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ON THE CALCULUS OF ORDER BOUNDED OPERATORS

НОВОСИБИРСК

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In memory of Yuri Abramovich

Annotation. This article deals with the Abramovich calculus of order bounded operators.

1. *Heuristic Preliminaries*

1.1. We start with a few well-known definitions and facts. Let E and F be vector lattices. The set of order bounded linear operators from E to F is denoted by $L^\sim(E,F)$. The order on $L^\sim(E,F)$ is introduced by the positive cone $L^+(E,F)$ comprising all $T \in L^\sim(E,F)$ such that $T(E_+) \subset F_+$. Throughout E_+ stands for the positive cone of E. The celebrated Riesz–Kantorovich theorem asserts that if F is a Dedekind complete vector lattice then so is $L^\sim(E,F)$. Moreover, several convenient formulas are available for calculating the lattice operations in $L^\sim(E,F)$. For instance, given $x \in E_+$ and $S,T \in L^\sim(E,F)$ we see

$$(S \vee T)x = \sup\{Sy + Tz : \ y,z \in E_+;\ x = y + z\}; \tag{1}$$

$$(S \wedge T)x = \inf\{Sy + Tz : \ y,z \in E_+;\ x = x_1 + x_2\}. \tag{2}$$

Analogous explicit expressions exist for finite and infinite lattice operations as well as for the modulus, positive and negative parts of an order bounded operator. There are bulkier versions of these formulas with suprema over upward directed sets and infima over downward directed sets which are sometimes essential in applications. The collection of these and all similar formulas is usually referred to as the *calculus of order bounded operator* or, in short, *order calculus* (cp. [4, 12]).

1.2. Yuri Abramovich outlined in [1] an instance of order calculus in which the suprema and infima are taken over disjoint decompositions of an argument (for example, in (1) and (2) we decompose $x \in E_+$ into y and z such that $y \wedge z = 0$). This collection of formulas we call the *Abramovich calculus.* W. Luxemburg and A. Zaanen independently found such a formula for the modulus of an order bounded operator in [18] (cp. [4]).

Consider two positive operators $S, T : E \to F$. Recall that S is a *component* or *fragment* of T if $S \wedge (T - S) = 0$. The set $\mathfrak{C}(T)$ of all fragments of T is a complete Boolean algebra. If E is a vector lattice with the projection property and $\mathfrak{P}(E)$ stands for the Boolean algebra of band projections in E then the versions of (1) and (2) in the Abramovich calculus may be written with the fragments $S \circ \pi$ and $T \circ \pi^{\perp}$ where π is a band projection in E and $\pi^{\perp} := I_E - \pi$:

$$(S \vee T)x = \sup\{S \circ \pi x + T \circ \pi^{\perp} x : \ \pi \in \mathfrak{P}(E)\}; \tag{3}$$

$$(S \wedge T)x = \inf\{S \circ \pi x + T \circ \pi^{\perp} x : \ \pi \in \mathfrak{P}(E)\}. \tag{4}$$

The supremum and infimum in (3) and (4) may be approximated by the elements of the shape $\rho_\xi(S \circ \pi_\xi x + T \circ \pi_\xi^{\perp} x)$, with (ρ_ξ) a family of disjoint band projections in F. We thus conclude that to validate order calculus it suffices to deal only with the simplest fragments.

1.3. Therefore, the "simplest fragments" or "elementary components" carry all information about an operator which is needed for the validity of the formulas of order calculus and, consequently, about all properties of an operator which may be obtained by using order calculus. Since the calculus of order bounded operators is one of the main tools for studying order bounded operators and since the Freudenthal spectral theorem guarantees that all operators in the principal order ideal generated by a given operator are uniformly approximated by the operators composed of the same fragments, we are in a position to state the following heuristic principle stemming from Yuri Abramovich:

1.3.1. *If the elementary fragments of a positive operator possess some property then this property or a duly modification of this property belongs to every operator in the principal order ideal generated by the original operator.*

The idea behind this principle is far from new. C. D. Aliprantis and O. Burkinshaw wrote in their book: "... the elementary components are the 'building blocks' for all components, and to a larger extent the building blocks for all positive operators in the ideal generated by T" (cp. [4; p.74]. Moreover, many tools corroborating and specifying the above heuristic principle were developed in [4]. Below we will outline an improved version of this machinery.

