Leonid Vital'evich Kantorovich
(1912–1986)
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NEW POSSIBILITIES OF K-SPACES

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L. V. Kantorovich considered the development of the foundation of the theory of ordered vector spaces among his most important achievements. Already in his pioneering works of the mid-1930s devoted to this subject, he singled out a class of spaces, which are now called *Kantorovich spaces*, or *K-spaces*. In modern language, K-spaces are order complete vector lattices. Ideas developed by Kantorovich and his students in the theory of K-spaces turned out to be extremely sagacious, very much ahead of their time. It is paradoxical that even now, almost half a century later, new deep relations are being revealed that establish the place of K-spaces in the same rank with mathematical objects of truly fundamental significance. This chapter is about some of the new possibilities of K-spaces related to the sphere of interests of students of Kantorovich working at the Institute of Mathematics of the Siberian Division of the USSR Academy of Sciences.

6.1 Boolean-valued analysis

Kantorovich always emphasized that the elements of an arbitrary K-space should be regarded as peculiar analogues of real numbers. Precisely related to this idea is the remarkable cycle of his works on carrying over the methods of functional analysis to the case of operators acting on general K-spaces. One of the summing-up works of this cycle turned out to be the classical monograph [1], which still retains its topicality.

The aforementioned idea of Kantorovich received a brilliant confirmation in connection with the method of Boolean-valued models of set theory developed by D. Scott, R. Solovay, and P. Wopenka during the memorable events caused by the fundamental result of P. Cohen on the classical continuum problem (see [2] and [3]).

Thus, let Y be a K-space and mY its maximal extension. We denote by B the base of Y or, equivalently, of mY (that is, the complete Boolean algebra

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of components of Y). We consider a separated Boolean-valued universum $V^{(B)}$ constructed over B. By the maximum principle, in $V^{(B)}$ there is an object R for which

$$[\![R \text{ is the } K\text{-space of real numbers}]\!] = \mathbb{1}$$

is true. Here $[\![\varphi]\!]$ is the evaluation of the truth of the formula φ of Zermelo-Fraenkel set theory, and $\mathbb{1}$ is the unity element of the Boolean algebra B. We denote by $R\downarrow$ the descent of R (as usual, we assume that R is the "carrier set" of the space of real numbers in the chosen model of set theory). Thus,

$$R\downarrow := \{t \in V^{(B)}: [\![t \in R]\!] = \mathbb{1}\}.$$

We effect the descent of structures from R into $R\downarrow$ according to the rules adopted in Boolean-valued analysis:

$$\begin{aligned} x + y = z &\leftrightarrow [\![x + y = z]\!] = \mathbb{1}, \\ x \leq y &\leftrightarrow [\![x \leq y]\!] = \mathbb{1}, \\ \lambda x = y &\leftrightarrow [\![\lambda\hat{}x = y]\!] = \mathbb{1}, \qquad (x,y,z \in R\downarrow,\ \lambda \in \mathbb{R}), \end{aligned}$$

where $\lambda\hat{}$ is the standard name of the number λ in $V^{(B)}$.

Theorem of Gordon [4]. The set $R\downarrow$ with descended structures is an extended K-space with base $B(R\downarrow)$ isomorphic to B. Such an isomorphism can be effected by identifying B with the descent of the field $\{0\hat{}, 1\hat{}\}$, that is, by the mapping ι: $B \to B(R\downarrow)$ determined by the mixing rule

$$[\![\iota(b) = 1\hat{}]\!] = b; \qquad [\![\iota(b) = 0\hat{}]\!] = b' \qquad (0, 1 \in \mathbb{R}).$$

Here, for any $x, y \in R\downarrow$ we have

$$\begin{aligned} &[\![\iota(b)x = \iota(b)y]\!] = b \Rightarrow [x = y], \\ &b\iota(b)x = bx, \qquad b'\iota(b)x = 0, \end{aligned}$$

where b' is the complementary element to b. In particular, the following equivalences hold:

$$\begin{aligned} \iota(b)x = \iota(b)y &\leftrightarrow [\![x = y]\!] \geq b, \\ \iota(b)x \geq \iota(b)y &\leftrightarrow [\![x > y]\!] \geq b. \end{aligned}$$

This theorem means that mY is indistinguishable from $R\downarrow$; that is, the elements of an abstract K-space depict the real numbers in a suitable model of set theory. A discussion of the problem of adapting the foregoing considerations to problems of functional analysis, as well as the literary notes, is given in [5]–[7].

6.2 Subdifferential calculus

The creative work of L. V. Kantorovich is characterized by extraordinary integrity. He always emphasized the unity of approaches to posing

mathematical and economical problems. These synthesizing tendencies found a certain reflection in the theory of subdifferentiation.

Let $f: X \to \mathbb{R}^{\cdot}$ be a convex function defined on a vector space X and acting to the semiextended line $\mathbb{R}^{\cdot} := \mathbb{R} \cup \{+\infty\}$. For $\varepsilon \geq 0$ we set

$$\partial^{\varepsilon} f(\bar{x}) := \{l \in X^{\#}: (\forall x \in X) l(x) - l(\bar{x}) \leq f(x) - f(\bar{x}) + \varepsilon\}.$$

The set $\partial^{\varepsilon} f(\bar{x})$ lying in the algebraically conjugate space $X^{\#}$ is called the *ε-subdifferential* of f at the point $\bar{x}$. The importance of ε-subdifferentials arrests our attention, because $0 \in \partial^{\varepsilon} f(\bar{x})$ if and only if $\bar{x}$ is an ε-solution of the extremal problem $f(x) \to \inf$; that is, $f(\bar{x}) \leq \inf f(X) + \varepsilon$. Of course, the notion of ε-subdifferential is useless if the rules for computing the ε-subdifferentials for composite functions are unknown. Here the formulation of the problem of finding the ε-subdifferential of a composition in itself requires a study of vector-valued analogues of convex functions, that is, of convex operators. The notion of a convex operator itself is naturally comprehended in the framework of the theory of ordered vector spaces.

