РОССИЙСКАЯ АКАДЕМИЯ НАУК • СИБИРСКОЕ ОТДЕЛЕНИЕ
ИНСТИТУТ МАТЕМАТИКИ им. С. Л. СОБОЛЕВА

Препринт 157
июль 2005

A.G. Kusraev and S.S. Kutateladze

BOOLEAN VALUED ANALYSIS AND POSITIVITY

НОВОСИБИРСК

УДК 517.11:517.98 Дата поступления 11 июля 2005 г.

Kusraev A.G. and Kutateladze S. S.
Boolean Valued Analysis and Positivity. — Новосибирск, 2005. — 30 с. — (Препринт / РАН. Сиб. отд-ние. Ин-т математики; № 157).

This is an overview of the recent results of the interaction of Boolean valued analysis and vector lattice theory. A short obituary of Saunders Mac Lane (1909–2005) is appended.

Ключевые слова и фразы:
Boolean valued analysis, vector lattice, positive operator

Адрес авторов:
Институт математики им. С. Л. Соболева СО РАН,
пр. Академика Коптюга, 4
630090, Новосибирск, Россия

E-mail: kusraev@alanianet.ru
sskut@math.nsc.ru

Институт математики им. С. Л. Соболева СО РАН   Препринт №157, июль 2005

BOOLEAN VALUED ANALYSIS AND POSITIVITY

In memory of Saunders Mac Lane (1909–2005)

Introduction

Boolean valued analysis is a general mathematical method that rests on a special model-theoretic technique. This technique consists generally in comparison between the representations of arbitrary mathematical objects and theorems in two different set-theoretic models whose constructions start with principally distinct Boolean algebras. We usually take as these models the cosiest Cantorian paradise, the von Neumann universe of Zermelo–Fraenkel set theory, and a special universe of Boolean valued "variable" sets trimmed and chosen so that the traditional concepts and facts of mathematics acquire completely unexpected and bizarre interpretations. The use of two models, one of which is *formally* nonstandard, is a family feature of *nonstandard analysis*. For this reason, Boolean valued analysis means an instance of nonstandard analysis in common parlance. By the way, the term *Boolean valued analysis* was minted by G. Takeuti.

Proliferation of Boolean valued models is due to P. Cohen's final breakthrough in Hilbert's Problem Number One. His method of forcing was rather intricate and the inevitable attempts at simplification gave rise to the Boolean valued models by D. Scott, R. Solovay, and P. Vopěnka.

Professor M. Weber had invited us to this Positivity Conference at the end of 2004 when we were completing our book "Introduction to Boolean Valued Analysis." The book was recently published and so this talk is a kind of presentation.

Another recent event of relevance to this talk is grievous. Saunders Mac Lane, a co-father of category theory, passed away in San Francisco on April 14, 2005. The power of mathematics rests heavily on the trick of socializing the objects and problems under consideration. The understanding of the social medium of set-theoretic models belongs to category theory.

A category is called an *elementary topos* provided that it is cartesian closed and has a subobject classifier. A. Grothendieck and F. W. Lawvere, the followers of Saunders Mac Lane, created topos theory in the course of "point elimination" and in the dream of invariance of the objects we operate in mathematics. It is on this road that we met

---

This talk is prepared for the International Conference "Positivity IV: Theory and Applications" to be held by Technische Universität Dresden on July 25–29, 2005.

the conception of variable sets, underlying the notion of topos and bringing about the understanding of the social medium of set-theoretic models. The Boolean valued models belong happily in the family of Boolean toposes enjoying the classical Aristotle logic.

Category theory, alongside set theory, serves as a universal language of modern mathematics. Categories, functors, and natural transformations are widely used in all areas of mathematics, allowing us to look uniformly and consistently on various constructions and formulate the general properties of diverse structures. The impact of category theory is irreducible to the narrow frameworks of its outstanding expressive conveniences. This theory has drastically changed our general outlook on the foundations of mathematics and widened the room of free thinking in mathematics.

Set theory, a great and ingenious creation of Georg Cantor, occupies in the common opinion of the 20th century the place of the sole solid base of modern mathematics. Mathematics becomes sinking into a section of the Cantorian set theory. Most active mathematicians, teachers, and philosophers view as obvious and undisputable the thesis that mathematics cannot be grounded on anything but set theory.

The set-theoretic stance transforms paradoxically into an ironclad dogma, a clear-cut forbiddance of thinking. This indoctrinated view of the foundations of mathematics is false and contradicts conspicuously to the leitmotif, nature, and pathos of the essence of all creative contribution of G. Cantor who wrote as far back as in 1883 that "denn das *Wesen* der *Mathematik* liegt gerade in ihrer *Freiheit*."

Topos theory provides a profusion of categories of which classical set theory is an ordinary member. Mathematics has thus acquired infinitely many new degrees of freedom. All these achievements rest on category theory.

> *There remains to us, then, the pursuit of truth, by way of proof, the concatenation of those ideas which fit, and the beauty which results when they do fit.*

So wrote Saunders Mac Lane, a great master and servant of mathematics, whose memory we reverently dedicate this talk to.

1. Boolean Requisites

We start with recalling some auxiliary facts about the construction and treatment of Boolean valued models.

1.1. Let $\mathbb{B}$ be a complete Boolean algebra. Given an ordinal α, put

$$\mathbb{V}_\alpha^{(\mathbb{B})} := \{x : x \text{ is a function } \wedge\ (\exists\beta)(\beta < \alpha\ \wedge \operatorname{dom}(x) \subset \mathbb{V}_\beta^{(\mathbb{B})}\ \wedge\ \operatorname{im}(x) \subset \mathbb{B})\}.$$

After this recursive definition the *Boolean valued universe* $\mathbb{V}^{(\mathbb{B})}$ or, in other words, the *class of* $\mathbb{B}$*-sets* is introduced by

$$\mathbb{V}^{(\mathbb{B})} := \bigcup_{\alpha\in\mathrm{On}} \mathbb{V}_\alpha^{(\mathbb{B})},$$

with On standing for the class of all ordinals.

In case of the two element Boolean algebra $2 := \{\mathbb{0}, \mathbb{1}\}$ this procedure yields a version of the classical *von Neumann universe* $\mathbb{V}$ (see 2.1 (2)).

Let φ be an arbitrary formula of ZFC, Zermelo–Fraenkel set theory with choice. The *Boolean truth value* $[\![\varphi]\!] \in \mathbb{B}$ is introduced by induction on the length of a formula φ by naturally interpreting the propositional connectives and quantifiers in the Boolean algebra $\mathbb{B}$ and taking into consideration the way in which this formula is built up from atomic formulas. The Boolean truth values of the *atomic formulas* $x \in y$ and $x = y$, with $x, y \in \mathbb{V}^{(\mathbb{B})}$, are defined by means of the following recursion schema:

$$[\![x \in y]\!] = \bigvee_{t \in \mathrm{dom}(y)} y(t) \wedge [\![t = x]\!],$$

$$[\![x = y]\!] = \bigvee_{t \in \mathrm{dom}(x)} x(t) \Rightarrow [\![t \in y]\!] \wedge \bigvee_{t \in \mathrm{dom}(y)} y(t) \Rightarrow [\![t \in x]\!].$$

The sign $\Rightarrow$ symbolizes the implication in $\mathbb{B}$; i.e., $a \Rightarrow b := a^* \vee b$ where a^* is as usual the *complement* of a.

The universe $\mathbb{V}^{(\mathbb{B})}$ with the Boolean truth value of a formula is a model of set theory in the sense that the following statement is fulfilled.

1.2. Transfer Principle. *For every theorem φ of* ZFC, *we have $[\![\varphi]\!] = \mathbb{1}$; i.e., φ is true inside $\mathbb{V}^{(\mathbb{B})}$.*

Enter into the next agreement: If x is an element of $\mathbb{V}^{(\mathbb{B})}$ and $\varphi(\cdot)$ is a formula of ZFC, then the phrase "x satisfies φ inside $\mathbb{V}^{(\mathbb{B})}$" or, briefly, "$\varphi(x)$ is true inside $\mathbb{V}^{(\mathbb{B})}$" means that $[\![\varphi(x)]\!] = \mathbb{1}$. This is sometimes written as $\mathbb{V}^{(\mathbb{B})} \models \varphi(x)$.

Given $x \in \mathbb{V}^{(\mathbb{B})}$ and $b \in \mathbb{B}$, define the function $bx : z \mapsto bx(z) \quad (z \in \mathrm{dom}(x))$. Here we presume that $b\varnothing := \varnothing$ for all $b \in \mathbb{B}$.

There is a natural equivalence relation $x \sim y \leftrightarrow [\![x = y]\!] = \mathbb{1}$ in the class $\mathbb{V}^{(\mathbb{B})}$. Choosing a representative of the smallest rank in each equivalence class or, more exactly, using the so-called "Frege–Russell–Scott trick," we obtain a *separated Boolean valued universe* $\overline{\mathbb{V}}^{(\mathbb{B})}$ in which $x = y \leftrightarrow [\![x = y]\!] = \mathbb{1}$.

The Boolean truth value of a formula φ remains unaltered if we replace in φ each element of $\mathbb{V}^{(\mathbb{B})}$ by one of its equivalents. In this connection from now on we take $\mathbb{V}^{(\mathbb{B})} := \overline{\mathbb{V}}^{(\mathbb{B})}$ without further specification.

Observe that in $\overline{\mathbb{V}}^{(\mathbb{B})}$ the element bx is defined correctly for $x \in \overline{\mathbb{V}}^{(\mathbb{B})}$ and $b \in \mathbb{B}$ since $[\![x_1 = x_2]\!] = \mathbb{1} \to [\![bx_1 = bx_2]\!] = b \Rightarrow [\![x_1 = x_2]\!] = \mathbb{1}$. For a similar reason, we often write $\mathbb{0} := \varnothing$, and in particular $\mathbb{0}\varnothing = \varnothing = \mathbb{0}x$ for $x \in \mathbb{V}^{(\mathbb{B})}$.

1.3. Mixing Principle. *Let $(b_\xi)_{\xi\in\Xi}$ be a partition of unity in $\mathbb{B}$, i.e.* $\sup_{\xi\in\Xi} b_\xi = \sup \mathbb{B} = \mathbb{1}$ *and $\xi \neq \eta \to b_\xi \wedge b_\eta = \mathbb{0}$. To each family $(x_\xi)_{\xi\in\Xi}$ in $\mathbb{V}^{(\mathbb{B})}$ there exists a unique element x in the separated universe such that $[\![x = x_\xi]\!] \geq b_\xi \quad (\xi \in \Xi)$.*

This element is called the *mixing* of $(x_\xi)_{\xi\in\Xi}$ by $(b_\xi)_{\xi\in\Xi}$ and is denoted by $\sum_{\xi\in\Xi} b_\xi x_\xi$.

1.4. Maximum Principle. *If φ is a formula of* ZFC *then there is a $\mathbb{B}$-valued set x_0 satisfying $[\![(\exists x)\varphi(x)]\!] = [\![\varphi(x_0)]\!]$.*

2. The Escher Rules

Boolean valued analysis consists primarily in comparison of the instances of a mathematical object or idea in two Boolean valued models. This is impossible to achieve without some dialog between the universes $\mathbb{V}$ and $\mathbb{V}^{(\mathbb{B})}$.

In other words, we need a smooth mathematical toolkit for revealing interplay between the interpretations of one and the same fact in the two models $\mathbb{V}$ and $\mathbb{V}^{(\mathbb{B})}$. The relevant *ascending-and-descending technique* rests on the functors of canonical embedding, descent, and ascent.

2.1. We start with the canonical embedding of the von Neumann universe $\mathbb{V}$.

Given $x \in \mathbb{V}$, we denote by $x^\wedge$ the *standard name* of x in $\mathbb{V}^{(\mathbb{B})}$; i.e., the element defined by the following recursion schema:

$$\varnothing^\wedge := \varnothing, \quad \mathrm{dom}(x^\wedge) := \{y^\wedge : y \in x\}, \quad \mathrm{im}(x^\wedge) := \{\mathbb{1}\}.$$

Observe some properties of the mapping $x \mapsto x^\wedge$ we need in the sequel.

(1) For an arbitrary $x \in \mathbb{V}$ and a formula φ of ZFC we have

$$[\![(\exists y \in x^\wedge)\, \varphi(y)]\!] = \bigvee_{z \in x} [\![\varphi(z^\wedge)]\!], \quad [\![(\forall y \in x^\wedge)\, \varphi(y)]\!] = \bigwedge_{z \in x} [\![\varphi(z^\wedge)]\!].$$

(2) If x and y are elements of $\mathbb{V}$ then, by transfinite induction, we establish $x \in y \leftrightarrow \mathbb{V}^{(\mathbb{B})} \models x^\wedge \in y^\wedge, \quad x = y \leftrightarrow \mathbb{V}^{(\mathbb{B})} \models x^\wedge = y^\wedge$. In other words, the standard name can be considered as an embedding of $\mathbb{V}$ into $\mathbb{V}^{(\mathbb{B})}$. Moreover, it is beyond a doubt that the standard name sends $\mathbb{V}$ onto $\mathbb{V}^{(2)}$, which fact is demonstrated by the next proposition:

(3) The following holds: $(\forall u \in \mathbb{V}^{(2)})\,(\exists! x \in \mathbb{V})\ \mathbb{V}^{(\mathbb{B})} \models u = x^\wedge$.

A formula is called *bounded* or *restricted* if each bound variable in it is restricted by a bounded quantifier; i.e., a quantifier ranging over a particular set. The latter means that each bound variable x is restricted by a quantifier of the form $(\forall x \in y)$ or $(\exists x \in y)$ for some y.

2.2. Restricted Transfer Principle. *For each bounded formula φ of* ZFC *and every collection $x_1, \dots, x_n \in \mathbb{V}$ the following holds: $\varphi(x_1, \dots, x_n) \leftrightarrow \mathbb{V}^{(\mathbb{B})} \models \varphi(x_1^\wedge, \dots, x_n^\wedge)$.*

Henceforth, working in the separated universe $\overline{\mathbb{V}}^{(\mathbb{B})}$, we agree to preserve the symbol $x^\wedge$ for the distinguished element of the class corresponding to x.

Observe for example that the restricted transfer principle yields:

$$\text{“}\Phi \text{ is a correspondence from } x \text{ to } y\text{”}$$
$$\leftrightarrow \mathbb{V}^{(\mathbb{B})} \models \text{“}\Phi^\wedge \text{ is a correspondence from } x^\wedge \text{ to } y^\wedge\text{”};$$
$$\text{“}f \text{ is a function from } x \text{ to } y\text{”} \leftrightarrow \mathbb{V}^{(\mathbb{B})} \models \text{“}f^\wedge \text{ is a function from } x^\wedge \text{ to } y^\wedge\text{”}$$

(moreover, $f(a)^\wedge = f^\wedge(a^\wedge)$ for all $a \in x$). Thus, the standard name can be considered as a covariant functor of the category of sets (or correspondences) in $\mathbb{V}$ to an appropriate subcategory of $\mathbb{V}^{(2)}$ in the separated universe $\mathbb{V}^{(\mathbb{B})}$.

2.3. A set X is *finite* if X coincides with the image of a function on a finite ordinal. In symbols, this is expressed as $\mathrm{fin}(X)$; hence,

$$\mathrm{fin}(X) := (\exists\, n)(\exists\, f)(n \in \omega \wedge f \text{ is a function} \wedge \mathrm{dom}(f) = n \wedge \mathrm{im}(f) = X)$$

(as usual $\omega := \{0, 1, 2, \dots\}$). Obviously, the above formula is not bounded. Nevertheless there is a simple transformation rule for the class of finite sets under the canonical embedding. Denote by $\mathscr{P}_{\mathrm{fin}}(X)$ the class of all finite subsets of X; i.e., $\mathscr{P}_{\mathrm{fin}}(X) := \{Y \in \mathscr{P}(X) : \mathrm{fin}(Y)\}$. For an arbitrary set X the following holds: $\mathbb{V}^{(\mathbb{B})} \models \mathscr{P}_{\mathrm{fin}}(X)^\wedge = \mathscr{P}_{\mathrm{fin}}(X^\wedge)$.

