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# NONSTANDARD METHODS FOR KANTOROVICH SPACES

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*Abstract.* A Kantorovich space or a $K$-space is an order complete vector lattice. Such spaces may be viewed as sublattices of the reals via Boolean-valued analysis. Infinitesimal analysis is also beneficial. Interaction of the two nonstandard ideas forms the bulk of exposition.

It is universally recognized that the thirties of the 20th century play a special role in modern science. Outlined at the turn of the century, the tendency towards drastic reorganization of mathematics has revealed itself since these years. It led to the creation of a number of new mathematical subjects, functional analysis being the first of them. Nowadays we realize the exceptional place of the seventies framed sweeping changes both in volume and in essence of mathematical theories. In the period mentioned, a qualitative leap forward was registered in understanding mutual interrelation and interdependence of mathematical subjects; outstanding advances took place in working out new synthetic approaches and in finding solutions to certain deep and profound problems unsolved for a long time.

The processes indicated above are characteristic of the theory of ordered vector spaces — the latter being one of the most actual and attractive branches of functional analysis.

This trend marking the beginning of the thirties and due to the influence of contributions of F. Riesz, L. V. Kantorovich, H. Freudenthal, G. Birkhoff, et al., experiences now a certain period of revival connected with assimilating mathematical ideas related to nonstandard models of set theory. Boolean-valued interpretations (which acquire a popularity in connection with P. J. Cohen's final solution to the continuum hypothesis) open new possibilities of interpreting L. V. Kantorovich's heuristic transfer principle.

A. Robinson's nonstandard analysis has in turn legitimized the development of infinitesimal methods, substantiating G. W. Leibniz's logical dream and opening broad vistas to general monadology of vector lattices. Brand-new nonstandard methods in $K$-space theory are under way.

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Expanding the well-known lines of N. S. Gumilëv, we can say that presently $K$-spaces "... are sloughing their skins to make room for souls to grow and mature ... ." Many of the arising lacunas are not filled in yet due to the short time for settling the corresponding problems rather than for the lack of proper understanding. At the same time a number of principal questions stand in line and waits for comprehension and novel ideas.

The aim of this paper is to present, to the broad community of specialists interested in methods of the theory of ordered vector spaces, a review of research into adaptation of the tools of nonstandard set theory for investigating $K$-spaces and classes of linear operators acting in them.

## 1. BASIC AXIOMATIC SET THEORIES

In this section we expose necessary information about formal set theories used in modern $K$-space theory. First of all the classical Zermelo–Fraenkel axiomatics is dealt with. Furthermore, Boolean-valued models ascending to the works of D. Scott, R. Solovay, and P. Vopenka are tackled. In addition, we present one of the most powerful and promising variants of external set theories which was recently proposed by T. Kawai and is widely used in modern infinitesimal analysis.

### *1.1. Zermelo–Fraenkel set theory*

The Zermelo–Fraenkel set theory (abbreviated as ZF) is commonly employed as an axiomatic foundation of mathematics nowadays. We shall briefly recall some of its notions and introduce necessary notation. The details can be found in [6, 16, 21, 27, 40, 46, 47].

**1.1.1.** *The language of the set theory* ZF uses the following symbols (called the *alphabet of* ZF): symbols of variables $x, y, z, \dots$; the parentheses ( , ); the propositional connectives (= the signs of propositional algebra) $\wedge, \vee, \to, \leftrightarrow, \neg$; the quantifiers $\forall, \exists$; the equality sign $=$ and the symbol of the special binary predicate of membership $\in$. Informally the domain of variables of ZF is thought of as *the world — universe — of sets.* The relation $\in(x, y)$ is written as $x \in y$ and read as "*$x$ is an element of $y$.*"

**1.1.2.** The formulas of ZF are defined by the usual recursive procedure. In other words, a *formula of* ZF is a finite text formed from atomic formulas such as $x = y$ and $x \in y$, where $x$ and $y$ are variables of ZF, with the help of sensible arrangement of parentheses, quantifiers, and propositional connectives. In addition, the ZF *theory* is the least set of formulas which contains the axioms of ZF and is closed under the rules of inference (see 1.1.4 below).

**1.1.3.** Common mathematical abbreviations are convenient in working with ZF. Some of them follow:

$$(\forall x \in y)\varphi(x) := \forall x\ (x \in y \to \varphi(x));$$

$$(\exists x \in y)\varphi(x) := \exists x\ (x \in y \vee \varphi(x));$$

$$\cup x := \{z : (\exists y \in x) z \in y\};$$

$$\cap x := \{z : (\forall y \in x) z \in y\};$$

$$x \subset y := \forall z\ (z \in x \to z \in y);$$

$$\mathcal{P}(x) := \text{“\textit{the class of all subsets of} } x\text{”} := \{z : z \subset x\};$$

$$\mathbb{V} := \text{“\textit{the class of all sets}”} := \{x : x = x\};$$

$$\text{“\textit{the class } } A \text{ \textit{is a set}”} := A \in \mathbb{V} := \exists x \forall y\ (y \in A \leftrightarrow y \in x);$$

$$f : X \to Y := \text{“} f \text{ \textit{is a function from} } X \text{ \textit{into} } Y\text{”};$$

$$\operatorname{dom}(f) := \text{“\textit{the domain of definition of} } f\text{”};$$

$$\operatorname{im}(f) := \text{“\textit{the image of} } f\text{”}.$$

**1.1.4.** The set theory ZF includes conventional axioms and rules of inference of first-order theory with equality. They fix the standard ways of classical resoning (syllogisms, the excluded middle, modus ponens, generalization, etc.). In addition, the following six special or nonlogical axioms are considered (they are written down with the standard abbreviations being accepted, cf. 1.1.3).

(1) *The axiom of extensionality*:

$$\forall x \forall y((x \subset y \vee y \subset x) \leftrightarrow x = y).$$

(2) *The axiom of union*: $\forall x(\cup x \in \mathbb{V})$.
(3) *The axiom of powerset*: $\forall x(\mathcal{P}(x) \in \mathbb{V})$.
(4) *The axiom schema of replacement*:

$$\forall x \forall y \forall z(\varphi(x, y) \wedge \varphi(x, z) \to y = z)$$

$$\to (\forall a)(\{v : (\exists u \in a)\varphi(u, v)\} \in \mathbb{V}).$$

(5) *The axiom of foundation*: $(\forall x)(x \neq \varnothing \to (\exists y \in x)(y \cap x = \varnothing))$.
(6) *The axiom of infinity*: $(\exists \omega)((\varnothing \in \omega) \wedge (\forall x \in \omega)(x \cup \{x\} \in \omega))$.

*The theory* ZFC (*Zermelo–Fraenkel theory with the axiom of choice*) can be obtained from ZF by adding the following postulate.

(7) *The axiom of choice*:

$$(\forall F)(\forall x)(\forall y)((x \neq \varnothing \wedge F : x \to \mathcal{P}(y))$$

$$\to ((\exists f)(f : x \to y) \wedge (\forall z \in x) f(z) \in F(z))).$$

**1.1.5.** *The Zermelo set theory* Z can be obtained from ZFC by deleting the axiom of foundation 1.1.4(5) and replacing the axiom schema of replacement 1.1.4(4) by its following consequences:

(1) *the axiom schema of comprehension*:

$$(\forall x)\{y \in x : \psi(y)\} \in \mathbb{V},$$

with $\psi$ being a ZF formula;

(2) *the axiom of pairing*: $(\forall x)(\forall y)\{x, y\} \in \mathbb{V}$.

Thus, the special axioms of the theory Z are 1.1.4(1–3, 6, 7), 1.1.5(1, 2).

So, the theories Z, ZF, and ZFC have one and the same language, one and the same logical axioms, and differ only in the collection of special axioms.

**1.1.6. Remarks.**

(1) The Zermelo–Fraenkel set theory slightly restricts a philistine mathematician by the axiom of foundation which, as a matter of fact, was proposed by J. von Neumann in 1925. At the same time it is the postulate which ensures the basis of the widely accepted set-theoretic view at the world of sets as at "the von Neumann universe" growing up hierarchically from the empty set — the mathematical proatom.

(2) The Zermelo–Fraenkel axiomatics did not ban all the ways of searching alternative set-theoretic foundations. In this regard, refer in particular to [7].

## *1.2. Boolean-valued set theory*

The theory of Boolean-valued models of set theory is outlined here in brief. In [27, 44, 45, 47] more complete introductions are available.

**1.2.1.** Let $B$ be a fixed complete Boolean algebra. *By a Boolean-valued interpretation of an $n$-ary predicate on a class $X$* we mean a suitably chosen mapping from $X^n$ into $B$. Suppose that $\mathbb{L}$ is a first-order language with the predicates $P_0, P_1, \dots, P_n$, and let $R_0, R_1, \dots, R_n$ be fixed Boolean-valued interpretations of these predicates on a class $X$. For an arbitrary formula $\varphi(u_1, \dots, u_m)$ of the language $\mathbb{L}$, and for elements $x_1, \dots, x_m \in X$, the truth value $[\![\varphi(x_1, \dots, x_m)]\!] \in B$ is defined by usual induction on the length of $\varphi$. For atomic formulas, write

$$[\![P_k(x_1, \dots, x_m)]\!] := R_k(x_1, \dots, x_m).$$

In induction steps the following rules are used:

$$[\![\varphi \vee \psi]\!] := [\![\varphi]\!] \vee [\![\psi]\!],$$

$$[\![\varphi \wedge \psi]\!] := [\![\varphi]\!] \wedge [\![\psi]\!],$$

$$[\![\varphi \to \psi]\!] := [\![\varphi]\!] \Rightarrow [\![\psi]\!],$$

$$[\![\neg\varphi]\!] := [\![\varphi]\!]^*,$$

$$[\![(\forall x)\varphi]\!] := \bigwedge_{x \in X} [\![\varphi(x)]\!],$$

$$[\![(\exists x)\varphi]\!] := \bigvee_{x \in X} [\![\varphi(x)]\!],$$

$\vee, \wedge, \Rightarrow, (\,\cdot\,)^*, \bigvee, \bigwedge$ in the right-hand sides of the equalities standing for the Boolean operations in $B (a \Rightarrow b := a^* \vee b)$.

**1.2.2.** Say that a statement $\varphi(x_1, \dots, x_m)$, with $x_1, \dots, x_m \in X$ and $\varphi(u_1, \dots, u_m)$ being a formula, is *valid* (*true*, *veritable*, etc.) in the system $\mathbb{X} := (X, R_0, \dots, R_n)$ and write $\mathbb{X} \models \varphi(x_1, \dots, x_m)$ if $[\![\varphi(x_1, \dots, x_m)]\!] = \mathbf{1}$. All logically true statements are valid in $\mathbb{X}$. If a predicate $P_0$ symbolizes equality, then the $B$-system $\mathbb{X} := (X, =, R_1, \dots, R_n)$ is required to satisfy equality axioms. If this requirement is fulfilled, then all logically true statements of the first-order logic with equality, which are expressible in the language $\mathbb{L} := \{=, P_1, \dots, P_n\}$, are valid in the $B$-system $\mathbb{X}$.

**1.2.3.** Now let us consider a Boolean-valued interpretation on a class $X$ of the language of the set theory ZFC ($\mathbb{L} := \{=, \in\}$), i.e., of the first-order language with two binary predicates $=$ and $\in$. We shall denote the interpretations of these predicates by $[\![\cdot = \cdot]\!]$ and $[\![\cdot \in \cdot]\!]$, respectively. Thus, $[\![\cdot = \cdot]\!], [\![\cdot \in \cdot]\!] : X \times X \to B$, and

$$[\![= (x, y)]\!] = [\![x = y]\!], \quad [\![\in (x, y)]\!] = [\![x \in y]\!] \quad (x, y \in X).$$

Our nearest aim is to characterize $B$-systems $\mathbb{X} := (X, [\![\cdot = \cdot]\!], [\![\cdot \in \cdot]\!])$, which are models of ZFC and such that $\mathbb{X} \models \mathrm{ZFC}$ . The last condition is equivalent to the fact that all the axioms of ZFC are valid in $\mathbb{X}$. So, for instance, by the rules of 1.2.1, the validity of the axiom of extensionality 1.1.4(1) means that, for all $x, y \in X$,

$$[\![x = y]\!] = \bigwedge_{z \in X} ([\![z \in x]\!] \Leftrightarrow [\![z \in y]\!]),$$

where $a \Leftrightarrow b = (a \Rightarrow b) \wedge (b \Rightarrow a)$ $(a, b \in B)$.

**1.2.4.** A $B$-system $\mathbb{X}$ is called *separated* whenever for all $x, y \in X$ the statement $[\![x = y]\!] = \mathbf{1}$ implies $x = y$. An arbitrary $B$-system $\mathbb{X}$ can be transformed into a separated one by taking the quotient with respect to the equivalence $\sim := \{(x, y) \in X^2 : [\![x = y]\!] = \mathbf{1}\}$. (A quotient class is defined with the help of the well-known method of Frege–Russell–Scott, see [46].) A $B$-system $\mathbb{X}$ is said to be *isomorphic* to a $B$-system $\mathbb{X}' := (X', [\![\cdot = \cdot]\!]', [\![\cdot \in \cdot]\!]')$, if there is a bijection $\beta : X \to X'$ such that $[\![x = y]\!] = [\![\beta x = \beta y]\!], [\![x \in y]\!] = [\![\beta x \in \beta y]\!]$ for all $x, y \in X$.

**1.2.5. Theorem.** *There exists a $B$-system $\mathbb{X}$ unique up to an isomorphism and such that*

(1) $\mathbb{X}$ *is separated (see* 1.2.4*);*

(2) *the equality axioms are valid in* $\mathbb{X}$*;*

(3) *the axiom of extensionality* 1.1.4(1) *and the axiom of foundation* 1.1.4(5) *are true in* $\mathbb{X}$ *(see* 1.2.3*);*

(4) *if a function* $f : \mathrm{dom}(f) \to B$ *satisfies* $\mathrm{dom}(f) \in \mathbb{V}$ *and* $\mathrm{dom}(f) \subset \mathbb{X}$*, then for some* $x \in \mathbb{X}$

$$[\![y \in x]\!] = \bigvee_{z \in \mathrm{dom}(f)} f(z) \wedge [\![z = y]\!] \quad (y \in \mathbb{X});$$

(5) *for each* $x \in \mathbb{X}$*, there exists a function* $f : \mathrm{dom}(f) \to B$ *with* $\mathrm{dom}(f) \in \mathbb{V}$, $\mathrm{dom}(f) \subset \mathbb{X}$*, the equality in* (4) *being valid for every* $y \in \mathbb{X}$*.*

**1.2.6.** A $B$-system meeting the requirements 1.2.5(1–5) is called a *Boolean-valued model of set theory* and is denoted by the symbol $\mathbb{V}^{(B)} := (\mathbb{V}^{(B)}, [\![\cdot = \cdot]\!], [\![\cdot \in \cdot]\!])$. The class $\mathbb{V}^{(B)}$ is also called *the Boolean-valued universe*. The basic properties of $\mathbb{V}^{(B)}$ are formulated in the following principles.

**(1) The transfer principle.** *Every axiom, and hence every theorem of* ZFC , *is valid in* $\mathbb{V}^{(B)}$*; in symbols:* $\mathbb{V}^{(B)} \models \mathrm{ZFC}$ .

**(2) The mixing principle.** *If $(b_\xi)_{\xi\in\Xi}$ is a partition of unity in $B$, and $(x_\xi)_{\xi\in\Xi}$ is a family of elements of $\mathbb{V}^{(B)}$, then there exists a unique element $x \in \mathbb{V}^{(B)}$ such that $b_\xi \le [\![x = x_\xi]\!]$ $(\xi \in \Xi)$. The element $x$ is called the mixture of the family $(x_\xi)_{\xi\in\Xi}$ with probabilities $(b_\xi)_{\xi\in\Xi}$ and is denoted by $\operatorname{mix}_{\xi\in\Xi} b_\xi x_\xi$ (the term "mixing" is also in use).*

**(3) The maximum principle.** *For every formula $\varphi(u)$ of ZFC (possibly with constants from $\mathbb{V}^{(B)}$) there exists an element $x_0 \in \mathbb{V}^{(B)}$ such that*

$$[\![(\exists u)\varphi(u)]\!] = [\![\varphi(x_0)]\!].$$

It follows, in particular, that if $[\![(\exists!x)\varphi(x)]\!] = \mathbf{1}$, then there exists a unique $x_0 \in \mathbb{V}^{(B)}$ such that $[\![\varphi(x_0)] = \mathbf{1}$.

**1.2.7.**There exists a unique mapping $x \to x^\wedge$ from $\mathbb{V}$ to $\mathbb{V}^{(B)}$ obeying the following conditions:

(1) $x = y \leftrightarrow [\![x^\wedge = y^\wedge]\!] = \mathbf{1}$; $x \in y \leftrightarrow [\![x^\wedge \in y^\wedge]\!] = \mathbf{1}$ $(x, y \in \mathbb{V})$,

(2) $[\![z \in y^\wedge]\!] = \bigvee_{x\in y} [\![x^\wedge = z]\!]$ $(z \in \mathbb{V}^{(B)}, y \in \mathbb{V})$.

This mapping is called the *canonical imbedding of* $\mathbb{V}$ *into* $\mathbb{V}^{(B)}$ and $x^\wedge$ is referred to as the *standard name of $x$*.

(3) **The restricted transfer principle.** Let $\varphi(u_1, \dots, u_n)$ be a restricted formula, i.e. with all the quantifiers in the form $(\forall u)(u \in v \to \dots)$ or $(\exists u)$ $(u \in v \wedge \dots)$ abbreviated to $(\forall u \in v), (\exists u \in v)$. Then for every $x_1, \dots, x_n \in \mathbb{V}$,

$$\varphi(x_1, \dots, x_n) \leftrightarrow \mathbb{V}^{(B)} \models \varphi(x_1^\wedge, \dots, x_n^\wedge).$$

**1.2.8.** For an element $X \in \mathbb{V}^{(B)}$, its *descent* $X\!\downarrow$ is defined by designating $X\!\downarrow= \{x \in \mathbb{V}^{(B)} : [\![x \in X]\!] = \mathbf{1}\}$. The set $X\!\downarrow$ is *cyclic*, i.e. closed under mixing its elements.

**1.2.9.** Let $F$ be a correspondence from $X$ to $Y$ inside $\mathbb{V}^{(B)}$, i.e., $X, Y, F \in \mathbb{V}^{(B)}$ and $[\![F \subset X \times Y]\!] = [\![F \ne \varnothing]\!] = \mathbf{1}$. There exists a unique correspondence $F\!\downarrow$ from $X\!\downarrow$ to $Y\!\downarrow$ such that for every set $A \subset X\!\downarrow$ inside $\mathbb{V}^{(B)}$ the equality $F(A)\!\downarrow= F\!\downarrow(A\!\downarrow)$ holds. Furthermore, $[\![F$ is a mapping from $X$ to $Y]\!] = \mathbf{1}$ if and only if $F\!\downarrow$ is a mapping from $X\!\downarrow$ to $Y\!\downarrow$.

