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SOBOLEV AND THE CALCULUS OF THE TWENTIETH CENTURY

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Sobolev and the Calculus of the Twentieth Century

Два доклада о С. Л. Соболеве (1908–1989) и его вкладе в формирование современных аналитических воззрений.

Two talks about Sergeĭ Sobolev (1908–1989) and his contribution to the modern views of mathematics.

1. Sobolev of the Euler School.

2. Sobolev and Schwartz: Two Fates and Two Fames.

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SOBOLEV OF THE EULER SCHOOL

Sergeĭ L′vovich Sobolev belongs to the Russian mathematical school and ranks among the scientists whose creativity has produced the major treasures of the world culture.

Mathematics studies the forms of reasoning. Generally speaking, differentiation discovers trends, and integration forecasts the future from trends. Mankind of the present day cannot be imagined without integration and differentiation. The differential and integral calculus was invented by Newton and Leibniz. The fluxions of Newton and the monads of Leibniz made these giants the forerunners of the classical analysis. Euler used the concepts by Newton and Leibniz to upbring and cultivate the new mathematics of variable quantities, while making quite a few phenomenal discoveries and creating his own inexhaustible collection of miraculous formulas and theorems. Mathematical analysis remained the calculus of Newton, Leibniz, and Euler for about two hundred years.

The classical calculus turned into the theory of distributions in the twentieth century. As the key objects of the modern analysis are ranked the integral in the sense of Lebesgue and the derivative in the sense of Sobolev which apply to the most general instances of interdependence that lie beyond the domains under the jurisdiction of the classical differentiation and integration. Lebesgue and Sobolev entered into history, suggesting the new approaches to the integral and derivative which expanded the sphere of influence and the scope of application of mathematics.

On the occasion of the centenary of the birth of S. L. Sobolev.

The historic figures and discoveries deserve the historical parallels and analysis. The gift of mathematics translates from teacher to student. The endless chain of alternating generations incarnates a mathematical tradition. Characterizing a scientific school, Luzin observed that "the elder school is more precious. Indeed, any school is the collections of the creative techniques, traditions, and narrations about the past and still living scientists as well as their manners of research and views of the object of research. These narrations are collected for ages but not intended for publication or revelation to those that seem undeserving. These narrations are treasures whose power is impossible to imagine or overrate . . . . If some analogy or comparison is welcome then the age of a school, together with the stock of its traditions and narrations, is nothing else but the energy of the school in implicit form."[1] Sobolev belongs to the school that originated with Leonhard Euler (1707–1783).[2]

Euler and Russia

Man is a physical object and as such can be partly represented by his worldline in the 4-dimensional Minkowski space-time. "Mathematics knows no races or geographic boundaries; for mathematics, the cultural world is one country," Hilbert said at the Congress in Bologna in 1928.[3]

No state is a physical object. In space-time we may identify a country with the funnel of the worldlines of its inhabitants. The longest part of the worldline of Euler belongs to Russia. There is neither Russian nor Swiss mathematics. However, there is mathematics in Russia, there is a national mathematical tradition, and there is a national mathematical school. Born in Switzerland, Euler found his second homeland in Russia and is buried in the soil of St. Petersburg. Da Vinci of mathematics, he had become part and parcel of the Russian spirit. Our compatriots are proud to acknowledge Euler as the founder of the Russian mathematical school.

The efforts of Euler made Petersburg the mathematical capital of the eighteenth century. Daniel Bernoulli wrote to Euler: "I fail to convey to you quite properly how greedily they ask everywhere for the Petersburg memoirs."[4] Implied were the celebrated *Commentarii Academiae Scientiarum Imperialis Petropolitanae* which became a leading scientific periodical of that epoch. The title of the journal changed many times and reads now as *Proceedings of the Russian Academy of Sciences (Mathematical Series)*. The journal of the Petersburg Academy of Sciences published 473 Euler's articles which were printed successively during many years after his death up to 1830.

From Ostrogradskiĭ to Sobolev

At the turn of the nineteenth century the center of mathematical thought shifted to France, the residence of Laplace, Poisson, Fourier, and Cauchy. The ideas of the new creators of mathematics were perceived by Ostrogradskiĭ who studied in Paris after he was deprived of his legitimate Graduation Diploma of Kharkov Imperial University.

[1] From a private letter of Luzin. Cited from [1].

[2] Cp. [2] about Euler.

[3] Cited in [3, Ch. 21].

[4] Cp. [4, p. 101].

Cauchy appraised Ostrogradskiĭ in one of his papers of 1825 as a youngster gifted with a keen vision and rather knowledgable in infinitesimal calculus.[5] The reputation of Ostrogradskiĭ in France, as well as a few memoirs submitted to the Academy of Sciences, led to the recognition of his merits in Russia. It was already in 1832 when Ostrogradskiĭ was elected as an ordinary academician in applied mathematics at the age of 32. Soon he became an undisputed leader of the Russian mathematical school.

Ostrogradskiĭ was fully aware of the importance of Euler to the science in Russia. He vigorously raised the question of publishing the legacy of Euler. In a relevant memo, Ostrogradskiĭ wrote: "Euler created the modern analysis, enriching it more than all his predecessors and making it the most powerful tool of human mind."[6] The collection of 28 volumes was to be completed in 10 years, but the Academy had found no finances neither then nor ever after . . . .

N. D. Brashman, N. E. Zhukovskiĭ, and S. A. Chaplygin are usually listed in the Moscow branch of the school of Ostrogradskiĭ. The Petersburg branch included P. L. Chebyshev, A. M. Lyapunov, V. A. Steklov, and A. N. Krylov. Many other Russian mathematicians and mechanists were influenced by the research, teaching, and personality of Ostrogradskiĭ.

Among the students of Chebyshev[7] we list A. N. Korkin and A. A. Markov who taught N. M. Günter, the future supervisor of the graduation thesis of Sobolev. As his second teacher, Sobolev acclaimed V. I. Smirnov, a student of V. A. Steklov who himself was supervised by A. M. Lyapunov. So is the brilliant chain of the scientific genealogy of Sobolev.[8]

Euler's archive belongs to Russia. However, the publication of the collected works of Euler was accomplished in Switzerland with the active participation of A. M. Lyapunov, A. N. Krylov, A. A. Markov, and V. I. Smirnov. The best minds of Russia strove to save the intellectual legacy of Euler. Smirnov rephrased the words of Goethe about Mozart as follows: "Euler will always remain a miracle beyond our ability to explain."[9] By now the 60 volumes of *Leonhardi Euleri Opera Omnia* are already published, and the whole collection of 72 volumes is planned to be completed this year.