2.4. Given an arbitrary element x of the (separated) Boolean valued universe $\mathbb{V}^{(\mathbb{B})}$, we define the *descent* $x{\downarrow}$ of x as $x{\downarrow} := \{y \in \mathbb{V}^{(\mathbb{B})} : [\![y \in x]\!] = \mathbb{1}\}$. We list the simplest properties of descending:

(1) The class $x{\downarrow}$ is a set, i.e., $x{\downarrow} \in \mathbb{V}$ for all $x \in \mathbb{V}^{(\mathbb{B})}$. If $[\![x \neq \varnothing]\!] = \mathbb{1}$ then $x{\downarrow}$ is a nonempty set.

(2) Let $z \in \mathbb{V}^{(\mathbb{B})}$ and $[\![z \neq \varnothing]\!] = \mathbb{1}$. Then for every formula φ of ZFC we have

$$[\![(\forall x \in z)\, \varphi(x)]\!] = \bigwedge_{x \in z\downarrow} [\![\varphi(x)]\!], \quad [\![(\exists x \in z)\, \varphi(x)]\!] = \bigvee_{x \in z\downarrow} [\![\varphi(x)]\!].$$

Moreover, there exists $x_0 \in z\downarrow$ such that $[\![\varphi(x_0)]\!] = [\![(\exists x \in z)\, \varphi(x)]\!]$.

(3) Let Φ be a correspondence from X to Y in $\mathbb{V}^{(\mathbb{B})}$. Thus, Φ, X, and Y are elements of $\mathbb{V}^{(\mathbb{B})}$ and, moreover, $[\![\Phi \subset X \times Y]\!] = \mathbb{1}$. There is a unique correspondence $\Phi\downarrow$ from $X\downarrow$ to $Y\downarrow$ such that $\Phi\downarrow(A\downarrow) = \Phi(A)\downarrow$ for every nonempty subset A of X inside $\mathbb{V}^{(\mathbb{B})}$. The correspondence $\Phi\downarrow$ from $X\downarrow$ to $Y\downarrow$ of the above proposition is called the *descent* of the correspondence Φ from X to Y inside $\mathbb{V}^{(\mathbb{B})}$.

(4) The descent of the composite of correspondences inside $\mathbb{V}^{(\mathbb{B})}$ is the composite of their descents: $(\Psi \circ \Phi)\downarrow = \Psi\downarrow \circ \Phi\downarrow$.

(5) If Φ is a correspondence inside $\mathbb{V}^{(\mathbb{B})}$ then $(\Phi^{-1})\downarrow = (\Phi\downarrow)^{-1}$.

(6) Let Id_X be the identity mapping inside $\mathbb{V}^{(\mathbb{B})}$ of a set $X \in \mathbb{V}^{(\mathbb{B})}$. Then $(\mathrm{Id}_X)\downarrow = \mathrm{Id}_{X\downarrow}$.

(7) Suppose that $X, Y, f \in \mathbb{V}^{(\mathbb{B})}$ are such that $[\![f : X \to Y]\!] = \mathbb{1}$, i.e., f is a mapping from X to Y inside $\mathbb{V}^{(\mathbb{B})}$. Then $f\downarrow$ is a unique mapping from $X\downarrow$ to $Y\downarrow$ satisfying $[\![f\downarrow(x) = f(x)]\!] = \mathbb{1}$ for all $x \in X\downarrow$.

By virtue of (1)–(7), we can consider the descent operation as a functor from the category of $\mathbb{B}$-valued sets and mappings (correspondences) to the category of the usual sets and mappings (correspondences) (i.e., in the sense of $\mathbb{V}$).

(8) Given $x_1, \dots, x_n \in \mathbb{V}^{(\mathbb{B})}$, denote by $(x_1, \dots, x_n)^{\mathbb{B}}$ the corresponding ordered n-tuple inside $\mathbb{V}^{(\mathbb{B})}$. Assume that P is an n-ary relation on X inside $\mathbb{V}^{(\mathbb{B})}$; i.e., $X, P \in \mathbb{V}^{(\mathbb{B})}$ and $[\![P \subset X^{n^\wedge}]\!] = \mathbb{1}$, where $n \in \omega$. Then there exists an n-ary relation P' on $X\downarrow$ such that $(x_1, \dots, x_n) \in P' \leftrightarrow [\![(x_1, \dots, x_n)^{\mathbb{B}} \in P]\!] = \mathbb{1}$. Slightly abusing notation, we denote the relation P' by the same symbol $P\downarrow$ and call it the *descent* of P.

2.5. Let $x \in \mathbb{V}$ and $x \subset \mathbb{V}^{(\mathbb{B})}$; i.e., let x be some set composed of $\mathbb{B}$-valued sets or, in other words, $x \in \mathscr{P}(\mathbb{V}^{(\mathbb{B})})$. Put $\varnothing\uparrow := \varnothing$ and $\mathrm{dom}(x\uparrow) := x$, $\quad \mathrm{im}(x\uparrow) := \{\mathbb{1}\}$ if $x \neq \varnothing$. The element $x\uparrow$ (of the separated universe $\mathbb{V}^{(\mathbb{B})}$, i.e., the distinguished representative of the class $\{y \in \mathbb{V}^{(\mathbb{B})} : [\![y = x\uparrow]\!] = \mathbb{1}\}$) is called the *ascent* of x.

(1) For all $x \in \mathscr{P}(\mathbb{V}^{(\mathbb{B})})$ and every formula φ we have the following:

$$[\![(\forall z \in x\uparrow)\, \varphi(z)]\!] = \bigwedge_{y \in x} [\![\varphi(y)]\!], \quad [\![(\exists z \in x\uparrow)\, \varphi(z)]\!] = \bigvee_{y \in x} [\![\varphi(y)]\!].$$

Introducing the ascent of a correspondence $\Phi \subset X \times Y$, we have to bear in mind a possible distinction between the domain of departure X and the domain $\mathrm{dom}(\Phi) := \{x \in X : \Phi(x) \neq \varnothing\}$. This circumstance is immaterial for the sequel; therefore, speaking of ascents, we always imply total correspondences; i.e., $\mathrm{dom}(\Phi) = X$.

(2) Let $X, Y, \Phi \in \mathbb{V}^{(\mathbb{B})}$, and let Φ be a correspondence from X to Y. There exists a unique correspondence $\Phi\uparrow$ from $X\uparrow$ to $Y\uparrow$ inside $\mathbb{V}^{(\mathbb{B})}$ such that $\Phi\uparrow(A\uparrow) = \Phi(A)\uparrow$ is valid for every subset A of $\mathrm{dom}(\Phi)$ if and only if Φ is *extensional*; i.e., satisfies the condition $y_1 \in \Phi(x_1) \to [\![x_1 = x_2]\!] \leq \bigvee_{y_2 \in \Phi(x_2)} [\![y_1 = y_2]\!]$ for $x_1, x_2 \in \mathrm{dom}(\Phi)$. In this event, $\Phi\uparrow = \Phi'\uparrow$, where $\Phi' := \{(x, y)^{\mathbb{B}} : (x, y) \in \Phi\}$. The element $\Phi\uparrow$ is called the *ascent* of the initial correspondence Φ.

(3) The composite of extensional correspondences is extensional. Moreover, the ascent of a composite is equal to the composite of the ascents inside $\mathbb{V}^{(\mathbb{B})}$: On assuming that $\mathrm{dom}(\Psi) \supset \mathrm{im}(\Phi)$ we have $\mathbb{V}^{(\mathbb{B})} \models (\Psi \circ \Phi)\uparrow = \Psi\uparrow \circ \Phi\uparrow$.

Note that if Φ and Φ^{-1} are extensional then $(\Phi\uparrow)^{-1} = (\Phi^{-1})\uparrow$. However, in general, the extensionality of Φ in no way guarantees the extensionality of Φ^{-1}.

(4) It is worth mentioning that if an extensional correspondence f is a function from X to Y then the ascent $f\uparrow$ of f is a function from $X\uparrow$ to $Y\uparrow$. Moreover, the extensionality property can be stated as follows: $[\![x_1 = x_2]\!] \leq [\![f(x_1) = f(x_2)]\!]$ for all $x_1, x_2 \in X$.

2.6. Given a set $X \subset \mathbb{V}^{(\mathbb{B})}$, we denote by the symbol $\operatorname{mix}(X)$ the set of all mixings of the form $\operatorname{mix}(b_\xi x_\xi)$, where $(x_\xi) \subset X$ and (b_ξ) is an arbitrary partition of unity. The following propositions are referred to as the *arrow cancellation rules* or *ascending-and-descending rules*. There are many good reasons to call them simply the *Escher rules* [17].

(1) Let X and X' be subsets of $\mathbb{V}^{(\mathbb{B})}$ and let $f : X \to X'$ be an extensional mapping. Suppose that $Y, Y', g \in \mathbb{V}^{(\mathbb{B})}$ are such that $[\![\, Y \neq \varnothing]\!] = [\![\, g : Y \to Y']\!] = \mathbb{1}$. Then $X\uparrow\downarrow = \operatorname{mix}(X)$, $Y\downarrow\uparrow = Y$, $f\uparrow\downarrow = f$, and $g\downarrow\uparrow = g$.

(2) From 2.3 (8) we easily infer the useful relation: $\mathscr{P}_{\rm fin}(X\uparrow) = \{\theta\uparrow : \theta \in \mathscr{P}_{\rm fin}(X)\}\uparrow$.

Suppose that $X \in \mathbb{V}$, $X \neq \varnothing$; i.e., X is a nonempty set. Let the letter ι denote the standard name embedding $x \mapsto x^\wedge$ $(x \in X)$. Then $\iota(X)\uparrow = X^\wedge$ and $X = \iota^{-1}(X^\wedge\downarrow)$. Using the above relations, we may extend the descent and ascent operations to the case in which Φ is a correspondence from X to $Y\downarrow$ and $[\![\Psi$ is a correspondence from $X^\wedge$ to $Y]\!] = \mathbb{1}$, where $Y \in \mathbb{V}^{(\mathbb{B})}$. Namely, we put $\Phi\Uparrow := (\Phi \circ \iota)\uparrow$ and $\Psi\Downarrow := \Psi\downarrow \circ \iota$. In this case, $\Phi\Uparrow$ is called the *modified ascent* of Φ and $\Psi\Downarrow$ is called the *modified descent* of Ψ. (If the context excludes ambiguity then we briefly speak of ascents and descents using simple arrows.) It is easy to see that $\Psi\Uparrow$ is a unique correspondence inside $\mathbb{V}^{(\mathbb{B})}$ satisfying the relation $[\![\Phi\Uparrow(x^\wedge) = \Phi(x)\uparrow]\!] = \mathbb{1}$ $\quad (x \in X)$. Similarly, $\Psi\Downarrow$ is a unique correspondence from X to $Y\downarrow$ satisfying the equality $\Psi\Downarrow(x) = \Psi(x^\wedge)\downarrow$ $\quad (x \in X)$. If $\Phi := f$ and $\Psi := g$ are functions then these relations take the form $[\![f\Uparrow(x^\wedge) = f(x)]\!] = \mathbb{1}$ and $g\Downarrow(x) = g(x^\wedge)$ for all $x \in X$.

2.7. Various function spaces reside in functional analysis, and so the problem is natural of replacing an abstract Boolean valued system by some function-space analog, a model whose elements are functions and in which the basic logical operations are calculated "pointwise." An example of such a model is given by the class $\mathbb{V}^Q$ of all functions defined on a fixed nonempty set Q and acting into $\mathbb{V}$. The truth values on $\mathbb{V}^Q$ are various subsets of Q: The truth value $[\![\varphi(u_1, \dots, u_n)]\!]$ of $\varphi(t_1, \dots, t_n)$ at functions $u_1, \dots, u_n \in \mathbb{V}^Q$ is calculated as follows: $[\![\varphi(u_1, \dots, u_n)]\!] = \big\{q \in Q : \ \varphi\big(u_1(q), \dots, u_n(q)\big)\big\}$.

A. G. Gutman and G. A. Losenkov solved the above problem by the concept of continuous polyverse which is a continuous bundle of models of set theory. It is shown that the class of continuous sections of a continuous polyverse is a Boolean valued system satisfying all basic principles of Boolean valued analysis and, conversely, each Boolean valued algebraic system can be represented as the class of sections of a suitable continuous polyverse. More details are collected in [35, Chapter 6].

2.8. Every Boolean valued universe has the collection of mathematical objects in full supply: available in plenty are all sets with extra structure: groups, rings, algebras, normed spaces, etc. Applying the descent functor to such *internal* algebraic systems of a Boolean valued model, we distinguish some bizarre entities or recognize old acquaintances, which leads to revealing the new facts of their life and structure.

This technique of research, known as *direct Boolean valued interpretation*, allows us to produce new theorems or, to be more exact, to extend the semantical content of the available theorems by means of slavish translation. The information we so acquire might fail to be vital, valuable, or intriguing, in which case the direct Boolean valued interpretation turns out into a leisurely game.

It thus stands to reason to raise the following questions: What structures significant for mathematical practice are obtainable by the Boolean valued interpretation of the most typical algebraic systems? What transfer principles hold true in this process? Clearly, the answers should imply specific objects whose particular features enable us to deal with their Boolean valued representation which, if understood duly, is impossible to implement for arbitrary algebraic systems.

An *abstract Boolean set* or *set with* $\mathbb{B}$*-structure* is a pair (X, d), where $X \in \mathbb{V}$, $X \neq \varnothing$, and d is a mapping from $X \times X$ to $\mathbb{B}$ such that $d(x, y) = \mathbb{0} \leftrightarrow x = y$; $d(x, y) = d(y, x)$; $d(x, y) \leq d(x, z) \vee d(z, y)$ all $x, y, z \in X$.

To obtain an easy example of an abstract $\mathbb{B}$-set, given $\varnothing \neq X \subset \mathbb{V}^{(\mathbb{B})}$ put

$$d(x, y) := [\![x \neq y]\!] = \neg[\![x = y]\!]$$

for $x,\ y \in X$.

Another easy example is a nonempty X with the *discrete* $\mathbb{B}$*-metric* d; i.e., $d(x, y) = \mathbb{1}$ if $x \neq y$ and $d(x, y) = \mathbb{0}$ if $x = y$.

Let (X, d) be some abstract $\mathbb{B}$-set. There exist an element $\mathscr{X} \in \mathbb{V}^{(\mathbb{B})}$ and an injection $\iota : X \to X' := \mathscr{X}{\downarrow}$ such that $d(x, y) = [\![\iota x \neq \iota y]\!]$ for all $x, y \in X$ and every element $x' \in X'$ admits the representation $x' = \mathrm{mix}_{\xi \in \Xi}(b_\xi \iota x_\xi)$, where $(x_\xi)_{\xi \in \Xi} \subset X$ and $(b_\xi)_{\xi \in \Xi}$ is a partition of unity in $\mathbb{B}$. The element $\mathscr{X} \in \mathbb{V}^{(\mathbb{B})}$ is referred to as the *Boolean valued realization* of X.

If X is a discrete abstract $\mathbb{B}$-set then $\mathscr{X} = X^\wedge$ and $\iota x = x^\wedge$ for all $x \in X$. If $X \subset \mathbb{V}^{(\mathbb{B})}$ then $\iota{\uparrow}$ is an injection from $X{\uparrow}$ to $\mathscr{X}$ (inside $\mathbb{V}^{(\mathbb{B})}$). A mapping f from a $\mathbb{B}$-set (X, d) to a $\mathbb{B}$-set (X', d') is said to be *contractive* if $d(x, y) \geq d'(f(x), f(y))$ for all $x, y \in X$.

We see that an abstract $\mathbb{B}$-set X embeds in the Boolean valued universe $\mathbb{V}^{(\mathbb{B})}$ so that the Boolean distance between the members of X becomes the Boolean truth value of the negation of their equality. The corresponding element of $\mathbb{V}^{(\mathbb{B})}$ is, by definition, the *Boolean valued representation* of X.

In case a $\mathbb{B}$-set X has some a priori structure we may try to furnish the Boolean valued representation of X with an analogous structure, so as to apply the technique of ascending and descending to the study of the original structure of X. Consequently, the above questions may be treated as instances of the unique problem of searching a well-qualified Boolean valued representation of a $\mathbb{B}$-set with some additional structure.

We call these objects *algebraic* $\mathbb{B}$*-systems*. Located at the epicenter of exposition, the notion of an algebraic $\mathbb{B}$-system refers to a nonempty $\mathbb{B}$-set endowed with a few contractive operations and $\mathbb{B}$-predicates, the latter meaning $\mathbb{B}$-valued contractive mappings.