All prerequisites, terminology, and notation are taken from the books by C. D. Aliprantis and O. Burkinshaw [4] and A. G. Kusraev [12].

2. *Building Blocks*

We must firstly find out what should be meant by the "building blocks" or "elementary components" of an operator. On the one hand, these should be simple modulo the operator under study. On the other hand, the set of these blocks should be sufficiently large so that every fragment of the operator is reconstructible from them.

2.1. The various approaches to this problem are unified by the following notion. A set of band projections $\mathscr{P} \subset \mathfrak{P}(L^{\sim}(E,F))$ is said to be *generating* if for all $T \in L^{+}(E,F)$ and for $x \in E$ we have

$$Tx^{+} = \sup\{pTx : p \in \mathscr{P}\}.$$

Below we exhibit two easy examples.

2.2. Let $E(e)$ and $\mathfrak{P}(E)$ denote the order ideal in E generating by $e \in E_{+}$ and the Boolean algebra of all band projections in E. Recall that a vector lattice E is said to possesses the *strong Freudenthal property* if every element $x \in E(e)$ can be e-uniformly approximated by linear combinations $\sum_{k=1}^{n} \lambda_k \pi_k e$ with $\lambda_1, \dots \lambda_n \in \mathbb{R}$ and $\pi_1, \dots \pi_n \in \mathfrak{P}(E)$.

To each band projection $\pi \in \mathfrak{P}(E)$ we assign the band projection $\hat{\pi} : T \mapsto T \circ \pi$ in $L^{\sim}(E,F)$ and denoted by $\mathscr{P}^{\circ}$ the set that comprises all these band projections $\hat{\pi}$.

2.2.1. *If a vector lattice E has the strong Freudenthal property then $\mathscr{P}^{\circ}$ is a generating set of band projections in $L^{\sim}(E,F)$.*

Proof. It can be shown [17] that a vector lattice E possesses the strong Freudenthal property if and only if every two disjoint elements in E are completely disjoint, i.e. belong to two disjoint projection bands of E. Thus, for every $x \in E$ there is a band projection π with $x^{+} = \pi x$. Therefore, $T(x^{+}) = \hat{\pi} T x$ for all $T \in L^{\sim}(E,F)$. □

2.3. Given an element $e \in E_{+}$, define the linear operator π_e in $L^{\sim}(E,F)$ by

$$\pi_e T x = \sup_n T(ne \wedge x) \quad (x \in E^{+},\, T \in L^{+}(E,F));$$
$$\pi_e T x = \pi_e T x^{+} - \pi_e T x^{-} \quad (x \in E,\, T \in L^{+}(E,F));$$
$$\pi_e T = \pi_e T^{+} - \pi_e T^{-} \quad (T \in L^{\sim}(E,F)).$$

Then the operator π_e is a band projection in $L^{\sim}(E,F)$ which agrees with T on $E(e)$ and vanishes on $E(e)^{\perp}$.

2.3.1. *Let $\mathscr{P}^{\pi} := \{\pi_e : e \in E_{+}\}$. Then $\mathscr{P}^{\pi}$ is a generating set of band projections in $L^{\sim}(E,F)$.*

Proof. Indeed, if $e := x^{+}$ then $\pi_e T x^{+} = T x^{+}$ and $\pi_e T x^{-} = 0$. Therefore $\pi_e T x = \pi_e T x^{+} - \pi_e T x^{-} = T x^{+}$. □

2.4. Theorem. *Let E and F be vector lattices with F Dedekind complete. A set $\mathscr{P}$ of band projections in $L^{\sim}(E,F)$ is generating if and only if for all $S, T \in L^{\sim}(E,F)$ and $x \in E^{+}$ the following hold:*

$$(T \wedge S)x = \inf\{pTx + p^{\perp}Sx :\, p \in \mathscr{P}\};$$
$$(T \vee S)x = \sup\{pTx + p^{\perp}Sx :\, p \in \mathscr{P}\}.$$