Mathematics of Russia in the 1930s

The great discoveries are the signposts of the inevitable which are not erected without efforts. Solving a problem presumes not only the statement of the problem but also some means and opportunities for solution. Necessity paves way through the impenetrable timberland of random events. Sobolev's contributions belong to the epoch of tremendous breakthroughs in the world science. The twentieth century is rightfully called the age of freedom. The development of the institutions of democracy was accompanied with the liberation of all aspects of the mental life of mankind. Mathematics has

[5] Gnedenko in [4, p. 60] gave a reference to an article of 1901 by E. F. Sabinin.

[6] Cited from [4, pp. 101–102] where the reference is given to the Archive of the Academy of Sciences of the USSR, Fond 2, Description 1844, pp. 13–14.

[7] About Chebyshev cp. [5].

[8] See an overview of the history of the Petersburg–Leningrad mathematical school in [6]. A few details of the green years of Sobolev are collected in [7].

[9] Cp. [8, p. 54].

revealed its essence of the science of the forms of free thinking. Freedom is a historical concept reflecting the manner of resolving the clashes between the individuals loose in diversity and the tight bonds of their collective coexistence. The historical entourage is an indispensable ingredient of any triumph and any tragedy.

Pondering over his achievements in 1957, Sobolev noticed:[10]

> In the study of the various problems of finding the functions that satisfy some partial differential equations, it turned out fruitful to use some class of the functions that fail to possess the continuous derivatives of appropriate order everywhere but serve in a sense as the limits of the genuine solutions of the equations. Naturally, we seek for these generalized solutions in various function spaces, sometimes complete and sometimes to be especially completed with the aid of new "ideal" elements.
>
> Science has traveled from an individual solution to studying the function spaces, operators between the spaces, and those elements that are solutions.
>
> The problem arises of importance in its own right of the conditions for these generalized solutions to be classical.

We see that Sobolev distinguished the close connection of his theory with the Hilbert idea of socializing mathematical problems. Hilbert's methodology rested on the Cantorian set theory.

The idea of revising the concept of solution of a differential equation was in the mathematical air of the early twentieth century. The interest of Sobolev in this topic is undoubtedly due to Günter. In the obituary by Sobolev and Smirnov, they emphasized the role of Günter in propounding the Lebesgue idea of the necessity of a new approach to the equations of mathematical physics on the basis of the theory of set functions.[11]

Sobolev learned the ideas of functional analysis in the seminar headed by Smirnov. The program of the seminar included the study of the classical book by J. von Neumann on the mathematical foundations of quantum mechanics. Von Neumann sharply criticized the approach by Dirac:

> Die "uneigentlichen" Gebilde (wie ($\delta(x)$, $\delta'(x)$, ...) spielen in ihnen eine entscheidende Rolle — sie liegen außerhalb des Rahmens der allgemein üblichen mathematischen Methoden ... .[12]

> The "improper" functions (such as $\delta(x)$, $\delta'(x)$, ...) play a decisive role in this development—they lie beyond the scope of mathematical methods generally used ... .[13]

The ideas of von Neumann attracted another participant of the Smirnov seminar, Leonid Kantorovich who became a friend of Sobolev in their university years. In 1935 Kantorovich published two articles in *Doklady AN SSSR* **4** (1935) which were devoted to introducing "certain new functions, 'ideal functions' that would not be functions in the strict sense of the word." His articles were written in the spirit of Friedrichs and con-

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[10] Cited from [9, p. 596] which is a reprint of an article in *Vesnik Drushtva Matematichara i Fizichara Narodne Republike Srbije (Jugoslavija)* also known as *Bulletin de la Société des Mathématiciens et Physiciens de la R. P. de Serbie (Yougoslavie)*, **9**, 215–244 (1957) (Zbl 0138.34503).

[11] In particular, cp. [10]. The original book by Günter appeared in French in 1934. The English translation by John R. Schulenberger was published in 1967 by the Frederick Ungar Publishing Co. in New York (Zbl 0164.41901).

[12] Cp. [11, p. 15]. Von Neumann remarked earlier that "Dirac fingierte trotzdem die Existenz einer solchen Funktion" (cp. [11, p. 14]).

[13] This translation belonging to R. Beyer was published by the Princeton University Press in 1957.

tained the distributional derivatives of periodic tempered distributions.[14] In 1991 Israel Gelfand appraised these articles as follows: "In essence, Leonid Vital′evich was the first who understood the importance of generalized functions and wrote about the matter much earlier than Laurent Schwartz."[15]

It seems absolutely improbable that Sobolev and Kantorovich, old cronies and members of the same seminar, could be unaware of the articles by one another which addressed the same topic. However, neither of the two had ever mentioned the episode in future. It becomes clear that the 1930s were the years of a temporary detachment between Sobolev and Kantorovich who cultivated a warm and cordial friendship up to their last days. The political events of the 1930s in the mathematical circles of Leningrad and Moscow seem helpful in understanding the predicament.

The "Leningrad mathematical front" was launched against the old mathematical professorate of the Northern capital of Russia. Günter, leading the Petrograd Mathematical Society from its reestablishment in 1920, was chosen as the main target of the offensive. Günter was not only accused in all instances of misconduct, idealism, and neglect of praxis but also branded as a "reactionary in social life" and "conservative in science." The "Declaration of the Initiative Group for Reorganization of the Leningrad Physical and Mathematical Society" as of March 10, 1931, containing dreadful accusations against Günter was endorsed by 13 persons, among them I. M. Vinogradov, B. N. Delaunay, L. V. Kantorovich, and G. M. Fikhtengolts. Günter was forced to resign as the chair of a department and had no choice but writing a letter of repentance which was nonetheless condemned by the "mathematician-materialists." Steklov, who had died in 1926, was ranked as a member of the band of idealists either.[16] Sobolev and Smirnov must be commended for abstaining from the public persecution of their teachers.[17] The antidote transpires in the definite affinity of the scientific views of the teachers and the students.

The situation in the mathematical community differed slightly from the routine of the epoch. The old professorate was pursued in Moscow either.[18] The Muscovites attempted to involve Kantorovich in their quandaries, since he was appraised among the top specialists in the descriptive theory of sets and functions. Kantorovich refrained from any offensive against Luzin, whereas Sobolev became an active member of the emergency Commission of the Academy of Sciences of the USSR on the "case of Academician Luzin."[19]

Omnipresent was the tragedy of mathematics in Russia. So were the triumphs.

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[14]Cp. [12, 13].