The Boolean valued representation of an algebraic $\mathbb{B}$-system appears to be a conventional two valued algebraic system of the same type. This means that an appropriate completion of each algebraic $\mathbb{B}$-system coincides with the descent of some two valued algebraic system inside $\mathbb{V}^{(\mathbb{B})}$.

On the other hand, each two valued algebraic system may be transformed into an algebraic $\mathbb{B}$-system on distinguishing a complete Boolean algebra of congruences of the original system. In this event, the task is in order of finding the formulas holding true in direct or reverse transition from a $\mathbb{B}$-system to a two valued system. In other words, we have to seek here for some versions of the transfer or identity preservation principle of long standing in some branches of mathematics.

3. Boolean Valued Numbers

Boolean valued analysis stems from the fact that each internal field of reals of a Boolean valued model descends into a universally complete Kantorovich space. Thus, a remarkable opportunity opens up to expand and enrich the treasure-trove of mathematical knowledge by translating information about the reals to the language of other noble families of functional analysis. We will elaborate upon the matter in this section.

3.1. Recall a few definitions. Two elements x and y of a vector lattice E are called *disjoint* (in symbols $x \perp y$) if $|x| \wedge |y| = 0$. A *band* of E is defined as the *disjoint complement* $M^{\perp} := \{x \in E : (\forall y \in M)\, x \perp y\}$ of a nonempty set $M \subset E$.

The inclusion-ordered set $\mathfrak{B}(E)$ of all bands in E is a complete Boolean algebra with the Boolean operations:

$$L \wedge K = L \cap K, \quad L \vee K = (L \cup K)^{\perp\perp}, \quad L^* = L^{\perp} \quad (L, K \in \mathfrak{B}(E)).$$

The Boolean algebra $\mathfrak{B}(E)$ is often referred as to the *base* of E.

A *band projection* in E is a linear idempotent operator in $\pi : E \to E$ satisfying the inequalities $0 \le \pi x \le x$ for all $0 \le x \in E$. The set $\mathfrak{P}(E)$ of all band projections ordered by $\pi \le \rho \Longleftrightarrow \pi \circ \rho = \pi$ is a Boolean algebra with the Boolean operations:

$$\pi \wedge \rho = \pi \circ \rho, \quad \pi \vee \rho = \pi + \rho - \pi \circ \rho, \quad \pi^* = I_E - \pi \quad (\pi, \rho \in (E)).$$

Let $u \in E_+$ and $e \wedge (u - e) = 0$ for some $0 \le e \in E$. Then e is a *fragment* or *component* of u. The set $\mathfrak{C}(u)$ of all fragments of u with the order induced by E is a Boolean algebra where the lattice operations are taken from E and the Boolean complement has the form $e^* := u - e$.

3.2. A Dedekind complete vector lattice is also called a *Kantorovich space* or *K-space*, for short. A K-space E is *universally complete* if every family of pairwise disjoint elements of E is order bounded.

(1) Theorem. *Let E be an arbitrary K-space. Then the correspondence $\pi \mapsto \pi(E)$ determines an isomorphism of the Boolean algebras $\mathfrak{P}(E)$ and $\mathfrak{B}(E)$. If there is an order unity $\mathbb{1}$ in E then the mappings $\pi \mapsto \pi\mathbb{1}$ from $\mathfrak{P}(E)$ into $\mathfrak{C}(E)$ and $e \mapsto \{e\}^{\perp\perp}$ from $\mathfrak{C}(E)$ into $\mathfrak{B}(E)$ are isomorphisms of Boolean algebras too.*

(2) Theorem. *Each universally complete K-space E with order unity $\mathbb{1}$ can be uniquely endowed by multiplication so as to make E into a faithful f-algebra and $\mathbb{1}$ into a ring unity. In this f-algebra each band projection $\pi \in \mathfrak{P}(E)$ is the operator of multiplication by $\pi(\mathbb{1})$.*

3.3. By a *field of reals* we mean every algebraic system that satisfies the axioms of an Archimedean ordered field (with distinct zero and unity) and enjoys the axiom of completeness. The same object can be defined as a one-dimensional K-space.

Recall the well-known assertion of ZFC: *There exists a field of reals* $\mathbb{R}$ *that is unique up to isomorphism.*

Successively applying the transfer and maximum principles, we find an element $\mathscr{R} \in \mathbb{V}^{(\mathbb{B})}$ for which $[\![\, \mathscr{R}$ is a field of reals$\,]\!] = \mathbb{1}$. Moreover, if an arbitrary $\mathscr{R}' \in \mathbb{V}^{(\mathbb{B})}$ satisfies the condition $[\![\, \mathscr{R}'$ is a field of reals$\,]\!] = \mathbb{1}$ then $[\![\,$the ordered fields $\mathscr{R}$ and $\mathscr{R}'$ are isomorphic$\,]\!] = \mathbb{1}$. In other words, there exists an internal field of reals $\mathscr{R} \in \mathbb{V}^{(\mathbb{B})}$ which is unique up to isomorphism.

By the same reasons there exists an internal field of complex numbers $\mathscr{C} \in \mathbb{V}^{(\mathbb{B})}$ which is unique up to isomorphism. Moreover, $\mathbb{V}^{(\mathbb{B})} \models \mathscr{C} = \mathscr{R} \oplus i\mathscr{R}$. We call $\mathscr{R}$ and $\mathscr{C}$ the *internal reals* and *internal complexes* in $\mathbb{V}^{(\mathbb{B})}$.

3.4. Consider another well-known assertion of ZFC: *If $\mathbb{P}$ is an Archimedean ordered field then there is an isomorphic embedding h of the field $\mathbb{P}$ into $\mathbb{R}$ such that the image $h(\mathbb{P})$ is a subfield of $\mathbb{R}$ containing the subfield of rational numbers. In particular, $h(\mathbb{P})$ is dense in $\mathbb{R}$.*

Note also that $\varphi(x)$, formally presenting the expressions of the axioms of an Archimedean ordered field x, is bounded; therefore, $[\![\, \varphi(\mathbb{R}^\wedge) \,]\!] = \mathbb{1}$, i.e., $[\![\, \mathbb{R}^\wedge$ is an Archimedean ordered field$\,]\!] = \mathbb{1}$. "Pulling" 3.2 (2) through the transfer principle, we conclude that $[\![\, \mathbb{R}^\wedge$ is isomorphic to a dense subfield of $\mathscr{R} \,]\!] = \mathbb{1}$. We further assume that $\mathbb{R}^\wedge$ is a dense subfield of $\mathscr{R}$ and $\mathbb{C}^\wedge$ is a dense subfield of $\mathscr{C}$. It is easy to note that the elements $0^\wedge$ and $1^\wedge$ are the zero and unity of $\mathscr{R}$.

Observe that the equalities $\mathscr{R} = \mathbb{R}^\wedge$ and $\mathscr{C} = \mathbb{C}^\wedge$ are not valid in general. Indeed, the axiom of completeness for $\mathbb{R}$ is not a bounded formula and so it may thus fail for $\mathbb{R}^\wedge$ inside $\mathbb{V}^{(\mathbb{B})}$.

3.5. Look now at the descent $\mathscr{R}{\downarrow}$ of the algebraic system $\mathscr{R}$. In other words, consider the descent of the underlying set of the system $\mathscr{R}$ together with descended operations and order. For simplicity, we denote the operations and order in $\mathscr{R}$ and $\mathscr{R}{\downarrow}$ by the same symbols $+$, $\cdot$, and $\le$. In more detail, we introduce addition, multiplication, and order in $\mathscr{R}{\downarrow}$ by the formulas

$$z = x + y \leftrightarrow [\![\, z = x + y \,]\!] = \mathbb{1}, \quad z = x \cdot y \leftrightarrow [\![\, z = x \cdot y \,]\!] = \mathbb{1},$$
$$x \le y \leftrightarrow [\![\, x \le y \,]\!] = \mathbb{1} \quad (x, y, z \in \mathscr{R}{\downarrow}).$$

Also, we may introduce multiplication by the usual reals in $\mathscr{R}{\downarrow}$ by the rule

$$y = \lambda x \leftrightarrow [\![\, \lambda^\wedge x = y \,]\!] = \mathbb{1} \quad (\lambda \in \mathbb{R},\ x, y \in \mathscr{R}{\downarrow}).$$

The fundamental result of Boolean valued analysis is Gordon's Theorem which reads as follows: *Each universally complete Kantorovich space is an interpretation of the reals in an appropriate Boolean valued model.* Formally, we have the following

3.6. Gordon Theorem. *Let $\mathscr{R}$ be the reals inside $\mathbb{V}^{(\mathbb{B})}$. Then $\mathscr{R}{\downarrow}$, with the descended operations and order, is a universally complete K-space with order unity 1. Moreover, there exists an isomorphism χ of $\mathbb{B}$ onto $\mathfrak{P}(\mathscr{R}{\downarrow})$ such that*

$$\chi(b)x = \chi(b)y \leftrightarrow b \le [\![\, x = y \,]\!], \quad \chi(b)x \le \chi(b)y \leftrightarrow b \le [\![\, x \le y \,]\!]$$

for all $x, y \in \mathscr{R}{\downarrow}$ and $b \in \mathbb{B}$.

The converse is also true: Each Archimedean vector lattice embeds in a Boolean valued model, becoming a vector sublattice of the reals (viewed as such over some dense subfield of the reals).

3.7. Theorem. *Let E be an Archimedean vector lattice, let $\mathscr{R}$ be the reals inside $\mathbb{V}^{(\mathbb{B})}$, and let $\jmath$ be an isomorphism of $\mathbb{B}$ onto $\mathfrak{B}(E)$. Then there is $\mathscr{E} \in \mathbb{V}^{(\mathbb{B})}$ such that*

(1) *$\mathscr{E}$ is a vector sublattice of $\mathscr{R}$ over $\mathbb{R}^\wedge$ inside $\mathbb{V}^{(\mathbb{B})}$;*

(2) *$E' := \mathscr{E}{\downarrow}$ is a vector sublattice of $\mathscr{R}{\downarrow}$ invariant under every band projection $\chi(b)$ ($b \in \mathbb{B}$) and such that each set of positive pairwise disjoint sets in it has a supremum;*

(3) *there is an o-continuous lattice isomorphism $\iota : E \to E'$ such that $\iota(E)$ is a coinitial sublattice of $\mathscr{R}{\downarrow}$;*

(4) *for every $b \in \mathbb{B}$ the band projection in $\mathscr{R}{\downarrow}$ onto $\{\iota(\jmath(b))\}^{\perp\perp}$ coincides with $\chi(b)$.*

Note also that $\mathscr{E}$ and $\mathscr{R}$ coincide if and only if E is Dedekind complete. Thus, each theorem about the reals within Zermelo–Fraenkel set theory has an analog in an arbitrary Kantorovich space. Translation of theorems is carried out by appropriate general functors of Boolean valued analysis. In particular, the most important structural properties of vector lattices such as the functional representation, spectral theorem, etc. are the ghosts of some properties of the reals in an appropriate Boolean valued model. More details and references are collected in [35].

3.8. The theory of vector lattices with a vast field of applications is thoroughly covered in many monographs (see [5, 7, 24, 25, 42, 53, 54, 64, 66]). The credit for finding the most important instance among ordered vector spaces, an order complete vector lattice or K-space, is due to L. V. Kantorovich. This notion appeared in Kantorovich's first article on this topic [23] where he wrote: "In this note, I define a new type of space that I call a semiordered linear space. The introduction of such a space allows us to study linear operations of one abstract class (those with values in such a space) as linear functionals."

Thus the *heuristic transfer principle* was stated for K-spaces which becomes the Ariadna thread of many subsequent studies. The depth and universality of Kantorovich's principle are explained within Boolean valued analysis.

3.9. Applications of Boolean valued models to functional analysis stem from the works by E. I. Gordon [11, 12] and G. Takeuti [58]. If $\mathbb{B}$ in 3.6 is the algebra of μ-measurable sets modulo μ-negligible sets then $\mathscr{R}{\downarrow}$ is isomorphic to the universally complete K-space $L^0(\mu)$ of measurable functions. This fact (for the Lebesgue measure on an interval) was already known to D. Scott and R. Solovay (see [35]). If $\mathbb{B}$ is a complete Boolean algebra of projections in a Hilbert space then $\mathscr{R}{\downarrow}$ is isomorphic to the space of selfadjoint operators $\mathfrak{A}(\mathbb{B})$. These two particular cases of Gordon's Theorem were intensively and fruitfully exploited by G. Takeuti (see [58] and the bibliography in [35]). The object $\mathscr{R}{\downarrow}$ for general Boolean algebras was also studied by T. Jech [18]–[20] who in fact rediscovered Gordon's Theorem. The difference is that in [21] a (complex) universally complete K-space with unity is defined by another system of axioms and is referred to as a *complete Stone algebra.* Theorem 3.7 was obtained by A. G. Kusraev [30]. A close result (in other terms) is presented in T. Jech's article [20] where some Boolean valued interpretation is revealed of the theory of linearly ordered sets. More details can be found in [35].

4. Band Preserving Operators

This section deals with the class of band preserving operators. Simplicity of these operators notwithstanding, the question about their order boundedness is far from trivial.

4.1. Recall that a complex K-space is the complexification $G_{\mathbb{C}} := G \oplus iG$ of a real K-space G (see [53]). A linear operator $T : G_{\mathbb{C}} \to G_{\mathbb{C}}$ is *band preserving* or *contractive* or a *stabilizer* if, for all $f, g \in G_{\mathbb{C}}$, from $f \perp g$ it follows that $Tf \perp g$. Disjointness in $G_{\mathbb{C}}$ is defined just as in G (see 3.1), whereas $|z| := \sup\{\mathrm{Re}(e^{i\theta} z) : 0 \le \theta \le \pi\}$ for $z \in G_{\mathbb{C}}$.

(1) Let $\mathrm{End}_N(G_{\mathbb{C}})$ stand for the set of all band preserving linear operators in $G_{\mathbb{C}}$, with $G := \mathscr{R}{\downarrow}$. Clearly, $\mathrm{End}_N(G_{\mathbb{C}})$ is a complex vector space. Moreover, $\mathrm{End}_N(G_{\mathbb{C}})$ becomes a faithful unitary module over the ring $G_{\mathbb{C}}$ if we define gT as $gT : x \mapsto g \cdot Tx$ for all $x \in G$. This follows from the fact that multiplication by a member of $G_{\mathbb{C}}$ is a band preserving operator and the composite of band preserving operators is band preserving too.

(2) Denote by $\mathrm{End}_{\mathbb{C}^\wedge}(\mathscr{C})$ the element of $\mathbb{V}^{(\mathbb{B})}$ representing the space of all $\mathbb{C}^\wedge$-linear mappings from $\mathscr{C}$ to $\mathscr{C}$. Then $\mathrm{End}_{\mathbb{C}^\wedge}(\mathscr{C})$ is a vector space over $\mathbb{C}^\wedge$ inside $\mathbb{V}^{(\mathbb{B})}$, and $\mathrm{End}_{\mathbb{C}^\wedge}(\mathscr{C}){\downarrow}$ is a faithful unitary module over $G_{\mathbb{C}}$.

4.2. Following [31] it is easy to prove that a linear operator T in the K-space $G_{\mathbb{C}}$ is band preserving if and only if T is extensional. Since each extensional mapping has an ascent, $T \in \mathrm{End}_N(G_{\mathbb{C}})$ has the ascent $\tau := T{\uparrow}$ which is a unique internal functional from $\mathscr{C}$ to $\mathscr{C}$ such that $[\![\tau(x) = Tx]\!] = \mathbb{1}$ $(x \in G_{\mathbb{C}})$. We thus arrive at the following assertion:

The modules $\mathrm{End}_N(G_{\mathbb{C}})$ *and* $\mathrm{End}_{\mathbb{C}^\wedge}(\mathscr{C}){\downarrow}$ *are isomorphic by sending each band preserving operator to its ascent.*

By Gordon's Theorem this assertion means that the problem of finding a band preserving operator in G amounts to solving (for $\tau : \mathscr{C} \to \mathscr{C}$) inside $\mathbb{V}^{(\mathbb{B})}$ the *Cauchy functional equation*: $\tau(x + y) = \tau(x) + \tau(y) \quad (x, y \in \mathscr{C})$ under the subsidiary condition $\tau(\lambda x) = \lambda\tau(x) \quad (x \in \mathscr{C}, \lambda \in \mathbb{C}^\wedge)$.

