mathfrak{K}$ there exists a $\mathfrak{L}$-reflector, then $\mathfrak{L}$ is said to be a reflective subcategory. It should, finally, be remarked that the subcategory $\mathfrak{L}$ is reflective iff the functor of an identical embedding $\mathfrak{L}\to\mathfrak{K}$ has a left conjugate functor $R:\mathfrak{K}\to\mathfrak{L}$. The functor R is called an $\mathfrak{L}$-reflector of the category $\mathfrak{K}$.

(2) *The functor* mix$:\mathfrak{PV}^{(B)}\to\mathfrak{TPV}^{(B)}$ *coincides with the superposition* $\mathfrak{F}^{\uparrow}\circ\mathfrak{F}^{\downarrow}$ *and is a* $\mathfrak{TPV}^{(B)}$*-reflector of the category* $\mathfrak{PV}^{(B)}$*. In particular,* $\mathfrak{TPV}^{(B)}$ *is a reflective subcategory in* $\mathfrak{PV}^{(B)}$.

⊲ The equality mix$:=\mathfrak{F}^{\uparrow}\circ\mathfrak{F}^{\downarrow}$ results from **3.3.3** (1) and **3.3.12** (2). Let us choose nonempty sets $A,C \in \mathcal{P}(\mathbf{V}^{(B)})$, and let C be cyclic. Then any extensional mapping $g: A \to C$ allows a unique extensional extension $\bar{g} = g \uparrow \downarrow : \mathrm{mix}A \to C$ (see **3.2.12**, **3.3.11** and **3.3.12** (2)). Therefore, the mapping of restriction $\theta_{A,C}: h \to h \uparrow A$ is a bijection of $\mathfrak{CPV}^{(B)}(\mathrm{mix}A,C)$ on $\mathfrak{PV}^{(B)}(A,C)$. Let us denote the family of the mappings $\theta_{A,C}$ by θ. Then θ is a conjugation from mix to the functor of the identical embedding of $\mathfrak{CPV}^{(B)}$ to $\mathfrak{PV}^{(B)}$. Indeed, if $A',C' \in \mathcal{P}(\mathbf{V}^{(B)})$ and C' is cyclic, then for any extensional mappings $f: \mathrm{mix}A \to C$, $g: A' \to A$, $h: C \to C'$ we get $(f \circ \mathrm{mix}(g)) | A' = (f|A) \circ g$. And obviously valid is the equality

$$h \circ (f \circ \mathrm{mix}(g))) | A' = h \circ (f|A) \circ g,$$

or, which is equivalent,

$$\theta_{A',C'}(h \circ f \circ \mathrm{mix}(g)) = h \circ \theta_A(f) \circ g. \ \triangleright$$

(3) *The superposition of a functor of the canonical embedding and that of descent is naturally isomorphic to the functor* B_0 *or, in symbols,* $\mathfrak{F}^{\downarrow} \circ \mathfrak{F}^{\wedge} \sim B_0$.

$\triangleleft$ For any set X the mapping

$$\theta_X: (b_X)_{x \in X} \to \underset{x \in X}{\mathrm{mix}}(b_X x^{\wedge}) \quad ((b_x)_{x \in X} \in B_0(X))$$

is a bijection of $B_0(X)$ on $X^{\wedge}\downarrow$. The mapping $\theta: X \to \theta_X(X \in \mathrm{Ob}\, \mathcal{V}_*)$ is an isomorphism of the functors B_0 and $\mathfrak{F}^{\downarrow} \circ \mathfrak{F}^{\wedge}$. To this end it suffices to remark that for $u \in B_0(X)$ and $v \in B_0(Y)$, $a = \theta_X(u)$ and $b := \theta_Y(v)$ we get $(a,b) \in \Phi^{\wedge}\downarrow$ iff $(x,y) \in \Phi$ whenever $u(x) \wedge v(x) \neq 0$. $\triangleright$

3.5.4. Theorem. *Let* (X,d_X) *be a* B*-set and* $X' := X^{\sim}\downarrow$. *Then the following statements are valid:*

(1) *there is an injection* $\iota_X: X \to X'$ *such that*

$$d_X(x_1,x_2) = [\iota_X x_1 \neq \iota_X x_2] \quad (x_1,x_2 \in X);$$

(2) *for any* $x' \in X'$ *there is a partition of unity* (b_ξ) *and a family* $(x_\xi) \subset X$ *such that* $x' = \mathrm{mix}(b_\xi \iota(x_\xi))$;

(3) *if* Φ *is a contraction correspondence from* X *to a* B*-set* Y, $Y' := Y^{\sim}\downarrow$ *and*

$\Phi' := \Phi^{\sim}\downarrow$, *then* $\Phi^{\sim}$ *is a unique fully extensional correspondence from* X' *to* Y', *for which* $\operatorname{dom}\Phi' = \operatorname{mix}\iota_X(\operatorname{dom}\Phi)$,

$$\Phi'(\iota_X x) = \operatorname{mix}\iota_Y(\Phi(x)) \quad (x \in \operatorname{dom}\Phi).$$

$\lhd$ (1) According to the definition of $X^{\sim}$ and π_X (see **3.4.12** (1-3)), for any $x \in X$ we have $[\pi_X x^{\wedge} \in X^{\sim}] = \mathbf{1}$, and therefore, there is a unique element $x' \in X'$ such that $[x' = \pi_X x^{\wedge}] = \mathbf{1}$. Let us put $\iota_X x := x'$. We thus defined the mapping $\iota := \iota_X : X \to X'$, in which case $[\iota x_1 = \pi_X x^{\wedge}] = \mathbf{1}\,(x \in X)$. Using the last relation and equality **3.4.12** (4), we deduce, for arbitrary $x_1, x_2 \in X$:

$$[\iota x_1 \neq \iota x_2] = [\pi_X x_1^{\wedge} = \pi_X x_2^{\wedge}]^* = [x_1 \sim x_2]^* = d_X(x_1, x_2),$$

which, in particular, implies that ι is injective.

(2) Let us first of all observe that the formula $[(\operatorname{im}\iota)\uparrow = \pi_X(X^{\wedge})] = \mathbf{1}$ holds. Indeed, for $t \in \mathbf{V}^{(B)}$ by the definition of the injection ι we have

$$[t \in (\operatorname{im}\iota)\uparrow] = \bigvee_{x \in X} [t = \iota x] = \bigvee_{x \in X} [t = \pi_X x^{\wedge}] = [t \in \pi_X(X^{\wedge})].$$

Now, making use of the cancellation rule **3.3.3** (1), we get

$$X' = \pi_X(X^{\wedge})\downarrow = (\operatorname{im}\iota)\uparrow\downarrow = \iota(X)\uparrow\downarrow = \operatorname{mix}\iota(X).$$

(3) Since $\Phi^{\sim}$ is a correspondence from $X^{\sim}$ to $Y^{\sim}$ inside $\mathbf{V}^{(B)}$, therefore, Φ' is a fully extensional correspondence from X' to Y' (see **3.4.7** (2)). Making use of the property of descent **3.2.13** (1), we write for arbitrary $x \in X$ and $y \in Y$:

$$\iota_Y y \in \Phi'(\iota_X x) \leftrightarrow [\iota_Y y \in \Phi^{\sim}(\iota_X x)] = \mathbf{1}.$$

In the right-hand part of this equivalence $\iota_X x$ can, by the construction of ι_X, be replaced with $\pi_X x^{\wedge}$. Then, by theorem **3.4.13**,

$$[\iota_Y y \in \Phi^{\sim}(\pi_X x^{\wedge})] = [\iota_Y y \in \pi_Y(\Phi(x)^{\wedge})].$$

All these facts imply that $\iota_Y y \in \Phi'(\iota_X x)$ iff $\iota_Y y \in \pi_Y(\Phi(x)^{\wedge})\downarrow$, which results in the required relation. Indeed, the facts proved in (1) and (2) allow one to conclude that $A^{\sim}\downarrow = \pi_Y(A^{\wedge})\downarrow = \operatorname{mix}\iota_Y(A)$ for any $A \subset Y$. Taking into account rule **3.2.13** (1) as well, we deduce:

$$\Phi'(\iota_X x) = \Phi^{\sim}\downarrow(\iota_X x) = \Phi^{\sim}(\pi_X(x^{\wedge}))\downarrow = \pi_Y(\Phi(x)^{\wedge}) = \operatorname{mix}\iota_Y(\Phi(x)),$$

where $x \in \operatorname{dom}\Phi$. Let us put $X_1 := \operatorname{im}\iota_X, Y_1 := \operatorname{im}\iota_Y$ and $\Phi_1 := \iota_Y^{-1} \circ \Phi' \circ \iota_X$. Then Φ_1 is an extensional correspondence from X_1 to Y_1 and the following equalities hold:

$$X' = \operatorname{mix}X_1, \quad Y' = \operatorname{mix}Y_1, \quad \Phi'(x) = \operatorname{mix}\Phi_1(x) \quad (x \in \operatorname{dom}\Phi_1).$$

Hence, $\Phi' = \operatorname{mix}\Phi_1$, and thus Φ' is unique. ▷

3.5.5. Let us describe modified descents and ascents of correspondences.

(1) Let X be a nonempty B-set, Y is an arbitrary element of $\mathbf{V}^{(B)}$ such that $[Y \neq \varnothing] = \mathbf{1}$. Let us consider an $\Phi \in \mathbf{V}^{(B)}$, for which $\mathbf{V}^{(B)} \models {}'\Phi = (F, X^{\sim}, Y)$ is a correspondence from $X^{\sim}$ to Y'. By theorem **3.2.13**, $\Phi\downarrow$ is a correspondence from $X' := X^{\sim}\downarrow$ to $Y\downarrow$. By definition, we put $\Phi\Downarrow := \Phi\downarrow \circ \iota_X$. The correspondence $\Phi\downarrow$ is termed the *modified descent of the correspondence* Φ. By virtue of theorems **3.2.13** and **3.5.4**, $\Phi\downarrow$ is the only fully contractive correspondence from X to $Y\downarrow$, for which

$$[y \in \Phi\Downarrow(x) \leftrightarrow [y \in \Phi(\iota_X x)]] = \mathbf{1} \quad (x \in X).$$

It should be also remarked that $\Phi\Downarrow = (F\downarrow^{-}, X, Y\downarrow)$, where

$$F\downarrow^{-} := \{(x, y) \in X \times Y\downarrow : (\iota_X x, y)^B \in F\}.$$

(2) Let us now assume that $\Psi := (F, X, Y\downarrow)$ is a contraction correspondence. The operation of ascent of **3.3** cannot be directly applied to Ψ. The correspondence $\Psi \circ \iota_X$ is, however, seen to be extensional and the ascent can be applied to it. Define $\Psi\Uparrow := (\Psi \circ \iota_X)\uparrow$ and call $\Psi\Uparrow$ the *modified ascent of the correspondence* Ψ. By virtue of theorems **3.3.10** and **3.5.4**, $\Psi\Uparrow$ is a unique correspondence from $X^{\sim}$ to Y inside $\mathbf{V}^{(B)}$ such that

$$[\operatorname{dom}\Psi\Uparrow = (\operatorname{dom}\Psi)^{\sim}] = \mathbf{1}, \ [\Psi\Uparrow(\iota_X x) = \Psi(x)\uparrow] = \mathbf{1} \quad (x \in \operatorname{dom}\Psi).$$

It should be again remarked that $\Psi\Uparrow = (F_{-}\uparrow, X^{\sim}, Y)$, where

$$F_{-} := \{(\iota_X x, y)^B : (x, y) \in F\}.$$

(3) Let us now assume that X is a discrete B-set. Then $\Phi\Downarrow$ is a correspondence from

X to $Y\downarrow$ and it is uniquely determined by the relation

$$y \in \Phi\Downarrow(x) \leftrightarrow [y \in \Phi(x^{\wedge})] = \mathbf{1} \quad (x \in X).$$

On the other hand, in this case any correspondence Ψ from X to $Y\downarrow$ is contractive, so that there is a unique correspondence $\Psi\Uparrow$ from $X^{\wedge}$ to Y, for which

$$[\Psi\Uparrow(x^{\wedge}) = \Psi(x)\uparrow] = \mathbf{1} \quad (x \in X).$$

3.5.6. Theorem. *Let $[X^{\sim},Y]$ be a set of elements, for which $[\Phi$ is a correspondence from $X^{\sim}$ to $Y] = \mathbf{1}$, while $[X,Y\downarrow]$ be a set of all fully contractive correspondences from X to $Y\downarrow$. The modified descent and ascent are mutually inverse mappings implementing a bijection between $[X^{\sim},Y]$ and $[X,Y\downarrow]$.*

⊲ Let us, for simplicity, denote $\iota := \iota_X$. By virtue of **3.5.4** (2) and **3.3.3** (1), $X^{\sim} = (\mathrm{im}\,\iota)\uparrow$. Hence, in line with **3.3.10** (3), we get $I_{X^{\sim}} = (I_{\mathrm{im}\,\iota})\uparrow$. Then, applying the rules of cancelling arrows for correspondences, we conclude that inside $\mathbf{V}^{(B)}$ the following equalities hold:

$$\Phi\downarrow\uparrow = \left((\Phi\downarrow\circ\iota)\circ\iota^{-1}\right)\uparrow = (\Phi\downarrow\circ I_{\mathrm{im}\,\iota})\uparrow = \Phi\downarrow\uparrow\circ(I_{\mathrm{im}\,\iota})\uparrow = \Phi\circ I_{X^{\sim}} = \Phi.$$

On the other hand, for fully contractive Ψ we have

$$\Psi\Uparrow\Downarrow(x) = (\Psi\circ\iota^{-1})\uparrow\downarrow(\iota x) = \mathrm{mix}\Psi\circ\iota^{-1}(\iota x)$$
$$= \mathrm{mix}\Psi(x) = \Psi(x)\;(x \in \mathrm{mix\,dom}\Psi = \mathrm{dom}\Psi).\ ⊳$$

3.5.7. Theorem. *The functor $\mathfrak{F}^{\downarrow}$ of descent is right-conjugate to the immersion functor $\mathfrak{F}^{\sim}$. In this case the modified descent $\Downarrow$ is a conjugation, while the modified ascent $\Uparrow$ is a co-conjugation.*

⊲ Let us consider functors $\mathfrak{H}^{\sim}$ and $\mathfrak{H}^{\downarrow}$ from the category $\mathrm{BSet}\times\mathcal{V}^{(B)}$ to the category $\mathcal{V}$ determined by the relations

$$\mathfrak{H}^{\sim}(X,Y) := \mathcal{V}^{(B)}(X^{\sim},Y),\quad \mathfrak{H}^{\downarrow}(X,Y) := \mathrm{BSet}_0(X,Y\downarrow);$$
$$\mathfrak{H}^{\sim}(\alpha,\beta) := \Phi' \leftrightarrow V^{(B)} \models \Phi' = \beta\circ\Phi\circ\alpha^{\sim};$$
$$\mathfrak{H}^{\downarrow}(\alpha,\beta) := \beta\downarrow\circ\Psi\circ\alpha,$$

where $X \in \operatorname{Ob} \mathrm{BSet}$, $Y \in \operatorname{Ob} \mathfrak{V}^{(B)}$, $\alpha \in \mathrm{BSet}(X_1, X)$, $\alpha \in \mathfrak{V}^{(B)}(Y, Y_1)$, $\Phi \in \mathfrak{H}^{\sim}(X, Y)$, $\Psi \in \mathfrak{H}^{\downarrow}(X, Y)$.

The statement to be proved is that the modified descent $\Downarrow$ is an isomorphism of the functors $\mathfrak{H}^{\sim}$ and $\mathfrak{H}^{\downarrow}$. By virtue of theorem **3.5.6**, we only have to establish that $\Downarrow$ is a functor morphism of the functor $\mathfrak{H}^{\sim}$ to the functor $\mathfrak{H}^{\downarrow}$ or, in other words, that the following diagram is commutative

$$\begin{array}{ccc}
\mathfrak{H}^{\sim}(X,Y) & \xrightarrow{\ \Downarrow\ } & \mathfrak{H}^{\downarrow}(X,Y) \\
{\scriptstyle \mathfrak{H}^{\sim}(\alpha,\beta)}\big\downarrow & & \big\downarrow{\scriptstyle \mathfrak{H}^{\downarrow}(\alpha,\beta)} \\
\mathfrak{H}^{\sim}(X_1,Y_1) & \xrightarrow[\ \Downarrow\ ]{} & \mathfrak{H}^{\downarrow}(X_1,Y_1)
\end{array}$$

for any X, X_1, Y, Y_1, α and β given above. The last result is equivalent to the fact that the equality $(\mathfrak{H}(\alpha,\beta)\Phi)\Downarrow = \mathfrak{H}^{\downarrow}(\alpha,\beta)(\Phi\Downarrow)$ is valid for every $\Phi \in \mathfrak{H}^{\sim}(X,Y)$ or, making use of the definition of $\mathfrak{H}^{\sim}$ and $\mathfrak{H}^{\downarrow}$, that the following conditions are compatible:

$$\Psi \in \mathfrak{H}^{\downarrow}(X,Y), \quad [\Psi = \beta \circ \Phi \circ \alpha^{\sim}] = \mathbf{1},$$
$$(\beta\downarrow) \circ (\Phi\Downarrow) \circ \alpha = \Psi\Downarrow.$$

The last equalities are fulfilled only if

$$[\beta \circ \Phi \circ \alpha^{\sim} = (\beta\downarrow \circ (\Phi\Downarrow) \circ \alpha)\Uparrow] = \mathbf{1}.$$

As is, however, seen from the rules of cancelling arrows and the definitions of modified descents and ascents, inside $\mathbf{V}^{(B)}$ the following equalities hold:

$$(\beta\downarrow \circ (\Phi\Downarrow) \circ \alpha)\Uparrow = (\beta\downarrow \circ (\Phi\downarrow) \circ \iota \circ \alpha \circ \iota^{-1})\uparrow$$
$$= \beta\downarrow\uparrow \circ (\Phi\downarrow\uparrow) \circ (\iota \circ \alpha \circ \iota^{-1})\uparrow = \beta \circ \Phi \circ (\iota \circ \alpha \circ \iota^{-1})\uparrow.$$

Now to prove the theorem, we only have to remark that $[(\iota \circ \alpha \circ \iota^{-1})\uparrow = \alpha^{\sim}] = \mathbf{1}$. ▷

3.5.8. Let us consider some important corollaries to theorem **3.5.4** (preserving the assumptions and notation assumed in it).

(1) *If (X, d_X) is an extended B-set, then ι_X is a bijection between X and X'.*

◁ We only have to remark that when $x = \operatorname{mix}(b_\xi x_\xi)$, for a partition of unity (b_ξ) and a family $(x_\xi) \subset X$ we have $\iota_X x = \operatorname{mix}(b_\xi \iota_X x_\xi)$. ▷

(2) *For any B-set (X,d_X) there is a triplet (X', d'_X, ι_X) termed a B-extension of (X,d_X) and obeying the following conditions:*

(a) (X', d'_X) *is an extended B-set, while ι_X is an isometric mapping of X to X';*

(b) $X' = \operatorname{mix}(\operatorname{im} \iota_X)$;

(c) *for any contractive correspondence Φ from X to an extended B-set Y there is a unique fully contractive correspondence Φ' from X' to Y such that* $\operatorname{dom}\Phi' = \operatorname{mix} \iota(\operatorname{dom}\Phi)$ *and*

$$\operatorname{mix}\Phi(x) = \Phi'(\iota_X x) \quad (x \in \operatorname{dom}\Phi);$$

(d) *if a triplet (X'', d''_X, ι''_X) obeys* (a) - (c), *then there is a B-isomorphism ι between X' and X'', for which $\iota \circ \iota_X = \iota'_X$.*

◁ For the proof one is to use an extended B-set instead of Y in **3.5.4** (3), and make use of corollary (1). ▷

(3) *If $X \in \operatorname{Ob} \mathfrak{V}^{(B)}$, then there is a $j_X \in \mathbf{V}^{(B)}$ such that $[j_X$ is an isomorphism (in the category $\mathfrak{V}^{(B)}$) of X on $X\!\downarrow^{\sim}] = 1$.*

◁ Indeed, if $Y := X\!\downarrow$, then, setting $j_X := \iota_Y\!\uparrow$, we get that j_X is an isomorphism between $Y\!\uparrow = X$ and $Y^{\sim} = X\!\downarrow^{\sim}$, since ι_Y is an isomorphism between Y and $Y^{\sim}\!\downarrow$. ▷

(4) *If X and Y are extended B-sets, and Φ is a correspondence from $X^{\sim}$ to $Y^{\sim}$ inside $\mathbf{V}^{(B)}$, then there is a unique fully contractive correspondence Ψ from X to Y such that $\Psi^{\sim} = \Phi$.*

◁ Indeed, $\Phi' := \Phi\!\downarrow$ is a fully extensional correspondence from $X' := X^{\sim}\!\downarrow$ to $Y' := Y^{\sim}\!\downarrow$. Hence, $\Psi := \iota_Y^{-1} \circ \Phi' \circ \iota_X$ is a fully contractive correspondence from X to Y. If $\Psi' := \Psi^{\sim}\!\downarrow$, then by **3.5.4** (3), we get $\iota_Y^{-1} \circ \Psi \circ \iota_X = \iota_Y^{-1} \circ \Psi' \circ \iota_X$. If we take account of (1), then $\Psi = \Psi'$ and, hence, $\Phi = \Phi'\!\uparrow = \Psi'\!\uparrow = \Psi\!\uparrow$. ▷

(5) *If X and Y are extended B-sets, then the mapping $\Phi \to \Phi^{\sim}$ sets a bijection between the sets of morphisms* $\mathrm{CBSet}_*(X,Y)$ *and* $\mathfrak{V}_*^{(B)}(X^{\sim}, Y^{\sim})$.

3.5.9. *Let X and Y be arbitrary B-sets and F be a fully contractive correspondence from X to Y. Then for any set $A \subset \operatorname{dom}\Phi$ we have*

$$\mathbf{V}^{(B)} \models \pi_\Phi(A)^{\sim} = \pi_{\Phi^\sim}(A^{\sim}).$$

$\lhd$ It should be observed that the relations $(\forall a \in A^{\wedge})(y \in \Phi^{\sim}(\pi_X a)$ and $y \in \pi_{\Phi^\sim}(A^{\sim})$ are equivalent, since $A^{\sim} = \pi_X(A^{\wedge})$. Making use of theorem **3.4.13** and of the fact that Φ is fully contractive, we can write out the following equivalences ($y \in Y$):

$$\begin{aligned} & y \in \pi_{\Phi^\sim}(A^{\sim}) \leftrightarrow \wedge\{[y \in \Phi^{\sim}(\pi_X a^{\wedge})]: a \in A\} = 1 \\ & \leftrightarrow (\forall a \in A)[y \in \pi_Y(\Phi(a)^{\wedge})] = 1 \leftrightarrow (\forall a \in A)(y \in \Phi(a)^{\sim}\downarrow) \\ & \leftrightarrow (\forall a \in A)\, y \in Y \cap \operatorname{mix} \iota_Y(\Phi(a)) = \iota_Y(\pi_\Phi(A)). \end{aligned}$$

Hence,

$$\pi_{\Phi^\sim}(A^{\sim}) = \iota_Y(\pi_\Phi(A))\uparrow = \pi_\Phi(A)^{\sim}. \rhd$$

3.5.10. *The functors $\mathfrak{F}^{\sim}$ and $\mathfrak{F}^{\downarrow}$ establish the equivalence of the categories* CBSet$_*$ *and* $\mathcal{V}_*^{(B)}$. *In particular, $\mathfrak{F}^{\sim}$ and $\mathfrak{F}^{\downarrow}$ are mutually conjugate complete univalent functors preserving inductive and projective limits (for the given categories).*

$\lhd$ It suffices to substantiate the validity of the following two statements:

(1) the functor $\mathfrak{F}^{\downarrow} \circ \mathfrak{F}^{\sim}$ is naturally isomorphic to the identical functor on the CBSet$_*$, while the isomorphism is implemented by the mappings $\iota_X: X \to X'$ ($X \in$ CBSet$_*$);

(2) the functor $\mathfrak{F}^{\sim} \circ \mathfrak{F}^{\downarrow}$ is naturally isomorphic to the identical functor on $\mathcal{V}_*^{(B)}$, the isomorphism is accomplished by the mappings $j_X \in \mathcal{V}^{(B)}(X, X\downarrow^{\sim})$ ($X \in \mathcal{V}_*^{(B)}$). To prove (1) we should use corollary **3.5.8** (1) and remark that, by virtue of **3.5.4** (3), for $X, Y \in \operatorname{Ob}$ CBSet$_*$ and $\Phi \in$ CBSet$_*(X,Y)$, the following diagram is commutative

$$\begin{array}{ccc} X & \xrightarrow{\ \iota_X\ } & X^{\sim}\downarrow \\ {\scriptstyle\Phi}\big\downarrow & & \big\downarrow{\scriptstyle\Phi^{\sim}\downarrow} \\ Y & \xrightarrow[\ \iota_Y\ ]{} & Y^{\sim}\downarrow \end{array}$$

Then, it follows from **3.5.8** (3,4) that for any $X, Y \in \operatorname{Ob} \mathcal{V}_*^{(B)}$ and $\Phi \in \mathcal{V}_*^{(B)}(X,Y)$ the diagram

$$\begin{array}{ccc} X & \xrightarrow{\ j_X\ } & X^{\sim}\downarrow \\ \Phi \downarrow & & \downarrow \Phi^{\sim}\downarrow \\ Y & \xrightarrow[\ j_Y\ ]{} & Y^{\sim}\downarrow \end{array}$$

is commutative. Hence, from we get (2). ▷

3.5.11. *For any* $X \in \mathrm{Ob}\,\mathrm{CBSet}_*$ *and* $Y \in \mathrm{Ob}\, \mathfrak{V}_*^{(B)}$ *the following relations are valid:*

$$(j_Y)\downarrow\ = \iota_{Y\downarrow},\quad \mathbf{V}^{(B)} \models (\iota_X)^{\sim} = j_{X^{\sim}}.$$

◁ The first equality results directly from the definitions $(j_Y)\downarrow\ = (\iota_{Y\downarrow})\uparrow\downarrow\ = \iota_{Y\downarrow}$. In order to prove the second equality, assign

$$b := [(\iota_X)^{\sim} = j_{X^{\sim}}],\ b_x := [\iota_{X^{\sim}}\pi_X x^{\wedge} = j_{X^{\sim}}\pi_X x^{\wedge}] \quad (x \in X).$$

It should be observed that $b = \wedge\{b_x : x \in X\}$, and, hence, we should prove that $b_x = \mathbf{1}$ for every $x \in X$. If, however, $x \in X$ then, by virtue of **3.4.13** and according to the definition of j_X we get $b_x = [\pi_{X^{\sim}\downarrow}(\iota_X x)^{\wedge} = (\iota_{X^{\sim}\downarrow})\uparrow \circ \pi_X(x^{\wedge})]$. Finally, applying the equalities

$$[\pi_X x^{\wedge} = \iota_X x] = [\pi_{X^{\sim}\downarrow} y^{\wedge} = \iota_{X^{\sim}\downarrow} y] = \mathbf{1}\,(x \in X, y \in X^{\sim}\downarrow),$$

and letting $y = \iota_X x$, we obtain, according to **3.4.13**,

$$b_x = [\pi_{X^{\sim}\downarrow}(\iota_X x)^{\wedge} = \iota_{X^{\sim}\downarrow}(\iota_X x)] = \mathbf{1},$$

which completes the proof. ▷

CHAPTER 4

BOOLEAN-VALUED ANALYSIS OF ALGEBRAIC SYSTEMS

In any Boolean-valued universe there are, in particular, sets of various structures: groups, rings, algebras, etc.. Applying the descent functor to algebraic systems in the Boolean-valued model singles out structures with new properties and results in discovering new facts about their structure and interrelations. Such a technique, called direct Boolean-valued interpretation, allows one to produce new theorems or, to be more exact, to extend the semantical volume of the theorems available by way of straightforward translating. The information arising in such a way, however, not always proves to be really new, expedient or interesting, so that the unsophisticated Boolean-valued interpretation sometimes becomes an aimless game.

In this respect it would be interesting to ask a question: what practically important mathematical structures can be obtained from a Boolean-valued interpretation of well-studied structures? What transfer principles are valid in this case? It is obvious that here we are speaking about a Boolean-valued implementation of specially structured sets, which is impossible for arbitrary objects. In the preceding chapter we have proved that an abstract B-set can be immersed in a Boolean-valued universe in such a way that the Boolean distance between objects becomes the Boolean truth-value of their noncoincidence. The corresponding element of the universe $\mathbf{V}^{(B)}$ is, by definition, a Boolean-valued implementation of the B-set under consideration. If the B-set is in some way *apriori* structured, one can try to furnish its Boolean-valued interpretation with an analogous structure in order to use the technique of descents and ascents to study the initial structure. Therefore, the questions formulated above can be treated as a problem of searching qualified Boolean-valued realizations of structured B-sets.

The present section will be devoted to an analysis of the aforementioned problem for general algebraic objects. The central notion under discussion will be that of an algebraic B-system, which is a nonempty B-set with contraction operations, having a certain quantity of B-predicates, i.e., B-valued contraction mappings, on it. The Boolean-valued realization of an algebraic B-system appears to be a conventional two-valued algebraic system of the same type. This implies that an appropriate extension of any algebraic B-system coincides with the descent of a two-valued algebraic system inside $\mathbf{V}^{(B)}$. On the other hand, a two-valued algebraic system can be turned into an algebraic B-system provided a complete Boolean algebra of congruences is singled out in it. In this case it is important to find out which formulas remain true under transition from a B-system to a two-valued one, and vice versa.

In other words, here some variations of the transfer principle or the 'principle of preserving relations' arise. General facts are illustrated by concrete examples of algebraic systems in which complete Boolean algebras of congruences are related with the relations of order and disjointness.

4.1. Algebraic B-Systems

Let us introduce a class of algebraic systems suitable for Boolean-valued interpretations of first-order languages. Such systems arise as B-sets furnished with contraction operations and predicates.

4.1.1. It should be recalled that a *signature* is a triplet $\sigma := (F, P, \mathfrak{a})$, where F and P are some (possibly, empty) sets, while $\mathfrak{a}$ is a mapping from $F \cup P$ to ω. An *n-ary operation* or an *n-ary predicate* on a B-set A is contraction mappings $f : A^n \to A$ or $p : A^n \to B$, respectively. By definition, the mappings f and p are contractions provided

$$d(f(a_0, \dots, a_{n-1}), f(a_0', \dots, a_{n-1}')) \le \bigvee_{k=0}^{n-1} d(a_k, a_k'),$$

$$d_s(p(a_0, \dots, a_{n-1}), p(a_0', \dots, a_{n-1}')) \le \bigvee_{k=0}^{n-1} d(a_k, a_k')$$

for all $a_0, a_0', \dots, a_{n-1}, a_{n-1}' \in A$, where d is the B-metric of the sets A, and d_s is the symmetric difference on B, i.e., $d_s(b_1, b_2) := b_1 \Delta b_2$ (see **1.1.4**).

An *algebraic B-system of signature* σ is a pair (A, v), where A is a nonempty B-set, while v is a mapping such that $\operatorname{dom}(v) = F \cup P$, and $v(f)$ is an $\mathfrak{a}(n)$-ary operation on A for all $f \in F$, while $v(p)$ is an $\mathfrak{a}(p)$-ary predicate on A for every $p \in P$. A contraction mapping from A^n to B is also termed a *B-predicate*, or a *B-valued predicate*. The mapping v is sometimes referred to as *interpretation* and presented, for convenience, as f^v and p^v instead of $v(f)$ and $v(p)$. The signature of an algebraic B-system $\mathfrak{A} := (A, v)$ will be often denoted by $\sigma(\mathfrak{A})$, while the carrier set A by $|\mathfrak{A}|$. As long as $A^0 = \{\varnothing\}$, then nullary operations and predicates on A are mappings from $\{\varnothing\}$ to the set A and to the algebra B, respectively. Let us identify the mapping $g : \{\varnothing\} \to A \cup B$ with an element $g(\varnothing)$. As a result, nullary operations on A are chosen A elements, while a set of all nullary predicates on A is a Boolean algebra B. If $F := \{f_1, \dots, f_n\}$ and $P := \{p_1, \dots, p_m\}$, then the algebraic B-system of signature σ is often represented as $(A, v(f_1), \dots, v(f_n), v(p_1), \dots, v(p_m))$, and even $(A, f_1, \dots, f_n, p_1, \dots, p_m)$, while instead of $\sigma(F, P, \mathfrak{a})$ use is made of the notation $\sigma = (f_1, \dots, f_n, p_1, \dots, p_m)$.

4.1.2. If B is the two-element Boolean algebra $\{0,1\}$, then instead of an algebraic B-system we speak about a *two-valued system* or simply about an *algebraic system.* In this case an arbitrary set can be treated as a B-set, while an n-ary operation and predicate on a B-set A are specialized, respectively, as an arbitrary mapping from A^n to A and a characteristic function $p: A^n \to \{0,1\}$ identified with the set $\{x \in A^n : p(x) = 1\}$. Therefore, an algebraic system of signature σ is a pair (A,v), where A is a nonempty set, while v is a function from $\mathrm{dom}(v) = F \cup P$ to $\mathbf{V}$ such that

$$v(f): A^{\mathfrak{a}(f)} \to A, \quad v(p) \subset A^{\mathfrak{a}(p)} \quad (f \in F, p \in P).$$

On the other hand, if (A,v) is an algebraic system of signature σ and $A \subset \mathbf{V}^{(B)}$, then, viewing A as a B-set (with the B-metric $d(a,a') := [a = a']^* = [a \neq a']$ $(a,a' \in A)$), for every $p \in P$ we can determine an $n := \mathfrak{a}(p)$-ary B-predicate $v'(p)$ on A by the following formula (see **3.4.5**):

$$v'(p) := (a_0,\dots,a_{n-1}) \to \mathrm{dist}((a_0,\dots,a_{n-1}), v(p)).$$

It is obvious that the mapping $v'(p): A^n \to B$ is a contraction. Let, besides, $v(f)$ be a contraction mapping for every $f \in F$. Let us put $v'(f) := v(f), f \in F$. Then (A,v') is an algebraic B-system.

4.1.3. An algebraic B-system $\mathfrak{A} := (A, v)$ is called *extended (decomposable)* provided A is an extended (decomposable) B-set (**3.4.3**). A B-valued predicate p on the set A is called *assertive* if there exists such an element $x \in A$ that $p(x) = 1$.

(1) *A contraction mapping p from an extended set A to B is an assertive B-valued predicate iff* $1 = \vee\{p(x): x \in A\}$.

