THE PATH AND SPACE OF KANTOROVICH

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Abstract. This talk overviews the life and mathematical legacy of L. V. Kantorovich (1912–1986). The Appendix contains details on intraction between K-space and the real axis.

Informatics, computational mathematics, and mathematical economics are in an obvious sense the successors and representatives of applied mathematics. Life testifies that, all their glaring defects notwithstanding, the terms "pure" and "applied" mathematics, provoking endless and vigorous discussions and controversies, not only persist in scientific usage but also incarnate distinct scientific phenomena. Scientometricians and ordinary mathematicians, pondering over this matter, usually state that the hallmark of the mathematics of this country as opposed to, say, the American mathematics, is a unifying tendency with intention when possible to distinguish and emphasize common features as well as developing a single mathematical culture and relevant infrastructure. As for the Russian specialists, any instance of division and contradistinction of pure and applied mathematics usually involves collisions, emotions, or at least discomfort. At the same time the separate existence of, say, the American Mathematical Society (AMS) and the Society for Industrial and Applied Mathematics (SIAM) is perfectly natural for American scholars.

It is rarely taken into account that dissemination of such trends in social life is linked with the stances and activities of particular individuals.

Leonid Vital′evich Kantorovich (1912–1986) belongs to those Russian classics of mathematics who maintained the trend to unity of pure and applied mathematics by an outstanding personal contribution.

I am pleased to recall some facts of the life of Kantorovich so as to present at least a rough scale of events and results.

Key words and phrases. Kantorovich, vector lattice, successive approximations, linear programming, Boolean valued analysis, Escher rules.

This talk is prepared for the meeting "Kantorovich Memorial. Mathematical Economics: Old Problems and New Approaches" to be held on January 8–13, 2004 in the Euler International Mathematical Institute at St. Petersburg.

Kantorovich was born in a family of a doctor at St. Petersburg on January 19, 1912 (January 6, according to the old Russian style). The boy's talent revealed itself very early. In 1926, just at the age of 14, he entered St. Petersburg (then Leningrad) State University (SPSU). Soon he started participating in a circle of G. M. Fikhtengolts for students and in a seminar on descriptive function theory. It is natural that the early student years formed his first environment: D. K. Faddeev, I. P. Natanson, S .L. Sobolev, S. G. Mikhlin, and a few others with whom Leonid Vital′evich was friendly during all his life also participated in Fikhtengolts's circle.

After graduation from SPSU in 1930, Kantorovich started teaching, combining it with intensive scientific research. Already in 1932 he became a professor at the Leningrad Institute of Civil Engineering and an assistant professor at SPSU.

In 1934 Kantorovich became a professor at his alma mater. His close connection with SPSU and the Leningrad Division of the Steklov Mathematical Institute of the Academy of Sciences lasted until his transition to Novosibirsk on the staff of the Institute of mathematics of the Siberian Division of the Academy of Sciences of the USSR (now, the Sobolev Institute) at the end of the 1950s.

Kantorovich had written practically all of his major mathematical works in his "Leningrad" period. Moreover, in the 1930s he published more articles on pure mathematics whereas the 1940s became his time of computational mathematics in which he was soon acknowledged as an established and acclaimed leader.

At the end of the 1930s Kantorovich revealed his outstanding gift of an economist. His booklet *Mathematical Methods of Production Planning and Organization* is a material evidence of the birth of linear programming. The economic works of Kantorovich were hardly visible at the surface of the scientific information flow in the 1940s. However, the problems of economics prevailed in his creative studies. During the Second World War he completed the first version of his book *The Best Use of Economic Resources* which led to the Nobel Prize awarded to him and Tjalling C. Koopmans in 1975.

In 1957 Kantorovich was invited to join the newly founded Siberian Division of the Academy of Sciences of the USSR. He was elected to the Division of Economics in the first elections of members of the Siberian Division. Since then his major publications were devoted to economics, with the exception of the celebrated course of functional analysis[1]—"Kantorovich and Akilov" in the students' jargon.

The 1960s became the age of his recognition. In 1964 he was elected a full member of the Department of Mathematics of the Academy of Sciences of the USSR, and in 1965 he was awarded the Lenin Prize. In these years he vigorously propounded and maintained his views of interplay between mathematics and economics and exerted great efforts to instil the ideas and methods of modern science into the top economic managers of the Soviet Union.

As the beginning of the 1970s Kantorovich left Novosibirsk for Moscow where he was still engaged in economic analysis, not seizing his efforts to influence the actual economic practice and decision making in the national economy. These years witnessed a considerable mathematical Renaissance of Kantorovich. Although he never resumed proving

[1]Kantorovich L. V. and Akilov G. P. *Functional Analysis.* Oxford etc., Pergamon Press, 1982.

theorems, his impact on the mathematical life of this country increased sharply due to the sweeping changes in the Moscow academic life on the eve of Gorbi's "perestroĭka." Cancer terminated his path in science on April 7, 1986.

Kantorovich started his scientific research in rather abstract and sophisticated sections of mathematics such as descriptive set theory, approximation theory and functional analysis. It should be stressed that at the beginning of the 1930s these areas were most topical, prestigious, and difficult. Kantorovich's fundamental contribution to theoretic mathematics, now indisputable and universally acknowledged, consists in his pioneering works in the above-mentioned areas. Note also that in the "mathematical" years of his career he was primarily famous for his research into approximate methods of analysis, the ancient euphemism for computational mathematics of today.

The first works of Kantorovich on computational mathematics were published in 1933. He suggested some approaches to approximate solution of the problem of finding a conformal mapping between domains. These methods used the idea of embedding the original domains into some one parameter family of domains. Expanding in a parameter, Kantorovich found out new explicit formulas for approximate calculation of conformal mappings between multiply-connected domains.

In 1933 one of Kantorovich's teachers, V. I. Smirnov included these methods in his multi-volume treatise *A Course of Higher Mathematics* which belongs now to the world-class desk-books.

Kantorovich paid much attention to direct variational methods. He suggested an original method for approximate solution of second order elliptic equations which was based on reduction of the initial problem to minimization of a functional over some functions of one variable. This technique is now called reduction to ordinary differential equations.

The variational method was developed in his subsequent works under the influence of other questions. For instance, his collocation method was suggested in an article about calculations for a beam on an elastic surface.

A few promising ideas were proposed by Kantorovich in the theory of mechanical quadratures which formed a basis for some numerical methods of solution of a general integral equation with a singularity.

This period of his research into applied mathematics was crowned with a joint book with V. I. Krylov *Methods for Approximate Solution of Partial Differential Equations* whose further expanded editions appeared under the title *Approximate Methods of Higher Analysis.*

Functional analysis occupies a specific place in the mathematical studies of Kantorovich. He was ranked among the classics of the theoretic sections of this area of research as one of the founders of ordered vector spaces. Also, he contributed much to making functional analysis a natural language of computational mathematics. His article "Functional analysis and applied mathematics" (*Russian Mathematical Surveys*, 1948) made a record in the personal file of Kantorovich as well as in the history of mathematics in this country. Kantorovich wrote in the introduction to this article[2]:

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[2]Kantorovich L. V. *Selected Works. Part II.* Gordon and Breach, 1996, pp. 171–280.

> *...there is now a tradition of viewing functional analysis as a purely theoretic discipline far removed from direct applications, a discipline which cannot deal with practical questions. This article is an attempt to break with this tradition, at least to a certain extend, and to reveal the relationship between functional analysis and questions of applied mathematics, an attempt to show that it can be useful to mathematicians dealing with practical applications.*
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> *Namely, we would try to show that the ideas and methods of functional analysis can be readily used to construct and analyze effective practical algorithms for solving mathematical problems with the same success as they were used for the theoretic studies of the problems.*

The mathematical ideas of the article remain classical for a long time: The method of finite-dimensional approximations, estimation of the inverse operator, and, last but not least, the Newton–Kantorovich method are well known to the majority of the persons recently educated in mathematics.

The general theory by Kantorovich for analysis and solution of functional equations bases on variation of "data" (operators and spaces) and provides not only estimates for the rate of convergence but also proofs of the very fact of convergence.

As an instance of incarnation of the idea of unity of functional analysis and computational mathematics Kantorovich suggested at the end of the 1940s that the Mechanics and Mathematics Department of SPSU began to prepare specialists in the area of computational mathematics for the first time in this country. He prolonged this line in Novosibirsk State University where he founded the chair of functional analysis which delivered graduate courses in functional analysis in the years when Kantorovich hold the chair. I had a privilege of specialization in functional analysis which was offered by the chair of computational mathematics in that span of time.

It should be emphasized that Kantorovich tied the progress of linear programming as an area of applied mathematics with the general demand for improving the functional-analytical techniques pertinent to optimization: the theory of topological vector spaces, convex analysis, the theory of extremal problems, etc. Several major sections of functional analysis (in particular, nonlinear functional analysis) underwent drastic changes under the impetus of new applications.