As another subsidiary condition we may consider the *Leibniz rule* $\tau(xy) = \tau(x)y + x\tau(y)$(in which case τ is called a $\mathbb{C}^\wedge$-*derivation*) or multiplicativity $\tau(xy) = \tau(x)\tau(y)$. These situations are addressed in 4.5.

4.3. An element $g \in G^+$ is *locally constant* with respect to $f \in G^+$ if $g = \bigvee_{\xi\in\Xi} \lambda_\xi \pi_\xi f$ for some numeric family $(\lambda_\xi)_{\xi\in\Xi}$ and a family $(\pi_\xi)_{\xi\in\Xi}$ of pairwise disjoint band projections. A universally complete K-space $G_{\mathbb{C}}$ is called *locally one-dimensional* if all elements of G^+ are locally constant with respect to some order unity of G (and hence each of them). Clearly, a K-space $G_{\mathbb{C}}$ is locally one-dimensional if each $g \in G_{\mathbb{C}}$ may be presented as $g = o\text{-}\sum_{\xi\in\Xi} \lambda_\xi \pi_\xi \mathbb{1} f$ with some family $(\lambda_\xi)_{\xi\in\Xi} \subset \mathbb{C}$ and partition of unity $(\pi_\xi)_{\xi\in\Xi} \subset \mathfrak{P}(G)$.

4.4. A σ-complete Boolean algebra $\mathbb{B}$ is called σ-*distributive* if

$$\bigvee_{n\in\mathbb{N}} \bigwedge_{m\in\mathbb{N}} b_{n,m} = \bigwedge_{\varphi\in\mathbb{N}^{\mathbb{N}}} \bigvee_{n\in\mathbb{N}} b_{n,\varphi(n)}.$$

for every double sequence $(b_{n,m})_{n,m\in\mathbb{N}}$ in $\mathbb{B}$. Other equivalent definitions are collected in [56]. As an example of a σ-distributive Boolean algebra we may take a complete atomic Boolean algebra, i.e., the boolean of a nonempty set. It is worth observing that there are nonatomic σ-distributive complete Boolean algebras (see [32, 5.1.8]).

We now address the problem which is often referred to in the literature as *Wickstead's problem*: Characterize the universally complete vector lattices spaces in which every band preserving linear operator is order bounded.

According to 4.2, Boolean valued analysis reduces Wickstead's problem to that of order boundedness of the endomorphisms of the field $\mathscr{C}$ viewed as a vector space and algebra over the field $\mathbb{C}^\wedge$.

4.5. Theorem. *Let* $\mathbb{P}$ *be an algebraically closed and topologically dense subfield of the field of complexes* $\mathbb{C}$. *The following are equivalent:*

(1) $\mathbb{P} = \mathbb{C}$;

(2) *every* $\mathbb{P}$*-linear function on* $\mathbb{C}$ *is order bounded;*

(3) *there are no nontrivial* $\mathbb{P}$*-derivations on* $\mathbb{C}$*;*

(4) *each* $\mathbb{P}$*-linear endomorphism on* $\mathbb{C}$ *is the zero or identity function;*

(5) *there is no* $\mathbb{P}$*-linear automorphism on* $\mathbb{C}$ *other than the identity.*

The equivalence (1) $\leftrightarrow$ (2) is checked by using a Hamel basis of the vector space $\mathbb{C}$ over $\mathbb{P}$. The remaining equivalences rest on replacing a Hamel basis with a transcendence basis (for details see [33]).

Recall that a linear operator $D : G_{\mathbb{C}} \to G_{\mathbb{C}}$ is a $\mathbb{C}$-*derivation* if $D(fg) = D(f)g + fD(g)$ for all $f, g \in G_{\mathbb{C}}$. It can be easily checked that every $\mathbb{C}$-derivation is band preserving.

Interpreting Theorem 4.5 in $\mathbb{V}^{(\mathbb{B})}$, we arrive at

4.6. Theorem. *If $\mathbb{B}$ is a complete Boolean algebra then the following are equivalent:*

(1) $\mathscr{C} = \mathbb{C}^{\wedge}$ *inside* $\mathbb{V}^{(\mathbb{B})}$*;*

(2) *every band preserving linear operator is order bounded in the complex vector lattice* $\mathscr{C}{\downarrow}$*;*

(3) *there is no nontrivial $\mathbb{C}$-derivation in the complex f-algebra $\mathscr{C}{\downarrow}$;*

(4) *each band preserving endomorphism is a band projection in $\mathscr{C}{\downarrow}$;*

(5) *there is no band preserving automorphism other than the identity in $\mathscr{C}{\downarrow}$.*

(6) *the K-space $\mathscr{R}{\downarrow}$ is locally one-dimensional;*

(7) $\mathbb{B}$ *is σ-distributive.*

4.7. The question was raised by A. W. Wickstead in [65] whether every band preserving linear operator in a universally complete vector lattice is automatically order bounded. The first example of an unbounded band preserving linear operator was suggested by Yu. A. Abramovich, A. I. Veksler, and A. V. Koldunov in [1, 2]. The equivalence (1) $\leftrightarrow$ (6) is trivial, whereas (2) $\leftrightarrow$ (6) combines a result of Yu. A. Abramovich, A. I. Veksler, and A. V. Koldunov [1, Theorem 2.1] and that of P. T. N. McPolin and A. W. Wickstead [44, Theorem 3.2]. The equivalence (6) $\leftrightarrow$ (7) was obtained by A. E. Gutman [15]; he also found an example of a purely nonatomic locally one-dimensional Dedekind complete vector lattice (see [14]). The equivalences (1) $\leftrightarrow$ (3) $\leftrightarrow$ (4) $\leftrightarrow$ (5) belong to A. G. Kusraev [33].

5. Boolean Valued Positive Functionals

A linear functional on a vector space is determined up to a scalar from its zero hyperplane. In contrast, a linear operator is recovered from its kernel up to a simple multiplier on a rather special occasion. Fortunately, Boolean valued analysis prompts us that some operator analog of the functional case is valid for each operator with target a Kantorovich space, a Dedekind complete vector lattice. We now proceed along the lines of this rather promising approach.

5.1. Let E be a vector lattice, and let F be a K-space with base a complete Boolean algebra $\mathbb{B}$. By 3.2, we may assume that F is a nonzero space embedded as an order dense ideal in the universally complete Kantorovich space $\mathscr{R}{\downarrow}$ which is the descent of the reals $\mathscr{R}$ inside the separated Boolean valued universe $\mathbb{V}^{(\mathbb{B})}$ over $\mathbb{B}$.

An operator T is F-*discrete* if $[0, T] = [0, I_F] \circ T$; i.e., for all $0 \le S \le T$ there is some $0 \le \alpha \le I_F$ satisfying $S = \alpha \circ T$. Let $L_a^{\sim}(E, F)$ be the band in $L^{\sim}(E, F)$ spanned by F-discrete operators and $L_d^{\sim}(E, F) := L_a^{\sim}(E, F)^{\perp}$. By analogy we define $(E^{\wedge\sim})_a$ and $(E^{\wedge\sim})_d$. The members of $L_d^{\sim}(E, F)$ are usually called F-*diffuse*.

5.2. As usual, we let $E^{\wedge}$ stand for the standard name of E in $\mathbb{V}^{(\mathbb{B})}$. Undoubtedly, $E^{\wedge}$ is a vector lattice over $\mathbb{R}^{\wedge}$ inside $\mathbb{V}^{(\mathbb{B})}$. Denote by $\tau := T{\uparrow}$ the ascent of T to $\mathbb{V}^{(\mathbb{B})}$. Clearly, τ acts from $E^{\wedge}$ to the ascent $F{\uparrow} = \mathscr{R}$ of F inside the Boolean valued universe $\mathbb{V}^{(\mathbb{B})}$. Therefore, $\tau(x^{\wedge}) = Tx$ inside $\mathbb{V}^{(\mathbb{B})}$ for all $x \in E$, which means in terms of truth values that $[\![\tau : E^{\wedge} \to \mathscr{R}]\!] = \mathbb{1}$ and $(\forall x \in E)\ [\![\tau(x^{\wedge}) = Tx]\!] = \mathbb{1}$.

Let $E^{\wedge\sim}$ stand for the space of all order bounded $\mathbb{R}^{\wedge}$-linear functionals from $E^{\wedge}$ to $\mathscr{R}$. Clearly, $E^{\wedge\sim} := L^{\sim}(E^{\wedge}, \mathscr{R})$ is a K-space inside $\mathbb{V}^{(\mathbb{B})}$. The descent $E^{\wedge\sim}{\downarrow}$ of $E^{\wedge\sim}$ is a K-space. Given $S, T \in L^{\sim}(E, F)$, put $\tau := T{\uparrow}$ and $\sigma := S{\uparrow}$.

5.3. Theorem. *For each $T \in L^{\sim}(E,F)$ the ascent $T{\uparrow}$ of T is an order bounded $\mathbb{R}^{\wedge}$-linear functional on $E^{\wedge}$ inside $\mathbb{V}^{(\mathbb{B})}$; i.e., $[\![T{\uparrow} \in E^{\wedge\sim}]\!] = \mathbb{1}$. The mapping $T \mapsto T{\uparrow}$ is a lattice isomorphism of $L^{\sim}(E,F)$ and $E^{\wedge\sim}{\downarrow}$. Moreover, the following hold:*
(1) $T \geq 0 \leftrightarrow [\![\, \tau \geq 0 \,]\!] = \mathbb{1}$;
(2) *S is a fragment of $T \leftrightarrow [\![\, \sigma$ is a fragment of $\tau \,]\!] = \mathbb{1}$;*
(3) *T is a lattice homomorphism $\leftrightarrow [\![\, \tau$ is a lattice homomorphism $]\!] = \mathbb{1}$;*
(4) *T is F-diffuse $\leftrightarrow [\![\, \tau$ is diffuse $]\!] = \mathbb{1}$;*
(5) $T \in L_a^{\sim}(E,F) \leftrightarrow [\![\, \tau \in (E^{\wedge\sim})_a \,]\!] = \mathbb{1}$;
(6) $T \in L_d^{\sim}(E,F) \leftrightarrow [\![\, \tau \in (E^{\wedge\sim})_d \,]\!] = \mathbb{1}$.

Since τ, the ascent of an order bounded operator T, is defined up to a scalar from $\ker(\tau)$, we infer the following analog of the Sard Theorem.

5.4. Theorem. *Let S and T be linear operators from E to F. Then $\ker(bS) \supset \ker(bT)$ for all $b \in \mathbb{B}$ if and only if there is an orthomorphism α of F such that $S = \alpha T$.*

We see that a linear operator T is, in a sense, determined up to an orthomorphism from the family of the kernels of the *strata* bT of T. This remark opens a possibility of studying some properties of T in terms of the kernels of the strata of T.

5.5. Theorem. *An order bounded operator T from E to F may be presented as the difference of some lattice homomorphisms if and only if the kernel of each stratum bT of T is a vector sublattice of E for all $b \in \mathbb{B}$.*

Straightforward calculations of truth values show that $T_+{\uparrow} = \tau_+$ and $T_-{\uparrow} = \tau_-$ inside $\mathbb{V}^{(\mathbb{B})}$. Moreover, $[\![\ker(\tau)$ is a vector sublattice of $X^{\wedge}]\!] = \mathbb{1}$ whenever so are $\ker(bT)$ for all $b \in \mathbb{B}$. Since the ascent of a sum is the sum of the ascents of the summands, we reduce the proof of Theorem 5.5 to the case of the functionals on using 5.3 (3).

5.6. Recall that a subspace H of a vector lattice E is a *G-space* or *Grothendieck subspace* (cp. [13, 40]) provided that H enjoys the following property:

$$(\forall x, y \in H)\ (x \vee y \vee 0 + x \wedge y \wedge 0 \in H).$$

By simple calculations of truth values we infer that $[\![\ker(\tau)$ is a Grothendieck subspace of $E^{\wedge}]\!] = \mathbb{1}$ if and only if the kernel of each stratum bT is a Grothendieck subspace of E. We may now assert that the following appears as a result of "descending" its scalar analog.

5.7. Theorem. *The modulus of an order bounded operator $T : E \to F$ is the sum of some pair of lattice homomorphisms if and only if the kernel of each stratum bT of T with $b \in \mathbb{B}$ is a Grothendieck subspace of the ambient vector lattice E.*

To prove the relevant scalar versions of Theorems 5.5 and 5.7, we use one of the formulas of subdifferential calculus (cp. [34]):

5.8. Decomposition Theorem. *Assume that $H_1, \dots, H_N$ are cones in a vector lattice E. Assume further that f and g are positive functionals on E. The inequality $f(h_1 \vee \dots \vee h_N) \geq g(h_1 \vee \dots \vee h_N)$ holds for all $h_k \in H_k$ $(k := 1, \dots, N)$ if and only if to each decomposition of g into a sum of N positive terms $g = g_1 + \dots + g_N$ there is a decomposition of f into a sum of N positive terms $f = f_1 + \dots + f_N$ such that $f_k(h_k) \geq g_k(h_k) \quad (h_k \in H_k;\ k := 1, \dots, N)$.*

5.9. Theorems 5.5 and 5.7 were obtained by S. S. Kutateladze in [38, 39]. Theorem 5.8 appeared in this form in [36].

Note that the sums of lattice homomorphisms were first described by S. J. Bernau, C. B. Huijsmans, and B. de Pagter in terms of n-disjoint operators in [9]. A survey of some conceptually close results on n-disjoint operators is given in [32].

6. Boolean Valued Banach Spaces

In this section we discuss the transfer principle of Boolean valued analysis in regard to lattice-normed spaces. It turns out that the interpretation of a Banach space inside an arbitrary Boolean valued model is a Banach–Kantorovich space. Conversely, the universal completion of each lattice-normed space becomes a Banach space on ascending in a suitable Boolean valued model. This open up an opportunity to transfer the available theorems on Banach spaces to analogous results on lattice-normed spaces by the technique of Boolean valued analysis.

6.1. Consider a vector space X and a real vector lattice E. Note that all vector lattices under consideration are assumed Archimedean. An *E-valued norm* is a mapping $\lvert\cdot\rvert : X \to E_+$ such that

(1) $\lvert x\rvert = 0 \Longleftrightarrow x = 0 \quad (x \in X)$;

(2) $\lvert\lambda x\rvert = |\lambda|\lvert x\rvert \quad (\lambda \in \mathbb{R},\ x \in X)$;

(3) $\lvert x + y\rvert \le \lvert x\rvert + \lvert y\rvert \quad (x, y \in X)$.

A vector norm is *decomposable* if

(4) for all $e_1, e_2 \in E_+$ and $x \in X$, from $\lvert x\rvert = e_1+e_2$ it follows that there exist $x_1, x_2 \in X$ such that $x = x_1 + x_2$ and $\lvert x_k\rvert = e_k$ $(k := 1, 2)$.

If (4) is valid only for disjoint $e_1, e_2 \in E_+$ then the norm is *d-decomposable*. A triple $\big(X, \lvert\cdot\rvert, E\big)$ as well as briefer versions is a *lattice-normed space* over E whenever $\lvert\cdot\rvert$ is an E-valued norm on X.

6.2. By a *Boolean algebra of projections* in a vector space X we mean a set $\mathscr{B}$ of commuting idempotent linear operators in X. Moreover, the Boolean operations have the following form:

$$\pi\wedge\rho := \pi\circ\rho = \rho\circ\pi, \quad \pi\vee\rho = \pi+\rho-\pi\circ\rho, \quad \pi^* = I_x-\pi \quad (\pi, \rho \in \mathscr{B}),$$

and the zero and identity operators in X serve as the zero and unity of the Boolean algebra $\mathscr{B}$.

Suppose that E is a vector lattice with the projection property and $E = \lvert X\rvert^{\perp\perp} := \{\lvert x\rvert : x \in X\}^{\perp\perp}$. If (X, E) is a d-decomposable lattice-normed space then there exists a complete Boolean algebra $\mathscr{B}$ of band projections in X and an isomorphism h from $\mathfrak{P}(E)$ onto $\mathscr{B}$ such that

$$b\lvert x\rvert = \lvert h(b)x\rvert \quad \big(b \in \mathfrak{P}(E),\ x \in X\big).$$

We identify the Boolean algebras $\mathfrak{P}(E)$ and $\mathscr{B}$ and write $\pi\lvert x\rvert = \lvert \pi x\rvert$ for all $x \in X$ and $\pi \in \mathfrak{P}(E)$.