◁ Indeed, if the above condition is fulfilled, then there is a family $(x_\xi) \subset A$ and a partition of unity $(b_\xi) \subset B$ such that $p(x_\xi) \geq b_\xi$. If $x := \mathrm{mix}(b_\xi x_\xi)$, then $p(x) = 1$. ▷

Every algebraic B-system $\mathfrak{A}$ can be related to an algebraic system $\overline{\mathfrak{A}}$ with the same carrier set $|\overline{\mathfrak{A}}| := |\mathfrak{A}|$, whose interpretation $\overline{v}$ is determined in the following way. If f is a function symbol, then $\overline{v}(f) := v(f)$; while if p is a predicate symbol and $n = \mathfrak{a}(p)$, then $\overline{v}(p) := \{(x_0,\dots,x_{n-1}) \in A^n : p(x_0,\dots,x_{n-1}) = 1\}$. It is obvious that the predicate $\overline{v}(p)$ can prove to be empty for a certain p. The algebraic system $\overline{\mathfrak{A}}$ is said to be a *purification* or *reduct* of $\mathfrak{A}$, or $\overline{\mathfrak{A}}$ is said to be obtained from $\mathfrak{A}$ by the *purification* or *reduction.*

(2) *If* (A,v) *is an algebraic* B*-system and* $(A,\bar{v})$ *is its purification, then for every assertive predicate* p^v *we have*

$$p^v: x \to \operatorname{dist}(x,\bar{v}(p))^* \quad (x \in A^{a(p)}).$$

◁ By virtue of the theorem on Boolean-valued realization of B-sets (see **3.5.8**), a B-set A admits an extension $A' \subset \mathbf{V}^{(B)}$, while p^v admits a unique extension $v'(p)$ to a B-valued predicate on A'. In this case $v'(p)(x) = \operatorname{dist}(x,\operatorname{mix}(\bar{v}(p)))^* = \operatorname{dist}(x,\bar{v}(p))^* = [x \in p^\gamma\uparrow]$ $(x \in A^{\mathfrak{a}(p)})$. This yields the required result, since the assumption $A \subset A'$ limits no generality. ▷

Proposition (2) makes it possible to identify an algebraic B-system with assertive predicates and some algebraic system, namely, its purification. It would be natural to ask a question: what algebraic systems are obtainable by purification of decomposable (extended) algebraic B-systems? The answer to this question will be formulated in terms of congruences of an algebraic system.

4.1.4. Let us consider an arbitrary algebraic system $\mathfrak{A} := (A,v)$ of signature $\sigma := (F,P,\mathfrak{a})$. An equivalence relation ρ on the set A is termed a *congruence* of the system $\mathfrak{A}$ provided for every $f \in F$ and for any $x_0,\dots,x_{n-1},y_0,\dots,y_{n-1} \in A$, $n = \mathfrak{a}(f)$ the relations $(x_0,y_0) \in \rho,\dots,(x_{n-1},y_{n-1}) \in \rho$ yield $(f^v(x_0,\dots,x_{n-1}), f^v(y_0,\dots,y_{n-1})) \in \rho$. The set of all congruences on the algebraic system $\mathfrak{A}$ is denoted by the symbol $\operatorname{Cong}(\mathfrak{A})$. Let us introduce an order relation in $\operatorname{Cong}(\mathfrak{A})$ by the formula

$$\rho_1 \le \rho_2 \leftrightarrow \rho_1 \subset \rho_2 \quad (\rho_1,\rho_2 \in \operatorname{Cong}(\mathfrak{A})).$$

The identity congruence $I_A := \{(x,x) : x \in A\}$ and the trivial congruence $A \times A$ are obviously the least and the greatest elements of $\operatorname{Cong}(\mathfrak{A})$.

(1) Theorem. *The ordered set* $\operatorname{Cong}(\mathfrak{A})$ *is a complete lattice. The greatest lower bound of the set* $\mathcal{P} \subset \operatorname{Cong}(\mathfrak{A})$ *coincides with the intersection* $\cap\{\rho : \rho \in \mathcal{P}\}$. *The least upper bound of the set* $\mathcal{P} \subset \operatorname{Cong}(\mathfrak{A})$ *is the union of all possible composits* $\rho_1 \circ \dots \circ \rho_n$, *where* $\{\rho_1,\dots,\rho_n\}$ *is an arbitrary finite subset in* $\mathcal{P}$.

The congruence $\rho_1 \vee \rho_2$, for ρ_1 and $\rho_2 \in \operatorname{Cong}(\mathfrak{A})$, as is seen from this theorem, coincides with the union of all possible relations of the form $\rho_1 \circ \rho_2 \circ \rho_1 \circ \dots \circ \rho_1 \circ \rho_2$. Hence, if ρ_1 and ρ_2 commute, i.e., $\rho_1 \circ \rho_2 = \rho_2 \circ \rho_1$, then $\rho_1 \vee \rho_2 = \rho_1 \circ \rho_2$. Conversely, if $\rho_1 \vee \rho_2 = \rho_1 \circ \rho_2$, then the congruences ρ_1 and ρ_2 commute.

A set of congruences Λ on the algebraic system $\mathfrak{A}$ is called *independent (finitely independent)* if for any family (finite family) $(\lambda_\xi)_{\xi\in\Xi}$ in Λ and $(a_\xi)_{\xi\in\Xi}$ in A there is such an element $a\in A$ that $(a,a_\xi)\in\lambda_\xi$ for all $\xi\in\Xi$. The set Λ is termed *complete* provided:

(a) $\inf(\Lambda):=\bigcap(\Lambda)=I_A$ and

(b) for any $p\in P$ and an arbitrary n-tuple $(x_0,\dots,x_{n-1})\in A^n, n=\mathfrak{a}(p)$ the relation $(x_0,\dots,x_{n-1})\notin v(p)$ yields the existence of such a congruence $\lambda\in\Lambda$ that $(y_0,\dots,y_{n-1})\notin v(p)$ as soon as $(x_0,y_0)\in\lambda,\dots,(x_{n-1},y_{n-1})\in\lambda$ (see [170]).

In the definition of a complete set of congruences it is convenient to formulate the condition (b) in terms of mixing. Let us consider a family $(a_\lambda)_{\lambda\in\Lambda}$ in a set A. If for some $a\in A$ and all $\lambda\in\Lambda$ we have $(a,a_\lambda)\in\lambda$, then it is natural to say that a is the *mixing of the family* (a_λ) *relative to* Λ. A set $U\subset A^n$ will be called *stable under* Λ*-mixing*, if for any family $((a_\lambda^0,\dots,a_\lambda^{n-1}))$ in U we have $(a_0,\dots,a_{n-1})\in U$, where a_k is the mixing of (a_λ^k) relative to Λ.

(2) *An independent set of congruences* Λ *of an algebraic system* $\mathfrak{A}$ *is complete iff* $\inf(\Lambda)=I_A$ *and any predicate* $v(p)$, $p\in P$ *is stable under* Λ*-mixing.*

◁ Indeed, let us assume that all the predicates are stable under Λ-mixing. Let $p\in P$, $n=\mathfrak{a}(p)$, $(x_0,\dots,x_{n-1})\notin v(p)$ and, nonetheless, for any $\lambda\in\Lambda$ net there exist such $(y_\lambda^0,\dots,y_\lambda^{n-1})\in v(p)$ that $(x_k,y_\lambda^k)\in\lambda$ $(k=0,\dots,n-1)$. Let y_k be the mixing of a family $(y_{\lambda,k})_{\lambda\in\Lambda}$ relative to Λ. Then $(y_0,\dots,y_{n-1})\in v(p)$. At the same time, $(x_k,y_k)\in\lambda$ for all $\lambda\in\Lambda$. Hence, $x_k=y_k$ $(k=0,\dots,n-1)$, since $\bigcap\Lambda=I_A$, and we thus come to a contradiction.

Assume conversely that Λ is a complete set. Choose a $p\in P$ and a family of n-tuples $(a_{\lambda,0},\dots,a_{\lambda,n-1})$ contained in $v(p)$. Let a_k be the mixing of a family $(a_{\lambda,k})_{\lambda\in\Lambda}$ relative to Λ. If $(a_0,\dots,a_{n-1})\notin v(p)$, then, since Λ is complete, there is a congruence $\lambda\in\Lambda$ for which $(a_{\lambda,0},\dots,a_{\lambda,n-1})\notin v(p)$. This, however, contradicts the choice of $(a_{\lambda,0},\dots,a_{\lambda,n-1})$ and, hence, $v(p)$ is stable under mixing. The necessity is seen to be true without the assumption that Λ is independent. ▷

4.1.5. Let us refer as *Boolean algebra of congruences* to any Boolean algebra $\mathcal{B}\subset\mathrm{Cong}(\mathfrak{A})$ if in $\mathcal{B}$ the least upper bounds of arbitrary sets are inherited from the lattice of $\mathrm{Cong}(\mathfrak{A})$ and the least congruence I_A serves as zero in $\mathcal{B}$. It should be emphasized that the Boolean extension ρ^* of an element $p\in\mathcal{B}$ can also not be an extension ρ in the lattice of $\mathrm{Cong}(\mathfrak{A})$, i.e., the least upper bound of ρ and ρ^* in $\mathrm{Cong}(\mathfrak{A})$ may be less than $A\times A$.

A *base of an algebraic system* $\mathfrak{A}$ will be any complete Boolean algebra of congruences $\mathcal{B}\subset\mathrm{Cong}(\mathfrak{A})$ provided each predicate $v(p)$ $(p\in P)$ is stable under Λ^*-mixing for any partition of unity $\Lambda\subset\mathcal{B}$, where $\Lambda^*:=\{b^*: b\in\Lambda\}$. An algebraic system with base $\mathscr{B}$ will be termed *extended (decomposable)* provided for any (any finite, respectively) partition of unity $\Lambda \subset \mathscr{B}$ the set of congruences Λ^* is independent. Then the following obvious statement is valid.

An algebraic system $\mathfrak{A}$ has a base $\mathscr{B}$ isomorphic to a complete Boolean algebra B iff there exists an injective mapping $h: B \to \mathrm{Cong}(\mathfrak{A})$ obeying the following conditions:

(a) *h preserves the greatest lower bounds of any sets and $h(0) = I_A$;*

(b) *any predicate $v(p)$ $(p \in P)$ is stable under $h(\Lambda^*)$-mixing for any partition of unity $\Lambda \subset B$.*

In this case $\mathfrak{A}$ is extended (decomposable) iff the set $h(\Lambda^*)$ is independent for every (for any finite) partition of unity $\Lambda \subset B$.

4.1.6. The algebraic B-system $\mathfrak{A}$ will be called *full* provided for any $0 \neq b \in B$ there are elements $x, y \in A$, $x \neq y$ such that $d(x,y) \leq b$. It is obvious that a decomposable B-system is full, but the converse statement is not valid in general.

Theorem. *An algebraic system $\mathfrak{A}$ is obtained by purification from a certain full algebraic B-system $\mathfrak{A}'$ iff $\mathfrak{A}$ has a base isomorphic to B. In this case both $\mathfrak{A}$ and $\mathfrak{A}'$ are extended (decomposable) or not simultaneously.*

$\lhd$ Let $\mathfrak{A}'$ be a full algebraic B-system. To every $b \in B$ let us put into correspondence the relation $h(b) := \{(x,y) \in A^2 : d(x,y) \leq b\}$. Since $v(t)$ is a contraction mapping for every $f \in F$; therefore, $h(b)$ is a congruence on A. It is obvious that $h(0) = I_A$ and that h preserves the greatest lower bounds. Since $\mathfrak{A}$ is full, h is injective. Let us assume that the algebraic system $\mathfrak{A}$ is obtained from $\mathfrak{A}'$ by purification. It should be remarked that a set of the type $\{z \in A : p(z) = 1\}$ is stable under any mixing in the B-set A. It is now seen, by virtue of **4.1.5**, that $\mathfrak{A}$ has a base isomorphic to B.

Conversely, let an algebraic system $\mathfrak{A}$ have a base $\mathscr{B}$ and let us assume that there exists a Boolean isomorphism h from B to $\mathscr{B}$. According to the definition, assign

$$d(x,y) := \wedge\{b \in B : (x,y) \in h(b)\} \quad (x, y \in A).$$

If $b_1, b_2 \in B$ are such that $(x,z) \in h(b_1)$ and $(z,y) \in h(b_2)$, then $(x,y) \in h(b_2) \circ h(b_1)$. However, $h(b_2) \circ h(b_1) \subset h(b_1 \vee b_2)$ and, hence, $d(x,y) \leq b_1 \vee b_2$. Taking the infimum over the given b_1 and b_2, and making use of the distributive law **1.1.5** (1), we get $d(x,y) \leq d(x,z) \vee d(z,y)$. It is now evident that d is a Boolean semimetric on A. Since h

preserves its greatest lower bounds; therefore,

$$h(d(x,y)) = \cap\{h(b) : b \in B \wedge (x,y) \in h(b)\}.$$

From here we deduce that $d(x,y) \leq b$ iff $(x,y) \in h(b)$. In particular, $d(x,y) = 0$ implies that $x = y$, while for $0 \neq b \in B$ we can find such $x, y \in A$ that $x \neq y$ and $d(x,y) \leq b$.

Now it remains to show that if Λ is a partition of unity in B, then for a family $(a_b)_{b\in\Lambda} \subset A$ the mixing relative to $h(\Lambda^*)$ coincides with that in the sense of the B-metric d, i.e., with $\text{mix}_{b\in\Lambda}(ba_b)$. This fact, however, follows trivially from the above: $(a, a_b) \in h(b^*) \leftrightarrow d(a,a_b) \leq b^* \leftrightarrow b \wedge d(a,a_b) = 0$. Let us now determine $\mathfrak{A}' := (A', v')$, setting $A' := A$, $v'(f) = v(f)$, $f \in F$ and

$$v'(p) := x \to \text{dist}(x, v(p)) \quad (p \in P, x \in A^{\mathfrak{a}(p)}).$$

If $f \in F$ and $n = \mathfrak{a}(f)$, then for any $b \in B$ and elements $x_0, y_0, \ldots, x_{n-1}, y_{n-1} \in A$ the relations $(x_k, y_k) \in h(b), k < n$ implies that $(f^v(x_0,\ldots,x_{n-1}), f^v(y_0,\ldots,y_{n-1})) \in h(b)$, which gives

$$d(f^v(x_0,\ldots,x_{n-1}), f^v(y_0,\ldots,y_{n-1})) \leq b.$$

Taking the greatest lower bound over b and observing that

$$\wedge\{b : (x_k,y_k) \in h(b), k < n\} = \bigvee_{k=0}^{n-1} d(x_k,y_k),$$

we conclude that the mappings $f^v = v(t)$ are contractions. Let us choose $p \in P$, $\mathfrak{a}(p) = m$ and elements $x := (x_0, \ldots, x_{m-1})$ and $y := (y_0, \ldots, y_{m-1})$ from A^m. Then

$$d(x,y) \wedge \text{dist}(x, v(p)) \leq \text{dist}(y, v(p)),$$

which implies that $v'(p)$ is a contraction. Besides, since $v(p)$ is stable (see **4.1.3** (2)), we get $v(p) = \{x \in A^m : v'(p)(x) = 1\}$. Hence, $\mathfrak{A}$ is a purification of the full algebraic B-system $\mathfrak{A}'$. The fact that the systems $\mathfrak{A}$ and $\mathfrak{A}'$ are extended implies that Λ^*, where Λ is a partition of unity in B, is an independent set and that in (A,d) there are any mixings, the last two statements being, however, equivalent. ▷

4.1.7. Let us consider some concrete examples of algebraic B-systems. An associative ring R is called a *Boolean ring* if its every element is idempotent, i.e., if $(\forall x \in R)\ (x \cdot x = 0)$. A Boolean ring with unity turns into a Boolean algebra if the operations in the former are determined as follows:

$$x \wedge y := x \cdot y,\, x \vee y := x + y - x \cdot y,\, x^* := 1 - x \quad (x,y \in R).$$

Conversely, any Boolean algebra is a Boolean ring with unity under the following definition of ring operations:

$$x + y := x \Delta y := (x \Leftrightarrow y)^*, \quad x \cdot y = x \wedge y \quad (x,y \in R).$$

In both cases zero and unity of the ring coincide with Boolean zero and unity, respectively.

(1) Let B_0 be a Boolean algebra and X be a unital module over the Boolean ring B_0. Let B be the completion of the algebra B_0, while j be an isomorphism of B_0 on a dense subalgebra in B. According to definition, assign

$$d_j(x,y) := \wedge\{j(b) : b^* x = b^* y \wedge b \in B_0\} \quad (x,y \in X).$$

As is easily seen, d_j is a B-semimetric on X. Let us check, for instance, the triangle inequality. If $b^* x = b^* y$ and $c^* z = c^* y$, then for $c := b^* \wedge c^* = (b \vee c)^*$ we have $cx = cz$ and $cy = cz$. Therefore, $d_j(x,y) \le j(b \vee c) = j(b) \vee j(c)$ and, since b and c are arbitrary, we get $d_j(x,y) \le d_j(x,z) \vee d_j(z,y)$. Call the module X *laterally exact* if for any partition of unity (b_ξ) in B_0 from $(b_\xi x = 0)$ we get $x = 0$ for any element $x \in X$. No doubt that for a laterally exact unital B_0-module X the semimetric d_j is a metric. By analogy with the the triangle inequality for d_j, we can check that the module operations are contractions:

$$d_j(x+u, y+v) \le d_j(x,y) \vee d_j(u,v) \quad (x,y,u,v \in X),$$
$$d_j(bx, cy) \le d_j(x,y) \vee d_s(b,c) \quad (x,y \in X; b,c \in B).$$

The last inequality, in particular, implies

$$d_j(bx, by) \le d_j(x,y) \quad (b \in B; x,y \in X).$$

Besides, it is obvious that $d_j(-x,-y) = d_j(x,y)$. Therefore, the set X with the operations $+,-$ and unary operations of multiplication by $b \in B_0$ is an algebraic B-system.

(2) Let R be a commutative ring with unity, and consider a set of all of its idempotent elements $B_0 := \{e \in R : e \cdot e = e\}$. Then B_0 is a Boolean ring with unity and R is a module over B_0. If B and j are the same as in (1), there arises a B-semimetric d_j on R. It is obvious that R is laterally exact over B_0. By virtue of (1) we deduce that the commutative ring R with unity, laterally exact over the subring of its idempotents B_0 is an algebraic B-

system of signature $(+,-,\cdot,1)$.

(3) Let C be a Boolean algebra, and ι be a homomorphism from a Boolean algebra B_0 to C. As long as $\iota(B_0)$ is a subring of the Boolean ring C, we can readily determine on C the structure of the unital module over B_0. If B and j are the same as in (1), then the B-semimetric d_j has the form

$$d_j(x,y):=\wedge\{j(b):\iota(b^*)x=\iota(b^*)y\}.$$

The module C is laterally exact if ι is a full monomorphism. In view of the relation between Boolean and ring operations discussed above, the Boolean algebra C is an algebraic B-system of signature $(\vee,\wedge,*,0,1)$ in the case of the full monomorphism ι. This system will be extended if, for instance, B_0 and C are complete Boolean algebras.

4.1.8. Let us now turn our attention to B-valued interpretation of first-order languages. Let us consider an algebraic B-system $\mathfrak{A}:=(A,v)$ of signature $\sigma:=\sigma(\mathfrak{A}):=(F,P,\mathfrak{a})$.

Let $\varphi(x_0,\dots,x_{n-1})$ be a formula of signature σ with n free variables, and $a_0,\dots,a_{n-1}\in A$. We can readily determine the truth-value $|\varphi|^{\mathfrak{A}}(a_0\dots,a_{n-1})\in B$ of the formula φ in the system $\mathfrak{A}$ for the given values $a_0,\dots,a_{n-1}$ of the variables $x_0,\dots,x_{n-1}$. The definition is, as usual, given by induction on the complexity of φ. Let us set for propositional connectives and quantifiers

$$|\varphi\wedge\psi|^{\mathfrak{A}}(a_0,\dots,a_{n-1}):=|\varphi|^{\mathfrak{A}}(a_0,\dots,a_{n-1})\wedge|\psi|^{\mathfrak{A}}(a_0,\dots,a_{n-1});$$
$$|\varphi\vee\psi|^{\mathfrak{A}}(a_0,\dots,a_{n-1}):=|\varphi|^{\mathfrak{A}}(a_0,\dots,a_{n-1})\vee|\psi|^{\mathfrak{A}}(a_0,\dots,a_{n-1});$$
$$|\neg\varphi|^{\mathfrak{A}}(a_0,\dots,a_{n-1}):=|\varphi|^{\mathfrak{A}}(a_0,\dots,a_{n-1})^*;$$
$$|(\forall x_0)\varphi|^{\mathfrak{A}}(a_1,\dots,a_{n-1}):=\bigwedge_{a_0\in A}|\varphi|^{\mathfrak{A}}(a_0,\dots,a_{n-1});$$
$$|(\exists x_0)\varphi|^{\mathfrak{A}}(a_1,\dots,a_{n-1}):=\bigvee_{a_0\in A}|\varphi|^{\mathfrak{A}}(a_0,\dots,a_{n-1}).$$

Now the case of atomic formulas remains to be considered. Let $p\in P$ be an m-ary predicate symbol, $q\in P$ be a nullary predicate symbol, and $t_1,\dots,t_{m-1}$ be terms of signature σ assuming values $b_0,\dots,b_{m-1}$ for the given values $a_0,\dots,a_{n-1}$ of the variables $x_0,\dots,x_{n-1}$. By definition, assign

$$|\varphi|^{\mathfrak{A}}(a_0,\dots,a_{n-1}):=v(q)\text{ if }\varphi:=q^v;$$
$$|\varphi|^{\mathfrak{A}}(a_0,\dots,a_{n-1}):=d(b_0,b_1)^*\text{ if }\varphi:=(t_0=t_1);$$
$$|\varphi|^{\mathfrak{A}}(a_0,\dots,a_{n-1}):=p^v(b_0,\dots,b_{m-1})\text{ if }\varphi:=p^v(t_0,\dots,t_{m-1}),$$

where d is a B-metric on the set A.

We say that $\varphi(x_0,\dots,x_{n-1})$ *is assertive in the system* $\mathfrak{A}$ *for the given values* $a_0,\dots,a_{n-1} \in A$ *of the variables* $x_0,\dots,x_{n-1}$ (or, to put it short, that $\varphi(a_0,\dots,a_{n-1})$ is *assertive in* $\mathfrak{A}$) and write $\mathfrak{A} \models \varphi(a_0,\dots a_{n-1})$ if $|\varphi|^{\mathfrak{A}}(a_0,\dots a_{n-1}) = \mathbf{1}_B$. When $B := \{\mathbf{0},\mathbf{1}\}$, we get the conventional definition of validity for a formula in an algebraic system (see [46, 170]).

It should be recalled that a closed formula φ of signature σ is termed *identically assertive* if it is fulfilled on any algebraic $\mathbf{2}$-system of signature σ.

4.1.9. Theorem. *Let* $\mathfrak{A}$ *be an arbitrary algebraic* B*-system. Then the following statements are valid:*

(1) *any theorem of first-order predicate calculus with equality is assertive in* $\mathfrak{A}$;

(2) *any identically assertive closed formula of signature* $\sigma(\mathfrak{A})$ *is assertive in* $\mathfrak{A}$.

$\triangleleft$ (1) Here we have to make sure that the axioms of first-order predicate calculus with equality are assertive in $\mathfrak{A}$, while the rules of inference do not violate validity in $\mathfrak{A}$. To this end, we have to check the corresponding calculations of Boolean truth-values (see [10, 46, 114, 121, 248, 249]).

(2) If the closed formula φ is not satisfied in $\mathfrak{A}$, then $b := |\varphi|^{\mathfrak{A}} < \mathbf{1}_B$. Let $h : B \to \mathbf{2} := \{\mathbf{0},\mathbf{1}\}$ be a full homomorphism, in which case $h(b) = \mathbf{0}$. Such an h does exist, since the ideal $[\mathbf{0},b]$ can be extended to a maximal ideal $h^{-1}(\mathbf{0})$. If v is an interpretation of $\mathfrak{A}$, then let us set $v'(f) := f^v$ for functional symbols and $v'(p) := h \circ p^v$ for predicate symbols. Then $\mathfrak{A}' := (|\mathfrak{A}|, v')$ is an algebraic $\mathbf{2}$-system and $|\varphi|^{\mathfrak{A}'} = h(b) = \mathbf{0}$, i.e., φ is not satisfied in $\mathfrak{A}'$ and cannot be identically assertive.

4.1.10. Let us consider algebraic $\mathcal{B}$-systems $\mathfrak{A} := (A, v)$ and $\mathfrak{C} := (C, \mu)$ of the same signature σ. The mapping $h : A \to C$ is called a *homomorphism of the* $\mathcal{B}$*-system* $\mathfrak{A}$ *to the algebraic* $\mathcal{B}$*-system* $\mathfrak{C}$, provided for any $a_0,\dots,a_{n-1} \in A$ we have:

(1) $d_B(h(a_1),h(a_2)) \le d_A(a_1,a_2)$;

(2) $h(f^v) = f^\mu, \quad \mathfrak{a}(f) = 0$;

(3) $h(f^{v}(a_0,\dots,a_{n-1})) = f^{\mu}(h(a_0),\dots,h(a_{n-1})),\quad 0 \neq n := \mathfrak{a}(f)$;

(4) $p^{v}(a_0,\dots,a_{n-1}) \leq p^{\mu}(h(a_0),\dots,h(a_{n-1})),\quad n := \mathfrak{a}(p)$.

The homomorphism h is called *strong* if

(5) for an arbitrary $p \in P, \mathfrak{a}(p) := n \neq 0$, and for any $b_0,\dots,b_{n-1} \in B$ the following inequality holds:

$$p^{\mu}(b_0,\dots,b_{n-1}) \leq \bigvee_{a_0,\dots a_{n-1} \in A} \{p^{v}(a_0,\dots,a_{n-1}) \wedge d_C(b_0, h(a_0)) \wedge \dots \wedge d_C(b_{n-1}, h(a_{n-1}))\}.$$

If the homomorphism h is injective, and conditions (1) and (4) are fulfilled with equality holding, then h is said to be a homomorphism from $\mathfrak{A}$ to $\mathfrak{C}$. Undoubtedly, any surjective isomorphism h and, in particular, the mapping $I_A : \mathfrak{A} \to \mathfrak{A}$ are strong homomorphisms. The composition of (strong) homomorphisms is a (strong) homomorphism. If h is a homomorphism and there is a mapping h^{-1} which is also a homomorphism, then h is an isomorphism.

It should be again observed that in the case of the two-element Boolean algebra $B := \{0, 1\}$ conventional notions of a homomorphism, strong homomorphism, and isomorphism arise (see 46, 170]).

4.1.11. Let us consider a certain set Φ of formulas of the same fixed signature σ. Let us determine the category B – AS(Φ) in the following way. The class Ob B – AS(Φ) consists of all algebraic B-systems of signature σ, on each of which all the formulas of Φ are assertive. The class Mor B – AS(Φ) is a class of all homomorphisms of algebraic B-systems of Ob B – AS(Φ) with the conventional composition as the composition of morphisms. The isomorphism in the category B – AS(Φ) is obvious to be a B-isometric strong homomorphism. By the symbol B – CAS(Φ) let us denote a complete subcategory of the category B – AS(Φ), the objects in which are extended algebraic B-systems.

4.1.12. According to **4.1.5** and **4.1.6**, the structure of an algebraic B-system is restored with the help of a complete Boolean algebra of congruences. On the other hand, one of most common methods of generating complete Boolean algebras is associated with disjointness. Let us consider some simplest interrelations among these notions, starting with some facts to be recalled.

Let us choose sets X and Y. Let Φ be a correspondence from X to Y. As before, $\pi_{\Phi}(A)$ and $\pi_{\Phi}^{-1}(C)$ are the polar of $A \subset X$ and the inverse polar of $C \subset Y$ relative to the correspondence Φ, respectively. A set $K \subset Y$ is called a *Φ-component* (or simply a *component* of Φ, when it is clear what Φ is meant), provided $K = \pi_\Phi(\pi_\Phi^{-1}(K))$ or, which is equivalent, $K = \pi_\Phi(A)$ for some $A \subset X$. The collection of all the Φ-components is denoted by the symbol $\mathfrak{K}_\Phi(Y)$. The least component containing a given set $C \subset Y$ is denoted by $[C]$, in which case $[C] = \pi_\Phi(\pi_\Phi^{-1})(C)$.

(1) Theorem. *The set $\mathfrak{K}_\Phi(Y)$ ordered by inclusion is a complete lattice. The supremum and infimum of a family $(K_\xi)_{\xi \in \Xi} \subset \mathfrak{K}_\Phi(Y)$ are calculated by the formulas*

$$\bigwedge_{\xi \in \Xi} K_\xi = \bigcap_{\xi \in \Xi} K_\xi, \quad \bigvee_{\xi \in \Xi} K_\xi = \Big[\bigcap_{\xi \in \Xi} K_\xi\Big].$$

The taking of the inverse polar $K \to \pi_\Phi^{-1}(K)$ is an antitonic bijection of $\mathfrak{K}_\Phi(Y)$ on $\mathfrak{K}_{\Phi^{-1}}(X)$.

(2) A correspondence Δ from X to X is termed a *disjointness relation* or *disjointness* (on the set X) provided the following conditions are met:

(a) $\Delta = \Delta^{-1}$, i.e., Δ is symmetric;

(b) $\Delta \cap I_X \subset \Theta \times \Theta$, where $\Theta := \pi_\Delta(X)$ is the least Δ-component;

(c) $[x] \cap [y] \subset \Theta \to (x,y) \in \Delta$.

A disjointness Δ is called *simple* if it obeys an additional requirement

(d) $(x,y) \in \Delta \to x \in \Theta \vee y \in \Theta$.

Since Δ is symmetric, the lattices $\mathfrak{K}_\Delta(X)$ and $\mathfrak{K}_{\Delta^{-1}}(X)$ coincide. If $A \subset X$, then the polar $\pi_\Delta(A)$ is called the *disjoint complement* of A and is also denoted by $A^\perp$. The relations $x \in \pi_\Delta(A)$ and $C \subset \pi_\Delta(A)$ are presented as $x^\perp A$ and $C^\perp A$. It should be also observed that $A^{\perp\perp} := (A^\perp)^\perp = [A]$.

(3) Theorem. *The set $\mathfrak{K}_\Delta(X)$ of all components of the disjointness Δ ordered by inclusion, is a complete Boolean algebra. The Boolean complement of a component coincides with its disjoint complement.*

$\lhd$ As mentioned in (1), $\mathfrak{K}_\Delta(X)$ is a complete lattice. The zero and unity of this lattice are the sets Θ and X, respectively. Applying elementary rules of operating polars and making use of distributivity of set-theoretic operations for arbitrary components K, L, M, can can write the following chain of equalities:

$$(K \vee L) \wedge M = \left((K \vee L)^{\perp} \cup M^{\perp}\right)^{\perp} = \left((K^{\perp} \cap L^{\perp}) \cup M^{\perp}\right)^{\perp}$$
$$= \left((K^{\perp} \cup M^{\perp}) \cap (L^{\perp} \cup M^{\perp})\right)^{\perp} = \left[(K^{\perp} \cup M^{\perp})^{\perp} \cup (L^{\perp} \cup M^{\perp})^{\perp}\right]$$
$$= (K^{\perp\perp} \cap M^{\perp\perp}) \vee (L^{\perp\perp} \cap M^{\perp\perp}) = (K \wedge M) \vee (L \wedge M).$$

Hence, the lattice $\mathfrak{K}_{\Delta}(X)$ is distributive. Obviously, $K \cap K^{\perp} = \Theta$. On the other hand,

$$K \vee K^{\perp} = \left[K \cup K^{\perp}\right] = (K^{\perp} \wedge K)^{\perp} = \Theta^{\perp} = X,$$

i.e., $K^{\perp}$ is the complement of K in the lattice $\mathfrak{K}_{\Delta}(X)$. ▷

4.1.13. Let us consider a set X with fixed disjointness Δ. Let j be an isomorphism of $\mathfrak{K}_{\Delta}(X)$ onto a complete Boolean algebra B. Let us introduce a mapping $s: X \to B$ by the formula $s(x) := j([x])$ $(x \in X)$. Let us assume that the least component is a singleton, i.e., $\Theta := \{\theta\} = [\theta]$ for some $\theta \in X$. We say that the B-metric d and disjointness Δ on the set X *are consistent* provided

$$d(x,\theta) = s(x) \quad (x \in X).$$

Let us consider another mapping

$$\delta: (x,y) \to (s(x) \wedge s(y))^{*} \quad (x,y \in X).$$

Theorem. *Let a set X be furnished with disjointness and a B-metric d consistent with it. Then the triplet $\mathfrak{X} := (X, \delta, \theta)$ is an algebraic B-system on which the axioms of simple disjointness (a) - (d) of* **4.1.12** (2) *are fulfilled.*

◁ First of all observe that

$$d(x,y)^{*} \wedge s(x) = d(x,y)^{*} \wedge d(x,\theta)$$
$$\leq d(x,y)^{*} \wedge (d(x,y) \vee d(y,\theta)) \leq d(y,\theta) = s(y).$$

Now this is obvious that s is a contraction mapping. Therefore, the mapping δ will be also a contraction and, hence, $\mathfrak{X}$ is an algebraic B-system with a binary predicate δ and a distinguished element θ. By definition **4.1.8**, we get

$$|x \delta y|^{\mathfrak{X}} = \delta(x,y), \quad |x \neq \theta|^{\mathfrak{X}} = s(x) \quad (x,y \in X).$$

Let us check the axioms of disjointness for δ. Obviously, δ is symmetric. The fact that $\{\theta\}$ is the least component is evident from the following relations:

$$|x \in \pi_\delta(X) \to x = \theta|^{\lambda} = \big(\bigwedge_{y \in X} \delta(x,y)\big) \Rightarrow s(x^*)$$
$$= \bigvee_{y \in X} (s(x) \wedge s(y)) \vee s(x)^* \vee \bigvee_{y \in X} s(y) = \mathbf{1}.$$

It is also obvious that for $x, y \in X$ we have

$$\delta(x,x) = |x\delta y|^{\lambda} = s(x^*) = |x = \theta|^{\lambda}.$$

Therefore, condition (b) of the definition of disjointness is fulfilled. It should then be observed that

$$|u \in [x]|^{\lambda} = s(u) \Rightarrow s(x) \quad (x, u \in X).$$

On the basis of this fact, we calculate:

$$|[x] \cap [y] = \{\theta\}|^{\lambda} = \Big(\bigwedge_{u \in X} (s(u) \Rightarrow s(x)) \wedge s(u) \Rightarrow s(y))\Big)$$
$$\Rightarrow s(u)^* = \bigwedge_{u \in X} s(u)^* \vee (s(x) \wedge s(y))^* = \delta(x,y).$$

Therefore, $|[x] \cap [y] = \{\theta\} \to x\delta y|^{\lambda} = \mathbf{1}$ and δ is a disjointness. The fact that δ is simple implies that for any $x, y \in X$ we have

$$|x\delta y \to x = \theta \vee y = \theta|^{\lambda} = \mathbf{1},$$

or, which is equivalent,

$$\delta(x,y) \Rightarrow s(x)^* \vee s(y)^* = \mathbf{1},$$

the last resulting from the definition of δ. ▷

Let $\mathfrak{A} := (A, v)$ be an algebraic B-system, while Δ be the same as in **4.1.13**. Assume that all the operations of the system $\mathfrak{A}$ preserve disjointness, i.e., for any functional symbol f and any elements $a \in A$, $x_0, \ldots, x_{n-1} \in A$ $(n := \mathfrak{a}(f))$ the relations $x_k \perp a$ $(k := 0, 1, \ldots, n-1)$ yield $f^v(x_0, \ldots, x_{n-1}) \perp a$. If, moreover, the B-metric and disjointness Δ are consistent, then the triplet (A, v, Δ) is called an *algebraic B-system with disjointness*.