[15]Cp. [14, p. 162]. Gelfand's article appeared firstly in the periodical collection of the Sobolev Institute of Mathematics—*Optimizatsiya* **50(67)** (1991), 131–134. There is a very rough English translation of the article in the first volume of the *Selected Works* of Kantorovich which was printed in 1996. Sobolev's article "The Cauchy problem in the space of functionals" was published in *Doklady AN SSSR* **3** (1935) and reprinted in [9, pp. 11–13].

[16]The "Declaration" and other documents of the "Leningrad mathematical front" are collected in the booklet [15].

[17]Also, Smirnov had his black mark as listed among the right-wing peacemakers and advocates of Günter [15, pp. 10, 33].

[18]Cp. [16] for relevant references.

[19]The historical details and shorthand minutes of the meetings of the Committee are collected in [17].

Sobolev and the A-bomb

Homo Sapiens reveals himself as *Homo Creatoris.* The power of man is his capability of creating and transferring intangible valuables. Mathematics saves the ancient technologies of impeccable intellectual conjurations. The art and science of provable calculuses, mathematics resides at the epicenter of culture. The freedom of reasoning is the *sine qua non* of the personal liberty of a human being. Mathematics in the foundations of mentality becomes the guarantee of freedom. The creative contributions of Euler as well as his best descendants exhibit uncountably many supreme examples. The fate of Sobolev made no exclusion.

In the twentieth century mankind came to the edge of the frontiers of its safe and serene existence, exhibiting the inability of halting the instigators of the First and Second World Wars. The weapon of deterrence arose as a warrant of freedom. The invention and production of the A-bomb in the USA and Russia demonstrate the tremendous power of science, the last resort of the survival of mankind. Mathematicians may be proud of the valor of their colleagues in these exploits. Von Neumann and Ulam participated in the Manhattan project. Sobolev and Kantorovich were involved in the Soviet project "Enormous."[20]

Most documents are declassified and published about the making of nuclear weapons, and so we may feel the tension of the heroic epoch.

The start of the atomic project in this country is traditionally marked with Directive No. 2352ss[21] of the SDC[22] which was entitled "Organization of the Works on Uranium" and dated September 24, 1942.[23] A few months later on February 1943, the SDC decided

[20]Transliterated in Russian like "Énormoz." This code name was used in the operative correspondence of the intelligence services of the USSR.

[21]The letters "ss" abbreviate the Russian for "top secret."

[22]This is the acronym of the State Defence Committee of the USSR. Another acronym was SDCO.

[23]The original was not signed by the Chairman of the SDC I. V. Stalin who had a habit of endorsing the front cover of the whole folder with a pile of documents. The appended mailing list indicates that the full text of the Directive was forwarded to V. M. Molotov, S. V. Kaftanov, A. F. Ioffe, V. L. Komarov, and Ya. E. Chadaev.

to organize Laboratory No. 2 of the Academy of Science of the USSR for studying nuclear energy. I. V. Kurchatov was entrusted with the supervision of the Laboratory as well as the management of all works related to the atomic problem. Sobolev was soon appointed one of the deputies of Kurchatov and joined the group of I. K. Kikoin which studied the problem of enriching uranium with cascades of diffusive membranes for isotope separation.

The Special Folder[24] saves the report by Kurchatov and Kikoin as of August 1945. The preamble of this document reads:

> The work on utilizing nuclear energy started in the USSR in 1943 when Laboratory No. 2 was arranged in the Academy of Sciences under the leadership of Academician Kurchatov I. V.
>
> Since the Laboratory has no premises, facilities, cadres, and uranium, the work was reduced to analyzing the secret materials about the investigations of the foreign scientists in the uranium problem as well as checking these data by calculation and performing of a few experiments.
>
> In the second half of 1944 and [in] the beginning of 1945, Laboratory No. 2 had received support by a decision of the SDCO with premises, facilities, materials, and cadres, which enables the Laboratory to launch its own research.
>
> A series of institutions as well as design and construction organizations of the USSR were assigned to work by the program of Laboratory No. 2 (including the Radium, Physical, and Energy Institutes of the Academy of Sciences of the USSR, the All-Union Institute of Mineral Resources, the State Rare Metal Institute, the State SRI[25]-42, etc.).
>
> As regards the methods for acquiring the atomic explosives (uranium-235 and plutonium-239) which are known abroad, namely, the method of the "uranium–graphite boiler," the method of the "uranium–deuterium boiler,"[26] the diffusion method, and the magnetic method, the top officials of Laboratory No. 2 (Academicians Kurchatov and Sobolev together with Corresponding Members of the Academy of Sciences Kikoin and Voznesenskiĭ) opine that the Laboratory has already the data on the first three of the methods which is enough for designing and erecting the facilities.[27]

It was already in 1946 that the first gaseous compressors were produced and put into the serial production. The tests began of enriching uranium hexafluoride. The work required solving incredibly many versatile scientific, technological, and managerial problems which became the main busyness of Sobolev for many years. It suffices to give the list of problems from a memo for L. P. Beriya as of August 15, 1946:[28]

1. Choice of the general scheme of the technological process of the industrial separation plant.
2. Raw materials.
3. The problem of filters.
4. Compressors.
5. The problem of the pressurization (hermetic sealing) of compressors and lubrication.
6. The problem of corrosive materials in uranium hexafluoride.
7. Analysis of the enrichment of the light isotope.
8. The problem of control and automation.

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[24] In those days the term "special folder" was also a formal top secrecy stamp.

[25] This is the acronym for the state research institute.

[26] The term "heavy water" is used in the original.

[27] The whole document is presented in [18, p. 307]. There is a handwritten note by Stalin: "Due for reading."

[28] Cp. [18, p. 567].

Sobolev joined the group for plutonium-239 and the group for uranium-235.[29] He organized and coordinated the work of the staff of calculators, solved the problem of control of the industrial isotope separation, and was responsible for minimizing the losses of production. His role in the atomic project became more important. In February of 1947 Kurchatov wrote to Beriya:

> By now Academician S. L. Sobolev was acquainted only with the documents of Bureau No. 2 which are related to the diffusion method. In regard of his appointment to the position of the Deputy Principal of Laboratory No. 2 of the Academy of Sciences of the USSR, I ask your permission to acquaint Academician Sobolev S. L. with the documents of Bureau No. 2 concerning all aspects of the problem.[30]

The test of the Joe-1[31] took place near Semipalatinsk at 8 a.m. local time on August 29, 1949. Exactly two months later more than eight hundred staff members of the atomic project were decorated with various state orders. Sobolev was awarded with the Order of Lenin. It was in the mid 1949 that Laboratory No. 2 was renamed to become the Laboratory of Measuring Tools of the Academy of Sciences, abbreviated as LIPAN in Russian. The efforts of Kikoin and Sobolev were focused on the manufacturing program of the diffusion plant. One of the items of Decree No. 5472-2086ss/op[32] of the Council of Ministers of the USSR as of December 1, 1949 reads:

> Entrust Comrade Sobolev S. L. (Deputy Principal of Laboratory No. 2 of the Academy of Sciences of the USSR) with the management of the theoretical calculation section of the Central Laboratory of Combine No. 813, on requesting that he be on duty at the combine for at least 50% of the whole working hours (on consent of Comrade Kurchatov I. V.).[33]

In the LIPAN Sobolev wrote the main book of his life, *Some Applications of Functional Analysis in Mathematical Physics.*[34]

The atomic project enriched the scientific and personal potential of Sobolev. Computational mathematics occupied a prominent place in his creative activities up to his last days. From 1952 to 1960 he held the chair of the department of computational mathematics at Lomonosov State University. Later in Siberia, Sobolev propounded the theory of cubature formulas which is wondrous in the beauty of its universality. Sobolev synthesized the ideas of the classical approximative methods and distribution theory. Sobolev suggested that calculations on a mesh should be considered as some integrals involving distributions. This was done within his deep belief in the indissoluble ties between functional analysis and the theory of computations.

The work in the LIPAN added many bright colors to Sobolev's views of mathematics. Those years brought to him the understanding that of importance in many cases is the actual presentation of a reasonable solution on the appointed time rather than the

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[29]See [18, p. 386].

[30]Cited from [19, p. 432]. This top secret document was handwritten in a sole copy and bears the resolution by Beriya: "Agreed. L. Beriya. 21/11–47."

[31]Joe-1 was the English nickname for the Soviet A-bomb No. 1. The Russian codename was RDS-1.

[32]The letters "op" imply the stamp "special folder" in Russian.

[33]Cp. [19, pp. 363–364].

[34]Published in 1950 by Leningrad State University, reprinted in 1962 by the Siberian Division of the Academy of Sciences of the USSR in Novosibirsk, and translated into English by the American Mathematical Society in 1963. The third Russian edition was printed by the Nauka Publishers in 1988.

abstract problem of existence of a solution. The civic courage of Sobolev in safeguarding the new ideas of genetics, cybernetics, and mathematical economics in the postwar years of the offensive of the obscurantists of "Marxism" ranks alongside his participation in the "Enormous" project and cultivation of the scientifically virgin lands of Siberia.

The contribution of Sobolev to the making of nuclear weapons is acknowledged and marked with not only the title of the Hero of the Socialist Labor but also the eternal gratitude of the people of this country to the famous and anonymous saviors of the freedom of the homeland.

New Derivation—New Calculus

Sobolev's contributions are connected with the reconsideration of the concept of solution to a differential equation. He suggested that the Cauchy problem be solved in the dual space, the space of functionals, which means the rejection of the classical view that any solution of any differential equation presents a function. Sobolev proposed to assume that a differential equation is solved provided that all integral characteristics are available of the behavior of the process under study. Moreover, the solution as a function of time may fail to exist at all rather than stay unknown for us temporarily. In actuality, science has acquired a new understanding of the key principles of prognosis.

It was as long ago as in 1755 that Euler gave the universal definition of function which was perceived as the most general and perfect. In his celebrated course in differential calculus, Euler wrote:[35]

> If, however, some quantities depend on others in such a way that if the latter are changed the former undergo changes themselves then the former quantities are called functions of the latter quantities. This is a very comprehensive notion and comprises in itself all the modes through which one quantity can be determined by others. If, therefore, x denotes a variable quantity then all the quantities which depend on x in any manner whatever or are determined by it are called its functions.

The generalized derivatives in the Sobolev sense do not obey the Eulerian definition of function. Differentiation by Sobolev implies the new conception of interrelation between mathematical quantities. A generalized function is determined implicitly from the integral characteristics of its action on each representative of some class of test functions that was chosen in advance.

The discoveries by Newton and Leibniz summarized the centenary-old prehistory of differential and integral calculus, opening the new areas of research. The achievements of Lebesgue and Sobolev continued the contemplations of their glorious predecessors and paved the turnpike for the present-day mathematicians.[36]

Sobolev was among the pioneers of application of functional analysis in mathematical physics, propounding his theory in 1935. In the articles by Laurent Schwartz[37] who

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[35] Cp. [20, p. 38] and [27, pp. 72–73].

[36] Consult [21] about the prehistory of distributions. The famous quandary between Euler and d'Alembert about the vibrating string was a harbinger of search into the abstractions of the concept of a solution to a differential equation (cp. [21, pp. 15–24] and [22]). Euler's liberal handling of divergent series reflected the flashes of the future theory of distributions (cp. [2, pp. 187–188]).

[37] Schwartz's views of the discovery of distribution theory are presented in his autobiography [23]. A few relevant references are given in [24].

came to the similar ideas a decade later, the new calculus became comprehensible and accessible for everyone in the elegant, powerful and rather transparent form of the theory of distributions which has utilized many progressive ideas of algebra, geometry, and topology.

Lavish was the Sobolev appraisal of the contribution of Schwartz into the elaboration of the technique of the Fourier transform for distributions:[38]

> The generalized functions, in much the same way as the ordinary functions, can be subjected to the Fourier transform. We may say even more: In the classical calculus, the Fourier transform was confronted with many considerable difficulties such as the divergence of integrals, the impossibility of interpreting the resultant infinite expressions in a definite sense, and so on. The theory of generalized functions eliminated most of these difficulties and made the Fourier transform a powerful tool of analysis.[39]

The differential calculus of the seventeenth century is inseparable from the general views of the classical mechanics. Distribution theory is tied with the mechanics of quanta.

We must emphasize that quantum mechanics is not a plain generalization of the mechanics of classics. Quantum mechanics presents the scientific outlook that bases on the new laws of thought. The classical determinism and continuity swapped placed with quantization and uncertainty. It was in the twentieth century that mankind raised to a completely new comprehension of the processes of nature.

Similar is the situation with the modern mathematical theories. The logic of these days is not a generalization of the logic of Aristotle. Banach space geometry is not an abstraction of the Euclidean plane geometry. Distribution theory, reigning as the calculus of today, has drastically changed the whole technology of the mathematical description of physical processes by means of differential equations.

Sobolev heard the call of the future and bequeathed his spaces to mankind.[40] His discoveries triggered many revolutionary changes in mathematics whose progress we are happy to observe and follow.

The terminal series of Sobolev's mathematical articles was devoted to the subtle properties of the roots of the Euler polynomials . . . .