6.3. A net $(x_\alpha)_{\alpha\in\mathrm{A}}$ in X is *bo-convergent* to $x \in X$ (in symbols: $x = bo\text{-}\lim x_\alpha$) if $(\lvert x - x_\alpha\rvert)_{\alpha\in\mathrm{A}}$ is o-convergent to zero. A lattice-normed space X is *bo-complete* if each net $(x_\alpha)_{\alpha\in\mathrm{A}}$ is bo-convergent to some element of X provided that $(\lvert x_\alpha - x_\beta\rvert)_{(\alpha,\beta)\in\mathrm{A}\times\mathrm{A}}$ is o-convergent to zero. A decomposable bo-complete lattice-normed space $(X, \lvert\cdot\rvert, E)$ is called a *Banach–Kantorovich space*. If E is a universally complete Kantorovich space then X is also referred to as universally complete. By a *universal completion* of a lattice-normed space (X, E) we mean a universally complete Banach–Kantorovich space $(Y, m(E))$ together with a linear isometry $\imath : X \to Y$ such that each universally complete bo-complete subspace of $(Y, m(E))$ containing $\imath(X)$ coincides with Y. Here $m(E)$ is a universal completion of E.

6.4. Theorem. *Let $(\mathscr{X}, \|\cdot\|)$ be a Banach space inside $\mathbb{V}^{(\mathbb{B})}$. Put $X := \mathscr{X}{\downarrow}$ and $|\cdot| := \|\cdot\|{\downarrow}(\cdot)$. Then $\big(X, |\cdot|, \mathscr{R}{\downarrow}\big)$ is a universally complete Banach–Kantorovich space. Moreover, X can be endowed with the structure of a faithful unitary module over the ring $\Lambda := \mathscr{C}{\downarrow}$ so that $|ax| = |a|\,|x|$ and $b \leq [\![\, x = 0 \,]\!] \leftrightarrow \chi(b)x = 0$ for all $a \in \mathscr{C}{\downarrow}$, $x \in X$, and $b \in \mathbb{B}$, where χ is an isomorphism of $\mathbb{B}$ onto $\mathfrak{P}(X)$.*

6.5. Theorem. *To each lattice-normed space $(X, |\cdot|)$, there exists a unique Banach space (up to a linear isometry) $\mathscr{X}$ inside $\mathbb{V}^{(\mathbb{B})}$, with $\mathbb{B} \simeq \mathfrak{B}\left(|X|^{\perp\perp}\right)$, such that the descent $\mathscr{X}{\downarrow}$ of $\mathscr{X}$ is a universal completion of X.*

As in 3.1, we call $x \in X$ and $y \in Y$ *disjoint* and write $x \perp y$ whenever $|x| \wedge |y| = 0$. Let X and Y be Banach–Kantorovich spaces over some K-space G. An operator T is *band preserving* if $x \perp y$ implies $Tx \perp y$ for all $x \in X$ and $y \in Y$. Denote by $\mathscr{L}_G(X, Y)$ the space of all band preserving operators $T : X \to Y$ that send all norm-o-bounded sets into norm-o-bounded sets.

6.6. Theorem. *Let $\mathscr{X}$ and $\mathscr{Y}$ be Boolean valued representations for Banach–Kantorovich spaces X and Y normed by some universally complete K-space $G := \mathscr{R}{\downarrow}$. Let $\mathscr{L}^{\mathbb{B}}(\mathscr{X}, \mathscr{Y})$ be the space of bounded linear operators from $\mathscr{X}$ into $\mathscr{Y}$ inside $\mathbb{V}^{(\mathbb{B})}$, where $\mathbb{B} := \mathfrak{B}(G)$. The descent and ascent mappings (for operators) implement linear isometries between the lattice-normed spaces $\mathscr{L}_G(X, Y)$ and $\mathscr{L}^{\mathbb{B}}(\mathscr{X}, \mathscr{Y}){\downarrow}$.*

6.7. The concept of lattice-normed space was suggested by L. V. Kantorovich in 1936 [23]. It is worth stressing that [23] is the fist article with the unusual decomposability axiom for an abstract norm. Paradoxically, this axiom was often omitted as inessential in the further papers by other authors. The profound importance of 6.1 (4) was revealed by Boolean valued analysis. The connection between the decomposability and existence of a Boolean algebra of projections in a lattice-normed space was discovered in [28, 29]. The theory of lattice-normed spaces and dominated operators is set forth in [32]. As regards the Boolean valued approach, see [35].

7. Boolean Valued Order Continuous Functionals

We now address the class of o-continuous order bounded operators that turn into o-continuous functionals on ascending to a suitable Boolean valued model.

7.1. Assume that a lattice-normed space X is simultaneously a vector lattice. The norm $|\cdot| : X \to E_+$ of X is *monotone* if from $|x| \leq |y|$ it follows that $|x| \leq |y|$ $(x, y \in X)$. In this event, X is a *lattice-normed vector lattice*. Moreover, if X is a Banach–Kantorovich space then X is called a *Banach–Kantorovich lattice*.

We say that the norm $|\cdot|$ in X is *additive* if $|x+y| = |x|+|y|$ for all $x, y \in X_+$; it is *order semicontinuous* or *o-semicontinuous* for short if $\sup|x_\alpha| = |\sup x_\alpha|$ for each increasing net $(x_\alpha) \subset X$ with the least upper bound $x \in X$; and it is *order continuous* or *o-continuous* if $\inf|x_\alpha| = 0$ for every decreasing net $(x_\alpha) \subset X$ with $\inf_\alpha x_\alpha = 0$.

The Boolean valued interpretation of Banach–Kantorovich lattices proceeds along the lines of the previous section.

7.2. Theorem. *Let $(X, |\cdot|)$ be a Banach–Kantorovich space and let $(\mathscr{X}, \|\cdot\|) \in \mathbb{V}^{(\mathbb{B})}$ stand for its Boolean valued realization. Then*

(1) *X is a Banach–Kantorovich lattice if and only if $\mathscr{X}$ is a Banach lattice inside $\mathbb{V}^{(\mathbb{B})}$;*

(2) *X is an order complete Banach–Kantorovich lattice if and only if $\mathscr{X}$ is an order complete Banach lattice inside $\mathbb{V}^{(\mathbb{B})}$;*

(3) *the norm* $|\cdot|$ *is o-continuous (order semicontinuous, monotone complete, or additive) if and only if the norm* $\|\cdot\|$ *is o-continuous (order semicontinuous, monotone complete, or additive) inside* $\mathbb{V}^{(\mathbb{B})}$.

7.3. Let E be a vector lattice, let F be some K-space, and let T be a positive operator from E to F. Say that T possesses the *Maharam property* if, for all $x \in E_+$ and $0 \leq f \leq Tx \in F_+$, there is some $0 \leq e \leq x$ satisfying $f = Te$. An o-continuous positive operator with the Maharam property is a *Maharam operator*. Observe that $T \in L(E, F)_+$ possesses the Maharam property if only if the equality $T([0, x]) = [0, Tx]$ holds for all $x \in E_+$. Thus, a Maharam operator is exactly an o-continuous order-interval preserving positive operator.

Let T be an essentially positive operator from E to F enjoying the Maharam property. Put $|e| := T(|x|)$ $(e \in E)$. Then $(E, |\cdot|)$ is a disjointly decomposable lattice-normed space over F.

Put $F_T := \{T(|x|) : x \in E\}^{\perp\perp}$, and let $\mathscr{D}_m(T)$ stand for the greatest order dense ideal of the universal completion $m(E)$ of E among those to which T can be extended by o-continuity. In other words, $z \in \mathscr{D}_m(T)$ if and only if $z \in m(E)$ and the set $\{T(x) : x \in E,\ 0 \leq x \leq |z|\}$ is bounded in F. In this event there exists a minimal extension of T to $\mathscr{D}_m(T)$ presenting an o-continuous positive operator.

Let E and F be some K-spaces, and let $T : E \to F$ be a Maharam operator. Put $X := \mathscr{D}_m(T)$ and $|x| := \Phi(|x|)$ $(x \in X)$, where Φ is an o-continuous extension of T to X. Then $(X, |\cdot|)$ is a Banach–Kantorovich lattice whose norm is o-continuous and additive.

7.4. Theorem. *Let X be an arbitrary K-space and let E be a universally complete K-space $\mathscr{R}{\downarrow}$. Assume that $\Phi : X \to E$ is a Maharam operator such that $X = X_\Phi = \mathscr{D}_m(\Phi)$ and $E = E_\Phi$. Then there are elements $\mathscr{X}$ and φ in $\mathbb{V}^{(\mathbb{B})}$ satisfying*

(1) $[\![\mathscr{X}$ *is a K-space,* $\varphi : \mathscr{X} \to \mathscr{R}$ *is a positive o-continuous functional, and* $\mathscr{X} = \mathscr{X}_\varphi = \mathscr{D}_m(\varphi)]\!] = \mathbb{1}$;

(2) *if* $X' := \mathscr{X}{\downarrow}$ *and* $\Phi' = \varphi{\downarrow}$ *then* X' *is a K-space and* $\Phi' : X' \to E$ *is a Maharam operator;*

(3) *there is a linear and lattice isomorphism h from X onto X' such that* $\Phi = \Phi' \circ h$;

(4) *for a linear operator Ψ, the containment* $\Psi \in \{\Phi\}^{\perp\perp}$ *is true if and only if there is* $\psi \in \mathbb{V}^{(\mathbb{B})}$ *such that* $\psi \in \{\varphi\}^{\perp\perp}$ *inside* $\mathbb{V}^{(\mathbb{B})}$ *and* $\Psi = (\psi{\downarrow}) \circ h$.

Theorem 7.4 enables us to claim that each fact about o-continuous positive linear functionals in K-spaces has a parallel version for Maharam operators which can be revealed by using 7.4. For instance, we state the abstract

7.5. Radon–Nikodým Theorem. *Let E and F be K-spaces. Assume further that S and T are o-continuous positive operators from E to F, with T enjoying the Maharam property. Then the following are equivalent:*

(1) $S \in \{T\}^{\perp\perp}$;

(2) $Sx \in \{Tx\}^{\perp\perp}$ *for all* $x \in E_+$;

(3) *there is an extended orthomorphism* $0 \leq \rho \in \mathrm{Orth}^\infty(E)$ *satisfying* $Sx = T(\rho x)$ *for all* $x \in E$ *such that* $\rho x \in E$;

(4) *there is a sequence of orthomorphisms* $(\rho_n) \subset \mathrm{Orth}(E)$ *such that* $Sx = \sup_n T(\rho_n x)$ *for all* $x \in E$.

7.6. A brief description for Maharam's approach to studying positive operators in the spaces of measurable functions and the main results in this area are collected in [43].

W. A. J. Luxemburg and A. R. Schep [41] extended a portion of Maharam's theory on the Radon–Nikodým Theorem to the case of positive operators in vector lattices.

Theorems 7.2 and 7.4 were obtained by A. G. Kusraev [27] and Theorem 7.5, by W. A. J. Luxemburg and A. R. Schep [41]. About various applications of the above results on Maharam operators and some extension of this theory to sublinear and convex operators see [29, 32, 34, 35].

8. Spaces with Mixed Norm

The definitions of various objects of functional analysis rest often on some blending of the norm and order properties. Among these are listed the spaces with mixed norm and the classes of linear operators between them.

8.1. If (X, E) is a lattice-normed space whose *norm lattice* E is a Banach lattice. Since, by definition, $|x| \in E$ for $x \in X$, we may introduce the *mixed norm* on X by the formula

$$|\!|\!|x|\!|\!| := \big\| |x| \big\| \quad (x \in X).$$

In this situation, the normed space $(X, |\!|\!|\cdot|\!|\!|)$ is called a *space with mixed norm.* A *Banach space with mixed norm* is a pair (X, E) with E a Banach lattice and X a *br*-complete lattice-normed space with E-valued norm. The following proposition justifies this definition.

Let E be a Banach lattice. Then $(X, |\!|\!|\cdot|\!|\!|)$ is a Banach space if and only if the lattice-normed space (X, E) is relatively uniformly complete.

8.2. Let Λ be the *bounded part* of the universally complete K-space $\mathscr{R}{\downarrow}$, i.e. Λ is the order-dense ideal in $\mathscr{R}{\downarrow}$ generated by the order unity $\mathbb{1} := 1^{\wedge} \in \mathscr{R}{\downarrow}$. Take a Banach space $\mathscr{X}$ inside $\mathbb{V}^{(\mathbb{B})}$. Put

$$\mathscr{X}{\Downarrow} := \{x \in \mathscr{X}{\downarrow} : |x| \in \Lambda\}, \quad |\!|\!|x|\!|\!| := \| |x| \|_{\infty} := \inf\{0 < \lambda \in \mathbb{R} : |x| \le \lambda \mathbb{1}\}.$$

Then $\mathscr{X}{\Downarrow}$ is a Banach–Kantorovich space called the *bounded descent* of $\mathscr{X}$. Since Λ is an order complete AM-space with unity, $\mathscr{X}{\Downarrow}$ is a Banach space with mixed norm over Λ.

Thus, we came to the following natural question: Which Banach spaces are linearly isometric to the bounded descents of internal Banach spaces? The answer is given in terms of $\mathbb{B}$-cyclic Banach spaces.

8.3. Let X be a normed space. Suppose that $\mathscr{L}(X)$ has a complete Boolean algebra of norm one projections $\mathscr{B}$ which is isomorphic to $\mathbb{B}$. In this event we will identify the Boolean algebras $\mathscr{B}$ and $\mathbb{B}$, writing $\mathbb{B} \subset \mathscr{L}(X)$. Say that X is a *normed* $\mathbb{B}$-*space* if $\mathbb{B} \subset \mathscr{L}(X)$ and for every partition of unity $(b_\xi)_{\xi\in\Xi}$ in $\mathbb{B}$ the two conditions hold:

(1) If $b_\xi x = 0$ $(\xi \in \Xi)$ for some $x \in X$ then $x = 0$;

(2) If $b_\xi x = b_\xi x_\xi$ $(\xi \in \Xi)$ for $x \in X$ and a family $(x_\xi)_{\xi\in\Xi}$ in X then $\|x\| \le \sup\{\|b_\xi x_\xi\| : \xi \in \Xi\}$.

Given a partition of unity (b_ξ), we refer to $x \in X$ satisfying the condition $(\forall\, \xi \in \Xi)\, b_\xi x = b_\xi x_\xi$ as a *mixing* of (x_ξ) by (b_ξ). If (1) holds then there is a unique mixing x of (x_ξ) by (b_ξ). In these circumstances we naturally call x the *mixing* of (x_ξ) by (b_ξ). Condition (2) maybe paraphrased as follows: *The unit ball U_X of X is closed under mixing.*

A normed $\mathbb{B}$-space X is $\mathbb{B}$-*cyclic* if we may find in X a mixing of each norm-bounded family by each partition of unity in $\mathbb{B}$. It is easy to verify that X is a $\mathbb{B}$-cyclic normed space if and only if, given a partition of unity $(b_\xi) \subset \mathbb{B}$ and a family $(x_\xi) \subset U_X$, we may find a unique element $x \in U_X$ such that $b_\xi x = b_\xi x_\xi$ for all ξ.

A linear operator (linear isometry) S between normed $\mathbb{B}$-spaces is $\mathbb{B}$-*linear* ($\mathbb{B}$-*isometry*) if S commutes with the projections in $\mathbb{B}$; i.e., $\pi \circ S = S \circ \pi$ for all $\pi \in \mathbb{B}$. Denote by $\mathscr{L}_{\mathbb{B}}(X,Y)$ the set of all bounded $\mathbb{B}$-linear operators from X to Y. We call $X^{\#} := \mathscr{L}_{\mathbb{B}}(X, \mathbb{B}(\mathbb{R}))$ the $\mathbb{B}$-*dual* of X. If $X^{\#}$ and Y are $\mathbb{B}$-isometric to each other then we say that Y is a $\mathbb{B}$-*dual space* and X is a $\mathbb{B}$-*predual* of Y.