In particular, a function $f : Z^\wedge \to Y$ inside $\mathbb{V}^{(B)}$, where $Z \in \mathbb{V}$, defines its *descent* $f\!\downarrow\colon Z \to Y\!\downarrow$ by $f\!\downarrow(z) = f(z^\wedge)$ $(z \in Z)$.

**1.2.10.** Suppose that $X \in \mathcal{P}(\mathbb{V}^{(B)})$. Define a function $f : \operatorname{dom}(f) \to B$ by declaring $\operatorname{dom}(f) = X$ and $\operatorname{im}(f) = \{\mathbf{1}\}$. By 1.2.5(4) there exists an element $X\!\uparrow\in \mathbb{V}^{(B)}$ such that

$$[\![y \in X]\!] = \bigvee_{x\in X} [\![x = y]\!] \quad (y \in \mathbb{V}^{(B)}).$$

The element $X\!\uparrow$ (which is unique by the axiom of extensionality) is called the *ascent of* $X$. Moreover, the following formulas are true:

(1) $Y\!\downarrow\uparrow= Y$ $(Y \in \mathbb{V}^{(B)})$,

(2) $X\!\uparrow\downarrow= \operatorname{mix}(X)$ $(X \in \mathcal{P}(\mathbb{V}^{(B)}))$,

where $\operatorname{mix}(X)$ consists of all mixtures of the form $\operatorname{mix} b_\xi x_\xi$, $(x_\xi) \subset X$, $(b_\xi)$ being a partition of unity in $B$.

**1.2.11** Assume that $X, Y \in \mathcal{P}(\mathbb{V}^{(B)})$ and let $F$ be a correspondence from $X$ to $Y$. The following statements are equivalent:

(1) there exists a unique correspondence $F{\uparrow}$ from $X{\uparrow}$ to $Y{\uparrow}$ inside $\mathbb{V}^{(B)}$ such that $\operatorname{dom}(F{\uparrow}) = \operatorname{dom}(F){\uparrow}$ and, for every subset $A$ of the set $\operatorname{dom}(F)$,

$$F{\uparrow}\,(A{\uparrow}) = F(A){\uparrow};$$

(2) the correspondence $F$ is *extensional*, i.e.,

$$y_1 \in F(x_1) \to [\![x_1 = x_2]\!] \leq \bigvee_{y_2 \in F(x_2)} [\![y_1 = y_2]\!].$$

The correspondence $F$ is a mapping from $X$ to $Y$ if and only if $[\![F{\uparrow}\colon X{\uparrow} \to Y{\uparrow}]\!] = \mathbf{1}$.

In particular, a mapping $f : Z \to Y{\downarrow}$ generates a function $f{\uparrow}\colon Z^{\wedge} \to Y$ such that $f{\uparrow}\,(x^{\wedge}) = f(x)(x \in Z)$.

**1.2.12.** Assume that a $B$-structure is defined on a nonempty set $X$, i.e., fix a mapping $d : X \times X \to B$ satisfying the "metric axioms":

(1) $d(x, y) = \mathbf{0} \leftrightarrow x = y$;

(2) $d(x, y) = d(y, x)$;

(3) $d(x, y) \leq d(x, z) \vee d(z, y)$.

Then there exists an element $\mathbb{X} \in \mathbb{V}^{(B)}$ and an injection $\iota : X \to X' := \mathbb{X}{\downarrow}$ such that $d(x, y) = [\![\iota(x) \neq \iota(y)]\!]$ and every element $x' \in X'$ can be represented in the form $x' = \operatorname{mix} b_\xi x_\xi$, with $(x_\xi) \subset X$ and $(b_\xi)$ being a partition of unity in $B$. This fact enables us to consider sets with $B$-structure as subsets of $\mathbb{V}^{(B)}$ and to handle them with the help of the rules described above.

**Remarks.**

(1) Boolean-valued analysis (the term was coined by G. Takeuti) is a branch of functional analysis which uses Boolean-valued models of set theory. It is interesting to note that at origination Boolean-valued models were not connected with the theory of vector lattices. The necessary language and technique were already formed within mathematical logic by 1960. Nevertheless, the main idea leading to rapid progress in the model theory was absent. Such idea emerged with P. J. Cohen's establishing the absolute unsolvability (in a precise mathematical sense) of the classical Continuum hypothesis. It was Cohen's forcing whose comprehension resulted in the invention of Boolean-valued models of set theory, the latter being connected with the names of P. Vopenka, D. Scott, and P. Solovay (see [61, 63, 65, 66]).

(2) The method of forcing is naturally divided into two parts — general and special. The general part is presented by the apparatus of Boolean-valued models of set theory. The Boolean algebra $B$ is absolutely arbitrary here. The special part consists in constructing a specific Boolean algebra $B$ in order to provide the necessary (often pathological, even exotic) properties of objects (e.g., of a $K$-space) sprouting from $B$. The both parts are of independent interest; but the most impressive results are obtained by combining the two. Most works on Boolean-valued analysis use only the general forcing. The future progress in Boolean-valued analysis will surely be connected with forcing in full strength (cf. [64]).

(3) The detailed information on this section can be found in [18, 21, 27, 64], see also [10, 23]. Various modifications of tools described in 1.2.8–1.2.11 are widely used

in investigations of the theory of Boolean-valued models. In [17, 22] the machinery is framed as the technique of descents and ascents which is more suitable for problems in analysis. The embedding 1.2.12 of sets with Boolean structure into a Boolean-valued universe is carried out in [17]. The motivation of such embedding is the Solovay–Tennenbaum method previously proposed for complete Boolean algebras [61].

**1.3. External set theory.**

From the viewpoint of applications, the existing variants of formally founding infinitesimal methods within axiomatic external set theory are almost equivalent. In this connection we present here one of the strongest versions of external set theory, the NST evoked by T. Kawai [53].

**1.3.1.** *The alphabet of the theory* NST is obtained from that of ZFC by adding two constants $\mathbb{V}^S$ and $\mathbb{V}^I$. Informally, $\mathbb{V}^S$ is thought of as the *universe of standard sets*, and $\mathbb{V}^I$ as the *world of internal sets* (in rany material interpretation). We point out that $\mathbb{V}^S$ and $\mathbb{V}^I$ are considered as concrete external sets, i.e., $\mathbb{V}^S \in \mathbb{V}^E$ and $\mathbb{V}^I \in \mathbb{V}^E$, where $\mathbb{V}^E := \{x : x = x\}$ is the class of all external sets. Sometimes instead of $x \in \mathbb{V}^S$ we write $\mathrm{St}(x)$ or "$x$ is a standard set." In the same way there is introduced the predicate $\mathrm{Int}\,(\,\cdot\,)$ which expresses the property of being an internal set.

The formulas are defined in the natural way. For $\varphi \in$ (ZFC ), by the symbol $\varphi^S$ (respectively, $\varphi^I$) we denote the *relativization of* $\varphi$ *on* $\mathbb{V}^S$ (*on* $\mathbb{V}^I$, respectively), i.e., the formula obtained by imagining all variables of $\varphi$ as ranging over standard (respectively, internal) sets.

If $\varphi \in$ (ZFC ) and $\varphi$ is considered as a formula of NST, then sometimes we write $\varphi^E$ and call this expression *E-formula.* The notions of *S-formula* and *I-formula* have the similar meaning.

Henceforth, we use the next conventional and convenient abbreviations of the form $(\forall^{\mathrm{St}} x)\varphi := (\forall x \in \mathbb{V}^S)\varphi$; $(\exists^{\mathrm{Int}}\ x)\varphi := (\exists x \in \mathbb{V}^I)\varphi$; $\mathrm{fin}(x) :=$ "$x$ is finite" (= not admitting a bijective mapping onto its proper subset); etc.

**1.3.2.** The special axioms of NST can be divided into three groups (the same situation is typical of other variants of external set theories). The first group consists of the rules for introducing external sets. The second includes the axioms on interrelations between the worlds of sets $\mathbb{V}^S$, $\mathbb{V}^I$, and $\mathbb{V}^E$. Finally, the third group contains the ordinary postulates of nonstandard analysis — the transfer, idealization, and standardization principles.

**1.3.3.** We begin with the structure of the universe $\mathbb{V}^E$.

(1) *The super-rule of introducing external sets*: if $\varphi$ is an axiom of ZFC, excluding the axiom of foundation, then $\varphi^E$ is an axiom of NST.

Thus, the axioms of the Zermelo theory Z act in NST, and the axiom schema of replacement is valid. Moreover, we assume

(2) The restricted axiom of foundation:

$$(\forall A)(A = \varnothing \vee A \cap \mathbb{V}^I = \varnothing) \to (\exists x \in A) x \cap A = \varnothing,$$

in other words, the regularity is postulated for external sets lacking in internal elements.

Emphasize that $\mathbb{V}^S \in \mathbb{V}^E$. In other words, the usual axiom of acceptability is valid (3.4.7 in [27]. We recall in this connection that an external set $A$ is said *to be of acceptable size* (or *S-size*), if there exists an external function that maps $\mathbb{V}^S$ onto $A$. In this case we write $A \in \mathbb{V}^{\text{a-size}}$.

**1.3.4.** The second group of axioms contains the following statements:

(1) *the modeling principle* (for standard sets) — the world $\mathbb{V}^S$ is the von Neumann universe, i.e., for each axiom $\varphi$ of ZFC the standardization $\varphi^S$ is an axiom of NST;

(2) *the axiom of transitivity* (for internal sets) — $(\forall x \in \mathbb{V}^I)x \subset \mathbb{V}^I$, i.e., internal sets are composed of internal elements only;

(3) *the axiom of embedding* — $\mathbb{V}^S \subset \mathbb{V}^I$, i.e., standard sets are internal.

**1.3.5.** The third group of postulates consists of the following axioms.

(1) *The transfer principle* —

$$(\forall^{\mathrm{St}} x_1) \dots (\forall^{\mathrm{St}} x_n)\varphi^S(x_1, \dots , x_n) \leftrightarrow \varphi^I(x_1, \dots , x_n)$$

for every formula $\varphi = \varphi(x_1, \dots , x_n), \varphi \in$ (ZFC ).

(2) *The standardization principle* —

$$(\forall A)(\exists^{\mathrm{St}} t)(\circ A \subset t) \to (\exists^{\mathrm{St}} a)(\forall^{\mathrm{St}} x) \quad (x \in A \leftrightarrow x \in a),$$

where ${}^\circ A = A \cap \mathbb{V}^S$ is the *standard core of* $A$. The appearing set $a$ is obviously unique. It is denoted by ${}^*A$ and is called the *standardization of* $A$.

(3) *The idealization principle* (= the axiom schema of saturation) —

$$(\forall^{\mathrm{Int}}\ x_1) \dots (\forall^{\mathrm{Int}}\ x_n)(\forall A \in \mathbb{V}^{\text{a-size}})[\![((\forall z) z \subset A \vee \mathrm{fin}^E(z)$$
$$\to (\exists^{\mathrm{Int}}\ x)(\forall y \in z)\varphi^I(x, y, x_1, \dots , x_n))$$
$$\to (\exists^{\mathrm{Int}}\ x)(\forall^{\mathrm{Int}}\ y \in A)\varphi^I(x, y, x_1, \dots , x_n)]\!]$$

for every formula $\varphi = \varphi(x, y, x_1, \dots , x_n), \varphi \in$ (ZFC ).

**1.3.6. Kawai's theorem.** NST *is a conservative extension of* ZFC .

**1.3.7.** As usual, inside $\mathbb{V}^E$ we can construct *the universe* $\mathbb{V}^C$ *consisting of classical* (= standard, or ordinary in Robinson's approach) sets, by using the class of standard ordinals, $\mathrm{On}^{\mathrm{St}}$. Namely,

$$\mathbb{V}^C_\beta := \{x : (\exists^{\mathrm{St}} \alpha \in \beta)x \subset \mathcal{P}(\mathbb{V}^C_\alpha)\},$$

$$\mathbb{V}^C := \bigcup_{\beta \in \mathrm{On}^{\mathrm{St}}} \mathbb{V}^C_\beta.$$

Robinson's standardization $* : \mathbb{V}^C \to \mathbb{V}^S$ appears in this situation via the recursion schema:

$$*\varnothing := \varnothing, \quad *A := {}^*\{*a : a \in A\}.$$

Robinson's standardization provides validity of Leibniz's principle in the form

$$(\forall x \in \mathbb{V}^C) \dots (\forall x_n \in \mathbb{V}^C)\varphi^C(x_1, \dots , x_n) \leftrightarrow \varphi^S(x_1, \dots , x_n)$$

for a formula $\varphi = \varphi(x_1, \dots, x_n) \in$ (ZFC ) and its relativizations $\varphi^C$ and $\varphi^S$ on $\mathbb{V}^C$ and $\mathbb{V}^S$ respectively.

**1.3.8.** The world of the radical (and classical) stance of nonstandard analysis also admits of an axiomatic description.

We shall describe the UNST theory analyzed by T. Kawai.

In UNST variables stand for external sets. There are constants $\mathbb{V}^C$, $\mathbb{V}^I$ and $*$. The corresponding external sets are naturally called the *classical world, the universe of internal sets* and *Robinson's standardization.*

The special axioms of UNST are similar to those of NST.

**1.3.9.** The structure of the UNST universe is defined by the following postulates.

(1) *The super-rule of introducing external sets* (similar to 1.3.3(1)).

(2) *The restricted axiom of foundation* (cf. 1.3.3(2)).

**1.3.10.** The axioms of interrelations between the worlds of sets contain such propositions:

(1) *the modeling principle* (for classical sets) — the world $\mathbb{V}^C$ is the von Neumann universe;

(2) *the axiom of transitivity* (for internal sets) — in the form 1.3.4(2);

(3) *the axiom of transitivity* (for classical sets) — $(\forall x \in \mathbb{V}^C) x \subset \mathbb{V}^C$, i.e. classical sets are composed of classical elements only;

(4) *the axiom of external assemblage* (= *the axiom of superstructure*) — external subsets of a classical set are classical;

(5) *the axiom of Robinson's standardization* — $*$ is an (external) mapping from $\mathbb{V}^C$ into $\mathbb{V}^I$.

Obviously, according to 1.3.10(5) there exists a unique set $\mathbb{V}^S$ consisting exactly of $\mathbb{V}^S := *(\mathbb{V}^C)$. In UNST elements of $\mathbb{V}^S$ are called *standard sets.* In analogy with 1.3.3(2), a set $A$ is said *to be of classical size* (or *of C-size*) whenever there exists an external function from $\mathbb{V}^C$ onto $A$. In this case we write $A \in \mathbb{V}^{\text{c-size}}$.

**1.3.11.** The postulates of nonstandard analysis in UNST are the following:

(1) *The transfer principle* in Leibniz's form, 1.3.5;

(2) *The idealization principle* in the form of the axiom schema of saturation for sets of classic size (cf. 1.3.5(3)).

Finally, the *standardization* ${}^*A$ (which is a subset of an element of $\mathbb{V}^S$) in UNST of a set $A$ presents the procedure

$$^*A := *(*^{-1}(A \cap \mathbb{V}^S)).$$

The following proposition is immediate from 1.3.6.

**1.3.12.** *The* UNST *theory is a conservative extension of* ZFC .

While working with analytic objects below we shall adopt a free stance close to the neoclassic and radical credos of nonstandard analysis. In particular, the field of reals will be considered as a standard element of the world of internal sets, the classic realization of $\mathbb{R}$ will be identified with the standard core ${}^\circ\mathbb{R}$. The symbols used in nonstandard analysis for infinitesimals, monads, etc., coincide with those from [27].

**1.3.13. Remarks.**

(1) The axiomatic approach to nonstandard analysis has started conquering popularity after the papers of E. Nelson [27, 56], who evoked an axiomatics of internal

set theory. As a result, views on the essence of infinitesimal methods have changed drastically (see [34]). The distinctive feature of changes undergone is a refusal of a "shy" approach viewing infinitesimals as monsters of some exotic status.

(2) The axiomatic theories of external sets were proposed by K. Hrbacek and T. Kawai (see [27, 34]). The above-indicated version of the theory follows [53]. Among the latest works we mention [14, 40] which expose, as a matter of fact, convenient formalisms of a "graded" theory of external sets connected with the conception of relative standardness.

## 2. BOOLEAN-VALUED ANALYSIS OF VECTOR LATTICES

An essentially new — nonstandard — possibility offered to the theory of ordered spaces is a formalization of the heuristic view on elements of a $K$-space as certain analogs of reals. Strictly speaking, points of a $K$-space mimic real numbers in a suitable model of set theory. The corresponding formalism introduced in this chapter presents one of the most fundamental and indispensable conceptions in the theory of ordered vector spaces.

### 2.1. Kantorovich spaces.

The theory of vector lattices is exposed in a number of excellent monographs, see, for instance, [10, 18, 19, 25, 26, 29, 31, 32, 38, 41, 42, 51]. Vector lattices used to be also called Riesz spaces. Here we shall briefly dwell on order complete vector lattices.

**2.1.1.** *A Kantorovich space* or, in brief, a *$K$-space* is such a vector lattice that its each order bounded nonvoid subset has a supremum and an infimum. If each countable order bounded nonempty subset of a vector lattice has a supremum and an infimum, then this vector lattice is called a $K_\sigma$-space. Everywhere below $E$ designates a $K$-space.

A *band* in $E$ is a set of the form

$$M^\perp := \{x \in E : (\forall y \in M)|x| \wedge |y| = 0\},$$

where $M \subset E$ and $M \neq \varnothing$. The collection of bands, ordered by inclusion, forms a complete Boolean algebra $\mathbb{B}(E)$, Boolean operations being as follows:

$$L \wedge K = L \cap K, L \wedge K = (L \cup K)^{\perp\perp}, L^* = L^\perp (L, K \in \mathbb{B}(E)).$$

The algebra $\mathbb{B}(E)$ is called the *base* of $E$.

**2.1.2.** To every band $K \subset E$ corresponds the unique *projection* $[\![K]\!]$ so as to ensure $0 \leq [\![K]\!]x \leq x$ for all $0 \leq x \in E$. The set $\mathfrak{P}(E)$ of all these projections is equipped with the order defined by $\rho \leq \pi \leftrightarrow \mathrm{im}\rho \subset \mathrm{im}\pi$.

