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INTERACTION OF ORDER AND CONVEXITY

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This is a talk for the Russian–German geometry meeting dedicated to the 95th anniversary of A. D. Alexandrov (1912–1999) to take place in St. Petersburg, June 18–23, 2007. The talk addresses the interaction between the techniques of positivity and abstract convexity in functional analysis and the extremal problems of convex geometry.

Appendix 1 is a short tribute to Alexandrov, and Appendix 2 is a short essay in memory of his general outlook on science.

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INTERACTION OF ORDER AND CONVEXITY

Alexandr Danilovich Alexandrov became the first and foremost Russian geometer of the twentieth century. He contributed to mathematics under the slogan: “Retreat to Euclid,” remarking that “the pathos of contemporary mathematics is the return to Ancient Greece.” Minkowski revolutionized the theory of numbers with the aid of the synthetic geometry of convex surfaces. The ideas and techniques of the geometry of numbers comprised the fundamentals of functional analysis which was created by Banach. The pioneering studies of Alexandrov continued the efforts of Minkowski and enriched geometry with the methods of measure theory and functional analysis. Alexandrov accomplished the turnround to the ancient synthetic geometry in a much deeper and subtler sense than it is generally acknowledged today. Geometry in the large reduces in no way to overcoming the local restrictions of differential geometry which bases upon the infinitesimal methods and ideas of Newton, Leibniz, and Gauss.

The works of Alexandrov [1, 2] made tremendous progress in the theory of mixed volumes of convex figures. He proved some fundamental theorems on convex polyhedra that are celebrated alongside the theorems of Euler and Cauchy. While discovering a solution of the Weyl problem, Alexandrov suggested a new synthetic method for proving the theorems of existence. The results of this research ranked the name of Alexandrov alongside the names of Euclid and Cauchy.

Alexandrov enriched the methods of differential geometry by the tools of functional analysis and measure theory, driving mathematics to its universal status of the epoch of Euclid. The mathematics of the ancients was geometry (there were no other instances of mathematics at all). Synthesizing geometry with the remaining areas of the today’s mathematics, Alexandrov climbed to the antique ideal of the universal science incarnated in mathematics. Return to the synthetic methods of *mathesis universalis* was inevitable and unavoidable as well as challenging and fruitful.

1. Minkowski Duality

1.1. A *convex figure* is a compact convex set. A *convex body* is a solid convex figure. The *Minkowski duality* identifies a convex figure S in $\mathbb{R}^N$ and its *support function* $S(z) := \sup\{(x, z) \mid x \in S\}$ for $z \in \mathbb{R}^N$. Considering the members of $\mathbb{R}^N$ as singletons, we assume that $\mathbb{R}^N$ lies in the set $\mathscr{V}_N$ of all compact convex subsets of $\mathbb{R}^N$.

1.2. The classical concept of support function gives rise to abstract convexity which focuses on the order background of convex sets.

Let $\overline{E}$ be a complete lattice E with the adjoint top $\top := +\infty$ and bottom $\bot := -\infty$. Unless otherwise stated, E is usually a *Kantorovich space* which is a Dedekind complete vector lattice in another terminology. Assume further that H is some subset of E which is by implication a (convex) cone in E, and so the bottom of E lies beyond H. A subset U of H is *convex relative to* H or H*-convex*, in symbols $U \in \mathscr{V}(H, \overline{E})$, provided that U is the H*-support set* $U_p^H := \{h \in H \mid h \leq p\}$ of some element p of $\overline{E}$.

Alongside the H-convex sets we consider the so-called H-convex elements. An element $p \in \overline{E}$ is H*-convex* provided that $p = \sup U_p^H$; i.e., p represents the supremum of the H-support set of p. The H-convex elements comprise the cone which is denoted by $\mathscr{C}nv(H, \overline{E})$. We may omit the references to H when H is clear from the context. It is worth noting that convex elements and sets are "glued together" by the *Minkowski duality* $\varphi : p \mapsto U_p^H$. This duality enables us to study convex elements and sets simultaneously.

Since the classical results by Fenchel [3] and Hörmander [4, 7] we know that the most convenient and conventional classes of convex functions and sets are $\mathscr{C}nv(\mathrm{Aff}(X), \overline{\mathbb{R}^X})$ and $\mathscr{V}(X', \overline{\mathbb{R}^X})$. Here X is a locally convex space, X' is the dual of X, and $\mathrm{Aff}(X)$ is the space of affine functions on X (isomorphic with $X' \times \mathbb{R}$).