Proof. Let $\mathscr{P}$ be a generating set of band projections. It suffices to show the equality $\inf\{pTx + p^{\perp}Sx\} = 0$ provided that $T \wedge S = 0$. Observe firstly that

$$
\begin{gathered}
Vx^{+} - \sup\{pVx : p \in \mathscr{P}\} = Vx^{+} \\
+ \inf\{-pVx^{+} + pVx^{-} : p \in \mathscr{P}\} = \inf\{p^{\perp}Vx^{+} + pVx^{-} : p \in \mathscr{P}\}.
\end{gathered}
$$

Thus, the identities $\inf\{pVx^{+} + p^{\perp}Vx^{-} : p \in \mathscr{P}\} = 0$ and $Vx^{+} = \sup\{pVx : p \in \mathscr{P}\}$ are equivalent for all $V \in L^{+}(E,F)$ and $x \in E$. Note now that

$$
\begin{gathered}
pTx = p(T(2y) + T(x - 2y)) \leq 2Ty + pT(x - 2y)^{-}; \\
p^{\perp}Sx = p^{\perp}(S(2x - 2y) + S(2y - x)) \leq 2S(x - y) + p^{\perp}S(2y - x)^{+}
\end{gathered}
$$

for $p \in \mathscr{P}$ and $0 \leq y \leq x$, $x \in E_{+}$. Adding these inequalities and taking the infimum, deduce

$$
\begin{gathered}
\inf_{p}\{pTx + p^{\perp}Sx\} \leq 2\inf_{y}\{Ty + S(x - y)\} \\
+ \inf_{p}\{p(T + S)(2y - x)^{-} + p^{\perp}(T + S)(2y - x)^{+}\} = 0,
\end{gathered}
$$

which yields sufficiency.

To prove necessity, take an operator $V \in L^{+}(E,F)$. Given $x \in E$, put $S := \pi V$ and $T := \pi^{\perp}V$, where $\pi := \pi_e$ and $e := x^{+}$ (cp. [2]). Then S and T are disjoint positive operators satisfying $Sx = Vx^{+}$ and $Tx = Vx^{-}$. It follows that

$$
\begin{aligned}
0 &= \inf\{p^{\perp}Sx + pTx : p \in \mathscr{P}\} = \inf\{p^{\perp}\pi Vx + p\pi^{\perp}Vx : p \in \mathscr{P}\} \\
&= \inf\{p^{\perp}Vx^{+} + pVx^{-} : p \in \mathscr{P}\},
\end{aligned}
$$

which shows that $\mathscr{P}$ is a generating set. $\square$

2.5. As an immediate corollary to what was proven we point out the following:

2.5.1. *If E has the strong Freudenthal property then the Abramovich calculus is valid in $L^{\sim}(E,F)$.*

2.6. Let G be an order ideal in E. Given $T \in L^{\sim}(E,F)$, put

$$
\begin{gathered}
\pi_G Tx := \sup\{T(\pi(g \wedge x)) : g \in G\} \quad (x \in E_{+}); \\
\pi_G Tx := \pi_G T(x^{+}) - \pi_G T(x^{-}) \quad (x \in E).
\end{gathered}
$$

Then π_G is a band projection in $L^{\sim}(E,F)$; moreover, $\pi_G(T)$ agrees with T on G and vanishes on $G^{\perp}$. Clearly, $\pi_e = \pi_G$ if $G = E(e)$. The operators of the form $\pi_G S$, with G an order ideal in E, have long history (for instance, see [4; theorem 6.7], [20; proposition 4.8], [22; lemma 5.12]), and [23; theorem IX.5.11]. However, they appeared as an explicit object of research in [21]. E. V. Kolesnikov suggested to use fragments of the form $\pi_G S$ and $\pi_e S$ instead of $S \circ \pi$ in [2, 9, 10].

2.7. The concept of a generating set of band projections as well as theorem 2.4 belongs to S. S. Kutateladze [16]. The main idea of [16] is as follows: The fragments of a positive operator S are the extreme points of the order interval $[0, S]$ which coincides with the subdifferential (at zero) ∂p of the sublinear operator $p(x) := S(x^+)$. Thereby, studying the fragments of a positive operator reduces to describing the extremal structure of this subdifferential. The latter was done for an abstract sublinear operator p by S. S. Kutateladze in [15] (for a detailed exposition, see [13]).

3. *Travelling Up and Down*

We now address the problem of approximating an arbitrary fragment by the elementary fragments. As above, E and F are vector lattices. Moreover, F is Dedekind complete.