4.1.14. Remarks

(1) In our proving the Stone theorem **1.2.4**, we find that a Boolean algebra B is isomorphic to the algebra of continuous functions $C(\mathfrak{T}(B), \mathbf{2})$, where $\mathfrak{T}(B)$ is an extremally disconnected compact space. In this statement one can try to substitute the two-element field **2** with an arbitrary universal algebra. This path leads us to an important example of an algebraic B-system, the Boolean power of a universal algebra introduced by P.F.Arens and I.Kaplanski [6] (see also [54, 55]).

(2) In the sequel we shall, as in the case of the present section, discuss only the problems pertaining to Boolean-valued realization of algebraic B-systems, to the specific methods of descents and ascents in the situation under consideration. Logical-algebraic aspects of algebraic B-systems are discussed in more detail elsewhere [8, 57].

4.2. Descents of Algebraic Systems

In the present section the operation of a descent is extended onto general algebraic systems, followed by some concrete examples.

4.2.1. Let $\sigma := (F, P, \mathfrak{a})$ be a signature. It follows from the general properties of the canonical embedding of sets $\mathbf{V}$ into the universe $\mathbf{V}^{(B)}$ (see **3.1.6, 3.1.9**) that $\mathbf{V}^{(B)} \models$ '$\mathfrak{a}^{\wedge}$ is a mapping from $F^{\wedge} \cup P^{\wedge}$ into the set of natural numbers'. Besides, $\mathbf{V}^{(B)} \models \sigma^{\wedge} = (F^{\wedge}, \Gamma^{\wedge}, \mathfrak{a}^{\wedge})$ and, hence,

$$\mathbf{V}^{(B)} \models \text{'}\sigma^{\wedge} \text{ is a signature'}.$$

If σ is a signature inside $\mathbf{V}^{(B)}$, then $\sigma\downarrow$ is not, in general, a signature in the conventional sense of the word. Indeed, let $\sigma = (F, P, \mathfrak{a})^{B} \in \mathbf{V}^{(B)}$ for certain $F, P, \mathfrak{a}^{B} \in \mathbf{V}^{(B)}$, in which case $[\![\mathfrak{a} : F \cup P \to \omega^{\wedge}]\!] = \mathbf{1}$. Then, for every $u \in F\downarrow \cup P\downarrow$, we can find a countable partition of unity $(b_n)_{n \in \omega} \subset B$ such that $\mathfrak{a}\downarrow(u) = \operatorname{mix}(b_n n^{\wedge})$. Therefore, under the descent of a system of arbitrary signature functional and predicate symbols of a 'mixed arity' arise. It goes without saying that it is possible to study a more general case of operations and predicates of a mixed arity, no principal difficulties arising on this way. Another direction of generalization is associated with algebraic systems having operations and predicates of infinite arity. These problems, however, will not be a subject for the discussions to follow.

4.2.2. Before giving general definitions, let us consider the descent of a very simple but important algebraic system, the two-element Boolean algebra. Let us choose two arbitrary elements, $\mathbf{0},\mathbf{1} \in \mathbf{V}^{(B)}$, for which $[0 \neq 1] = \mathbf{1}_B$. We can for instance, assume that $0 := \mathbf{0}_B{}^{\wedge}$ and $1 := \mathbf{1}_B{}^{\wedge}$.

The descent C of the two-element Boolean algebra $\{0,1\}^B \in \mathbf{V}^{(B)}$ is a complete Boolean algebra isomorphic to B. The isomorphism $\chi\colon B \to C$ can be chosen in such a way that

$$[\chi(b) = 1] = b, \quad [\chi(b) = 0] = b^* \quad (b \in B).$$

$\lhd$ Since $0,1 \in C$, for every $b \in B$ the mixing $c := \operatorname{mix}(b1, b^*0)$ also belongs to C, in which case $[c = 1] \geq b$ and $[c = 0] \geq b^*$. On the other hand,

$$[c = 1] \wedge [c = 0] = [c = 1 \wedge c = 0] \leq [0 = 1] = \mathbf{0}_B,$$

and, hence, $[c = 1] = b$ and $[c = 0] = b^*$. Setting $\chi(b) := c$, we get a mapping $\chi\colon B \to C$. It is obvious that χ is injective. Now let us check that χ is surjective. Indeed, if $c \in C$ then for $b := [c = 1]$ we have

$$[\chi(b) = 0] = b^* = [c = 0], \quad [\chi(b) = 1] = b,$$

and, hence,

$$[\chi(b) = c] \geq [\chi(b) = 1] \wedge [c = 1] = b.$$

Analogously, $[\chi(b) = c] \geq b^*$ and, hence, $\chi(b) = c$.

Let us now implement the descent of the Boolean operations of $\{0,1\}^{(B)}$. In this case for any $x, y, z \in C$ the following equalities are valid:

$$z = x \wedge y \leftrightarrow [z = 1 \leftrightarrow x = 1 \wedge y = 1] = \mathbf{1},$$
$$z = x \vee y \leftrightarrow [z = 0 \leftrightarrow x = 0 \wedge y = 0] = \mathbf{1},$$
$$z = y^* \leftrightarrow [x = 1 \leftrightarrow y = 0] = \mathbf{1}.$$

Using these relations, we can easily prove that C is a Boolean algebra, while χ is a Boolean isomorphism. Let us, for instance, show that χ preserves the least upper bounds of any pair of elements. Let $b_1, b_2 \in B$, $b_0 := b_1 \vee b_2$ and $c_l := \chi(b_l)$ for $l = 0,1,2$. Then, by definition, we get

$$[c_l = 1] = b_l, \quad [c_l = 0] = b_l^* \quad (l = 0,1,2),$$

and, hence,

$$[c_0 = 0] = b_0^* = b_1^* \wedge b_2^* = [c_1 = 0] \wedge [c_2 = 0],$$

or, which is the same, $[c_0 = 0 \leftrightarrow c_1 = 0 \wedge c_2 = 0] = 1$. Therefore, $c_0 = c_1 \vee c_2$, or $\chi(b_0) = \chi(b_1) \vee \chi(b_2)$. In an analogous way we can deduce that the greatest lower bounds and complements are preserved, too. ▷

4.2.3. Let us now consider an algebraic system $\mathfrak{A}$ of signature $\sigma^\wedge$ inside $\mathbf{V}^{(B)}$, and let $[\mathfrak{A} = (A,v)^B] = 1$ for some $A,v \in \mathbf{V}^{(B)}$. The descent of the algebraic system is the pair $\mathfrak{A}\downarrow := (A\downarrow, \mu)$, where μ is a function determined by the relations

$$\mu : f \to (v \Downarrow (f))\downarrow \quad (f \in F),$$
$$\mu : p \to \chi^{-1} \circ (v \Downarrow (p))\downarrow \quad (p \in P).$$

Here χ is the isomorphism of the Boolean algebras B and $\{0,1\}^B$ determined in **4.2.2.**

In more detail, the modified descent $v \Downarrow$ is a mapping with the domain of definition $\mathrm{dom}(v\Downarrow) = F \cup P$. For every $f \in F$ we have $[\mathfrak{a}(f)^\wedge = a^\wedge(f^\wedge)] = 1$, $[v \Downarrow (f) = v(f^\wedge)] = 1$ and, hence,

$$\mathbf{V}^{(B)} \models v \Downarrow (p) : A^{\mathfrak{a}(f)^\wedge} \to \{0,1\}^B.$$

It is now obvious that $(v \Downarrow (p))\downarrow : A^{\mathfrak{a}(f)} \to \{0,1\}^B \downarrow$ and we can put $\mu(p) := \chi^{-1} \circ (v \downarrow (p))\downarrow$.

Let $\varphi(x_0,\dots,x_{n-1})$ be a fixed formula of signature σ in n free variables. Let us construct a formula $\Phi(x_0,\dots,x_{n-1},\mathfrak{A})$ in the language of set theory which formalizes the statement $\mathfrak{A} \models \varphi(x_0,\dots,x_{n-1})$. It should be recalled that the relation $\mathfrak{A} \models \varphi(x_0,\dots,x_{n-1})$ determines an n-ary predicate in A or, which is equivalent, a mapping from A^n to $\{0,1\}$. Then, by the maximum and transfer principles, there is a unique element $|\varphi|^{\mathfrak{A}} \in \mathbf{V}^{(B)}$ such that

$$[|\varphi|^{\mathfrak{A}} : A^{n^\wedge} \to \{0,1\}^B] = 1,$$
$$[|\varphi|^{\mathfrak{A}} (a\uparrow) = 1] = [\Phi(a(0),\dots,a(n-1),\mathfrak{A})] = 1$$

for every $a : n \to A\downarrow$. Henceforth instead of $|\varphi|^{\mathfrak{A}}(a\uparrow)$ we shall write $|\varphi|^{\mathfrak{A}}(a_0,\dots,a_{n-1})$, where $a_l := a(l)$. Therefore, the relation

$$\mathbf{V}^{(B)} \models \text{`}\varphi(a_0,\ldots,a_{n-1}) \text{ holds in the model'}$$

is valid iff $[\![\Phi(a_0,\ldots,a_{n-1},a)]\!] = \mathbf{1}$.

4.2.4. Theorem. *Let $\mathfrak{A}$ be an algebraic system of signature $\sigma^{\wedge}$ inside $\mathbf{V}^{(B)}$. Then $\mathfrak{A}\downarrow$ is an extended algebraic B-system of signature σ. In this case for any formula φ of signature σ we have*

$$\chi \circ |\varphi|^{\mathfrak{A}\downarrow} = |\varphi|^{\mathfrak{A}}\downarrow.$$

$\triangleleft$ As we already know, $A\downarrow$ is an extended B-set. Then, the modified descent v' of the element $v \in \mathbf{V}^{(B)}$ is a mapping, in which case $\mathrm{dom}(v') = F \cup P$ (see **3.5.5** (3)). Besides,

$$[\![v'(f): A^{\mathfrak{a}(f)^{\wedge}} \to A]\!] = \mathbf{1} \quad (f \in F),$$
$$[\![v'(p): A^{\mathfrak{a}(p)^{\wedge}} \to \{0,1\}]\!] = \mathbf{1} \quad (p \in P).$$

The above relations show $v'(f)\downarrow$ and $v'(p)\downarrow$ to be contraction mappings from $A^{\mathfrak{a}(f)}$ to A and from $A^{\mathfrak{a}(p)}$ to $C := \{0,1\}^{B}\downarrow$, respectively. Hence, $(A\downarrow,\mu)$ is an extended algebraic B-system. Let now φ be a formula of signature σ and let us show that

$$[\![|\varphi|^{\mathfrak{A}}(a_0,\ldots,a_{n-1}) = 1]\!] = |\varphi|^{\mathfrak{A}\downarrow}(a_0,\ldots,a_{n-1})$$

for any $a_0,\ldots,a_{n-1} \in A\downarrow$. Use induction on the complexity of the formula φ. At first let φ be an atomic formula. If $q \in P$ and $\mathfrak{a}(q) = 0$, then $[\![v(q^{\wedge}) = 0 \vee v(q^{\wedge}) = 1]\!] = \mathbf{1}$, so that $v'(q) \in C$ and $\mu(q) = \chi^{-1}(v'(q)) \in B$. According to **4.2.2**, $\mu(q) = [\![\chi \circ \mu(q) = \mathbf{1}]\!] = [\![\mathbf{1} = v(q^{\wedge})]\!]$. Now consider the terms $t_0,\ldots,t_{m-1}$ of signature σ assuming the values $b_0,\ldots,b_{m-1}$ for the values $a_0,\ldots,a_{n-1}$ of the variables $x_0,\ldots,x_{n-1}$. Let $p \in P$ and $\mathfrak{a}(p) = m$. If $\varphi(x_0,\ldots,x_{n-1}) := p(t_0,\ldots,t_{m-1})$, then

$$[\![|\varphi|^{\mathfrak{A}}(a_0,\ldots,a_{n-1}) = 1]\!] = [\![v\downarrow(p)(b_0,\ldots,b_{m-1}) = \mathbf{1}]\!]$$
$$[\![\chi \circ p^{\mu}(b_0,\ldots,b_{m-1}) = 1]\!] = p^{\mu}(b_0,\ldots,b_{m-1}).$$

Whereas if $\varphi(x_0,\ldots,x_{n-1}) := (t_0(x_0,\ldots,x_{n-1}) = t_1(x_0,\ldots,x_{n-1}))$, then

$$[\![|\varphi|^{\mathfrak{A}}(a_0,\ldots,a_{n-1}) = 1]\!] = [\![b_0 = b_1]\!] = d(b_0,b_1)^{*}.$$

Let us now assume that φ_1 and φ_2 have the forms $\varphi \wedge \psi$ and $(\forall x_0)\varphi$, respectively. In this case

$$[|\varphi_1|^{\mathfrak{A}}(a_0,\dots,a_{n-1})=1]=[|\varphi|^{\mathfrak{A}}(a_0,\dots,a_{n-1})=1\wedge\psi|^{\mathfrak{A}}(a_0,\dots,a_{n-1})=1]$$
$$=[|\varphi|^{\mathfrak{A}}(a_0,\dots,a_{n-1})=1]\wedge[|\psi|^{\mathfrak{A}}(a_0,\dots,a_{n-1})=1]=|\varphi_1|^{\mathfrak{A}\downarrow}(a_0,\dots,a_{n-1})=1;$$
$$[|\varphi_2|^{\mathfrak{A}}(a_0,\dots,a_{n-1})=1]=[(\forall x_0\in A)|\varphi|^{\mathfrak{A}}(a_0,\dots,a_{n-1})=1]$$
$$=\bigwedge_{a_0\in A\downarrow}[|\varphi|^{\mathfrak{A}}(a_0,\dots,a_{n-1})=1]=|\varphi_2|^{\mathfrak{A}\downarrow}(a_0,\dots,a_{n-1}).$$

The case of the universal quantifier and the remaining propositional connectives is considered analogously. ▷

4.2.5. Theorem. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be algebraic systems of the same signature $\sigma^\wedge$ inside $\mathbf{V}^{(B)}$. Assign $\mathfrak{A}':=\mathfrak{A}\downarrow$ and $\mathfrak{B}':=\mathfrak{B}\downarrow$. Then, if h is a homomorphism (strong homomorphism) inside $\mathbf{V}^{(B)}$ from the system $\mathfrak{A}$ to the system $\mathfrak{B}$, then $h':=h\downarrow$ is a homomorphism (strong homomorphism) of the B-systems $\mathfrak{A}'$ and $\mathfrak{B}'$. Conversely, if $h':\mathfrak{A}'\to\mathfrak{B}'$ is a homomorphism (strong homomorphism) of algebraic B-systems, then $h:=h'\uparrow$ is a homomorphism (strong homomorphism) inside $\mathbf{V}^{(B)}$ from the system $\mathfrak{A}$ to the system $\mathfrak{B}$.*

◁ Let us confine ourselves to substantiating **4.1.10** (3) of the definition of homomorphism, i.e., let us consider only the case of a non-nullary functional symbol, the considerations for other symbols of signature σ being analogous. Let $\mathfrak{A}:=(A,v)^B$ for some $A,v\in\mathbf{V}^{(B)}$ and $\mathfrak{A}'=(A',v')$. Let us assume that $\mu\in\mathbf{V}^{(B)}$ and $\mu'\in\mathbf{V}$ be interpreting mappings of the systems $\mathfrak{B}$ and $\mathfrak{B}'$, respectively. Consider a functional symbol f of the arity $n=\mathfrak{a}(f)$ and elements $a_0,\dots,a_{n-1}\in A'$. As before, the presentation $t=g(a_0,\dots,a_{n-1})$ for $g\in\mathbf{V}^{(B)}$ will denote the formula $t=g(a)$, where $a\in\mathbf{V}^{(B)}$ is such an element from $\mathbf{V}^{(B)}$ that $[a:n^\wedge\to A]=\mathbf{1}$ and $a\downarrow(l)=a_l$ $(l<n)$. If $h\in\mathbf{V}^{(B)}$ is a homomorphism inside $\mathbf{V}^{(B)}$ from $\mathfrak{A}$ to $\mathfrak{B}$, then

$$[h(v(f^\wedge)(a_0,\dots,a_{n-1}))=\mu(f^\wedge)(h(a_0),\dots,h(a_{n-1}))]=\mathbf{1}.$$

Besides, in accord with the definition of descents (see **3.5.5** (3)),

$$[v(f^\wedge)=v\downarrow(f)]=[\mu(f^\wedge)=\mu\downarrow(f)]=\mathbf{1};$$
$$[v\downarrow(f)(a_0,\dots,a_{n-1})=v'(f)(a_0,\dots,a_{n-1})]=\mathbf{1};$$
$$[\mu\downarrow(f)(b_0,\dots,b_{n-1})=\mu'(f)(b_0,\dots,b_{n-1})]=\mathbf{1};$$
$$[h(t)=h'(t)]=\mathbf{1}\quad(t\in A').$$

Summing up all the above relations and bearing in mind that $\mathbf{V}^{(B)}$ is separated, we get

$$h'(v'(f)(a_0,\dots,a_{n-1})) = \mu'(f)(h(a_0),\dots,h(a_{n-1})).$$

Assume, conversely, that the last equality holds. By replacing in it h' with $h := h'\uparrow$, we get a formula which is assertive inside $\mathbf{V}^{(B)}$. Substituring in it $v\downarrow(f)$ for $v'(f)$ and $v(f^{\wedge})$ for $v\downarrow(f)$, and then $\mu\downarrow(f)$ for $\mu'(f)$ and $\mu(f^{\wedge})$ for $\mu\downarrow(f)$, we get a new formula assertive inside $\mathbf{V}^{(B)}$. It is this new formula that is the required property inside $\mathbf{V}^{(B)}$. ▷

Corollary. *In terms of theorem* **4.2.5** [*h is a homomorphism between the algebraic systems* $\mathfrak{A}$ *and* $\mathfrak{B}$|] = **1** *iff* h' *is an isomorphism between the algebraic B-systems* $\mathfrak{A}'$ *and* $\mathfrak{B}'$.

4.2.6. As observed in **4.1.3**, an extended algebraic B-system $\mathfrak{A} := (A, v)$ can be viewed as a conventional (i.e., {0,1}-valued) algebraic system $\mathfrak{A}' := (A, v')$ of the same signature provided the B-valued predicates p^v are replaced with the sets $v'(p) := \{(x_0,\dots,x_{n-1}) \in A^n : p^v(x) = \mathbf{1}\}$. This, however, does not at all mean that if $\mathfrak{A}$ is a B-model of the formula φ of signature $\sigma(\mathfrak{A})$, then $\mathfrak{A}'$ is a $\{\mathbf{0},\mathbf{1}\}$-valued model, i.e., a model in the conventional sense of the word for the same formula φ. Moreover, for some formulas this is exactly the case. This problem will be considered in more detail in the section to follow, while here we shall confine ourselves to some concrete examples of algebraic B-systems obtained with the help of descent. If the formula φ is the conjunction of the axioms of a group , a ring, a module, etc., and the algebraic system $\mathfrak{A}$ is a two-valued model for φ, then, as usual, $\mathfrak{A}$ is said to be a group, a ring, a module, etc.. Whereas if $\mathfrak{A}$ is a B-model for φ, then $\mathfrak{A}$ is said to be a B-group, a B-ring, a B-module, etc..

Let us consider an arbitrary group G. An endomorphism $\pi : G \to G$ is termed a *projection* if $\pi \circ \pi = \pi$. We say that $\mathbf{B}$ is a Boolean algebra of projections in the group G, if $\mathbf{B}$ consists of mutually commuting projections in G and forms a Boolean algebra with zero, $\mathbf{0}_{\mathbf{B}} := 0$, and unit, $\mathbf{1}_{\mathbf{B}} := I_G$ if furnished with:

$$\pi_1 \vee \pi_2 := \pi_1 + \pi_2 - \pi_1 \circ \pi_2, \quad \pi_1 \wedge \pi_2 := \pi_1 \circ \pi_2, \pi^* := 1 - \pi$$
$$(\pi_1, \pi_2, \pi \in \mathbf{B}).$$

The order in $\mathbf{B}$ is such that $\pi_1 \le \pi_2$ iff $\pi_1(G) \subset \pi_2(G)$. A group G with a Boolean algebra of projections $\mathbf{B}$ will be called *extended* if $\mathbf{B}$ is order-complete and for any partition of unity $(\pi_\xi) \subset \mathbf{B}$ there is a unique element $x \in G$ such that $\pi_\xi x_\xi = \pi_\xi x$ for all ξ. Let $(G, \mathbf{B})$ and

$(G',\mathbb{B}')$ be groups with Boolean algebras of projections (or, to put it short, groups with projections). A group isomorphism $h: G \to G'$ is called a *homomorphism of groups with projections*, if there is a Boolean isomorphism $j: \mathbb{B} \to \mathbb{B}'$ such that $h \circ \pi = j(\pi) \circ h$ for all $\pi \in \mathbb{B}$.

Let now R be a ring, and let the additive group of this ring have a Boolean algebra of projections $\mathbb{B}$. If, moreover, each projection $\pi \in \mathbb{B}$ is a ring homomorphism, then $(R,\mathbb{B})$ is said to be a *ring with a Boolean algebra of projections*, or a *ring with projections*. The projection $[x] := \wedge\{\pi \in \mathbb{B} : \pi x = x\}$ will be termed the *carrier of an element* $x \in R$.. It is obvious that if the carriers $[x]$ and $[y]$ are disjoint (as elements of the Boolean algebra $\mathbb{B}$), then $x \cdot y = 0$, the converse statement being, generally speaking, invalid. If $x \cdot y = 0$, then x and y are said to *be orthogonal*. An element is called *regular* if it is orthogonal only to the zero element. A *zero divisor* is any element orthogonal to a nonzero element.

Let S be a multiplicative subset of a ring with unity K, i.e., $1 \in S$ and $xy \in S$ for any x and $y \in S$. In the set $K \times S$ let us introduce the equivalence relation, by letting

$$(x,s) \sim (x',s') \leftrightarrow (\exists t \in S) \quad (t(sx' - s'x) = 0).$$

Let $S^{-1}K := K \times S/\sim$, and $(x,s) \to x/s$ be the canonical factor-mapping. The set $S^{-1}K$ can be endowed with a ring structure using the equalities

$$(x/s) + (y/t) := (tx + sy)/st, \quad (x/s)(y/t) := (xy)/(st).$$

The mapping $x \to x/1$, $x \in K$, is a homomorphism from K to $S^{-1}K$ called *canonical*. The ring $S^{-1}K$ is termed the *ring of fractions of K relative to S*.

4.2.7. Theorem. *Let* $\mathcal{G}$ *be a group inside* $\mathbf{V}^{(B)}$ *and* $G := \mathcal{G}\downarrow$. *Then* G *is a group and in it there is a complete Boolean algebra of projections* $\mathbb{B}$ *and an isomorphism* $j: B \xrightarrow{on} \mathbb{B}$ *such that*

$$b \leq [x = 0] \leftrightarrow j(b)x = 0 \quad (x \in G, b \in B).$$

Moreover, $(G,\mathbb{B})$ *is an extended group with projections, and the following equivalences hold:*

(1) $\mathbf{V}^{(B)} \models$ '$\mathcal{G}$ *is commutative*' $\leftrightarrow$ 'G *is commutative*';

(2) $\mathbf{V}^{(B)} \models$ '$\mathcal{G}$ *is torsion-free*' $\leftrightarrow$ 'G *is torsion-free*'.

◁ By theorem **4.2.4**, $\mathcal{G}\downarrow$ is an extended algebraic B-system, being a B-group. Let us determine the descent of addition + by the same symbol. Let us show that G is a group, limiting ourselves to the case when there are inverse elements. Let $\varphi := (\forall x)(\exists! y)(x + y = 0)$. Then, according to **4.1.8**,

$$|\varphi|^G := \bigwedge_{x\in G} \bigvee_{y\in G} |x + y = 0|^G = \mathbf{1}.$$

Since the B-set G is extended, for every $x \in G$ there is a $y \in G$ such that

$$\mathbf{1} = |x + y = 0|^G = d(x + y, 0)^* = [x + y = 0],$$

and, hence, $x + y = 0$. If $x + z = 0$ for a certain $z \in G$, then $|x + z = 0|^G = \mathbf{1}$. As G is a B-group, we have

$$\mathbf{1} = |x + y = 0 \wedge x + z = 0|^G \Rightarrow |y = z|^G,$$

and, hence, $|y = z|^G = [z = y] = \mathbf{1}$ and $z = y$.

The congruences of the group G are exactly the equivalences determined by its different normal subgroups. Therefore, by virtue of theorem **4.16**, there is an isomorphism j from B on a certain complete Boolean algebra $\mathfrak{B}'$ of normal subgroups of the group G such that

$$b \le [x = 0] \leftrightarrow x \in j(b^*) \quad (b \in B, x \in G).$$

If $b \in B$, then $j(b) \cap j(b^*) = 0$. On the other hand, for every $x \in G$ there are $x_1 := \operatorname{mix}\{bx, b^*0\}$, $x_2 := \operatorname{mix}\{b^*x, b0\}$ and, since $b^* \le [x_1 = 0]$, $b \le [x_2 = 0]$, we have $x_1 \in j(b)$, $x_2 \in j(b^*)$. Besides, $[x = x_1 + x_2] \ge [x_1 = x] \wedge [x_2 = 0] \ge b$ and $[x = x_1 + x_2] \ge [x_1 = 0] \wedge [x_2 = x] \ge b^*$, which gives $x = x_1 + x_2$. Therefore, any subgroup of the type $j(b)$ is singled out into a direct addent and is corresponded to by the operator of projecting π_b on $j(b)$ along the complementary subgroup $j(b^*)$. To be more exact, π_b is determined by the conditions: $\pi_b x = x$ for all $x \in j(b)$ and $\pi_b x = 0$ for $x \in j(b^*)$. Let the same letter j denote the isomorphism $b \to \pi_b, b \in B$ and put $\mathfrak{B} := j(B)$. Obviously, $\mathfrak{B}$ and j obey the required conditions. The fact that the group G is extended is equivalent to the fact the corresponding B-set is also extended, as $x = \operatorname{mix}(b_\xi x_\xi)$ iff $j(b_\xi)x = j(b_\xi)x_\xi$ for all ξ.

Let us assume that $\mathcal{G}$ is torsion-free. Then

$$[(\exists x \in \mathcal{G})(\exists n \in \omega^\wedge)(nx = 0) \wedge (0 \ne x) \wedge (0 < n)] = \mathbf{1},$$

and, hence, there is an element $0 \ne x \in G$ and a partition of unity $(b_n)_{n\in\omega}$ in B such that $b_n \le [n^\wedge x = 0]$ for all $n \in \omega$. It should be observed that $[n^\wedge x = nx] = \mathbf{1}$ and, hence,

$b_n \le [x \ne 0]$, $b_n \le [nx = 0]$ and $j(b_n)(nx) = nj(b_n)x = 0$. For at least one $0 \ne n \in \omega$ the projection $j(b_n)$ is non-zero which implies that G is not torsion-free. Conversely, if $nx = 0$ for some $0 \ne x \in G$ and $n \in \omega$, then $[(\exists n \in \omega^{\wedge})(nx = 0) \wedge (n > 0)] = \mathbf{1}$, i.e., $[\mathcal{G}$ is not free-torsion $] = \mathbf{1}$. The statement referring to commutativity is obvious. ▷

4.2.8. Theorem. *Let $\mathfrak{R}$ be a ring inside $\mathbf{V}^{(B)}$ and $K := \mathfrak{R}\downarrow$. Then $\mathfrak{R}$ is an extended ring with Boolean algebra of projections $\mathbf{B}$ and there is an isomorphism $j: K \xrightarrow{on} \mathbf{B}$ such that*

$$b \le [x = 0] \leftrightarrow j(b)x = 0 \quad (x \in K, b \in B).$$

In this case the following equivalences are valid:

(1) $\mathbf{V}^{(B)} \models$ *'$\mathfrak{R}$ is commutative (semiprimitive)' $\leftrightarrow$ 'K is commutative (semiprimitive)' ;*

(2) $\mathbf{V}^{(B)} \models$ *'$\mathfrak{R}$ has no zero divisors' $\leftrightarrow$ 'any two elements of K are orthogonal only if their carriers are disjoint';*

(3) $\mathbf{V}^{(B)} \models$ *'$\mathfrak{T}$ is a multiplicative subset of $\mathfrak{R}$' $\leftrightarrow$ '$S := \mathfrak{T}\downarrow$ is a multiplicative subset in K', in which case $(\mathfrak{T}^{-1}K)\downarrow \simeq S^{-1}K$ (here $\simeq$ denotes a ring isomorphism);*

(4) $\mathbf{V}^{(B)} \models$ *'$\mathfrak{R}$ is a field' $\leftrightarrow$ 'K is semiprimitive, the orthogonality of the elements of K is equivalent to disjointness of their carriers and any regular element in it is invertible';*

(5) $\mathbf{V}^{(B)} \models$ *'$\mathbf{R}$ is the radical of a ring with unity $\mathfrak{R}$' $\leftrightarrow$ '$\mathfrak{R}\downarrow$ is the radical of the ring with unity $\mathfrak{R}$'; in other words, if $\mathfrak{R}$ has unity, then $\mathfrak{R}(\mathfrak{R})\downarrow = \mathbf{R}(K)$;*

(6) $\mathbf{V}^{(B)} \models$ *'$(\mathfrak{R}, \mathfrak{D})$ is a ring with projections' $\leftrightarrow$ 'the mapping $\pi \to \pi\downarrow$ $(\pi \in \mathfrak{D}\downarrow)$ is an isomorphism $\mathfrak{D}\downarrow$ on a certain Boolean algebra of projections D to K, in which case $\mathbf{B}$ is a regular subalgebra in D, and (K, D) is a ring with projections'.*

◁ According to theorem **4.2.7**, K is an extended group with projections, and there is an isomorphism j from B onto the complete Boolean algebra $\mathbf{B}$ of additive projections obeying the necessary condition. Let us supply K with the operation of multiplication in accordance with the general definition **4.2.3**, i.e., for elements $x, y \in K$, we have $[x, y \in \mathfrak{R}] = \mathbf{1}$ and, hence, in the model $\mathbf{V}^{(B)}$ there is a product z of these elements: $[z \in \mathfrak{R}] = [z = x \cdot y] = \mathbf{1}$. Let us assume z to be the product of x and y in K. Therefore,

$$z = x \cdot y \leftrightarrow [z = x \cdot y] = \mathbf{1} \quad (x, y, z \in K).$$

The fact that in this case we get a ring is easily deduced using theorem **4.2.4**. Let us choose an arbitrary element $b \in B$ and show that the projection $j(b)$ is a ring homomorphism. Indeed, the operation of multiplication in K is the descent of the corresponding operation in $\mathcal{R}$ and, hence, extensional, and thus it preserves mixing. Therefore, by the definition of the projection $j(b)$ (see **4.2.7**), for any $x, y \in K$ we get

$$j(b)xy = \operatorname{mix}\{bxy, b * 0\} =$$
$$= \operatorname{mix}\{bx, b * 0\} \cdot \operatorname{mix}\{by, b * 0\} = j(b)x \cdot j(b)y.$$

Let us now turn our attention to proving statements (1) - (6).

(1) The proof is conducted by analogy with **4.2.7** (1).

(2) The statement $\mathbf{V}^{(B)} \models$ '$\mathcal{R}$ *has no zero divisors*' is equivalent to the fact that for any $x, y \in \mathcal{R} \downarrow$ we have $b : [xy = 0] = [x = 0] \vee [y = 0]$. If the latter relation is fulfilled and $xy = 0$, then $b = \mathbf{1}$ and, hence, for $e := [x = 0]$ and $c := [y = 0]$ we have $e^* \wedge c^* = 0$. Besides, $j(e^*)x = x$ and $j(c^*)y = y$ and, therefore, $[x] \leq j(e^*)$ and $[y] \leq j(c^*)$. The carriers $[x]$ and $[y]$ are thus seen to be disjoint. If, however, $[x] \circ [y] = 0$, then, as has been earlier remarked in **4.2.6**, $x \cdot y = 0$.

Conversely, assume that the equality $xy = 0$ is equivalent to the fact that the carriers $[x]$ and $[y]$ are disjoint. Then for $b := [xy = 0]$ the equalities $0 = j(b)xy = (j(b)x) \cdot (j(b)y)$ yield that the projections $\pi := [j(b)x]$ and $\rho := [j(b)y]$ are disjoint. It should be observed that $j(b) \circ \pi^* x = 0$ and $j(b) \circ \rho^* y = 0$ and, hence,

$$[x = 0] \vee [y = 0] \geq (b \wedge j^{-1}(\pi^*)) \vee (b \wedge j^{-1}(\rho^*)) = b.$$

(3) The statement concerning multiplication is evident. Let us prove that the descent of a ring of fractions is a ring of fractions. Let us first remark that $(\mathcal{T} \times K)\downarrow = S \times K$. Let us consider an equivalence relation $\mathcal{P} \in \mathbf{V}^{(B)}$ such that for $x, x' \in K$ and $s, s' \in S$ we have

$$\mathbf{V}^{(B)} \models (x, s)\mathcal{P}(x', s') \leftrightarrow (\exists t \in \mathcal{T})(t(sx' - s'x) = 0).$$

If $P := \mathcal{P} \downarrow$, then P is an equivalence relation in $K \times S$, in which case

$$(x, s)P(x', s') \leftrightarrow (\exists t \in S) \quad (t(sx' - s'x) = 0).$$

Then, the descent of the factor-set $\mathcal{T} \times \mathcal{R} / \mathcal{P}$ is bijective with the set $KS \times K / P$. And, finally, for $x, y \in K$ and $s, t \in S$ the equalities

$$(x / s) + (y / t) = (tx + sy) / st, \; (x / s)(y / t) = (xy / st)$$

are valid iff they are true inside $\mathbf{V}^{(B)}$. All we have to do now is to correlate the above-proved with the definition of a ring of fractions.

(4) Let us assume that $[\mathfrak{K}$ is a field$] = \mathbf{1}$. In this case K is semiprimitive and $xy = 0$ yields that $[x] \circ [y] = 0$ for all $x, y \in K$ (see (1) and (2)). For any regular element $x \in K$ we get $j(b)xy = 0 \to j(b)y = 0$ whatever $b \in B$ and $y \in K$ are. In this case, however, $[xy = 0] \le [y = 0]$, i.e., $[x \ne 0] = \mathbf{1}$. Therefore, there is an element $u \in K$ such that $[xu = ux = 1] = \mathbf{1}$ and, hence, $xu = ux = 1$, i.e., x is invertible in the ring K. Conversely, let K be semiprimitive, any regular element in it be invertible and the orthogonality of the elements in K be equivalent to the disjointness of their carriers. Then $\mathbf{V}^{(B)} \models$ '$\mathfrak{K}$ is a commutative ring', and, hence, $[\mathfrak{K}$ is a field$] = [(\forall x)(x \in \mathfrak{K} \wedge x \ne 0 \to$ 'x is invertible ')$] = \wedge\{[\exists x^{-1}] : x \in K \wedge \wedge [x \ne 0] = \mathbf{1}\}$. Therefore, it suffices to show that if $[x \ne 0] = \mathbf{1}$, then $[x$ is convertible $] = \mathbf{1}$, whatever an $x \in K$. Let us assume that $[x \ne 0] = \mathbf{1}$ and $xy = 0$ for some $y \in K$. Then for $\pi := [x]$ and $\rho = [y]$ we have $\pi \circ \rho = 0$. On the other hand, $j(b)x = 0$ implies $b \le [x = 0] = [x \ne 0]^* = \mathbf{1}^* = \mathbf{0}$, and, hence, $\rho = j(1) = I_K$. Therefore, we get $\pi \le \rho^* = 0$, or $y = 0$, and, hence the element x is invertible in the ring K. This immediately results in the relation $[x$ is invertible in $\mathfrak{K}\,] = \mathbf{1}$.