Let $\mathbf{1}$ be a unity in $E$, i.e. $\{\mathbf{1}\}^{\perp\perp} = E$. An element $e \in E$ is called *unit*, or a *fragment* (= *component*) *of unity* if $e \wedge (\mathbf{1} - e) = 0$. The set $\mathcal{E}(E) := \mathcal{E}(\mathbf{1})$ of all unit elements is furnished with the order induced from $E$. The ordered sets $\mathfrak{P}(E)$ and $\mathcal{E}(E)$ are complete Boolean algebras.

**2.1.3. Theorem.** *The mapping $K \to [\![K]\!]$ is an isomorphism between the Boolean algebras $\mathbb{B}(E)$ and $\mathfrak{P}(E)$. And if there is a unity in $E$, then the mappings $\pi \to \pi\mathbf{1}$ from $\mathbb{P}(E)$ into $\mathcal{E}(E)$ and $e \to \{e\}^{\perp\perp}$ from $\mathcal{E}(E)$ into $\mathbb{B}(E)$ are also isomorphism s between the Boolean algebras.*

**2.1.4.** A $K$-space $E$ is called *universally complete* (or *extended* in the Russian terminology) if every nonempty set of mutually disjoint elements of $E$ has a supremum. (Elements $x$ and $y$ are *disjoint* if $|x| \wedge |y| = 0$.) We shall give the most important examples of universally complete $K$-spaces. For the sake of brevity we restrict ourselves to the case of real scalars.

(1) The space $M(\Omega, \Sigma, \mu)$ of equivalence classes of measurable functions, where $(\Omega, \Sigma, \mu)$ is a measure space, and $\mu$ is $\sigma$-finite (or, more generally, $\mu$ possesses the direct sum property, cf. [46]). The base of the $K$-space $M(\Omega, \Sigma, \mu)$ is isomorphic to the Boolean algebra $\Sigma/\mu^{-1}(0)$ of measurable sets modulo sets of measure zero.

(2) The space $C_\infty(Q)$ of continuous functions defined on an extremally disconnected compactum $Q$ with values in the extended real line and taking the values $\pm\infty$ only on a rare (= nowhere dense) set [10]. The base of this $K$-space is isomorphic to the Boolean algebra of clopen (= closed and open) subsets of $Q$.

(3) The space $\mathrm{Bor}(Q)$ of equivalence classes of Borel functions defined on a topological space $Q$. Two functions are equivalent if they coincide on the complement of a meager set. The base of the $K$-space $\mathrm{Bor}(Q)$ is isomorphic to the Boolean algebra of Borel subsets of $Q$ modulo meager sets.

(4) The space $\overline{\mathbb{A}}$ of hermitian (not necessarily bounded) operators in a Hilbert space which are adjoint to a commutative von Neumann algebra $\mathbb{A}$ (see [6, 30, 62]). The base of the $K$-space $\overline{\mathbb{A}}$ is isomorphic to the Boolean algebra of all projections in $\mathbb{A}$.

**2.1.5.** Let $E$ and $F$ be vector lattices. An operator $T : E \to F$ is said to be *positive* if $Tx \geq 0$ for every $0 \leq x \in E$, and *regular* if $T = T_1 - T_2$, with $T_1, T_2$ being positive. The operator $T$ is said *to be order bounded* (or *o-bounded*) if $T(M)$ is a order bounded set in $F$ for every order bounded $M \subset E$. If $F$ is a $K$-space then the classes of regular and order bounded operators coincide. Moreover, the following is true.

**2.1.6. Riesz–Kantorovich theorem.** *If $E$ is a vector lattice and $F$ is a $K$-space then the space $L^\sim(E, F)$ of all regular operators from $E$ into $F$ is also a $K$-space.*

**2.1.7. Remarks.**

(1) The invention of the theory of vector lattices is customarily connected with research of G. Birkhoff, L. V. Kantorovich, M. G. Kreĭn, H.Nakano, F.Riesz, H. Freudenthal, et al. Nowadays the theory and applications of vector lattices form a vast domain of mathematics. It is painstakingly charted in monographs [3, 5, 10, 18, 19, 38, 42, 43, 51]. For the prerequisite of Boolean algebras, see [8, 27, 30, 64].

(2) The class of order complete vector lattices, i.e. of $K$-spaces, was introduced by L. V. Kantorovich in his first fundamental paper [17]. There he has also suggested the heuristic transfer principle for $K$-spaces, the crux of the idea is that elements of a $K$-space serve as generalized numbers. Later on this principle has acquired a good deal of clarification in investigations of the author himself and his disciples. As a matter of fact, the principle became one of the basal ideas playing an organizing and directing role and led to more deep and elegant theory of $K$-spaces abundant in various applications.

(3) During the first period of the theory development some attempts have already been made which were intended to formalizing the heuristic observation mentioned. In this direction so-called theorems on preservation of relations (sometimes a less exact term "conservation" is also employed) appeared. They assert that if an expression containing a finite number of functional relations is proved for reals, then the similar fact is preserved as being also true for elements of a $K$-space (see [10, 19]). Unfortunately, there was no satisfactory explanation for the internal mechanism controlling the phenomenon, the preservation of relations. Limits to applying the above assertions and the general background of similarity and parallelism between them and their analogs in the classical function theory had not been sufficiently clarified. The depth and universality of Kantorovich's principle were fully explicated within the frames of Boolean-valued analysis (see [13]).

**2.2. Reals within Boolean-valued models.**

Boolean-valued analysis began with the representation of "genuine" reals in a Boolean-valued model. Such representation happened to be a universally complete $K$-space. On selecting a Boolean algebra $B$ (algebra of measurable sets, of regular open sets, or of projections in a Hilbert space) assumed as initial for the Boolean-valued model $\mathbb{V}^{(B)}$, different universally complete $K$-spaces appear (the spaces of measurable functions, or of semicontinuous functions, or of selfadjoint operators). Thus there was open a broad way of transfering all the knowledge about numbers to many classical objects of analysis.

**2.2.1.** Under the field of real numbers we understand such an algebraic system that the axioms of an archimedean ordered field (with distinct zero and unity) and the axiom of completeness are valid. Recall two well-known statements.

(1) There exists a field of real numbers $\mathbb{R}$ being unique up to an isomorphism.

(2) If $P$ is an Archimedean ordered field, then there exists an isomorphic embedding, $h$ of $P$ into $\mathbb{R}$, such that the image $h(P)$ is a subfield of $\mathbb{R}$ containing the subfield of rational numbers. In particular, $h(P)$ is dense in $\mathbb{R}$.

**2.2.2.** On applying successively the transfer principle and the maximum principle to 2.2.1(1), find such an element $\mathcal{R} \in \mathbb{V}^{(B)}$ that $[\![\mathcal{R}$ is a field of real numbers$]\!] = \mathbf{1}$. Moreover, for every $\mathcal{R}' \in \mathbb{V}^{(B)}$ satisfying the condition $[\![\mathcal{R}'$ is a field of real numbers$]\!] = \mathbf{1}$, the equality $[\![$ the ordered fields $\mathcal{R}$ and $\mathcal{R}'$ are isomorphic $]\!] = \mathbf{1}$ also holds. In other words, in the model $\mathbb{V}^{(B)}$ there is located a field of real numbers, $\mathcal{R}$, which is unique up to an isomorphism.

**2.2.3.** Note also that the formula $\varphi(\mathbb{R})$, recording the axioms of an Archimedean ordered field, is restricted. So, $[\![\varphi(\mathbb{R}^\wedge)]\!] = \mathbf{1}$, i.e. $[\![\mathbb{R}^\wedge$ is an Archimedean ordered field$]\!] = \mathbf{1}$. Putting the statement 2.2.1(2) into the transfer principle, we obtain the fact that $[\![\mathbb{R}^\wedge$ is isomorphic to a dense subfield of $\mathcal{R}]\!] = \mathbf{1}$. On these grounds we infer below that $\mathcal{R}$ is the field of real numbers in the model $\mathbb{V}^{(B)}$, and $\mathbb{R}^\wedge$ is a dense subfield of it.

Consider now the descent $\mathcal{R}{\downarrow}$ of the algebraic system $\mathcal{R}$. In other words, we look at the descent of the carrier set of $\mathcal{R}$ equipped with the descended operations and order. For the sake of simplicity, the operations and order in $\mathcal{R}$ and $\mathcal{R}{\downarrow}$ will be denoted by the same symbols $+, \cdot, \leq$.

**2.2.4. Gordon's theorem.** *Let $\mathcal{R}$ be the ordered field of reals in the model $\mathbb{V}^{(B)}$. Then $\mathcal{R}{\downarrow}$ (equipped with the descended operations and order) is a universally* *complete $K$-space with unity* $\mathbf{1}$*. Furthermore, there exists an isomorphism* $\chi$ *of the Boolean algebra* $B$ *onto the base* $\mathfrak{P}(\mathcal{R}{\downarrow})$ *such that the equivalences*

$$\chi(b)x = \chi(b)y \leftrightarrow b \leq [\![x = y]\!],$$

$$\chi(b)x \leq \chi(b)y \leftrightarrow b \leq [\![x \leq y]\!]$$

*hold for all* $x, y \in \mathcal{R}$ *and* $b \in B$*.*

**2.2.5.** The universally complete $K$-space $\mathcal{R}{\downarrow}$ is at the same time a faithful $f$-algebra with the ring unity $\mathbf{1}$, and for every $b \in B$ the projection $\chi(b)$ is the operator of multiplication by the unit element $\chi(b)\mathbf{1}$. It is clear, therefore, that the mapping $b \to \chi(b)\mathbf{1}$ $(b \in B)$ is a Boolean isomorphism of $B$ and the algebra of unit elements $\mathcal{E}(\mathcal{R}{\downarrow})$. This isomorphism is denoted by the same letter $\chi$.

**2.2.6.** Recall that if $E$ is a $K$-space with unity and $x \in E$, then the projection of the unity onto the band $\{x\}^{\perp\perp}$ is called the *trace of* $x$ and is denoted by the symbol $e_x$. For a real $\lambda$, denote by $e^x_\lambda$ the trace of the positive part of $\lambda\mathbf{1} - x$, i.e. $e^x_\lambda := e_{(\lambda\mathbf{1}-x)^+}$. The mapping $\lambda \to e^x_\lambda (\lambda \in \mathbb{R})$ is called the *spectral function* or the *characteristic of* $x$.

For every element $x \in \mathcal{R}{\downarrow}$, the following relations hold:

$$e_x = \chi([\![x \neq 0]\!]),\ e^x_\lambda = \chi([\![x < \lambda^\wedge]\!]) \quad (\lambda \in \mathbb{R}).$$

The next result states that every Archimedean vector lattice is realizable as a sublattice of $\mathcal{R}$ in a suitable Boolean-valued model.

**2.2.7. Theorem.** *Let* $E$ *be an Archimedean vector lattice, then let* $\mathcal{R}$ *be the field of reals in the model* $\mathbb{V}^{(B)}$*, and finally let* $j$ *be an isomorphism of* $B$ *onto the base* $\mathbb{B}(E)$*. There exists an element* $\mathcal{E} \in \mathbb{V}^{(B)}$ *satisfying the following conditions:*

(1) $\mathbb{V}^{(B)} \models \mathcal{E}$ *is a vector sublattice of the field* $\mathcal{R}$ *considered as a vector lattice over* $\mathbb{R}^\wedge$*;*

(2) $E' := \mathcal{E}{\downarrow}$ *is a vector sublattice of* $\mathcal{R}{\downarrow}$*,* $E'$ *is invariant with respect to every projection* $\chi(b)(b \in B)$*, and each set of positive mutually disjoint elements of* $E'$ *has a supremum;*

(3) *there exists an o-continuous lattice isomorphism* $\iota : E \to E'$ *such that* $\iota(E)$ *is a coinitial sublattice in* $\mathcal{R}{\downarrow}$ *(that is* $(\forall 0 < x \in \mathcal{R}{\downarrow})(\exists y \in \iota(E))0 < y \leq x)$*;*

(4) *for every* $b \in B$ *the projection onto the band generated in* $\mathcal{R}{\downarrow}$ *by the set* $\iota(j(b))$ *coincides with* $\chi(b)$*.*

**2.2.8.** The element $\mathcal{E} \in \mathbb{V}^{(B)}$ in Theorem 2.2.7 is called a *Boolean-valued realization of the vector lattice* $E$. Thus, Boolean-valued realizations of Archimedean vector lattices are vector sublattices of the field of reals, $\mathcal{R}$, considered as a vector lattice over the field $\mathbb{R}^\wedge$.

Notice now several corollaries of 2.2.4 and 2.2.7, keeping the previous notations.

(1) If $E$ is a $K$-space, then $\mathcal{E} = \mathcal{R}, E' = \mathcal{R}{\downarrow}$ and $\iota(E)$ is a foundation (= order dense ideal) of the $K$-space $\mathcal{R}{\downarrow}$. Furthermore, $\iota^{-1} \circ \chi(b) \circ \iota$ is the projection onto the band $j(b)$ for every $b \in B$.

(2) The image $\iota(E)$ coincides with the whole $\mathcal{R}{\downarrow}$ if and only if $E$ is a universally complete $K$-space.

(3) Universally complete $K$-spaces are isomorphic if and only if their bases are isomorphic.

(4) Let $E$ be a universally complete $K$-space with unity $\mathbf{1}$. Then in $E$ a multiplication can be uniquely defined in such a way that $E$ becomes a faithful $f$-algebra with $\mathbf{1}$ as a multiplication unity.

**2.2.9.** Not only Boolean-valued realizations of Archimedean vector lattices lead to subsystems of the field $\mathcal{R}$, see 2.2.7. We formulate, for instance, several statements from [27].

**Theorem.** (1) *The Boolean-valued realization of an Archimedean lattice-ordered group is a subgroup of the additive group of the field* $\mathcal{R}$.

(2) *An Archimedean $f$-ring contains two mutually complementing bands, one being a group with zero multiplication and realizable as in* (1)*, and the other possessing a subring of* $\mathcal{R}$ *as its Boolean-valued realization.*

(3) *An Archimedean $f$-algebra contains two mutually complementing bands, one being a vector lattice with zero multiplication and realizable as in* 2.2.7*, and another being representable as a subring and sublattice* $\mathcal{R}$ *considered as an $f$-algebra over* $\mathbb{R}^\wedge$.

**2.2.10. Remarks.**

(1) The Boolean-valued status of the notion of $K$-space was exposed by Gordon's theorem obtained in [11]. This fact can be reformulated as follows: a universally complete $K$-space is an interpretation of the field of reals in a suitable Boolean-valued model. Furthermore, every theorem (within ZFC) about real numbers has its complete analog for the corresponding $K$-space. The transfer of one theorem into the other is fulfilled by the instrumentality of precisely-defined procedures: ascent, descent, canonic embedding, etc., i.e. by an algorithm, as a matter of fact. Thus, Kantorovich's motto: "elements of a $K$-space are generalized numbers" finds a precise mathematical formulation within Boolean-valued analysis. On the other hand, the heuristic transfer principle which has played an auxiliary role in many investigations of the pre-Boolean-valued theory of $K$-spaces becomes a precise method of research in Boolean-valued analysis.

(2) If, in 2.2.4, $B$ is the $\sigma$-algebra of measurable sets modulo sets of measure zero, then $\mathcal{R}{\downarrow}$ is isomorphic to the universally complete $K$-space of measurable functions $M(\Omega, \Sigma, \mu)$. This fact (for the Lebesgue measure on an interval) was known as far back as Scott and Solovay's article (see [60]). If $B$ is a complete Boolean algebra of projections in a Hilbert space, then $\mathcal{R}{\downarrow}$ is isomorphic to the space of selfadjoint operators whose spectral function act into $B$. The two particular cases of Gordon's theorem were intensively and fruitfully exploited by G. Takeuti, see [62] and the references in [27]. T. Jech [50, 51] also considered the object $\mathcal{R}{\downarrow}$ for general Boolean algebras and rediscovered Gordon's theorem at that. The notable distinction is in the fact that in [51] a (complex) universally complete $K$-space with unity is defined by another axiom system and is called a complete Stone algebra.

(3) The realization theorem 2.2.7 was obtained by A. G. Kusraev [25]. A close result (stated in somewhat different terms) appeared in the paper [52] which develops the Boolean-valued interpretation of the theory of linearly ordered sets. Corollaries 2.2.7(3,4) are well known (see [10, 19]). The notion of a univesal completion of a $K$-space was introduced by A. G. Pinsker in a slightly different way. He also

proved the existence of a unique, up to an isomorphism, universal completion for an arbitrary $K$-space. The existence of an order completion of an Archimedean vector lattice was established by A. I. Yudin. The corresponding references are in [19]. All these facts can be easily derived from 2.2.4 and 2.2.7 (see [27] for details).

(4) As was already mentioned in 2.2.10(1), the initial attempts at formalizing heuristic Kantorovich's principle led to the theorems on preservation of relations (see [10, 19]). The modern forms of such theorems are presented in [13, 52], see also [27, 38, 39].

**2.3. Functional calculus in $K$-spaces.**

The most important structure properties of vector lattices — representability by function spaces, the spectral theorem, the functional calculus, etc. — replicate analogous properties of the field of reals in a suitable Boolean-valued model. We shall briefly outline the Boolean-valued approach to the functional calculus in $K$-spaces.

**2.3.1.** Further we shall need the notion of the integral with respect to a spectral measure. Let $(\Omega, \Sigma)$ be a measurable space; i.e., $\Omega$ is a nonempty set, and $\Sigma$ is a fixed $\sigma$-algebra of subsets of $\Omega$. A mapping $\mu : \Sigma \to B$ is said to be *a spectral measure* if $\mu(\Omega - A) = \mathbf{1} - \mu(A)$ and

$$\mu(\bigcup_{n=1}^{\infty} A_n) = \bigvee_{n=1}^{\infty} \mu(A_n)$$

for every sequence $(A_n)$ of elements of the $\sigma$-algebra $\Sigma$.