3.1. Given a set M in a lattice Z, denote by $M^{\downarrow}$ the set of all elements $z \in Z$, of the form $z = \inf z_\alpha$, where $(z_\alpha) \subset M$ is a downward directed net. The set $M^{\uparrow}$ is defined similarly on using upward directed nets. If we take sequences instead of nets in these definitions then the corresponding sets are denoted by $M^{\downharpoonleft}$ and $M^{\upharpoonleft}$. More precisely, $M^{\downharpoonleft}$ is the set of all $y = \inf z_n$ in Z, where $(z_n) \subset M$ is a decreasing sequence. Finally, we put $M^{\downarrow\uparrow} := (M^{\downarrow})^{\uparrow}$, $M^{\downharpoonleft\upharpoonleft} := (M^{\downharpoonleft})^{\upharpoonleft}$, etc.

3.2. Let $\mathscr{P}$ be some set of band projections in $L^{\sim}(E, F)$. Given an operator $T \in L^{+}(E, F)$, define the set of *elementary fragments* $\mathscr{P}^{\vee}(T)$ of T by the formula

$$\mathscr{P}^{\vee}(T) = \left\{ \sum_{k=1}^{n} \pi_k p_k T : \ p_1, \dots, p_n \in \mathscr{P};\ \pi_1, \dots, \pi_n \in \mathfrak{P}(F); \pi_k \circ \pi_l = 0\,(k \neq l) \right\}.$$

Clearly, it is possible to replace summation with the taking of the supremum and eliminate the condition for the band projections π_k to be disjoint in this definition. Put $\mathscr{C}(T) := \mathscr{P}^{\vee}(T)$ provided that $\mathscr{P} := \mathscr{P}^{\circ}$ (cp. 2.2); and $\mathscr{S}(T) := \mathscr{P}^{\vee}(T)$ provided that $\mathscr{P} := \mathscr{P}^{\pi}$ (cp. 2.3).

3.3. Theorem. *A set $\mathscr{P}$ of band projections in $L^{\sim}(E, F)$ is generating if and only if for every positive operator $T : E \to F$ the following holds*

$$\mathfrak{E}(T) = \mathscr{P}^{\vee}(T)^{\uparrow\downharpoonleft\uparrow}.$$

Proof. Necessity results on using the principal scheme by C. D. Aliprantis and O. Burkinshaw [3] together with the modifications by A. G. Kusraev and V. Z. Strizhevskiĭ [14].

Consider the set $\mathscr{P}_0^{\vee}(T)$ of all sums $o\text{-}\sum_{\alpha \in \mathrm{A}} \pi_\alpha p_\alpha T$, where $(p_\alpha) \subset \mathscr{P}$, $(\pi_\alpha) \subset \mathfrak{P}(F)$, the elements π_α are disjoint, and $o\text{-}\sum_{\alpha \in \mathrm{A}} \pi_\alpha = I_E$. It is easy to see that $\mathscr{P}_0^{\vee}(T) \subset \mathscr{P}^{\vee}(T)^{\uparrow}$.

Let S be an arbitrary fragment of T. Proceeding along the lines of [3] and [14], we infer

(1) for all $0 < \varepsilon \in \mathbb{R}$ and $x \in E_+$ there is an element $S' \in \mathscr{P}_0^{\vee}(T)$ satisfying $|S' - S|x \leq \varepsilon Tx$;
(2) for all $x \in E_+$ there are some $S_1 \in \mathscr{P}_0^{\vee}(T)^{\downarrow\uparrow}$ and $S_2 \in \mathscr{P}_0^{\vee}(T)^{\uparrow\downarrow}$ such that $|S_k - S|x = 0$, $k = 1, 2$;
(3) for all $x \in E_+$ there are some $S_1 \in \mathscr{P}_0^{\vee}(T)^{\downarrow\uparrow}$ and $S_2 \in \mathscr{P}_0^{\vee}(T)^{\uparrow\downarrow}$ such that $S_1 \geq S \geq S_2$ and $(S_1 - S_2)x = 0$.