(5) The element x is in the radical of a ring iff for any y the element $1 - yx$ is left-invertible. Now we have to remark that $1 - yx$ is left-invertible in K iff $[1 - yx$ is left-invertible in $\mathfrak{K}\,] = \mathbf{1}$.

(6) If $[(\mathfrak{K}, \mathfrak{D})$ is a ring with projections $] = \mathbf{1}$ and $\pi \in \mathfrak{D}\downarrow$, then, by **4.2.7**, $\pi\downarrow : K \to K$ is a homomorphism. On the other hand, $[\pi \circ \pi = \pi] = \mathbf{1}$ and, hence, $(\pi\downarrow) \circ (\pi\downarrow) = (\pi \circ \pi)\downarrow = \pi\downarrow$, i.e., $\pi\downarrow$ is a projection. The fact that D is a Boolean algebra will be established in **4.2.9**. Therefore, (K, D) is a ring with projections. By definition, $\mathfrak{B} = \{\pi\downarrow : \pi \in \{\mathbf{0}_{\mathfrak{D}}, \mathbf{1}_{\mathfrak{D}}\}^B\downarrow\}$ (see **4.2.7**) and, hence, $\mathfrak{B} \subset D$. The converse implication is established analogously. ▷

4.2.9. Theorem. *Let $\mathfrak{D}$ be a complete Boolean algebra inside $\mathbf{V}^{(B)}$ and $D := \mathfrak{D}\downarrow$. Then D is a complete Boolean algebra and there is a full monomorphism $\iota : B \to D$ such that*

$$b \le [x \le y] \leftrightarrow \iota(b)x \le \iota(b)y$$

for all $x, y \in D$ and $b \in B$.

◁ By virtue of **4.2.4**, D is an extended algebraic B-system of signature $(\vee, \wedge, *, 0, 1)$. The fact that D is a Boolean algebra also follows from **4.2.4**. Let us temporarily denote

Boolean operations in D by $\tilde{\vee}$, $\tilde{\wedge}$, and check distributivity, for instance. Let us consider the terms $t_1(x,y,z) := (x \wedge y) \vee z$, $t_2(x,y,z) := (x \vee z) \wedge (x \vee y)$ and the formula $\Psi := (\forall x,y,z)\varphi(x,y,z)$, where $\varphi(x,y,z) := (t_1(x,y,z) = t_2(x,y,z))$. In this case

$$[|\Psi|^{\mathfrak{D}} = 1] = |\Psi|^{D} = \bigwedge_{a,b,c \in D} |\varphi|^{D}(a,b,c) = \mathbf{1},$$

and, hence, $|\varphi|^D(a,b,c) = \mathbf{1}$ for all $a,b,c \in D$. Then

$$\mathbf{1} = |\varphi|^D(a,b,c) = d(t_1(a,b,c), t_2(a,b,c))^*$$
$$= [t_1(a,b,c) = t_2(a,b,c)] = [(a \tilde{\wedge} b) \tilde{\vee} c = (a \tilde{\vee} c) \tilde{\wedge} (b \tilde{\vee} c)].$$

Hence, since $\mathbf{V}^{(B)}$ is separated, we get $(a \tilde{\wedge} b) \tilde{\vee} c = (a \tilde{\vee} c) \tilde{\wedge} (b \tilde{\vee} c)$. In the same way we check the validity of the remaining axioms of Boolean algebras. Therefore, D is a Boolean algebra.

The completeness of D is not a first-order property and, hence, it cannot be deduced by using the above scheme. Let $\leq \in \mathbf{V}^{(B)}$ be the conventional order relation in $\mathfrak{D}$, i.e.,

$$\mathbf{V}^{(B)} \models (\forall x \in \mathfrak{D})(\forall y \in \mathfrak{D})(x \leq y \leftrightarrow x \wedge y = x).$$

Let us put $\tilde{\leq} := (\leq)\downarrow$. Then for $x, y \in D$ $x \tilde{\leq} y$ is fulfilled iff $x \tilde{\wedge} y = x$. Consider a correspondence $\Phi := (\tilde{\leq}, D, D)$. It is obvious that Φ is a full contraction. Then, if $A \subset D$, then $\pi_\Phi(A)$ $(\pi_\Phi^{-1}(A))$ is a set of all upper (lower, respectively) bounds of the set A (relative to the order $\tilde{\leq}$). Therefore,

$$\{\sup(A)\} = \pi_\Phi(A) \cap \pi_\Phi^{-1}(\pi_\Phi(A)),$$

provided $\sup(A)$ exists. If $\Psi := (\leq, \mathfrak{D}, \mathfrak{D})^B$, then Ψ is a correspondence inside $\mathbf{V}^{(B)}$ and $\Phi = \Psi\downarrow$. Since $\mathfrak{D}$ is complete, there is such an element $a \in D$ such that $[a = \sup(A)] = \mathbf{1}$ or $[\pi_\Psi(A) \cap \pi_\Psi^{-1}(\pi_\Psi(A)) = \{a\}] = \mathbf{1}$. Employing the rule for descending polars (see **3.2.13** (2)), we can easily calculate

$$\{a\} = [\pi_{\Psi^{-1}}(\pi_\Psi(A\uparrow)) \cap \pi_\Psi(A\uparrow)]\downarrow =$$
$$= \pi_{\Psi^{-1}}(\pi_\Psi(A\uparrow\downarrow)) \cap \pi_\Phi(A\uparrow\downarrow) = \sup(\mathrm{mix}(A)) = \sup(A).$$

Therefore, $a = \sup(A)$ and we have substantiated the fact that D is complete. Let $\lambda \in \mathbf{V}^{(B)}$ be the identical embedding of the algebra $\{\mathbf{0}_{\mathfrak{D}}, \mathbf{1}_{\mathfrak{D}}\}^B$ in $\mathfrak{D}$ inside $\mathbf{V}^{(B)}$. Let us put $\iota_1 = \lambda\downarrow$ and $\iota := \iota_1 \circ \iota_2$, where ι_2 is an isomorphism of B on $\{\mathbf{0}_{\mathfrak{D}}, \mathbf{1}_{\mathfrak{D}}\}^B\downarrow$. In this case ι is a monomorphism. The monomorphism ι is full since for $A \subset B$ we have

$\iota(\pi_{\Phi'}(A)) \subset \pi_{\Phi}(\iota(A))$, where $\Phi' := \iota^{-1} \circ \Phi \circ \iota$.

Then, by virtue of the obvious relation

$$\mathbf{V}^{(B)} \models (\forall x, y \in \mathcal{D})(\forall c \in \{\mathbf{0}_{\mathcal{D}}, \mathbf{1}_{\mathcal{D}}\})(\lambda(c)x = \lambda(c)y \leftrightarrow (c = \mathbf{0}_{\mathcal{D}}) \vee (c = \mathbf{1}_{\mathcal{D}} \wedge x = y))$$

for any $x, y \in D$ and $b \in B$ we have

$$[\iota(b)x = \iota(b)y] = b^* \vee (b \wedge [x = y]).$$

Hence, we get

$$\iota(b)x = \iota(b)y \leftrightarrow b \le [x = y],$$

and, therefore,

$$d(x,y)^* = [x = y] = \vee\{b \in B : \iota(b)x = \iota(b)y\}.$$

It is now evident that if $\varphi(x, y) := x \le y$, then

$$|\varphi|^D(x,y) = \vee\{b \in B : \iota(b)x \le \iota(b)y\},\ [|\varphi|^D(x,y) = 1] = [x \le y],$$

which yields the equivalence in question. ▷

4.3. Immersion of Algebraic B-systems into a Boolean-Valued Universe

In the present section the immersion functor studied in **3.4** is extended onto the category of algebraic B-systems.

4.3.1. Let $\mathfrak{A} := (A, v)$ be an algebraic B-system of signature $\sigma := (F, P, \mathfrak{a})$. Let us consider a mapping $v' : F \cup P \to \mathbf{V}^{(B)}$ operating by the rule

$$v' : s \to v(s)^{\sim} := \mathfrak{F}^{\sim}(v(s)) \quad (s \in F \cup P),$$

where $\mathfrak{F}^{\sim}$ is the descent functor (see **3.4.12** - **3.4.16**). In accordance with the general definition of correspondences **3.4.13**, for every $f \in F, \mathfrak{a}(f) = n$ the mapping $\lambda'(f) : (A^{\sim})^{n^{\wedge}} \to A^{\sim}$ inside $\mathbf{V}^{(B)}$ is governed by the relation

$$[v'(f)(\iota_A(x_0),\ldots,\iota_A(x_{n-1})) = \iota_A \circ v(f)(x_0,\ldots,x_{n-1})] = \mathbf{1},$$

where ι_A is the canonical embedding of A to $A' := A^{\sim}\downarrow$ (see **3.5.4**). Analogously, for $p \in P, \mathfrak{a}(p) = m$ the element $v'(p) \in \mathbf{V}^{(B)}$ is such a mapping from $(A^{\sim})^{m^{\wedge}}$ to $\{0,1\}^B \in \mathbf{V}^{(B)}$ that

$$[v'(p)(\iota_A(x_0),\ldots,\iota_A(x_{m-1})) = \iota_B \circ v(p)(x_0,\ldots,x_{m-1})] = \mathbf{1}.$$

The modified ascent $\mu := (v')\uparrow$ of the mapping $v' : F \cup P \to \mathrm{im}(v')$ is seen to be an interpretation inside $\mathbf{V}^{(B)}$. The pair $(A^{\sim},\mu)$ or the element $(A^{\sim},\mu)^B \in \mathbf{V}^{(B)}$ is termed the *Boolean-valued realization of the algebraic B-system $\mathfrak{A}$* and denoted by the symbol $\mathfrak{A}^{\sim}$.

4.3.2. Theorem. *For any algebraic B-system $\mathfrak{A}$ of signature σ its Boolean-valued realization $\mathfrak{A}^{\sim}$ is an algebraic system of signature $\sigma^{\wedge}$ inside $\mathbf{V}^{(B)}$. In this case for any formula φ of signature σ with n free variables and for arbitrary $a_0,\ldots,a_{n-1} \in A := |\mathfrak{A}|$ we have*

$$|\varphi|^{\mathfrak{A}}(a_0,\ldots,a_{n-1}) = [\![|\varphi|^{\mathfrak{A}^{\sim}}(\iota_A(a_0),\ldots,\iota_A(a_{n-1})) = 1]\!].$$

$\lhd$ It should be recalled that while considering an arbitrary set as a B-set we mean the discrete B-metric in it. Therefore, $\sigma^{\sim} = \sigma^{\wedge}$ (see **3.4.12**). By virtue of **3.5.5**, it holds that

$$\mathbf{V}^{(B)} \models \text{`}\mu \text{ is a function and } \mathrm{dom}(\mu) = F^{\wedge} \cup P^{\wedge}\text{'}.$$

By theorem **3.4.14**, $\mathbf{V}^{(B)} \models \text{`}\mu(f^{\wedge})$ is a mapping from $(A^{\sim})^{\mathfrak{a}(f)^{\wedge}}$ to $A^{\sim}\text{'}$ for all $f \in F$ and $\mathbf{V}^{(B)} \models \text{`}\mu(p)$ is a mapping from $(A^{\sim})^{\mathfrak{a}(p)^{\wedge}}$ to $\{0,1\}\text{'}$ for every $p \in P$. Hence, $\mathbf{V}^{(B)} \models \text{`}\mathfrak{A}^{\sim}$ is an algebraic system of signature $\sigma^{\wedge}\text{'}$.

Let us now consider a formula φ of signature σ. By theorem **3.5.5** (3) for $f \in F$ and $p \in P$ we have

$$\iota_A \circ f^{v}(a_0,\ldots,a_{n-1}) = \mu(f^{\wedge})\downarrow(\iota_A(a_0),\ldots,\iota_A(a_{n-1})) \quad (a_l \in A),$$
$$\iota_B \circ p^{v}(a_0,\ldots,a_{n-1}) = \mu(p^{\wedge})\downarrow(\iota_A(a_0),\ldots,\iota_A(a_{n-1})) \quad (a_l \in A).$$

Using the above equalities, we deduce by induction on the complexity of the formula φ:

$$|\varphi|^{\mathfrak{A}}(a_0,\dots,a_{n-1}) = |\varphi|^{\mathfrak{A}'}(\iota_A(a_0),\dots,\iota_A(a_{n-1})) \quad (a_0,\dots,a_{n-1}\in A),$$

where $\mathfrak{A}' := \mathfrak{A}^{\sim}\downarrow$. Now we have to use theorem **4.2.4.** ▷

4.3.3. Theorem. *Let now $\mathfrak{A} := (A, v)$ be an algebraic B-system of signature σ. Then there are such $\mathcal{A}$ and $\mu \in \mathbf{V}^{(B)}$ that the following conditions are fulfilled:*

(2) $\mathbf{V}^{(B)} \models$ '$(\mathcal{A}, \mu)$ *is an algebraic system of signature* $\sigma^{\wedge}$';

(2) *if $\mathfrak{A}' := (A', v')$ is the descent of the system $(\mathcal{A}, \mu)$, then $\mathfrak{A}'$ is an extended algebraic $\mathcal{B}$-system of signature σ;*

(3) *there is an isomorphism ι from $\mathfrak{A}$ to $\mathfrak{A}'$ such that* $A' = \mathrm{mix}(\iota(A))$;

(4) *for any formula φ of signature σ in n free variables it holds that*

$$|\varphi|^{\mathfrak{A}}(a_0,\dots,a_{n-1}) = |\varphi|^{\mathfrak{A}'}(\iota(a_0),\dots,\iota(a_{n-1}))$$
$$= \chi^{-1}\circ\left(|\varphi|^{\mathfrak{A}^{\sim}}\right)\downarrow(\iota(a_0),\dots,\iota(a_{n-1}))$$

for all $a_0,\dots,a_{n-1}\in A$.

◁ Assign $\mathcal{A} := A^{\sim}$, $\iota := \iota_A$, while letting μ be determined as in **4.3.1**. Then the required statements result from **3.5.5** (3), **4.2.4** and **4.3.2.** ▷

4.3.4. Theorem. *Let us consider algebraic B-systems $\mathfrak{A}$ and $\mathcal{B}$ of the same signature.*

(1) *Let h be a contraction mapping from $|\mathfrak{A}|$ to $|\mathcal{B}|$. Then h is a homomorphism (strong homomorphism, isomorphism) iff* $\mathbf{V}^{(B)} \models$ '$h^{\sim}$ *is a homomorphism (strong homomorphism, isomorphism) from $\mathfrak{A}^{\sim}$ to $\mathcal{B}^{\sim}$*'. *The monomorphism $h^{\sim}$ is surjective inside $\mathbf{V}^{(B)}$ iff* $|\mathcal{B}| = \mathrm{mix}(h(|\mathfrak{A}|))$.

(2) *Let $g \in \mathbf{V}^{(B)}$ and* $\mathbf{V}^{(B)} \models$ '$g : \mathfrak{A}^{\sim} \to \mathcal{B}^{\sim}$ *is a homomorphism of algebraic B-systems*'. *If in this case $\mathcal{B}$ is an extended algebraic B-system, then there is a unique homomorphism $h : \mathfrak{A} \to \mathcal{B}$ such that $g = h^{\sim}$.*

◁ (1) If $h' := h^{\sim}\downarrow$, $\mathfrak{A}' := \mathfrak{A}^{\sim}\downarrow$, $\mathcal{B}' := \mathcal{B}^{\sim}\downarrow$, $i := \iota_{|\mathfrak{A}|}$ and $j := \iota_{|\mathcal{B}|}$, then $h'\circ i = j\circ h$

(see **3.5.4** (3)). Let us show that h is a homomorphism iff h' is a homomorphism. In this case we shall limit ourselves to substantiating **4.1.10** (3) with $n = 1$. In other words, we must show that h and h' either preserve or do not preserve unary operations simultaneously. Let $v, \lambda, \mu(v)$ and $\mu(\lambda)$ be interpretations of the systems $\mathfrak{A}, \mathfrak{B}, \mathfrak{A}^\sim$ and $\mathfrak{B}^\sim$. If h is a homomorphism, then $h \circ f^v = f^\lambda \circ h$. Besides, $i \circ f^v = (f^{\mu(v)}\downarrow) \circ i$ and $j \circ f^\lambda = (f^{\mu(\lambda)}\downarrow) \circ j$ and, hence,

$$h' \circ (f^{\mu(v)}\downarrow) \circ i = j \circ h \circ f^v = j \circ f^\lambda \circ h = (f^{\mu(\lambda)}\downarrow) \circ h' \circ i.$$

Making use of the relation $|\mathfrak{A}^\sim\downarrow| = \mathrm{mix}(i(|\mathfrak{A}|))$, we get $h' \circ (f^{\mu(v)}\downarrow) = (f^{\mu(\lambda)}\downarrow) \circ h'$. Conversely, if the last equality holds, then, reasoning in the opposite direction, we find $h \circ f^v = f^\lambda \circ h$. The case of arbitrary operations, as well as that of arbitrary predicates is more cumbersome but causes no principal difficulties. Therefore, h is a homomorphism, a strong homomorphism or an isomorphism between the algebraic B-systems $\mathfrak{A}$ and $\mathfrak{B}$ iff the mapping h' from $\mathfrak{A}'$ to $\mathfrak{B}'$ has the corresponding property. Therefore, the required property results from **4.2.5** and **4.3.3**. ▷

4.3.5. Let us note some corollaries to theorems **4.3.3** and **4.3.4**.

(1) **Theorem.** *If $\mathfrak{A}$ is an algebraic system of finite signature σ, then $\mathbf{V}^{(B)} \models {}'\mathfrak{A}^\wedge$ is an algebraic system of signature $\sigma^{\wedge}{}'$. In this case for any formula of signature φ with n free variables we have*

$$\mathfrak{A} \models \varphi(a_0,\dots,a_{n-1}) \leftrightarrow [\mathfrak{A}^\wedge \models \varphi(a_0^\wedge,\dots,a_{n-1}^\wedge)] = 1,$$

for all $a_0,\dots,a_{n-1} \in A$.

◁ To prove the theorem, we only have to remark that if $\mathfrak{A} := (A, f_0, \dots, f_{k-1}, p_0, \dots, p_{m-1})$, then the statement $\mathfrak{A} \models \varphi(a_0,\dots,a_{n-1})$ is written as a bounded formula of set theory $\Psi(A^\wedge, f_0^\wedge, \dots, f_{m-1}^\wedge, a_0^\wedge, \dots, a_{n-1}^\wedge)$ and refer to **2.2.9**. ▷

(2) **Theorem.** *For any algebraic B-system $\mathfrak{A}$ there is an extended algebraic B-system $\mathfrak{A}'$ of signature $\sigma(\mathfrak{A})$ and an isomorphism ι from $\mathfrak{A}$ to $\mathfrak{A}'$ such that*

(a) $|\mathfrak{A}'| = \mathrm{mix}(\iota(|\mathfrak{A}|))$;

(b) *if h is a homomorphism from $\mathfrak{A}$ to the extended algebraic B-system $\mathfrak{B}$, then there is a unique homomorphism $h' : \mathfrak{A}' \to \mathfrak{B}$ such that $h' \circ \iota = h$;*

(c) *if $\mathfrak{A}''$ is an extended algebraic B-system, and the isomorphism $\iota' : \mathfrak{A} \to \mathfrak{A}''$ obeys condition (a) (with $\mathfrak{A}'$ substituted for $\mathfrak{A}''$), then there is a unique isomorphism h from $\mathfrak{A}'$ on $\mathfrak{A}''$ such that $h \circ \iota = \iota'$.*

◁ Let $(\mathcal{A}, \mu)$ be a Boolean-valued realization of the algebraic B-system $\mathfrak{A}$. Then the descent $\mathfrak{A}' := (\mathcal{A}, \mu)\downarrow$ obeys all the required conditions. Indeed, by virtue of **4.3.3** (3,4) the canonical embedding $\iota := \iota_{|\mathfrak{A}|}$ is an isomorphism, in which case condition (a) is fulfilled. If h and $\mathfrak{B}$ are the same as in (b), then by theorem **4.3.4**, $g := h^{\sim}\downarrow$ is a homomorphism from $\mathfrak{A}'$ to $\mathfrak{B}' := \mathfrak{B}^{\sim}\downarrow$. Since $\mathfrak{B}$ is extended, the canonical mapping $j := \iota_{|\mathfrak{B}|}$ is an isomorphism 'onto'. It is obvious that $h' := j^{-1} \circ g$ is the sought homomorphism. It is expedient to remark that if $a \in |\mathfrak{A}'|$ and $a = \text{mix}(b_\xi \iota(a_\xi))$, then $h'(a) = \text{mix}(b_\xi h \circ \iota(a_\xi))$. Statement (c) results from (a) and theorem **4.3.4**. ▷

Any pair $(\mathfrak{A}', \iota)$ where $\mathfrak{A}'$ is an extended algebraic B-system, while ι is an isomorphism from $\mathfrak{A}$ to $\mathfrak{A}'$ obeying condition (a) of theorem (2), can be naturally called a *maximal extension* of $\mathfrak{A}$. Then theorem (2) yields the following statement.

(3) *Any algebraic B-system has a maximal extension unique up to isomorphism.*

Let us choose a full homomorphism π from B to a complete Boolean algebra C. Let $\mathfrak{A} := (A, f_0, \dots, f_{k-1}, p_0, \dots, p_{m-1})$ is an algebraic system of finite signature inside $\mathbf{V}^{(B)}$. Let us denote

$$\pi^*(\mathfrak{A}) := (\pi^*(A), \pi^*(f_0), \dots, \pi^*(p_{m-1}))^C, \quad \pi^*(\mathfrak{A}) \in \mathbf{V}^{(C)},$$

where $\pi^* : \mathbf{V}^{(B)} \to \mathbf{V}^{(C)}$ is a mapping associated with π (see **2.2**).

(4) **Theorem.** *An element $\pi^*(\mathfrak{A})$ is an algebraic system of finite signature $\sigma(\mathfrak{A})$ inside $\mathbf{V}^{(C)}$. The mapping $a \to \pi^*(a) (a \in A\downarrow)$ is a homomorphism from $\mathfrak{A}\downarrow$ to $\pi^*(\mathfrak{A})\downarrow$. For any formula φ of signature $\sigma(\mathfrak{A})$ with n free variables and for arbitrary $a_0, \dots, a_{n-1} \in |\mathfrak{A}\downarrow|$ the following formula is fulfilled*

$$\mathfrak{A}\downarrow \models \varphi(a_0, \dots, a_{n-1}) \to \pi^*(\mathfrak{A})\downarrow \models \varphi(\pi^*(a_0), \dots, \pi^*(a_{n-1})).$$

In particular, if $\mathfrak{B}$ is an algebraic B-system of finite signature and $\mathfrak{A} = \mathfrak{B}^{\sim}$, then for $a_0, \dots, a_{n-1} \in |\mathfrak{B}|$ we have

$$\mathfrak{B} \models \varphi(a_0, \dots, a_{n-1}) \to \pi^*(\mathfrak{A})\downarrow \models \varphi(\pi^* \circ \iota(a_0), \dots, \pi^* \circ \iota(a_{n-1}))$$

where $\iota := \iota_{|\mathfrak{B}|}$. If π is an isomorphism, then π^* is an isomorphism from $\mathfrak{A}\downarrow$ to

$\pi^*(\mathfrak{A})\downarrow$ and in the above formulas the reverse implication is also valid. If π is an isomorphism, then π^* is an isomorphism of algebraic B-systems.

◁ To prove this fact we must combine **2.2.4**, **2.2.5**, **4.1.10** and **4.2.5** and make use of the considerations of (1). ▷

(5) *For any algebraic system $\mathfrak{A}$ inside $\mathbf{V}^{(B)}$ we have $[\mathfrak{A}\downarrow^{\sim}$ is isomorphic to $\mathfrak{A}] = 1$.*

(6) Theorem. *The Boolean-valued realization $(\mathcal{A},\mu,\delta)$ of an algebraic B-system with disjointness (A,v,Δ) is an algebraic system with simple disjointness inside $\mathbf{V}^{(B)}$. If $(A',v') := (\mathcal{A},\mu)\downarrow$ and $\Delta' := \{(x,y)\in A'\times A' : \delta\downarrow(x,y) = 1\}$, then (A',v',Δ') is an extended algebraic B-system with disjointness, and for any $x,y\in A$ the following equivalences are valid:*

$$x\perp y \leftrightarrow \iota x\perp \iota y \leftrightarrow [\iota x = \theta \vee \iota y = \theta] = 1,$$

where $\iota = \iota_A : A \to A'$ is the canonical embedding.

◁ It suffices to use **4.1.13** and **4.3.3**. ▷

4.3.6. Let us now consider in more detail the important problem mentioned in **4.2.6**. Choose an algebraic B-system $\mathfrak{A}$ of signature σ. For the formula φ of the same signature and elements $a_0,\dots,a_{n-1}\in|\mathfrak{A}|$ we shall temporarily employ a more informative presentation $\mathfrak{A}\models_B \varphi(a_0,\dots,a_{n-1})$ instead of $\mathfrak{A}\models \varphi(a_0,\dots,a_{n-1})$. Using the B-system $\mathfrak{A}$ let us evolve the two-valued algebraic system $\bar{\mathfrak{A}}$ through the procedure of purification described in **4.1.3.** We can say that $\varphi(a_0,\dots,a_{n-1})$ is assertive both in $\mathfrak{A}$ and $\bar{\mathfrak{A}}$ as $|\mathfrak{A}|=|\bar{\mathfrak{A}}|$ and $\sigma(\bar{\mathfrak{A}}) = \sigma$. There arises a natural question about an interrelation between the statements $\mathfrak{A}\models_B \varphi(a_0,\dots,a_{n-1})$ and $\bar{\mathfrak{A}}\models \varphi(a_0,\dots,a_{n-1})$. Theorems **4.2.7** and **4.2.8** provide examples of such formulas φ for which $\bar{\mathfrak{A}}\models\varphi$ results from $\mathfrak{A}\models_B\varphi$. On the other hand, we can easily give an example violating this implication. Indeed, let $B := \mathcal{P}([0,1])$ and $A := \mathbf{R}^{[0,1]}$ be the set of all real functions on the interval $[0,1]$ with the B-metric

$$d(f,g) := \{t\in[0,1] : f(t)\neq g(t)\}\quad (f,g\in A).$$

Let us introduce a B-valued binary predicate $[\cdot\leq\cdot]$ on A by the formula

$$[f\leq g] = \{t\in[0,1] : f(t)\leq g(t)\}\quad (f,g\in A).$$

Then $\mathfrak{A} := (A, [\cdot \le \cdot])$ is an algebraic B-system and $\mathfrak{A} \models_B \varphi$, where $\varphi := (\forall x)(\forall y)(x \le y \vee y \le x)$. Besides, it is obvious that $\overline{\mathfrak{A}} := (A, \le)$ is a purification of $\mathfrak{A}$ if we set

$$f \le g \leftrightarrow (\forall t \in [0,1])\ f(t) \le g(t).$$

Obviously, $\overline{\mathfrak{A}} \models \neg\varphi$. Therefore, if $\mathfrak{T}^B(\mathfrak{A})$ and $\mathfrak{T}(\overline{\mathfrak{A}})$ are sets of all the formulas (with the constants ranging over $|\mathfrak{A}|$) true in the systems $\mathfrak{A}$ and $\overline{\mathfrak{A}}$, respectively, then none of these two sets is not, generally speaking, a subset of the other. It stands to reason, therefore, to expect that for a certain class Φ of formulas of signature σ there exist only relations of the type $\mathfrak{T}^B(\mathfrak{A}) \cap \Phi (?) \mathfrak{T}(\overline{\mathfrak{A}}) \cap \Phi$. For exact formulations a definite syntactic analysis of texts is needed.

4.3.7. **(1)** Let us introduce a class of generic and strictly generic formulas. The definition is given by recursion over the length of a formula. The rules of formulation are as follows.

(a) Any atomic formula is strictly generic.

(b) If φ and ψ are strictly generic formulas, then $\varphi \wedge \psi$, $(\exists x)\varphi$, $(\forall x)\varphi$ are also strictly generic.

(c) A strictly generic formula is generic.

(d) If φ and ψ are generic formulas, then $\varphi \wedge \psi$, $(\exists x)\varphi$, $(\forall x)\varphi$ are also generic.

(e) If φ is a strictly generic formula, then $\neg\varphi$ is a generic formula.

(f) If φ is a strictly generic formula, and ψ is a generic formula, then $\varphi \to \psi$ is a generic formula.

(2) *A basic Horn formula* is a disjunction $\theta_1 \wedge \ldots \wedge \theta_n$, where at most one of the formulas θ_k is basic, while the rest formulas are negations of atomic formulas. A formula is termed *a Horn one* if it is built of basic Horn formulas with the connectives $\wedge, \exists$ and $\forall$.

(3) *Any generic formula is logically equivalent to a Horn formula and vice versa.*

4.3.8. Examples.

(**1**) Let φ be a formula of signature $\{\leq\}$ with the only predicate symbol. If φ are axioms of a lattice-ordered set (= lattice; see **1.1.1**), then φ is a generic formula. In the signature in question distributivity is not written as a generic formula. If, however, we choose a signature $\sigma := \{\wedge,\vee\}$, where $\wedge$ and $\vee$ are binary functional symbols, then the formula $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ is atomic and, hence, strictly generic. Moreover, the property of being a distributive lattice is a strictly generic formula of signature $\{\wedge,\vee\}$.

(**2**) Let us choose formulas φ and ψ of signature $\{\wedge,\vee,*,0,1\}$. Let φ be axioms of a Boolean algebra (see **1.1.2**), while $\psi :=$' there exists at least one atom', i.e.,

$$\psi := (\exists x)(\forall y)(x \neq 0 \wedge x \wedge y = y \rightarrow x = y \vee y = 0).$$

Then φ is a strictly generic formula, then ψ is not generic.

(**3**) Let $\sigma := \{+,0\}$, where $+$ is a binary functional symbol and 0 is a constant symbol. If φ are the axioms of a group (associativity of a group operation, the axiom of zero, existence of an inverse element), then φ is a strictly generic formula of signature σ.

(**4**) Let $\sigma := \{+,\cdot,0,1\}$, where + and $\cdot$ are binary functional symbols, and 0 and 1 are constant symbols. Let φ be the axioms of a ring, while ψ be the axioms of an integral domain, i.e., $\psi := \varphi \wedge \theta$, where

$$\theta := (\forall x)(\forall y)(x \cdot y = 0 \rightarrow x = 0 \vee y = 0).$$

Then φ is a strictly generic formula, while ψ is a generic formula.

4.3.9. Theorem. *Let $\mathfrak{A}$ be an extended algebraic B-system, while φ be a formula of signature $\sigma(\mathfrak{A})$.*

(**a**) *if φ is strictly generic, then*

$$\mathfrak{A} \models_B \varphi(a_0,\ldots,a_{n-1}) \leftrightarrow \overline{\mathfrak{A}} \models \varphi(a_0,\ldots,a_{n-1})\ (a_0,\ldots,a_{n-1} \in |\mathfrak{A}|).$$

(**b**) *if φ is generic, then*

$$\mathfrak{A} \models_B \varphi(a_0,\ldots,a_{n-1}) \rightarrow \overline{\mathfrak{A}} \models \varphi(a_0,\ldots,a_{n-1})\ (a_0,\ldots,a_{n-1} \in |\overline{\mathfrak{A}}|).$$

$\lhd$ The proof is carried out by induction on length of the formula φ. In relation with theorem **4.3.3**, one can assume $\mathfrak{A} = \mathcal{A}\downarrow$, where $\mathcal{A}$ is an algebraic system of signature $\sigma^\wedge$

inside $\mathbf{V}^{(B)}$.

(a) If φ is an atomic formula, then the statement directly follows from the definition of purification, since for a predicate symbol $p = \sigma(\mathfrak{A})$, $\varkappa(p) = n$ we have

$$p^{v}(a_0,\dots,a_{n-1}) = 1 \leftrightarrow (a_0,\dots,a_{n-1}) \in \bar{v}(p)\ (a_0,\dots,a_{n-1} \mathrel{\varepsilon} |\mathfrak{A}|).$$

For the conjunction $\varphi := \psi \wedge \theta$ we have, making use of definition **4.1.8** and the induction hypothesis,

$$|\psi \wedge \theta|^{\mathfrak{A}} = 1 \leftrightarrow |\psi|^{\mathfrak{A}} = 1 \wedge |\theta|^{\mathfrak{A}} = 1 \leftrightarrow \bar{\mathfrak{A}} \models \psi \wedge \bar{\mathfrak{A}} \models \theta \leftrightarrow \bar{\mathfrak{A}} \models \psi \wedge \theta.$$

Analogous is the case of the universal quantifier $\varphi := (\forall x)\,\psi$:

$$|(\forall x)\varphi|^{\mathfrak{A}} = 1 \leftrightarrow (\forall a \mathrel{\varepsilon} |\mathfrak{A}|)\,|\psi(a)|^{\mathfrak{A}} = 1$$
$$\leftrightarrow (\forall a \mathrel{\varepsilon} |\mathfrak{A}|)\ \bar{\mathfrak{A}} \models \psi(a) \leftrightarrow \bar{\mathfrak{A}} \models (\forall x)\psi.$$

Let us consider the case of the existential quantifier $\varphi := (\exists x)\,\psi$. By virtue of the maximum principle, there is an element $z \in \mathbf{V}^{(B)}$ such that

$$[\mathcal{A} \models (\exists x)\psi\,] = [z \mathrel{\varepsilon} |\mathcal{A}| \wedge \mathcal{A} \models \psi(z)].$$

According to theorem **4.3.3**, this formula can be rewritten as

$$[z \mathrel{\varepsilon} |\mathcal{A}|] \wedge |\psi(z)|^{\mathfrak{A}} = |(\exists x)\,\psi|^{\mathfrak{A}}.$$

Hence, in line with the induction hypothesis, we can deduce that the following equivalences are valid:

$$|(\exists x)\psi|^{\mathfrak{A}} = 1 \leftrightarrow (\exists z \in |\mathfrak{A}|)\,|\psi(z)|^{\mathfrak{A}} = 1$$
$$\leftrightarrow (\exists z \in |\bar{\mathfrak{A}}|)\ \bar{\mathfrak{A}} \models \psi(z) \leftrightarrow \bar{\mathfrak{A}} \models (\exists x)\psi,$$

since, by definition **4.2.3** $|\mathfrak{A}| = |\mathcal{A}|{\downarrow}$. Therefore, in each of the cases considered the induction step is realizable and the proof is completed by referring to items (a) and (b) of definition **4.3.7** (1).