Let $B := \mathcal{E}(E)$ be the Boolean algebra of unit elements of a $K$-space $E$ with a fixed unity $\mathbf{1}$. Consider a measurable function $f : \Omega \to \mathbb{R}$. For an arbitrary partition of the real line $\beta := (\lambda_k)_{k\in\mathbb{Z}}$, $\lambda_k < \lambda_{k+1} (k \in \mathbb{Z})$, $\lim_{n\to\pm\infty} \lambda_n = \pm\infty$, denote by $A_k$ the inverse image $f^{-1}([\lambda_k, \lambda_{k+1}))$ and arrange the integral sums

$$\underline{\sigma}(f, \beta) := \sum_{-\infty}^{\infty} \lambda_k \mu(A_k), \quad \overline{\sigma}(f, \beta) := \sum_{-\infty}^{\infty} \lambda_{k+1} \mu(A_k),$$

where summation is performed in $E$. If there exists an element $x \in E$ such that $\{\underline{\sigma}(f, \beta)\} = x \inf\{\overline{\sigma}(f, \beta)\}$, where the supremum and infimum are taken over all partitions $\beta := (\lambda_k)$ of the real line, then the function $f$ is said to be *integrable with respect to the spectral measure* $\mu$, or we say that the *spectral integral* $I_\mu(f)$ exists and write

$$I_\mu(f) := \int_\Omega f d\mu := \int_\Omega f(t) d\mu(t) := x.$$

**2.3.2. Theorem.** *Set $E := \mathcal{R}{\downarrow}$ and let $\mu$ be a spectral measure with values in $B := \mathcal{E}(E)$. Then for every measurable function $f$, the integral $I_\mu(f)$ is the unique element of the $K$-space $E$ satisfying the following condition*

$$[\![ I_\mu(f) < \lambda^\wedge ]\!] = \mu(\{f < \lambda\}) (\lambda \in \mathbb{R}),$$

*where $\{f < \lambda\} := \{t \in \Omega : f(t) < \lambda\}$.*

It is clear from this theorem that if the integral $I_\mu(f) \in E$ exists, then the mapping $\lambda \to \mu(\{f < \lambda\})$ coincides with the spectral function of $I_\mu(f)$. In particular, if

$E$ is universally complete, then $I_\mu(f)$ exists for every measurable function $f$. Moreover, on using elementary properties of the field $\mathcal{R}$, the next result easily follows from Theorem 2.3.2.

**2.3.3. Theorem.** *Let $E$ be a universally complete $K$-space, and let $\mu : \Sigma \to B := \mathcal{E}(E)$ be a spectral measure. The spectral integral $I_\mu(\cdot)$ is a sequentially o-continuous (linear, multiplication, and lattice) homomorphism from the $f$-algebra of measurable functions $\mathbb{M}(\Omega, \Sigma)$ into $E$.*

**2.3.4.** Let $e_1, \dots, e_n : \mathbb{R} \to B$ be a finite collection of spectral functions with values in the $\sigma$-algebra $B$. Then there exists a unique $B$-valued spectral measure $\mu$ defined on the Borel $\sigma$-algebra $\mathbb{B}(\mathbb{R}^n)$ of the space $\mathbb{R}^n$, for which

$$\mu(\prod_{k=1}^{n}(-\infty, \lambda_k)) = \bigwedge_{k=1}^{n} e_k(\lambda_k),$$

whenever $\lambda_1, \dots, \lambda_n \in \mathbb{R}$.

**2.3.5.** Consider now an ordered collection of elements ($n$-tuple) $x_1, \dots, x_n$ of a $K$-space $E$ with unity $\mathbf{1}$. Let $e^{x_k} : \mathbb{R} \to B := \mathcal{E}(E)$ be the spectral function of $x_k$. According to the proposition indicated above, there exists a spectral measure $\mu : \mathbb{B}(\mathbb{R}^n) \to B$ such that

$$\mu(\prod_{k=1}^{n}(-\infty, \lambda_k)) = \bigwedge_{k=1}^{n} e^{x_k}(\lambda_k).$$

It is clear that the measure $\mu$ is uniquely defined by the $n$-tuple $\mathbb{X} := (x_1, \dots, x_n) \in E^n$. This allows us to write $\mu_{\mathbb{X}} := \mu$ and to say that $\mu_{\mathbb{X}}$ is the spectral measure of $\mathbb{X}$. For the integral of a measurable function $f : \mathbb{R}^n \to \mathbb{R}$ with respect to the spectral measure $\mu_{\mathbb{X}}$ the following notations are convenient

$$\widehat{\mathbb{X}}(f) := f(\mathbb{X}) := f(x_1, \dots, x_n) := I_\mu(f).$$

If $\mathbb{X} := \{x\}$ then we also write $\widehat{x}(f) := f(x) := I_\mu(f)$; in addition, the measure $\mu_x := \mu_{\mathbb{X}}$ is said to be the *spectral measure of the element* $x$. For a function $f(t) = t$ ($t \in \mathbb{R}$), Freudenthal's spectral theorem follows from 2.3.2:

$$x = \int_{\mathbb{R}} t d\mu_x(t) = \int_{-\infty}^{\infty} t de_t^x.$$

We recall that the space $\mathbb{B}(\mathbb{R}^n, \mathbb{R})$ of all Borel functions in $\mathbb{R}^n$ is a universally complete $K_\sigma$-space and a faithful $f$-algebra.

**2.3.6. Theorem.** *The spectral measures of an $n$-tuple $\mathbb{X} := (x_1, \dots, x_n)$ and of an element $f(\mathbb{X})$ are interrelated as follows*

$$\mu_{f(\mathbb{X})} = \mu_{\mathbb{X}} \circ f^{\leftarrow},$$

*with* $f^{\leftarrow} : \mathbb{B}(\mathbb{R}) \to \mathbb{B}(\mathbb{R}^n)$ *being the homomorphism defined by* $A \to f^{-1}(A)$*. In particular, for arbitrary measurable functions,* $f \in \mathbb{B}(\mathbb{R}^n, \mathbb{R})$ *and* $g \in \mathbb{B}(\mathbb{R}, \mathbb{R})$*, the next identity,* $(g \circ f)(\mathbb{X}) = g(f(\mathbb{X}))$*, holds; with* $f(\mathbb{X})$ *and* $g(f(\mathbb{X}))$ *being existent.*

*Proof.* By 2.3.2, for every $\lambda \in \mathbb{R}$, we have

$$\mu_{\mathbb{X}}(-\infty, \lambda) = e_{\lambda}^{f(\mathbb{X})} = [\![f(\mathbb{X}) < \lambda^{\wedge}]\!] = \mu_{\mathbb{X}} \circ f^{-1}(-\infty, \lambda).$$

Hence, the spectral measures, $\mu_{f(\mathbb{X})}$ and $\mu_{\mathbb{X}} \circ f^{-1}$, defined on $\mathbb{B}(\mathbb{R})$, coincide on intervals of the form $(-\infty, \lambda)$. Using standard arguments of measure theory, we derive that these measures coincide everywhere. To prove the second part, it suffices to note that $(g \circ f)^{\leftarrow} = f^{\leftarrow} \circ g^{\leftarrow}$, and to use twice the statement obtained.

The next fact follows from 2.3.3 and 2.3.6.

**2.3.7. Theorem.** *For each* $n$*-tuple* $\mathbb{X} := (x_1, \dots, x_n)$ *of elements of a universally complete* $K$*-space* $E$*, the mapping*

$$\widehat{\mathbb{X}} : f \to \widehat{\mathbb{X}}(f) \quad (f \in \mathbb{B}(\mathbb{R}^n, \mathbb{R}))$$

*is a unique sequentially o-continuous homomorphism of the* $f$*-algebra* $\mathbb{B}(\mathbb{R}^n, \mathbb{R})$ *into* $E$ *such that the following condition holds*

$$\widehat{\mathbb{X}}(d_k^{\lambda}) = x_k \quad (k := 1, \dots, n),$$

*with* $d_k^{\lambda} : (\lambda_1, \dots, \lambda_n) \to \lambda_k$ *being a coordinate function in* $\mathbb{R}^n$*.*

**2.3.8.** We will briefly discuss two realizations of the universally complete $K$-space $\mathcal{R}{\downarrow}$ which can be obtained with the help of 2.2.4. We recall the necessary definitions. For a compactum $Q$, the symbol $C_{\infty}(Q)$ stands for the set of all continuous functions from $Q$ into $\overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty, -\infty\}$ which take the values $\pm\infty$ only on a rare (= nowhere dense) set.

*A resolution of unity* in a Boolean algebra $B$ is a mapping $e : \mathbb{R} \to B$ meeting the following conditions:

(1) $s \le t \to e(s) \le e(t)$ $(s, t \in \mathbb{R})$;
(2) $\bigvee_{t \in \mathbb{R}} e(t) = \mathbf{1}, \bigwedge_{t \in \mathbb{R}} e(t) = \mathbf{0}$;
(3) $\bigvee_{s \in \mathbb{R}, s < t} e(s) = e(t)$ $(t \in \mathbb{R})$.

Let $\mathbb{R}(B)$ be the set of all resolutions of unity in $B$.

**2.3.9. Theorem.** *Let* $B$ *be a complete Boolean algebra. The set* $\mathbb{R}(B)$ *with suitable operations and order is a universally complete* $K$*-space. The function which maps an element* $x \in \mathcal{R}{\downarrow}$ *into the resolution of unity* $\lambda \to [\![x < \lambda^{\wedge}]\!] (\lambda \in \mathbb{R})$*, is an isomorphism between the* $K$*-spaces* $\mathcal{R}{\downarrow}$ *and* $\mathbb{R}(B)$*.*

**2.3.10. Theorem.** *Let* $Q$ *be the Stone compactum of a complete Boolean algebra* $B$*, and let* $\mathcal{R}$ *be the field of reals in the model* $\mathbb{V}^{(B)}$*. The vector lattice* $C_{\infty}(Q)$ *is isomorphic to the universally complete* $K$*-space* $\mathcal{R}{\downarrow}$*. The isomorphism is defined by sending an element* $x \in \mathcal{R}{\downarrow}$ *to the function* $\hat{x} : Q \to \overline{\mathbb{R}}$ *according to the formula:*

$$\hat{x}(q) = \inf\{\lambda \in \mathbb{R} : [\![x < \lambda^{\wedge}]\!] \in q\}.$$

**2.3.11. Remarks.**

(1) The notions of unity, unit element, and characteristic (spectral function of an element) were introduced by H. Freudenthal. He also established the spectral theorem, see 2.3.5 and [10, 19]. It follows from Theorem 2.3.9 that, for a complete Boolean algebra $B$, the set of resolutions of unity is a universally complete $K$-space with base isomorphic to $B$. This fact belongs to L. V. Kantorovich [19]. The realization of an arbitrary $K$-space as a foundation of a universally complete $K$-space was fulfilled by A. G. Pinsker (see [10, 19]). The possibility of realizing an arbitrary $K$-space as a foundation in $C_\infty(Q)$ follows from 2.2.8(1) and 2.3.10. This possibility was first established by B. Z. Vulikh and T. Ogasawara independently (see [10, 19]).

(2) It follows from 2.3.4 that every spectral function with values in a $\sigma$-algebra defines a spectral measure on the Borel $\sigma$-algebra of the real line. This fact was first mentioned by V. I. Sobolev in [41]. Nevertheless, it was supposed in [41] that such a measure can be obtained by the Carathéodory extension. As was shown by D. A. Vladimirov, for a complete Boolean algebra of countable type the Carathéodory extension is possible only if the algebra is regular. Thus, the extension method leading to 2.3.4 considerably differs from the Carathéodory extension and is based on the Loomis–Sikorki representation of Boolean $\sigma$-algebras. M. Wright derived Proposition 2.3.4 as a consequence of a Riesz theorem established for operators with values in a $K$-space.

(3) Apparently, Borel functions of elements of an arbitrary $K$-space with unity were first considered by V. I. Sobolev (see [10, 39]). Theorem 2.3.6 in full generality is presented in [29, 30]. The Borel functional calculus of (countable and uncountable) collections of elements of an arbitrary $K$-space is also constructed in [29, 30]. A Boolean-valued proof of Theorem 2.3.7 is also available (see [5]).

(4) For other aspects of Boolean-valued analysis of vector lattices see [5, 23, 24, 27, 51, 52].

## 3. INFINITESIMALS IN VECTOR LATTICES

The apology to the Infinitesimal offered by A. Robinson immediately opened new possibilities for the theory of Banach spaces. The central construction here is the nonstandard hull of a space, the latter being the quotient space of the external subspace of elements with finite norm over the monad of the space (= the set of elements with infinitesimal norm). The adaptation of nonstandard hulls to the theory of lattices is discussed in the first section of the current chapter. The rest of it is devoted to a scantily explored theme of combining Boolean-valued and infinitesimal methods. Theoretically two approaches are feasible. The first consists in studying a Boolean-valued model realized within the internal world of external set theory. This approach is outlined in Section 3.2. The second approach deals with exploring a suitable fragment of nonstandard set theory (for instance, in the form of ultraproduct or ultralimit) located inside the corresponding Boolean-valued universe. The latter approach is introduced in the third section of the chapter. It is important to note to this end that in spite of superficial resemblance, the formalisms considered lead to tremendously different constructions in $K$-space theory. We illustrate the arising peculiarities with the theories of cyclic topologies and cyclic compactness which are of great importance to the Boolean-valued analysis.

**3.1. Nonstandard hulls and Banach lattices.**

In the geometric theory of Banach spaces the notion of the nonstandard hull is fundamental.

**3.1.1.** Let $(E, |\cdot|)$ be an internal normed space. An element $x \in E$ is called *finite* (*infinitesimal*) if $|x|$ is finite (infinitesimal). Denote by $\mathrm{fin}(E)$ and $\mu(E)$ the external sets of all finite and, respectively, infinitesimal elements of $E$. Then $\mathrm{fin}(E)$ is an (external) vector space over the field ${}^\circ\mathbb{R}$, and $\mu(E)$ is its subspace. The quotient space $\mathrm{fin}(E)/\mu(E)$ is denoted by the symbol $\widehat{E}$. A norm is defined in $\widehat{E}$ by the formula

$$\|\pi x\| = \mathrm{st}\ (\|x\|) \in \circ^{\mathbb{R}} \quad (x \in \mathrm{fin}(E)),$$

where $\pi : \mathrm{fin}(E) \to \widehat{E}$ is the quotient homomorphism. Furthermore, $(\widehat{E}, \|\cdot\|)$ is an external normed space, and is called the *nonstandard hull of* $E$. If the internal dimension of $E$ is finite, then $\widehat{E}$ is called a *hyperfinite dimensional space.* If the space $(E, \|\cdot\|)$ is standard, then ${}^\circ E$ with the induced norm from $E$ is an external normed space, and the restriction of $\pi$ on ${}^\circ E$ is an isometric embedding of ${}^\circ E$ into $\widehat{E}$. The inclusion ${}^\circ E \subset \widehat{E}$ is usually assumed.

**3.1.2. Theorem.** *The space $\widehat{E}$ is a Banach space for every internal (not necessarily complete) normed space $E$.*

*Proof.* Let $B_X(a, r)$ be a closed ball in $X$ with center $a$ and radius $r$. Consider a decreasing sequence of balls $B_{\widehat{E}}(\widetilde{x}_n, r_n)$ in $X$ such that $(x_n)_{n\in{}^\circ\mathbb{N}} \subset E, \widetilde{x}_n = \pi x_n$, $(r_n)_{n\in{}^\circ\mathbb{N}} \subset {}^\circ\mathbb{R}$, and $\lim_{n\to\infty} r_n = 0$. We may assume that $r_n$ decrease. Then the sequence of internal closed balls $B_E(x_n, r_n + r_n/2^{n+1}) \subset E$ decreases. By idealization principle, there exists an element $x \in E$ lying in every of these balls. The element $\widetilde{x} = \pi^x$ is a common point of the balls $B_{\widehat{E}}(\widetilde{x}_n, r_n)$.

**3.1.3.** Suppose that $E$ is an internal normed lattice. Then we can define an order relation in $\widehat{E}$ so that the quotient homomorphism $\pi$ is positive. More precisely, if $\widetilde{x} := \pi x$ and $y := \pi y$, then we assume by definition

$$\widetilde{x} \le \widetilde{y} \leftrightarrow (\exists z \in \mu(E))x \le y + z.$$

*The nonstandard hull $\widehat{E}$ is a Banach lattice with sequentially o-continuous norm. Moreover, every increasing and norm bounded sequence in $\widehat{E}$ is order bounded.*

At the same time, it is worth noting that the nonstandard hull of an internal norm lattice is not necessarily a $K$-space (not even a $K_\sigma$-space: for example, $\widehat{c}_0$, where $c_0$ is the lattice of vanishing sequences).

**3.1.4. Theorem.** *For an internal normed lattice $E$, the following statements are equivalent:*

(1) *$E$ is a $K$-space;*

(2) *$\widehat{E}$ is a $K_\sigma$-space;*

(3) *$\widehat{E}$ has an o-continuous norm;*

(4) *in $\widehat{E}$, there is no closed sublattice, isometric and order isomorphic to the Banach lattice $c_0$.*

**3.1.5.** A normed lattice is said to be *rich in finite-dimensional sublattices*, if for every finite collection $x_1, \dots, x_n \in {}^\circ E, n \in {}^\circ\mathbb{N}$, and for arbitrary $0 < \varepsilon \in {}^\circ\mathbb{R}$ there exists a finite-dimensional sublattice $E_0 \subset {}^\circ E$ and elements $y_1, \dots, y_n \in E_0$ such that the inequality $\|x_k - y_k\| < \varepsilon$ holds for all $k := 1, \dots, n$.

A standard Banach lattice $E$ is rich in finite-dimensional sublattices if and only if ${}^\circ E$ is contained in a hyperfinite-dimensional subspace of the hull $\widehat{E}$.

**3.1.6.** Suppose now that $E$ and $F$ are internal normed spaces and $T : E \to F$ is an internal linear bounded operator. The set

$$c(T) := \{C \in \mathbb{R} : (\forall x \in E)\|Tx\| \le C\|x\|\}$$

is internal and bounded from above. Hence, $\|T\| := \inf c(T)$ exists. If $\|T\|$ is finite then it follows from the inequality $\|Tx\| \le \|T\| \cdot \|x\| (x \in E)$ that $T(\mathrm{fin}(E)) \subset \mathrm{fin}(E)$ and $T(\mu(E)) \subset \mu(E)$. Thus, an external operator $\widehat{T} : \widehat{E} \to \widehat{F}$ is well defined by the formula

$$\widehat{T}\pi x = \pi T x \quad (x \in E).$$

The operator $\widehat{T}$ is linear (over ${}^\circ\mathbb{R}$) and bounded; moreover, $\|\widehat{T}\| = \mathrm{st}\ \|T\|$. It is natural to call $\widehat{T}$ the *nonstandard hull of* $T$.

If $E$ and $F$ are normed lattices, and the operator $T$ is positive; then $\widehat{T}$ is a positive sequentially $o$-continuous operator.

**3.1.7.** It can be easily seen that, for bounded operators $S$ and $T$, the equality $(S \circ T)^\wedge = \widehat{S} \circ \widehat{T}$ holds; and, in addition, $\widetilde{I}_E = I_{\widehat{E}}$, with $I_X$ being the identity operator on $X$. Thus, the operation of passing to nonstandard hull is a covariant functor (in the suitable categories of normed spaces). A good deal of questioning about general properties of this functor arises. How does the nonstandard hull functor cooperate with other functors of the theory of Banach spaces (lattices)? How do the well-known properties in the geometrical theory of Banach spaces (Radon–Nikodým property, Kreĭn–Mil′man property, etc.) transform under this functor? What is the structure of nonstandard hulls of concrete spaces? Analogous questions can be formulated for operators as well (see [24]), so forth and so on. The main ideas and methods are exposed in the surveys [1, 4, 5, 27]. Here we shall briefly outline the three important directions and formulate simple illustrative statements.