The proof is complete on applying (3) to $\pi_x S$ and noting that $S = \sup_{x \in E_+} \pi_x S$. □

3.4. In 3.3 every fragment of a positive operator is obtained from some simpler fragments of the latter by the up and down procedures. Similar assertions are sometimes referred to as *up-down theorems*. A general up-down theorem 3.3 was obtained by S. S. Kutateladze [16] on using some nonstandard (infinitesimal and Boolean-valued) tools. If we insert $\mathscr{P} := \mathscr{P}^{\circ}$ in 3.3 (cp. 2.2) then we arrive at the formula

$$\mathfrak{E}(T) = \mathscr{C}(T)^{\uparrow\downarrow\uparrow}.$$

This was the first up-down theorem established by B. de Pagter [19] with the two essential constraints: it was assumed that F admits a total set of o-continuous functionals and E is Dedekind complete or at least possesses the principal projection property. The first constraint was eliminated by A. G. Kusraev and V. Z. Strizhevskiĭ [14] by replacing approximation in the absolute weak topology $|\sigma|(F, F_n^{\sim})$ with piecewise uniform approximation.

3.5. The second constraint was eliminated by G. P. Akilov, E. V. Kolesnikov, and A. G. Kusraev [2]. Of course, in this case the set of simple fragments $\mathscr{P} := \mathscr{P}^{\pi}$ is essentially different and the corresponding formula takes the shape

$$\mathfrak{E}(T) = \mathscr{S}(T)^{\downarrow\uparrow}.$$

Proof. Applying 3.4 to the operator $\pi_x S$, with $x \in E_+$, find some operators $S'_x \in \mathscr{P}_0^{\vee}(T)^{\downarrow\uparrow}$ and $S_x \in \mathscr{P}_0^{\vee}(T)^{\uparrow\downarrow}$ such that $S'_x \geq \pi_x S \geq S_x$ and $(S'_x - S_x)x = 0$. Therefore, $0 = \pi_x(S'_x - S_x) = \pi_x S'_x - S_x$ and $\pi_x S'_x = S_x = \pi_x S$. Moreover, on assuming that $\mathscr{P}_0^{\vee}(\pi_x T) \subset \mathscr{P}_0^{\vee}(T)^{\downarrow}$ we may deduce

$$S = \sup_{x \geq 0} \pi_x S'_x \in (\mathscr{P}_0^{\vee}(\pi_x T)^{\downarrow\uparrow})^{\uparrow} \subset (\mathscr{P}^{\vee}(T)^{\downarrow})^{\downarrow\uparrow\uparrow} = \mathscr{P}^{\vee}(T)^{\downarrow\uparrow},$$

as required.

The above assumption is verifies as follows: Take a family of disjoint band projections $(\rho_\alpha) \subset \mathfrak{P}(F)$ and a family of elements $(e_\alpha) \subset E_+$. Consider the band projection

$$\pi := o\text{-}\sum_{\alpha \in \mathrm{A}} \rho_\alpha p_\alpha \pi_x T \in \mathscr{P}_0^{\vee}(\pi_x T)$$

with $p_\alpha := \pi_{e_\alpha}$ and $o\text{-}\sum_\alpha \rho_\alpha = I_F$. Let Θ stand for the family of all finite subsets of A and take $\theta \in \Theta$. Put $\sigma_\theta := I_F - \sum_{\alpha \in \theta} \rho_\alpha$. Since $p_\alpha \circ \pi_x \leq \pi_x$; therefore,

$$\pi = \inf_{\theta \in \Theta} \left\{ \sum_{\alpha \in \theta} \rho_\alpha p_\alpha \pi_x T + \sigma_\theta \pi_x T \right\}$$

which yields the claim. □

4. *The Method in Action*

4.1. Consider a positive operator $T \in L^{\sim}(E,F)$ and let $\mathscr{I}(T)$ stand for the order ideal generated by T. Assume that $\Phi(\cdot)$ is some property of T and we are interested in knowing whether $\Phi(S)$ holds for every $S \in \mathscr{I}(T)$. The above prompts the following approach to this problem:

(1) Choose some set of band projections $\mathscr{P} \subset \mathfrak{P}(L^{\sim}(E,F))$ and show that $\mathscr{P}$ is a generating set;

(2) Demonstate that $\Phi(S)$ with $S := \sum_{k=1}^{n} \pi_k p_k T$ holds for every choice of $p_1,\dots,p_n \in \mathscr{P}$ and disjoint band projections $\pi_1,\dots,\pi_n \in \mathfrak{P}(F)$.