(b) The case of the connectives $\wedge$, $\exists$ and $\forall$ is considered in the same way as above in (a). Let $\varphi := \neg\,\psi$, where ψ is a strictly generic formula. If $\varphi|^{\mathfrak{A}} = 1$, then $|\varphi|^{\mathfrak{A}} = \mathbf{0}$ and, by virtue of what has been proved in (a), φ cannot be true in $\bar{\mathfrak{A}}$. In this case, however,

$\overline{\mathfrak{A}} \models \varphi$. And finally, let us consider a formula of the type $\varphi := \theta \to \psi$, where θ is a strictly generic formula, while ψ is a generic formula. Let us assume that $|\theta \to \psi|^{\mathfrak{A}} = \mathbf{1}$. If $\overline{\mathfrak{A}} \models \theta$, then the facts proved in (a) yield $|\theta|^{\mathfrak{A}} = \mathbf{1}$ and, hence, $|\psi|^{\mathfrak{A}} = \mathbf{1}$. By the induction hypothesis we get $\overline{\mathfrak{A}} \models \theta \to \psi$. ▷

It should be remarked that the Jech theorem makes it possible to replace the proofs of some fragments of theorems **4.2.7** - **4.2.9** with a syntactical analysis of corresponding sentences. It goes without saying that a general fact of the kind can also be formulated.

(2) **Corollary.** *Let $\mathcal{A}$ and $\overline{\mathfrak{A}}$ be some Boolean-valued realization and purification of an extended algebraic B-system, respectively. For any Horn sentence φ we have*

$$[\mathcal{A} \models \varphi] = \mathbf{1} \to \overline{\mathfrak{A}} \models \varphi .$$

4.3.10. Let Φ be a certain set of formulas of the same signature σ. Let us introduce the category $\mathrm{AS}^{(B)}(\Phi)$ in the following way:

$$\mathrm{ObAS}^{(B)}(\Phi) := \{\mathfrak{A} \in \mathbf{V}^{(B)} : [\mathfrak{A} \text{ is an algebraic system of signature } \sigma^{\wedge} \text{ and } \mathfrak{A} \models \Phi] = \mathbf{1}\};$$

$$\mathrm{AS}^{(B)}(\mathfrak{A}, \mathcal{B}) := \{h \in \mathbf{V}^{(B)} : [h \text{ is a homomorphism from } \mathfrak{A} \text{ to } \mathcal{B}] = \mathbf{1}\};$$
$$\mathrm{Com}(f, g) = h \leftrightarrow [h = g \circ f] = \mathbf{1}.$$

The fact that the above conditions do determine a category results from the principle of transfer, that of maximum, theorem **4.3.2**, as well as from the other properties of the embedding functor. As before, by the symbols $\mathfrak{F}^{\sim}$ and $\mathfrak{F}^{\downarrow}$ we shall denote, respectively, the mappings of the embedding and descent, which operate in the categories of algebraic systems: $\mathfrak{F}^{\sim} : B - \mathrm{AS}(\Phi) \to \mathrm{AS}^{(B)}(\Phi)$, $\mathfrak{F}^{\downarrow} : \mathrm{AS}^{(B)}(\Phi) \to B - \mathrm{AS}(\Phi)$.

Theorem. *The following statements are valid:*

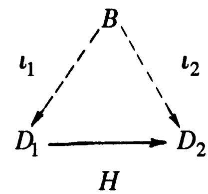
(1) *the mapping $\mathfrak{F}^{\downarrow}$ is a covariant functor from the category $\mathrm{AS}^{(B)}(\Phi)$ into the category $B - \mathrm{CAS}^{(B)}(\Phi)$;*

(2) *the mapping $\mathfrak{F}^{\sim}$ is a covariant functor from the category $B - \mathrm{AS}(\Phi)$ (as well as from $B - \mathrm{CAS}(\Phi)$) into the category $\mathrm{AS}^{(B)}(\Phi)$;*

(3) *the functors* $\mathfrak{F}^{\downarrow}$ *and* $\mathfrak{F}^{\sim}$ *establish the equivalence of the categories* $\mathrm{AS}^{(B)}(\Phi)$ *and* $B-\mathrm{CAS}(\Phi)$.

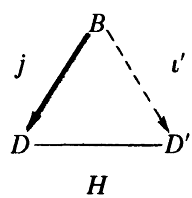
Let us now note some corollaries for rings with projections and Boolean algebras whose proofs are in essence contained in **4.2.5**, **4.2.7**, **4.2.8** and **4.3.2.** We consider rings with projections K_1 and K_2, and let j_1 and j_2 be isomorphisms of B on the Boolean algebras of projections in K_1 and K_2, respectively. The homomorphism $h:K_1 \to K_2$ will be termed *B-homogeneous* if $h \circ j_1(b) = j_2(b) \circ h$ $(b \in B)$. We shall also say that *K_l is a ring with Boolean algebra of projections B and that h commutes with projections of B.*

4.3.11. (1) **Theorem.** *Let* $\mathfrak{D}_1$ *and* $\mathfrak{D}_2$ *be complete Boolean algebras inside* $\mathbf{V}^{(B)}$. *Let us set* $D_k := \mathfrak{D}_k \downarrow$ *and let* $\iota_k : B \to D_k$ *be a canonical monomorphism for* $k = 1,2$ *(see* **4.2.9***). If* $h \in \mathbf{V}^{(B)}$ *is an isomorphism of* $\mathfrak{D}_1$ *on* $\mathfrak{D}_2$ *inside* $\mathbf{V}^{(B)}$, *then there is an isomorphism H of the algebra* D_1 *on* D_2, *for which the following diagram is commutative:*



Conversely, if $H:D_1 \to D_2$ *is an isomorphism of Boolean algebras such that the above diagram is commutative, then the algebras* $\mathfrak{D}_1$ *and* $\mathfrak{D}_2$ *are isomorphic inside* $\mathbf{V}^{(B)}$.

(2) **Theorem.** *Let D be a complete algebra and* $j: B \to D$ *be a full monomorphism. Then there is a complete Boolean algebra* $\mathfrak{D}$ *inside* $\mathbf{V}^{(B)}$, *and an isomorphism H from D on* $D' := \mathfrak{D} \downarrow$ *such that the following diagram is commutative:*



where ι' *is the canonical monomorphism from B to* D'.

4.3.12. (1) **Theorem.** *Let* $\mathfrak{R}_1$ *and* $\mathfrak{R}_2$ *be rings with algebra of projections* $\mathfrak{D}$ *inside* $\mathbf{V}^{(B)}$. *Assign* $D := \mathfrak{D} \downarrow$, $K_l := \mathfrak{R}_l \downarrow$ *and* $l := 1,2$. *Then* K_1 *and* K_2 *are rings with algebra of projections D, and if inside* $\mathbf{V}^{(B)}$ *it is true that h is a homomorphism from the ring* $\mathfrak{R}_1$ *to* *the ring* $\mathfrak{K}_2$ *commuting with projections of* $\mathfrak{D}$, *then* $h\downarrow$ *is a homomorphism of the ring* K_1 *to the ring* K_2 *commuting with projections of* D. *If* h *is an isomorphism of* $\mathfrak{K}_1$ *on* $\mathfrak{K}_2$, *then* $h\downarrow$ *is an isomorphism of* K_1 *on* K_2.

(2) **Theorem.** *Let* (K,D) *be a ring with projections and let* $j: B \to D$ *be a full homomorphism. Then there is a ring with projections* $(\mathfrak{K},\mathfrak{D})$ *inside* $\mathbf{V}^{(B)}$, *and an isomorphism* h *of the ring* K *to the ring* $K' := \mathfrak{K}\downarrow$ *such that for any* $b \in B$ *the following diagram is commutative:*

$$\begin{array}{ccc} & h & \\ K & \dashrightarrow & K' \\ j(b)\ \downarrow & & \downarrow\ \iota'(b) \\ K & \longrightarrow & K' \\ & h & \end{array}$$

where ι' *is the canonical monomorphism from* B *to* D'.

Analogous results are also valid for groups with projections.

4.3.13. Remarks

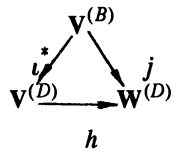
(1) Let C and D be Boolean algebras, while P and Q be their Stone spaces. Determine the tensor product $C \otimes D$ of the algebras C and D as the Boolean algebra of clopen subsets of the Cartesian product $P \times Q$ (see **1.1.6** (6) and **1.2.6** (8)). Let $C \hat{\oplus} D$ be the completion of the Boolean algebra $C \otimes D$ (see **1.1.6** (7) and **1.2.6.** (9)). If D is a Boolean algebra, and an element $\mathfrak{D} \in \mathbf{V}^{(B)}$ is such that $\mathbf{V}^{(B)} \models$ "$\mathfrak{D}$ is the completion of the Boolean algebra $D^{\wedge}$", then the algebras $\mathfrak{D}\downarrow$ and $B \hat{\oplus} D$ are isomorphic (see **[236]**).

(2) The Solovay-Tennenbaum theorems (see **4.3.11**) can form the cornerstone of iterating the construction of a Boolean-valued model. Let $\mathfrak{D} \in \mathbf{V}^{(B)}$ and $\mathbf{V}^{(B)} \models$ "$\mathfrak{D}$ is a complete Boolean algebra". According to scheme **2.1**, inside $\mathbf{V}^{(B)}$ it is possible to construct $\mathbf{V}^{(B)}$-classes, i.e., the Boolean-valued universe $(\mathbf{V}^{(B)})^{\mathfrak{D}}$, corresponding Boolean truth-values $[\cdot = \cdot]^{\mathfrak{D}}$ and $[\cdot \in \cdot]^{\mathfrak{D}}$, as well as the canonical embedding $(\cdot)^{\wedge}$ of the universal class $\mathbf{U}_B$ in $(\mathbf{V}^{(B)})^{\mathfrak{D}}$. Let us set $D := \mathfrak{D}\downarrow$, $\mathbf{W}^{(D)} := (\mathbf{V}^{(B)})^{\mathfrak{D}}\downarrow$, $[\cdot = \cdot]^{D} := \left([\cdot = \cdot]^{\mathfrak{D}}\right)\downarrow$, $[\cdot \in \cdot]^{D} := \left([\cdot \in \cdot]^{\mathfrak{D}}\right)\downarrow$, $j := (\cdot)^{\wedge}\downarrow$. Let $\iota: B \to D$ be the canonical monomorphism, while $\iota^{*}: \mathbf{V}^{(B)} \to \mathbf{V}^{(D)}$ be the corresponding injection (see **2.2**).

Then there is a unique bijection $h: \mathbf{V}^{(D)} \to \mathbf{W}^{(D)}$ such that

$$[x = y]^D = [h(x) = h(y)]^{\mathfrak{D}}, \ [x \in y]^D = [h(x) \in h(y)]^{\mathfrak{D}},$$

whatever x and $y \in \mathbf{V}^{(D)}$ might be. In this case the diagram



is commutative (for details see [**236**]).

(**3**) Further iterations of the construction presented above result in a transfinite sequence of Boolean-valued extensions. In this way there appears an efficient method, iterated forcing, which has been used to establish relative consistency of the Suslin hypothesis with ZFC (see **236**]).

4.4. Ordered Algebraic Systems

A complete Boolean algebra of congruences necessary for Boolean-valued realization of an algebraic system is often generated by a relation of order. This peculiarity brings about the possibility of the Boolean-valued realization for ordered algebraic systems. In the next section we shall present some results in this direction. All the necessary additional information can be found in [**13, 110, 12, 59**].

4.4.1. *An ordered group* is an algebraic system $(G,+,0,\leq)$ for which the following conditions are met:

(**1**) $(G,+,0)$ is a group;

(**2**) $(G,\leq)$ is a (partially) ordered set;

(**3**) the group structure and the order structure are compatible, which means that group translations are isotonic mappings, i.e., G is a model for

$$(\forall x)(\forall y)(\forall a)(\forall b)\quad (x \le y \leftrightarrow a + x + b \le a + y + b).$$

(An additive presentation of the group operation does not imply that it is commutative). We say that G is *a linearly ordered group* in the case when in addition to (**1**) - (**3**) the following condition is also met:

(**4**) $(G,\le)$ is a linearly ordered set, i.e., the formula $(\forall x)(\forall y)\quad (x \le y \vee y \le x)$ holds on G. The element $x \in G$ is termed *positive* if $x \ge 0$. A set of all positive elements is called *the positive cone* and denoted by G^+. A subset K of the group G is a positive cone of some order on G provided the following conditions are met:

(a) $K \cap (-K) = \{0\}$;

(b) $K + K = K$;

(c) $x + K = K + x \quad (x \in G)$.

In this case the cone K and the order corresponding to it are related as follows:

$$x \le y \leftrightarrow y - x \in K \leftrightarrow -x + y \in K.$$

The group G is linearly ordered iff

(d) $G = G^+ \cup (-G^+)$.

The cone of positive elements is called *reproducing* provided $G = G^+ - G^+$. When this condition is fulfilled, G is also said to be *a directed group.* By definition, an ordered group G is *integrably-closed (Archimedean)* iff for any $x, y \in G$ it follows from the inequalities $nx \le y$, $n \in \omega$ (respectively, $nx \le y$, $\pm n \in \omega$) that $x \le 0$ (respectively, $x = 0$). The homomorphism $h: G \to G'$ of ordered groups is positive if $h(x) \ge 0$ for every $0 \le x \in G$.

4.4.2. *A lattice-ordered group* is an ordered group G, in which every nonempty finite set $\{x_0,\dots,x_{n-1}\} \subset G$ has a least upper bound $x_0 \vee \dots \vee x_{n-1} := \sup\{x_0,\dots,x_{n-1}\}$ and a greatest lower bound $x_0 \wedge \dots \wedge x_{n-1} := \inf\{x_0,\dots,x_{n-1}\}$. For any element x of the lattice-ordered group G determined are elements $|x| := x \vee (-x)$, $x^+ := x \vee 0$ and $x^- := (-x)^+ = -x \wedge 0$ called, respectively, *the modulus, the positive part and the negative part* of x. In any lattice-ordered group the following relations hold:

(**1**) $x = x^+ - x^-$, $|x| = x^+ + x^-$, $x^+ \wedge x^- = 0$;

(**2**) $(x + y)^+ \le x^+ + y^+$, $(x + y)^- \le x^- + y^-$;

(3) $(nx)^+ = nx^+$, $(nx)^- = nx^-$, $|nx| = n|x| (n \in \omega)$;

(4) $|x+y| \le |x| + |y| + |x|$;

(5) $|x+y-x| = x + |y| - x$; $(x+y-x)^- = x + y^- - x$;

(6) $u \wedge x = 0 \wedge u \wedge y = 0 \rightarrow u \wedge (x+y) = 0$.

The lattice-ordered group G is commutative iff instead of (4) we have $|x+y| \le |x| + |y|$ for all $x, y \in G$. Among the other properties of the group G it should be recalled that it is torsion-free, is a distributive lattice and the following relations are valid:

$$a + (\vee x_a) + b = \vee(a + x_a + b),$$
$$a + (\wedge x_a) + b = \wedge(a + x_a + b).$$

A subgroup G_0 of a lattice-ordered group is called an *o-ideal* (or *a convex subgroup*) if for any x and y it follows from $|x| \le |y|$ and $y \in G_0$ that $x \in G_0$. If, moreover, the subgroup G_0 is normal, than it is termed an *l-ideal.*

4.4.3. Let us from now on assume G to be a lattice-ordered group and introduce in it the disjointness $\perp$ by the following formula:

$$\perp := \{(x,y) \in G \times G : |x| \wedge |y| = 0\}.$$

There is no doubt that $\perp$ obeys all the axioms of disjointness of **4.1.2.**(2). The complete Boolean algebra of $\perp$-components $\mathfrak{K}_\perp(G)$ is called *the base* of G and is denoted by $\mathfrak{B}(G)$. Let us assume that a component $K \in \mathfrak{B}(G)$ is singled out by a direct addent of the group G. Then the corresponding projection π_K is a positive endomorphism in G, in which case $\pi_K x \le x$ for all $0 \le x \in G$. If any component in K is singled out by a direct addent, then the set $\mathfrak{Pr}(G)$ of all projections of the kind $\pi_K (K \in \mathfrak{B}(G))$ is a complete Boolean algebra isomorphic to $\mathfrak{B}(G)$. Under these circumstances G is said to be *a group with projections on components*or a *group with band projections.* A lattice-ordered group G' with projections on components is said to be *extended* or *orthogonally complete* or *laterally complete* provided it is extended relative to the algebra of projections $\mathfrak{Pr}(G)$. The *maximal or universal extension of the lattice-ordered group G* is an extended lattice-ordered group G' combined with an o-isomorphism $\iota : G \rightarrow G'$ such that $G' = \mathrm{mix}(\iota(G))$ and for any $0 < x' \in G'$ we can be find $0 < x \in G$, $\iota(x) \le x'$ (here mix is calculated relative to the Boolean algebra $\mathfrak{Pr}(G)$).

It should be recalled that $[x]$ denotes the least component that contains x. The properties listed in **4.4.2** yield the following conclusions.

(1) *The following relations are valid:*

$$[x+y]=[x\vee y]=[x]\vee[y]\quad (x,y\in G^{+});$$
$$[x]=[|x|]=[x^{+}]\vee[x^{-}]\quad (x\in G);$$
$$[x+y-x]=x+[y]-x\quad (x,y\in G);$$
$$x\perp y\rightarrow x+y=y+x\quad (x,y\in G).$$

(2) *Any component* $K\in\mathfrak{B}(G)$ *is an* o*-ideal.*

⊲ Indeed, if x and $y\in A^{\perp}$ for some $A\subset G$, then, by virtue of the second relation of **(1)** and **4.4.2** (4), we can write

$$\{x+y\}^{\perp}\supset\{x\}^{\perp}\wedge\{y\}^{\perp}\wedge\{x\}^{\perp}\supset A,$$

and, hence, $x+y\in\{x+y)^{\perp\perp}\subset A^{\perp}$. We thus have established that if $y\in A^{\perp}$ and $|x|\leq|y|$, then $\{x\}^{\perp}\supset\{y\}^{\perp}\supset A$ and, hence, $x\in\{x\}^{\perp\perp}\subset A^{\perp}$, which completes the proof. ⊳

4.4.4. If the group G is not commutative, then the components in it are not necessarily normal subgroups, i.e., they are not, generally speaking, l-ideals. Therefore, the following notion can be introduced. The component $K\in\mathfrak{B}(G)$ is called *invariant* if $x+K-x\subset K$ for any $x\in G$. By virtue of **4.4.3** (2) it is equivalent to the fact that K is an l-ideal. The set of all invariant components will be denoted by the symbol $\mathfrak{B}_i(G)$.

(1) *The set of all invariant components* $\mathfrak{B}_i(G)$ *is a regular subalgebra of the Boolean algebra of all the components.*

⊲ It is obvious that the intersection of any set of invariant components will be an invariant component. It is therefore sufficient to prove that the disjoint complement of each invariant component is an invariant component. Let us choose $K\in\mathfrak{B}_i(G)$ and $x\in K^{\perp}$. Then for any $y\in K$ and $a\in G$ we have $0=(a+|y|-a)\wedge|x|=-a+(a+|y|-a)\wedge|x|+a=$ $=|y|\wedge(-a+|x|+a)$ and, hence, $-a+|x|+a\in K^{\perp}$, which is the proof that the component $K^{\perp}$ is invariant. ⊳

(2) *The following statements are valid for a lattice-ordered group* G:

(a) *any component is invariant, i.e.,* $\mathfrak{B}(G)=\mathfrak{B}_i(G)$;

(b) *for any* $x,y\in G$ *the following equality holds:*

$$\{x\}^{\perp} = y + \{x\}^{\perp} - y;$$

(c) *if an element $x \in G$ is disjoint from any of its conjugates $y + x - y$, then $x = 0$.*

$\lhd$ The condition (b) is an obvious corollary to (a). Let us assume that (b) is fulfilled and $x^{\perp}(y + x - y)$ for some x and $y \in G$. Then

$$x \in \{y + x - y\}^{\perp} = y + \{x\}^{\perp} - y = \{x\}^{\perp},$$

which immediately yields $x = 0$. Let, finally, (c) be fulfilled and the component K have the form $A^{\perp}$, $A \subset G$. Let us arbitrarily choose $x \in K$, $y \in G$, $a \in A$ and set $z := (y + |x| - y) \wedge |a|$. It is obvious that $0 \le z \wedge (-y + z + y) \le |x| \wedge |a| = 0$, so that $z = 0$, which does imply that $|y + x - y| = y + |x| - y \in A^{\perp} = K$, i.e., that $y + K - y \subset K$. $\rhd$

Introduce a symmetric relation Δ in G by the formula

$$\Delta := \{(x,y) \in G \times G : (\forall a,b \in G)\,(a + |x| - a) \wedge (b + |y| - b) = 0\}.$$

If for some $x, y \in G$ it is invalid that $x \Delta y$, then there are such $a_0, b_0 \in G$ that $u_0 := (a_0 + |x| - a_0) \wedge (b_0 + |y| - b_0) \ne 0$. It is obvious that $u_0 \in \{a_0 + |x| - a_0\}^{\Delta\Delta}$ but, on the other hand, $\{a_0 + |x| - a_0\}^{\Delta\Delta} = \{x\}^{\Delta\Delta}$. Therefore, $u_0 \in \{x\}^{\Delta\Delta}$ and, analogously, $u_0 \in \{y\}^{\Delta\Delta}$. It should be also remarked that the least Δ-component is $\{0\}$ and $\Delta \cap I_G \subset \perp \cap I_G = \{(0,0)\}$. Hence, Δ is a disjointness on G (see **4.1.12** (2)).

(3) *The set of all Δ-components coincides with the complete Boolean algebra of invariant $\perp$-components: $\mathfrak{K}_{\Delta}(G) = \mathfrak{B}_i(G)$.*

4.4.5. Now assume that the group G has invariant base, i.e., its all components are invariant. This exactly implies that $\Delta = \perp$. The commutative lattice-ordered group obviously has an invariant base. Under these circumstances G can be turned into an algebraic B-system. Let now j be an isomorphism of a complete Boolean algebra B on the (invariant) basis $\mathfrak{B}(G)$. By definition, assign

$$p(x) := j^{-1}(\{x^{-}\}^{\Delta}) \quad (x \in G).$$

The mapping $p: G \to B$ has a number of important properties.

For any $x, y \in G$ the following relations are valid:

(1) $0 \le x \to p(x) = 1$;

(2) $p(x) \wedge p(-x) = j^{-1}(\{x\}^{\perp})$;
(3) $p(x) \wedge p(y) \leq p(x+y)$;
(4) $p(x) = p(y+x-y)$;
(5) $p(x) \vee p(-x) = \mathbf{1}$.

◁ The first statement is obvious. To prove (2), we should remark that $\{x\}^{\perp} = \{x^{+}\}^{\perp} \wedge \{x^{-}\}^{\perp} = \{x^{-}\}^{\perp} \wedge \{(-x)^{-}\}^{\perp}$ due to the fact that x^{+} and x^{-} are disjoint. It is then clear that $j^{-1}(\{x\}^{\perp}) = j^{-1}(\{x^{\cdot}-\}^{\perp}) \wedge j^{-1}(\{-x\}^{-}) = p(x) \wedge p(-x)$. Analogous considerations are used to establish (3) provided we begin with relations **4.4.2** (2,6). Relation (4) results from **4.4.2** (5) since the components are invariant. Taking it again into account that the elements x^{+} and x^{-} are disjoint, we can write

$$(\{x^{+}\}^{\perp} \vee \{x^{-}\}^{\perp})^{\perp} = \{x^{+}\}^{\perp\perp} \wedge \{x^{-}\}^{\perp\perp} = \{0\}.$$

Hence, we deduce $\{x^{+}\}^{\perp} \vee \{x^{-}\}^{\perp} = G$, which is equivalent to the required result. ▷

Let us introduce two mappings, σ and $d := G \times G \to B$ by the following formulas:

$$\sigma(x,y) := p(y-x), \quad d(x,y) := j^{-1}(\{x-y\}^{\Delta}) \quad (x,y \in G).$$

From (1) - (5) the following properties of the mapping σ are straightforward:

(a) $\sigma(x,x) = 0$ (symmetry);
(b) $\sigma(x,y) \wedge \sigma(y,z) \leq \sigma(x,z)$ (transitivity);
(c) $\sigma(x,y) = \sigma(a+x-b, a+y-b)$ (invariance);
(d) $\sigma(x,y) \wedge \sigma(y,x) = d(x,y)^{*}$ (reflexivity).

By virtue of (d), $d(x,y) = \sigma(x,y)^{*} \vee \sigma(y,x)^{*}$ and, hence, d is a B-metric on G invariant relative to left and right transitions, while σ is a B-predicate. Finally, it is obvious that $d(x,0) = j^{-1}(\{x\}^{\perp\perp})$, i.e., the B-metric d agrees with the disjointness $\perp$ (see **4.1.13**).

4.4.6. Theorem. *Let G be a lattice-ordered group with invariant base. Then G, when considered with a B-predicate σ and with the corresponding B-metric d, is an algebraic B-system of signature $(+,0,\leq)$, on which the axioms of a linearly ordered group are fulfilled.*

◁ As has been remarked above, a B-metric d is invariant under translations. Taking this fact into consideration, we can write

$$d(x+y, u+v) = d(x, -y+u+v) \leq d(x,u) \vee d(u, -y+u+v),$$

$$d(u,-y+u+v) = d(u+y-u,v) \leq d(v,y) \vee d(u+y-u,y),$$
$$d(u+y-u,y) = d(u+y,u+y) = 0.$$

These relations show that $d(x+y,u+v) \leq d(x,u) \vee d(y,v)$, i.e., the operation of the summation is a contraction mapping. Then, by virtue of **4.4.5** (3), the definition of d gives

$$d(x,y)^* \wedge p(x) = p(x) \wedge p(x-y) \wedge p(y-x) \leq p(y),$$

whatever $x,y \in G$ might be. From here we can easily deduce that $\sigma(x,y) \wedge d(x,u)^* \wedge d(y,v)^* \leq \sigma(u,v)$, which implies that the mapping σ is contractive. Hence, $(G,+,0,\leq,\sigma)$ is an algebraic B-system of signature $(+,0,\leq)$. In this case the following interpretation of the symbol $\leq$ is implied: if $x,y \in G$, then $|x \leq y|^G := \sigma(x,y)$. Then the unary B-predicate p on G will, obviously, be an interpretation of the property of being a positive element, i.e., $|0 \leq x|^G = p(x)$. The fact that G is a B-model for the axioms of a linearly ordered group is just a different interpretation of properties **4.4.5** (1-5). Let us check, for instance, that the order of σ agrees with group structure and is total.

If φ is a closed formula from **4.4.1** (3), then presenting the Boolean truth-values for quantifiers according to **4.1.8**, we get

$$|\varphi|^G = \bigwedge_{x,y,a,b \in G} |x \leq y \rightarrow a+x+b \leq a+y+b|^G.$$

Then, making use of the fact that σ serves as an interpretation of the symbol $\leq$, we write:

$$|x \leq y \rightarrow a+x+b \leq a+y+b|^G = \sigma(x,y) \Rightarrow \sigma(a+x+b,a+y+b).$$

By virtue of **4.4.5** (4), however, we have

$$\sigma(a+x+b,a+y+b) = p(a+y+b-(a+x+b))$$
$$= p(a+(y-x)-a) = p(y-x) = \sigma(x,y).$$

Therefore, $\mathbf{1} = \sigma(x,y) \Rightarrow \sigma(a+x+b,a+y+b)$ and, hence, $|\varphi|^G = \mathbf{1}$.

Let now φ be the axiom expressing the linearity of order **4.4.1** (4). Let us again use rules **4.1.8** and write

$$|\varphi|^G = \bigwedge_{x,y \in G} |x \leq y \vee y \leq x|^G = \bigwedge_{x,y \in G} \sigma(x,y) \vee \sigma(y,x).$$

Observe that by virtue of **4.4.5** (5) we have

$$\sigma(x,y) \vee \alpha(y,x) = p(y-x) \vee p(x-y) = \mathbf{1},$$

and, hence, $|\varphi|^G = \mathbf{1}$. ▷

4.4.7. Let us now consider the case of lattice-ordered rings. An algebraic system $(A,+,\cdot;0,\le)$ is called *an ordered ring* if the following statements are valid:

(1) $(K,+,0,\le)$ is a commutative ordered group;

(2) $(K,+,\cdot;0)$ is a ring (not necessarily commutative or associative);

(3) the multiplication of the ring K is compatible with the order in such a way that $0 \le x, y \in K$ yields $0 \le xy$, i.e., K is a model for $(\forall x)(\forall y)(x \ge 0 \wedge y \ge 0 \to x \cdot y \ge 0)$.

Therefore, an ordered ring is a ring such that its additive group is ordered and, moreover, the ring homotheties corresponding to positive elements are positive endomorphisms of the ordered group in question. We shall often ascribe to a ring the properties of the corresponding ordered group. Thus, for instance, the notions of a lattice- or linearly ordered ring, of the positive cone, etc., refer to the ordered group of a ring and need no further specifications. The order of K is called a *ring order* provided it obeys all the conditions from **(1)** to **(3)**.

The ordered ring K is termed *commutative* if alongside with **(1)** - **(3)** the following axiom is also fulfilled

(4) $(\forall x)(\forall y)(xy = yx)$.

A subset P of the ring K is the positive cone of a certain ring order iff

$$P \cap (-P) = \{0\}; \quad P + P \subset P; \quad P \cdot P \subset P.$$

In the lattice-ordered ring K alongside with the relations considered in **4.4.2**, the following relations also hold: $(xy)^+ \le x^+y^+ + x^-y^-$; $(xy)^- \le x^+y^- + x^-y^+$; $|xy| \le |x| \cdot |y|$.

4.4.8. Any lattice-ordered ring K can be transformed into an ordered B-group, in which case K will not, in general, be a B-ring. The point is that the ring multiplication cannot obligatory be a contractive operation relative to the corresponding B-metric. In order to exclude this undesirable phenomenon, a more close interrelation between the multiplication and order is required. The lattice-ordered ring K is called an *f-ring* provided it satisfies the following condition: if $x, y \in K$ and $x \wedge y = 0$, then $(ax) \wedge y = 0$ and $(xa) \wedge y = 0$ for any $a \in K$. It should be observed that in any f-ring the following condition is fulfilled: $|x| \wedge |y| = 0 \to xy = 0$. If in an f-ring there are no nilpotent elements, then the converse statement is also valid or, as it is often said, the *f-ring is exact.* In particular, an f-ring without zero divisors is linearly ordered, and a linearly ordered ring without nilpotent elements is known to contain no zero divisors. Among the other properties of f-rings let us recall the following:

$$(x \vee y)z = (xz) \vee (yz); \quad z(x \vee y) = (zx) \vee (zy);$$
$$(x \wedge y)z = (xz) \wedge (yz); \quad z(x \wedge y) = (zx) \wedge (zy);$$
$$|xy| = |x||y|.$$

For any lattice-ordered ring K the following statements are equivalent:

(1) *K is an f-ring;*

(2) $\{xy\}^{\perp\perp} \leq \{x\}^{\perp\perp} \wedge \{y\}^{\perp\perp}$;

(3) $d(xy,uv) \leq d(x,u) \vee d(y,v)$.

◁ Assume that K is an f-ring. If $|x| \wedge |u| = 0$ or $|y| \wedge |u| = 0$, then $|xy| \wedge |u| = (|x||y|) \wedge |u| = 0$. Therefore, either $u \in \{x\}^{\perp}$ or $u \in \{y\}^{\perp}$ affords $u \in \{x \cdot y\}^{\perp}$, i.e., $\{x\}^{\perp} \cup \{y\}^{\perp} \subset \{xy\}^{\perp}$. Hence, $\{xy\}^{\perp\perp} \leq (\{x\}^{\perp} \cup \{y\}^{\perp})^{\perp} = \{x\}^{\perp\perp} \wedge \{y\}^{\perp\perp}$. Let now condition (2) be fulfilled. It should be observed that $|xy - uv| = |x(y-v) + (x-u)v| \leq \leq |x||y-v| + |x-u||v|$ and, hence,

$$\{xy - uv\}^{\perp\perp} \leq \{y - v\}^{\perp\perp} \vee \{x - u\}^{\perp\perp}.$$

This inequality is equivalent to (3). Let us, finally, assume that the mapping $(x,y) \to xy$ is a contraction. In (3) let us put $u = 0$, $v = -y$ and rewrite it as $\{x \cdot y\}^{\perp\perp} \subset \{x\}^{\perp\perp} \vee \{0\}^{\perp\perp} = \{x\}^{\perp\perp}$, or $\{xy\}^{\perp} \supset \{x\}^{\perp}$. Analogously, $\{xy\}^{\perp} \supset \{y\}^{\perp}$ and, hence, K is an f-ring. ▷

4.4.9. *Any (associative, commutative) f-ring K together with a B-predicate σ and corresponding B-metric d is an algebraic B-system which is a B-model for the axioms of an (associative, commutative) totally ordered ring. In this case the element $0 \neq e \in K$ is a ring unity of the B-ring under discussion iff e is an order and ring unity of the ring K.*

◁ As has been proved in **4.4.6**, K is a linearly ordered B-group with σ and d specified above. Adjoin to this group a contraction mapping $(x,y) \to xy$ and prove that the the algebraic B-system obtained is an f-ring. Associativity, commutativity and distributivity in the B-system of K trivially follow from the corresponding properties of the ring K. Let us check the compatibility condition **4.4.7** (3). To this end let us observe that by virtue of **4.4.7** and **4.4.8** (2), we have

$$\{(xy)^-\}^\perp \geq \{x^+y^-\}^\perp \wedge \{x^-y^+\}^\perp \geq \{x^-\}^\perp \wedge \{y^-\}^\perp .$$

According to the definition of d, we conclude that $p(x) \wedge p(y) \leq p(xy)$. The task now is to calculate the Boolean truth-values by rules **4.1.8** :

$$\begin{aligned} &|(\forall x)(\forall y)(x \geq 0 \wedge y \geq 0 \to xy \geq 0)|^K \\ &= \bigwedge_{x,y \in K} |x \geq 0|^K \wedge |y \geq 0|^K \Rightarrow |xy \geq 0|^K \\ &= \bigwedge_{x,y \in K} p(x) \wedge p(y) \Rightarrow p(x \cdot y) = \mathbf{1}. \end{aligned}$$

It should be further observed that for $e \in K$ the equality $\mathbf{1} := |\theta < e|^K = |e \geq 0 \wedge e \neq 0|^K$ implies $p(e) \wedge d(e,0) = \mathbf{1}$, i.e., $e \geq 0$ and e is not an order unit. On the other hand,

$$|(\forall x)(xe = ex = x)|^K = \bigwedge_{x \in K} d(x,ex)^* \wedge d(x,xe)^*,$$

and, hence, e is a unit of the B-ring iff e is an order unit in K and for any $x \in K$ we have $d(xe,x) = d(ex,x) = 0$. The last fact implies that $x = ex = xe$, which is the required proof. ⊳

4.4.10. Theorem. *Let $\mathcal{G}$ be an ordered group in the model $\mathbf{V}^{(B)}$ and $G := \mathcal{G}\downarrow$. Then G is an ordered group extended relative to the Boolean algebra of projections $\mathcal{B}$, and there is an isomorphism j from $\mathcal{B}$ to B such that*

$$b \leq [0 \leq x] \leftrightarrow 0 \leq j(b)x \quad (x \in G, b \in B).$$

In this case the following equivalences hold:

(1) $\mathbf{V}^{(B)} \models$ *"$\mathcal{G}$ is directed (integrally-closed, Archimedean)" ↔ "G is directed (integrally-closed, Archimedian)";*

(2) $\mathbf{V}^{(B)} \models$ *"$\mathcal{G}$ is lattice-ordered (order-complete)" ↔ "G is lattice-ordered (order-complete)";*

(3) $\mathbf{V}^{(B)} \models$ *"$\mathcal{G}$ is an ordered ring" ↔ "G is an extended ordered ring with the Boolean algebra of projections $\mathcal{B}$ ";*

(4) $\mathbf{V}^{(B)} \models$ *"$\mathcal{G}$ is a linearly ordered skew field" ↔ "G is an extended f-ring without*

nilpotent elements, $\mathcal{B}$ is an algebra of projections on all possible components of $\mathcal{G}$, and any regular element in G is invertible".