**3.1.8. Analytical description for nonstandard hulls.** The most complete investigation of this question is carried out for the classical Banach spaces.

**Theorem.** (1) *If $E$ is an internal $AL_p$-space, where $p, 1 \le p$, is a finite element of $\mathbb{R}$, then $\widehat{E}$ is an $AL_r$-space, for $r = \mathrm{st}\ (p)$.*

(2) *If $E$ is in internal $AL_p$-space, with $p$ being an infinite element of $\mathbb{R}$, or if $E$ is an internal $AM$-space, then $\widehat{E}$ is an $AM$-space.*

(3) *If $Q$ is an internal compactum, and $C(Q)$ is the internal space of continuous functions from $Q$ to $\mathbb{R}$, then $C(Q)^\wedge$ is linearly isometric to $C(\widehat{Q})$, where $\widehat{Q}$ is the external completion of $Q$ in some uniformity.*

In axiomatic external set theory only general results of this type can be obtained. Nevertheless, while working in a frame of classical stance of nonstandard analysis (i.e. in a finite fragment of the von Neumann universe), a more detailed description of nonstandard hulls is possible.

**3.1.9. Local geometry of normed spaces.** Some properties of a normed space are “local” in the sense that they are defined by the structure and localization of finite-dimensional subspaces of the space. In this regard, nonstandard hulls have much more preferable structure. For instance, it often appears that if a condition is satisfied “approximately” on finite-dimensional subspaces, then this condition is satisfied “exactly” in the nonstandard hull.

Let $E$ and $F$ be Banach lattices. The lattice $E$ is said to be *finitely representable in $F$ (as a Banach sublattice)*, if for every finite-dimensional sublattice $E_0 \subset E$ and every number $\varepsilon 0$ there exists a linear and lattice isomorphism $T : E_0 \to F$ such that $\|x\| \le \|Tx\| \le (1+\varepsilon)\|x\|$ $(x \in E_0)$.

**Theorem.** *Let $E$ be a standard Banach lattice rich in finite-dimensional sublattices* (3.1.5)*, and let $F$ be an internal Banach lattice. Then ${}^\circ E$ is finitely representable in $F$ if and only if ${}^\circ E$ is linearly isometric and lattice isomorphic to a sublattice of $\widehat{F}$.*

**3.1.10. Model-theoretic properties.** We will introduce a first-order language $\mathbb{L}_B$. The signature of the language is $\{=, +, P, Q\} \cup \mathbb{Q}$, where $\mathbb{Q}$ is the set of rationals. Every Banach space $E$ can be considered as a model of $\mathbb{L}_B$ via interpreting $=$ and $+$ respectively as equality and addition, $P$ as $\{x \in E : \|x\| \le 1\}$, $Q$ as $\{x \in E : \|x\| \ge 1\}$, and finally, every $r \in \mathbb{Q}$ as the operation of multiplying by $r$. A formula $\varphi$ of $\mathbb{L}_B$ of the form $(Sx_1)\dots(Sx_n)(\varphi_1 \wedge \dots \wedge \varphi_n)$, where $S$ is a restricted quantifier and $\varphi_k$ is a conjunction of formulas of the form $u = v, P(u), Q(u)$, is called *a restricted positive formula.* If $\varphi$ is such a formula and $m$ is a natural $(\ne 0)$, then $\varphi^m$ is a new formula constructed as follows: in the subformulas $\varphi_1, \dots, \varphi_n$ the expression $u = v$ should be replaced by $P(m(u-v))$, $P(u)$ by $P((1-1/m)u)$, $Q(u)$ by $Q((1+1/m)u)$. If $\varphi^m$ is valid in $E$ for all $m \in \mathbb{N}$, then $\varphi$ is said to be *approximately valid in $E$.* Banach spaces $E$ and $F$ are called *approximately equivalent*, if the same restricted positive formulas are approximately valid in them.

**Theorem.** (1) *Banach spaces are approximately equivalent if and only if their nonstandard hulls are isometric.*

(2) *Let $\mu$ and $\nu$ be $\sigma$-finite measures, and $1 \le p < \infty$. The spaces $L_p(\mu)$ and $L_p(\nu)$ are approximately equivalent if and only if the measures $\mu$ and $\nu$ either have the same finite number of atoms or both possess an infinite number of atoms.*

**3.1.11. Remarks.**

(1) The nonstandard hull of a Banach space was introduced by W. A. J. Luxemburg [27]. The ultraproducts of Banach spaces introduced by D. Dacunha—Castelle and I. L. Krivine [45] are very similar to nonstandard hulls. About the role of these notions in the theory of Banach spaces, the most important results, and further references, consult [27, 57].

(2) The first-order language described in 3.1.10 was employed first by W. Henson and later on by J. Stern (see [57]). The notion of finite representability had come into the theory of Banach spaces long before appealing to the set-theoretic technique. It was introduced by A. Dvoretsky (the term is due to R. C. James).

(3) About 3.1.4, 3.1.5 and 3.1.9 see W. Henson’s article in [57].

**3.2. Boolean-valued modeling in the nonstandard universe.**

In Boolean-valued analysis a new important class of mathematical objects is introduced — the class of structures with cyclic property (= closure under mixing, see 1.2.6(2)). These objects are the descents of corresponding formations in $\mathbb{V}^{(B)}$, see 1.2.8. As a matter of fact, the methodology developed by infinitesimal analysis is connected with inventing a special machinery for studying filters — monadology.

Let $\mathcal{F}$ be a standard filter, let ${}^{\circ}\mathcal{F}$ be its standard core, and let ${}^{a}\mathcal{F} := \mathcal{F} \setminus {}^{\circ}\mathcal{F}$ be the external set of *astray or distant elements* of $\mathcal{F}$. If

$$\mu(\mathcal{F}) := \cap^{\circ}\mathcal{F} = \cup^{a}\ \mathcal{F}$$

is the *monad of* $\mathcal{F}$, then $\mathcal{F} = {}^{*}ss(\{\mu(\mathcal{F})\})$, with $ss(A)$ standing for the collection of all supersets of $A$. The notion of monad is central to the external set theory. In this connection the development of combined methods, in particular, applying infinitesimals and ascents simultaneously in $K$-space theory requires adaptation of the notion of monad for filters and their images. In this section we pursue an approach in which the ordinary monadology is applied to descents of objects. An alternative way — applying the standard monadology inside $\mathbb{V}^{(B)}$ with further descending — will be considered in the next section.

**3.2.1.** We shall recall some constructions from the theory of filters in $\mathbb{V}^{(B)}$.

Let $\mathcal{G}$ be a filterbase on $X$, and $X \in \mathcal{P}(\mathbb{V}^{(B)})$. Write

$$\mathcal{G}' := \{F \in \mathcal{P}(X{\uparrow}){\downarrow} \colon (\exists G \in \mathcal{G})[\![F \supset G{\uparrow}]\!] = \mathbf{1}\};$$

$$\mathcal{G}'' := \{G{\uparrow} \colon G \in \mathcal{G}\}.$$

Then $\mathcal{G}'{\uparrow}$ and $\mathcal{G}''{\uparrow}$ are bases of one and the same filter $\mathcal{G}^{\uparrow}$ on $X{\uparrow}$ inside $\mathbb{V}^{(B)}$. The filter $\mathcal{G}^{\uparrow}$ is called the *ascent of* $\mathcal{G}$. If $\operatorname{mix}(\mathcal{G})$ is the set of all the mixtures of nonempty families formed by elements of $\mathcal{G}$, and $\mathcal{G}$ consists of cyclic sets, then $\operatorname{mix}(\mathcal{G})$ is a filterbase on $X$ and $\mathcal{G}^{\uparrow} = \operatorname{mix}(\mathcal{G})^{\uparrow}$.

If $\mathcal{F}$ is a filter on $X$ inside $\mathbb{V}^{(B)}$, then write $\mathcal{F}^{\downarrow} := ss(\{F{\downarrow} \colon F \in \mathcal{F}{\downarrow}\})$. The filter $\mathcal{F}^{\downarrow}$ on $X{\downarrow}$ is called the *descent of* $\mathcal{F}$. A filterbase $\mathcal{G}$ on $X{\downarrow}$ is called *extensional*, if there is a filter $\mathcal{F}$ on $X$ such that $ss(\mathcal{G}) = \mathcal{F}$. Finally, the descents of ultrafilters in $X$ are called *proultrafilters on* $X{\downarrow}$. A filter having a base of cyclic sets is called *cyclic*. Proultrafilters are maximal cyclic filters.

**3.2.2.** Fix a standard complete Boolean algebra $B$ and the corresponding Boolean-valued universe $\mathbb{V}^{(B)}$ thought of as being composed of internal sets. If $A$ is an external set, then the cyclic hull, $\operatorname{mix}(A)$, is introduced as follows. Say that an element $x \in \mathbb{V}^{(B)}$ belongs to $\operatorname{mix}(A)$, if there is an internal family $(a_\xi)_{\xi\in\Xi}$ of elements of $A$ and an internal partition $(b_\xi)_{\xi\in\Xi}$ of unity in $B$, such that $x$ is the mixture of $(a_\xi)_{\xi\in\Xi}$ with probabilities $(b_\xi)_{\xi\in\Xi}$, i.e. $b_\xi^x = b_\xi a_\xi$ for $\xi \in \Xi$ or, equivalently, $x = \operatorname{mix}_{\xi\in\Xi}(b_\xi a_\xi)$.

**3.2.3. Theorem.** *For a filter* $\mathcal{F}$ *on* $X{\downarrow}$*, consider*

$$\mathcal{F}{\uparrow}{\downarrow} := ss(\{F{\uparrow}{\downarrow} \colon F \in \mathcal{F}\}).$$

*Then* $\operatorname{mix}(\mu(\mathcal{F})) = \mu(\mathcal{F}{\uparrow}{\downarrow})$*, and* $\mathcal{F}{\uparrow}{\downarrow}$ *is the greatest cyclic filter coarser than* $\mathcal{F}$*.*

In connection with this theorem, the monad of $\mathcal{F}$ is called *cyclic*, if $\mu(\mathcal{F}) = \operatorname{mix}(\mu(\mathcal{F}))$. Unfortunately, cyclicity of a monad is not completely responsible for extensionality of a filter. In this connection, the *cyclic monad hull* $\mu_c(U)$ of an external set $U$ should be introduced. Namely, we are compelled to set

$$x \in \mu_c(U) \leftrightarrow (\forall^{\mathrm{St}} V = V{\uparrow}{\downarrow}) V \supset U \to x \in \mu(V).$$

In particular, if $B = \{0,1\}$, then $\mu_c(U)$ coincides with the monad of the standardization of the external filter of supersets of $U$, i.e. with the so-called (*discrete*)*monad hull* $\mu_d(U)$ (the word "monadic" is also employed).

**3.2.4.** *The cyclic monad hull of a set is the cyclic hull of its monad hull*

$$\mu_c(U) = \mathrm{mix}(\mu_d(U)).$$

A special role is played by *essential points* of $X{\downarrow}$ constituting the external set ${}^eX$. By definition, ${}^eX$ consists of elements of proultrafilter monads in $X{\downarrow}$.

**Criterion for essentiality.** *A point is essential if and only if it can be separated by a standard set from every standard cyclic set not containing the point.*

**3.2.5.** If in the monad of ultrafilter $\mathcal{F}$ there is a an essential point, then $\mu(\mathcal{F}) \subset {}^eX$, and moreover, $\mathcal{F}{\uparrow}{\downarrow}$ is a proultrafilter.

The next statement follows from the constructions and considerations presented above.

**3.2.6. Criterion for filter extensionality.** *A filter is extensional if and only if its monad is the cyclic monad hull of the set of its own essential points.*

**3.2.7.** A standard set is cyclic if and only if it is the cyclic monad hull of its own essential points.

**3.2.8. Nonstandard criterion for a mixture of filters.** *Let $(\mathcal{F}_\xi)_{\xi\in\Xi}$ be a standard family of extensional filters, and let $(b_\xi)_{\xi\in\Xi}$ be a standard partition of unity. The filter $\mathcal{F}$ is the mixture of $(\mathcal{F}_\xi)_{\xi\in\Xi}$ with probabilities $(b_\xi)_{\xi\in\Xi}$ if and only if*

$$(\forall^{\mathrm{St}} \xi \in \Xi) b_\xi \mu(\mathcal{F}) = b_\xi \mu(\mathcal{F}_\xi).$$

A peculiarity of the approach presented exposes itself in applications to the descents of topological spaces through a special new role of essential points. In this connection, we note some properties of the latter.

**3.2.9.** *The following statements are true:*

(1) *the image of an essential point with respect to an extensional mapping is an essential point of the image;*

(2) *let $E$ be a standard set, and let $X$ be a standard element of $\mathbb{V}^{(B)}$. Consider the product $X^{E^\wedge}$ inside $\mathbb{V}^{(B)}$, where $E^\wedge$ is the standard name of $E$ in $\mathbb{V}^{(B)}$. If $x$ is an essential point of $X^{E^\wedge}{\downarrow}$, then for every standard $e \in E$ the point $x{\downarrow}(e)$ is essential in $X{\downarrow}$;*

(3) *let $\mathcal{F}$ be a cyclic filter in $X{\downarrow}$, and let ${}^e\mu(\mathcal{F}) := \mu(\mathcal{F}) \cap {}^eX$ be the set of essential points of its monad. Then*

$${}^e\mu(\mathcal{F}) = {}^e\mu(\mathcal{F}^{\uparrow\downarrow}).$$

Let $(X,\mathcal{U})$ be a uniform space inside $\mathbb{V}^{(B)}$. The uniform space $(X{\downarrow},\mathcal{U}^{\downarrow})$ is called *procompact* (= *cyclic compact*), if $(X,\mathcal{U})$ is compact inside $\mathbb{V}^{(B)}$. The similar sense is implied in the notion of *pro-total-boundedness* and so on.

**3.2.10. Nonstandard criterion for procompactness.** *Every essential point of $X{\downarrow}$ is nearstandard (i.e. infinitesimally close to a standard point) if and only if $X{\downarrow}$ is procompact.*

It is easily seen from Theorem 3.2.10, that the Boolean-valued criterion of procompactness differs from the usual one: "a compact space is a space formed by nearstandard points." The existence of a great number of procompact and noncompact spaces provides the variety of examples of nonessential points. We note here that a combined application of 3.2.10 and 3.2.9(2), of course, allows us to produce a nonstandard proof of a natural analog of Tychonoff's theorem for a product of procompact spaces — "the descent of Tychonoff's theorem in $\mathbb{V}^{(B)}$."

**3.2.11. Nonstandard criterion for proprecompactness.** *A standard space is a descent of a totally bounded uniform space if and only if its every essential point is prenearstandard (i.e. is in the monad of a Cauchy filter).*

We shall apply the approach presented to description for $o$-convergence in a $K$-space $Y$. To save space, we restrict ourselves to the consideration of filters containing order intervals (or, equivalently, filters with bounded monads). In addition, in accordance with the same end, the $K$-space $Y$ is assumed to be universally complete. By Gordon's theorem, the space $Y$ can be assumed to be realized as the descent $\mathcal{R}{\downarrow}$ of the element $\mathcal{R}$ representing the field of reals $\mathbb{R}$ in the Boolean-valued universe $\mathbb{V}^{(B)}$ constructed over the base $B$ of the space $Y$. Let us denote by $\mathcal{E}$ the filter of order units in $Y$, i.e. the set $\mathcal{E} := \{\varepsilon \in Y_+ : [\![\varepsilon = 0]\!] = \mathbf{0}\}$. We write $x \approx y$ whenever elements $x, y \in Y$ are infinitely close with respect to the descent of the natural topology of $\mathcal{R}$ in $\mathbb{V}^{(B)}$, i.e. $x \approx y \leftrightarrow (\forall^{\mathrm{St}}\varepsilon \in \mathcal{E})|x - y| < \varepsilon$. For $a, b \in Y$, we write $a < b$ if $[\![a < b]\!] = \mathbf{1}$, in other words, $a > b \leftrightarrow a - b \in \mathcal{E}$. Thus, there is a deviation from the understanding of the theory of ordered vector spaces. Of course, this circumstance is necessitated if we want to follow the principles of introducing notations for descents and ascents. Let ${}^{\approx}Y$ be the nearstandard part of $Y$. For $y \in {}^{\approx}Y$, denote by ${}^{\circ}y$ (or by st $(y)$) the standard part of $y$, i.e. the unique standard element infinitely closed to $y$.

**3.2.12. Theorem.** *For a standard filter $\mathcal{F}$ in $Y$ and a standard $z \in Y$, the following statements are true:*

(1) $\inf_{F\in\mathcal{F}} \sup F \le z \leftrightarrow (\forall y \in {}^{\circ}\mu(\mathcal{F}{\uparrow}{\downarrow}))\,{}^{\circ}y \le z \leftrightarrow (\forall y \in {}^{e}\mu(\mathcal{F}{\uparrow}{\downarrow}))\,{}^{\circ}y \le z$;

(2) $\sup_{F\in\mathcal{F}} \inf F \ge z \leftrightarrow (\forall y \in {}^{\circ}\mu(\mathcal{F}{\uparrow}{\downarrow}))\,{}^{\circ}y \ge z \leftrightarrow (\forall y \in {}^{e}\mu(\mathcal{F}{\uparrow}{\downarrow}))\,{}^{\circ}y \ge z$;

(3) $\inf_{F\in\mathcal{F}} \sup F \ge z \leftrightarrow (\exists y \in {}^{\circ}\mu(\mathcal{F}{\uparrow}{\downarrow}))\,{}^{\circ}y \ge z \leftrightarrow (\exists y \in {}^{e}\mu(\mathcal{F}{\uparrow}{\downarrow}))\,{}^{\circ}y \ge z$;

(4) $\sup_{F\in\mathcal{F}} \inf F \le z \leftrightarrow (\exists y \in {}^{\circ}\mu(\mathcal{F}{\uparrow}{\downarrow}))\,{}^{\circ}y \le z \leftrightarrow (\exists y \in {}^{e}\mu(\mathcal{F}{\uparrow}{\downarrow}))\,{}^{\circ}y \le z$;

(5) $\mathcal{F} \overset{o}{\to} z \leftrightarrow (\forall y \in {}^{e}\mu(\mathcal{F}{\uparrow}{\downarrow}))y \approx z \leftrightarrow (\forall y \in \mu(\mathcal{F}^{\uparrow\downarrow}))y \approx z$. *Here* ${}^{\circ}\mu(\mathcal{F}{\uparrow}{\downarrow}) := \mu(\mathcal{F}{\uparrow}{\downarrow}) \cap {}^{\approx}Y$, *and, as usual,* ${}^{e}\mu(\mathcal{F}{\uparrow}{\downarrow})$ *is the set of essential points of the monad* $\mu(\mathcal{F}{\uparrow}{\downarrow})$, *i.e.* ${}^{e}\mu(\mathcal{F}{\uparrow}{\downarrow}) = \mu(\mathcal{F}{\uparrow}{\downarrow}) \cap {}^{e}\mathcal{R}$.