We are then establish that the property Φ is preserved under the up-down procedure, i.e., the following are valid:

(3) If $(S_\alpha)_{\alpha\in\mathrm{A}}$ is an increasing (decreasing) net in $L^{+}(E,F)$ and $S = \sup_{\alpha\in\mathrm{A}} S_\alpha$ ($S = \inf_{\alpha\in\mathrm{A}} S_\alpha$) and $\Phi(S_\alpha)$ is true for every $\alpha \in \mathrm{A}$ then so is $\Phi(S)$;

(4) If a net $(S_\alpha)_{\alpha\in\mathrm{A}}$ in $L^{\sim}(E,F)$ converges relatively uniformly to S and $\Phi(S_\alpha)$ is true for every $\alpha \in \mathrm{A}$ then so is $\Phi(S)$.

We provide two example by way of illustration. The first is the celebrated Dodds–Fremlin theorem, and the second is a very important although less-known decomposability property of the norm of a dominated operator.

4.2. Recall that a vector lattice E is a *Banach lattice* if E is a Banach space with monotone norm, i.e., $|x| \le |y|$ implies that $\|x\| \le \|y\|$. The norm of a Banach lattice is *order continuous* provided that $\|x_\alpha\| \to 0$ as $x_\alpha \searrow 0$.

We now state the Dodds–Fremlin theorem [5] whose enthralling history is outlined in [4; p. 279].

Theorem. *Let E and F be Banach lattices with E' and F having order continuous norms. If $S,T \in L^{\sim}(E,F)$, $0 \le S \le T$, and T is a compact operator then S is also a compact operator.*

Proof. We give the scheme of demonstration which was suggested by E. V. Kolesnikov [11]. Let $T \in L^{\sim}(E,F)$ be a compact operator.

To begin with, show that all operators $\pi_e(T)$, $e \in E_+$, are compact. Put $B_+ := \{x \in E_+ : \|x\| \le 1\}$ and take a finite set $x_1,\dots,x_n \in B_+$. If $W := \{Tx_1,\dots,Tx_n\}$ is an ε-net for $T(B_+)$ then W is also an ε-net for $(\pi_e T)(B_+)$. Indeed, given $x \in B_+$, note that $(\pi_e T)(x) = \lim_{k\to\infty} T(x \wedge ke)$. Hence, $\|(\pi_e T)(x) - T(x \wedge k_0 e)\| \le \varepsilon/2$ for an appropriate $k_0 \in \mathbb{N}$. On the other hand $\|T(x_j) - T(x \wedge k_0 e)\| \le \varepsilon/2$ for some x_j since $x \wedge k_0 e \in B_+$. Therefore, $\|(\pi_e T)(x) - T(x_j)\| \le \varepsilon$. It is obvious now that the fragments in $\mathscr{S}(T)$ are all compact.

From the order continuity of the norms of E' and F it is easy that the operator norm is order continuous on the ideal generated by T; i.e., if $T_\alpha \le T$ and $T_\alpha \searrow 0$ then $\|T_\alpha\| \to 0$. Indeed, put $u := T(x_1 + \dots + x_n)$, where $\{Tx_1,\dots,Tx_n\}$ is the above ε-net for $T(B_+)$. Then $\|(T_\alpha x - u)^+\| < \varepsilon$ for all $x \in B_+$ by [4; Lemma 16.12]. Proceed with using the order continuity of the norm of F. By [4; Theorem 12.14] there exists a positive linear functional $f \in F'$ that is strictly positive on $[0,u]$, and by [4; Lemma

12.15] there exists $\delta > 0$ such that $\|y\| < \varepsilon$ for every $y \in [-u, u]$ with $f(|y|) < \delta$. Since the norm of E' is order continuous, the net $(f \circ T_\alpha) \subset E'$ vanishes in the norm; therefore, $\sup\{f(T_{\alpha_0}x) : x \in B_+\} < \delta$ for some α_0. Consequently, for this α_0 an all $x \in B_+$ we have

$$\|T_{\alpha_0}x\| \leq \|(T_{\alpha_0}x - u)^+\| + \|u \wedge T_{\alpha_0}x\| < 2\varepsilon.$$