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[38]Sobolev dated the theory of generalized functions from his paper of 1935 and wrote: "The theory of generalized functions was further developed by L. Schwartz [21] who has in particular considered and studied the Fourier transform of a generalized function" (cp. [25, p. 355]). This is a curious misprint: the correct reference to Schwartz's two-volume set should be [47].

[39]Cp. [25, p. 415].

[40]In an untitled poem with the first line "It is not seemly to be famous" as of 1956, Boris Pasternak wrote (as rendered in English by Lydia Pasternak Slater, cp. [26, p. 255]):

Try not to live as a pretender,
But so to manage your affairs
That you are loved by wide expanses,
And hear the call of future years.

The verbatim translation of the Russian original of the third line of the above excerpt contains the marvelous expression "love of space." So, the above verses might be rendered as follows:

Avoid conceit and self-pretense,
But pace up life and make your soul
Attract to you the love of space
And read the future's distant call.

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SOBOLEV AND SCHWARTZ: TWO FATES AND TWO FAMES

In the history of mathematics there are quite a few persons whom we prefer to recollect in pairs. Listed among them are Euclid and Diophant, Newton and Leibniz, Bolyai and Lobachevskiĭ, Hilbert and Poincaré, as well as Bourbaki and Arnold. In this series we enroll Sobolev and Schwartz who are inseparable from one of the most brilliant discoveries of the twentieth century, the theory of generalized functions or distribution theory providing a revolutionary new approach to partial differential equations.

The most vibrant and lasting achievements of mathematics reside in formulas and lists. There are pivotal distinctions between lists and formulas. The former deposit that which was open for us. The lists of platonic solids, elementary catastrophes, and finite simple groups are next of kin to the *Almagest* and herbaria. They are the objects of admiration, tremendous and awe-struck. The article of the craft of mathematics is a formula. Every formula enters into life as an instance of materialization of mathematical creativity. No formula serves only the purpose it was intended to. In part, any formula is reminiscent of household appliances, toys, or software. It is a very rare event that somebody reads the user's guide of a new TV set or the manual for running a new computer program. Usually everyone utilizes his or her new gadgets experimentally by pressing whatever keys and switches. In much the same way we handle formulas. We painstakingly "twist and turn" them, audaciously insert new parameters, willfully interpret symbols, and so on.

Mathematics is the craft of formulas and the art of calculus. If someone considers this claim as feeble and incomplete, to remind is in order that, logically speaking, set theory is just an instance of the first order predicate calculus.

Distribution theory has become the calculus of today. Of such a scale and scope is the scientific discovery by Sobolev and Schwartz.

1. Sergeĭ L′vovich Sobolev

Sobolev was born in St. Petersburg on October 6, 1908 in the family of Lev Aleksandrovich Sobolev, a solicitor. Sobolev's grandfather on his father's side descended from a family of Siberian Cossacks.

Sobolev was bereaved of his father in the early childhood and was raised by his mother Natal′ya Georgievna who was a highly-educated teacher of literature and history. His mother also had the second speciality: she graduated from a medical institute and worked

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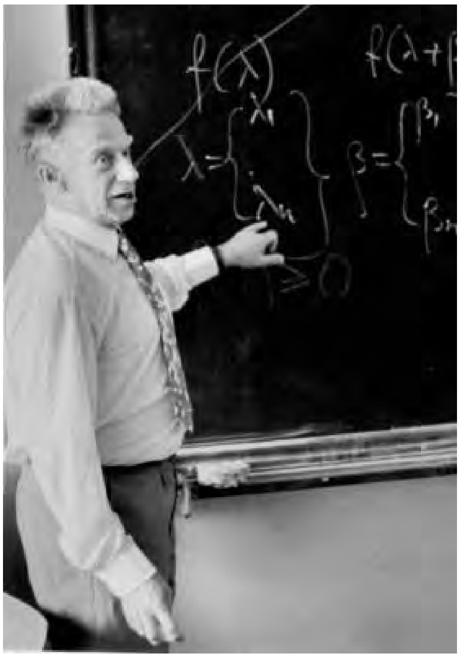
Partly printed in [1] with unauthorized omissions.

The author thanks V. A. Aleksandrov and V. P. Golubyatnikov who helped him in better understanding of French sources. The author is especially grateful to Yu. L. Ershov who was persistent in inviting the author to make a talk at the special session of the Academic Council of the Sobolev Institute on October 14, 2003. The present article bases on this talk. The author acknowledges the subtle and deep comments of V. I. Arnold and V. S. Vladimirov on the preprint of a draft of the talk which led to many improvements.

as a tutor at the First Leningrad Medical Institute. She cultivated in Sobolev the decency, indefatigability, and endurance that characterized him as a scholar and personality.



Sobolev fulfilled the program of secondary school at home, revealing his great attraction to mathematics. During the Civil War he and his mother lived in Kharkov. When living there, he studied at the preparatory courses of a evening technical school for one semester. At the age of 15 he completed the obligatory programs of secondary school in mathematics, physics, chemistry, and other natural sciences, read the classical pieces of the Russian and world literature as well as many books on philosophy, medicine, and biology. After the family had transferred from Kharkov to Petersburg in 1923, Sobolev entered the graduate class of School No. 190 and finished with honors in 1924, continuing his study at the First State Art School in the piano class. At the same year he entered the Faculty of Physics and Mathematics of Leningrad State University (LSU) and attended the lectures of Professors N. M. Günter, V. I. Smirnov, G. M. Fikhtengolts, and others. He made his diploma on analytic solutions of a system of differential equations with two independent variables under the supervision of Günter.

Günter propounded the idea that the set functions are inevitable in abstracting the concept of solution to a differential equation. Günter's approach influenced the further train of thought of Sobolev.[1]

After graduation from LSU in 1929, Sobolev started his work at the Theoretical Department of the Leningrad Seismological Institute. In a close cooperation with Smirnov he then solved some fundamental problems of wave propagation. It was Smirnov whom Sobolev called his teacher alongside Günter up to his terminal days.

Since 1932 Sobolev worked at the Steklov Mathematical Institute in Leningrad, and since 1934, in Moscow. He continued the study of hyperbolic equations and proposed a new method for solving the Cauchy problem for a hyperbolic equation with variable coefficients. This method was based on a generalization of the Kirchhoff formula. Research into hyperbolic equations led Sobolev to revising the classical concept of a solution to a differential equation. The concept of a generalized or weak solution of a differential equation was considered earlier.