In the first case the Minkowski duality is the mapping $f \mapsto \mathrm{epi}(f^*)$ where

$$f^*(y) := \sup_{x\in X}(\langle y, x\rangle - f(x))$$

is the *Young–Fenchel transform* of f or the *conjugate function* of f. In the second case we return to the classical identification of U in $\mathscr{V}(X', \overline{\mathbb{R}^X})$ and the standard support function that uses the canonical pairing $\langle \cdot, \cdot \rangle$ of X' and X.

This idea of abstract convexity lies behind many current objects of analysis and geometry. Among them we list the "economical" sets with boundary points meeting the Pareto criterion, capacities, monotone seminorms, various classes of functions convex in some generalized sense, for instance, the Bauer convexity in Choquet theory, etc. It is curious that there are ordered vector spaces consisting of the convex elements with respect to narrow cones with finite generators. Abstract convexity is traced and reflected, for instance, in [8]–[11].

2. Positive Functionals over Convex Objects

2.1. The Minkowski duality makes $\mathscr{V}_N$ into a cone in the space $C(S_{N-1})$ of continuous functions on the Euclidean unit sphere S_{N-1}, the boundary of the unit ball $\mathfrak{z}_N$. This yields the so-called *Minkowski structure* on $\mathscr{V}_N$. Addition of the support functions of convex figures amounts to taking their algebraic sum, also called the *Minkowski addition*. It is worth observing that the *linear span* $[\mathscr{V}_N]$ of $\mathscr{V}_N$ is dense in $C(S_{N-1})$, bears a natural structure of a vector lattice and is usually referred to as the *space of convex sets*. The study of this space stems from the pioneering breakthrough of Alexandrov in 1937 and the further insights of Radström [5], Hörmander [4], and Pinsker [6].

2.2. It was long ago in 1954 that Reshetnyak suggested in his Ph. D. thesis [12] to compare positive measures on S_{N-1} as follows.

A measure μ *linearly majorizes* or *dominates* a measure ν provided that to each decomposition of S_{N-1} into finitely many disjoint Borel sets $U_1, \dots, U_m$ there are measures $\mu_1, \dots, \mu_m$ with sum μ such that every difference $\mu_k - \nu|_{U_k}$ annihilates all restrictions to S_{N-1} of linear functionals over $\mathbb{R}^N$. In symbols, we write $\mu \gg_{\mathbb{R}^N} \nu$.

Reshetnyak proved that

$$\int_{S_{N-1}} p d\mu \geq \int_{S_{N-1}} p d\nu$$

for each sublinear functional p on $\mathbb{R}^N$ if $\mu \gg_{\mathbb{R}^N} \nu$. This gave an important trick for generating positive linear functionals over various classes of convex surfaces and functions.

2.3. A similar idea was suggested by Loomis [13] in 1962 within Choquet theory:

A measure μ *affinely majorizes* or *dominates* a measure ν, both given on a compact convex subset Q of a locally convex space X, provided that to each decomposition of ν into finitely many summands $\nu_1, \dots, \nu_m$ there are measures $\mu_1, \dots, \mu_m$ whose sum is μ and for which every difference $\mu_k - \nu_k$ annihilates all restrictions to Q of affine functionals over X. In symbols, $\mu \gg_{\mathrm{Aff}(Q)} \nu$.

Cartier, Fell, and Meyer [14] proved in 1964 that

$$\int_Q f d\mu \geq \int_Q f d\nu$$

for each continuous convex function f on Q if and only if $\mu \gg_{\mathrm{Aff}(Q)} \nu$. An analogous necessity part for linear majorization was published in 1970; cf. [15].