Thus, all operators in $\mathscr{S}(T)^{\downarrow\uparrow}$ are compact too. We are done on noting that each operator $0 \leq S \leq T$ is the relatively uniform limit (and hence the limit in the operator norm) of some net of elements in the linear span of $\mathfrak{E}(T) = \mathscr{S}(T)^{\downarrow\uparrow}$. $\square$

4.3. Consider a vector space X and a real vector lattice E. All vector lattices will be assumed Archimedean. A mapping $|\cdot| : X \to E_+$ is a *vector* (*E-valued*) *norm* provided that $|\cdot|$ satisfies the following axioms:

$$|x| = 0 \Longleftrightarrow x = 0 \quad (x \in X);$$
$$|\lambda x| = |\lambda||x| \quad (\lambda \in \mathbb{R},\, x \in X);$$
$$|x + y| \leq |x| + |y| \quad (x, y \in X).$$

A vector norm is called a *decomposable* or *Kantorovich norm* provided that for all $e_1, e_2 \in E_+$ and $x \in X$ the following holds:

$$|x| = e_1 + e_2 \Longrightarrow (\exists\, x_1, x_2 \in X)\,(x = x_1 + x_2,\ |x_1| = e_1,\ |x_2| = e_2).$$

In the case when the last implication is valid only for the disjoint $e_1, e_2 \in E_+$, we call the norm *disjointly-decomposable* or, in short, *d-decomposable*.

A triple $(X, |\cdot|, E)$ (in brief (X, E), $(X, |\cdot|)$, or X with the default parameters omitted) is a *lattice-normed space* over E provided that $|\cdot|$ is an E-valued norm in the vector space X.

A lattice-normed space X is *bo-complete* provided that to every net $(x_\alpha) \subset X$ with $o\text{-}\lim_{\alpha,\beta}|x_\alpha - x_\beta| = 0$ there is an element $x \in X$ such that $o\text{-}\lim_\alpha|x_\alpha - x| = 0$. A decomposable *bo*-complete lattice-normed space is a *Banach–Kantorovich space*.

4.4. Consider two lattice-normed spaces (X, E) and (Y, F), a linear operator $T : X \to Y$, and a positive operator $S : E \to F$. If

$$|Tx| \leq S\big(|x|\big) \quad (x \in X)$$

then we say that S *dominates* or *majorizes* T and call S a *dominant* or *majorant* of T. In this situation, T is called a *dominated* or *majorizable* operator. Let $\operatorname{maj}(T)$ be the set of all dominants of T. It is clear that $\operatorname{maj}(T)$ is a convex set in the Dedekind complete vector lattice $L^\sim(E, F)$. If there is a least element in $\operatorname{maj}(T)$ with respect to the order induced from $L^\sim(E, F)$ then this unique element $|T|$ is called the *least* or *exact dominant* of T. Consequently, $|T|$ is a positive operator from E to F; moreover, $|T| = \inf \operatorname{maj}(T) \in \operatorname{maj}(T)$ and the inequality

$$|Tx| \leq |T|\big(|x|\big) \quad (x \in X)$$

holds. The set of all dominated operators from X into Y is denoted by $M(X, Y)$. Thus,

$$T \in M(X, Y) \iff \operatorname{maj}(T) \neq \varnothing.$$

4.5. Theorem. *Let X be a decomposable lattice-normed space and let Y be a Banach–Kantorovich space. Then the space of dominated operators $M(X,Y)$ is also a Banach–Kantorovich space.*

Proof. Show firstly that if $e \in E_+$ and $T \in M(X,Y)$ then there exists a unique operator $\pi_e T \in M(X,Y)$ such that $|\pi_e T| = \pi_e |T|$ and $|T - \pi_e T| = |T| - \pi_e |T|$. Moreover, for each $x \in X$, the following hold:

$$(\pi_e T)x = bo\text{-}\lim\{Tx_n : \ |x - x_n| = |x| - |x_n|,\ |x_n| = |x| \wedge (ne)\}.$$