8.4. Theorem. *A Banach space X is linearly isometric to the bounded descent of some Banach space $\mathscr{X}$ inside $\mathbb{V}^{(\mathbb{B})}$ (called a Boolean valued representation of X) if and only if X is $\mathbb{B}$-cyclic. If X and Y are $\mathbb{B}$-cyclic Banach spaces and $\mathscr{X}$ and $\mathscr{Y}$ stand for some Boolean valued representations of X and Y, then the space $\mathscr{L}_{\mathbb{B}}(X,Y)$ is $\mathbb{B}$-isometric to the bounded descent of the internal space $\mathscr{L}(\mathscr{X},\mathscr{Y})$ of all bounded linear operators from $\mathscr{X}$ to $\mathscr{Y}$.*

8.5. Let Λ be a Stone algebra with unity $\mathbb{1}$ (= an order complete complex AM-space with strong order unity $\mathbb{1}$ and uniquely defined multiplicative structure) and consider a unitary Λ-module X. The mapping $\langle \cdot \,|\, \cdot \rangle : X \times X \to \Lambda$ is a Λ-*valued inner product*, if for all $x, y, z \in X$ and $a \in \Lambda$ the following are satisfied:

(1) $\langle x \,|\, x \rangle \geq \mathbb{0}$; $\langle x \,|\, x \rangle = \mathbb{0} \leftrightarrow x = \mathbb{0}$;

(2) $\langle x \,|\, y \rangle = \langle y \,|\, x \rangle^*$;

(3) $\langle ax \,|\, y \rangle = a\langle x \,|\, y \rangle$;

(4) $\langle x + y \,|\, z \rangle = \langle x \,|\, z \rangle + \langle y \,|\, z \rangle$.

Using a Λ-valued inner product, we may introduce the norm in X by $|\!|\!| x |\!|\!| := \sqrt{\| \langle x|x \rangle \|}$ and the decomposable vector norm $|x| := \sqrt{\langle x|x \rangle}$ ($x \in X$). Obviously, $|\!|\!| x |\!|\!| = \big\| |x| \big\|$ for all $x \in X$, and so X is a space with mixed norm.

8.6. Let X be a Λ-module with an inner product $\langle \cdot \,|\, \cdot \rangle : X \times X \to \Lambda$. If X is complete with respect to the mixed norm $|\!|\!| \cdot |\!|\!|$ then X is called a C^*-*module* over Λ. It can be proved (see [32]) that for a C^*-module X the pair $(X, |\!|\!| \cdot |\!|\!|)$ is a $\mathbb{B}$-cyclic Banach space if and only if $(X, |\cdot|)$ is a Banach–Kantorovich space over Λ. If a unitary C^*-module satisfies one of these equivalent conditions then it is called a *Kaplansky–Hilbert module.*

8.7. Theorem. *The bounded descent of an arbitrary Hilbert space in $\mathbb{V}^{(\mathbb{B})}$ is a Kaplansky–Hilbert module over the Stone algebra $\mathscr{C}{\Downarrow}$. Conversely, if X is a Kaplansky–Hilbert module over $\mathscr{C}{\Downarrow}$, then there is a Hilbert space $\mathscr{X}$ in $\mathbb{V}^{(\mathbb{B})}$ whose bounded descent is unitarily equivalent with X. This space is unique up to unitary equivalence inside $\mathbb{V}^{(\mathbb{B})}$.*

8.8. Theorem. *Suppose that $\mathscr{X}$ and $\mathscr{Y}$ are Hilbert spaces inside $\mathbb{V}^{(\mathbb{B})}$. Let X and Y stand for the bounded descents of $\mathscr{X}$ and $\mathscr{Y}$. Then the space $\mathscr{L}_{\mathbb{B}}(X,Y)$ of all $\mathbb{B}$-linear bounded operators is a $\mathbb{B}$-cyclic Banach space $\mathbb{B}$-isometric to the bounded descent of the internal Banach space $\mathscr{L}^{\mathbb{B}}(\mathscr{X},\mathscr{Y})$ of bounded linear operators from $\mathscr{X}$ to $\mathscr{Y}$.*

8.9. Boolean valued analysis approach gives rise to an interesting concept of cyclically compact operator in a Banach $\mathbb{B}$-space [32, 8.5.5]. Without plunging into details we formulate a result on the general form of cyclically compact operators in Kaplansky–Hilbert modules.

Theorem. *Let X and Y be Kaplansky–Hilbert modules over a Stone algebra Λ and let T be a cyclically compact operator from X to Y. There are orthonormal families $(e_k)_{k\in\mathbb{N}}$ in X, $(f_k)_{k\in\mathbb{N}}$ in Y, and a family $(\mu_k)_{k\in\mathbb{N}}$ in Λ such that the following hold:*

(1) $\mu_{k+1} \leq \mu_k$ ($k \in \mathbb{N}$) *and* $o\text{-}\lim_{k\to\infty} \mu_k = 0$;

(2) *there exists a projection π_∞ in Λ such that $\pi_\infty \mu_k$ is a weak order unity in $\pi_\infty \Lambda$ for all $k \in \mathbb{N}$;*

(3) *there exists a partition $(\pi_k)_{k=0}^\infty$ of the projection $\pi_\infty^\perp$ such that $\pi_0 \mu_1 = 0$, $\pi_k \leq \mu_k$, and $\pi_k \mu_{k+1} = 0$ for all $k \in \mathbb{N}$;*

(4) *the representation is valid*

$$T = \pi_\infty \, bo\text{-}\sum_{k=1}^{\infty} \mu_k e_k^\# \otimes f_k + bo\text{-}\sum_{n=1}^{\infty} \pi_n \sum_{k=1}^{n} \mu_k e_k^\# \otimes f_k.$$

8.10. The bounded descent of 8.2 appeared in the research by G. Takeuti into von Neumann algebras and C^*-algebras within Boolean valued models [60, 61] and in the research by M. Ozawa into Boolean valued interpretation of the theory of Hilbert spaces [45]. Theorems 8.4 and 8.9 were obtained by A. G. Kusraev in [28, 29, 32]. Theorems 8.7 and 8.8 were proved by M. Ozawa [45].

9. Banach Algebras

The possibility of applying Boolean valued analysis to operator algebras rests on the following observation: If the center of an algebra is properly qualified and perfectly located then it becomes a one-dimensional subalgebra after ascending in a suitable Boolean valued universe. This might lead to a simpler algebra. On the other hand, the transfer principle implies that the scope of the formal theory of the initial algebra is the same as that of its Boolean valued representation.

9.1. An *AW^*-algebra* is a C^*-algebra presenting a Baer $*$-algebra. More explicitly, an AW^*-algebra is a C^*-algebra A whose every *right annihilator* $M^\perp := \{y \in A : (\forall x \in M)\ xy = 0\}$ has the form pA, with p a projection. A *projection* p is a hermitian ($p^* = p$) idempotent ($p^2 = e$) element. An element $z \in A$ is said to be *central* if it commutes with every member of A. The *center* of an AW^*-algebra A is the set $\mathscr{Z}(A)$ of all central elements. Clearly, $\mathscr{Z}(A)$ is a commutative AW^*-subalgebra of A, with $\lambda \mathbb{1} \in \mathscr{Z}(A)$ for all $\lambda \in \mathbb{C}$. If $\mathscr{Z}(A) = \{\lambda \mathbb{1} : \lambda \in \mathbb{C}\}$ then the AW^*-algebra A is called an *AW^*-factor*.

The symbol $\mathfrak{P}(A)$ stands for the set of all projections of an involutive algebra A. Denote the set of all central projections by $\mathfrak{P}_c(A)$.

9.2. Theorem. *Assume that $\mathscr{A}$ is an AW^*-algebra inside $\mathbb{V}^{(\mathbb{B})}$ and A is the bounded descent of $\mathscr{A}$. Then A is also an AW^*-algebra and, moreover, $\mathfrak{P}_c(A)$ has an order-closed subalgebra isomorphic with $\mathbb{B}$. Conversely, let A be an AW^*-algebra such that $\mathbb{B}$ is an order-closed subalgebra of the Boolean algebra $\mathfrak{P}_c(A)$. Then there is an AW^*-algebra $\mathscr{A}$ in $\mathbb{V}^{(\mathbb{B})}$ whose bounded descent is $*$-$\mathbb{B}$-isomorphic with A. This algebra $\mathscr{A}$ is unique up to isomorphism inside $\mathbb{V}^{(\mathbb{B})}$.*

Observe that if $\mathscr{A}$ is an AW^*-factor inside $\mathbb{V}^{(\mathbb{B})}$ then the bounded descent A of $\mathscr{A}$ is an AW^*-algebra whose Boolean algebra of central projections is isomorphic with $\mathbb{B}$. Conversely, if A is an AW^*-algebra and $\mathbb{B} := \mathfrak{P}_c(A)$ then there is an AW^*-factor $\mathscr{A}$ inside $\mathbb{V}^{(\mathbb{B})}$ whose bounded descent is isomorphic with A.

9.3. Take an AW^*-algebra A. Clearly, the formula

$$q \leq p \leftrightarrow q = qp = pq \quad (q, p \in \mathfrak{P}(X))$$

(sometimes reads as "p contains q") specifies some order $\leq$ on the set of projections $\mathfrak{P}(A)$. Moreover, $\mathfrak{P}(A)$ is a complete lattice and $\mathfrak{P}_c(A)$ is a complete Boolean algebra.

The classification of AW^*-algebras into types is determined from the structure of its lattice of projections [32, 52]. It is important to emphasize that Boolean valued representation preserves this classification. We recall only the definition of type I AW^*-algebra. A projection $\pi \in A$ is called *abelian* if the algebra $\pi A \pi$ is commutative. An algebra A has *type* I, if each nonzero projection in A contains a nonzero abelian projection.

We call an AW^*-algebra *embeddable* if it is $*$-isomorphic with the double commutant of some type I AW^*-algebra. Each embeddable AW^*-algebra admits a Boolean valued representation, becoming a von Neumann algebra or factor. A C^*-algebra A is called $\mathbb{B}$-*embeddable* if there is a type I AW^*-algebra N and a $*$-monomorphism $\imath : A \to N$ such that $\mathbb{B} = \mathfrak{P}_c(N)$ and $\imath(A) = \imath(A)''$, where $\imath(A)''$ is the bicommutant of $\imath(A)$ in N. Note that in this event A is an AW^*-algebra and $\mathbb{B}$ is a regular subalgebra of $\mathfrak{P}_c(A)$. In particular, A is a $\mathbb{B}$-cyclic algebra (see 8.3).

Say that a C^*-algebra A is *embeddable* if A is $\mathbb{B}$-embeddable for some regular subalgebra $\mathbb{B} \subset \mathfrak{P}_c(A)$. If $\mathbb{B} = \mathfrak{P}_c(A)$ and A is $\mathbb{B}$-embeddable then A is called a *centrally embeddable algebra*.

9.4. Theorem. *Let $\mathscr{A}$ be a C^*-algebra inside $\mathbb{V}^{(\mathbb{B})}$ and let A be the bounded descent of $\mathscr{A}$. Then A is a $\mathbb{B}$-embeddable AW^*-algebra if and only if $\mathscr{A}$ is a von Neumann algebra inside $\mathbb{V}^{(\mathbb{B})}$. The algebra A is centrally embeddable if and only if $\mathscr{A}$ is a von Neumann factor inside $\mathbb{V}^{(\mathbb{B})}$.*

Using this representation, we can obtain characterizations of embeddable AW^*-algebras. In particular, an AW^*-algebra A is embeddable if and only if the center-valued normal states of A separate A.

9.5. Theorem. *For an AW^*-algebra A the following are equivalent:*
(1) *A is embeddable;*
(2) *A is centrally embeddable;*
(3) *A has a separating set of center-valued normal states;*
(4) *A is a $\mathfrak{P}_c(A)$-predual space.*

9.6. Combining the results about the Boolean valued representations of AW^*-algebras with the analytical representations for dominated operators (see [32]), we come to some functional representations of AW^*-algebras.

Suppose that Q is an extremally disconnected compact space, H is a Hilbert space, and $B(H)$ is the space of bounded linear endomorphisms of H. Denote by $\mathfrak{C}(Q, B(H))$ the set of all operator-functions $u : \mathrm{dom}(u) \to B(H)$ on the comeager sets $\mathrm{dom}(u) \subset Q$ and continuous in the strong operator topology. Introduce some equivalence on $\mathfrak{C}(Q, B(H))$ by putting $u \sim v$ if and only if u and v agree on $\mathrm{dom}(u) \cap \mathrm{dom}(v)$.

If $u \in \mathfrak{C}(Q, B(H))$ and $h \in H$ then the vector-function $uh : q \mapsto u(q)h$ $(q \in \mathrm{dom}(u))$ is continuous thus determining a unique element $\widetilde{uh} \in C_\infty(Q, H)$ from the condition $uh \in \widetilde{uh}$. If $\tilde{u}$ is the coset of the operator-function $u : \mathrm{dom}(u) \to B(H)$ then $\tilde{u}h := \widetilde{uh}$ $(h \in H)$ by definition. Denote by $SC_\infty(Q, B(H))$ the set of all cosets $\tilde{u}$ such that $u \in \mathfrak{C}(Q, B(H))$ and the set $\{|\tilde{u}h| : \|h\| \leq \mathbb{1}\}$ is bounded in $C_\infty(Q)$. Put $|\tilde{u}| := \sup\{|\tilde{u}h| : \|h\| \leq \mathbb{1}\}$, where the supremum is taken in $C_\infty(Q)$. We naturally furnish $SC_\infty(Q, B(H))$ with the structure of a $*$-algebra and unitary $C_\infty(Q)$-module. We now introduce the following normed $*$-algebra

$$SC_\#(Q, B(H)) := \{v \in SC_\infty(Q, B(H)) : |v| \in C(Q)\},$$
$$\|v\| = \||v|\|_\infty \quad (v \in SC_\#(Q, B(H))).$$

9.7. Theorem. *To each type I AW^*-algebra A there exists a family of nonempty extremally disconnected compact spaces $(Q_\gamma)_{\gamma\in\Gamma}$ such that*

(1) *Γ is a set of cardinals and Q_γ is γ-stable for every $\gamma \in \Gamma$;*

(2) *there is a $*$-$\mathbb{B}$-isomorphism:*

$$A \simeq \sum\nolimits_{\gamma\in\Gamma}^{\oplus} SC_{\#}(Q_\gamma, B(l_2(\gamma))).$$

This family is unique up to congruence.

A cardinal number γ is *Q-stable* if $\gamma^\wedge$ is a cardinal number inside $\mathbb{V}^{(\mathbb{B})}$ and Q is the Stone space of $\mathbb{B}$.

9.8. The study of C^*-algebras and von Neumann algebras by Boolean valued models was started by G. Takeuti with [60, 61]. Theorems 9.2, 9.4, and 9.5 were obtained by M. Ozawa [47, 48, 50]. Theorem 9.7 was established by A. G. Kusraev.

Boolean valued analysis of AW^*-algebras yields a negative solution to the I. Kaplansky problem of unique decomposition of a type I AW^*-algebra into the direct sum of homogeneous bands. M. Ozawa gave this solution in [48, 49]. The lack of uniqueness is tied with the effect of the cardinal shift that may happens on ascending into a Boolean valued model $\mathbb{V}^{(\mathbb{B})}$. The cardinal shift is impossible in the case when the Boolean algebra of central idempotents $\mathbb{B}$ under study satisfies the countable chain condition, and so the decomposition in question is unique. I. Kaplansky established uniqueness of the decomposition on assuming that B satisfies the countable chain condition and conjectured that uniqueness fails in general [26].

10. JB-Algebras

We also consider similar problems for the so-called JB-algebras presenting some real nonassociative analogs of C^*-algebras.

10.1. Recall that a JB-algebra is a real Banach space A which is a unital Jordan algebra satisfying the conditions:

(1) $\|xy\| \le \|x\| \cdot \|y\| \quad (x, y \in A)$;

(2) $\|x^2\| = \|x\|^2 \quad (x \in A)$;

(3) $\|x^2\| \le \|x^2 + y^2\| \quad (x, y \in A)$.