The scientific legacy of Kantorovich is immense. His research in the areas of functional analysis, computational mathematics, optimization, and descriptive set theory has had a dramatic impact on the foundation and progress of these disciplines. Kantorovich is rightfully ranked among the founders of the modern economic-mathematical methods. Linear programming, his most popular discovery, has changed the image of economics.

Kantorovich authored more than 300 articles. When we discussed with him the first edition of an annotated bibliography of his publications at the early 1980s, he suggested to combine them in the nine sections:

(1) descriptive function theory and set theory;
(2) constructive function theory;
(3) approximate methods of analysis;
(4) functional analysis;
(5) functional analysis and applied mathematics;
(6) linear programming;

(7) hardware and software;
(8) optimal planning and optimal prices;
(9) economic problems of planned economy.

The impressive versatility of these areas of research is united by not only the personality of Kantorovich but also the unity of his methodological views. He always emphasized the innate integrity of his scientific research as well as mutual penetration and syntheses of the methods and techniques he used in solving the most diverse theoretic and applied problems of mathematics and economics. I leave a thorough analysis of the methodology of Kantorovich's contribution a challenge to professional scientometricians. It deserves mentioning right away only that the abstract ideas of Kantorovich in the theory of Dedekind complete vector lattices, now called *Kantorovich spaces* or K-*spaces*, turn out to be closely connected with the techniques of linear programming and the approximate methods of analysis.

Kantorovich told me in the fall of 1983 that his main mathematical achievement is the development of K-space theory within functional analysis while remarking that his most useful deed is linear programming. K-space, a beautiful pearl of his scientific legacy, deserves a special discussion.

Let us look back at the origin of K-space. The first work of Kantorovich in the area of ordered vector appeared in 1935 as a short note in "Doklady."[3] Therein he treated the members of a K-space as generalized numbers and propounded the *heuristic transfer principle*. He wrote:

> *In this note, I define a new type of space that I call a semiordered linear space. The introduction of such a space allows us to study linear operations of one abstract class (those with values in these spaces) in the same way as linear functionals.*

It is worth noting that his definition of a semi-ordered linear space contains the axiom of Dedekind completeness which was denoted by I_6. Therefore, Kantorovich selected the class of K-spaces, now named after him, in his first article on ordered vector spaces. He tied the study of K-spaces with the scope of the fundamental Hahn–Banach Theorem and stated Theorem 3 which is now known as the Hahn–Banach–Kantorovich Theorem. This theorem claims in fact that the heuristic transfer principle is applicable to the classical Dominated Extension Theorem; i.e., one may replace reals in the Hahn–Banach Theorem with elements of an arbitrary K-space and replace linear functionals with operators acting into this space.

The diversity of Kantorovich's contributions combines with methodological integrity. It is no wonder so that Kantorovich tried to apply semi-ordered spaces to numerical methods in his earliest papers. In a note[4] of 1936 he described the background for his approach as follows:

> *The method of successive approximations is often applied to proving existence of solutions to various classes of functional equations; moreover, the proof of convergence of these approximations leans on the fact the equation under study may be majorized by another equation of a simple kind.*

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[3]Kantorovich L. V. *Selected Works. Part 1.* Gordon and Breach, 1996, pp. 213–216.

[4]Kantorovich L. V. On one class of functional equations. *Dokl. Akad. Nauk SSSR*, **4**:5 (1936), pp. 211–216.

> *Similar proofs may be encountered in the theory of infinitely many simultaneous linear equations and in the theory of integral and differential equations. Consideration of semi-ordered spaces and operations between them enables us to easily develop a complete theory of such functional equations in abstract form.*

There is no denying that the classical method of majorants which stems from the works of A. Cauchy acquires its natural and final form with K-space theory.

Inspired by some applied problems, Kantorovich propounded the idea of a *lattice-normed space* or *B_K-space* and introduce a special decomposability axiom for the lattice norm of a B_K-space. This axiom looked bizarre and was often omitted in the subsequent publications of other authors as definitely immaterial. The principal importance of this axiom was revealed only within Boolean valued analysis in the 1980s. As typical of Kantorovich, the motivation of B_K-space, now called *Banach–Kantorovich space*, was deeply rooted in abstractions as well as in applications. The general domination method of Kantorovich was essentially developed by himself and his students and followers and occupies a noble place in the theoretic toolkit of computational mathematics.

The above-mentioned informal principle was corroborated many times in the works of Kantorovich and his students and followers. Attempts at formalizing the heuristic ideas by Kantorovich started at the initial stages of K-space theory, resulting in the so-called identity preservation theorems. They assert that if a proposition with finitely many function variables holds for the reals then a similar fact is valid for the members of an arbitrary K-space.

Unfortunately, no satisfactory explanation was suggested for the internal mechanism behind the phenomenon of identity preservation. Rather obscure remained the limits on the heuristic transfer principle. The same applies to the general reasons for similarity and parallelism between the reals and their analogs in K-space which reveal themselves every now and then. The omnipotence and omnipresence of Kantorovich's transfer principle found its full explanation only within Boolean valued analysis in the 1970s.

Boolean valued analysis (the term was coined by G. Takeuti) is a branch of functional analysis which uses a special model-theoretic technique of the so-called Boolean valued models of set theory. Since recently this term has been treated in a broader sense implying the tools that rest on simultaneous use of two distinct Boolean valued models.

Note that the invention of Boolean valued analysis was not connected with the theory of vector lattices. The necessary language and technique had already been available within mathematical logic by 1960. Nevertheless, the main idea was still absent for rapid progress in model theory and its applications. This idea emerged when P. J. Cohen demonstrated in 1960 that the classical continuum hypothesis is undecidable in a rigorous mathematical sense. It was the Cohen method of forcing whose comprehension led to the invention of Boolean valued models of set theory which is attributed to the efforts by D. Scott, R. Solovay, and P. Vopěnka.

The Boolean valued status of the notion of Kantorovich space was first demonstrated by Gordon's Theorem[5] dated from the mid 1970s. This fact can be reformulated as

[5]Gordon E. I. "Real numbers in Boolean-valued models of set theory and K-spaces," *Dokl. Akad. Nauk SSSR*, **237**:4 (1977), pp. 773–775.

follows: *A universally complete K-space serves as interpretation of the reals in a suitable Boolean valued model.* Furthermore, every theorem (within Zermelo–Fraenkel set theory ZFC) about real numbers has a full analog for the corresponding K-space. Translation of one theorem into the other is fulfilled by some precisely-defined procedures: ascent, descent, canonical embedding, etc., i.e., by algorithm, as a matter of fact. Thus, Kantorovich's motto: "The elements of a K-space are generalized numbers" acquires a rigorous mathematical formulation within Boolean valued analysis. On the other hand, the heuristic transfer principle which played an auxiliary role in many studies of the pre-Boolean-valued K-space theory becomes a powerful and precise method of research in Boolean valued analysis.

Further progress of Boolean valued analysis revealed that this translation (transfer or interpretation) making new theorems from available facts is possible not only for K-spaces but also for practically all objects related to them such as B_K-spaces, various classes of linear and nonlinear operators, operator algebras, etc. The heuristic transfer principle for B_K-spaces was realized by A. G. Kusraev[6] to within elementary stipulations as follows: *Each Banach–Kantorovich space embeds in a Boolean valued model, becoming a Banach space.* In other words, a B_K-space is a Boolean valued interpretation of a Banach space. Moreover, it is the "bizarre" decomposability axiom of Kantorovich that guarantees the possibility of this embedding.

Returning back to the background ideas of K-space theory in his last mathematical paper,[7] he had finished just before his death, Kantorovich wrote:

> *One aspect of reality was temporarily omitted in the development of the theory of function spaces. Of great importance is the relation of comparison between practical objects, alongside algebraic and other relations between them. Simple comparison applicable to every pair of objects is of a depleted character; for instance, we may order all items by weight which is of little avail. That type of ordering is more natural which is defined or distinguished when this is reasonable and which is left indefinite otherwise (partial ordering or semi-order). For instance, two sets of goods must undoubtedly be considered as comparable and one greater than the other if each item of the former set is quantitatively greater than its counterpart in the latter. If some part of the goods of one set is greater than and another part is less than the corresponding part of the other then we can avoid prescribing any order between these sets. It is with this in mind that the theory of ordered vector spaces was propounded and, in particular, the theory of the above-defined K-spaces. It found various applications not only in theoretic problems of analysis but also in construction of some applied methods, for instance the theory of majorants in connection with the study of successive approximations. At the same time the opportunities it offers are not revealed fully yet. The importance is underestimated of this branch of functional analysis for economics. However, the comparison and correspondence relations play an extraordinary role in economics and it was definitely clear even at the cradle of K-spaces that they will find their place in economic analysis and will yield luscious fruits.*
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> *The theory of K-spaces has another important feature: their elements can be treated as numbers. In particular, we may use elements of such a space, finite- or infinite-dimensional, as a norm in*

[6]Kusraev A. G. "On Banach–Kantorovich spaces," *Sibirsk. Mat. Zh.*, **26**:2 (1985), pp. 119–126.