Next, apply the up-down procedure (cp. [12] for details). □

4.6. The above method works in a more general setting. Let p be some sublinear operator from a vector space X to a Dedekind complete vector lattice E. Considering the subdifferential $\partial p := \{T \in L(X,E) : (\forall x \in X)\ (Tx \le p(x))\}$, distinguish $\mathscr{E}_0(p)$, the set of o-extreme points of ∂p (cp. [13, 16]). Recall that ∂p may be reconstructed from $\mathscr{E}_0(p)$. Namely, we have the representation $\partial p = \mathrm{cl}\,(\mathrm{mix}\,(\mathrm{co}\,(\mathscr{E}_0(p)))$, where $\mathrm{co}\,(\mathscr{U})$ stands for the convex hull of $\mathscr{U}$; $\mathrm{mix}(\mathscr{U})$ denotes the set of all *mixings* of $\mathscr{U}$, i.e., the set of all operators of the shape $Sx := o\text{-}\sum_\xi \pi_\xi S_\xi(x)$ $(x \in X)$, with $(S_\xi) \subset \mathscr{U}$ and (π_ξ) a family of disjoint band projections in E whose sum equals the identity operator I_E. The set $\mathrm{cl}\,(\mathscr{U})$ consists of the operators $S \in L(X,E)$ represented as the pointwise o-limit $Sx := o\text{-}\lim_\alpha S_\alpha x$ $(x \in X)$ of some net $(S_\alpha) \subset \mathscr{U}$. Thus, the property Φ is valid for every operator in ∂p provided that Φ holds for all o-extreme points of ∂p and, moreover, Φ is preserved under the taking of convex hulls, mixings, and pointwise o-limits. Thus, paraphrasing C. D. Aliprantis and O. Burkinshaw, we may observe that the o-extreme points are the "building blocks" of all extreme points, and to a larger extent the building blocks for all operators in the subdifferential of the sublinear operator under study.

4.7. The concepts of lattice-normed space and dominated operator were introduced for the first time by L. V. Kantorovich in 1936 [7]. These notions were motivated by the two reasons: the request of the general theory of operations in "semiordered" spaces and the request of the so-called "approximate methods of analysis." The "bizarre" decomposability axiom of a lattice-valued norm was firstly formulated in [7].

Since the second half of the 1930s it has been well known (cp. [6–8]) that the space of dominated operators $M(X,Y)$ is a bo-complete lattice-normed space if X is decomposable and Y is bo-complete. However, the question of decomposability remained open for a long time. The decomposability of $M(X,Y)$ was established by A. G. Kusraev and V. Z. Strizhevskiĭ [14] in 1987.

5. *Concluding Remarks*

We hope that the following remarks nontypical of a research article are justified by the extraordinary reasons behind this special issue of the journal.

5.1. Yuri Abramovich was a noble representative of the scientific school of L. V. Kantorovich, one of the founding fathers of the theory of vector lattices and positive operators. Yuri left many beautiful theorems and some of them are now classical in the field. Much higher is the number of the ideas evoked by his papers or conversations with him. His ideas still leave (seemingly, in the whereabouts of human soles) and participate in the intricate transformations and interactions of the world of ideas, enchanting and inspiring novices and adepts. We tried to track the virtual path of these ideas. Clearly, someone could arrive at the same or similar ideas by another way not appealing to the contributions of Abramovich. This will increase the value of each of the ways since there is nothing more dubious and even perilous sometimes that the real or would-be truth that could be reached only by a unique road.

5.2. Yuri Abramovich was an open-hearted, amiable, and charming person. He readily volunteered for the difficult mission of helping those who strived for a living on the cracked and wrecked Soviet Pangaea. Many can tell about the indispensable help Yuri lent to them in a hard moment. Some persons will never know who was behind so vital and timely support. There are those who never tried his support but who knew that he would come and help if need be. This is a part of legacy he bequeathed to the world mathematical community.

5.3. In the summer of 1994 there was an international conference on "Interaction Between Functional Analysis, Harmonic Analysis, and Probability" in Missouri, Columbia. During this meeting the authors discussed with Yuri the idea of an international journal covering the most diverse areas of mathematics which are unified by the concept of comparison and order. Yuri was very enthusiastic about this idea and in a short time he managed to find and attract those who were in a position to implement such a challenging project. So Yuri stood at the cradle of what is now our *Positivity*. This is also a part of his memory...

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