The intersection of all maximal associative subalgebras of A is called the *center* of A and denoted by $\mathscr{Z}(A)$. Evidently $\mathscr{Z}(A)$ is an associative JB-algebra and every such algebra is isometrically isomorphic to the real Banach algebra $C(K)$ of continuous functions on a compact space K. If $\mathscr{Z}(A) = \mathbb{R}\cdot\mathbb{1}$ then A is said to be a *JB-factor.*

The idempotents of JB-algebras are also called *projections*. The set $\mathfrak{P}(A)$ of projections naturally underlies a complete lattice. The set $\mathfrak{P}_c(A)$ of all projections belonging to the center makes a Boolean algebra. Assume that $\mathbb{B}$ is a subalgebra of $\mathfrak{P}_c(A)$. Then we say that A is a *$\mathbb{B}$-JB-algebra* if to each partition of unity $(e_\xi)_{\xi\in\Xi}$ in $\mathbb{B}$ and each family $(x_\xi)_{\xi\in\Xi}$ in A there exists a unique $\mathbb{B}$-*mixing* $x := \mathrm{mix}_{\xi\in\Xi}(e_\xi x_\xi)$; i.e., a unique $x \in A$ such that $e_\xi x_\xi = e_\xi x$ for all $\xi \in \Xi$. If $\mathbb{B}(\mathbb{R}) := \mathscr{R}{\downarrow} = \mathscr{Z}(A)$ then a $\mathbb{B}$-JB-algebra is also called a *centrally extended JB-algebra.*

Clearly, the unit ball of a $\mathbb{B}$-JB-algebra A is closed under $\mathbb{B}$-mixing and so A is a $\mathbb{B}$-cyclic Banach space. Therefore, from 7.4 we can arrive to

10.2. Theorem. *The bounded descent of a JB-algebra inside $\mathbb{V}^{(\mathbb{B})}$ is a $\mathbb{B}$-JB-algebra. Conversely, to each $\mathbb{B}$-JB-algebra A there exists a unique JB-algebra $\mathscr{A}$ (up to isomorphism) whose bounded descent is isometrically $\mathbb{B}$-isomorphic to A. Moreover, $[\![\mathscr{A}$ is a JB-factor$]\!] = \mathbb{1}$ if and only if $\mathbb{B}(\mathbb{R}) = \mathscr{Z}(A)$.*

10.3. Let A be a $\mathbb{B}$-JB-algebra and $\Lambda := \mathbb{B}(\mathbb{R})$. An operator $\Phi \in A^{\#}$ is called a *Λ-valued state* if $\Phi \geq 0$ and $\Phi(\mathbb{1}) = \mathbb{1}$. A state Φ is *normal* if $\Phi(x) = o\text{-}\lim \Phi(x_\alpha)$ for every increasing net (x_α) in A with $x := \sup x_\alpha$. If $\mathscr{A}$ is a Boolean valued representation of A then the ascent $\phi := \Phi{\uparrow}$ of Φ is a normal state on $\mathscr{A}$. Conversely, if $[\![\phi$ is a normal state on $\mathscr{A}]\!] = \mathbb{1}$ then the restriction to A of $\phi{\downarrow}$ is a Λ-valued normal state. We now give a characterization of the $\mathbb{B}$-JB-algebras that are $\mathbb{B}$-dual spaces.

10.4. Theorem. *For a $\mathbb{B}$-JB-algebra A the following are equivalent:*

(1) *A is a $\mathbb{B}$-dual space;*

(2) *A is monotone complete and admits a separating set of Λ-valued normal states.*

If one of these conditions holds then the part of $A_{\#}$ consisting of o-continuous operators serves as a $\mathbb{B}$-predual of A.

This is just a Boolean valued interpretation of the following theorem by F. W. Shultz [55]: *a JB-algebra A is a dual Banach space if and only if A is monotone complete and has a separating family of normal states.*

10.5. An algebra A satisfying one of the equivalent conditions 10.4 (1,2), is called a *$\mathbb{B}$-JBW-algebra*. If, moreover, $\mathbb{B}$ coincides with the set of all central projections then A is a *$\mathbb{B}$-JBW-factor*. It follows from Theorem 10.4 that A is a $\mathbb{B}$-JBW-algebra ($\mathbb{B}$-JBW-factor) if and only if its Boolean valued representation $\mathscr{A} \in \mathbb{V}^{(\mathbb{B})}$ is a JBW-algebra (JBW-factor).

Consider one example. Let X be a Kaplansky–Hilbert module over the algebra $\overline{\Lambda} := \mathbb{B}(\mathbb{C}) := \mathscr{C}{\downarrow}$. Then X is a $\mathbb{B}$-cyclic Banach space and $\mathscr{L}_{\mathbb{B}}(X)$ is a type I AW^*-algebra. Given $x, y \in X$, define the seminorm $p_{x,y}(a) := \|\langle ax, y\rangle\|_\infty \quad (a \in \mathscr{L}_{\mathbb{B}}(X))$, where $\langle\cdot,\cdot\rangle$ is the inner product on X with values in $\overline{\Lambda}$. Denote by σ_∞ the topology on $\mathscr{L}_{\mathbb{B}}(X)$ that is generated by the system of seminorms $p_{x,y}$. Then a σ_∞-closed $\mathbb{B}$-JB-algebra of selfadjoint operators presents an example of a $\mathbb{B}$-JBW-algebra.

10.6. Let A be an associative algebra over a field of characteristic $\neq 2$. Define the new multiplication $a \circ b := 1/2(ab + ba)$ on the vector space of A. Denote the resulting algebra by A^J. This A^J is a Jordan algebra. If the subspace $A_\circ$ of A is closed under $a \circ b$ then $A_\circ$ is a subalgebra of A^J and so $A_\circ$ is Jordan. Such a Jordan algebra $A_\circ$ is called *special*. The nonspecial Jordan algebras are referred to as *exceptional*.

Let $\mathbb{O}$ be the *Cayley* or *octonian algebra*. Let $M_n(\mathbb{O})$ be the algebra of $n \times n$-matrices with entries in $\mathbb{O}$. The involution $*$ on $M_n(\mathbb{O})$ is as usual the transposition of a matrix followed by conjugation of every entry. The set $M_n(\mathbb{O})_{\mathrm{sa}} := \{x \in M_n(\mathbb{O}) : x^* = x\}$ of hermitian matrices is closed in $M_n(\mathbb{O})$ under the Jordan multiplication $x \circ y = 1/2(xy + yx)$. The real vector space $M_n(\mathbb{O})_{\mathrm{sa}}$ is a Jordan algebra under $\circ$ only for $n \leq 3$. In case $n = 1, 2$ we arrive at special Jordan algebras. The Jordan algebra $M_3(\mathbb{O})_{\mathrm{sa}}$ is special and denoted by M_3^8.

10.7. Theorem. *A special $\mathbb{B}$-JB-algebra A is a $\mathbb{B}$-JBW-algebra if and only if A is isomorphic to a σ_∞-closed $\mathbb{B}$-JB-subalgebra of $\mathscr{L}_{\mathbb{B}}(X)_{sa}$ for some Kaplansky–Hilbert module X.*

10.8. Theorem. *Each $\mathbb{B}$-JBW-factor A admits a unique decomposition $A = eA \oplus e^*A$ with a central projection $e \in \mathbb{B}$, $e^* := \mathbb{1} - e$, such that the algebra eA is special and the algebra e^*A is purely exceptional. Moreover, eA is $\mathbb{B}$-isomorphic to a σ_∞-closed subalgebra of selfadjoint endomorphisms of some AW^*-module and e^*A is isomorphic to $C(Q, M_3^8)$, where Q is the Stone compact space of the Boolean algebra $e^*\mathbb{B} := [0, e^*]$.*

10.9. The JB-algebras are nonassociative real analogs of C^*-algebras and von Neumann operator algebras. The theory of these algebras stems from the article of P. Jordan, J. von Neumann, and E. Wigner [22] and exists as a branch of functional analysis since the mid 1960s, when D. M. Topping [63] and E. Størmer [57] have started the study of the nonassociative real analogs of von Neumann algebras, the JW-algebras presenting weakly closed Jordan algebras of bounded selfadjoint operators in a Hilbert space. The steps of development are reflected in [4, 6, 16]. The Boolean valued approach to JB-algebras is outlined by A. G. Kusraev. More details and references are collected in [35].

Список литературы

[1] Abramovich Yu. A., Veksler A. I., and Koldunov A. V., "On disjointness preserving operators," Dokl. Akad. Nauk SSSR, **289**, No. 5, 1033–1036 (1979).

[2] Abramovich Yu. A., Veksler A. I., and Koldunov A. V., "Disjointness preserving operators, their continuity, and multiplicative representation," in: Linear Operators and Their Applications [in Russian], Leningrad: Leningrad Ped. Inst., 1981.

[3] Aczél J. and Dhombres J., *Functional Equations in Several Variables*, Cambridge etc.: Cambridge Univ. Press, 1989.

[4] Ayupov Sh. A., "Jordan operator algebras," in: Modern Problems of Mathematics. Recent Advances, Moscow: Inst. Nauchn. i Tekhn. Informatsii, 1985, **27**, pp. 67–97.

[5] Akilov G. P. and Kutateladze S. S., *Ordered Vector Spaces* [in Russian], Novosibirsk: Nauka, 1978.

[6] Alfsen E. M., Shultz F. W., and Størmer E., "A Gelfand–Neumark theorem for Jordan algebras," Adv. in Math., **28**, 11–56 (1978).

[7] Aliprantis C. D. and Burkinshaw O., *Positive Operators*, New York: Academic Press, 1985.

[8] Bell J. L., *Boolean-Valued Models and Independence Proofs in Set Theory*, New York etc.: Clarendon Press, 1985.

[9] Bernau S. J., Huijsmans C. B., and de Pagter B., "Sums of lattice homomorphisms," Proc. Amer. Math. Soc., **115**, No. 1, 51–156 (1992).

[10] Goodearl K. R., *Von Neumann Regular Rings*, London: Pitman, 1979.

[11] Gordon E. I., "Real numbers in Boolean valued models of set theory and K-spaces," Dokl. Akad. Nauk SSSR, **237**, No. 4, 773–775 (1977).

[12] Gordon E. I., "K-spaces in Boolean valued models of set theory," Dokl. Akad. Nauk SSSR, **258**, No. 4, 777–780 (1981).

[13] Grothendieck A., "Une caractérisation vectorielle-métrique des espaces L^1," Canad. J. Math., **4**, 552–561 (1955).

[14] Gutman A. E., "Locally one-dimensional K-spaces and σ-distributive Boolean algebras," Siberian Adv. Math., **5**, No. 2, 99–121 (1995).

[15] Gutman A. E., "Disjointness preserving operators," in: Vector Lattices and Integral Operators (Ed. S. S. Kutateladze), Dordrecht etc.: Kluwer Academic Publishers, 1996, pp. 361–454.

[16] Hanshe-Olsen H. and Størmer E., *Jordan Operator Algebras*, Boston etc.: Pitman Publ. Inc., 1984.

[17] Hofstedter D. R., *Gödel, Escher, Bach: an Eternal Golden Braid*, New York: Basic Books, 1999.

[18] Jech T. J., "Abstract theory of abelian operator algebras: an application of forcing," Trans. Amer. Math. Soc., **289**, No. 1, 133–162 (1985).

[19] Jech T. J., "First order theory of complete Stonean algebras (Boolean valued real and complex numbers)," Canad. Math. Bull., **30**, No. 4, 385–392 (1987).

[20] Jech T. J., "Boolean-linear spaces," Adv. in Math., **81**, No. 2, 117–197 (1990).

[21] Jech T., *Lectures in Set Theory with Particular Emphasis on the Method of Forcing*, Berlin: Springer-Verlag, 1971.

[22] Jordan P., von Neumann J., and Wigner E., "On an algebraic generalization of the quantum mechanic formalism," Ann. Math., **35**, 29–64 (1944).

[23] Kantorovich L. V., "On semiordered linear spaces and their applications to the theory of linear operations," Dokl. Akad. Nauk SSSR, **4**, No. 1–2, 11–14 (1935).

[24] Kantorovich L. V. and Akilov G. P., *Functional Analysis* [in Russian], Moscow: Nauka, 1984.

[25] Kantorovich L. V., Vulikh B. Z., and Pinsker A. G., *Functional Analysis in Semiordered Spaces* [in Russian], Moscow; Leningrad: Gostekhizdat, 1950.

[26] Kaplansky I., "Modules over operator algebras," Amer. J. Math., **75**, No. 4, 839–858 (1953).

[27] Kusraev A. G., "General disintegration formulas," Dokl. Akad. Nauk SSSR, **265**, No. 6, 1312–1316 (1982).

[28] Kusraev A. G., "On Banach–Kantorovich spaces," Sibirsk. Mat. Zh., **26**, No. 2, 119–126 (1985).

[29] Kusraev A. G., *Vector Duality and Its Applications* [in Russian], Novosibirsk: Nauka, 1985.

[30] Kusraev A. G., "Numeric systems in Boolean valued models of set theory," in: Proceedings of the VIII All-Union Conference in Mathematical Logic (Moscow), Moscow, 1986, p. 99.

[31] Kusraev A. G., "On band preserving operators," Vladikavkaz Math. J., **6**, No. 3, 48–58 (2004).

[32] Kusraev A. G., *Dominated Operators*, Dordrecht: Kluwer Academic Publishers, 2000.

[33] Kusraev A. G., "Automorphisms and derivations in extended complex f-algebras," Siberian Math. J., **46**, No. 6 (2005).

[34] Kusraev A. G. and Kutateladze S. S., *Subdifferentials: Theory and Applications*, Novosibirk: Nauka, 1992; Dordrecht: Kluwer Academic Publishers, 1995.

[35] Kusraev A. G. and Kutateladze S. S., *Introduction to Boolean Valued Analysis* [in Russian], Moscow: Nauka, 2005.

[36] Kutateladze S. S., "Choquet boundaries in K-spaces," Russ. Math. Surveys, **30**, No. 4, 115–155 (1975).

[37] Kutateladze S. S., "Nonstandard tools for convex analysis," Math. Japon., **43**, No. 2, 391–410 (1996).

[38] Kutateladze S. S., "On differences of Riesz homomorphisms," Siberian Math. J. **46**, No. 2, 305–307 (2005).

[39] Kutateladze S. S., "On Grothendieck subspaces," Siberian Math. J., **46**, No. 3, 489–493 (2005).

[40] Lindenstrauss J. and Wulbert D. E., "On the classification of the Banach spaces whose duals are L_1-spaces," J. Funct. Anal., **4**, No. 3, 322–249 (1969).

[41] Luxemburg W. A. J. and Schep A., "A Radon–Nikodým type theorem for positive operators and a dual," Indag. Math., **40**, 357–375 (1978).

[42] Luxemburg W. A. J. and Zaanen A. C., *Riesz Spaces.* Vol. 1, Amsterdam; London: North-Holland, 1971.

[43] Maharam D., "On positive operators," Contemporary Math., **26**, 263–277 (1984).

[44] McPolin P. T. N. and Wickstead A. W., "The order boundedness of band preserving operators on uniformly complete vector lattices," Math. Proc. Cambridge Philos. Soc., **97**, No. 3, 481–487 (1985).

[45] Ozawa M., "Boolean valued interpretation of Hilbert space theory," J. Math. Soc. Japan, **35**, No. 4, 609–627 (1983).

[46] Ozawa M., "Boolean valued analysis and type I AW^*-algebras," Proc. Japan Acad. Ser. A Math. Sci., **59A**, No. 8, 368–371 (1983).

[47] Ozawa M., "A classification of type I AW^*-algebras and Boolean valued analysis," J. Math. Soc. Japan, **36**, No. 4, 589–608 (1984).

[48] Ozawa M., “A transfer principle from von Neumann algebras to AW^*-algebras,” J. London Math. Soc. (2), **32**, No. 1, 141–148 (1985).

[49] Ozawa M., “Nonuniqueness of the cardinality attached to homogeneous AW^*-algebras,” Proc. Amer. Math. Soc, **93**, 681–684 (1985).

[50] Ozawa M., “Boolean valued analysis approach to the trace problem of AW^*-algebras,” J. London Math. Soc. (2), **33**, No. 2, 347–354 (1986).

[51] Ozawa M., “Boolean valued interpretation of Banach space theory and module structures of von Neumann algebras,” Nagoya Math. J., **117**, 1–36 (1990).