[7]Kantorovich L. V. Functional analysis (the main ideas). *Siberian Math. J.*, **28**:1 (1987), pp. 7–16.

construction of analogs of Banach spaces. This choice of norms for objects is much more accurate. Say, a function is normed not by its maximum on the whole interval but a dozen of numbers, its maxima on parts of this interval.

Observe that this excerpt of the Kantorovich article draws attention to the close connection of K-spaces with the theory of inequalities and economic topics. It is also worth noting that the ideas of linear programming are immanent to K-space theory in the following rigorous sense: The validity of each of the universally accepted formulation of the duality principle with prices in some algebraic structure necessitates that this structure is a K-space.

Magically prophetic happens to be the claim of Kantorovich that the elements of a K-space are generalized numbers. The heuristic transfer principle of Kantorovich found a brilliant justification in the framework of modern mathematical logic. The Kantorovich spaces providing many new equivalent but unpredictable models of the real axis will stay for ever in the treasure-trove of the world science.

It was explicitly stated in Alfred Marshall's *Principles of Economics* that this science leaves no room for long chains of deduction. At the same time, there is no gainsay in ascribing the beauty and power of mathematics to the axiomatic method that, in its most cherished although unattainable form, consists in deriving the sought truths via arbitrarily lengthy sequences of formal implications. This conspicuous discrepancy between economists and mathematicians in mentality hinders their mutual understanding and cooperation. The resultant partitions in ratiocination, invisible but ubiquitous, isolate the economic community from its mathematical counterpart and vice versa. This situation rooted deeply in history was always a challenge to Kantorovich, contradicting his views on interaction between mathematics and economics. Most of the precious gems of his legacy convey the message: "Mathematicians and Economists of the World, Unite!"

The years of the Kantorovich life dim in the past. Yet the scales of his personality and unifying ideas become clearer and brighter, showing us new roads to the future.

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APPENDIX: THE REAL NATURE OF K-SPACE

This Appendix briefs the reader on the representation of a K-space in the framework of Boolean valued analysis. [1]

1. Boolean valued universes

The universe of discourse of Boolean valued analysis is a Boolean valued model of ZFC. To sketch its structure, we start with a complete Boolean algebra B. Given an ordinal α, put

$$\mathbb{V}_\alpha^{(B)} := \{x \mid x \text{ is a function} \wedge (\exists\beta)(\beta < \alpha \wedge \operatorname{dom}(x) \subset \mathbb{V}_\beta^{(B)} \wedge \operatorname{im}(x) \subset B)\}.$$

Thus, in more detail we have

$$\mathbb{V}_0^{(B)} := \varnothing,$$

$$\mathbb{V}_{\alpha+1}^{(B)} := \{x \mid x \text{ is a function with domain in } \mathbb{V}_\alpha^{(B)} \text{ and range in } B\};$$

$$\mathbb{V}_\alpha^{(B)} := \bigcup_{\beta<\alpha} \mathbb{V}_\beta^{(B)} \quad (\alpha \text{ is a limit ordinal}).$$

The class

$$\mathbb{V}^{(B)} := \bigcup_{\alpha\in\mathrm{On}} \mathbb{V}_\alpha^{(B)}$$

is a *Boolean valued universe*. An element of the class $\mathbb{V}^{(B)}$ is a *B-valued set*. It is necessary to observe that $\mathbb{V}^{(B)}$ consists only of functions. In particular, $\varnothing$ is the function with domain $\varnothing$ and range $\varnothing$. Hence, the "lower" levels of $\mathbb{V}^{(B)}$ are organized as follows:

$$\mathbb{V}_0^{(B)} = \varnothing, \quad \mathbb{V}_1^{(B)} = \{\varnothing\}, \quad \mathbb{V}_2^{(B)} = \{\varnothing, (\{\varnothing\}, b) \mid b \in B\}.$$

It is worth stressing that $\alpha \le \beta \to \mathbb{V}_\alpha^{(B)} \subset \mathbb{V}_\beta^{(B)}$ for all ordinals α and β. Moreover, the following *induction principle* is valid in $\mathbb{V}^{(B)}$:

$$(\forall x \in \mathbb{V}^{(B)})\,((\forall y \in \operatorname{dom}(x))\ \varphi(y) \to \varphi(x)) \to (\forall x \in \mathbb{V}^{(B)})\,\varphi(x),$$

where φ is a formula of ZFC.

Take an arbitrary formula $\varphi = \varphi(u_1, \dots, u_n)$ of ZFC. If we replace the elements $u_1, \dots, u_n$ by elements $x_1, \dots, x_n \in \mathbb{V}^{(B)}$ then we obtain some statement about the objects $x_1, \dots, x_n$. It is to this statement that we intend to assign some *truth-value*. Such a value $[\![\psi]\!]$ must be an element of the algebra B. Moreover, it is naturally desired that the theorems of ZFC be true, i.e., attain the greatest truth-value, unity.

We must obviously define the truth-value of a well-formed formula by double induction, on considering the way in which this formula is built up from the atomic formulas $x \in y$ and $x = y$, while assigning truth-values to the latter when x and y range over $\mathbb{V}^{(B)}$ on using the recursive definition of this universe.

[1] Extra details may be found in the book Kusraev A. G. and Kutateladze S. S., *Boolean Valued Analysis*. Kluwer Academic Publishers, Dordrecht (1996).

It is clear that if φ and ψ are evaluated formulas of ZFC and $[\![\varphi]\!] \in B$ and $[\![\psi]\!] \in B$ are their truth-values then we should put

$$[\![\varphi \wedge \psi]\!] := [\![\varphi]\!] \wedge [\![\psi]\!],$$
$$[\![\varphi \vee \psi]\!] := [\![\varphi]\!] \vee [\![\psi]\!],$$
$$[\![\varphi \to \psi]\!] := [\![\varphi]\!] \to [\![\psi]\!],$$
$$[\![\neg\varphi]\!] := \neg[\![\varphi]\!],$$
$$[\![(\forall x)\, \varphi(x)]\!] := \bigwedge_{x\in\mathbb{V}^{(B)}} [\![\varphi(x)]\!],$$
$$[\![(\exists x)\, \varphi(x)]\!] := \bigvee_{x\in\mathbb{V}^{(B)}} [\![\varphi(x)]\!],$$

where the right-hand sides involve the Boolean operations corresponding to the logical connectives and quantifiers on the left-hand sides: $\wedge$ is the meet, $\vee$ is the join, $\neg$ is the complementation, while the implication $\to$ is introduced as follows: $a \to b := \neg a \vee b$ for $a, b \in B$. Only such definitions provide the value "unity" for the classical tautologies.

We turn to evaluating the atomic formulas $x \in y$ and $x = y$ for $x, y \in \mathbb{V}^{(B)}$. The intuitive idea consists in the fact that a B-valued set y is a "(lattice) fuzzy set," i.e., a "set that contains an element z in $\mathrm{dom}(y)$ with probability $y(z)$." Keeping this in mind and intending to preserve the logical tautology of $x \in y \leftrightarrow (\exists z \in y)\,(x = z)$ as well as the axiom of extensionality, we arrive at the following definition by recursion:

$$[\![x \in y]\!] := \bigvee_{z\in\mathrm{dom}(y)} y(z) \wedge [\![z = x]\!],$$

$$[\![x = y]\!] := \bigwedge_{z\in\mathrm{dom}(x)} x(z) \to [\![z \in y]\!] \wedge \bigwedge_{z\in\mathrm{dom}(y)} y(z) \to [\![z \in x]\!].$$

Now we can attach some meaning to formal expressions of the form $\varphi(x_1, \dots, x_n)$, where $x_1, \dots, x_n \in \mathbb{V}^{(B)}$ and φ is a formula of ZFC; i.e., we may define the exact sense in which the set-theoretic proposition $\varphi(u_1, \dots, u_n)$ is valid for the assignment of $x_1, \dots, x_n \in \mathbb{V}^{(B)}$.

Namely, we say that the *formula* $\varphi(x_1, \dots, x_n)$ *is valid inside* $\mathbb{V}^{(B)}$ or the *elements* $x_1, \dots, x_n$ *possess the property* φ if $[\![\varphi(x_1, \dots, x_n)]\!] = \mathbb{1}$. In this event, we write $\mathbb{V}^{(B)} \vDash \varphi(x_1, \dots, x_n)$.

It is easy to convince ourselves that the axioms and theorems of the first-order predicate calculus are valid in $\mathbb{V}^{(B)}$. In particular,

(1) $[\![x = x]\!] = \mathbb{1}$,

(2) $[\![x = y]\!] = [\![y = x]\!]$,

(3) $[\![x = y]\!] \wedge [\![y = z]\!] \le [\![x = z]\!]$,

(4) $[\![x = y]\!] \wedge [\![z \in x]\!] \le [\![z \in y]\!]$,

(5) $[\![x = y]\!] \wedge [\![x \in z]\!] \le [\![y \in z]\!]$.