2.4. Majorization is a vast subject [16]. We only site one of the relevant abstract claims of subdifferential calculus [17]:

2.5. Theorem. Assume that $H_1, \dots, H_N$ are cones in a Riesz space X. Assume further that f and g are positive functionals on X. The inequality

$$f(h_1 \vee \dots \vee h_N) \geq g(h_1 \vee \dots \vee h_N)$$

holds for all $h_k \in H_k$ $(k := 1, \dots, N)$ if and only if to each decomposition of g into a sum of N positive terms $g = g_1 + \dots + g_N$ there is a decomposition of f into a sum of N positive terms $f = f_1 + \dots + f_N$ such that

$$f_k(h_k) \geq g_k(h_k) \quad (h_k \in H_k;\ k := 1, \dots, N).$$

3. Alexandrov Measures and the Blaschke Structure

The celebrated *Alexandrov Theorem* [1, p. 108] proves the unique existence of a translate of a convex body given its surface area function. Each surface area function is an *Alexandrov measure*. So we call a positive measure on the unit sphere which is supported by no great hypersphere and which annihilates singletons. The last property of a measure is referred to as translation invariance in the theory of convex surfaces. Thus, each Alexandrov measure is a translation-invariant additive functional over the cone $\mathscr{V}_N$.

This yields some abstract cone structure that results from identifying the coset of translates $\{z + \mathfrak{x} \mid z \in \mathbb{R}^N\}$ of a convex body $\mathfrak{x}$ the corresponding Alexandrov measure on the unit sphere which we call the *surface area function* of the coset of $\mathfrak{x}$ and denote by $\mu(\mathfrak{x})$. The soundness of this parametrization rests on the Alexandrov Theorem.

The cone of positive translation-invariant measures in the dual $C'(S_{N-1})$ of $C(S_{N-1})$ is denoted by $\mathscr{A}_N$. We now agree on some preliminaries.

Given $\mathfrak{x}, \mathfrak{y} \in \mathscr{V}_N$, we let the record $\mathfrak{x} =_{\mathbb{R}^N} \mathfrak{y}$ mean that $\mathfrak{x}$ and $\mathfrak{y}$ are equal up to translation or, in other words, are translates of one another. We may say that $=_{\mathbb{R}^N}$ is the associate equivalence of the preorder $\geq_{\mathbb{R}^N}$ on $\mathscr{V}_N$ which symbolizes the possibility of inserting one figure into the other by translation. Arrange the factor set $\mathscr{V}_N/\mathbb{R}^N$ which consists of the cosets of translates of the members of $\mathscr{V}_N$. Clearly, $\mathscr{V}_N/\mathbb{R}^N$ is a cone in the factor space $[\mathscr{V}_N]/\mathbb{R}^N$ of the vector space $[\mathscr{V}_N]$ by the subspace $\mathbb{R}^N$. There is a natural bijection between $\mathscr{V}_N/\mathbb{R}^N$ and $\mathscr{A}_N$. Namely, we identify the coset of singletons with the zero measure. To the straight line segment with endpoints x and y, we assign the measure $|x-y|(\varepsilon_{(x-y)/|x-y|}+\varepsilon_{(y-x)/|x-y|})$, where $|\cdot|$ stands for the Euclidean norm and the symbol ε_z for $z \in S_{N-1}$ stands for the *Dirac measure* supported at z. If the dimension of the affine span $\operatorname{Aff}(\mathfrak{x})$ of a representative $\mathfrak{x}$ of a coset in $\mathscr{V}_N/\mathbb{R}^N$ is greater than unity, then we assume that $\operatorname{Aff}(\mathfrak{x})$ is a subspace of $\mathbb{R}^N$ and identify this class with the surface area function of $\mathfrak{x}$ in $\operatorname{Aff}(\mathfrak{x})$ which is some measure on $S_{N-1}\cap\operatorname{Aff}(\mathfrak{x})$ in this event. Extending the measure by zero to a measure on S_{N-1}, we obtain the member of $\mathscr{A}_N$ that we assign to the coset of all translates of $\mathfrak{x}$. The fact that this correspondence is one-to-one follows easily from the Alexandrov Theorem.

The vector space structure on the set of regular Borel measures induces in $\mathscr{A}_N$ and, hence, in $\mathscr{V}_N/\mathbb{R}^N$ the structure of an abstract cone or, strictly speaking, the structure of a commutative $\mathbb{R}_+$-operator semigroup with cancellation. This structure on $\mathscr{V}_N/\mathbb{R}^N$ is called the *Blaschke structure* (cp. [18] and the references therein). Note that the sum of the surface area functions of $\mathfrak{x}$ and $\mathfrak{y}$ generates a unique class $\mathfrak{x}\#\mathfrak{y}$ which is referred to as the *Blaschke sum* of $\mathfrak{x}$ and $\mathfrak{y}$.