⊲ The fact that G is an extended group with complete Boolean algebra of projections $\mathcal{B}$ has been established in **4.2.7**. Let $\mathcal{G}^+$ be the positive cone of the group $\mathcal{G}^+$ inside $\mathbf{V}^{(B)}$. Then

$$[\mathcal{G}^+ + \mathcal{G}^+ \subset \mathcal{G}^+] = [\mathcal{G}^+ \cap -\mathcal{G}^+ = \{0\}] =$$
$$[((\forall x) \in \mathcal{G})(x + \mathcal{G}^+ = \mathcal{G}^+ + x)] = \mathbf{1}.$$

Assign $G^+ := \mathcal{G}^+ \downarrow$ and observe that, according to the rules of descents of intersection and image, $G^+ + G^+ \subset G^+$, $G^+ \cap -G^+ = \{0\}$. Then, for any $x \in G$ we have $[x + \mathcal{G}^+ = \mathcal{G}^+ + x] = \mathbf{1}$, i.e., $x + \mathcal{G}^+ = \mathcal{G}^+ + x$. In this case, however,

$$(x + G^+) = (x + \mathcal{G}^+)\downarrow = (\mathcal{G}^+ + x)\downarrow + x.$$

Therefore, G is an ordered group with positive cone G^+. In **4.2.7** we have proved that the isomorphism $j: B \to \mathcal{B}$ does exist, in which case the relations $b \leq [x = y]$ and $j(b)x = j(b)y$ are equivalent. Let us choose $x \in G$ and remark that $[0 \leq x \leftrightarrow (\exists y \in \mathcal{G}^+)(x = y)] = \mathbf{1}$. This implies that $b \leq [0 \leq x]$ iff $b \leq [(\exists y \in \mathcal{G}^+)(x = y)]$. The last statement is equivalent to the existence of an $y \in \mathcal{G}^+\downarrow =: G^+$ such that either $b \leq [x = y]$ or $j(b)x = j(b)y \geq 0$. Let us now prove equivalences **(1)** - **(4)**.

(1) If $\mathcal{G}$ is directed, then $[\mathcal{G}^+ - \mathcal{G}^+ = \mathcal{G}] = \mathbf{1}$, which is equivalent to the fact that G is directed, since $(\mathcal{G}^+ - \mathcal{G}^+)\downarrow = \mathcal{G}^+\downarrow - \mathcal{G}^+\downarrow = G^+ - G^+$. When $\mathcal{G}$ is integrally-closed, this implies nothing but

$$\wedge\{[x \leq 0] : [(\exists y \in \mathcal{G})(\forall n \in \omega^\wedge)(nx \leq y)] = 1\} = \mathbf{1}.$$

Hence, $\mathcal{G}$ is integrally-closed iff for every $x \in G$ the following implication is valid:

$$(\exists y \in G)([(\forall n \in \omega^\wedge)(nx \leq y)] = \mathbf{1} \to [x \leq 0] = \mathbf{1}),$$

or

$$((\exists y \in G)(\forall n \in \omega)[n^\wedge x \leq y] = \mathbf{1}) \to [x \leq 0] = \mathbf{1}.$$

The last line is an equivalent presentation of the fact that the group G is integrally-closed. The statement that G is Archimedean is proved analogously.

(2) Let $\mathcal{G}$ be lattice-ordered. Let us prove that on the algebraic system G the closed

formula $(\forall x)(\forall y)(\exists z)(z = \sup\{x, y\})$ is valid, i.e., in G for any two elements there is a least upper bound. If x and $y \in G$, then $[\{x,y\} \subset \mathcal{G}] = \mathbf{1}$, and, therefore, $[(\exists u \in \mathcal{G})(u = \sup\{x, y\})] = \mathbf{1}$. By virtue of the maximum principle, there is a $z \in \mathbf{V}^{(B)}$ such that

$$[z \in \mathcal{G}] \wedge [z = \sup\{x, y\}] = \mathbf{1}.$$

This implies that, on the one hand, $z \in G$, while on the other,

$$|z = \sup\{x,y\}|^{\mathcal{G}\downarrow} = \mathbf{1}.$$

By virtue of the definition of order, we deduce from here that $z = x \vee y$. Analogous considerations make us conclude that there is a least upper bound $x \wedge y$.

Assume now that [$\mathcal{G}$ is an order-complete group] = $\mathbf{1}$. Let us show that in this case G is also order-complete. First we recall the following equivalent definition of the least upper bound $\sup(A)$ of a set A in an arbitrary ordered set

$$\{\sup(A)\} = \pi_{\leq}(A) \cap \pi_{\leq}^{-1}(\pi_{\leq}(A)).$$

Let us now choose in the system $\mathcal{G}\downarrow$ an arbitrary subset A bounded from above. This implies that $\pi_{\leq}(A) \neq \varnothing$. In this case, however, according to the rules of descents and ascents of polars, $[\pi_{\leq}(A\uparrow) \neq \varnothing] = \mathbf{1}$, or, which is equivalent, [$A\uparrow$ is a subset bounded from above in $\mathcal{G}$] = $\mathbf{1}$. Hence, according to the maximum principle, we deduce that for a certain $a \in \mathcal{G}\downarrow$ we have

$$[\{a\} = \{\sup(A\uparrow)\} = \pi_{\leq}(A\uparrow) \cap \pi_{\leq}^{-1}(\pi_{\leq}(A\uparrow))] = \mathbf{1}.$$

Applying now the required rules of descents and ascents again, we deduce $a = \sup(\operatorname{mix}(A))$. Finally, making use of the fact that the relation $\leq$ is fully extensional, we conclude $\sup(\operatorname{mix}(A)) = \sup(A)$. Therefore, A has a least upper bound and, hence, G is an order-complete group.

(3) This statement follows from **4.2.8** and the properties of G established earlier.

(4) Let $\mathbf{V}^{(B)} \models$ "$\mathcal{G}$ is a linearly ordered skew field". By virtue of (3) and **4.2.8**, we can conclude that G is an ordered extended associative ring with the Boolean algebras of positive projections $\mathcal{B}$, having no nilpotent elements. Since $\mathcal{G}$ is a model for $(\forall x)(\forall y)(x \wedge y = 0 \rightarrow x = 0 \vee y)$, for any $x, y \in G$ we have $[x \wedge y = 0] \leq (x = 0) \vee \vee (y = 0)$. If $x \wedge y = 0$, then $b^* \leq [x = 0]$ and $b \leq [y = 0]$, or $j(b)x = x$ and $j(b)y = 0$ for a suitable $b \in B$. Hence, we easily deduce that $\mathcal{B}$ is a Boolean algebra of projections on

components. In this case, however, the fact that G is orthogonally complete is equivalent to the statement that G is extended relative to $\mathcal{B}$. Since the projections $j(b)$ $(b \in B)$ are multiplicative (see **4.2.8**), the kernel of any projections is a ring ideal. This directly leads us to the conclusion that the characteristic property of an f-ring is valid in G (see **4.4.8** (2)). Conversely, if G obeys the conditions specified in (4), then, by virtue of (2), $[\mathcal{G}$ is a lattice-ordered ring$] = \mathbf{1}$. As is readily seen, $\mathcal{G}$ is also an f-ring without nilpotent elements inside $\mathbf{V}^{(B)}$. In this case, however, for $x, y \in G$ it follows from $[xy = 1] = \mathbf{1}$ that $[|x| \wedge |y| = 0] = \mathbf{1}$, or $|x| \wedge |y| = 0$, and, hence, there is such an element $b \in B$ that $j(b)x = 0$ and $j(b^*)y = 0$. Therefore, $b \leq [x = 0]$ and $b^* \leq [y = 0]$ and, hence, $[x = 0 \vee y = 0] \geq b \vee b^* = \mathbf{1}$. We thus established that $\mathbf{V}^{(B)} \models$ "$\mathcal{G}$ has no zero divisors". An f-ring with no zero divisors is, however, known to be linearly ordered, so that $\mathbf{V}^{(B)} \models$ "$\mathcal{G}$ is linearly ordered". Finally, by virtue of **4.2.8**, the nonzero elements of $\mathcal{G}$ are invertible and, hence, $\mathbf{V}^{(B)} \models$ "$\mathcal{G}$ is a linearly ordered skew field". ⊳

4.4.11. Therefore, both linearly ordered groups and f-rings are in a certain way transformed into B-groups and B-rings. By virtue of **4.3**, this implies that they have Boolean-valued realizations which are linearly ordered groups and rings, respectively. Hence, any information on the structure of linearly ordered groups and rings can be used for studying more general classes of groups and rings. Let us demonstrate this statement with examples of the well-known facts presented below (see [**13, 59**]).

(1) **Hölder theorem.** *Any Archimedean linearly ordered group is isomorphic to a subgroup of the additive group of real numbers.*

(2) *Any Archimedean directed group is commutative.*

(3) **Theorem.** *An Archimedean linearly ordered ring is either zero (i.e., the product of any two elements is zero), or order and algebraically isomorphic to a uniquely determined subring of the field of real numbers.*

4.4.12. Theorem. *Let G be an Archimedean lattice-ordered group with its base isomorphic to a Boolean algebra B. Then in the Boolean-valued model $\mathbf{V}^{(B)}$ there is a subgroup $\mathcal{G}$ of the additive group of the field of real numbers such that the lattice-ordered group $G' := \mathcal{G}\downarrow$ is the maximal extension of the group G.*

◁ By **4.4.6**, the group G can be transformed into an ordered B-group. Let $\mathcal{G}$ be a Boolean-valued realization of this algebraic B-system. Then, by **4.3.3**, $\mathcal{G}$ is a linearly ordered group inside $\mathbf{V}^{(B)}$. In line with theorem **4.4.10**, $G' := \mathcal{G}\downarrow$ is a lattice-ordered group, in which case, as is known, $G' = \mathrm{mix}(\iota(G))$, where ι is the canonical isomorphism from G to G'. If $b \in B$, while $L_b \in \mathfrak{B}(G)$ and $\pi_b \in \mathfrak{Pr}(G')$ are the component and projection, respectively, then the conditions $x \in L_b$ and $(I - \pi_b)(\iota(x)) = 0$ are equivalent for any $x \in G$. Indeed, by the definition of a B-metric on G (see **4.4.5**), $\pi_b \iota(x) = \iota(x)$ is fulfilled iff $b^* \le [\iota(x) = 0]$. In this case, as is known,

$$[\iota(x) = 0] = [\iota(x) \neq 0]^* = d(x,0)^*.$$

We thus established that the correspondence $L' \mapsto \iota^{-1}L' \cap G$, $L' \in \mathfrak{B}(G')$ is an isomorphism of the bases $\mathfrak{B}(G')$ and $\mathfrak{B}(G)$. Let us now choose $0 < x \in G'$. If $x = \mathrm{mix}(\pi_\xi \iota(x_\xi))$, then $0 < \pi_\xi \circ \iota(x_\xi) \le \iota(x_\xi)$ for a certain ξ. According to the isomorphism of bases presented above, there is a $0 < z \in G$, for which $z \in \{\pi_\xi \circ \iota(x_\xi)\}^{\perp\perp}$. Now for $x_0 := x_\xi \wedge z$ we have

$$0 < \iota(x_0) \le \iota(z) \wedge \pi_\xi \circ \iota(x_\xi) \le \pi_\xi \circ \iota(x_\xi) \le x.$$

Therefore, $\iota(G)$ is minorant in G'. Let us now assume that for some $x, y \in G'$ it is fulfilled that $n|x| \le y$ $(n \in \omega)$. Let $y = \mathrm{mix}(\pi_\xi \iota(y_\xi))$ and $x = \mathrm{mix}(\pi_\xi \iota(x_\xi))$ for some families (x_ξ) and (y_ξ) in G and a partition of unity (π_ξ) in $\mathfrak{Pr}(G')$. Designate $\Xi_0 := \{\xi \in \Xi : \pi_\xi \circ \iota(|x_\xi|) = 0\}$. Since $\iota(G)$ is minorant, for any $\xi \in \Xi \setminus \Xi_0$ there is a $0 < u_\xi \in G$, for which $\iota(u_\xi) \le \pi_\xi(\iota|x_\xi|)$. Then, for the same ξ and for all $n \in \omega$ we get

$$\iota(nu_\xi) \le \pi_\xi \circ \iota(n|x_\xi|) = \pi_\xi(n|x|) \le \pi_\xi y = \pi_\xi \circ \iota(y_\xi) \le \iota(y_\xi),$$

or $nu_\xi \le y_\xi$. Since G is Archimedean, we get $u_\xi = 0$, which implies that $\Xi_0 = \Xi$ and, hence, $x = 0$. Therefore, the group G' is Archimedean and, by **4.4.10**, [$\mathcal{G}$ is Archimedean] $= \mathbf{1}$. In line with the Hölder theorem **4.4.11** (1) $\mathcal{G}$ is isomorphic to an additive subgroup of the group of real numbers $\mathcal{R}$. According to theorem **4.3.4**, $\mathcal{G}$ can be assumed to be a linearly ordered subgroup in $\mathcal{R}$. ▷

4.4.13. Theorem. *Let K be an Archimedean f-ring. Then in K there are two components K_0 and K_1 complementary to each other such that if the bases $\mathfrak{B}(K_0)$ and $\mathfrak{B}(K_1)$ are isomorphic to the Boolean algebras B_0 and B_1, respectively, then the following statements are valid:*

(1) *in the Boolean-valued model $\mathbf{V}^{(B)}$ there is a subgroup $\mathcal{R}_0$ of the group of real*

numbers such that the lattice-ordered group $K_0' := \mathscr{R}_0\downarrow$ *with zero multiplication is the maximal extension of the* f*-ring* K_0*;*

(2) *in the Boolean-valued model* $\mathbf{V}^{(B)}$ *there is a subring* $\mathscr{R}_1$ *of the ring of real numbers such that an* f*-ring* $K_1' := \mathscr{R}_1'\downarrow$ *is the maximal extension of* K*. In this case an* f*-ring* $K_0' \oplus K_1'$ *is the maximal extension of the* f*-ring* K*.*

◁ As we have seen in **4.4.12**, the realization of the additive group of the f-ring K in the model $\mathbf{V}^{(B)}$, $B = \mathfrak{B}(K)$, will be a subgroup of the additive group of real numbers. According to **4.4.9**, however, K is a B-ring, while by theorem **4.3.3**, $[\mathscr{R}$ is a ring$] = \mathbf{1}$. Let us set $b_0 := [\mathscr{R}$ is a zero ring$]$ and $b_1 := [\mathscr{R}$ is a subring of the ring of real numbers$]$. By the transfer principle and theorem **4.4.11** (3), $b_0 \vee b_1 = \mathbf{1}$. On the other hand, $b_0 \wedge b_1 = \mathbf{0}$, since a ring cannot be simultaneously both zero and a subring of the ring of real numbers. Let K_0 and K_1 be components in K corresponding to the elements b_0 and b_1, i.e., K_0 and K_1 are determined by the conditions

$$x \in K_l \leftrightarrow d(x,0) \le b_l \quad (l = 0,1),$$

where d is the B-metric of the B-system K. Assign $B_l := [0,b_l]$ and observe that the base $\mathfrak{B}(K_l)$ is isomorphic to B_l, in which case b_l is the unity of the algebra B_l. Let us introduce the denotation $\mathscr{R}_l := \overset{*}{\pi}_l(\mathscr{R}) \in \mathbf{V}^{(B_l)}$, where $\pi_l : b \to b \wedge b_l$, $b \in B$. Since π_l is an isomorphism of B on B_l; therefore, $\mathbf{V}^{(B_0)} \models$ "$\overset{*}{\pi}_0(\mathscr{R})$ is a subgroup of the additive group of real numbers" and $\mathbf{V}^{(B_1)} \models$ "$\overset{*}{\pi}_1(\mathscr{R})$ is a subring of the ring of real numbers". By theorem **4.4.12**, $K' := K\downarrow$ is an extension of the ordered group K. As far as $b_l = [\overset{*}{\pi}_l(\mathscr{R}) \cong \mathscr{R}]$, where $K_l' := \mathscr{R}_l\downarrow \cong j(b_l)(K_l)$ and, hence, $K' \cong K_0' \oplus K_1'$. Therefore, K' is the maximal extension of K. ▷

CHAPTER 5

BOOLEAN REPRESENTATIONS IN FUNCTIONAL ANALYSIS

We have already convinced ourselves that the Boolean-valued universe $\mathbf{V}^{(B)}$ associated with a fixed Boolean algebra B is one of the arenas where mathematical events occur. Indeed, by virtue of the transfer and maximum principles, in $\mathbf{V}^{(B)}$ there are numbers and groups, Lebesque and Riemann integrals, the Radon-Nikodym theorems are fulfilled, and the Jordan expansion of a matrix is implementable. The elementary technique of descents and ascents, which we got acquainted with when considering algebraic systems, shows each of mathematical objects in $\mathbf{V}^{(B)}$ to be a realization of an analogous classical object with an additional structure determined by the algebra B. In particular, this consideration refers to functional-analytical objects as well.

In the present chapter facts associated with Boolean-valued realization of the last objects are considered. The most important and remarkable achievement of Boolean-valued analysis is the fact that it establishes a inseparable immanent interrelation of the realizations discussed above with the concepts of the theory of ordered vector spaces and, above all, with K-spaces introduced by L.V.Kantorovich at the beginning of the thirties.

5.1 Vector Lattices

In this section the basic notions of the theory of vector lattices are briefly presented, a more detailed presentation can be found elsewhere [**1, 103, 159, 279**].

5.1.1. Let $\mathbf{F}$ be a linearly ordered field. An *ordered vector space* over $\mathbf{F}$ is an algebraic system E of signature $(+,0,\leq\ \lambda)$, where λ ranges over the set of elements of the field $\mathbf{F}$, and denotes, when λ is fixed, a unary operation on F such that the following conditions are met:

(1) $(E,+,0,\leq)$ is a commutative ordered group;

(2) $(E,+,0,\lambda)$ is a vector space over $\mathbf{F}$;

(3) every positive element of the field $\mathbf{F}$ determines the respective positive endomorphism of the ordered group $(E,+,0,\leq)$ by its action of scalar multiplication.

Therefore, an ordered vector space can be defined as a pair $(E,\leq)$, where E is a vector space over the field $\mathbf{F}$, while $\leq$ is a *vector order* in E, i.e., a relation of order in E which agrees with the structure of a vector space. The last, speaking informally, implies that inequalities in E "can be added and multiplied by the positive elements of the field $\mathbf{F}$". Introducing a vector order in a vector space E over the field $\mathbf{F}$ is equivalent to defining a set (*positive cone*) $E^+ \subset E$ with the following properties: $E^+ + E^+ \subset E^+$; $\lambda E^+ \subset E^+$ $(0 \leq \lambda \in \mathbf{F})$; $E^+ \cap E^+ = \{0\}$. In this case the order $\leq$ and the cone E^+ are connected by the relation

$$x \leq y \leftrightarrow y - x \in E^+ \quad (x,y \in E).$$

The notions and results of the theory of ordered groups are undoubtedly applicable to ordered vector spaces. It is, for instance, obvious that for an ordered vector space the notions of being Archimedean, linearly ordered, of the σ-ideal, etc., refer to the corresponding ordered group.

5.1.2. *A vector space* is an ordered vector space which is a lattice-ordered group. It should be recalled that disjointness of the vector lattice E is introduced through the formula

$$\perp := \{(x,y) \in E \times E : |x| \wedge |y| = 0\}.$$

A component (or *band*) of the vector lattice E is a set of the type

$$M^{\perp} := \{x \in E : (\forall y \in M) x \perp y\},$$

where M is an arbitrary nonempty set in E. The totality of all components of the vector lattice ordered by inclusion forms a complete Boolean algebra $\mathfrak{B}(E)$, with its Boolean operations having the following form:

$$L \wedge K = L \cap K, \quad L \vee K = (L \cup K)^{\perp\perp}, \quad L^* = L^{\perp} \quad (l,k \in \mathfrak{B}(E)).$$

The algebra $\mathfrak{B}(E)$ is called the *base of* E.

Let K be a component (or band) of the vector lattice E and $0 \leq x \in E$. If in E there is an element $\sup\{u \in K : 0 \in u \leq x\}$, then it is termed the *projection* of x on the component K and denoted by $[K]x$ (or $\mathrm{Pr}_K x$). For an arbitrary $x \in E$ we set $[K]x : [K]x^+ - [K]x^-$. The projection of an element $x \in E$ on the component K exists iff the expansion $x = y + z$ is valid, where $y \in K$ and $z \in K^{\perp}$, in which case $y = [K]x$ and $z = [K^{\perp}]x$. Let us assume that any element $x \in E$ has a projection on K. Then the operator $x \to [K]x$ $(x \in E)$ is linear, idempotent, and $0 \le [K]x \le x$ for all $0 \le x \in E$. The vector lattice is said *to admit projections on components* (on principal components) if for any component (principal component) K the operator of projection $[K]$ is determined. If the vector lattice allows projections on components and any disjoint set of positive elements in it has a supremum, then it is called *extended* (or *universally complete*).

5.1.3. An element $\mathbf{1} \in E$ is called an (order) *unit* if $\{\mathbf{1}\}^{\perp\perp} = E$, i.e., if in E there are no nonzero elements disjoint from $\mathbf{1}$. Let for a certain $0 \le e \in E$ we have $e \wedge (\mathbf{1} - e) = 0$. In this case we say that e is a *unit element* (relative to $\mathbf{1}$). The set $\mathfrak{E}(\mathbf{1}) := \mathfrak{E}(E)$ of all unit elements with the order induced from E is a Boolean algebra. The lattice operations in $\mathfrak{E}(\mathbf{1})$ are inherited from E, while the Boolean extension has the form $e^{*} = \mathbf{1} - e$ $(e \in \mathfrak{E}(\mathbf{1}))$.

Henceforth, unless the field $\mathbf{F}$ is explicitly indicated, the vector lattice over the linearly ordered field of real numbers $\mathbf{R}$ is implied. In the ideal $I(u) := \bigcup_{n=1}^{\infty} [-nu, nu]$ generated by an element $0 \le u \in E$ the following seminorm can be introduced:

$$\|x\|_u := \inf\{\lambda \in \mathbf{R} : |x| \le \lambda u\} \quad (x \in I(u)).$$

If $I(u) = E$, then we say that u is a *strong unit*, while E is the *vector lattice* of *bounded elements*. The seminorm $\|\cdot\|_u$ is a norm iff E is Archimedean.

An element $x \ge 0$ of the vector lattice is called *discrete* if $[0, x] = [0,1]x$, i.e., it follows from $0 \le y \le x$ that $y = \lambda x$ for some $0 \le \lambda \le 1$. A vector lattice E is termed *discrete* if for any $0 < y \in E$ there is a discrete element $x \in E$ such that $0 < x \le y$. In the case when E has no nonzero discrete elements, it is said to be *continuous*.

5.1.4. *A Kantorovich space*, or, in brief, a K-*space* is a vector lattice such that every order-bounded nonempty subset has exact bounds. Sometimes, instead of this term, K-space, a more expanded term is used, i.e., a conditionally order-complete vector lattice. If in a vector lattice there are exact bounds only of countable bounded sets, then it is called a K_σ-*space*. Any K_σ-space and, moreover, any K-space is Archimedean.

The set of the projections on all possible components in E is denoted by the symbol $\mathfrak{Pr}(E)$. For projections π and ρ we set $\pi \le \rho$ iff $\pi x \le \rho x$ at all $0 \le x \in E$.

Theorem. *Let E be an arbitrary K-space. Then projecting onto a component determines an isomorphism $K \to [K]$ of the Boolean algebras $\mathfrak{K}(E)$ and $\mathfrak{Pr}(E)$. If there a unit in E, then the mapping $\pi \to \pi\mathbf{1}$ from $\mathfrak{Pr}(E)$ to $\mathfrak{E}(E)$ and $e \to \{e\}^{\perp\perp}$ from $\mathfrak{E}(E)$ to $\mathfrak{K}(E)$ are also* *isomorphisms of Boolean algebras.*

The projection π_u on a component of the type $\{u\}^{\perp\perp}$, where $0 \le u \in E$, can be obtained by a simpler rule than that in **5.1.2**, namely,

$$\pi_e x = \sup\{x \wedge (nu) : n \in \mathbb{N}\} \quad (0 \le x \in E).$$

In particular, in an K_σ-space there is a projection of any element on any principal component.

Let E be a K_σ-space with a unit $\mathbf{1}$. The projection of the unit on the component $\{x\}^{\perp\perp}$ is called the *trace* of the element x and denoted by the symbol e_x. Therefore, $e_x := \sup\{\mathbf{1} \wedge (n|x|) : n \in \mathbb{N}\}$. The trace e_x serves both as a unit in $\{x\}^{\perp\perp}$ and a unit element in E. For every real number λ the trace of the positive part of an element $\lambda\mathbf{1} - x$ is denoted by e_λ^x, i.e., $e_\lambda^x := e_{(\lambda\mathbf{1}-x)^+}$. The function $\lambda \to e_\lambda^x$ ($\lambda \in \mathbf{R}$) arising in this case is called the *spectral function* or *characteristic* of the element x.

5.1.5. **(1)** *An ordered algebra over* $\mathbf{F}$ (or *an ordered* $\mathbf{F}$*-algebra*) is any ordered ring E with operations such that the external law has the field $\mathbf{F}$ as the set of its operators and, together with addition given in E, determines the structure of an order vector space in E. Therefore, an ordered algebra can be defined as an algebraic system of signature $(+, 0, \le, \lambda, \cdot)$, where λ runs over the set of elements of the field $\mathbf{F}$ and denotes a unary operation for every λ. In this case the following conditions are met:

(a) $(E, +, 0, \le, \lambda)$ is an ordered vector space;
(b) $(E, +, \mathbf{0}, \le, \cdot)$ is an ordered ring.

Let us say that E is a *lattice-ordered algebra* (f*-algebra*) if E is an ordered algebra and the corresponding ordered ring is lattice-ordered (is an f-ring). *Exact* is such an f-algebra wherein for any two elements x and y it follows from $x \cdot y = 0$ that $x \perp y$. It can be easily proved that an f-algebra is exact iff there no nonzero nilpotent elements in it. The fact that an f-algebra is exact is equivalent to the absence of positive elements whose square is nonzero (see **4.4.8**).

(2) *A complex vector lattice* is the complexification $E \oplus iE$ (i is the imaginary unity) of a real vector lattice E. In this case it is often additionally required that any element $z \in E \oplus iE$ should have the modulus

$$|z| := \sup(\operatorname{Re}(e^{i\theta} z) : 0 \le \theta \le \pi\}.$$

For a K-space this requirement is excessive, so that *a complex* K*-space* is the complexification of a real K-space. Speaking about the order properties of a complex vector lattice $E\oplus iE$, we mean its real part E. The notions of a sublattice, ideal, projection band, etc., are naturally extended to the case of a complex vector lattice by way of appropriate complexification.

5.1.6. The order relation in a vector lattice is related with various types of convergence. Let $(A,\leq)$ be a set filtered upwards, and let us consider a net $(x_\alpha):=(x_\alpha)_{\alpha\in A}$ in E, which is called *increasing (decreasing)* if $x_\alpha \leq x_\beta$ $(x_\beta \leq x_\alpha)$ for $\alpha \leq \beta$ $(\alpha,\beta \in A)$.

The net (x_α) is said to be *o-convergent* to an element $x \in E$ if in E there is a decreasing net $(e_\alpha)_{\alpha\in A}$ with the properties $\inf_{\alpha\in A} e_\alpha = 0$ and $|x - x_\alpha| \leq e_\alpha$ $(\alpha \in A)$. In this case x is termed the *o-limit* of the net (x_α) and we write $x = o - \lim x_\alpha$, or $x_\alpha \xrightarrow{(o)} x$. In a K-space for an order-bounded net E also introduces the *upper and lower o-limits* (or *limit superior* and *limit interior*) through the formulas:

$$\limsup_{\alpha\in A} x_\alpha := \overline{\lim_{\alpha\in A}}\, x_\alpha := \inf_{\alpha\in A} \sup_{\beta\geq\alpha} x_\beta,$$
$$\liminf_{\alpha\in A} x_\alpha := \underline{\lim}_{\alpha\in A}\, x_\alpha := \sup_{\alpha\in A} \inf_{\beta\geq\alpha} x_\beta.$$

There is an evident relation between these objects:

$$x = o - \lim x_\alpha \leftrightarrow \limsup x_\alpha = x = \liminf x_\alpha.$$

The net $(x_\alpha)_{\alpha\in A}$ is said *to r-converge* (*to converge with regulator*) to $x \in X$ if there is an element $0 \leq u \in E$ termed the *regulator of convergence*, and a numerical net $(\lambda_\alpha)_{\alpha\in A} \subset \mathbf{R}$ with the properties $\lim \lambda_\alpha = 0$ and $|x - x_\alpha| \leq \lambda_\alpha u$ $(\alpha \in A)$. In this case x is termed the *r-limit* of the net (x_α) and we write $x = r\text{-}\lim x_\alpha$, or $x_\alpha \xrightarrow{(r)} x$. As is seen, the convergence with regulator is that in the normed space $(I(u), \|\cdot\|_u)$.

The presence of o-convergence in a K-space allows one to determined the sum of an infinite family $(x_\xi)_{\xi\in\Xi}$. Indeed, for $\theta := \{\xi_1,\dots,\xi_n\} \in \mathcal{P}_{\text{fin}}(\Xi)$ let us denote $y_\theta := x_{\xi_1} + \dots + x_{\xi_n}$. Then there arises a net $(y_\theta)_{\alpha\in\Theta}$, where $\Theta := \mathcal{P}_{\text{fin}}(\Xi)$ is naturally ordered by inclusion. If there is an $x = o - \lim_{\theta\in\Theta} y_\theta$, then the element x is termed the *o-sum* of the family (x_ξ) and we write $x = \sum_{\xi\in\Xi} x_\xi$. It is obvious that, given $x_\xi \geq 0$ $(\xi \in \Xi)$, for the o-sum of the family (x_ξ) to exist it is necessary and sufficient that the net $(y_\theta)_{\theta\in\Theta}$ be order-bounded, in which case $o\text{-}\sum_{\xi\in\Xi} x_\xi = \sup_{\theta\in\Theta} y_\theta$. If the elements of the family (x_ξ) are pairwise disjoint, then

$$o\text{-}\sum_{\xi\in\Xi} x_\xi = \sup_{\xi\in\Xi} x_\xi^+ - \sup_{\xi\in\Xi} x_\xi^-.$$

Any K-space E is *o-complete* in the following sense. If the net $(x_\alpha)_{\alpha\in A}$ in E satisfies the condition $\limsup |x_\alpha - x_\beta| = \inf_{\gamma\in A} \sup_{\alpha,\beta\geq\gamma} |x_\alpha - x_\beta| = 0$, then there is such an element $x\in E$, that $x = o - \lim x_\alpha$.

5.1.7. Examples

(1) Let $(E_\alpha)_{\alpha\in A}$ be a family of vector lattices (f-algebras) over the same ordered field $\mathbf{F}$. Then the Cartesian product $E := \prod_{\alpha\in A} E_\alpha$, considered with the coordinatewise operations and order, is a vector lattice (f-algebra) over the field $\mathbf{F}$. In this case the lattice E is order-complete, extended or discrete iff all the cofactors E_α have this property. The basis $\mathfrak{E}(E)$ is isomorphic to the product of the family of the Boolean algebras $(\mathfrak{E}(E))_{\alpha\in A}$. An element $e\in E$ is a unit iff $e(\alpha)$ is a unit in E_α for all $\alpha\in A$. In particular, the set $\mathbf{R}^A$ ($\mathbf{C}^A$) of all real (complex) functions on a nonempty set A is an extended discrete K-space (complex K-space). It should be emphasized that an extended K-space is often referred to as *universally complete* in the western literature.

(2) Any ideal and, therefore, the foundation of a vector lattice (of a K-space) is a vector lattice (a K-space). The base of a vector lattice is isomorphic to that of each of its foundations (a foundation is recalled to be an order-dense o-ideal). In particular, $l_p(A)$ is a K-space for any $1\leq p\leq\infty$ (see (1)).

(3) Let N be an o-ideal of a vector lattice E. Then the factor-space $\tilde{E} := E/N$ is also a vectors lattice, provided the order relation in it is determined by the positive cone $\varphi(E^+)$, where $\varphi: E\to\tilde{E}$ is the canonical factor-homomorphism. The vector lattice $\tilde{E}$ is Archimedean iff N is closed relative to convergence with regulator. If E is an f-algebra, and the o-ideal N is a ring ideal as well, then $\tilde{E}$ is an f-algebra. If E is a K_σ-space and N is sequentially o-closed, then $\tilde{E}$ is a K_σ-space, while the homomorphism φ is sequentially o-continuous. The bases of the vector lattice $\tilde{E}$ is isomorphic to the complete Boolean algebra of Δ-components $\mathfrak{K}_\Delta(E)$, where $\Delta := \{(x,y)\in E\times E : |x|\wedge|y|\in N\}$.

(4) Let $(\Omega,\mathfrak{A})$ be a measurable space, i.e., Ω is a nonempty set and $\mathfrak{A}$ is a σ-algebra of its subsets. Let us denote by $\mathfrak{M}(\Omega,\mathfrak{A})$ the set of all real (complex) measurable functions on Ω with the operations and order induced from $\mathbf{R}^\Omega$ (from $\mathbf{C}^\Omega$). Let us choose a σ-ideal $\mathfrak{N}$ of the algebra $\mathfrak{A}$. Let N be a set of such functions $f\in\mathfrak{M}(\Omega,\mathfrak{A})$ that $\{t\in\Omega : f(t)\neq 0\}\in\mathfrak{N}$, and assign $M(\Omega,\mathfrak{A},\mathfrak{N}) := \mathfrak{M}(\Omega,\mathfrak{A})/\mathfrak{N}$. Then $\mathfrak{M}(\Omega,\mathfrak{A})$ and $M(\Omega,\mathfrak{A},\mathfrak{N})$ are real (complex) K_σ-spaces and and f-algebras at the same time. Let us assume that $\mu:\mathfrak{A}\to\mathbf{R}\cup\{+\infty\}$ is a countably additive positive measure. The vector lattice

$M(\Omega,\mathfrak{A},\mu) := M(\Omega,\mathfrak{A},\mu^{-1}(0))$ will be an extended K-space provided the measure μ is either finite or σ-finite. Generally speaking, the order completeness of the lattice $M(\Omega,\mathfrak{A},\mu)$ is related with the direct sum property for the measure μ [**159**, **227**, **279**]. For simplicity, however, we shall confine ourselves to a σ-finite measure μ. The space $M(\Omega,\mathfrak{A},\mu)$ is continuous iff μ has no atoms. It should be recalled that the ***atom of a measure*** is a set $A \in \mathfrak{A}$ such that $0 < \mu(A)$ and if $A' \in \mathfrak{A}$, $A' \subset A$, then either $\mu(A') = 0$ or $\mu(A') = \mu(A)$. If $M(\Omega,\mathfrak{A},\mu)$ is discrete then the measure μ is *purely atomic*, i.e., any set of nonzero measure contains an atom of μ. The class of equivalence of a function which is identically equal to unity will be an order and ring unit in $M(\Omega,\mathfrak{A},\mu)$.