It is worth observing that for each formula φ we have

$$\mathbb{V}^{(B)} \models x = y \wedge \varphi(x) \to \varphi(y),$$

i.e., in a detailed notation

(6) $[\![x = y]\!] \wedge [\![\varphi(x)]\!] \le [\![\varphi(y)]\!]$.

2. Principles of Boolean valued analysis

In a Boolean valued universe $\mathbb{V}^{(B)}$, the relation $[\![x = y]\!] = \mathbb{1}$ in no way implies that the functions x and y (considered as elements of $\mathbb{V}$) coincide. For example, the function equal to zero on each level $\mathbb{V}_\alpha^{(B)}$, where $\alpha \geq 1$, plays the role of the empty set in $\mathbb{V}^{(B)}$. This circumstance may complicate some constructions in the sequel.

In this connection, we pass from $\mathbb{V}^{(B)}$ to the *separated Boolean valued universe* $\overline{\mathbb{V}}^{(B)}$ often preserving for it the same symbol $\mathbb{V}^{(B)}$; i.e., we put $\mathbb{V}^{(B)} := \overline{\mathbb{V}}^{(B)}$. Moreover, to define $\overline{\mathbb{V}}^{(B)}$, we consider the relation $\{(x, y) \mid [\![x = y]\!] = \mathbb{1}\}$ on the class $\mathbb{V}^{(B)}$ which is obviously an equivalence. Choosing an element (a representative of the least rank) in each class of equivalent functions, we arrive at the separated universe $\overline{\mathbb{V}}^{(B)}$. Note that

$$[\![x = y]\!] = \mathbb{1} \rightarrow [\![\varphi(x)]\!] = [\![\varphi(y)]\!]$$

is valid for an arbitrary formula φ of ZFC and elements x and y in $\mathbb{V}^{(B)}$.

Therefore, in the separated universe we can calculate the truth-values of formulas paying no attention to the way of choosing representatives. Furthermore, working with the separated universe, for the sake of convenience we (exercising due caution) often consider some particular representative of an equivalence class rather than the whole class as it is customary, for example, while dealing with function spaces.

The most important properties of a Boolean valued universe $\mathbb{V}^{(B)}$ are stated in the following three principles:

(1) Transfer Principle. *All theorems of* ZFC *are true in* $\mathbb{V}^{(B)}$*; in symbols,*

$$\mathbb{V}^{(B)} \models \textit{a theorem of } \mathrm{ZFC}.$$

The transfer principle is established by laboriously checking that all axioms of ZFC have truth-value $\mathbb{1}$ and the rules of inference increase the truth-values of formulas. Sometimes, the transfer principle is worded as follows: "$\mathbb{V}^{(B)}$ is a Boolean valued model of ZFC."

(2) Maximum Principle. *For each formula* φ *of* ZFC *there exists* $x_0 \in \mathbb{V}^{(B)}$ *for which*

$$[\![(\exists x)\, \varphi(x)]\!] = [\![\varphi(x_0)]\!].$$

In particular, if it is true in $\mathbb{V}^{(B)}$ that there is an x for which $\varphi(x)$ then there is an element x_0 in $\mathbb{V}^{(B)}$ (in the sense of $\mathbb{V}$) for which $[\![\varphi(x_0)]\!] = \mathbb{1}$. In symbols,

$$\mathbb{V}^{(B)} \models (\exists x)\, \varphi(x) \rightarrow (\exists x_0)\, \mathbb{V}^{(B)} \models \varphi(x_0).$$

Thus, the *maximum principle* reads:

$$(\exists x_0 \in \mathbb{V}^{(B)})\, [\![\varphi(x_0)]\!] = \bigvee_{x \in \mathbb{V}^{(B)}} [\![\varphi(x)]\!]$$

for each formula φ of ZFC.

The last equality accounts for the origin of the term "maximum principle." The proof of the maximum principle is a simple application of the following

(3) Mixing Principle. Let $(b_\xi)_{\xi \in \Xi}$ be a *partition of unity* in B, i.e. a family of elements of a Boolean valued algebra B such that

$$\bigvee_{\xi \in \Xi} b_\xi = \mathbb{1}, \quad (\forall \xi, \eta \in \Xi)\, (\xi \neq \eta \rightarrow b_\xi \wedge b_\eta = \mathbb{0}).$$

For each family of elements $(x_\xi)_{\xi\in\Xi}$ of the universe $\mathbb{V}^{(B)}$ and each partition of unity $(b_\xi)_{\xi\in\Xi}$ there exists a (unique) mixing of (x_ξ) by (b_ξ); i.e. an element x of the separated universe $\mathbb{V}^{(B)}$ such that $b_\xi \leq [\![x = x_\xi]\!]$ for all $\xi \in \Xi$.

The *mixing* x of a family (x_ξ) by (b_ξ) is denoted as follows:

$$x = \operatorname*{mix}_{\xi\in\Xi}(b_\xi x_\xi) = \operatorname{mix}\{b_\xi x_\xi \mid \xi \in \Xi\}.$$

3. Ascending and descending

Boolean valued analysis consists primarily in comparative analysis of a mathematical object or idea simultaneously in two Boolean valued models. This is impossible to achieve without some dialog between the universes $\mathbb{V}$ and $\mathbb{V}^{(B)}$. In other words, we need a rigorous mathematical apparatus that would allow us to reveal the interplay between the interpretations of one and the same fact in the two models $\mathbb{V}$ and $\mathbb{V}^{(B)}$. The relevant technique rests on by the operations of canonical embedding, descent, and ascent to be presented below. We start with the canonical embedding of the von Neumann universe. Given $x \in \mathbb{V}$, we denote by the symbol $x^\wedge$ the *standard name* of x in $\mathbb{V}^{(B)}$; i.e., the element defined by the following recursion schema:

$$\varnothing^\wedge := \varnothing, \quad \operatorname{dom}(x^\wedge) := \{y^\wedge \mid y \in x\}, \quad \operatorname{im}(x^\wedge) := \{\mathbb{1}\}.$$

Observe some properties of the mapping $x \mapsto x^\wedge$ we need in the sequel.

(1) For an arbitrary $x \in \mathbb{V}$ and a formula φ of ZFC we have

$$[\![(\exists y \in x^\wedge)\,\varphi(y)]\!] = \bigvee\{[\![\varphi(z^\wedge)]\!] : z \in x\},$$

$$[\![(\forall y \in x^\wedge)\,\varphi(y)]\!] = \bigwedge\{[\![\varphi(z^\wedge)]\!] : z \in x\}.$$

(2) If x and y are elements of $\mathbb{V}$ then, by transfinite induction, we establish

$$x \in y \leftrightarrow \mathbb{V}^{(B)} \models x^\wedge \in y^\wedge,$$

$$x = y \leftrightarrow \mathbb{V}^{(B)} \models x^\wedge = y^\wedge.$$

In other words, the standard name can be considered as an embedding of $\mathbb{V}$ into $\mathbb{V}^{(B)}$. Moreover, it is beyond a doubt that the standard name sends $\mathbb{V}$ onto $\mathbb{V}^{(2)}$, which fact is demonstrated by the next proposition:

(3) The following holds:

$$(\forall u \in \mathbb{V}^{(2)})\,(\exists! x \in \mathbb{V})\ \mathbb{V}^{(B)} \models u = x^\wedge.$$

(4) A formula is called *bounded* or *restricted* if each bound variable in it is restricted by a bounded quantifier; i.e., a quantifier ranging over a specific set. The latter means that each bound variable x is restricted by a quantifier of the form $(\forall x \in y)$ or $(\exists x \in y)$ for some y.

Restricted Transfer Principle. *For each bounded formula φ of* ZFC *and every collection $x_1, \dots, x_n \in \mathbb{V}$ the following equivalence holds:*

$$\varphi(x_1, \dots, x_n) \leftrightarrow \mathbb{V}^{(B)} \models \varphi(x_1^\wedge, \dots, x_n^\wedge).$$

Henceforth, working in the separated universe $\overline{\mathbb{V}}^{(B)}$, we agree to preserve the symbol $x^\wedge$ for the distinguished element of the class corresponding to x.

(5) Observe for example that the restricted transfer principle yields the following assertions:

$$\text{“}\Phi \text{ is a correspondence from } x \text{ to } y\text{”}$$
$$\leftrightarrow \mathbb{V}^{(B)} \models \text{“}\Phi^\wedge \text{ is a correspondence from } x^\wedge \text{ to } y^\wedge\text{”};$$

$$\text{“}f \text{ is a function from } x \text{ to } y\text{”} \leftrightarrow \mathbb{V}^{(B)} \models \text{“}f^\wedge \text{ is a function from } x^\wedge \text{ to } y^\wedge\text{”}$$

(moreover, $f(a)^\wedge = f^\wedge(a^\wedge)$ for all $a \in x$). Thus, the standard name can be considered as a covariant functor of the category of sets (or correspondences) in $\mathbb{V}$ to an appropriate subcategory of $\mathbb{V}^{(2)}$ in the separated universe $\mathbb{V}^{(B)}$.