Let $C(S_{N-1})/\mathbb{R}^N$ stand for the factor space of $C(S_{N-1})$ by the subspace of all restrictions of linear functionals on $\mathbb{R}^N$ to S_{N-1}. Denote by $[\mathscr{A}_N]$ the space $\mathscr{A}_N-\mathscr{A}_N$ of translation-invariant measures. It is easy to see that $[\mathscr{A}_N]$ is also the linear span of the set of Alexandrov measures. The spaces $C(S_{N-1})/\mathbb{R}^N$ and $[\mathscr{A}_N]$ are made dual by the canonical bilinear form

$$\langle f,\mu\rangle=\frac{1}{N}\int_{S_{N-1}} f\,d\mu \quad (f\in C(S_{N-1})/\mathbb{R}^N,\ \mu\in[\mathscr{A}_N]).$$

For $\mathfrak{x}\in\mathscr{V}_N/\mathbb{R}^N$ and $\mathfrak{y}\in\mathscr{A}_N$, the quantity $\langle\mathfrak{x},\mathfrak{y}\rangle$ coincides with the *mixed volume* $V_1(\mathfrak{y},\mathfrak{x})$. The space $[\mathscr{A}_N]$ is usually furnished with the weak topology induced by the above indicated duality with $C(S_{N-1})/\mathbb{R}^N$.

4. Cones of Feasible Directions

4.1. By the *dual* K^* of a given cone K in a vector space X in duality with another vector space Y, we mean the set of all positive linear functionals on K; i.e., $K^*:=\{y\in Y \mid (\forall x\in K)\ \langle x,y\rangle\ge 0\}$. Recall also that to a convex subset U of X and a point $\bar{x}$ in U there corresponds the cone

$$U_{\bar{x}}:=\operatorname{Fd}(U,\bar{x}):=\{h\in X \mid (\exists\alpha\ge 0)\ \bar{x}+\alpha h\in U\}$$

which is called the *cone of feasible directions* of U at $\bar{x}$. Fortunately, description is available for all dual cones we need.

4.2. Let $\bar{\mathfrak{x}}\in\mathscr{A}_N$. Then the dual $\mathscr{A}^*_{N,\bar{\mathfrak{x}}}$ of the cone of feasible directions of $\mathscr{A}_N n$ at $\bar{\mathfrak{x}}$ may be represented as follows

$$\mathscr{A}^*_{N,\bar{\mathfrak{x}}}=\{f\in\mathscr{A}^*_N \mid \langle\bar{\mathfrak{x}},f\rangle=0\}.$$

4.3. Let $\mathfrak{x}$ and $\mathfrak{y}$ be convex figures. Then
(1) $\mu(\mathfrak{x}) - \mu(\mathfrak{y}) \in \mathscr{V}_N^* \leftrightarrow \mu(\mathfrak{x}) \gg_{\mathbb{R}^N} \mu(\mathfrak{y})$;
(2) If $\mathfrak{x} \geq_{\mathbb{R}^N} \mathfrak{y}$ then $\mu(\mathfrak{x}) \gg_{\mathbb{R}^N} \mu(\mathfrak{y})$;
(3) $\mathfrak{x} \geq_{\mathbb{R}^2} \mathfrak{y} \leftrightarrow \mu(\mathfrak{x}) \gg_{\mathbb{R}^2} \mu(\mathfrak{y})$;
(4) If $\mathfrak{y} - \bar{\mathfrak{x}} \in \mathscr{A}_{N,\bar{\mathfrak{x}}}^*$ then $\mathfrak{y} =_{\mathbb{R}^N} \bar{\mathfrak{x}}$;
(5) If $\mu(\mathfrak{y}) - \mu(\bar{\mathfrak{x}}) \in \mathscr{V}_{N,\bar{\mathfrak{x}}}^*$ then $\mathfrak{y} =_{\mathbb{R}^N} \bar{\mathfrak{x}}$.
It stands to reason to avoid discriminating between a convex figure, the respective coset of translates in $\mathscr{V}_N/\mathbb{R}^N$, and the corresponding measure in $\mathscr{A}_N$.

5. Comparison Between the Blaschke and Minkowski Structures

The isoperimetric-type problems with subsidiary constraints on location of convex figures comprise a unique class of the challenging extremal problems with two essentially different parametrizations. The principal features of the latter are seen from the table.