*Proof.* To illustrate we shall prove (3).

Suppose, at first, that in the greater set ${}^{\circ}\mu(\mathcal{F}{\uparrow}{\downarrow})$ there is an element $y$ such that ${}^{\circ}y \ge z$. For every standard $F \in \mathcal{F}$ we have $y \in F{\uparrow}{\downarrow}$. Hence, if $\varepsilon \in {}^{\circ}\mathcal{E}$ then $y\ z - \varepsilon$ and $\sup F = \sup F{\uparrow}{\downarrow}\ z - \varepsilon$. By Leibniz's principle we obtain: $(\forall^{\mathrm{St}} F \in \mathcal{F})(\forall^{\mathrm{St}}\varepsilon > 0) \sup F \ge z$, i.e. $(\forall F \in \mathcal{F}) \sup F \ge z$ and $\inf_{F\in\mathcal{F}} \sup F \ge z$.

To prove the rest, begin with noting that by properties of the upper limit in $\mathbb{R}$ and by the transfer principle of Boolean-valued analysis we have

$$[\![(\exists \mathcal{G})(\mathcal{G} \text{ is an ultrafilter in } \mathcal{R} \wedge \mathcal{G} \supset \mathcal{F}^{\uparrow} \wedge \inf_{G \in \mathcal{G}} \sup G \geq z)]\!] = \mathbf{1}.$$

According to the maximum principle, there is a proultrafilter $\mathcal{G}$ such that $\mathcal{G} \supset \mathcal{F}^{\uparrow\downarrow}$ and $\inf_{G\in\mathcal{G}} \sup G \geq z$. Using the transfer and idealization principles, we obtain successively

$$(\forall^{\mathrm{St}} G \in \mathcal{G}) \sup G \geq z \leftrightarrow (\forall^{\mathrm{St}} G \in \mathcal{G})[\![\sup G{\uparrow}= z]\!] = \mathbf{1}$$

$$\leftrightarrow (\forall^{\mathrm{St}} G \in \mathcal{G})[(\exists \varepsilon > 0)(\exists g \in G{\uparrow}) g > z - \varepsilon]\!] = \mathbf{1}$$

$$\leftrightarrow (\forall^{\mathrm{St}} G \in \mathcal{G})(\forall \varepsilon > 0)(\exists g \in G{\uparrow\downarrow}) g > z - \varepsilon$$

$$\leftrightarrow (\forall^{\mathrm{St}} G \in \mathcal{G})(\forall^{\mathrm{St}} \varepsilon > 0)(\exists g \in G{\uparrow\downarrow}) g > z - \varepsilon$$

$$\leftrightarrow (\forall^{\mathrm{St\,fin}} \mathcal{G}_0 \supset \mathcal{G})(\forall^{\mathrm{St\,fin}} \mathcal{E}_0 \subset \mathcal{E})(\exists g)$$

$$(\forall G \in \mathcal{G}_0)(\forall \varepsilon \in \mathcal{E}_0)(g \in G{\uparrow\downarrow} \vee g > z - \varepsilon)$$

$$\leftrightarrow (\exists g)(\forall^{\mathrm{St}} G \in \mathcal{G})(\forall^{\mathrm{St}} \varepsilon 0)(g \in G{\uparrow\downarrow} \vee g > z - \varepsilon)$$

$$\leftrightarrow (\exists g \in \mu(\mathcal{G}^{\uparrow\downarrow}))\,{}^{\circ}g \geq z \leftrightarrow (\exists g \in \mu(\mathcal{G}))\,{}^{\circ}g = z.$$

The observation,

$$\mu(\mathcal{G}) \subset {}^{e}\mu(\mathcal{F}^{\uparrow\downarrow}) = {}^{e}\mu(\mathcal{F}{\uparrow\downarrow}) \subset {}^{\circ}\mu(\mathcal{F}{\uparrow\downarrow}),$$

completes the proof.

**3.2.13. Remarks.**

(1) The monadology as a philosophical doctrine was developed by G. W. Leibniz [37]. The general theory of monads of filters was proposed by W. A. J. Luxemburg. Cyclic topologies are widely used in Boolean-valued analysis. The theory of cyclic compactness and the principles of dealing with images of filters are presented in [24, 33, 35]. Our introducing the cyclic monadology follows the general line of [33, 35].

(2) Considering ultraproducts inside the Boolean-valued universe causes no difficulty in principle and was adopted in several papers. We do not discuss here how Robinson's standardization is introduced in $\mathbb{V}^{(B)}$; as a matter of fact, an axiomatic approach is also possible. The decisive element is the appearance of excrescences on $K$-spaces sprouting, generally speaking, out of the uncustomary way of standardizing the source space (effect of essential points). Our presentation follows [28].

### 3.3. Infinitesimal modeling inside the Boolean-valued universe.

In this section we assume a complete Boolean algebra $B$ and a separated universe $\mathbb{V}^{(B)}$ to be fixed.

Applying methods of infinitesimal analysis we adopt the classical A. Robinson's approach realized inside $\mathbb{V}^{(B)}$. In other words, in concrete situations the classical and internal universes and the corresponding *-map (Robinson's standardization) are understood as elements of $\mathbb{V}^{(B)}$. Moreover, the nonstandard world is supposed to be saturated to a degree required.

**3.3.1.** The descent of the $^*$-map will be called the *descent standardization*. Side by side with the term of descent standardization we also use the expressions as "$B$-standardization," "prostandardization," etc. Furthermore, for Robinson's standardization of a $B$-set $A$ the symbol $^*A$ is used. Respectively, the descent standardization of a set $A$ with $B$-structure (i.e. a subset of $\mathbb{V}^{(B)}$) is defined as $(^*(A{\uparrow})){\downarrow}$ and is denoted by the symbol $_*A$ (it is meant here that $A{\uparrow}$ is an element of the standard world located in $\mathbb{V}^{(B)}$). Thus, $^*a \in {}_*A \leftrightarrow a \in A{\uparrow}{\downarrow}$. The descent standardization $_*\Phi$ of an extensional correspondence $\Phi$ is also defined in a natural way. While considering descent standardizations of standard names of elements of the von Neumann universe $\mathbb{V}$, for convenience we shall use the abbreviations in writing $^*x := {}^*(x^{\wedge})$ and respectively $_*x := (^*x){\downarrow}$ for $x \in \mathbb{V}$. The rules of placing and omitting (by default) asterisks in using the descent standardization are also assumed to be as free as those for Robinson's $*$-map.

**3.3.2. The transfer principle.** Let $\varphi = \varphi(x,y)$ be a formula of the Zermelo–Fraenkel theory (without any free variables except $x$ and $y$). For a nonempty element $F$ in $\mathbb{V}^{(B)}$ and for every $z$ we have:

$$(\exists x \in {}^*F)[\![\varphi(x, {}^*z)]\!] = \mathbf{1} \leftrightarrow (\exists x \in F{\downarrow})[\![\varphi(x,z)]\!] = \mathbf{1};$$
$$(\forall x \in {}^*F)[\![\varphi(x, {}^*z)]\!] = \mathbf{1} \leftrightarrow (\forall x \in F{\downarrow})[\![\varphi(x,z)]\!] = \mathbf{1}.$$

If $G$ is a subset of $\mathbb{V}^{(B)}$, then the following equivalences are true:

$$(\exists x \in {}^*G)[\![\varphi(x, {}^*z)]\!] = \mathbf{1} \leftrightarrow (\exists x \in G{\uparrow}{\downarrow})[\![\varphi(x,z)]\!] = \mathbf{1};$$
$$(\forall x \in {}^*G)[\![\varphi(x, {}^*z)]\!] = \mathbf{1} \leftrightarrow (\forall x \in G{\uparrow}{\downarrow})[\![\varphi(x,z)]\!] = \mathbf{1}.$$

**3.3.3. The idealization principle.** Let $X{\uparrow}$ and $Y$ be classical elements of $\mathbb{V}^{(B)}$ and let $\varphi = \varphi(x,y,z)$ be a formula of the Zermelo–Fraenkel theory. For an internal element $z$ in $\mathbb{V}^{(B)}$ we have:

$$(\forall^{\mathrm{fin}} A \subset X)(\exists y \in {}^*Y)(\forall x \in A)[\![\varphi({}^*x, y, z)]\!] = \mathbf{1}$$
$$\leftrightarrow (\exists y \in {}_*Y)(\forall x \in X)[\![\varphi({}^*x, y, z\}]\!] = \mathbf{1}.$$

For a filter $\mathcal{F}$ of sets with $B$-structure, its *descent monad* $m(\mathcal{F})$ is defined as follows:

$$m(\mathcal{F}) := \bigcap_{F \in \mathcal{F}} {}^*F.$$

**3.3.4. Theorem.** *Let $\mathcal{E}$ be a set of filters and let $\mathcal{E}^{\uparrow} := \{\mathcal{F}^{\uparrow} : \mathcal{F} \in \mathcal{E}\}$ be its ascent in $\mathbb{V}^{(B)}$. The following statements are equivalent:*

(1) *the set of cyclic hulls $\mathcal{E}{\uparrow}{\downarrow} := \{\mathcal{F}{\uparrow}{\downarrow} : \mathcal{F} \in \mathcal{E}\}$ is bounded above;*

(2) *the set $\mathcal{E}^{\uparrow}$ is bounded above inside $\mathbb{V}^{(B)}$;*

(3) $\cap\{m(\mathcal{F}) : \mathcal{F} \in \mathcal{E}\} \neq \varnothing$.

*Moreover, if the conditions* (1)–(3) *are satisfied, then*

$$m(\sup \mathcal{E}{\uparrow}{\downarrow}) = \cap\{m(\mathcal{F}) : \mathcal{F} \in \mathcal{E}\};$$
$$\sup \mathcal{E}^{\uparrow\downarrow} = (\sup \mathcal{E})^{\uparrow}.$$

It is useful to note that for an infinite set of descent monads, its union (and even the cyclic hull of this union) is not a descent monad in general. The situation here is the same as for ordinary monads.

**3.3.5. Nonstandard criteria for a proultrafilter.** *The following statements are equivalent:*

(1) $\mathcal{A}$ *is a proultrafilter;*

(2) $\mathcal{A}$ *is an extensional filter with inclusion-minimal descent monad;*

(3) *the representation* $\mathcal{A} = (x)^{\downarrow} := ss(\{A{\uparrow}{\downarrow} \colon x \in {}^*A\})$ *holds for each point* $x$ *of the descent monad* $m(\mathbb{A})$*;*

(4) $\mathcal{A}$ *is such an extensional filter that its descent monad can be easily caught by a cyclic set, i.e. for every* $U = U{\uparrow}{\downarrow}$ *we have either* $m(\mathcal{A}) \subset {}_*U$ *or* $m(\mathcal{A}) \subset {}_*(X \setminus U)$*;*

(5) $\mathcal{A}$ *is a cyclic filter satisfying the condition: for every cyclic* $U$*, if* ${}_*U \cap m(\mathcal{A}) \neq \varnothing$ *then* $U \in \mathcal{A}$*.*

**3.3.6. Nonstandard criterion for a mixture of filters.** *Let* $(\mathcal{F}_\xi)_{\xi\in\Xi}$ *be a family of filters, let* $(b_\xi)_{\xi\in\Xi}$ *be a partition of unity, and let* $\mathcal{F} = \operatorname{mix}_{\xi\in\Xi}(b_\xi \mathcal{F}_\xi^{\uparrow})$ *be the mixture of* $(\mathcal{F}_\xi^{\uparrow})_{\xi\in\Xi}$ *with probabilities* $(b_\xi)_{\xi\in\Xi}$*. Then*

$$m(\mathcal{F}^{\downarrow}) = \operatorname{mix}_{\xi\in\Xi} \quad (b_\xi^m(\mathcal{F}_\xi)).$$

It is useful to compare 3.3.6 with 3.2.8.

A point $y$ of the set ${}_*X$ is called *descent-nearstandard* (or simply *nearstandard* if there is no danger of misunderstanding) whenever for some $x \in X{\downarrow}$ one has ${}^*x \approx y$ (i.e. $(x, y) \in m(\mathbb{U}^{\downarrow})$, with $\mathcal{U}$ being the uniformity on $X$).

**3.3.7. Nonstandard criterion for procompactness.** *The set* $A{\uparrow}{\downarrow}$ *is procompact if and only if every point of* ${}_*A$ *is descent-nearstandard.*

It is reasonable to compare 3.3.7 with 3.2.10.

Finally, we will formulate certain general principles of using the descent standardization.

**3.3.8.** Let $\varphi = \varphi(x)$ be a formula of the Zermelo–Fraenkel theory. The truth value of $\varphi$ is constant on the descent monad of every proultrafilter $\mathcal{A}$, i.e.

$$(\forall x, y \in m(\mathcal{A}))[\![\varphi(x)]\!] = [\![\varphi(y)]\!].$$

**3.3.9. Theorem.** *Let* $\varphi = \varphi(x, y, z)$ *be a formula of the Zermelo–Fraenkel theory and let* $\mathcal{F}, \mathcal{G}$ *be filters of sets with* $B$*-structure. The following quantification rules are valid (for internal* $y, z$ *in* $\mathbb{V}^{(B)}$*):*

(1) $(\exists x \in m(\mathcal{F}))[\![\varphi(x, y, z)]\!] = \mathbf{1}$
$\leftrightarrow (\forall F \in \mathcal{F})(\exists x \in {}^*F)[\![\varphi(x, y, z)]\!] = \mathbf{1}$;

(2) $(\forall x \in m(\mathcal{F}))[\![\varphi(x, y, z)]\!] = \mathbf{1}$
$\leftrightarrow (\exists F \in \mathcal{F}^{\uparrow\downarrow})(\forall x \in {}_*F)[\![\varphi(x, y, z)]\!] = \mathbf{1}$;

(3) $(\forall x \in m(\mathcal{F}))(\exists y \in m(\mathcal{G}))[\![\varphi(x, y, z)]\!] = \mathbf{1}$
$\leftrightarrow (\forall G \in \mathcal{G})(\exists F \in \mathcal{F}^{\uparrow\downarrow})(\forall x \in {}^*F)(\exists y \in {}^*G)[\![\varphi(x, y, z)]\!] = \mathbf{1}$;

(4) $(\exists x \in m(\mathcal{F}))(\forall y \in m(\mathcal{G}))[\![\varphi(x, y, z)]\!] = \mathbf{1}$
$\leftrightarrow (\exists G \in \mathcal{G}^{\uparrow\downarrow})(\forall F \in \mathcal{F})(\exists x \in {}^*F)(\forall y \in {}^*G)[\![\varphi(x, y, z)]\!] = \mathbf{1}$.

*Moreover, for standardized free variables, we have:*

(1) $(\exists x \in m(\mathcal{F}))[\![\varphi(x, {}^*y, {}^*z)]\!] = \mathbf{1}$
$\leftrightarrow (\forall F \in \mathcal{F})(\exists x \in F{\uparrow}{\downarrow})[\![\varphi(x, y, z)]\!] = \mathbf{1}$;

(2) $(\forall x \in m(\mathcal{F}))[\![\varphi(x, {}^*y, {}^*z)]\!] = \mathbf{1}$

$\leftrightarrow (\exists F \in \mathcal{F}^{\uparrow\downarrow})(\forall x \in F)[\![\varphi(x, y, z)]\!] = \mathbf{1}$;
(3) $(\forall x \in m(\mathcal{F}))(\exists y \in m(\mathcal{G}))[\![\varphi(x, y, {}^*z)]\!] = \mathbf{1}$
$\leftrightarrow (\forall G \in \mathcal{G})(\exists F \in \mathcal{F}^{\uparrow\downarrow})(\forall x \in F)(\exists y \in G{\uparrow\downarrow})[\![\varphi(x, y, z)]\!] = \mathbf{1}$;
(4) $(\exists x \in m(\mathcal{F}))(\forall y \in m(\mathcal{G}))[\![\varphi(x, y, {}^*z)]\!] = \mathbf{1}$
$\leftrightarrow (\exists G \in \mathcal{G}^{\uparrow\downarrow})(\forall F \in \mathcal{F})(\exists x \in F{\uparrow\downarrow})(\forall y \in G)[\![\varphi(x, y, z)]\!] = \mathbf{1}$.

## 4. NONSTANDARD ANALYSIS OF POSITIVE OPERATORS

Positive operators are central to the theory of ordered vector spaces.

The principal possibility, provided by nonstandard methods, is that the formalisms appeared allows us to considerably simplify the analysis of operators by reducing the situation to functionals and sometimes even to numbers. In the current chapter we will illustrate the general methods of nonstandard analysis of operators in connection with the problems of their extension and decomposition and in dealing with the structure of homomorphisms and Maharam operators.

We shall also pay attention to the problem of generating fragments of a positive operator, because the complete description of the latter was successfully and easily obtained by a consistent usage of nonstandard analysis both in Boolean-valued and in infinitesimal variants.

### 4.1. Extension and decomposition of positive operators.

We shall demonstrate in this section that many questions of the theory of order bounded operators can be reduced to the case of functionals with the help of Boolean-valued models.

**4.1.1.** The statement that $E$ is a vector lattice can be rewritten as a restricted formula, say, $\varphi(E, \mathbb{R})$. Hence, by the restricted transfer principle, we have the identity $[\![\varphi(E^\wedge, \mathbb{R}^\wedge)]\!] = \mathbf{1}$, i.e. $E^\wedge$ is a vector lattice over the ordered field $\mathbb{R}^\wedge$ inside $\mathbb{V}^{(B)}$. Let $E^{\wedge\sim}$ be the space of all $\mathbb{R}^\wedge$-linear regular functionals from $E^\wedge$ to $\mathcal{R}$. It can be easily seen that $E^{\wedge\sim} := L^\sim(E^\wedge, \mathcal{R})$ is a $K$-space in the model $\mathbb{V}^{(B)}$. The descent $E^{\wedge\sim}{\downarrow}$, as the descent of a $K$-space, is a $K$-space. Consider the universally complete $K$-space $F := \mathcal{R}{\downarrow}$ (see 2.2.4). We recall that for every operator $T \in L^\sim(E, F)$ the ascent $T{\uparrow}$ is defined by the equality $[\![T^x = T{\uparrow}\,(x^\wedge)]\!] = \mathbf{1}(x \in E)$. Note that if $\tau \in E^{\wedge\sim}$, then $[\![\tau : E^\wedge \to \mathcal{R}]\!] = \mathbf{1}$; hence, the operator $\tau{\downarrow}\colon E \to F$ is determined. Moreover, $\tau{\downarrow\uparrow} = \tau$. On the other hand, $T{\uparrow\downarrow} = T$.