The base of the K-space $M(\Omega,\mathfrak{A},\mu)$ is isomorphic to the Boolean algebra $\mathfrak{A}/\mu^{-1}(0)$ of measurable sets modulo sets of zero measure. By virtue of (2), the spaces $L_p(\Omega,\mathfrak{A},\mu)$ $(1 \le p \le \infty)$, which are foundations of $M(\Omega,\mathfrak{A},\mu)$, are also K-spaces.

(5) Let H be a complex Hilbert space and $\mathfrak{U}$ be a strongly-closed commutative algebra of selfadjoint bounded operators in H. By the letter B we shall denote the set of all ortho-projections in H belonging to the algebra $\mathfrak{U}$. Then B is a complete Boolean algebra. Let $\mathfrak{U}_\infty$ be a set of all densely defined selfadjoint operators a in H such that the spectral function $\lambda \to e_\lambda^a$ $(\lambda \in \mathbf{R})$ of the operator a assumes its values in B. Let $\overline{\mathfrak{U}}_\infty$ be a set of densely defined normal operators a in H such that if $a = u|a|$ is the polar decomposition of a, then $|a| \in \mathfrak{U}_\infty$. In the sets $\mathfrak{U}_\infty$ and $\overline{\mathfrak{U}}_\infty$ the structure of an ordered vector space is introduced in a natural way. Thus, for a and $b \in \mathfrak{U}_\infty$ the sum $a + b$ and the product $a \cdot b$ are determined as the unique selfadjoint extensions of the operators $h \to ah + bh$, and $h \to a \cdot bh$, $h \in \mathfrak{D}(a) \cap \mathfrak{D}(b)$, where $\mathfrak{D}(c)$ is the domain of definition for c. Moreover, for $a \in \mathfrak{U}_\infty$ we set $a \ge 0$ iff $\langle ah,h \rangle \ge 0$ for all $h \in \mathfrak{D}(a)$. The operations and order in $\overline{\mathfrak{U}}_\infty$ are obtained by complexifying $\mathfrak{U}_\infty$.

The sets $\mathfrak{U}_\infty$ and $\overline{\mathfrak{U}}_\infty$ with the operations and order discussed above are an extended K-space and a complex extended K-space, respectively, with the base of unit elements B. In this case $\mathfrak{U}$ is a K-space of bounded elements in $\mathfrak{U}_\infty$.

(6) Let Q be a topological space, while $\mathcal{B}(Q) := \mathcal{B}(Q,\mathbf{R})$ be the set of all Borel functions from Q to $\mathbf{R}$ with pointwise operations of addition and multiplication, as well as with the pointwise order relation. Then $\mathcal{B}(Q,\mathbf{R})$ is a K_σ-space. By N we denote the set of such Borel functions $f \in \mathcal{B}(Q)$ that $\{t \in Q : f(t) \ne 0\}$ is a meager set (i.e., a set of the first category). Let $B(Q)$ be the factor-space of $\mathcal{B}(Q) / N$ with the operations and order induced from $B(Q)$. Then $B(Q)$ is a K-space whose base is isomorphic to the Boolean algebra of Borel subsets Q modulo the first-category sets. If the topological space Q is *Baire* (i.e., any open set in Q is not meager), then the base $\mathfrak{E}(B(Q))$ is isomorphic to the Boolean algebra of all regular open (or regular closed) subsets Q. Each of the spaces $\mathcal{B}(Q)$ and $B(Q)$ is an exact f-algebra. The function identically equal to unity serves as an order and ring unit in them. Having replaced $\mathbf{R}$ with $\mathbf{C}$, we get the complex K-space $B(Q)$.

(7) Let Q be again a topological space, while $C(Q)$ be the space of all continuous real functions on Q. Then $C(Q)$ is a sublattice and a subalgebra in $\mathcal{B}(Q)$. In particular, $C(Q)$ is an exact Archimedean f-algebra. Generally speaking, $C(Q)$ is not a K-space. The order completeness of $C(Q)$ is associated with the total disconnectedness of the space Q (see **1.2.5**) For a uniformizable topological space Q the base of the vector space $C(Q)$ is isomorphic to the algebra of regular open sets.

Let now $LSC(Q)$ be the set of (the equivalence classes of) lower semicontinuous functions $f: Q \to$ R$:= \mathbf{R} \cup \{\pm\infty\}$ such that $f^{-1}(-\infty)$ is nowhere dense, while the interior of the set $f^{-1}([-\infty,\infty))$ is dense in Q. As usual, two functions are considered equivalent if their values differ only on a meager set. The sum $f+g$ (the product $f \cdot g$) of the elements $f, g \in LSC(Q)$ will be determined as the lower semicontinuous regularization of the pointwise sum $t \to f(t)+g(t)$ $(t \in Q_0)$ (of the pointwise product $t \to f(t) \cdot g(t)$ $(t \in Q_0)$), where Q_0 is a dense subset of Q on which f and g are finite. Therefore, $LSC(Q)$ turns into an extended K-space and an f-algebra, in which case the base of $LSC(Q)$ is isomorphic to the algebra of regular open sets. Therefore, when Q is Baire, the K-spaces $B(Q)$ and LSC(Q) are isomorphic, while if Q uniformizable, then $C(Q)$ is a (order) dense sublattice in $LSC(Q)$.

5.1.8. A special role in the theory of vector lattices is played by spaces of continuous functions assuming infinite values on a nowhere dense set. In order to introduce such a space, some additional facts are to be employed. For an arbitrary function $f: Q \to \overline{\mathbf{R}}$ and a number $\lambda \in \overline{\mathbf{R}}$, we shall denote

$$\{f < \lambda\} := \{t \in Q : f(t) < \lambda\}, \quad \{f \leq \lambda\} := \{t \in Q : f(t) \leq \lambda\}.$$

(1) *Let Q be an arbitrary topological space, Λ be a dense set in $\overline{\mathbf{R}}$ and $\lambda \to U_\lambda$ $(\lambda \in \Lambda)$ be an increasing mapping from Λ to the set $\mathcal{P}(Q)$ ordered by inclusion. Then the following statements are equivalent:*

(a) *there is a unique and continuous function $f: Q \to \overline{\mathbf{R}}$ such that*

$$\{f < \lambda\} \subset U_\lambda \subset \{f \leq \lambda\} \quad (\lambda \in \Lambda),$$

(b) *for any $\lambda, \mu \in \Lambda$, it follows from $\lambda < \mu$ that*

$$\mathrm{cl}(U_\lambda) \subset \mathrm{int}(U_\lambda).$$

$\triangleleft$ The implication (a) $\Rightarrow$ (b) is trivial. Let us prove (b) $\Rightarrow$ (a). For every $t \in Q$, put $f(t) := \inf\{\lambda \in \Lambda : t \in U_\lambda\}$. We, thus, have defined the function $f: Q \to \overline{\mathbf{R}}$ and can now easily prove that $\{f < \lambda\} \subset U_\lambda \subset \{f \le \lambda\}$. It is also obvious that

$$\{f < \lambda\} = \cup\{U_\mu : \mu < \lambda, \mu \in \Lambda\}, \quad \{f \le \lambda\} = \cap\{U_v : \mu < v, v \in \Lambda\}.$$

It should be observed that so far we have made use of only the isotonicity of the mapping $\lambda \to U_\lambda$. Let us also consider the mappings

$$\lambda \to V_\lambda := \mathrm{int}(U_\lambda),\ \lambda \to W_\lambda := \mathrm{cl}(U_\lambda) \quad (\lambda \in \Lambda).$$

These mappings are seen to increase as well and, hence, by what was above proved above, there are such functions g and $h: Q \to \overline{\mathbf{R}}$ that

$$\{g < \lambda) \subset V_\lambda \subset \{g \le \lambda\}, \{h < \lambda) \subset W_\lambda \subset \{h \le \lambda\} \quad (\lambda \in \Lambda).$$

It follows from the definition of W_λ that for $\mu < \lambda$ $U_\mu \subset W_\lambda$. Since Λ in $\mathbf{R}$ is dense, for any $t \in Q$ and $v > f(t)$ there are such $\lambda, \mu \in \Lambda$ that $f(t) < \mu < \lambda < v$ and, hence, $t \in U_\mu \subset W_\lambda$ and $h(t) < \lambda < v$. Tending v to $f(t)$, we get $h(t) \le f(t)$, the same inequality being obvious for $f(t) = +\infty$, too. Writing the relation (b) as $W_\mu \subset V_\lambda$ $(\mu < \lambda)$, we again conclude, using the considerations presented above, that $g(t) \le h(t)$ for all $t \in Q$. Therefore, $f = g = h$. The fact that f is continuous follows from the inequalities

$$\{f < \lambda\} = \{g < \lambda\} = \cup\{V_\mu : \mu < \lambda, \mu \in \Lambda\},$$
$$\{f \le \lambda\} = \{h \le \lambda\} = \cap\{W_\mu : \mu < \lambda, \mu \in \Lambda\},$$

since V_μ is open, while W_μ is closed for all $\mu \in \Lambda$. ▷

(2) Let Q be a totally disconnected compactum, i.e., Q is a compact topological space wherein the closure of every open set is open (and closed). Let Q_0 be an open dense subset of Q and let $f: Q_0 \to \mathbf{R}$ is a continuous function. Then there is a unique continuous function $\bar{f}: Q_0 \to \overline{\mathbf{R}}$ such that $f(t) = \bar{f}(t)$ $(t \in Q_0)$.

◁ Indeed, if U_μ: cl$\{f < \mu\}$, then the mapping $\mu \to U_\mu$ $(\mu \in \mathbf{R})$ increases and meets the condition (b) of (1). Therefore, there is a unique function $\bar{f}: Q \to \overline{\mathbf{R}}$ with the properties $\{\bar{f} < \mu\} \subset U_\mu \subset \{\bar{f} \le \mu\}$ $(\mu \in \mathbf{R})$. It is obvious that in this case $\bar{f} \upharpoonright Q_0 = f$. ▷

(3) Let us denote by the symbol $C_\infty(Q)$ the set of all continuous functions $x: Q \to \overline{\mathbf{R}}$ that can assume the values $\pm\infty$ only on a nowhere dense set. Introduce into $C_\infty(Q)$ a relation of order by setting $x \le y$ iff $x(t) \le y(t)$ for all $t \in Q$. Then, choose $x, y \in C_\infty(Q)$ and set $Q_0 := \{|x| < +\infty\} \cap \{|y| < +\infty\}$. In this case Q_0 is open and dense in Q. According to (2), there is a unique continuous function $z: Q \to \overline{\mathbf{R}}$ such that $z(t) = x(t) + y(t)$ for $t \in Q_0$. It this function of z that we shall assume to be the sum of the elements x and y. In an analogous way one can determine the product of each pair of elements. Identifying the number λ with the function identically equal to λ on Q, we get the product of any $x \in C_\infty(Q)$ and $\lambda \in \mathbf{R}$.

One can easily see that $C_\infty(Q)$ with the operations and order introduced above is a vector lattice and an exact f-algebra simultaneously. Below we shall prove that $C_\infty(Q)$ is an extended K-space. The function identically equal to unity is a ring and order unit. The base of the vector space of $C_\infty(Q)$ is isomorphic to the Boolean algebra of all clopen subsets of the compact set Q.

5.2. Boolean-Valued Analysis of Vector Lattices

In the present section we prove that Archimedean vector lattices are realizable as subgroups of an additive group of real numbers in an appropriate Boolean-valued model. Such an approach enables one to deduce the basic structural properties of vector lattices, such as functional calculus, integral presentation of elements, presentation by spaces of functions, etc..

5.2.1. Let $\mathbf{R}$ be the linearly ordered field of real numbers, while $\mathbf{R}^\wedge$ be its image under the canonical embedding of the class of all sets into the universe $\mathbf{V}^{(B)}$ (see **2.2.7**). Since $\mathbf{R}$ is an algebraic system of signature $\sigma := (+,\cdot,0,1,\leq)$, then, by virtue of corollary **4.3.5** (1), $\mathbf{R}^\wedge$ is an algebraic system of signature $\sigma^\wedge$ inside $\mathbf{V}^{(B)}$. Moreover, for any formula $\varphi(u_0,\dots,u_{n-1})$ of signature σ and for any $x_0,\dots,x_{n-1} \in \mathbf{R}$, $\varphi(x_0,\dots,x_{n-1})$ is fulfilled on $\mathbf{R}$ iff inside $\mathbf{V}^{(B)}$ $\varphi(x_0^\wedge,\dots,x_{n-1}^\wedge)$ is fulfilled on $\mathbf{R}^\wedge$. In particular, we can choose as φ the axioms of an Archimedean linearly ordered field. Hence, $\mathbf{V}^{(B)} \models$ "$\mathbf{R}^\wedge$ is an Archimedean linearly ordered field". However, $\mathbf{R}^\wedge$ cannot be claimed to be the field of real numbers inside $\mathbf{V}^{(B)}$. The point is that the axiom of completeness for the field of real numbers is not expressed by a bounded formula. Here is one of the equivalent formulations of the axiom of completeness:

$$(\forall A)\,(A \subset \mathbf{R} \wedge \neq \varnothing \wedge \pi_{\leq}(A) \neq \varnothing \to (\exists x \in \mathbf{R})(x = \sup(A))),$$

i.e., any nonempty set of real numbers bounded above has a least upper bound as well. In this axiom the universal quantifier runs over the powerset of $\mathbf{R}$.

It should be recalled (see **3.1.1**) that $B_0(\mathbf{R}) := \mathbf{R}^{\wedge\downarrow}$ consists of all mixings of the type $\operatorname{mix}_{t \in \mathbf{R}}(b_t t^\wedge)$, where $(b_t)_{t \in \mathbf{R}}$ is a partition of unity in B. According to theorem **4.4.10**, $B_0(\mathbf{R})$ is an extended exact f-ring. The f-ring $B_0(\mathbf{R})$ can be identified with the f-ring of all continuous functions x from the Stone compactum Q of the algebra B to the set

$\overline{\mathbf{R}} := \mathbf{R} \cup \{\pm\infty\}$ with the discrete topology, which assume the values $\pm\infty$ on a nowhere dense set. Obviously, $B_0(\mathbf{R})$ is indeed an f-algebra, since we can assume $\mathbf{R} \subset B_0(\mathbf{R})$ when identifying the number λ with the function identically equal to λ on Q.

5.2.2. By virtue of the principles of transfer and maximum, there is such an element $\mathfrak{R} \in \mathbf{V}^{(B)}$ that $\mathbf{V}^{(B)} \models$ "$\mathfrak{R}$ is an ordered field of real numbers". It is obvious that inside $\mathbf{V}^{(B)}$ the field $\mathfrak{R}$ is unique up to isomorphism, i.e., in $\mathfrak{R}'$ is another field of real numbers inside $\mathbf{V}^{(B)}$, then $\mathbf{V}^{(B)} \models$ "$\mathfrak{R}$ and $\mathfrak{R}'$ are isomorphic". As has been pointed out above, $\mathbf{R}^\wedge$ is an Archimedean ordered field inside $\mathbf{V}^{(B)}$ and, hence, we can assume that $\mathbf{V}^{(B)} \models$ "$\mathbf{R}^\wedge \in \mathfrak{R}$ and $\mathfrak{R}$ is the (metric) completion of the field $\mathbf{R}^\wedge$ ". In this case for the unit 1 of the field $\mathbf{R}^\wedge$ we get $\mathbf{V}^{(B)} \models$ "$1 := 1^\wedge$ is a unit of the field $\mathfrak{R}'$ ".

Let us now consider the descent $\mathfrak{R}\downarrow$ of the algebraic system $\mathfrak{R} := (|\mathfrak{R}|, +, \cdot, 0, 1, \leq)$. In other words, the descent of the carrier of the system $\mathfrak{R}$ is considered together with the descended operations and order in $\mathfrak{R}$. In more detail, addition, multiplication and order in $\mathfrak{R}$ are introduced by the following rules (see **4.2.3**):

$$
\begin{aligned}
&x + y = z \leftrightarrow [x + y = z] = \mathbf{1},\\
&xy = z \leftrightarrow [xy = z] = \mathbf{1},\\
&x \leq y \leftrightarrow [x \leq y] = \mathbf{1},\\
&\lambda x = y \leftrightarrow [\lambda^\wedge x = y] = \mathbf{1},\\
&(x, y, z \in \mathfrak{R}\downarrow, \lambda \in \mathbf{R}).
\end{aligned}
$$

Gordon theorem. *Let $\mathfrak{R}$ be the field of real numbers in the model $\mathbf{V}^{(B)}$. The algebraic system $\mathfrak{R}\downarrow$ (i.e., a set $|\mathfrak{R}|\downarrow$ with the operations and order descended) is an extended K-space. In this case there is an isomorphism χ from the Boolean algebra B onto the Boolean algebra of projections $\mathfrak{Pr}(\mathfrak{R}\downarrow)$ (or of unit elements $\mathfrak{E}(\mathfrak{R}\downarrow)$) such that the following equivalences are valid:*

$$
\begin{aligned}
&\chi(b)x = \chi(b)y \leftrightarrow b \leq [x = y],\\
&\chi(b)x \leq \chi(b)y \leftrightarrow b \leq [x \leq y]
\end{aligned}
$$

for all $x, y \in \mathfrak{R}\downarrow$ and $b \in B$.

⊲ The proof of this result can actually be found in **4.4.10**. Indeed, by **4.4.10** (2,4), $\mathfrak{R}\downarrow$ is an extended and order-complete f-ring with unit $\mathbf{1} := 1^\wedge$. The mapping $\lambda \to \lambda^\wedge \cdot 1$ is an isomorphism of the field $\mathbf{R}$ in $\mathfrak{R}\downarrow$. Letting $\lambda x := \lambda^\wedge x$ $(x \in \mathfrak{R}\downarrow, \lambda \in \mathbf{R})$, we obtain the required vector structure on $\mathfrak{R}\downarrow$. Therefore, $\mathfrak{R}\downarrow$ is an extended K-space. ⊳

5.2.3. With the same notations as in **5.2.2**, let us elucidate the sense of some statements in terms of the K-space $\mathfrak{R}\downarrow$.

(1) *Let* $(b_\xi)_{\xi\in\Xi}$ *be a partition of unity in* B *and* $(x_\xi)_{\xi\in\Xi}$ *be an arbitrary family in* $\mathfrak{R}\downarrow$. *Then*

$$\underset{\xi\in\Xi}{mix}(b_\xi x_\xi) = o\text{-}\sum_{\xi\in\Xi}\chi(b_\xi)x_\xi.$$

◁ Indeed, if $x = mix(b_\xi x_\xi)$, then the definition of mixing, with theorem **5.2.2** taken into account, implies that $\chi(b_\xi)x = \chi(b_\xi)x_\xi$ for every ξ. Summing this relation over ξ, we get the required result. ▷

(2) *For a set* $A\subset\mathfrak{R}\downarrow$ *and arbitrary* $a\in\mathfrak{R}\downarrow$ *and* $b\in B$ *the following equivalence is valid:*

$$\chi(b)a = \sup(\chi(b)(A)) \leftrightarrow b \le [a = \sup(A\uparrow)].$$

◁ Indeed, by virtue of **5.2.2**, the equality $\chi(b)a = \sup\{\chi(b)x\colon x\in A\}$ holds iff $b\le[x\le a]$ for all $a\in A$ and for every $y\in\mathfrak{R}\downarrow$ the relation $(\forall x\in A)\times(b\le[x\le y])$ implies $b\le[a\le y]$. The last statement can be easily seen to be a different expression of the relation $b\le[\sup(A\uparrow)=a]$. ▷

(3) *Let us consider a net* $s\colon \mathrm{A}\to\mathfrak{R}\downarrow$, *where* A *is a directed set. Then the modified ascent* $s\uparrow\colon \mathrm{A}^\wedge\to\mathfrak{R}$ *is a net inside* $\mathbf{V}^{(B)}$, *in which case for any* $x\in\mathfrak{R}\downarrow$ *and* $b\in B$ *we have*

$$\chi(b)x = o\text{-}\lim(\chi(b)\circ s) \leftrightarrow b\le[x=\lim(s\uparrow)].$$

◁ The relation $\chi(b)x = o\text{-}\lim(\chi(b)\circ s)$ is equivalent to the existence of a net $r\colon\mathrm{A}\to\mathfrak{R}\downarrow$ such that $r(\alpha)\le r(\beta)$ for $\alpha\le\beta$, $\inf\{r(\alpha)\colon\alpha\in\mathrm{A}\}=0$ and $|\chi(b)x-\chi(b)s(\alpha)|\le\chi(b)r(\alpha)$ for all $\alpha\in\mathrm{A}$. By virtue of **5.2.3** (2) and the equality $r(\mathrm{A})\uparrow = r\uparrow(\mathrm{A}^\wedge)$, the last three relations imply that the following inequalities are valid:

$$b\le[(\forall\alpha\in\mathrm{A}^\wedge)(|x-s\uparrow(\alpha)|\le r(\alpha))],$$
$$b\le[\inf(r\uparrow(\mathrm{A}^\wedge)=0)],$$
$$b\le[(\forall\alpha,\beta\in\mathrm{A}^\wedge)(\alpha\le\beta\to r\uparrow(\alpha)\le r\uparrow(\beta)],$$

or, in short, $b\le[x=\lim(s\uparrow)]$, which was required. ▷

The following proposition is proved in an absolutely analogous way.

(4) *Let* s *and* $\mathrm{A}\in\mathbf{V}^{(B)}$ *be such that* $[s\colon\mathrm{A}\to\mathfrak{R}$ *is a net* $]=1$. *Then the descent*

$s{\downarrow}: A{\downarrow} \to \mathcal{R}{\downarrow}$ *is a net, in which case, for any* $x \in \mathcal{R}{\downarrow}$ *and* $b \in B$, *we have*

$$\chi(b)x = o - \lim(\chi(b) \circ s{\downarrow}) \leftrightarrow b \le [x = \lim(s)].$$

(5) *For every element* $x \in \mathcal{R}{\downarrow}$ *the following equalities are valid:*

$$e_x = \chi([x \ne 0]), \quad e_\lambda^x = \chi([x < \lambda]) \quad (\lambda \in \mathbf{R}).$$

⊲ It should be observed that a real number t is distinct from zero iff the least upper bound of the set $\{1 \wedge (n|t|): n \in \omega\}$ is 1. Hence, according to the principle of transfer, for $x \in \mathcal{R}{\downarrow}$ we have $[x \ne 0] = [\sup\{1^\wedge \wedge (n|x|): n \in \omega^\wedge\} = 1]$. If $A := \{1 \wedge (n|x|): n \in \omega\}$, then $[\sup(A{\uparrow}) = \sup\{1^\wedge \wedge (n|x|): n \in \omega^\wedge\}] = 1$ and $e_x = \sup(A)$. Therefore, $b := [x \ne 0] = [e_x = 1]$ and, analogously, $b^* = [e_x = 0]$. Making use of the properties of χ, we deduce $e_x = \chi(b)$. Let us now choose an arbitrary number $\lambda \in \mathbf{R}$ and note that $\lambda^\wedge = \lambda^\wedge 1$ and, hence, $e_\lambda^x = e_{(\lambda^\wedge - x)^+}$. Using the facts proved earlier, we get

$$\chi^{-1}(e_\lambda^x) = [(\lambda^\wedge - x) \vee 0 \ne 0] = [(\lambda^\wedge - x) > 0] = [x < \lambda^\wedge]. \ ⊳$$

5.2.4. Theorem. *Let* X *be an Archimedean vector lattice with base* $B := \mathfrak{E}(X)$. *Let* $\mathcal{R}$ *be the field of real numbers in the model* $\mathbf{V}^{(B)}$. *Then there is a linear and lattice isomorphism* i *from* X *into an extended* K*-space* $\mathcal{R}{\downarrow}$ *such that the following conditions are met:*

(1) *the isomorphism* i *preserves the least upper and greatest lower bounds of any nonempty bounded sets;*

(2) *the order ideal* $J(i(X))$ *generated by a set* $i(X)$ *is a foundation of* $\mathcal{R}{\downarrow}$;

(3) *for any* $y \in J(i(X))$ *the following equalities are valid:*

$$\inf\{i(x): x \in X \wedge i(x) \ge y\} = y = \sup\{i(x): x \in X \wedge i(x) \le y\};$$

(4) *for* $x \in X$ *and* $b \in B$, *we have* $b \le [i(x) = 0]$, *iff* $x \in b^\perp$.

⊲ In theorem **4.4.12** we have already proved that there is a subgroup $\mathcal{X}$ of the additive group of the field of real numbers $\mathcal{R} \in \mathbf{V}^{(B)}$ as well as an additive and lattice isomorphism $i := \iota_X$ from X to $\mathcal{X}$. Let e be a nonzero positive element of the group $\mathcal{X}$. Replacing, when necessary, $\mathcal{X}$ with the group $e^{-1}\mathcal{X}$ isomorphic to it, we can assume $e = 1 \in \mathcal{X}$. It should be recalled that $X^\wedge$ is a vector space over the field $\mathbf{R}^\wedge$. It is obvious that under these circumstances the factor-mapping $\varphi := \varphi_X : X^\wedge \to \mathcal{X}$ is $\mathbf{R}^\wedge$-linear. In particular, $[\varphi((\lambda x)^\wedge) = \lambda^\wedge \varphi(x^\wedge)] = \mathbf{1}$ $(\varphi \in \mathbf{R}, x \in X)$. Therefore, $[i(\lambda x) = \lambda^\wedge i(x)] = \mathbf{1}$, or $i(\lambda x) = \lambda i(x)$ (see **5.2.2**). Now for $\mathbf{1} = \mathrm{mix}(b_\xi i(e_\xi))$, $(e_\xi) \subset X$, and for $\lambda \in \mathbf{R}$ we can write

$$b_\xi \le [\lambda^\wedge = \lambda^\wedge \cdot ie_\xi] \wedge [\lambda^\wedge \cdot ie_\xi = i(\lambda e_\xi)] \wedge [i(\lambda e_\xi) \in \mathcal{X}] \le [\lambda^\wedge \in \mathcal{X}].$$

Therefore, $\lambda^\wedge \in \mathcal{X}$ and, hence, $[\mathbf{R}^\wedge \subset \mathcal{X} \subset \mathfrak{R}] = \mathbf{1}$. Moreover, $\mathbf{V}^{(B)} \models$ "$\mathcal{X}$ is a vector sublattice of the field $\mathfrak{R}$ considered as a vector lattice over $\mathbf{R}^\wedge$". In this case, however, $\mathcal{X}\downarrow$ is a vector sublattice of the extended K-space $\mathfrak{R}\downarrow$, while i can be considered as an embedding of X in $\mathfrak{R}\downarrow$. The task now is to check whether (1) - (4) are valid.

(1) Let us choose such $A \subset X$ and $a \in X$ that $a = \sup(A)$. Let $z = \sup(i(A))$, where the supremum is calculated in $\mathfrak{R}\downarrow$.

From an obvious relation $[\mathcal{X}$ is minorant in $\mathfrak{R}] = \mathbf{1}$ we can easily deduce that $\mathcal{X}\downarrow$ is minorant in $\mathfrak{R}\downarrow$. In this case, however, $i(X)$ is also minorant in $\mathfrak{R}\downarrow$ (see **4.4.12**). If $i(a) > z$, then for a certain $0 < x \in X$ we have $i(x) \le i(a) - z$, or $z \le i(a - x)$, which implies that $a - x$ is an upper bound of the set A and, by virtue of the equality $a = \sup(A)$ we must get $a - x > a$ or $x \le 0$. This contradiction shows that $z = i(a)$.

(2) As far as $z = i(a)$ i(X) is minorant in $\mathfrak{R}\downarrow$, we have $\mathfrak{R}\downarrow = i(X)^{\perp\perp}$. Moreover, the equality $\mathfrak{R}\downarrow = J(i(X))^{\perp\perp}$ holds, where $J(i(X))$ is the order ideal generated by the set $i(X)$.

(3) The relation $[\mathbf{R}^\wedge \subset \mathcal{X} \subset \mathfrak{R}] = \mathbf{1}$ allows one to conclude that $\mathbf{V}^{(B)} \models$ "$\mathcal{X}$ is a dense subgroup in $\mathfrak{R}$". Hence, for any $x \in \mathfrak{R}\downarrow$ inside $\mathbf{V}^{(B)}$ we have

$$\inf\{x' \in X : x' \ge x\} = x = \sup\{x' \in X : x' \le x\}.$$

Applying **5.2.3** (2), we directly deduce from the above relation

$$\inf\{x' \in \mathcal{X}\downarrow : x' \ge x\} = x = \sup\{x' \in \mathcal{X}\downarrow : x' \le x\}$$

and all we have to do now to complete the proof is to take into account the fact that $i(X)$ is minorant in $\mathcal{X}\downarrow$.

(4) This statement has been proved in **4.4.12**. ▷

5.2.5. Notice some corollaries to the theorem proved above.

(1) *Let X be an Archimedean vector lattice, the base of which, $\mathfrak{B}(X)$, is isomorphic to the Boolean algebra B. There is an element $\mathcal{X} \in \mathbf{V}^{(B)}$ which obeys the following conditions:*

(a) $\mathbf{V}^{(B)} \models$ "*$\mathcal{X}$ is a vector sublattice of the field of real numbers $\mathcal{R}$ considered as a vector space over $\mathbf{R}^{\wedge}$*";

(b) *$X' := \mathcal{X}\downarrow$ is an extended vector lattice with projections, which is an r-dense sublattice of the K-space $\mathcal{R}\downarrow$;*

(c) *There is a linear and lattice isomorphism $\iota: X \to X'$ preserving suprema and infima, in which case for any $x \in X'$ there is a partition of unity $(\pi_\xi)_{\xi\in\Xi}$ in $\mathfrak{B}(X')$, and a family $(x_\xi)_{\xi\in\Xi}$ in X such that*

$$x = o - \sum_{\xi\in\Xi} \pi_\xi \circ i(x_\xi).$$

◁ All the statements above can, in essence, be found in **5.2.4.** Let us, for instance, prove that X' is r-dense in $\mathcal{R}\downarrow$. If $x \in \mathcal{R}\downarrow$, then $\mathbf{V}^{(B)} \models$ "x is a real number and it can be approximated within any accuracy by the elements of $\mathcal{X}$". In other words, the following equality is valid:

$$[\![(\forall \varepsilon \in \mathbf{R}^{\wedge})(\varepsilon > 0 \to (\exists \lambda \in \mathcal{X})(|\lambda - x| < \varepsilon))]\!] = \mathbf{1}.$$

Writing out Boolean truth-values for the quantifiers, for any $\varepsilon > 0$ we can find a $\lambda \in X'$ such that $|\lambda - x| \le \varepsilon \mathbf{1}$, which completes the proof. ▷.

(2) *If X is a K-space, then $\mathcal{X} = \mathcal{R}$, while $i(X)$ is a foundation in $\mathcal{R}\downarrow$. The whole of $\mathcal{R}\downarrow$ serves as the image of X under the isomorphism i iff X is an extended K-space.*

◁ A proof results from **5.2.2** and **5.2.4** (2,3). ▷

(3) *Extended K-spaces are order-isomorphic iff they have isomorphic bases.*

◁ Indeed, if X and Y are extended K-spaces, while h is an order isomorphism of X on Y, then the correspondence $K \to h(K)$ $(K \in \mathfrak{B}(X))$ is an isomorphism of the bases. Vice versa, if $\mathfrak{B}(X)$ and $\mathfrak{B}(Y)$ are isomorphic to a Boolean algebra B then, making use of (2), X and Y are order-isomorphic to the extended K-space $\mathcal{R}\downarrow$. ▷

(4) An *extension* of a K-space X is a pair (Y,i), where Y is also a K-space, while i is an isomorphism of X on a certain foundation in Y. Let us introduce a preorder in the class $\mathrm{Ext}(X)$ of all extensions of the K-space as follows. For (Y,i) and $(Z,j) \in \mathrm{Ext}(X)$ we shall set $(Y,i) \prec (Z,j)$ provided there is an isomorphism h of the space Y on a certain foundation in Z such that $h \circ i = j$. The maximal elements of the preordered class $\mathrm{Ext}(X)$ are called *maximal extensions* of X (sometimes the term *universal completion* is used). The following result ensues from (1) and (2).

Any K-space has a maximal extension. Such a maximal extension is unique up to an order isomorphism and presents an extended K-space.

(5) *Let X be an extended K-space with a fixed order unit* $\mathbf{1}$.. *In this case it is a unique possibility of determining multiplication in X in such a manner that X becomes an exact f-ring, while* $\mathbf{1}$ *becomes the identity of multiplication.*

$\lhd$ Let us identify a number $\lambda \in \mathbf{R}$ with the element $\lambda \cdot \mathbf{1}$. By virtue of (2), X is isomorphic to $\mathfrak{R}\downarrow$ and under such an isomorphism $\mathbf{1}$ is transformed to $1 := 1^{\wedge} \in \mathfrak{R}\downarrow$, since $[1^{\wedge}$ is a unit of the field $\mathfrak{R}\downarrow] = \mathbf{1}$. The descent of the operation of multiplication in $\mathfrak{R}$ supplies us with the required multiplicative structure. If $\times : X^2 \to X$ is another multiplication in X satisfying the above conditions, then it is extensional and its ascent $(\times)\uparrow$ is a multiplication in $\mathfrak{R}$ with the unit $\mathbf{1}$. It is obvious that in this case $\times = \cdot$, since the multiplicative structure of the field $\mathfrak{R}$ is unique. $\rhd$

(6) For any Archimedean vector lattice X there is a K-space Y, unique up to linear and lattice isomorphism, as well as a linear isomorphism $j: X \to Y$ preserving suprema and infima such that

$$\sup\{j(x) : x \in X, j(x) \le y\} = y = \inf\{j(x) : x \in X, j(x) \ge y\}.$$

$\lhd$ Let $\mathfrak{R}$ and $J(i(X))$ be the same as in **5.2.4**. Then a pair $(J(i(X)), i)$ obeys all the properties required. If (Y,j) is a pair with the same properties, then the bases $\mathfrak{B}(Y)$ and $\mathfrak{B}(\mathfrak{R}\downarrow)$ are mutually isomorphic and, hence, by virtue of (2), the K-spaces $m\,Y$ and $\mathfrak{R}\downarrow$ are also isomorphic. Therefore, we can assume that $i(X) \subset Y \subset \mathfrak{R}\downarrow$, in which case Y is a foundation of $\mathfrak{R}\downarrow$. Then $(J(i(X)) \subset Y$. For every $y \in Y$, however, there must exist such x' and $x'' \in X$, that $i(x') \le y \le i(x'')$, i.e., there must be $Y \subset j(i(X))$. $\rhd$

5.2.6. Theorem. *Let X be an arbitrary K_{σ}-space with order unit* 1. *The spectral function $\lambda \to e_{\lambda}^{x}$ $(\lambda \in \mathbf{R})$ of an element $x \in X$ has the following properties:*

(1) $e_{\lambda}^{x} \le e_{\mu}^{x}$ for $\lambda \le \mu$;

(2) $e^x_{+\infty} := \bigvee_{\mu \in \mathbf{R}} e^x_\mu = 1, \quad e^x_{-\infty} := \bigwedge_{\mu \in \mathbf{R}} e^x_\mu = 0$;

(3) $\bigvee_{\mu<\lambda} e^x_\mu = e^x_\lambda \quad (\lambda \in \mathbf{R})$;

(4) $x \le y \leftrightarrow (\forall \lambda \in \mathbf{R})(e^y_\lambda \le e^x_\lambda)$;

(5) $e^{x+y}_\lambda = \vee\{e^x_\mu \cdot e^y_v : \mu, v \in \mathbf{R}, \mu + v = \lambda\}$;

(6) $e^{xy}_\lambda = \vee\{e^x_\mu \cdot e^y_v : 0 \le \mu, v \in \mathbf{R}, \mu v = \lambda\}\ (x \ge 0, y \ge 0)$;

(7) $e^{-x}_\lambda = \vee\{1 - e^x_{-\mu} : \mu \in \mathbf{R}, \mu < \lambda\} = (1 - e^x_{-\lambda}) \cdot e_{x+\lambda 1}$;

(8) $x = \inf(A) \leftrightarrow (\forall \lambda \in \mathbf{R})(e^x_\lambda = \vee\{e^a_\lambda : a \in A\})$;

(9) $e^{x \vee y}_\lambda = e^x_\lambda \cdot e^y_\lambda$;

(10) $e^{cx}_\lambda = ce^x_\lambda + c^*$ for $\lambda > 0$,
$e^{cx}_\lambda = ce^x_\lambda$ for $\lambda \le 0\ (c \in \mathfrak{S}(X))$.