Object of Parametrization	Minkowski's Structure	Blaschke's Structure
cone of sets	$\mathscr{V}_N/\mathbb{R}^N$	$\mathscr{A}_N$
dual cone	$\mathscr{V}_N^*$	$\mathscr{A}_N^*$
positive cone	$\mathscr{A}_N^*$	$\mathscr{A}_N$
typical linear functional	$V_1(\mathfrak{z}_N, \cdot\,)$ (width)	$V_1(\,\cdot\,, \mathfrak{z}_N)$ (area)
concave functional (power of volume)	$V^{1/N}(\,\cdot\,)$	$V^{(N-1)/N}(\,\cdot\,)$
simplest convex program	isoperimetric problem	Urysohn's problem
operator-type constraint	inclusion of figures	inequalities on "curvatures"
Lagrange's multiplier	surface	function
differential of volume at a point $\bar{\mathfrak{x}}$ is proportional to	$V_1(\bar{\mathfrak{x}}, \cdot\,)$	$V_1(\,\cdot\,, \bar{\mathfrak{x}})$

This table shows that the classical isoperimetric problem is not a convex program in the Minkowski structure for $N \geq 3$. In this event a necessary optimality condition leads to a solution only under extra regularity conditions. Whereas in the Blaschke structure this problem is a convex program whose optimality criterion reads: "Each solution is a ball."

The problems are challenging that contain some constrains of inclusion type: for instance, the isoperimetric problem or Urysohn problem with the requirement that the solutions lie among the subsets or supersets of a given body. These problems can be solved in a generalized sense "modulo" the Alexandrov Theorem. Clearly, some convex combination of the ball and a tetrahedron is proportional to the solution of the Urysohn problem in this tetrahedron. If we replace the condition on the integral width which is typical of the Urysohn problem [19, 20] by a constraint on the surface area or other mixed volumes of a more general shape then we come to possibly nonconvex programs for which a similar reasoning yields only necessary extremum conditions in general. Recall that in case $N = 2$ the Blaschke sum transforms as usual into the Minkowski sum modulo translates.

The task of choosing an appropriate parametrization for a wide class of problems is practically unstudied in general. In particular, those problems of geometry remain unsolved which combine constraints each of which is linear in one of the two vector structures on the set of convex figures. The simplest example of an unsolved "combined" problem is the internal isoperimetric problem in the space $\mathbb{R}^N$ for $N \geq 3$. The only instance of progress is due to Pogorelov who found in [21] the form of a soap bubble inside a three-dimensional tetrahedron. This happens to be proportional to the Minkowski convex combination of the ball and the solution to the internal Urysohn problem in the tetrahedron.

The above geometric facts make it reasonable to address the general problem of parametrizing the important classes of extremal problems of practical provenance.

5.1. By way of example, consider the *external Urysohn problem*: Among the convex figures, circumscribing $\mathfrak{x}_0$ and having integral width fixed, find a convex body of greatest volume.

5.2. Theorem. A feasible convex body $\bar{\mathfrak{x}}$ is a solution to the external Urysohn problem if and only if there are a positive measure μ and a positive real $\bar{\alpha} \in \mathbb{R}_+$ satisfying

(1) $\bar{\alpha}\mu(\mathfrak{z}_N) \gg_{\mathbb{R}^N} \mu(\bar{\mathfrak{x}}) + \mu$;
(2) $V(\bar{\mathfrak{x}}) + \frac{1}{N}\int_{S_{N-1}} \bar{\mathfrak{x}} d\mu = \bar{\alpha} V_1(\mathfrak{z}_N, \bar{\mathfrak{x}})$;
(3) $\bar{\mathfrak{x}}(z) = \mathfrak{x}_0(z)$ for all z in the support of μ.

5.3. If, in particular, $\mathfrak{x}_0 = \mathfrak{z}_{N-1}$ then the sought body is a *spherical lens*, that is, the intersection of two balls of the same radius; while the critical measure μ is the restriction of the surface area function of the ball of radius $\bar{\alpha}^{1/(N-1)}$ to the complement of the support of the lens to S_{N-1}. If $\mathfrak{x}_0 = \mathfrak{z}_1$ and $N = 3$ then our result implies that we should seek a solution in the class of the so-called spindle-shaped constant-width surfaces of revolution.