**4.1.2. Theorem.** *For every $T \in L^\sim(E, F)$ the descent $T{\uparrow}$ is a regular $\mathbb{R}^\wedge$-form on $E^\wedge$ inside $\mathbb{V}^{(B)}$, i.e. $[\![T{\uparrow} \in E^{\wedge\sim}]\!] = \mathbf{1}$. The mapping $T \to T{\uparrow}$ is a linear and lattice isomorphism of the $K$-spaces $L^\sim(E, F)$ and $E^{\wedge\sim}{\downarrow}$.*

**4.1.3.** We shall formulate certain corollaries to 4.1.2. First of all, let us introduce necessary definitions. An operator $S \in L^\sim(E, F)$ is called a *fragment of an operator* $T \geq 0$, if $S \wedge (T - S) = 0$. We shall say that the operator $T$ is *$F$-discrete*, whenever $[\![0, T]\!] = [\![0, I_F]\!] \circ T$; i.e., for every $0 \leq s \leq t$ there exists an operator $0 \leq \alpha \leq I_F$ such that $S = \alpha T$. Let $L_a^\sim(E, F)$ be the band of the space $L^\sim(E, F)$ generated by $F$-discrete operators, and write $L_d^\sim(E, F) := L_a^\sim(E, F)^\perp$. The bands $(E^{\wedge\sim})_a$ and $(E^{\wedge\sim})_d$ are introduced similarly. Elements of $L_d^\sim(E, F)$ are usually referred to as *$F$-spread* or *$F$-diffuse operators*. $\mathbb{R}$-discrete or $\mathbb{R}$-diffuse operators are called, for the sake of simplicity, *discrete* or *diffuse functionals*. Consider $S, T \in L^\sim(E, F)$ and write $\tau := T{\uparrow}, \sigma := S{\uparrow}$. The following equivalences are true:

(1) $T \geq 0 \leftrightarrow [\![\tau \geq 0]\!] = \mathbf{1}$;

(2) "$S$ is a fragment of $T$" $\leftrightarrow [\![\sigma$ is a fragment of $\tau]\!] = \mathbf{1}$;

(3) "$T$ is $F$-discrete" $\leftrightarrow [\![\tau$ is discrete$]\!] = \mathbf{1}$;

(4) $T \in L_a^{\sim}(E,F) \leftrightarrow [\![\tau \in (E^{\wedge\sim})_a]\!] = \mathbf{1}$;

(5) $T \in L_d^{\sim}(E,F) \leftrightarrow [\![\tau \in (E^{\wedge\sim})_d]\!] = \mathbf{1}$.

We need one more fact which follows from 4.1.2 by direct computation of Boolean truth-values.

(6) "$T$ is a lattice homomorphism" $\leftrightarrow [\![\tau$ is a lattice homomorphism $]\!] = \mathbf{1}$.

**4.1.4. Theorem.** *Let $E$ be a vector lattice, let $F$ be a $K$-space, and suppose that $T \in L^{\sim}(E,F)$. The following statements are equivalent:*

(1) *$T$ is an $F$-discrete element of the $K$-space $L^{\sim}(E,F)$;*

(2) *$T$ is a lattice homomorphism;*

(3) *$T$ preserves disjointness, i.e. if $x, y \in E$ and $x \perp y$ then $Tx \perp Ty$.*

*Proof.* Draw 4.1.2, 4.1.3 and use the well-known result on characterization of discrete functionals (= Theorem 4.1.4 for $F = \mathbb{R}$).

**4.1.5.** It is easy to verify that if a functional $f \in E^{\sim}$ preserves disjointness, then $|f|$ has the same property. By 4.1.4(1) the functionals $f^+$ and $f^-$ are proportional to $|f|$, and since $f^+ \perp f^-$, either $f^+ = 0$ or $f^- = 0$. This means that either $f \geq 0$ or $f \leq 0$. In particular, for the functional $\tau := T\!\uparrow$ we have $[\![\tau \geq 0]\!] \vee [\![\tau \leq 0]\!] = \mathbf{1}$. If $\pi := \chi[\![\tau \geq 0]\!]$ then $\pi^{\perp} \leq \chi[\![\tau \leq 0]\!]$ and the inequalities $\pi\tau \geq 0$ and $\pi^{\perp}\tau \leq 0$ are true. The application of descents leads to the following result.

*For a regular disjointness preserving operator $T \in L^{\sim}(E,F)$ there exists a projection $\pi \in \mathbb{P}(F)$ such that $\pi T = T^+$ and $\pi^{\perp}T = T^-$. In particular, for all $0 \leq x$, $y \in E$ we have $(Tx)^+ \perp (Ty)^-$.*

**4.1.6.** A subspace $E_0 \subset E$ is called *massive*, or *coinitial*, or even *cofinal* whenever for every $x \in E$ there exist $\underline{x}$ and $\overline{x} \in E_0$ such that $\underline{x} \leq x \leq \overline{x}$. Suppose that $T_0 \in L(E_0, E)$ and write $\tau_0 := T_0\!\uparrow$. Obviously, the following take place:

(1) "$E_0$ is massive in $E$" $\leftrightarrow [\![E_0^{\wedge}$ is massive in $E^{\wedge}]\!] = \mathbf{1}$;

(2) "$T$ is an extension of $T_0$" $\leftrightarrow [\![\tau$ is an extension of $\tau_0]\!] = \mathbf{1}$.

The Kreĭn–Rutman theorem states that a positive functional defined on a massive subspace admits a positive extension to the whole space. The theorem remains valid if the word "positive" be replaced in it by "discrete." Putting these facts into $\mathbb{V}^{(B)}$ and using the statements (1), (2) and 4.1.3(3), we obtain the following results.

**4.1.7. Kantorovich's theorem.** *Let $F$ be an arbitrary $K$-space. If $E_0$ is a massive subspace of $E$, then every positive operator $T_0 : E_0 \to F$ admits a positive extension $T \in L^{\sim}(E,F)$.*

**4.1.8. Theorem.** *Under the conditions of* 4.1.6, *every $F$-discrete operator $T_0 : E_0 \to F$ admits an $F$-discrete extension $T : E \to F$. In particular, if $E_0$ is a massive sublattice, then, for a lattice homomorphism $T_0 : E_0 \to F$, there exists a lattice homomorphism extending $T_0$.*

**4.1.9. Theorem.** *For a positive operator $T : E \to F$, the following statements are equivalent:*

(1) *$T$ is $F$-diffuse;*

(2) *for all* $0 \le x \in E$, $0 \le \varepsilon \in F$, *and* $b \in B$ *such that* $b_\varepsilon \neq 0$, *there exists a nonzero projection* $\rho \le b$ *and mutually disjoint positive operators* $T_1, \dots, T_n$ *such that*

$$T = T_1 + \ldots + T_n, \quad |\rho T_k x| \le \varepsilon \quad (k := 1, \dots, n);$$

(3) *for all* $0 \le x \in E$, $0 \le \varepsilon \in F$, *and* $b \in B$ *such that* $b_\varepsilon \neq 0$, *there exists a countable partition of unity* $(b_n)$ *such that for every* $n \in \mathbb{N}$ *the operator* $T$ *can be decomposed into the sum of mutually disjoint positive operators* $T_{1,n}, \dots, T_{k_n,n}$ *satisfying the inequalities* $b_n|T_{k,n}x| \le \varepsilon$ $(k := 1, \dots, k_n)$.

*Proof.* The proof is obtained by interpreting inside $\mathbb{V}^{(B)}$ the following scalar fact: a positive functional $f$ is diffuse if for all $x \ge 0$ and $0 < \varepsilon \in \mathbb{R}$ there are positive mutually disjoint functionals $f_1, \dots, f_n$ such that $f = f_1 + \ldots + f_n$ and $|f_k(x)| < \varepsilon$ $(k := 1, \dots, n)$.

**4.1.10. Theorem.** *For every positive operator* $T : E \to F$, *the following representation holds:*

$$Tx = T_0 x + \sum_{\xi \in \Xi} T_\xi x \quad (x \in E)$$

*with* $T_0$ *being an* $F$*-diffuse operator and* $(T_\xi)_{\xi \in \Xi}$ *being a family of mutually disjoint lattice homomorphisms.*

The operator $T_0$ is uniquely determined and the family $(T_\xi)$ is unique up to "transposing" and "mixing." To prove the theorem, we need the fact that, by transfer principle in $\mathbb{V}^{(B)}$, every $K$-space ($E^{\wedge\sim}$ in our case) can be decomposed into the direct sum of the band of diffuse operators and the band generated by discrete elements; the latter being the concatenation of one-dimensional bands, i.e. bands generated by discrete elements. After this 4.1.3(3–5) are easy to be inferred.

**4.1.11. Remarks.**

(1) The material of this section can be viewed as an illustration to the following heuristic principle formulated by L. V. Kantorovich in the article [17], where he has introduced $K$-spaces: "Introduction of these spaces allows us to study linear operations of one general class (operations with values in such spaces) as linear functionals."

(2) The elementary theorem, 4.1.2, serves as the basic technical method raising the heuristic principle formulated to the level of investigating precisely (within the range of problems considered). Other variants are in [5, 12, 13, 27, 38].

(3) The equivalence (1)$\leftrightarrow$(2) in Theorem 4.1.4 was obtained by S. S. Kutateladze (see [26]) by standard methods. The scalar case ($F = \mathbb{R}$) is well known. As regards 4.1.5, see [43].

(4) A standard proof of Theorem 4.1.7 is presented in many monographs. The theorem is also valid if $E$ is an ordered vector space. Extending a positive operator with additional properties (discreteness or preserving the lattice operations as in 4.1.8) is a rather dominant theme. We will only note here that this topic is close to studying the extreme structure of special convex sets, see, e.g., [4, 5, 26].

## *Fragments of positive operators*

In this section we shall dwell on the problem of computing fragments of positive operators. This problem can be scrutinized rather deeply with the help of consistent usage of nonstandard methods. As in the previous section, $E$ denotes a vector lattice and $F$ stands for a $K$-space.

**4.2.1.** A set $\mathfrak{P}$ of projections in the $K$-space $L^{\sim}(E,F)$ is said to *generate fragments of an operator* $T$, $0 \le T \in L^{\sim}(E,F)$, if $Tx^+ = \sup\{PTx : P \in \mathfrak{P}\}$ for all $x \in E$. The last being valid for every $0 \le T \in L^{\sim}(E,F)$, the set $\mathfrak{P}$ is called *generating*. Write $F := \mathcal{R}{\downarrow}$ and let $P$ be a projection in $L^{\sim}(E,F)$. Then

(1) there exists a unique element $P{\uparrow} \in \mathbb{V}^{(B)}$ such that $[\![P{\uparrow}$ is a projection in $E^{\wedge\sim}]\!] = \mathbf{1}$ and $(PT){\uparrow} = P{\uparrow}\, T{\uparrow}$ for all $T \in L^{\sim}(E,F)$.

Now consider a set $\mathfrak{P}$ of projections in $L^{\sim}(E,F)$ and a positive operator $T \in L^{\sim}(E,F)$. Write $\tau := T{\uparrow}$ and $\mathfrak{P}{\uparrow} := \{P{\uparrow} : P \in \mathfrak{P}\}{\uparrow}$. Then $[\![\mathfrak{P}{\uparrow}$ is a set of projections in $E^{\wedge\sim}]\!] = \mathbf{1}$ and the following statements are true:

(2) "$\mathfrak{P}$ generates fragments of $T$" $\leftrightarrow$ $[\![\mathfrak{P}{\uparrow}$ generates fragments of$\tau]\!] = \mathbf{1}$;

(3) "$\mathfrak{P}$ is a generating set" $\leftrightarrow$ $[\![\mathfrak{P}{\uparrow}$ is a generating set$]\!] = \mathbf{1}$.

**4.2.2.** For a set $A$ in a $K$-space denote by $A^{\wedge}$ the result of adjoining to $A$ suprema of its every nonempty finite subsets. The symbol $A^{\uparrow}$ is used for the result of adjoining to $A$ suprema of nonempty increasing nets of elements of $A$. The symbols $A^{\uparrow\downarrow}$ and $A^{\uparrow\downarrow\uparrow}$ are naturally interpreted. The sign $\approx$ in a $K$-space $F$ has the ordinary meaning: $x \approx y$ for $x, y \in F$ symbolizes that $\forall^{\mathrm{St}} e \in \mathcal{E})|x-y| \le e$, where $\mathcal{E}$ is the filter of units in $F$. It is clear that if $F := \mathbb{R}$ then this stands for infinitesimality of the number $x - y$.

Our results on positive operators will be obtained by the same schema as in 4.1 with the help of Boolean-valued models. At first it is necessary to investigate the case of functionals. We shall use the notation $\mathfrak{P}(f) := \{Pf : P \in \mathfrak{P}\}$. Henceforth, in the subsections 4.2.3–4.2.5 $E$ is a vector lattice over a dense subfield of $\mathbb{R}$, $\mathfrak{P}$ is a set of projections in $E^{\sim}$.

**4.2.3. Theorem.** *The following statements are equivalent:*

**(1)** $\mathfrak{P}(f)^{\vee(\uparrow\downarrow\uparrow)} = \mathcal{E}(f)$*;*

**(2)** $\mathfrak{P}$ *generates fragments of* $f$*;*

**(3)** $(\forall x \in {}^{\circ}E)(\exists p \in \mathfrak{P})pf(x) \approx f(x^+)$*;*

**(4)** *a functional* $g$ *in* $[0,f]$ *is a fragment of* $f$ *if and only if for every* $0 \le x \in E$ *the following identity holds*

$$\inf_{p \in \mathfrak{P}} (pg(x) + p(f-g)(x)) = 0;$$

**(5)** $(\forall g \in {}^{\circ}\mathcal{E}(f))(\forall x \in {}^{\circ}E_+)(\exists p \in \mathfrak{P})|pf - g|(x) \approx 0$*;*

**(6)** $\inf\{|pf - g|(x) : p \in \mathfrak{P}\} = 0$ *for all fragments* $g \in \mathcal{E}(f)$ *and positive elements* $x \ge 0$*;*

**(7)** *for* $x \in E_+$ *and* $g \in \mathcal{E}(f)$ *there is an element* $p \in \mathfrak{P}(f)^{\vee(\uparrow\downarrow\uparrow)}$ *satisfying the identity* $|pf - g|(x) = 0$.

*Proof.* The implications (1)$\to$(2)$\to$(3) are obvious.

(3)$\to$(4). We shall work within standard entourage. Note first of all, that validity of the required identity for all functionals $g$ and $f$ such that $0 \le g \le f$ provides, for a standard $x \ge 0$, the existence of such $p \in \mathbb{P}$ that $p^{\perp}g(x) \approx 0$ and $p(f-g)(x) \approx 0$. Thus, ${}^{\circ}p(g \wedge (f-g))(x) \le {}^{\circ}p(f-g)(x) = 0$ and ${}^{\circ}p^{\perp}((f-g) \wedge g)(x) \le {}^{\circ}p^{\perp}g(x) = 0$, i.e. $g \wedge (f-g) = 0$.

Prove now that under conditions (3) the required identity is provided by the ordinary criterion of disjointness:

$$\inf\{g(x_1)+(f-g)(x_2) : x_1 \geq 0,\ x_2 \geq 0, x_1+x_2=x\}=0.$$

For a fixed standard $x$, find internal positive $x_1$ and $x_2$ such that $x=x_1+x_2$ and, moreover, $g(x_1)\approx 0$ and $f(x_2)\approx g(x_2)$. By the condition (3), it follows from the Kreĭn – Mil′man theorem that the fragment $g$ belongs to the weak closure of $\mathfrak{P}(f)$. In particular, there is an element $p\in\mathfrak{P}$ satisfying $g(x_1)=pf(x_1)$ and $g(x_2)\approx pf(x_2)$. Thus, $p^{\perp}g(x_2)\approx 0$ because $p^{\perp}g\leq p^{\perp}f$. Finally, $p^{\perp}g(x)\approx 0$. Hence,

$$p(f-g)(x)=pf(x_2)+pf(x_1)-pg(x)\approx g(x_2)+g(x_1)-pg(x)\approx p^{\perp}g(x)\approx 0.$$

This provides the needed identity.

(4)⇒(5). By the equality $|pf-g|(x)=p^{\perp}g(x)+p(f-g)(x)$ we can select $p\in\mathfrak{P}$ such that $p^{\perp}g(x)\approx 0$ and $p(f-g)(x)\approx 0$. This provides the required statement.

The equivalence (5)↔(6) is clear. The implications (5)⇒(7)⇒(1) can be proved with the help of methods presented in [43].

**4.2.4. Theorem.** *For positive functionals $f$ and $g$ and for a generating set of projections $\mathfrak{P}$, the following statements are equivalent:*

**(1)** $g\in\{f\}^{\perp\perp}$;

**(2)** *for every finite* $x\in {}^{\mathrm{fin}}E:=\{x\in E\colon(\exists\overline{x}\in\ {}^{\circ}E)|x|\leq\overline{x}\}$ *one has* $pg(x)\approx 0$ *whenever* $pf(x)\approx 0$ *for* $p\in\mathfrak{P}$;

**(3)** $(\forall x\in E_+)(\forall\varepsilon>0)(\exists\delta>0)(\forall p\in\mathfrak{P})pf(x)\leq\delta\to pg(x)\leq\varepsilon$.