(6) A set X is *finite* if X coincides with the image of a function on a finite ordinal. In symbols, this is expressed as $\operatorname{Fin}(X)$; hence,

$$\operatorname{Fin}(X) := (\exists\, n)(\exists\, f)(n \in \omega \wedge f \text{ is a function} \wedge \operatorname{dom}(f) = n \wedge \operatorname{im}(f) = X)$$

(as usual $\omega := \{0, 1, 2, ...\}$). Obviously, the above formula is not bounded. Nevertheless there is a simple transformation rule for the class of finite sets under the canonical embedding. Denote by $\mathscr{P}_{\mathrm{Fin}}(X)$ the class of all finite subsets of X:

$$\mathscr{P}_{\mathrm{Fin}}(X) := \{Y \in \mathscr{P}(X) \mid \operatorname{Fin}(Y)\}.$$

For an arbitrary set X the following holds:

$$\mathbb{V}^{(B)} \models \mathscr{P}_{\mathrm{Fin}}(X)^\wedge = \mathscr{P}_{\mathrm{Fin}}(X^\wedge).$$

Given an arbitrary element x of the (separated) Boolean valued universe $\mathbb{V}^{(B)}$, we define the *descent* $x{\downarrow}$ of x as

$$x{\downarrow} := \{y \in \mathbb{V}^{(B)} \mid [\![y \in x]\!] = \mathbb{1}\}.$$

We list the simplest properties of descending:

(1) The class $x{\downarrow}$ is a set, i.e., $x{\downarrow} \in \mathbb{V}$ for each $x \in \mathbb{V}^{(B)}$. If $[\![x \neq \varnothing]\!] = \mathbb{1}$ then $x{\downarrow}$ is a nonempty set.

(2) Let $z \in \mathbb{V}^{(B)}$ and $[\![z \neq \varnothing]\!] = \mathbb{1}$. Then for every formula φ of ZFC we have

$$[\![(\forall x \in z)\, \varphi(x)]\!] = \bigwedge\{[\![\varphi(x)]\!] \mid x \in z{\downarrow}\},$$

$$[\![(\exists x \in z)\, \varphi(x)]\!] = \bigvee\{[\![\varphi(x)]\!] \mid x \in z{\downarrow}\}.$$

Moreover, there exists $x_0 \in z{\downarrow}$ such that $[\![\varphi(x_0)]\!] = [\![(\exists x \in z)\, \varphi(x)]\!]$.

(3) Let Φ be a correspondence from X to Y in $\mathbb{V}^{(B)}$. Thus, Φ, X, and Y are elements of $\mathbb{V}^{(B)}$ and, moreover, $[\![\Phi \subset X \times Y]\!] = \mathbb{1}$. There is a unique correspondence $\Phi{\downarrow}$ from $X{\downarrow}$ to $Y{\downarrow}$ such that

$$\Phi{\downarrow}(A{\downarrow}) = \Phi(A){\downarrow}$$

for every nonempty subset A of the set X inside $\mathbb{V}^{(B)}$. The correspondence $\Phi{\downarrow}$ from $X{\downarrow}$ to $Y{\downarrow}$ involved in the above proposition is called the *descent* of the correspondence Φ from X to Y in $\mathbb{V}^{(B)}$.

(4) The descent of the composite of correspondences inside $\mathbb{V}^{(B)}$ is the composite of their descents:

$$(\Psi \circ \Phi){\downarrow} = \Psi{\downarrow} \circ \Phi{\downarrow}.$$

(5) If Φ is a correspondence inside $\mathbb{V}^{(B)}$ then

$$(\Phi^{-1}){\downarrow} = (\Phi{\downarrow})^{-1}.$$

(6) Let Id_X be the identity mapping inside $\mathbb{V}^{(B)}$ of a set $X \in \mathbb{V}^{(B)}$. Then

$$(\mathrm{Id}_X){\downarrow} = \mathrm{Id}_{X{\downarrow}}.$$

(7) Suppose that $X, Y, f \in \mathbb{V}^{(B)}$ are such that $[\![f : X \to Y]\!] = \mathbb{1}$, i.e., f is a mapping from X to Y inside $\mathbb{V}^{(B)}$. Then $f{\downarrow}$ is a unique mapping from $X{\downarrow}$ to $Y{\downarrow}$ for which

$$[\![f{\downarrow}(x) = f(x)]\!] = \mathbb{1} \quad (x \in X{\downarrow}).$$

By virtue of (1)–(7), we can consider the descent operation as a functor from the category of B-valued sets and mappings (correspondences) to the category of the usual sets and mappings (correspondences) (i.e., in the sense of $\mathbb{V}$).

(8) Given $x_1, \dots, x_n \in \mathbb{V}^{(B)}$, denote by $(x_1, \dots, x_n)^B$ the corresponding ordered n-tuple inside $\mathbb{V}^{(B)}$. Assume that P is an n-ary relation on X inside $\mathbb{V}^{(B)}$; i.e., $X, P \in \mathbb{V}^{(B)}$ and $[\![P \subset X^{n^\wedge}]\!] = \mathbb{1}$, where $n \in \omega$. Then there exists an n-ary relation P' on $X{\downarrow}$ such that

$$(x_1, \dots, x_n) \in P' \leftrightarrow [\![(x_1, \dots, x_n)^B \in P]\!] = \mathbb{1}.$$

Slightly abusing notation, we denote the relation P' by the same symbol $P{\downarrow}$ and call it the *descent* of P.

Let $x \in \mathbb{V}$ and $x \subset \mathbb{V}^{(B)}$; i.e., let x be some set composed of B-valued sets or, in other words, $x \in \mathscr{P}(\mathbb{V}^{(B)})$. Put $\varnothing{\uparrow} := \varnothing$ and

$$\operatorname{dom}(x{\uparrow}) = x, \quad \operatorname{im}(x{\uparrow}) = \{\mathbb{1}\}$$

if $x \neq \varnothing$. The element $x{\uparrow}$ (of the separated universe $\mathbb{V}^{(B)}$, i.e., the distinguished representative of the class $\{y \in \mathbb{V}^{(B)} \mid [\![y = x{\uparrow}]\!] = \mathbb{1}\}$) is called the *ascent* of x.

(1) The following equalities hold for every $x \in \mathscr{P}(\mathbb{V}^{(B)})$ and every formula φ:

$$[\![(\forall z \in x{\uparrow})\, \varphi(z)]\!] = \bigwedge_{y \in x} [\![\varphi(y)]\!],$$

$$[\![(\exists z \in x{\uparrow})\, \varphi(z)]\!] = \bigvee_{y \in x} [\![\varphi(y)]\!].$$

Introducing the ascent of a correspondence $\Phi \subset X \times Y$, we have to bear in mind a possible difference between the domain of departure X and the domain $\operatorname{dom}(\Phi) := \{x \in X \mid \Phi(x) \neq \varnothing\}$. This difference is inessential for our further goals; therefore, we assume that, speaking of ascents, we always consider total correspondences; i.e., $\operatorname{dom}(\Phi) = X$.

(2) Let $X, Y, \Phi \in \mathbb{V}^{(B)}$, and let Φ be a correspondence from X to Y. There exists a unique correspondence $\Phi{\uparrow}$ from $X{\uparrow}$ to $Y{\uparrow}$ inside $\mathbb{V}^{(B)}$ such that

$$\Phi{\uparrow}(A{\uparrow}) = \Phi(A){\uparrow}$$

is valid for every subset A of the set $\operatorname{dom}(\Phi)$ if and only if Φ is *extensional*; i.e., satisfies the condition

$$y_1 \in \Phi(x_1) \to [\![x_1 = x_2]\!] \leq \bigvee_{y_2 \in \Phi(x_2)} [\![y_1 = y_2]\!]$$

for $x_1, x_2 \in \operatorname{dom}(\Phi)$. In this event, $\Phi{\uparrow} = \Phi'{\uparrow}$, where $\Phi' := \{(x, y)^B \mid (x, y) \in \Phi\}$. The element $\Phi{\uparrow}$ is called the *ascent* of the initial correspondence Φ.

(3) The composite of extensional correspondences is extensional. Moreover, the ascent of a composite is equal to the composite of the ascents (inside $\mathbb{V}^{(B)}$): On assuming that $\operatorname{dom}(\Psi) \supset \operatorname{im}(\Phi)$ we have

$$\mathbb{V}^{(B)} \vDash (\Psi \circ \Phi){\uparrow} = \Psi{\uparrow} \circ \Phi{\uparrow}.$$

Note that if Φ and Φ^{-1} are extensional then $(\Phi{\uparrow})^{-1} = (\Phi^{-1}){\uparrow}$. However, in general, the extensionality of Φ in no way guarantees the extensionality of Φ^{-1}.