If $\mathfrak{x}_0$ is an equilateral triangle then the solution $\bar{\mathfrak{x}}$ looks as follows:

Note that $\bar{\mathfrak{x}}$ is the union of $\mathfrak{x}_0$ and three congruent slices of a circle of radius $\bar{\alpha}$ and centers O_1–O_3. The critical measure μ is the restriction of $\mu(\mathfrak{z}_2)$ to the subset of S_1 comprising the endpoints of the unit vectors of the shaded zone.

5.4. We turn now to consider the *internal Urysohn problem with a current hyperplane* (cp. [22]): Find two convex figures $\bar{\mathfrak{x}}$ and $\bar{\mathfrak{y}}$ lying in a given convex body $\mathfrak{x}_o$, separated by a hyperplane with the unit outer normal z_0, and having the greatest total volume of $\bar{\mathfrak{x}}$ and $\bar{\mathfrak{y}}$ given the sum of their integral widths.

5.5. Theorem. A feasible pair of convex bodies $\bar{\mathfrak{x}}$ and $\bar{\mathfrak{y}}$ solves the internal Urysohn problem with a current hyperplane if and only if there are convex figures $\mathfrak{x}$ and $\mathfrak{y}$ and positive reals $\bar{\alpha}$ and $\bar{\beta}$ satisfying

(1) $\bar{\mathfrak{x}} = \mathfrak{x}\#\bar{\alpha}\mathfrak{z}_N$;
(2) $\bar{\mathfrak{y}} = \mathfrak{y}\#\bar{\alpha}\mathfrak{z}_N$;
(3) $\mu(\mathfrak{x}) \geq \bar{\beta}\varepsilon_{z_0}$, $\mu(\mathfrak{y}) \geq \bar{\beta}\varepsilon_{-z_0}$;
(4) $\bar{\mathfrak{x}}(z) = \mathfrak{x}_0(z)$ for all $z \in \operatorname{supp}(\mathfrak{x}) \setminus \{z_0\}$;
(5) $\bar{\mathfrak{y}}(z) = \mathfrak{x}_0(z)$ for all $z \in \operatorname{supp}(\mathfrak{x}) \setminus \{-z_0\}$,

with $\operatorname{supp}(\mathfrak{x})$ standing for the *support* of $\mathfrak{x}$, i.e. the support of the surface area measure $\mu(\mathfrak{x})$ of $\mathfrak{x}$.

The internal isoperimetric problem and its analogs seem indispensable since we have no adequate means for expressing their solutions. The new level of understanding is in order in convexity that we may hope to achieve with the heritage of Alexandrov, the teacher of universal freedom in geometry.

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Appendix 1

ALEXANDROV PAR EXCELLENCE

Many years ago some visiting professor persuaded Alexandrov in my presence to make a demarche at the public maintenance of a mediocre geometric thesis. The defendant was a Jew, and the final argument of the Muscovite professor was xenophobia. He pointed to the picturesque and rather gloomy figure of a complete stranger to the event at the end of a corridor and began blaming us in the sense that if we disagreed to his proposal then "everyone would be like that down here." By the vicissitudes of fate, the lad in the corner was Perelman, a postgraduate of Alexandrov, a student of Professor Yuriĭ Bugaro who was himself a student of Alexandrov.

The love and hatred to Alexandrov stemmed from the same sources. His reviews and opinions were welcome and appreciated, but his approaches and areas of research were silenced if not scorned. He was accused of zionism, but many bet and counted upon his antisemitism. His communistic beliefs were blasphemed obscenely, but he was humbly requested to write a letter or two to the Central Committee of the Communist Party of the USSR or *The Communist* magazine. His philosophical essays were spit upon furiously, but the same despisers required that their students used Alexandrov's popular writings at the final examinations in philosophy which were obligatory for admittance to the public maintenance of theses. The professorship of St. Petersburg is full of raptures about the palace, fountain, and park ensemble of Peterhof, but most of Alexandrov's colleagues will never forgive the sage decision of Rector Alexandrov who suggested to build a new university campus in Peterhof. During the years of Gorbi's "perestroika" Alexandrov was accused in confessing lysenkoism but decorated with the Order of the Red Banner of Labor for his efforts in safeguarding and propelling genetics and selection in the USSR. So were the scales of Alexandrov's personality.