**4.2.5. Theorem.** *Let $f$, $g$ be positive functionals on $E$ and let $x$ be a positive element of $E$. The following representations of the projection $\pi_f$ onto the band $\{f\}^{\perp\perp}$ are valid:*

(1) $\pi_f g(x)\ \risingdotseq\ \inf{}^*\{{}^{\circ}pg(x) : p^{\perp}f(x)\approx 0,\ p\in\mathfrak{P}\}$ *(the sign $\risingdotseq$ means that the formula is exact, i.e. the equality is accessible);*

(2) $\pi_f g(x)=\sup_{\varepsilon 0}\inf\{{}^{\circ}pg(x) : p^{\perp}f(x)\leq\varepsilon, p\in\mathfrak{P}\}$;

(3) $\pi_f g(x)\ \risingdotseq \inf{}^*\{{}^{\circ}g(y) : f(x-y)\approx 0,\ 0\leq y\leq x\}$;

(4) $(\forall\varepsilon>0)(\exists\delta>0)(\forall p\in\mathfrak{P})pf(x)<\delta\to\pi_f g(x)\leq p^{\perp}g(x)+\varepsilon$;
$(\forall\varepsilon>0)(\forall\delta>0)(\exists p\in\mathfrak{P})pf(x)<\delta\vee p^{\perp}g(x)\leq\pi_f g(x)+\varepsilon$;

(5) $(\forall\varepsilon>0)(\exists\delta>0)(\forall 0\leq y<x)f(x-y)\leq\delta\to\pi_f g(x)\leq g(y)+\varepsilon$;
$(\forall\varepsilon>0)(\forall\delta>0)(\exists 0\leq y\leq x)f(x-y)\leq\delta\vee g(y)\leq\pi_f g(x)+\varepsilon$.

Putting the statements 4.2.3–4.2.5 inside $\mathbb{V}^{(B)}$, while using 4.2.1, we infer the items 4.2.6–4.2.9 presented below.

**4.2.6.** For a set of projections $\mathfrak{P}$ in $L^{\sim}(E,F)$ and $0\leq S\in L^{\sim}(E,F)$, the following statements are equivalent:

(1) $\mathfrak{P}(S)^{\vee(\uparrow\downarrow\uparrow)}=\mathcal{E}(S)$;

(2) $\mathfrak{P}$ generates fragments of $S$;

(3) an operator $T\in[\![0,S]\!]$ is a fragment of $S$ if and only if for every $0\leq x\in E$ the identity holds:

$$\inf_{P\in\mathfrak{P}}(P^{-}Tx+P(S-T)x)=0;$$

(4) $(\forall x \in {}^{\circ}E)(\exists P \in \mathfrak{P}{\uparrow}{\downarrow})PSx \approx Sx^{+}$.

**4.2.7.** For positive operators $S$ and $T$ and a generating set $\mathfrak{P}$ of projections in $L^{\sim}(E,F)$, the following statements are equivalent:

(1) $T \in \{S\}^{\perp\perp}$;

(2) $(\forall x \in {}^{\mathrm{fin}}E)(\forall P \in \mathfrak{P})(\forall \pi \in B)\pi PSx \approx 0 \to \pi PTx \approx 0$;

(3) $(\forall x \in {}^{\mathrm{fin}}E)(\forall \pi \in B)\pi Sx \approx 0 \to \pi Tx \approx 0$;

(4) $(\forall x \geq 0)(\forall \varepsilon \in \mathcal{E})(\exists \delta \in \mathcal{E})(\forall P \in \mathfrak{P})(\forall \pi \in B)\pi PSx \leq \delta \to \pi PTx \leq \varepsilon$;

(5) $(\forall x \geq 0)(\forall \varepsilon \in \mathcal{E})(\exists \delta \in \mathcal{E})(\forall \pi \in B)\pi Sx \leq S \to \pi Tx \leq \varepsilon$.

**4.2.8. Theorem.** *Let $E$ be a vector lattice and let $F$ be a $K$-space with the filter of units $\mathcal{E}$ and the base $B$. Suppose that $S, T$ are positive operators in $L^{\sim}(E,F)$ and $R$ is the projection of $T$ onto the band $\{S\}^{\perp\perp}$. For a positive $x \in E$, the following representations are valid:*

*(1) $Rx = \sup_{\varepsilon\in\mathcal{E}} \inf\{\pi Ty + \pi^{\perp}Sx : 0 \leq y \leq x,\ \pi \in B,\ \pi S(x-y) \leq \varepsilon\}$;*

*(2) $Rx = \sup_{\varepsilon\in\mathcal{E}} \inf\{(\pi P)^{\perp}Tx : \pi PSx \leq \varepsilon,\ P \in \mathfrak{P}, \pi \in B\}$, where $\mathfrak{P}$ is a generating set of projections in $L^{\sim}(E,F)$.*

**4.2.9.** For an element $0 \leq e \in E$ define the operator $\pi_e S$ by the formulas:

$$(\pi_e S)x := \sup_{n\in\mathbb{N}} S(x \wedge ne) \quad (x \in E_{+}),$$

$$(\pi_e S)x := (\pi_e S)x^{+} - (\pi_e S)x^{-} \quad (x \in E).$$

It is easy to see that $\pi_e S \in L^{\sim}(E,F)$. Moreover, $\pi_e S$ is a fragment of the operator $S$ and the mapping $S \to \pi_e S$ $(S \geq 0)$ can be naturally extended onto $L^{\sim}(E,F)$ as an order projection. The set of projections $\mathfrak{P} := \{\pi_e : 0 \leq e \in E\}$ is generating. Hence, from 4.2.6 follows the formula

$$\mathcal{E}(S) = \{(\rho \circ \pi_e)S : \rho \in \mathfrak{P}(F), 0 \leq e \in E\}^{\wedge({\uparrow}{\downarrow}{\uparrow})}.$$

**4.** *.2.10. Remarks.*

(1) The formulas for projections in the form 4.2.8(1,2) have been constructed gradually. A survey of this history can be found with the help of [31, 43]. The general approach proposed in [36] is taken as basic to this section. It allows us to derive various formulas of projections via choosing concrete generating sets.

(2) A formula like 4.2.9(1) was established for the first time by B. de Pagter (see [31, 45]) with two essential restrictions: $F$ has a total set of $o$-continuous functionals and $E$ is order complete. The first restriction was removed in [31], the second – in [36]. All these cases correspond to different generating sets of projections.

(3) The main idea proposed in [36] is as follows. The fragments of a positive operator $T$ are the extreme points of the order interval $[0,T]$. The latter set coincides with the subdifferential at zero (the supporting set) $\partial P$ of the sublinear operator $Px = Tx_{+}$. Thus, studying fragments of a positive operator can be reduced to describing the extreme structure of subdifferentials. Such a description for general sublinear operators was obtained for the first time in the paper of S. S. Kutateladze (see the details in [26]). Note that the approach developed in [36] solves, in particular, the problem of an extreme extension of a positive operator (for the references on this subject, see [4, 26]).

**4.3. Order continuous operators.**

The methods exposed in the two previous sections can not be directly applied to order continuous operators; because, while ascending an operator (see 4.1.2), the property of order continuity is lost. We shall consider here another approach based on D. Maharam's ideas.

**4.3.1.** A positive operator $T : E \to F$ is said to satisfy *Maharam's condition*, if for every $0 \le x \in E$ one has $T[\![0, x]\!] = [\![0, Tx]\!]$, i.e. if for all $0 \le x \in E$ and $0 \le z \le Tx$ there exists a $0 \le y \in E$ such that $Ty = z$ and $0 \le y \le x$. A positive order continuous operator satisfying this condition is customarily called a *Maharam operator*.

Everywhere in this section $E$ and $F$ are $K$-spaces, and for simplicity we assume $F$ to be universally complete. By the symbol $E_T$ we denote the *carrier* of the operator $T$, i.e. the set $\{x \in E : T(|x|) = 0\}^{\perp}$. Write $F_T := (\mathrm{im} T)^{\perp\perp}$ and let $\mathbb{D}_m(T)$ stand for the greatest foundation in a universal completion of $E$, onto $\mathbb{D}_m(T)$ the operator $T$ being extendable with preserving order continuity. If $E_T = E$ and $T \ge 0$ then the operator $T$ is said to be *essentially positive.*

**4.3.2. Theorem.** *Let $E$ be a $K$-space, $F := \mathcal{R}{\downarrow}$, and let $T : E \to F$ be a Maharam operator such that $E = E_T = \mathbb{D}_m(T)$ and $F = F_T$. Then there exist $\mathcal{E} \in \mathbb{V}^{(B)}$ and $\tau \in \mathbb{V}^{(B)}$ satisfying the following conditions:*

(1) $\mathbb{V}^{(B)} \models$ *"$\mathcal{E}$ is a $K$-space and $\tau : \mathcal{E} \to \mathcal{R}$ is an essentially order continuous functional";*

(2) *$\mathcal{E}{\downarrow}$ is also a $K$-space, $\tau{\downarrow}: \mathcal{E}{\downarrow} \to \mathcal{R}{\downarrow}$ is a Maharam operator, furthermore, $\mathcal{E}{\downarrow} = \mathbb{D}_m(\tau{\downarrow})$;*

(3) *there exists a linear and lattice isomorphism $h$ from $\mathcal{E}$ onto $\mathcal{E}{\downarrow}$ such that $T = \tau{\downarrow} \circ h$.*

**4.3.3.** For a Maharam operator the decomposition 4.1.10 can be elaborated. Let $e$ be an order unit in $E$. Then $[\![e \text{ is an order unit in } \mathcal{E}]\!] = \mathbf{1}$. The functional $\tau$ is representable as $\tau = \tau_0 - \sum_{k=1}^{\infty} \tau_k$ where $\tau_0$ is a diffuse functional and $\tau_k$ are order continuous lattice homomorphisms. All these functionals are uniquely defined by measures on the base of unit elements. Furthermore, an atomless measure corresponds to $\tau_0$, and two-valued measures correspond to $\tau_k$. Interpreting this situation in $\mathbb{V}^{(B)}$, we obtain the result presented below. Recall that Maharam's condition for a positive vector measure $\mu : \mathcal{E}(e) \to F$ has precisely the same meaning as in 4.2.1, i.e. $\mu[\![0, a]\!] = [\![0, \mu(a)]\!]$ $(a \in \mathcal{E}(e))$. If $\mu$ is an isomorphism of Boolean algebras then $\mu^*$ stands for the resulting isomorphism of the corresponding universally complete $K$-spaces.

**4.3.4. Theorem.** *Let $E$ be a $K$-space with the unity $e$ and let $T : E \to F$ be an essentially positive Maharam operator. Then there exist sequences $(e_k)_{k=0}^{\infty}$, $(c_k)_{k=1}^{\infty}$, $(\mu_k)_{k=0}^{\infty}$, $(\alpha_k)_{k=1}^{\infty}$ such that*

(1) *$(e_k)$ is a partition of unity of the Boolean algebra $\mathcal{E}(e)$ and $(c_k)$ is a sequence of fragments of the element $c = Te$;*

(2) *$\mu : \mathcal{E}(e_0) \to F$ is a strictly positive order continuous measure satisfying Maharam's condition;*

(3) *$\mu_k : \mathcal{E}(e_k) \to \mathcal{E}(c_k)$ is a Boolean isomorphism, $\alpha_k$ is a positive invertible orthomorphism in $\{c_k\}^{\perp\perp}$;*

(4) *the representation*

$$Tx = \int_{-\infty}^{\infty} \lambda \cdot d\mu_0(e_\lambda^{x_0}) + \sum_{k=1}^{\infty} \alpha_k \mu_k^*(x_k)$$

*holds, with $x_k$ being the projection of the element $x$ onto the band $\{e_k\}^{\perp\perp}$.*

For Maharam operators the dual analogs of 4.1.4 and 4.1.5 are valid.

**4.3.5. Theorem.** *Let $T : E \to F$ be a positive order continuous operator. The following statements are equivalent:*

(1) *$T$ satisfies Maharam's condition;*

(2) *for every operator $0 \le S \le T$ there exists an orthomorphism $\alpha : E \to E$, $0 \le \alpha \le I_E$ such that $Sx = S\alpha x (x \in E)$;*

(3) *if $Tx = f_1 + f_2$ for some $0 \le x \in E$ and $0 \le f_1, f_2 \in F$, and if $f_1 \perp f_2$, then there are $0 \le x_1, x_2 \in E$ such that $x = x_1 + x_2$, $x_1 \perp x_2$, and $Tx_k = f_k (k = 1, 2)$.*

*Proof.* Without loss of generality we may assume $T$ to be essentially positive. If (1) is true then $T = \tau{\downarrow} \circ h$ (see 4.3.2). Since $\tau$ is $\mathcal{R}$-linear, then $T$ is $\mathcal{R}{\downarrow}$-linear. If $0 \le S \le T$ then $S$ is also $\mathcal{R}{\downarrow}$-linear and, hence, a Maharam operator. By 4.3.2, $S = \sigma{\downarrow} \circ h$ where $[\![\sigma \in \mathcal{E}^\sim]\!] = [\![0 \le \sigma \le \tau]\!] = \mathbf{1}$. The statement (2) for the functionals $\tau$ and $\sigma$ follows from the Radon–Nikodým theorem. Taking descents, we obtain (2) for the operators $T$ and $S$. The rest implications are plain.

**4.3.6.** *Let $S : E \to F$ be a regular operator such that $T := |S|$ is a Maharam operator. Then there exists a projection $\pi \in \mathbb{R}(F)$ such that $S^+ = S\pi$ and $S^- = S\pi^\perp$.*

*Proof.* Again we may assume that $T := \tau{\downarrow}$, where $\tau$ is an essentially positive $o$-continuous functional inside $\mathbb{V}^{(B)}$. As in 3.4.5 we obtain the fact that there exists a regular functional $\sigma \in \mathcal{E}$ such that $\tau = |\sigma|$. If $p$ is a projection in $\mathcal{E}$ onto the carrier (= the band of essential positivity) of $\sigma^+$. Order continuous functionals are disjoint if and only if their carriers are disjoint. Hence, $\sigma^+ = \sigma p$ and $\sigma^- = \sigma p^\perp$. Writing $\pi := p{\downarrow}$ and taking descents, we complete the proof.

**4.3.7.** Thus, the general properties of Maharam operators can be deduced from the corresponding facts about functionals with the help of Theorem 4.3.2. Nevertheless, the methods described may be also useful in studying arbitrary regular operators.

Fix a positive operator $\Phi$ which acts from a vector lattice $X$ into $F$. By Theorem 4.1.2, there exists a positive $\mathbb{R}^\wedge$-linear functional $\varphi : X^\wedge \to \mathcal{R}$ such that the identity $[\![\Phi(x) = \varphi(x^\wedge)]\!] = \mathbf{1}$ holds for all elements $x \in X$. Equip $X^\wedge$ with the seminorm $\rho(x) := \varphi(|x|)$. Let $\mathbb{X}$ be the completion of the quotient lattice $X^\wedge / \rho^{-1}(0)$ with respect to the quotient norm. Then $\mathbb{X}$ is a Banach lattice and there exists a unique positive ($\mathcal{R}$-linear) functional $\overline{\varphi} : \mathbb{X} \to \mathcal{R}$ such that $\varphi = \overline{\varphi} \circ \iota$, where $\iota : X^\wedge \to \mathbb{X}$ is the quotient homomorphism. Moreover, $\overline{\varphi}$ is order continuous and essentially positive. Now, working with descents and ascents, one can obtain the following result.

**4.3.8. Theorem.** *There exists a $K$-space $\overline{X}$ and an essentially positive Maharam operator $\overline{\Phi} : \overline{X} \to F$ satisfying the following conditions:*

(1) *there are lattice homomorphisms* $i : X \to \overline{X}$ *and* $j : Z(X) \to Z(\overline{X})$ *(with* $Z(X)$ *being the ideal of* $X$ *generated by the identity operator) such that* $j$ *is also a ring homomorphism and* $\alpha\Phi x = \overline{\Phi}(j(\alpha)i(x))$ *for all elements* $x \in X$ *and* $\alpha \in Z(F)$*; in particular,* $\Phi(x) = \overline{\Phi}(i(x))$*;*

(2) $i(X)$ *is a massive sublattice in* $\overline{X}$ *and* $j(Z(F))$ *is an o-closed sublattice and subring of* $Z(\overline{X})$*;*

(3) $\overline{X} = b(X \otimes Z(F))^{\downarrow\uparrow}$*, with* $b : X \otimes Z(F) \to \overline{X}$ *being the lattice operator defined by the relation* $b(x \otimes \alpha) := j(\alpha)i(x)(x \in X, \alpha \in Z(F))$*.*

The pair $(\overline{X}, \overline{\Phi})$ is defined uniquely up to an isomorphism. Moreover, if an essentially positive Maharam operator $\overline{\Phi}_1 : \overline{X}_1 \to F$ and a lattice homomorphism $i_1 : \overline{X}_1 \to F$ satisfy the condition $\Phi = \overline{\Phi}_1 \circ i_1$, then there exists an isomorphism $h$ from $\overline{X}$ onto an $o$-closed sublattice in $\overline{X}$ such that $\overline{\Phi} = \overline{\Phi}_1 \circ h$ and $h \circ i = i_1$.

Denote by $m\overline{X}$ a universal completion of a $K$-space $\overline{X}$. Fix, in $m\overline{X}$, an order unity, thus an $f$-algebra structure is uniquely defined. Let $L_1(\Phi)$ be the greatest foundation in $m\overline{X}$, onto which $\overline{\Phi}$ can be extended with preserving $o$-continuity. The following result is a variant of the Radon–Nikodým theorem for positive operators.

**4.3.9. Theorem.** *For every operator* $T \in \{\Phi\}^{\perp\perp}$ *there exists a unique element* $z \in m\overline{X}$ *such that*

$$Tx = \overline{\Phi}(z \cdot i(x)) \quad (x \in X).$$

The correspondence $T \to z$ establishes a linear and lattice isomorphism between the band $\{\Phi\}^{\perp\perp}$ and a foundation in $m\overline{X}$ defined by the formula $\{z \in m\overline{X} : z \cdot i(X) \subset L_1(\Phi)\}$.

**4.3.10. Remarks.**

(1) In a long sequence of papers published in the 1950, D. Maharam proposed an original approach to studying positive operators. The concept of a Maharam operator and the idea of extending a positive operator to a Maharam operator (see 4.1.8) ascend to those papers. It is worth to note that within Boolean-valued analysis D. Maharam's approach is notable for clarity of the idea and for certain simplicity, because a considerable part of the theory can be reduced to manipulating numerical measure and integral in a suitable Boolean-valued model.

(2) Several results of D. Maharam were transfered to vector lattices by W. A. J. Luxemburg and A. R. Schep (see [26]). Theorem 4.3.2 is due to A. G. Kusraev.

(3) The equivalence (1)$\leftrightarrow$(2) in 4.3.5 is a restricted version of the Radon–Nikodým theorem for a Maharam operator. The complete form of this theorem was proved in [24] by standard methods, and in [5] — with the help of 4.3.2. The statement 4.3.6 is an operator variant of the Hahn theorem on the decomposition of measure. For an operator acting between spaces of measurable functions, D. Maharam established Theorem 4.3.4 by her original method.

(4) The problem of extending a positive operator to a Maharam operator was thoroughly studied (see [5, 24, 26]). In these papers the details of 4.3.8 and 4.3.9 can also be found. The structure of such extension can be rather complicated, but sometimes admits a functional description.

## KÏŸЖϘΘΓ́ŸYΘΓ́

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