(4) It is worth mentioning that if an extensional correspondence f is a function from X to Y then its ascent $f{\uparrow}$ is a function from $X{\uparrow}$ to $Y{\uparrow}$. Moreover, the extensionality property can be stated as follows:

$$[\![x_1 = x_2]\!] \leq [\![f(x_1) = f(x_2)]\!] \quad (x_1, x_2 \in X).$$

Given a set $X \subset \mathbb{V}^{(B)}$, we denote by the symbol $\operatorname{mix} X$ the set of all mixings of the form $\operatorname{mix}(b_\xi x_\xi)$, where $(x_\xi) \subset X$ and (b_ξ) is an arbitrary partition of unity. The following propositions are referred to as the *rules for cancelling arrows* or the “*descending-and-ascending*” and “*ascending-and-descending” rules.* It seems smarter to call them *Escher rules.* [2]

(5) Let X and X' be subsets of $\mathbb{V}^{(B)}$ and $f : X \to X'$ be an extensional mapping. Suppose that $Y, Y', g \in \mathbb{V}^{(B)}$ are such that $[\![Y \neq \varnothing]\!] = [\![g : Y \to Y']\!] = \mathbb{1}$. Then

$$X{\uparrow}{\downarrow} = \operatorname{mix} X, \quad Y{\downarrow}{\uparrow} = Y; \quad f{\uparrow}{\downarrow} = f, \quad g{\downarrow}{\uparrow} = g.$$

(6) From (6) follows the useful relation:

$$\mathscr{P}_{\mathrm{Fin}}(X{\uparrow}) = \{\theta{\uparrow} \mid \theta \in \mathscr{P}_{\mathrm{Fin}}(X)\}{\uparrow}\ .$$

Suppose that $X \in \mathbb{V}$, $X \neq \varnothing$; i.e., X is a nonempty set. Let the letter ι denote the standard name embedding $x \mapsto x^\wedge$ $(x \in X)$. Then $\iota(X){\uparrow} = X^\wedge$ and $X = \iota^{-1}(X^\wedge{\downarrow})$. Using the above relations, we may extend the descent and ascent operations to the case in which Φ is a correspondence from X to $Y{\downarrow}$ and $[\![\Psi$ is a correspondence from $X^\wedge$ to $Y]\!] = \mathbb{1}$, where $Y \in \mathbb{V}^{(B)}$. Namely, we put $\Phi{\Uparrow} := (\Phi \circ \iota){\uparrow}$ and $\Psi{\Downarrow} := \Psi{\downarrow} \circ \iota$. In this case, $\Phi{\Uparrow}$ is called the *modified ascent* of the correspondence Φ and $\Psi{\Downarrow}$ is called the *modified descent* of the correspondence Ψ. (If the context excludes ambiguity then we briefly speak of ascents and descents using simple arrows.) It is easy to see that $\Psi{\Uparrow}$ is a unique correspondence inside $\mathbb{V}^{(B)}$ satisfying the relation

$$[\![\Phi{\Uparrow}(x^\wedge) = \Phi(x){\uparrow}]\!] = \mathbb{1} \quad (x \in X).$$

Similarly, $\Psi{\Downarrow}$ is a unique correspondence from X to $Y{\downarrow}$ satisfying the equality

$$\Psi{\Downarrow}(x) = \Psi(x^\wedge){\downarrow} \quad (x \in X).$$

If $\Phi := f$ and $\Psi := g$ are functions then these relations take the form

$$[\![f{\Uparrow}(x^\wedge) = f(x)]\!] = \mathbb{1}, \quad g{\Downarrow}(x) = g(x^\wedge) \quad (x \in X).$$

(1) A *Boolean set* or a *set with B-structure* or just a *B-set* is a pair (X, d), where $X \in \mathbb{V}$, $X \neq \varnothing$, and d is a mapping from $X \times X$ to the Boolean algebra B such that for all $x, y, z \in X$ the following hold:

(a) $d(x, y) = \mathbb{0} \leftrightarrow x = y$;

(b) $d(x, y) = d(y, x)$;

(c) $d(x, y) \le d(x, z) \vee d(z, y)$.

An example of a B-set is given by each $\varnothing \neq X \subset \mathbb{V}^{(B)}$ if we put

$$d(x, y) := [\![x \neq y]\!] = \neg [\![x = y]\!] \quad (x,\ y \in X).$$

Another example is a nonempty X with the “discrete B-metric” d; i.e., $d(x, y) = \mathbb{1}$ if $x \neq y$ and $d(x, y) = \mathbb{0}$ if $x = y$.

(2) Let (X, d) be some B-set. There exist an element $\mathscr{X} \in \mathbb{V}^{(B)}$ and an injection $\iota : X \to X' := \mathscr{X}{\downarrow}$ such that $d(x, y) = [\![\iota x \neq \iota y]\!]$ for all $x, y \in X$ and every element $x' \in X'$ admits the representation $x' = \operatorname{mix}_{\xi \in \Xi}(b_\xi \iota x_\xi)$, where $(x_\xi)_{\xi \in \Xi} \subset X$ and $(b_\xi)_{\xi \in \Xi}$ is a partition of unity in B. The element $\mathscr{X} \in \mathbb{V}^{(B)}$ is referred to as the *Boolean valued realization* of the B-set X. If X is a discrete B-set then $\mathscr{X} = X^\wedge$ and $\iota x = x^\wedge$ for all $x \in X$. If $X \subset \mathbb{V}^{(B)}$ then $\iota{\uparrow}$ is an injection from $X{\uparrow}$ to $\mathscr{X}$ (inside $\mathbb{V}^{(B)}$).

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[2]Cf. Hofstedter D. R., *Gödel, Escher, Bach: an Eternal Golden Braid. (20th Anniversary Edition).* Basic Books, New York (1999).

A mapping f from a B-set (X,d) to a B-set (X',d') is said to be *nonexpanding* if $d(x,y) \geq d'(f(x),f(y))$ for all $x,y \in X$.

(3) Let X and Y be some B-sets, $\mathscr{X}$ and $\mathscr{Y}$ be their Boolean valued realizations, and ι and $\varkappa$ be the corresponding injections $X \to \mathscr{X}{\downarrow}$ and $Y \to \mathscr{Y}{\downarrow}$. If $f : X \to Y$ is a nonexpanding mapping then there is a unique element $g \in \mathbb{V}^{(B)}$ such that $[\![g : \mathscr{X} \to \mathscr{Y}]\!] = \mathbb{1}$ and $f = \varkappa^{-1} \circ g{\downarrow} \circ \iota$. We also accept the notations $\mathscr{X} := \mathscr{F}^{\sim}(X) := X^{\sim}$ and $g := \mathscr{F}^{\sim}(f) := f^{\sim}$.

(4) Moreover, the following are valid:

(1) $\mathbb{V}^{(B)} \models f(A)^{\sim} = f^{\sim}(A^{\sim})$ for $A \subset X$;

(2) If $g : Y \to Z$ is a contraction then $g \circ f$ is a contraction and $\mathbb{V}^{(B)} \models (g \circ f)^{\sim} = g^{\sim} \circ f^{\sim}$;

(3) $\mathbb{V}^{(B)} \models$ "$f^{\sim}$ is injective" if and only if f is a B-isometry;

(4) $\mathbb{V}^{(B)} \models$ "$f^{\sim}$ is surjective" if and only if $\bigvee\{d(f(x),y) \mid x \in X\} = \mathbb{1}$ for all $y \in Y$.

Recall that a *signature* is a 3-tuple $\sigma := (F,P,\mathfrak{a})$, where F and P are some (possibly, empty) sets and $\mathfrak{a}$ is a mapping from $F \cup P$ to ω. If the sets F and P are finite then σ is a *finite signature.* In applications we usually deal with algebraic systems of finite signature.

An *n-ary operation* and an *n-ary predicate* on a B-set A are contractive mappings $f : A^n \to A$ and $p : A^n \to B$ respectively. By definition, f and p *are contractive mappings* provided that

$$d(f(a_0,\dots,a_{n-1}), f(a'_0,\dots,a'_{n-1})) \leq \bigvee_{k=0}^{n-1} d(a_k,a'_k),$$

$$d_s\big(p(a_0,\dots,a_{n-1}), p(a'_0,\dots,a'_{n-1})\big) \leq \bigvee_{k=0}^{n-1} d(a_k,a'_k)$$

for all $a_0,\ a'_0,\dots,a_{n-1},\ a'_{n-1} \in A$, where d is the B-metric of A, and d_s is the *symmetric difference* on B; i.e., $d_s(b_1,b_2) := (b_1 x \wedge \neg b_2) \vee (\neg b_1 \wedge b_2)$. .

Clearly, the above definitions depend on B and it would be cleaner to speak of B-operations, B-predicates, etc. We adhere to a simpler practice whenever it entails no confusion.