Alexandrov often remarked that a man is what he does. Alexandrov did the business of geometry. It is more appropriate to speak about geometry as his most adorable part of the universal science, mathematics. Mac Lane, a cofather of category theory, had coined the term "working mathematician" which is rather close to the routine "workman" or even "toiler." Mac Lane wrote not only about work in mathematics but also about criteria of excellence in mathematics. Excellent mathematics by Mac Lane should be inevitable, illuminating, deep, relevant, responsive, and timely. Excellent mathematics is exercised by excellent mathematicians, mathematicians *par excellence*. Such was Alexandrov.

Alexandrov contributed to mathematics under the slogan: "Retreat to Euclid." He remarked that "the pathos of contemporary mathematics is the return to Ancient Greece." Hermann Minkowski revolutionized the theory of numbers with the aid of the synthetic geometry of convex figures. The ideas and techniques of the geometry of numbers comprised the fundamentals of functional analysis which was created by Banach. The pioneering studies of Alexandrov continued the efforts of Minkowski and enriched geometry with the methods of measure theory and functional analysis. Alexandrov accomplished the turnround to the ancient synthetic geometry in a much deeper and subtler sense than it is generally acknowledged today. Geometry in the large reduces in no way to overcoming the local restrictions of differential geometry which bases upon the infinitesimal methods and ideas of Newton, Leibniz, and Gauss.

The works of Alexandrov made tremendous progress in the theory of mixed volumes of convex figures. He proved some fundamental theorems on convex polyhedra that are celebrated alongside the theorems of Euler and Cauchy. While discovering a solution of the Weyl problem, Alexandrov suggested a new synthetic method for proving the theorems of existence. The results of this research ranked the name of Alexandrov alongside the names of Euclid and Cauchy.

Another outstanding contribution of Alexandrov to science is the creation of the intrinsic geometry of irregular surfaces. He suggested his amazingly visual and powerful method of cutting and gluing. This method enabled him to solve many extremal problems of the theory of manifolds of bounded curvature.

Alexandrov developed the theory of metric spaces with one-sided constraints on curvature. This gave rise to the class of metric spaces generalizing the Riemann spaces in the sense that these spaces are furnished with some curvature, the basic concept of Riemannian geometry. The research of Alexandrov into the theory of manifolds with bounded curvature prolongates and continues the traditions of Gauss, Lobachevskiĭ, Poincaré, and Cartan.

Alexandrov enriched the methods of differential geometry by the tools of functional analysis and measure theory, driving mathematics to its universal status of the epoch of Euclid. The mathematics of the ancients was geometry (there were no other instances of mathematics at all). Synthesizing geometry with the remaining areas of the today's mathematics, Alexandrov climbed to the antique ideal of the universal science incarnated in mathematics. Return to the synthetic methods of *mathesis universalis* was inevitable and unavoidable as illustrated in geometry with the beautiful results of the disciples and followers of Alexandrov like Gromov, Perelman, Pogorelov, and Reshetnyak.

The first and foremost Russian geometer of the nineteenth century was Nikolaĭ Ivanovich Lobachevskiĭ. Alexandr Danilovich Alexandrov became the first and foremost Russian geometer of the twentieth century.

Appendix 2

НАУКА, ЛЖЕНАУКА И СВОБОДА

Памяти А. Д. Александрова

Наука — система знаний и основанных на них представлений о той или иной сфере действительности. Цели науки — объяснение прошлого, нахождение решений проблем настоящего и предвидение будущего. Она отнюдь не единственная сфера деятельности человека, преследующая эти цели. Псевдонаука, религия, здравый смысл предлагают свои методы достижения целей и задач науки.

Особенность науки в её абсолютной критичности, вечном служении и безграничной преданности истине. Наука объясняет «как оно есть на самом деле» с величием и скромностью, основываясь на опыте, фактах и логике. Наука чужда всякой предвзятости и доктринёрства, открыта критике, но не легкомысленна, не руководствуется симпатиями, модой или веяниями времени. Наука требовательна, несварлива и незлоблива. Наука надёжна и солидна, сохраняет здравый консерватизм, но восприимчива ко всему новому и легко отказывается от заблуждений. Наука открыта для всех, не творит кумиров и не поклоняется авторитетам. Наука следует фактам и логике. Наука может мечтать, фантазировать и творить чудеса, но чужда мистике и вере в сверхъестественное. Истина, логика, опыт и факты — вот фетиши и инструменты науки.