There is yet another immanent connection of the theory of extremal problems with convexity and hence with order in vector spaces. In optimization one is often interested in the value of the problem $f(x) \to \inf$, that is, the quantity $\inf f(X)$. The question arises: What is the behavior of the values under a change of variable? In the theoretical analysis of this question, an assumption is adopted that the problem $f(x) \to \inf$ is as complicated as any of the problems $f(x) - l(x) \to \inf$, where $l \in X^{\#}$. In other words, when we encounter the problem of minimizing f, we assume that the Young–Fenchel transformation f^* of the function f is known—that is, the mapping associating to $l \in X^{\#}$ the quantity

$$f^*(l) := \sup_{x \in X} (l(x) - f(x)).$$

Since f^* is convex, posing the question of change of variables in the Young–Fenchel transformation leads immediately to the necessity of studying its analogues for mappings acting on K-spaces.

We note the useful equivalence

$$l \in \partial^{\varepsilon} f(x) \leftrightarrow f^*(l) + f(\bar{x}) \leq l(\bar{x}) + \varepsilon,$$

which shows the interrelation between the problem of subdifferentiation and the problem of finding the Young–Fenchel transformations.

By now practically all the main formulas of the calculus have been found. One can familiarize himself with the status of the problem from [7]–[11]. We note here that the central role in subdifferential calculus is played by the Hahn–Banach–Kantorovich theorem discovered by Kantorovich at the dawn of K-spaces. In modern form this theorem can be written as the following particular formula of subdifferentiation:

$$\partial(P \circ \iota) = \partial P \circ \iota,$$

where $P: X \to Y$ is a sublinear operator acting from the vector space X to

the K-space Y, whereas $\iota: X_0 \to X$ is the identity embedding of the subspace X_0 in the space X, and $\partial P := \partial P(0)$ is the subdifferential at zero or a supporting set of P, that is, the collection of linear operators A from X to Y such that $Ax \leq Px$ for every $x \in X$.

Note that in their turn K-spaces are practically the only objects for which nearly any subdifferentiation formula is valid in the mass formulation. Transition to objects more general than vector spaces—to modules—actually does not take us out of the class of K-spaces.

6.3 Infinitesimal programming

Kantorovich promoted the role of functional analysis in applied mathematics, emphasizing the possibilities thus arising of obtaining not only quantitative but also qualitative results. In the last years of his life he was considerate toward and supported research on the application of nonstandard methods in analysis, in particular "neoclassical" and "radical" ideas related to the theory of internal sets of E. Nelson and the theories of external sets of K. Hrbacek and T. Kawai (see [12]). We shall dwell here on a recent idea in this direction related to convex programming. It is no secret that the theory of ε-subdifferentials is, in a certain sense, too subtle from the point of view of those who perform real calculations. Qualitatively it appears that by checking an optimality criterion with the "utmost available accuracy" we arrive at a "solution with utmost accuracy." However, the ε-subdifferentiation formulas require recalculation of accuracy estimates. For example,

$$\partial^{\varepsilon}(f_1 + f_2)(\bar{x}) = \bigcup_{\substack{\varepsilon_1 \geq 0,\ \varepsilon_2 \geq 0 \\ \varepsilon_1 + \varepsilon_2 = \varepsilon}} \partial^{\varepsilon_1} f_1(\bar{x}) + \partial^{\varepsilon_2} f_2(\bar{x}).$$

The rules for calculating ε-subdifferentials of more complicated mappings are even more intricate, which leads to unmanageable growth of the number of corrections in optimality criteria.

In the light of ideas of nonstandard analysis, it appears to be expedient to study the infinitesimal subdifferential (of an internal function)

$$Df(\bar{x}) := \bigcup_{\substack{\varepsilon \geq 0 \\ \varepsilon \approx 0}} \partial^{\varepsilon} f(\bar{x}),$$

where $\varepsilon \approx 0$ means that the parameter ε is infinitesimal. In other words, $\varepsilon \approx 0$ holds for a positive ε if and only if ε is less than any "determinable" standard ε'. To put it differently,

$$Df(\bar{x}) = \bigcap_{\substack{\varepsilon' > 0 \\ \varepsilon' \text{ is standard}}} \partial^{\varepsilon'} f(\bar{x}).$$

Note that an infinitesimal subdifferential is, as a rule, an external set. It is clear that $0 \in Df(\bar{x})$ if and only if $\bar{x}$ is an infinitesimal solution (i.e., such that $f(\bar{x}) \approx \inf f(X)$).

It is easy to understand, for example, that

$$D(f_1 + f_2)(\bar{x}) = Df_1(\bar{x}) + Df_2(\bar{x}).$$

And indeed the complementary slackness approximately satisfied serves as a necessary and sufficient condition of infinitesimal optimality in convex programs [13].

There exist fascinating connections between the methods of K-spaces and nonstandard methods of analysis (in the Boolean-valued and the infinitesimal variants [14]). Now all these new possibilities will be investigated without attention and participation of the unforgettable Leonid Vital'evich Kantorovich.

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