An *algebraic B-system* $\mathfrak{A}$ of signature σ is a pair (A,ν), where A is a nonempty B-set, the *underlying set* or *carrier* or *universe* of $\mathfrak{A}$, and ν is a mapping such that (a) $\mathrm{dom}(\nu) = F \cup P$; (b) $\nu(f)$ is an $\mathfrak{a}(f)$-ary operation on A for all $f \in F$; and (c) $\nu(p)$ is an $\mathfrak{a}(p)$-ary predicate on A for all $p \in P$.

It is in common parlance to call ν the *interpretation* of $\mathfrak{A}$ in which case the notation f^{ν} and p^{ν} are common substitutes for $\nu(f)$ and $\nu(p)$.

The signature of an algebraic B-system $\mathfrak{A} := (A,\nu)$ is often denoted by $\sigma(\mathfrak{A})$; while the carrier A of $\mathfrak{A}$, by $|\mathfrak{A}|$. Since $A^0 = \{\varnothing\}$, the nullary operations and predicates on A are mappings from $\{\varnothing\}$ to the set A and to the algebra B respectively. We agree to identify a mapping $g : \{\varnothing\} \to A \cup B$ with the element $g(\varnothing)$. Each nullary operation on A thus transforms into a unique member of A. Analogously, the set of all nullary predicates on A turns into the Boolean algebra B. If $F := \{f_1,\dots,f_n\}$ and $P := \{p_1,\dots,p_m\}$ then an algebraic B-system of signature σ is often written down as $(A,\nu(f_1),\dots,\nu(f_n),\ \nu(p_1),\dots,\nu(p_m))$ or even $(A,f_1,\dots,f_n,p_1,\dots,p_m)$. In this event, the expression $\sigma = (f_1,\dots,f_n,p_1,\dots,p_m)$ is substituted for $\sigma = (F,P,\mathfrak{a})$.

We now address the B-valued interpretation of a first-order language. Consider an algebraic B-system $\mathfrak{A} := (A,\nu)$ of signature $\sigma := \sigma(\mathfrak{A}) := (F,P,\mathfrak{a})$.

Let $\varphi(x_0,\dots,x_{n-1})$ be a formula of signature σ with n free variables. Assume given $a_0,\dots,a_{n-1}\in A$. We are ready to define the truth-value $|\varphi|^{\mathfrak A}(a_0,\dots,a_{n-1})\in B$ of a formula φ in the system $\mathfrak A$ for the given values $a_0,\dots,a_{n-1}$ of the variables $x_0,\dots,x_{n-1}$. The definition proceeds as usual by induction on the complexity of φ: Considering propositional connectives and quantifiers, we put

$$|\varphi\wedge\psi|^{\mathfrak A}(a_0,\dots,a_{n-1}):=|\varphi|^{\mathfrak A}(a_0,\dots,a_{n-1})\wedge|\psi|^{\mathfrak A}(a_0,\dots,a_{n-1});$$
$$|\varphi\vee\psi|^{\mathfrak A}(a_0,\dots,a_{n-1}):=|\varphi|^{\mathfrak A}(a_0,\dots,a_{n-1})\vee|\psi|^{\mathfrak A}(a_0,\dots,a_{n-1});$$
$$|\neg\varphi|^{\mathfrak A}(a_0,\dots,a_{n-1}):=\neg|\varphi|^{\mathfrak A}(a_0,\dots,a_{n-1});$$
$$|(\forall\,x_0)\varphi|^{\mathfrak A}(a_1,\dots,a_{n-1}):=\bigwedge_{a_0\in A}|\varphi|^{\mathfrak A}(a_0,\dots,a_{n-1});$$
$$|(\exists\,x_0)\varphi|^{\mathfrak A}(a_1,\dots,a_{n-1}):=\bigvee_{a_0\in A}|\varphi|^{\mathfrak A}(a_0,\dots,a_{n-1}).$$

Now, the case of atomic formulas is in order. Suppose that $p\in P$ symbolizes an m-ary predicate, $q\in P$ is a nullary predicate, and $t_0,\dots,t_{m-1}$ are terms of signature σ assuming values $b_0,\dots,b_{m-1}$ at the given values $a_0,\dots,a_{n-1}$ of the variables $x_0,\dots,x_{n-1}$. By definition, we let

$$|\varphi|^{\mathfrak A}(a_0,\dots,a_{n-1}):=\nu(q),\ \text{ if }\varphi=q^\nu;$$
$$|\varphi|^{\mathfrak A}(a_0,\dots,a_{n-1}):=\neg d(b_0,b_1),\quad\text{ if }\varphi=(t_0=t_1);$$
$$|\varphi|^{\mathfrak A}(a_0,\dots,a_{n-1}):=p^\nu(b_0,\dots,b_{m-1}),\quad\text{ if }\varphi=p^\nu(t_0,\dots,t_{m-1}),$$

where d is a B-metric on A.

Say that $\varphi(x_0,\dots,x_{n-1})$ is *valid* in $\mathfrak A$ at the given values $a_0,\dots,a_{n-1}\in A$ of $x_0,\dots,x_{n-1}$ and write $\mathfrak A\models\varphi(a_0,\dots,a_{n-1})$ provided that $|\varphi|^{\mathfrak A}(a_0,\dots,a_{n-1})=\mathbb 1_B$. Alternative expressions are as follows: $a_0,\dots,a_{n-1}\in A$ *satisfies* $\varphi(x_0,\dots,x_{n-1})$; or $\varphi(a_0,\dots,a_{n-1})$ *holds true* in $\mathfrak A$. In the case of the two-element Boolean algebra $B:=\mathbb 2:=\{\mathbb 0,\mathbb 1\}$, we arrive at the conventional definition of the validity of a formula in an algebraic system.

Recall that a closed formula φ of signature σ is a *tautology* if φ is valid on every algebraic $\mathbb 2$-system of signature σ.

Consider algebraic B-systems $\mathfrak A:=(A,\nu)$ and $\mathfrak D:=(D,\mu)$ of the same signature σ. The mapping $h:A\to D$ is a *homomorphism* of $\mathfrak A$ to $\mathfrak D$ provided that, for all $a_0,\dots,a_{n-1}\in A$, the following are valid:

(1) $d_B(h(a_1),h(a_2))\le d_A(a_1,a_2)$;

(2) $h(f^\nu)=f^\mu$ if $\mathfrak a(f)=0$;

(3) $h(f^\nu(a_0,\dots,a_{n-1}))=f^\mu(h(a_0),\dots,h(a_{n-1}))$ if $0\neq n:=\mathfrak a(f)$;

(4) $p^\nu(a_0,\dots,a_{n-1})\le p^\mu(h(a_0),\dots,h(a_{n-1}))$, with $n:=\mathfrak a(p)$.

A homomorphism h is called *strong* if

(5) $\mathfrak a(p):=n\neq 0$ for $p\in P$, and, for all $d_0,\dots,d_{n-1}\in D$ the following inequality holds:

$$p^\mu(d_0,\dots,d_{n-1})$$
$$\ge\bigvee_{a_0,\dots,a_{n-1}\in A}\{p^\nu(a_0,\dots,a_{n-1})\wedge d_D(d_0,h(a_0))\wedge\dots\wedge d_D(d_{n-1},h(a_{n-1}))\}.$$

If a homomorphism h is injective and (1) and (4) are fulfilled with equality holding; then h is said to be a *isomorphism from* $\mathfrak A$ *to* $\mathfrak D$. Undoubtedly, each surjective isomorphism h and, in particular, the identity mapping $\mathrm{Id}_A:A\to A$ are strong homomorphisms. The composite of (strong) homomorphisms is a (strong) homomorphism. Clearly, if h is a homomorphism and h^{-1} is a homomorphism too, then h is an isomorphism.

Note again that in the case of the two-element Boolean algebra we come to the conventional concepts of homomorphism, strong homomorphism, and isomorphism.

Before giving a general definition of the descent of an algebraic system, consider the descent of the two-element Boolean algebra. Choose two arbitrary elements, 0, $1 \in \mathbb{V}^{(B)}$, satisfying $[\![0 \neq 1]\!] = \mathbb{1}_B$. We may for instance assume that $0 := \mathbb{0}_B^{\wedge}$ and $1 := \mathbb{1}_B^{\wedge}$.

(1) *The descent D of the two-element Boolean algebra $\{0,1\}^B \in \mathbb{V}^{(B)}$ is a complete Boolean algebra isomorphic to B. The formulas*

$$[\![\chi(b) = 1]\!] = b, \quad [\![\chi(b) = 0]\!] = \neg b \quad (b \in B)$$

defines an isomorphism $\chi : B \to D$.

Consider now an algebraic system $\mathfrak{A}$ of signature $\sigma^{\wedge}$ inside $\mathbb{V}^{(B)}$, and let $[\![\mathfrak{A} = (A,\nu)^B]\!] = \mathbb{1}$ for some A, $\nu \in \mathbb{V}^{(B)}$. The *descent* of $\mathfrak{A}$ is the pair $\mathfrak{A}{\downarrow} := (A{\downarrow}, \mu)$, where μ is the function determined from the formulas:

$$\mu : f \mapsto (\nu\!\downarrow\!\!\downarrow(f)){\downarrow} \quad (f \in F),$$

$$\mu : p \mapsto \chi^{-1} \circ (\nu\!\downarrow\!\!\downarrow(p)){\downarrow} \quad (p \in P).$$

Here χ is the above isomorphism of the Boolean algebras B and $\{0,1\}^B{\downarrow}$.