Разумеется, наука может быть стерильной и неинтересной. Признаки стерильности и неинтересности куда как субъективнее нежели критерии истинности. Именно поэтому учёные по убеждениям воздерживаются от крайних обвинений в бесплодности не только в погромном стиле приснопамятной сессии ВАСХНИЛ, но и в многочисленных благопристойных по форме и оскорбительных по существу противопоставлениях теоретических и прикладных исследований в науке. Стоит подчеркнуть, что совсем немало людей, заметно обогативших науку, учёными по убеждениям не являются. Учёный по убеждениям внутренне свободен и потому не может быть источником негодного, причинять зло. Вклад в науку внесли и отъявленные негодяи. Это обстоятельство никак не опровергает классический тезис о несовместности гения и злодейства, а только доказывает, что свойство быть учёным — это разрывная функция времени. Учёными по убеждениям даже лучшие представители науки бывают далеко не всегда. К счастью, раз найденная истина не зависит от личных качеств обнаружившего её человека. Наука делает любую истину вечным достоянием человечества.

Человечность, ответственность и научность — таковы составляющие полноты нравственности по Александрову. Человек — источник и цель всего. Таково содержание универсального гуманизма. Человек — в ответе за всё. Таков смысл ответственности. Научность, как человеческое суждение, отвлечённое от субъективизма, лежит в основе нравственности.

В наше время наиболее активно науке противостоит псевдонаука и её корыстная составляющая — лженаука. Лженаука всегда обслуживает властные интересы. Так было и есть в России. Так было и в СССР (Кольман, Лепешинская, Лысенко и прочие проходимцы, попы и инквизиторы от марксизма). Власть противостоит свободе, а наука освобождает человека материально и раскрепощает его интеллектуально. Поэтому наука власти чужда, а лженаука — приятна.

Лженаука всегда ненадёжна, неглубока, неосновательна и обманчива. Она игнорирует факты и логику, творит кумиров и млеет от создаваемых миражей и фантомов. Наличие аргументов, превосходящих по силе факты и логику, характеризует веру. В этой связи лженаука часто прикрывается религией и пытается последнюю приручать или обслуживать. Между тем современные богословы и теологи выделяют особую роль науки в восхождении к истине. Покойный Папа Иоанн Павел II в своей знаменитой энциклике «Разум и вера» от 15 октября 1998 года констатировал, что «Наука и религия подобны двум крылам, на которых человеческий дух возносится к созерцанию истины...». Между тем еще в 1941 году во второй части эссе “Science and Religion” Эйнштейн писал: “...science without religion is lame, religion without science is blind”.

Лженаука, в отличие от религии, рядится в тогу науки и выдаёт свои глупости за научные достижения. Она эксплуатирует авторитет науки и дискредитирует научные знания и методы. Лженаука разрушает базу мировоззрения людей и тем препятствует их интеллектуальному раскрепощению. Поэтому лженаука не просто примитивный завистник и антагонист науки. Лженаука — враг истины, а стало быть, свободы и независимости, высших человеческих ценностей.

Бывают гениальные теоремы и формулы, а злодейских теорем и формул не бывает. Между тем гениальные теории и эксперименты соседствуют в истории человечества с расизмом и вивисекцией. Здесь проходит демаркационная линия между наукой и лженаукой. Наука злодейству чужда. Зло — клеймо лженауки.

Как бы ни мимикрировала и ни раздувалась лженаука, в конечном счете она обречена. Наука не отступала прежде и не отступит в будущем от своих принципов. Каким бы консервативным и склонным к лености человечество ни казалось и ни бывало временами, оно весьма прагматично, даже прижимисто и нажитыми ценностями дорожит. Люди по отдельности любят командовать, но все вместе сохраняют осмотрительность и недоверие ко всякой власти, к любой попытке одного человека или группы лиц манипулировать другими, навязывать им свои представления и волю. Люди небезупречны, но далеко небезнадежны. Их скепсис, любознательность и свободное мышление — вечные источники неиссякаемой силы и несказанных чудес науки.

Это внушает надежду...

Кутателадзе Семён Самсонович

Interaction of Order and Convexity

Препринт № 191

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