However, it was exactly in the works by Sobolev that this concept was elaborated and applied systematically. Sobolev posed and solved the Cauchy problem in spaces of functionals, which was based on the revolutionary extension of the Eulerian concept of function and declared 1935 as the date of the birth of the theory of distributions. Suggesting his definition of generalized derivative, Sobolev enriched mathematics with the spaces of functions whose weak derivatives are integrable to some power. These are now called *Sobolev spaces*.

[1]It was A. M. Vershik and V. I. Arnold who attracted the author's attention to the especial role of Günter in the prehistory of distribution theory.

Let f and g be locally summable functions on an open subset G of $\mathbb{R}^n$, and let α be a multi-index. A function g, denoted by $D^\alpha f$, is the *generalized derivative in the Sobolev sense* or *weak derivative* of f of order α provided that

$$\int\limits_G f(x) D^\alpha \varphi(x)\, dx = (-1)^{|\alpha|} \int\limits_G g(x)\varphi(x)\, dx,$$

for every *test function* φ, i.e. such that the support of φ is a compact subset of G and φ is $|\alpha| = \alpha_1 + \dots + \alpha_n$ times continuously differentiable in G, where $D^\alpha \varphi$ is the classical derivative of φ of order α. The vector space W_p^l, with $p \geq 1$, of the (cosets of) locally summable f on G, whose all weak derivatives $D^\alpha f$ with $|\alpha| \leq l$ are p-summable in G becomes a Banach space under the norm:

$$\|f\|_{W_p^l} = \Bigg(\int\limits_G |f|^p\, dx\Bigg)^{1/p} + \sum_{|\alpha|=l} \Bigg(\int\limits_G |D^\alpha f|^p\, dx\Bigg)^{1/p}.$$

Sobolev found the general criteria for equivalence of various norms on W_p^l and showed that these spaces are the natural environment for posing the boundary value problems for elliptic equations. This conclusion was based on his thorough study of the properties of Sobolev spaces. The most important facts are *embedding theorems*. The content of an embedding theorem consists in estimating the operator norm of the embedding; in other words, in the special inequalities between the norms of one and the same function considered as inside various spaces.

The contributions of Sobolev brought him recognition in the USSR. In 1933 Sobolev was elected a Corresponding Member of the Academy of Sciences at the age of 24 years. In 1939 he became a Full Member of the Academy and remained the youngest academician for many years.

Inspired by military applications in the 1940s, Sobolev studying the system of differential equations describing small oscillations of a rotating fluid. He obtained the conditions for stability of a rotating body with a filled-in cavity which depend on the shape and parameters of the cavity Moreover, he elaborated the cases in which the cavity is a cylinder or an ellipsoid of rotation. This research by Sobolev signposted another area of the general theory which concerns to the Cauchy and boundary value problems for the equations and systems that are not solved with respect to the higher time derivatives. In the grievous years of the Second World War from 1941 to 1944 Sobolev occupied the position of the director of the Steklov Mathematical Institute.

Sobolev was one of the first scientists who foresaw the future of computational mathematics and cybernetics. From 1952 to 1960 he held the chair of the first national department of computational mathematics at Moscow State University. This department has played a key role in the development of many important areas of the today's mathematics.

Addressing the problems of computational mathematics, Sobolev lavishly applied the apparatus of the modern sections of the theoretical core of mathematics. It is typical for him to pose the problems of computational mathematics within functional analysis. Winged are his words that "to conceive the theory of computations without Banach spaces is impossible just as trying to conceive it without computers."

It is worthwhile to emphasize the great role in the uprise of cybernetics, genetics, and other new areas of research in this country which was played by the publications and speeches of Sobolev who valiantly defended the new trends in science from the ideologized obscurantism.

It is difficult to overrate the contribution of Sobolev to the design of the nuclear shield of this country. From the first stages of the atomic project of the USSR he was listed among the top officials of Laboratory No. 2 which was renamed for secrecy reasons into the Laboratory of Measuring Instruments (abbreviated as LIPAN in Russian). Now LIPAN lives as the Kurchatov Center. The main task of the joint work with I. K. Kikoin was the implementation of gaseous diffusive uranium enrichment for creation of an nuclear explosive device.

Sobolev administered and supervised various computational teams, studied the control of the industrial processes of isotope separation, struggled for the low costs of production and made decisions on many managerial and technological matters. For his contribution to the A-bomb project, Sobolev twice gained a Stalin Prize of the First Degree. In January of 1952 Sobolev was awarded with the highest title of the USSR: he was declared the Hero of the Socialist Labor for exceptional service to the state.

Sobolev's research was inseparable from his management in science. At the end of the 1950s M. A. Lavrent′ev, S. L. Sobolev, and S. A. Khristianovich came out with the initiative to organize a new big scientific center, the Siberian Division of the Academy of Sciences. For many scientists of the first enrolment to the Siberian Division it was the example of Sobolev, his authority in science, and the attraction of his personality that yielded the final argument in deciding to move to Novosibirsk.

The Siberian period of Sobolev's life in science was marked by the great achievements in the theory of cubature formulas. The problem of approximate integration of functions is one of the main problems in the theory of computations—the cost of computation of multidimensional integrals is extremely high. Optimizing the formulas of integration is understood now to be the problem of minimizing the norm of the error on some function space. Sobolev suggested the new approaches to the problem and discovered the new classes of optimal cubature formulas.

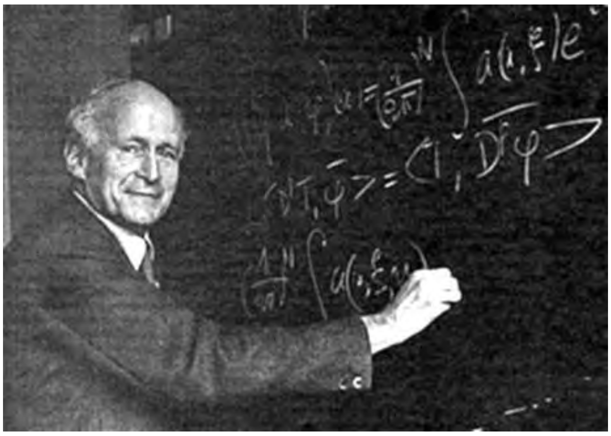
Sobolev merits brought him many decorations and signs of distinction. In 1988 he was awarded the highest prize of the Russian Academy of Sciences, the Lomonosov Gold Medal.

Sobolev passed away in Moscow on January 3, 1989.