[52] Sakai S., *C^*-algebras and W^*-algebras*, Berlin etc.: Springer-Verlag, 1971.

[53] Schaefer H. H., *Banach Lattices and Positive Operators*, Berlin etc.: Springer-Verlag, 1974.

[54] Schwarz H.-V., *Banach Lattices and Operators*, Leipzig: Teubner, 1984.

[55] Shultz F. W., “On normed Jordan algebras which are Banach dual spaces,” J. Funct. Anal., **31**, 360–376 (1979).

[56] Sikorski R., *Boolean Algebras*, Berlin etc.: Springer-Verlag, 1964.

[57] Størmer E., “Jordan algebras of type I,” Acta Math., **115**, No. 3–4, 165–184 (1966).

[58] Takeuti G., *Two Applications of Logic to Mathematics*, Princeton: Princeton Univ. Press, 1978.

[59] Takeuti G., “Boolean valued analysis,” in: Applications of Sheaves (Proc. Res. Sympos. Appl. Sheaf Theory to Logic, Algebra and Anal., Univ. Durham, Durham, 1977), Berlin etc.: Springer-Verlag, 1979, pp. 714–731 (Lecture Notes in Math., 753.)

[60] Takeuti G., “Von Neumann algebras and Boolean valued analysis,” J. Math. Soc. Japan, **35**, No. 1, 1–21 (1983).

[61] Takeuti G., “C^*-algebras and Boolean valued analysis,” Japan. J. Math. (N.S.), **9**, No. 2, 207–246 (1983).

[62] Takeuti G. and Zaring W. M., *Axiomatic Set Theory*, New York: Springer-Verlag, 1973.

[63] Topping D. M., “Jordan algebras of self-adjoint operators,” Mem. Amer. Math. Soc., **53** (1965).

[64] Vulikh B. Z., *Introduction to the Theory of Partially Ordered Spaces* [in Russian], Moscow: Fizmatgiz, 1961.

[65] Wickstead A. W., “Representation and duality of multiplication operators on Archimedean Riesz spaces,” Compositio Math., **35**, No. 3, 225–238 (1977).

[66] Zaanen A. C., *Riesz Spaces*. Vol. 2, Amsterdam etc.: North-Holland, 1983.

SAUNDERS MAC LANE, THE KNIGHT OF MATHEMATICS

Abstract. This is a short obituary of Saunders Mac Lane (1909–2005)[2].

San Francisco and April 14, 2005 form the terminal place and date of the marvellous almost centennial life of the prominent American mathematician Saunders Mac Lane who shared with Samuel Eilenberg (1913–1998) the honor of creation of category theory which ranks among the most brilliant, controversial, ambitious, and heroic mathematical achievements of the 20th century.

Category theory, alongside set theory, serves as a universal language of modern mathematics. Categories, functors, and natural transformations are widely used in all areas of mathematics, allowing us to look uniformly and consistently on various constructions and formulate the general properties of diverse structures. The impact of category theory is irreducible to the narrow frameworks of its great expressive conveniences. This theory has drastically changed our general outlook on the foundations of mathematics and widened the room of free thinking in mathematics.

Set theory, a great and ingenious creation of Georg Cantor, occupies in the common opinion of the 20th century the place of the sole solid base of modern mathematics. Mathematics becomes sinking into a section of the Cantorian set theory. Most active mathematicians, teachers, and philosophers consider as obvious and undisputable the thesis that mathematics cannot be grounded on anything but set theory. The set-theoretic stance transforms paradoxically into an ironclad dogma, a clear-cut forbiddance of thinking (as L. Feuerbach once put it wittily). Such an indoctrinated view of the foundations of mathematics is false and conspicuously contradicts the leitmotif, nature, and pathos of the essence of all creative contribution of Cantor who wrote as far back as in 1883 that "denn das *Wesen* der *Mathematik* liegt gerade in ihrer*Freiheit*."

It is category theory that one of the most ambitious projects of the 20th century mathematics was realized within in the 1960s, the project of socializing set theory. This led to topos theory providing a profusion of categories of which classical set theory is an ordinary member. Mathematics has acquired infinitely many new degrees of freedom. All these rest on category theory originated with the article by Mac Lane and Eilenberg "General Theory of Natural Equivalences," which was presented to the American Mathematical Society on September 8, 1942 and published in 1945 in the Transactions of the AMS.

Mac Lane authored or coauthored more than 100 research papers and 6 books: *A Survey of Modern Algebra* (1941, 1997; with Garrett Birkhoff); *Homology* (1963); *Algebra* (1967; with Garrett Birkhoff); *Categories for the Working Mathematician* (1971, 1998); *Mathematics, Form and Function* (1985); *Sheaves in Geometry and Logic: A First Introduction to Topos Theory* (1992; with Ieke Moerdijk).

Mac Lane was the advisor of 39 Ph.D. theses. Alfred Putman, John Thompson, Irving Kaplansky, Robert Solovay, and many other distinguished scientists are listed as his students. He was elected to the National Academy of Sciences of the USA in 1949 and received the National Medal of Science, the highest scientific award of the USA in 1989. Mac Lane served as vice-president of the National Academy of Sciences and the American Philosophical Society. He was elected as president of the American Mathematical Society and Mathematical Association of America. He contributed greatly to modernization of the teaching programs in mathematics. Mac Lane received many signs of honor from the leading universities of the world and possessed an impressive collection of mathematical awards and prizes. Mac Lane became a living legend of the science of the USA.

Mac Lane was born on August 4, 1909 in Norwich near Taftville, Connecticut in the family of a Congregationalist minister and was christened as Leslie Saunders MacLane. The name Leslie was suggested by his nurse, but his mother disliked the name. A month later, his father put a hand on the head of the son, looked up to the God, and said: "Leslie forget." His father and uncles changed the spelling of their surname and began to write MacLane instead of MacLean in order to avoid sounding Irish. The space in Mac Lane was added by Saunders himself at request of his first wife Dorothy. That is how Mac Lane narrated about his name in *A Mathematical Biography* which was published soon after his death.

Saunders's father passed away when the boy was 15 and it was Uncle John who supported the boy and paid for his education in Yale. Saunders was firstly fond of chemistry but everything changed

[1]To appear in Scientiae Mathematicae Japonicae, **62**:1 (2005).

[2]A Russian version is online in Siberian Electronic Mathematical Reports **2** (2005), A5–A9.

after acquaintance with differential and integral calculus by the textbook of Longley and Wilson (which reminds of the later book by Granville, Smith, and Longley). The university years revealed Mac Lane's attraction to philosophy and foundations of mathematics. He was greatly impressed by the brand-new three volumes by Whitehead and Russell, the celebrated *Principia Mathematica*. The mathematical tastes of Mac Lane were strongly influenced by the lectures of a young assistant professor Oystein Ore, a Norwegian mathematician from the Emmy Noether's school. After graduation from Yale, Mac Lane continued education in the University of Chicago. At that time he was very much influenced by the personalities and research of Eliakim Moore, Leonard Dickson, Gilbert Bliss, Edmund Landau, Marston Morse, and many others. Mac Lane was inclined to wrote a Ph.D. thesis in logic but this was impossible in Chicago and so Saunders decided to continue education in Göttingen.

The stay in Germany in 1931–1933 was decisive for the maturity of Mac Lane's gift and personality. Although David Hilbert had retired, he still delivered weekly lectures on philosophy and relevant general issues. The successor of Hilbert was Hermann Weyl who had recently arrived from Zürich and was in the prime of his years and talents. Weyl advised Saunders to attend the lectures on linear associative algebras by Emmy Noether whom Weyl called "the equal of each of us." In the Mathematical Institute Mac Lane met and boiled with Edmund Landau, Richard Courant, Gustav Herglotz, Otto Neugebauer, Oswald Teichmüller, and many others. Paul Bernays became the advisor of Mac Lane's Ph.D thesis "Abbreviated Proofs in Logic Calculus."

The Nazis gained power in Germany in February 1933. The feast of antisemitism started immediately and one of the first and fiercest strokes fell upon the Mathematical Institute. The young persons are welcome to read as an antidote Mac Lane's masterpiece "Mathematics at Göttingen under the Nazis" in the Notices of the AMS, **42**:10, 1134–1138 (1995).

In the fall of 1933 Mac Lane returned to the States with Dorothy Jones Mac Lane whom he had married recently in Germany. The further academic career of Mac Lane was mainly tied with Harvard and since 1947 with Chicago.

To evaluate the contribution of Mac Lane to mathematics is an easy and pleasant task. It suffices to cite the words A. G. Kurosh, a renowned Russian professor of Lomonosov State University. In the translator's preface to the Russian edition of the classical *Homology* book, Kurosh wrote:

> *The author of this book, a professor of Chicago University, is one of the most prominent American algebraists and topologists. His role in homological algebra as well as category theory is the role of one of the founders of this area.*

Homological algebra implements a marvelous project of algebraization of topological spaces by assigning to such a space X the sequence of (abelian) homology groups $H_n(X)$. Moreover, each continuous map $f : X \to Y$ from X to Y induces a family of homomorphisms of the homology groups $f_n : H_n(X) \to H_n(Y)$. The aim of homological algebra consists in calculation of homologies.

In his research into homological algebra and category theory Mac Lane cooperated with Eilenberg whom he met in 1940. Eilenberg had arrived from Poland two years earlier. He saw the affinity of the algebraic calculations of Mac Lane with those he encountered in algebraic topology. Eilenberg offered cooperation to Mac Lane. The union of Eilenberg and Mac Lane lasted for 14 years and resulted in 15 joint papers which noticeably changed the mathematical appearance of the 20th century.

The pearl of this cooperation was category theory. Mac Lane always considered category theory "a natural and perhaps inevitable aspect of the 20th century mathematical emphasis on axiomatic and abstract methods—especially as those methods when involved in abstract algebra and functional analysis." He stressed that even if Eilenberg and he did not propose this theory it will necessarily appear in the works of other mathematicians. Among these potential inventors of the new conceptions Mac Lane listed Claude Chevalley, Heinz Hopf, Norman Steenrod, Henri Cartan, Charles Ehresmann, and John von Neumann.

In Mac Lane's opinion, the conceptions of category theory were close to the methodological principles of the project of Nicholas Bourbaki. Mac Lane was sympathetic with the project and was very close to joining in but this never happened (the main obstacles were in linguistic facilities). However, even the later membership of Eilenberg in the Bourbaki group could not overcome a shade of slight disinclination and repulsion. It turned out impossible to "categorize Bourbaki" with a theory of non-French origin as Mac Lane had once phrased the matter shrewdly and elegantly. It is worth noting in this respect that the term "category theory" had roots in the mutual interest of its authors in philosophy and, in particular, in the works of Immanuel Kant.

Set theory rules in the present-day mathematics. The buffoon's role of "abstract nonsense" is assigned in mathematics to category theory. History and literature demonstrate to us that the relations between the ruler and the jester may be totally intricate and unpredictable. Something very similar transpires in the interrelations of set theory and category theory and the dependency of one of them on the other.

From a logic standpoint, set theory and category theory are instances of a first order theory. The former deals with sets and the membership relation between them. The latter speaks of objects and morphisms (or arrows). Of course, there is no principle difference between the atomic formulas $a \in b$ and $a \to b$. However, the precipice in meaning is abysmal between the two concepts that are formalized by the two atomic formulas. The stationary universe of Zermelo-Fraenkel, cluttered up with uncountably many copies of equipollent sets confronts the free world of categories, ensembles of arbitrary nature that are determined by the dynamics of their transformations.

The individual dualities of set theory, dependent on the choice of particular realizations of the pairs of objects under study, give up their places to the universal *natural transformations* of category theory. One of the most brilliant achievements of category theory was the development of axiomatic homology theory. Instead of the homological diversity for topological spaces (the simplicial homology for a polyhedron, singular and Čech homology, Vietoris homology, etc.) Eilenberg and Steenrod suggested as far back as in 1952 the new understanding of each homology or cohomology theory as a functor from the category of spaces under consideration to the category of groups. The axiomatic approach to defining such a functor radically changed the manner of further progress in homological algebra and algebraic topology. The study of the homology of Eilenberg–Mac Lane spaces and the method of acyclic models demonstrated the strength of the ideas of category theory and led to universal use of simplicial sets in K-theory and sheaves.

In 1948 Mac Lane proposed the concept of abelian category abstracting the categories of abelian groups and vector spaces which played key roles in the first papers on axiomatic homology theory. The abelian categories were rediscovered in 1953 and became a major tool in research into homological algebra by Cartan, Eilenberg, and their followers.

Outstanding advances in category theory are connected with the names of Alexander Grothendieck and F. William Lawvere. Topos theory, their aesthetic creation, appeared in the course of "point elimination" called upon by the challenge of invariance of the objects we study in mathematics. It is on this road that we met the conception of variable sets which led to the notion of topos and the understanding of the social medium of set-theoretic models.

A category is called an *elementary topos* provided that it is cartesian closed and has a suboject classifier. The sources of toposes lie in the theory of sheaves and Grothendieck topology. Further progress of the concept of topos is due to search for some category-theoretic axiomatization of set theory as well as study into forcing and the nonstandard set-theoretic models of Dana Scott, Robert Solovay, and Petr Vopěnka. The new frameworks provide a natural place for the Boolean valued models that are viewed now as the toposes with Aristotle logic and which pave king's ways to the solution of the problem of the continuum by Kurt Gödel and Paul Cohen. These toposes are the main arena of Boolean valued analysis.

Bidding farewell to Mac Lane, reading his sincere and openhearted autobiography, enjoying his vehement polemics with Freeman J. Dyson, and perusing his deep last articles on general mathematics, anyone cannot help but share his juvenile devotion and love of mathematics and its creators. His brilliant essays "Despite Physicists, Proof Is Essential in Mathematics" and "Proof, Truth, and Confusion" form an anthem of mathematics which is only possible by proof.

> *Let me summarize where we have come. As with any branch of learning, the real substance of mathematics resides in the ideas. The ideas of mathematics are those which can be formalized and which have been developed to fit issues arising in science or in human activity. Truth in mathematics is approached by way of proof in formalized systems. However, because of the paradoxical kinds of self-reference exhibited by the barn door and Kurt Gödel, there can be no single formal system which subsumes all mathematical proof. To boot, the older dogmas that "everything is logic" or "everything is a set" now have competition— "everything is a function." However, such questions of foundation are but a very small part of mathematical activity, which continues to try to combine the right ideas to attack substantive problems. Of these I have touched on only a few examples: Finding all simple groups, putting groups together by extension, and characterizing spheres by their connectivity. In such cases, subtle ideas, fitted by hand to the problem, can lead to triumph.*
>
> *Numerical and mathematical methods can be used for practical problems. However, because of political pressures, the desire for compromise, or the simple desire for more publication, formal ideas may be applied in practical cases where the ideas simply do not fit. Then confusion arises—whether from misleading formulation of questions in opinion surveys, from nebulous calculations of airy benefits, by regression, by extrapolation, or otherwise. As the case of fuzzy sets indicates, such confusion is not fundamentally a trouble caused by the organizations issuing reports, but is occasioned by academicians making careless use of good ideas where they do not fit.*
>
> *As Francis Bacon once said, "Truth ariseth more readily from error than from confusion." There remains to us, then, the pursuit of truth, by way of proof, the concatenation of those ideas which fit, and the beauty which results when they do fit.*

So wrote Saunders Mac Lane, a great genius, creator, master, and servant of mathematics. His unswerving devotion to the ideals of truth and free thinking of our ancient science made him the eternal and tragicomical mathematical Knight of the Sorrowful Figure...

S. S. KUTATELADZE

JUNE 11, 2005

Кусраев Анатолий Георгиевич
Кутателадзе Семён Самсонович

Булевозначный анализ и положительность

Препринт № 157

Ответственный за выпуск
академик Ю. Г. Решетняк

Издание подготовлено с использованием макропакета $\mathcal{AMS}$-TEX
разработанного Американским математическим обществом

This publication was typeset using $\mathcal{AMS}$-TEX,
the American Mathematical Society's TEX macro package

Подписано в печать 11.07.05. Формат 60 × 84 1/8. Печать офсетная.
Усл. печ. л. 4,0. Уч.-изд. л. 2,1. Тираж 100 экз. Заказ № 90.

Отпечатано в ООО «Омега Принт»
пр. Академика Лаврентьева, 6, 630090 Новосибирск