When calculating exact bounds in (2), (3) *and* (5)-(7), *one can consider* μ *and* v *to assume values in a certain dense subfield* $\mathbf{P} \subset \mathbf{R}$.

◁ By virtue of theorem **5.2.4**, we can assume, without loss of generality, that $X = \mathfrak{R}\downarrow$. In this case, however, the required statements are easily deduced from **5.2.3** (5) and properties of numbers. Let us prove, for instance, (6) and (8). Let $x \ge 0$, $y \ge 0$ and assume that there is a product $x \cdot y$. Then x and y are nonnegative numbers inside $\mathbf{V}^{(B)}$. According to **5.2.3** (5), $e^x_\lambda = \chi([\![x \cdot y < \lambda^\wedge]\!])$, $e^x_\lambda = \chi([\![x < \lambda^\wedge]\!])$ and $e^y_\lambda = \chi([\![y < \lambda^\wedge]\!])$. Inside $\mathbf{V}^{(B)}$, however, we have

$$(\forall x, y \in \mathfrak{R})(x \ge 0 \wedge y \ge 0 \rightarrow (x \cdot y < \lambda$$
$$\leftrightarrow (\exists 0 < \mu, v \in \mathbf{P}^\wedge)(x < \mu) \wedge (y < v) \wedge (\lambda = \mu v))),$$

and, hence,

$$[\![x \cdot y < \lambda^\wedge]\!] = \bigvee_{\substack{0<\mu, v \in \mathbf{P} \\ \lambda = \mu v}} \{[\![x < \mu^\wedge]\!] \wedge [\![y < v^\wedge]\!]\},$$

whence we get the required result.

Let now $A \subset X$ and let us assume that $x = \inf(A)$. Then $e^x_\lambda = \chi([\![x < \lambda^\wedge]\!]) = \chi([\![\inf(A\uparrow) < \lambda^\wedge]\!])$ (see **5.2.3** (1,5)). However, $A\uparrow$ is a certain nonempty set of real numbers inside $\mathbf{V}^{(B)}$, and, hence,

$$\mathbf{V}^{(B)} \models \inf(A\uparrow) < \lambda^{\wedge} \leftrightarrow (\exists a \in A\uparrow)(a < \lambda^{\wedge}).$$

Calculating Boolean truth-values, we find

$$[x < \lambda^{\wedge}] = \bigvee_{a \in A} [a < \lambda^{\wedge}],$$

and, hence,

$$e_{\lambda}^{x} = \vee\{\chi([a < \lambda^{\wedge}]) : a \in A\} = \vee\{e_{\lambda}^{a} : a \in A\}.$$

Conversely, let us assume that e_{λ}^{x} is the supremum of the set $\{e_{\lambda}^{a} : a \in A\}$, for $\lambda \in \mathbf{R}$. Then

$$[x < \lambda^{\wedge}] = [(\exists a \in A\uparrow)(a < \lambda^{\wedge})] = [\inf(A\uparrow) < \lambda^{\wedge}]$$

for every $\lambda \in \mathbf{R}$ and, hence,

$$[(\forall \lambda \in \mathbf{R}^{\wedge})(x < \lambda \leftrightarrow \inf(A\uparrow) < \lambda)] = \mathbf{1}.$$

The preceding expression implies $[x = \sup(A\uparrow)] = \mathbf{1}$ and, applying **5.2.3** (2), we get $x = \inf(A)$.

The last claim of the theorem results from the fact that if $\mathbf{P}$ is a dense subfield of $\mathbf{R}$, then $\mathbf{V}^{(B)} \models$ "the field $\mathbf{P}^{\wedge}$ is dense in $\mathfrak{R}$". ▷

5.2.7. Now establish the following three useful characteristics of o-convergence.

(1) *Let X be again a K-space with order unit* $\mathbf{1}$.. *Choose an order-bounded net* $(x_{\alpha})_{\alpha} \in A$ *of positive elements in* X. *Then* (x_{α}) *o-converges to zero iff for any* $0 < \varepsilon \in \mathbf{R}$ *the net of unit elements* $\left(e_{\varepsilon}^{x_{\alpha}}\right)$ *o-converges to* $\mathbf{1}$.

Indeed, by theorem **5.2.4**, x_{α} can be considered as positive elements of the K-space $\mathfrak{R}\downarrow$. The mapping $s: \alpha \to s_{\alpha}$ has the modified ascent $\delta := s\uparrow$, which is a set in $\mathfrak{R}$, i.e., a numerical net inside $\mathbf{V}^{(B)}$. According to **5.2.3** (3), $o - \lim(x_{\alpha}) = 0$ iff $[\lim(\delta) = 0] = \mathbf{1}$, which can be rewritten in an equivalent form as

$$\mathbf{V}^{(B)} \models (\forall \varepsilon \in \mathbf{R}^{\wedge})(\varepsilon > 0 \to (\exists \alpha \in \mathrm{A}^{\wedge})(\forall \beta \in \mathrm{A}^{\wedge})(\beta \geq \alpha \to x_{\beta} < \varepsilon)).$$

Writing out Boolean truth-values for quantifiers, we find another equivalent presentation:

$$(\forall \varepsilon > 0)(\exists (b_\alpha))(\forall \beta \in \mathrm{A})(\alpha \le \beta \to b_\alpha \le [x_\beta < \varepsilon^\wedge]),$$

where (b_α) is a partition of unity in B. Finally, applying **5.2.3** (5), we get

$$(\forall \varepsilon > 0)(\exists (b_\alpha)_{\alpha \in \mathrm{A}})(\forall \beta \in \mathrm{A})(\alpha \le \beta \to \chi(b_\alpha) \le e_\varepsilon^{x_\beta})$$

or

$$(\forall \varepsilon > 0)(\exists (b_\alpha)_{\alpha \in \mathrm{A}})(\chi(b_\alpha) \le \wedge \{e_\varepsilon^{x_\beta} \ge \alpha\}).$$

Since $\vee(b_\alpha) = \mathbf{1}$, the expression o-lim $x_\alpha = 0$ is seen to be equivalent to the following statement: for every $\varepsilon > 0$ we have

$$o - \lim(e_\varepsilon^{x_\alpha}) = \liminf (e_\varepsilon^{x_\alpha}) = \bigvee_{\alpha \in \mathrm{A}} \wedge \{e_\varepsilon^{x_\beta} : \beta \ge \alpha\} = \mathbf{1}. \ \rhd$$

(2) *An order-bounded net (x_α) in a K-space X with order unit o-converges to an element $x \in X$ iff for any $\varepsilon > 0$ there is a partition of unity $(\pi_\alpha)_{\alpha \in \mathrm{A}}$ in $\mathfrak{Pr}(X)$ such that*

$$\pi_\alpha |x - x_\beta| \le \varepsilon \mathbf{1} \quad (\alpha, \beta \in \mathrm{A}, \beta \ge \alpha).$$

$\lhd$ For the proof we again use **5.2.4.** Let s, δ be the same as in (1). Reasoning as above, we find out that $x_\alpha \xrightarrow{o} x$ is equivalent to the following condition: for any $\varepsilon > 0$ there is a partition of unity $(b_\alpha)_{\alpha \in \mathrm{A}}$ in B such that

$$b_\alpha \le [|x_\beta - x| \le \varepsilon^\wedge] \quad (\alpha, \beta \in \mathrm{A}, \beta \ge \alpha).$$

If $\pi_\alpha := \chi(b_\alpha)$ (see **5.2.2.**), then the last relation implies that

$$\pi_\alpha |x_\beta - x| \le \varepsilon \mathbf{1} \quad (\alpha, \beta \in \mathrm{A}, \beta \ge \alpha). \ \rhd$$

(3) *An order-bounded net (x_α) in the K-space X with order unit o-converges to an element $x \in X$ iff for any $\varepsilon > 0$ there is an increasing net of projections (ρ_α) such that $o - \lim(\rho_\alpha) = I_X$ and*

$$\rho_\alpha |x - x_\beta| \le \varepsilon \mathbf{1} \quad (\alpha, \beta \in \mathrm{A}, \beta \ge \alpha).$$

$\lhd$ Indeed, this is true if we put $\rho_\alpha := \vee\{\pi_\beta : \beta \ge \alpha\}$ in (2). $\rhd$

5.2.8. Let us now turn our attention to results on functional realization of vector spaces.

(1) Let B be a complete Boolean algebra. A *resolution of unity* (not to be mixed with partition!) in the algebra B is a mapping $e:\mathbf{R}\to B$ having the properties **5.2.6** (1-3) of a spectral function. The set of all resolutions of unity in B will be denoted by the symbol $\mathfrak{S}(B)$. In this set let us introduce addition, multiplication by real numbers and the order according to the following rules (cf. **5.2.6** (4-6)):

$$(e_1+e_2)(\lambda):=\vee\{e_1(\mu)\cdot e_2(v):\mu,v\in\mathbf{R};\ \ \mu+v=\lambda\};$$
$$(\alpha e)(\lambda):=e(\lambda/\alpha)\quad(\alpha>0);$$
$$(-e)(\lambda):=\bigvee_{\mu<\lambda}1-e(-\mu)=1-\bigwedge_{\mu<\lambda}e(-\mu);$$
$$(0\cdot e)(\lambda):=\mathbf{0}(\lambda):=\begin{cases}1,\text{if } \lambda>0,\\ 0,\text{if } \lambda\le 0;\end{cases}$$
$$e_1\le e_2\leftrightarrow(\forall\lambda\in\mathbf{R})\,e_1(\lambda)\ge e_2(\lambda).$$

The set $\mathfrak{S}(B)$ with the operations and order introduced above is an extended K-space isomorphic to $\mathfrak{R}\downarrow$.

◁ In line with **5.2.2**, without loss of generality, we can assume B to be the base of unit elements of the K-space $\mathfrak{R}\downarrow$. Let us put in correspondence to an element $X\in\mathfrak{R}\downarrow$ its spectral function $\lambda\to e_\lambda^x$ $(\lambda\in\mathbf{R})$. Thus, we have obtained an injective lattice homomorphism from $\mathfrak{R}\downarrow$ to $\mathfrak{S}(B)$, as is seen from theorem **5.2.6**. It is now necessary to substantiate the fact that this homomorphism is surjective. Let us choose an arbitrary resolution of unity e: $\mathbf{R}\to B$. Let Σ be a set of all partitions of the numerical straight lime, i.e., $\sigma\in\Sigma$ if $\sigma:\mathbf{Z}\to\mathbf{R}$ is a strictly increasing function, $\lim\limits_{n\to\infty}\sigma(n)=\infty$ and $\lim\limits_{n\to\infty}\sigma(-n)=-\infty$ (as usual, $\mathbf{Z}$ is the set of integer numbers). In an extended K-space $\mathfrak{R}\downarrow$ there is a sum $x_\sigma:=\sum\limits_{n\in\mathbf{Z}}\overline{\sigma}(n+1)b_{n\sigma}$, where $b_{n\sigma}:=e(\sigma(n+1))-e(\sigma(n))$. Let us set $A:=\{x_\sigma:\ \sigma\in\Sigma\}$ and $x=\inf A$. The infimum does exist since $x_\sigma\ge\sum\limits_{n\in\mathbf{Z}}\overline{\sigma}(n)b_{n\sigma}$ for a fixed partition $\overline{\sigma}\in\Sigma$. Let us also remark that $x_\sigma=\operatorname{mix}(b_{n\sigma}\sigma(n)^\wedge)$ and

$$[x_\sigma<\lambda^\wedge]=\vee\{b_{n\sigma}:\sigma(n)<\lambda\}=\vee\{e(\sigma(n)):\sigma(n)<\lambda\}.$$

Since $[x=\inf(A\uparrow)]=\mathbf{1}$, the following calculations are valid:

$$[x<\lambda^\wedge]=[(\exists a\in A\uparrow)a<\lambda^\wedge]$$
$$=\bigvee_{a\in A}[a<\lambda^\wedge]=\bigvee_{\sigma\in\Sigma}\ \bigvee_{\sigma(n)<\lambda}b_{n\sigma}$$
$$=\bigvee_{\sigma\in\Sigma}\ \bigvee_{\sigma(n)<\lambda}e(\sigma(n))=\bigvee_{\mu<\lambda}e(\mu)=e(\lambda).$$

Therefore, e is the spectral function of the element x. ▷

(2) **Theorem.** *Let Q be the Stone space of a complete Boolean algebra B, while $\mathfrak{R}$ be the field of real numbers in the model $\mathbf{V}^{(B)}$. The vector lattice $C_\infty(Q)$ is an extended K-space which is linear and lattice isomorphic to $\mathfrak{R}$t. An isomorphism can be established by assigning the function $\hat{x}: Q \to \overline{\mathbf{R}}$ to an element $x \in \mathfrak{R}\downarrow$ by the formula*

$$\hat{x}(q) := \inf\{\lambda \in \mathbf{R}: [x < \lambda^\wedge] \in q\}.$$

◁ We have already established in (1) that the K-space $\mathfrak{R}\downarrow$ is isomorphic to the space of all B-valued spectral functions, in which case the function $\lambda \to [x < \lambda^\wedge]$ $(\lambda \in \mathbf{R})$ corresponds to an element $x \in \mathfrak{R}\downarrow$. Let an element $[x < \lambda^\wedge] \in B$ be assigned to a clopen set U_λ of the Stone space Q. Then, by virtue of **5.8.1** (2), to every element $x \in \mathfrak{R}\downarrow$ there corresponds a unique continuous function $\hat{x}: Q \to \overline{\mathbf{R}}$ such that $\{\hat{x} < \lambda\} \subset U_\lambda \subset \{\hat{x} \leq \lambda\}$. In this case, however, $\hat{x}(q) = \inf\{\lambda \in \mathbf{R}: q \in U_\lambda\} = \ = \inf\{\lambda \in \mathbf{R}: [x < \lambda^\wedge] \in q\}$. The relations $\wedge\{[x < \lambda^\wedge]\} = 0$ and $\vee\{[x < \lambda^\wedge]\} = 1$ imply that the closed set $\bigcap\{U_\lambda : \lambda \in \mathbf{R}\}$ has an empty interior, while the set $\bigcup\{U_\lambda : \lambda \in \mathbf{R}\}$ is dense in Q. Therefore, the function $\hat{x}$ can assume the values $\pm\infty$ only on a nowhere dense set and, hence, $\hat{x} \in C_\infty(Q)$. An elementary checking if $\hat{x}$ is a linear and lattice isomorphism is omitted. ▷

5.2.9. Let us note some corollaries to the theorem just proved.

(1) *Let X be an arbitrary K-space and $\{e_\xi\}_{\xi \in \Xi}$ be a complete set of pairwise disjoint positive elements in X Let Q be the Stone space of the Boolean algebra of bands of $\mathfrak{E}(X)$. Then there is a unique linear and lattice isomorphism of X on a foundation of the K-space $C_\infty(Q)$ such that e_ξ transforms into the characteristic function of a certain clopen set $Q_\xi \subset Q$, This isomorphism puts the function $\hat{x}: Q \to \overline{\mathbf{R}}$ into correspondence to an element $x \in X$ by the rule*

$$\hat{x}(q) := \inf\{\lambda \in \mathbf{R}: \{e_\lambda^\xi\}^{\perp\perp} \in q\} \quad (q \in Q_\xi),$$

where (e_λ^ξ) is the characteristic of the band projection of x on $\{e_\lambda^\xi\}^{\perp\perp}$ with respect to the unit e_ξ.

(2) *The space X is an extended (K-space of bounded elements) iff under the given isomorphism its image is all $C_\infty(Q)$ (the subspace $C(Q)$ of all continuous finite functions on the compactum Q).*

(3) *Any Archimedean vector lattice (f-algebra) X is both linearly and lattice-isomorphic to a vector sublattice (and a subalgebra) of the space $C_\infty(Q)$, where Q is the Stone space of the base $\mathfrak{E}(X)$.*

By the symbol $C_\infty(Q,S\mathbf{Z})$ let us denote the subset of the functions of $C_\infty(Q)$ assuming integer values on an clopen set $S \subset Q$. It is obvious that $C_\infty(Q,S\mathbf{Z})$ is an extended f-ring.

(4) *A complete lattice-ordered group G is isomorphic to a foundation of the extended lattice-ordered group $C_\infty(Q,S\mathbf{Z})$, where Q is the Stone space of the base $\mathfrak{E}(G)$.*

$\triangleleft$ If $\mathcal{G}$ is the Boolean-valued realization of G then, by virtue of **4.4.10** and **4.4.12**, $\mathcal{G}$ is a complete totally ordered group. In this case, however, $\mathcal{G}$ is either isomorphic to $\mathfrak{R}$ or is an infinite cyclic group. Therefore, there is such a $b \in B$ that $b = [\mathcal{G} \simeq \mathbf{Z}^\wedge]$ and $b^* = [\mathcal{G} \simeq \mathfrak{R}]$. In the same way as in **4.4.13** we establish that G is expanded into a direct sum of two components, one of which is implemented as $\mathfrak{R}$ in $\mathbf{V}^{([0,b^*])}$, while the other as $\mathbf{Z}$ in $\mathbf{V}^{([0,b])}$. Now we have to apply theorem (1) and observe that $\mathbf{Z}^\wedge\downarrow \simeq B_0(\mathbf{Z}) \simeq C_\infty(Q,S\mathbf{Z})$, where S is an clopen set in Q corresponding to an element $b \in B$. $\triangleright$

In an analogous way we can deduce the following statement.

(5) *Any f-ring is o-isomorphic to the product of two f-rings K_1 and K_2 such that K_1 is a foundation and subring of an extended f-ring $C_\infty(Q_1,S_1\mathbf{Z})$, while K_2 is a foundation of the extended group $C_\infty(Q_2,S_2\mathbf{Z})$ with zero multiplication, where Q_l is the Stone space of the algebra $\mathfrak{E}(K_l)$ and $S_l \in \mathfrak{E}(Q_l)$ $(l = 1,2)$.*

5.2.10. Let us construct an integral of Stieltjes type over a spectral measure. Let Ω be an arbitrary nonempty set, while Σ be a certain σ-algebra of subsets of Ω. Let us consider a Boolean algebra B of unit elements of a fixed K_σ-space X. A *spectral measure* is a σ-continuous Boolean homomorphism μ from Σ to B. Here σ-continuity means that for any sequence $(e_n)_{n\in\omega}$ of elements of the σ-algebra Σ we have $\mu\left(\bigvee_{n=0}^{\infty}\right) = \bigvee_{n=0}^{\infty} \mu(e_n)$.

Let us choose a measurable function $f\colon \Omega \to \mathbf{R}$. For an arbitrary partition of a numerical straight line $\Lambda := (\lambda_k)_{k\in\mathbf{Z}}$, $-\infty \leftarrow \ldots \lambda_{-1} < \lambda_0 < \lambda_1 < \ldots \to +\infty$ let us compose the integral sums

$$\underline{\sigma}(f,\Lambda) = \sum_{-\infty}^{\infty} \lambda_n \mu(e_n), \quad \overline{\sigma}(f,\Lambda) = \sum_{-\infty}^{\infty} \lambda_{n+1} \mu(e_n),$$

where the sums are calculated in X. For any choice of $t_n \in e_n$ $(n \in \mathbf{Z})$ we obviously have

$$\underline{\sigma}(f,\Lambda) \le \sum_{-\infty}^{\infty} f(t_n)\mu(e_n) \le \overline{\sigma}(f,\Lambda).$$

It is also clear that, while partitions are refined, $\underline{\sigma}(f,\Lambda)$ increases and $\overline{\sigma}(f,\Lambda)$ decreases. If there is such an element $x \in X$, that $\underline{\sigma}(f,\Lambda) = x = \inf \overline{\sigma}(f,\Lambda)$, where the exact bounds are taken over all possible partitions $\Lambda := (\lambda_l)_{l \in \mathbf{Z}}$ of the numerical straight line as $\delta(\Lambda) := \sup\{\lambda_n - \lambda_{n-1}\} \to 0$, then the function f is said *to be integrable with respect to the spectral measure* and we write

$$I(f) := I_\mu(f) := \int_\Omega f(t)\, d\mu(t) := x.$$

Since $0 \le \overline{\sigma}(f,\Lambda) - \underline{\sigma}(f,\Lambda) \le \sum_{n=-\infty}^{\infty} \delta\mu(e_k) = \delta\mathbf{1}$, where $\delta := \delta(\Lambda)$, for the integral $I_\mu(f)$ to exist it is necessary and sufficient that there exist $\overline{\sigma}(f,\Lambda)$ and $\underline{\sigma}(f,\Lambda)$ for at least one partition of Λ. In particular, a bounded measurable spectral function is integrable.

(1) *Let $X = \mathfrak{R}\downarrow$ and μ be a spectral measure with values in $B := \mathfrak{S}(X)$. Then for any measurable function f we have*

$$[I_\mu(f) < \lambda^\wedge)] = \mu(\{f < \lambda\}) \quad (\lambda \in \mathbf{R}),$$

in which case $I_\mu(f)$ is the only element of the K-space X obeying this condition.

$\lhd$ Let us choose an arbitrary number $\lambda \in \mathbf{R}$, and let $b \le [\lambda^\wedge \le I_\mu(f)]$. According to theorem **5.2.2**, for any partition Λ we have $b\lambda \le bI_\mu(f) \le b\overline{\sigma}(f,\Lambda)$. If the partition $\Lambda := (\lambda_l)_{l \in \mathbf{Z}}$ is such that $\lambda_0 = \lambda$ and $c_n := \{u \in \Omega\colon \lambda_n \le f(u) < \lambda_{n+1}\}$, then for $n < -1$ we get $\lambda b \wedge \mu(c_n) \le \lambda_{n+1} b \wedge \mu(c_n)$ and, hence, $b \wedge \mu(c) = 0$, or $b \le \mu(c)^* = \mu(\Omega - c) = \mu(\{f \ge \lambda\})$. Therefore, $[I_\mu(f) \ge \lambda^\wedge] = \mu(\{f < \lambda\})$, which is equivalent to the relation sought.

Let us assume that $[x < \lambda^\wedge] = \mu(\{f < \lambda\})$ for some $x \in X$. By virtue of the above-stated property of $I_\mu(f)$ we can write

$$[(\forall \lambda \in \mathbf{R}^\wedge)\,(I_\mu(f) < \lambda \leftrightarrow x < \lambda)]$$
$$= \bigwedge_{\lambda \in \mathbf{R}} [I_\mu(f) < \lambda] \Leftrightarrow [x < \lambda^\wedge] = \mathbf{1}.$$

By the density of $\mathbf{R}^\wedge$ in $\mathfrak{R}$, we get $x = I_\mu(f)$. $\rhd$

(2) *Under the conditions of proposition* (2) *the function* $\lambda \to \mu(\{f < \lambda\})$, $\lambda \in \mathbf{R}$ *is the characteristic of the element* $I_\mu(f)$.

5.2.11. Theorem. *Let X be an extended K_σ-space, while $\mu: \Sigma \to B = \mathfrak{S}(X)$ be a spectral measure. The spectral integral $I_\mu(\cdot)$ is a sequentially o-continuous (linear, multiplicative and lattice) homomorphism from the f-algebra of measurable functions $\mathfrak{M}(\Omega, \Sigma)$ to X.*

◁ Without loss of generality, we can assume $X \subset \mathcal{R}{\downarrow}$. The sums $\bar{\sigma}(f,\Lambda)$ and $\underline{\sigma}(f,\Lambda)$ do exist, since pairwise disjoint elements are summed, while the space X is extended. This fact, as has been remarked earlier, implies that there exists the integral $I_\mu(f)$. It is obvious that the operator I_μ is both linear and positive. Let us prove that it is sequentially o-continuous. Choose a decreasing sequence $(f_n)_{n\in\omega}$ of measurable functions for which $\lim\limits_{n\to\infty} f_n(t) = 0$ for all $t \in \Omega$. Assign $x_n := I_\mu(f_n)$ $(n \in \omega)$ and $0 < \varepsilon \in \mathbf{R}$. If we denote $c_n := \{t \in \Omega: f_n(t) < \varepsilon\}$, then $\Omega = \bigcup\limits_{n=0}^{\infty} c_n$. By virtue of propositions **5.2.3** (5) and **5.2.10** (2), we can write

$$o-\lim_{n\to\infty} e_\varepsilon^{x_n} = o-\lim_{n\to\infty} \mu(c_n) = \bigvee_{n\in\omega} \mu(c_n) = \mu(\Omega) = 1.$$

By virtue of the test for o-convergence **5.2.7** (1), we get $o-\lim\limits_{n\to\infty} x_n = 0$. Then, for any measurable functions f and $g: \Omega \to \mathbf{R}$ it follows from **5.2.6** (9) and **5.2.10** (2) that

$$\begin{aligned} e_\lambda^{f\vee g} &= \mu(\{f \vee g < \lambda\}) = \mu(\{f < \lambda\} \cap \{g < \lambda\}) \\ &= \mu(\{f < \lambda\}) \wedge \mu(\{g < \lambda\}) = e_\lambda^{I(f)} \wedge e_\lambda^{I(g)} = e_\lambda^{I(f)\vee I(g)}. \end{aligned}$$

Therefore, $I(f \vee g) = I(f) \vee I(g)$, which implies that $I := I_\mu$ is a lattice homomorphism. Analogously, for $f \geq 0$ and $g \geq 0$ it follows from **5.2.6** (6) and **5.2.8** (2) that for $\lambda \in \mathbf{Q}$ we get

$$\begin{aligned} e_\lambda^{I(fg)} &= \mu(\{fg < \lambda\}) = \vee\{\mu(\{f < \aleph\}) \wedge (\{g < \upsilon\}): \lambda = \upsilon\aleph, 0 \leq \aleph, \upsilon \in \mathbf{Q}\} \\ &= \vee\{e_\aleph^{I(f)} \cdot e_\upsilon^{I(g)}: 0 \leq \aleph, \upsilon \in \mathbf{Q}, \upsilon\aleph = \lambda\} = e_\lambda^{I(f)\cdot I(g)}. \end{aligned}$$

Hence, $I(f) \cdot I(g) = I(fg)$. For arbitrary f and g the last equality follows from the earlier established properties of the spectral integral:

$$I_\mu(fg) = I_\mu(f^+g^+) + I_\mu(f^-g^-) - I_\mu(f^+g^-) - I_\mu(f^-g^+)$$
$$= I_\mu(f)^+ I_\mu(g)^+ + I_\mu(f)^- I_\mu(g)^- - I_\mu(f)^- I_\mu(g)^+ - I_\mu(f)^+ I_\mu(g)^-$$
$$= I_\mu(f)\cdot I_\mu(g).\ \triangleright$$

5.2.12. *Let $e_0,\dots,e_{n-1}:\mathbf{R}\to B$ be an arbitrary finite set of spectral functions with values in a σ-algebra B. Then there is a unique B-valued spectral measure μ defined on the Borel σ-algebra $\mathcal{B}(\mathbf{R}^n)$ of the space $\mathbf{R}^n$ such that*

$$\mu\left(\prod_{l=0}^{n-1}(-\infty,\lambda_l)\right) = \bigwedge_{l=0}^{n-1} e_l(\lambda_l),$$

for arbitrary $\lambda_0,\dots,\lambda_{n-1}\in\mathbf{R}$.

$\triangleleft$ Without loss of generality, we can assume that $B = \mathfrak{A}(Q)$, where Q is the Stone space of B. According to **5.2.8** (2), there are continuous functions $x_l: Q\to\overline{\mathbf{R}}$ $(l := 0,\dots,n-1)$ such that $e_l(\lambda) = \{x_\lambda < \lambda\}$ for all $\lambda\in\mathbf{R}$ and $l = 0,\dots,n-1$. Assign $f(t) := (x_0(t),\dots,x_{n-1}(t))\in\mathbf{R}^n$, if all $x_l(t)$ are finite and $f(t) = \infty$ provided $x_l(t) = +\infty$ for at least one index l. We, thus, have defined a continuous mapping $f: Q\to\mathbf{R}^n\cup\{\infty\}$ (a filter base of neighbourhoods of the point ∞ consists of all the complements of arbitrary balls centred at zero). It is obvious that f is measurable relative to the Borel algebras $\mathcal{B}(Q)$ and $\mathcal{B}(\mathbf{R}^n)$. Let $\mathfrak{E}_\sigma(Q)$ be the σ-algebra of the Q subsets generated by the algebra $\mathfrak{A}(Q)$, while Δ be the σ-ideal in $\mathfrak{E}_\sigma(Q)$ consisting of meager sets. In this case there is an isomorphism h of the factor-algebra $\mathfrak{E}_\sigma(Q)/\Delta$ on a σ-algebra $B := \mathfrak{E}(Q)$. Let us denote by $[A]_\Delta$ the class of equivalence of a set $A\in\mathfrak{E}_\sigma(Q)$. Now we determine $\mu:\mathcal{B}(\mathbf{R}^n)\to B$ by the formula

$$\mu(A) := h([f^{-1}(A)]_\Delta)\quad (A\in\mathcal{B}(\mathbf{R}^n)).$$

It is obvious that μ is a spectral measure. If $A = \prod_{l=0}^{n-1}(-\infty,\lambda_l)$, then

$$f^{-1}(A) = \bigcap_{l=0}^{n-1}\{x_l < \lambda_l\} = \bigwedge_{l=0}^{n-1} e_l(\lambda_l),$$

and, hence, $\mu(A) = e_0(\lambda_0)\wedge\dots\wedge e_{n-1}(\lambda_{n-1})$. If μ' is another spectral measure with the same properties as μ, then the set $\mathcal{B} := \{A\subset\mathbf{R}^n : \mu(A) = \mu'(A)\}$ is a σ-algebra and contains all sets of the type $(-\infty,\lambda_0)\times\dots\times(-\infty,\lambda_{n-1})$. Therefore, $\mathcal{B}(\mathbf{R}^n)\subset\mathcal{B}$ and $\mu = \mu'$. $\triangleright$

Now let us choose elements $x_0,\dots,x_{n-1}$ of a K_σ-space X with unit $\mathbf{1}$. Let

$e^{x_l}: \mathbf{R} \to B := \mathfrak{S}(1)$ be the characteristic of the element x_l. In accord with the proposition proved above, there is a spectral measure $\mu: \mathcal{B}(\mathbf{R}^n) \to B$ for which

$$\mu\left(\prod_{l=0}^{n-1}(-\infty,\lambda_l)\right) = \bigwedge_{l=0}^{n-1} e^{x_l}(\lambda_l).$$

The integral of a measurable function $f: \mathbf{R}^n \to \mathbf{R}$ with respect to measure μ will be denoted by $I(f,\mathfrak{x}) := I(f, x_0, \dots, x_{n-1})$, where $\mathfrak{x} := (x_0, \dots, x_{n-1})$. It should be recalled that $\mathcal{B}(\mathbf{R}^n, \mathbf{R})$, which is the space of all Borel functions from $\mathbf{R}^n$ to $\mathbf{R}$, is a K_σ-space and an exact f-algebra.

5.2.13. Theorem. *For any ordered set* $\mathfrak{x} := (x_0, \dots, x_{n-1})$ *of elements of an extended* K_σ-*space* X *the mapping* $f \to I(f,\mathfrak{x})$ $(f \in \mathcal{B}(\mathbf{R}^n, \mathbf{R}))$ *is a homomorphism of the* f-*algebra* $\mathcal{B}(\mathbf{R}^n, \mathbf{R})$ *in* X *which obeys the following conditions:*

(1) $I(d\lambda_l, \mathfrak{x}) = x_l$ $(l < n)$, *where* $d\lambda_l: \mathbf{R}^n \to \mathbf{R}$ *is the* l-*th coordinate function* $(\lambda_0, \dots, \lambda_{n-1}) \to \lambda_l$;

(2) *if a sequence* $(f_k) \subset \mathcal{B}(\mathbf{R}^n, \mathbf{R})$ *is such that* $\lim_{n\to\infty} f_k(t) = f(t)$ *for all* $t \in \mathbf{R}^n$, *then* $o\text{-}\lim_{n\to\infty}(f_k, \mathfrak{x}) = I(f, \mathfrak{x})$.

$\triangleleft$ According to theorem **5.2.11**, it suffices to prove only statement (1). For simplicity, we limit ourselves to the case when $n = 1$.

So, let $x \in X$, and μ be a spectral measure associated with the characteristic $(e^x_\lambda)_{\lambda \in \mathbf{R}}$ of the element x. Prove then that

$$x = \int_{\mathbf{R}} \lambda \, d\mu(\lambda) := \int_{\mathbf{R}} \lambda \, de^x_\lambda.$$

Choose an arbitrary number $\varepsilon > 0$. Let a partition $\Lambda := (\lambda_l)$ of the numerical straight line be such that $\lambda_{l+1} - \lambda_l < \varepsilon$ for all $l \in \mathbf{Z}$. Assign

$$\sigma := \sum_{-\infty}^{\infty} \xi_n \mu([\lambda_{n-1}, \lambda_n)) = \sum_{-\infty}^{\infty} \xi_n (e^x_{\lambda_n} - e^x_{\lambda_{n-1}}),$$

where $\xi_n \in [\lambda_{n-1}, \lambda_n)$. By virtue of **5.2.3** (2),

$$b_n := e^x_{\lambda_n} - e^x_{\lambda_{n-1}} = e^x_{\lambda_n} \wedge (e^x_{\lambda_{n-1}})^* = [\lambda^\wedge_{n-1} \le x < \lambda^\wedge_n].$$

It should be observed that $b_n = [\xi_n^\wedge = \sigma]$ (see **5.2.2**). On the other hand,

$$b_n = [\lambda_{n-1}^\wedge \le x < \lambda_n^\wedge] \wedge [\lambda_n^\wedge - \lambda_{n-1}^\wedge \le \varepsilon^\wedge]$$
$$\wedge [\lambda_{n-1}^\wedge \le \xi_n < \lambda_n^\wedge] \le [|x - \xi_n^\wedge| \le \varepsilon^\wedge],$$

and, hence, $[|x - \sigma| \le \varepsilon^\wedge] = \mathbf{1}$, or $|x - \sigma| < \varepsilon \mathbf{1}$. This implies that x is the r-limit of the integral sums in question.

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