2. Laurent Schwartz

Schwartz was born in Paris on March 5, 1915 in the family of Anselme Schwartz, a surgeon. There were quite a few prominent persons among his next of kin. J. Hadamard was his granduncle. Many celebrities are list in his mother's line Claire Debrés: Many famous Gaullist politicians are listed among the Debérs. In 1938 Schwartz married Marie-Hélèn Lévy, the daughter of the outstanding mathematician P. Lévy who was one of the forefathers of functional analysis. Marie-Hélèn had become a professional mathematician and gained the position of a full professor in 1963.

The munificent gift of Schwartz was revealed in his lecée years. He won the most prestigious competition for high school students, Concours Général in Latin. Schwartz was unsure about his future career, hovering between geometry and "classics" (Greek and Latin). It is curious that Hadamard had a low opinion of Schwartz mathematical plans, since the sixteen-years old Laurent did not know the Riemann zeta function. By a startling contrast, Schwartz was boosted to geometry by the pediatrician Robert Debré and one of his teachers of classics.



In 1934 Schwartz passed examinations to the École Normale Supérieure (ENS) after two years of preparation. He was admitted together with Gustave Choquet, a winner of the Concours Général in mathematics, and Marie-Hélèn, one of the first females in the ENS. The mathematical atmosphere of those years in the ENS was determined by É. Borel, É. Cartan, A. Denjoy, M. Fréchet, and P. Montel. The staff of the neighboring Collège de France included H. Lebesgue who delivered lectures and Hadamard who conducted a seminar. It was in his student years that Schwartz had acquired the insoluble and permanent love to probability theory which grew from conversations with his future father-in-law Lévy.

After graduation from the ENS Schwartz decided to be drafted in the compulsory military service for two years. He had to stay in the army in 1939–1940 in view of war times. These years were especially hard for the young couple of the Schwartzes. It was unreasonable for Jews to stay in the occupied zone. The Schwartzes had to escape from the native north and manage to survive on some modest financial support that was offered in particular by Michelin, a world-renowned tire company. In 1941 Schwartz was in Toulouse for a short time and met H. Cartan and J. Delsarte who suggested that the young couple should move to Clermont-Ferrand, the place of temporary residence of the group of professors of Strasbourg University which had migrated from the German occupation. These were J. Dieudonné, Ch. Ehresmann, A. Lichnerowicz, and S. Mandelbrojt. In Clermont-Ferrand Schwartz completed his Ph. D. Thesis on approximation of a real function on the axis by sums of exponentials.

Unfortunately, the war had intervened into the mathematical fate of Schwartz and his family had to change places under false identities. Curiously, in the time of the invention of distributions in November of 1944 Schwartz used the identity of Selimartin. The basics of Schwartz's theory was published in the *Annales* of the University of Grenoble in 1945 [8]. Schwartz described the process of invention as "cerebral percolation." After a year's stay in Grenoble, Schwartz acquired a position in Nansy, some plunging in the center of "Bourbakism." It is well known that N. Bourbaki resided in Nancago, a mixture of Nancy and Chicago. A. Weyl was in Chicago, while Delsarte and Dieudonné were in Nancy. Before long Schwartz was enrolled in the group of Bourbaki. In 1950 he was awarded with the Fields medal for distribution theory. His famous two-volume set *Théorie des Distributiones* was printed a short time later.

In 1952 Schwartz returned back to Paris and began to lecture in Sorbonne; and since 1959, in the École Politechnique in company with his father-in-law Lévy. Many celebrities were the direct students of Schwartz. Among them we list A. Grothendieck, J.-L. Lions, B. Malgrange, and A. Martineau.

Schwartz wrote: "To discover something in mathematics is to overcome an inhibition and a tradition. You cannot move forward if you are not subversive." This statement is in good agreement with the very active and versatile public activities of Schwartz. He joined Trotskists in his green years, protesting against the monstrosities of capitalism and

Stalin's terror of the 1930s. Since then he had never agreed with anything that he view as violation of human rights, oppression, or injustice. He was very active in struggling against the American war in Vietnam and the Soviet invasion in Afghanistan. He fought for liberation of a few mathematicians that were persecuted by political reasons, among them Jose Luis Massera, Vaclav Benda, et al.

Schwartz was an outstanding lepidopterist and had collected more than 20 000 butterflies. It is not by chance that a few butterflies are depicted on the soft covers of the second edition of his *Théorie des Distributiones*.

Schwartz passed away in Paris on June 4, 2002.

3. Advances of Distribution Theory

Distribution theory stems from the intention to apply the technology of functional analysis to studying partial differential equations. Functional analysis rests on algebraization, geometrization, and socialization of analytical problems. By socialization we usually mean the inclusion of a particular problem in an appropriate class of its congeners. Socialization enables us to erase the "random features,"[2] eliminating the difficulties of the insurmountable specifics of a particular problem. In the early 1930s the merits of functional analysis were already demonstrated in the area of integral equations. The time was ripe for the differential equations to be placed on the agenda.

It is worth observing that the contemplations about the nature of integration and differentiation underlie most of the theories of the present-day functional analysis. This is no wonder at all in view of the key roles of these remarkable linear operations. Everyone knows that integration possesses a few more attractive features than differentiation: the integral is monotone and raises smoothness. Derivation lacks these nice properties completely. Everyone knows as well that the classical derivative yields a closed yet unbounded operator (with respect to the natural uniform convergence topology that is induces by the Chebyshev sup-norm. The series of smooth functions cannot be differentiated termwise in general, which diminishes the scope of applications of analysis to solving differential equations.

There is practically no denying today that the concept of generalized derivative occupies a central place in distribution theory. Derivation is now treated as the operator that acts on the nonsmooth functions according to the same integral was as the procedure of taking the classical derivative. It is exactly this approach that was explicitly demonstrated by Sobolev. The open way enabled mathematicians to enlarge the stock of the formulas of differentiation. It turned out that each distribution possesses derivatives of all orders, every series of distributions may be differentiated termwise however often, and many "traditionally divergent" Fourier series admit presentations by explicit formulas. Mathematics has acquired additional fantastic degrees of freedom, which makes immortal the name of Sobolev as a pioneer of the new calculus.

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[2]This is a cliché with a century-old history. The famous Russian symbolist Alexander Blok (1880–1921) used the concept of random feature in his incomplete poem "Revenge" as of 1910 [7, p. 482]. The prologue of this poem contains the lines that are roughly rendered in English as follows:

> You share the gift of prudent measure
> For what keen vision might perceive.
> Erasing random features, treasure
> The world of beauty to receive.

The detailed expositions of the new theory had appeared practically at the same time. In 1950 the first volume of *Théorie des Distributiones* was published in Paris, while Sobolev's book *Some Applications of Functional Analysis in Mathematical Physics* was printed in Leningrad. In 1962 the Siberian Division of the Academy of Sciences of the USSR reprinted the book, while in 1963 it was translated into English by the American Mathematical Society. The second edition of the Schwartz book was slightly enriched with a generalized version of the de Rham currents. Curiously, Schwartz left the historical remarks practically the same as before in the introduction to the second edition.