In more detail, the modified descent $\nu\!\downarrow\!\!\downarrow$ is the mapping with domain $\mathrm{dom}(\nu\!\downarrow\!\!\downarrow) = F \cup P$. Given $p \in P$, observe $[\![\mathfrak{A}(p)^{\wedge} = \mathfrak{A}^{\wedge}(p^{\wedge})]\!] = \mathbb{1}$, $[\![\nu\!\downarrow\!\!\downarrow(p) = \nu(p^{\wedge})]\!] = \mathbb{1}$ and so

$$\mathbb{V}^{(B)} \models \nu\!\downarrow\!\!\downarrow(p) : A^{\mathfrak{a}(f)^{\wedge}} \to \{0,1\}^B.$$

It is now obvious that $(\nu\!\downarrow\!\!\downarrow(p)){\downarrow} : (A{\downarrow})^{\mathfrak{a}(f)} \to D := \{0,1\}^B{\downarrow}$ and we may put $\mu(p) := \chi^{-1} \circ (\nu\!\downarrow\!\!\downarrow(p)){\downarrow}$.

Let $\varphi(x_0,\dots,x_{n-1})$ be a fixed formula of signature σ in n free variables. Write down the formula $\Phi(x_0,\dots,x_{n-1},\mathfrak{A})$ in the language of set theory which formalizes the proposition $\mathfrak{A} \models \varphi(x_0,\dots,x_{n-1})$. Recall that the formula $\mathfrak{A} \models \varphi(x_0,\dots,x_{n-1})$ determines an n-ary predicate on A or, which is the same, a mapping from A^n to $\{0,1\}$. By the maximum and transfer principles, there is a unique element $|\varphi|^{\mathfrak{A}} \in \mathbb{V}^{(B)}$ such that

$$[\![|\varphi|^{\mathfrak{A}} : A^{n^{\wedge}} \to \{0,1\}^B]\!] = \mathbb{1},$$

$$[\![|\varphi|^{\mathfrak{A}}(a{\uparrow}) = 1]\!] = [\![\Phi(a(0),\dots,a(n-1),\mathfrak{A})]\!] = \mathbb{1}$$

for all $a : n \to A{\downarrow}$. Instead of $|\varphi|^{\mathfrak{A}}(a{\uparrow})$ we will write $|\varphi|^{\mathfrak{A}}\ (a_0,\dots,a_{n-1})$, where $a_l := a(l)$. Therefore, the formula

$$\mathbb{V}^{(B)} \models \text{“}\varphi(a_0,\dots,a_{n-1}) \text{ is valid in } \mathfrak{A}\text{”}$$

holds true if and only if $[\![\Phi(a_0,\dots,a_{n-1},\mathfrak{A})]\!] = \mathbb{1}$.

(2) Let $\mathfrak{A}$ be an algebraic system of signature $\sigma^{\wedge}$ inside $\mathbb{V}^{(B)}$. Then $\mathfrak{A}{\downarrow}$ is a universally complete algebraic B-system of signature σ. In this event,

$$\chi \circ |\varphi|^{\mathfrak{A}\downarrow} = |\varphi|^{\mathfrak{A}}{\downarrow}$$

for each formula φ of signature σ.

(3) Let $\mathfrak{A}$ and $\mathfrak{B}$ be algebraic systems of the same signature $\sigma^{\wedge}$ inside $\mathbb{V}^{(B)}$. Put $\mathfrak{A}' := \mathfrak{A}{\downarrow}$ and $\mathfrak{B}' := \mathfrak{B}{\downarrow}$. Then, if h is a homomorphism (strong homomorphism) inside $\mathbb{V}^{(B)}$ from $\mathfrak{A}$ to $\mathfrak{B}$ then $h' := h{\downarrow}$ is a homomorphism (strong homomorphism) of the B-systems $\mathfrak{A}'$ and $\mathfrak{B}'$.

Conversely, if $h' : \mathfrak{A}' \to \mathfrak{B}'$ is a homomorphism (strong homomorphism) of algebraic B-systems then $h := h'\!\uparrow$ is a homomorphism (strong homomorphism) from $\mathfrak{A}$ to $\mathfrak{B}$ inside $\mathbb{V}^{(B)}$.

Let $\mathfrak{A} := (A, \nu)$ be an algebraic B-system of signature σ. Then there are $\mathscr{A}$ and $\mu \in \mathbb{V}^{(B)}$ such that the following are fulfilled:

(1) $\mathbb{V}^{(B)} \models$ *"$(\mathscr{A}, \mu)$ is an algebraic system of signature $\sigma^{\wedge}$";*

(2) *If $\mathfrak{A}' := (A', \nu')$ is the descent of $(\mathscr{A}, \mu)$ then $\mathfrak{A}'$ is a universally complete algebraic B-system of signature σ;*

(3) *There is an isomorphism $\imath$ from $\mathfrak{A}$ to $\mathfrak{A}'$ such that $A' = \mathrm{mix}(\imath(A))$;*

(4) *For every formula φ of signature σ in n free variables, the equalities hold*

$$|\varphi|^{\mathfrak{A}}(a_0, \dots, a_{n-1}) = |\varphi|^{\mathfrak{A}'}(\imath(a_0), \dots, \imath(a_{n-1}))$$
$$= \chi^{-1} \circ (|\varphi|^{\mathfrak{A}^{\sim}})\!\downarrow\!(\imath(a_0), \dots, \imath(a_{n-1}))$$

for all $a_0, \dots, a_{n-1} \in A$ and χ the same as above.

4. The reals descend into a K-space

We are now ready to apply the technique of Boolean valued analysis to the algebraic system of paramount importance for mathematics, the field of real numbers.

By the transfer and maximum principles, there is an element $\mathscr{R} \in \mathbb{V}^{(B)}$ such that $\mathbb{V}^{(B)} \models$ "$\mathscr{R}$ is an ordered field of the reals." It is obvious that inside $\mathbb{V}^{(B)}$ the field $\mathscr{R}$ is unique up to isomorphism; i.e., if $\mathscr{R}'$ is another field of the reals inside $\mathbb{V}^{(B)}$ then $\mathbb{V}^{(B)} \models$ "$\mathscr{R}$ and $\mathscr{R}'$ are isomorphic."

It is an easy matter to show that $\mathbb{R}^{\wedge}$ is an Archimedean ordered field inside $\mathbb{V}^{(B)}$ and so we may assume that $\mathbb{V}^{(B)} \models$ "$\mathbb{R}^{\wedge} \subset \mathscr{R}$ and $\mathscr{R}$ is the (metric) completion of $\mathbb{R}^{\wedge}$." Regarding the unity 1 of $\mathbb{R}$, notice that $\mathbb{V}^{(B)} \models$ "$1 := 1^{\wedge}$ is an order unit of $\mathscr{R}$."

Consider the descent $\mathscr{R}\!\downarrow$ of the algebraic system $\mathscr{R} := (|\mathscr{R}|, +, \cdot, 0, 1, \leq)$. By implication, we equip the descent of the underlying set of $\mathscr{R}$ with the descended operations and order of $\mathscr{R}$. In more detail, the addition, multiplication, and order on $\mathscr{R}\!\downarrow$ appear in accord with the following rules:

$$x + y = z \leftrightarrow [\![x + y = z]\!] = \mathbb{1}, \quad xy = z \leftrightarrow [\![xy = z]\!] = \mathbb{1},$$
$$x \leq y \leftrightarrow [\![x \leq y]\!] = \mathbb{1}, \quad \lambda x = y \leftrightarrow [\![\lambda^{\wedge} x = y]\!] = \mathbb{1}$$
$$(x, y, z \in \mathscr{R}\!\downarrow,\ \lambda \in \mathbb{R}).$$

Gordon Theorem. *Let $\mathscr{R}$ be the reals in $\mathbb{V}^{(B)}$. Assume further that $\mathscr{R}\!\downarrow$ stands for the descent $|\mathscr{R}|\!\downarrow$ of the underlying set of $\mathscr{R}$ equipped with the descended operations and order. Then the algebraic system $\mathscr{R}$ is a universally complete K-space.*

Moreover, there is a (canonical) isomorphism χ from the Boolean algebra B onto the base $H_{\mathscr{R}\downarrow}$ of $\mathscr{R}\!\downarrow$ such that the following hold:

$$\chi(b)x = \chi(b)y \leftrightarrow b \leq [\![x = y]\!],$$
$$\chi(b)x \leq \chi(b)y \leftrightarrow b \leq [\![x \leq y]\!]$$

for all $x, y \in \mathscr{R}\!\downarrow$ and $b \in B$.