The new methods of distribution theory turned out so powerful as to enable mathematicians to write down, in explicit form, the general solution of an arbitrary partial differential equation with constant coefficients. In fact, everything reduces to existence of fundamental solutions; i.e., to the case of the Dirac delta-function on the right-hand side of the equation under consideration. The existence of these solutions was already established in 1953 and 1954 by B. Malgrange and L. Ehrenpreis independently of each other. However, it was only in 1994 that some formula for a fundamental solution was exhibited by H. König. Somewhat later N. Ortner and P. Wagner found a more elementary formula. The formulation of their result is as follows:[3]

Theorem. *Assume that $P(\partial) \in \mathbb{C}[\partial]$, where P is a polynomial of degree m. Assume further that $\eta \in \mathbb{R}^n$ and $P_m(\eta) \neq 0$, where P_m is the principal part of P; i.e., $P_m = \sum_{|\alpha|=m} a_\alpha \partial^\alpha$. Then the distribution E given as*

$$E := \frac{1}{\overline{P_m(\eta)}} \int\limits_{\mathbb{T}} \lambda^m e^{\lambda \eta x} \mathfrak{F}^{-1}_{\xi \to x} \left(\frac{\overline{P(i\xi + \lambda\eta)}}{P(i\xi + \lambda\eta)} \right) \frac{d\lambda}{2\pi i \lambda}$$

is a fundamental solution of the operator $P(\partial)$. Moreover, $E/\cosh(\eta x) \in \mathscr{S}'(\mathbb{R}^n)$.

It stands to reason to inspect the structure of the formula which reveals the role of the distributional Fourier transform $\mathfrak{F}$ and the Schwartz space $\mathscr{S}'(\mathbb{R}^n)$ comprising tempered distributions.[4]

The existence of a fundamental solution of an arbitrary partial differential equation with constant coefficients is reverently called the *Malgrange–Ehrenpreis Theorem.* It is hard to overestimate this splendid achievement which remains one of the triumphs of the abstract theory of topological vector spaces.

The road from generalized solutions to standard solutions lies through Sobolev spaces. Study of the embeddings and traces of Sobolev spaces has become one of the main sections of the modern theory of real functions. Suffice it to mention S. M. Nikol′skiĭ, O. V. Besov, G. Weiss, V. P. Il′in, and V. G. Mazya in order to conceive the sizes of this area of mathematical research. The titles of dozen books mention Sobolev spaces, which is far from typical in the present-day science.

The broad stratum of modern studies deals with applications of distributions in mathematical and theoretical physics, complex analysis, the theory of pseudodifferential operators, Tauberian theorems, and other sections of mathematics.

The physical sources of distribution theory, as well as the ties of the latter with theoretical physics, are the topics of paramount importance. They require a special scrutiny that

[3] Cp. [12] and [13, Theorem 2.3].

[4] Also known as "generalized functions of slow growth."

falls beyond the scope of this article.[5] We will confine exposition to the concise historical comments by V. S. Vladimirov:[6]

> It was already the creators of this theory, S. L. Sobolev [5] and L. Schwartz [24] who studied the applications of the theory of generalized functions in mathematical physics. After a conversation with S. L. Sobolev about generalized functions, N. N. Bogolyubov used the Sobolev classes [3] of test and generalized functions C^m_{comp} and $(C^m_{\text{comp}})^*$ in constructing his axiomatic quantum field theory [25]–[27]. The same related to the Wightman axiomatics [28]. Moreover, it is impossible in principle to construct any axiomatics of quantum field theory without generalized functions. Furthermore, in the theory of the dispersion relations [29] that are derived from the Bogolyubov axiomatics, the generalized functions, as well as their generalizations–hyperfunctions, appear as the boundary values of holomorphic functions of (many) complex variables. This fact, together with the related aspects such as Bogolyubov's "Edge-of-the-Wedge" Theorem, essentially enriches the theory of generalized functions.

4. Various Opinions About the History of Distributions

J. Leray was one of the most prominent French mathematicians of the twentieth century. He was awarded with the Lomonosov Gold Medal together with Sobolev in 1988. Reviewing the contributions of Sobolev from 1930 to 1955 in the course of Sobolev's election to the Academy of Sciences of the Institute of France in 1967, J. Leray wrote:

> Distribution theory is now well developed due to the theory of topological vector spaces and their duality as well as the concept of tempered distribution which is one of the important achievements of L. Schwartz (Paris) which enabled him to construct the beautiful theory of the Fourier transform for distributions; G. de Rham supplied the concept of distribution with that of current which comprises the concepts of differential form and topological chain; L. Hörmander (Lund, Princeton), B. Malgrange (Paris), J.-L. Lions (Paris) used the theory of distributions to renew the theory of partial differential equations; while P. Lelong (Paris) established one of the fundamental properties of analytic sets. The comprehensive two-volume treatise by L. Schwartz and even more comprehensive five-volume treatise[7] by Gelfand and Shilov (Moscow) are the achievements of so great an importance that even the French contribution deserves the highest awards of our community. The applications of distribution theory in all areas of mathematics, theoretical physics, and numerical analysis remind of the dense forest hiding the tree whose seeds it has grown from. However, we know that if Sobolev had fail to make his discovery about 1935 in Russian it would be done in France by 1950 and somewhat later in Poland; the USA also flatters itself that they would make this discovery in the same years: The science and art of mathematics would be late only by 15 years as compared with Russia. . . .

In sharp contrast with this appraisal, we cite F. Tréves who wrote in the memorial article about Schwartz in October 2003 as follows:[8]

[5] Some historical details are collected in [31]. Also see [32]. J.-M. Kantor kindly made his article available to the author before publication with a courteous cooperation of Ch. Davis, Editor-in-Chief of *The Mathematical Intelligencer*. It was the proposal of Ch. Davis that the article by J.-M. Kantor is supplemented with the short comments [33] and [34].

[6] Cited from the handwritten review for the *Herald of the Russian Academy of Sciences*, dated as of December 10, 2003.

[7] In fact, the series consists of 6 volumes [16]–[21].

[8] Cp. [15, p. 1076].

> The closest any mathematician of the 1930s ever came to the general definition of a distribution is Sobolev in his articles [Sobolev, 1936] and [Sobolev, 1938][9] (Leray used to refer to "distributions, invented by my friend Sobolev"). As a matter of fact, Sobolev truly defines the distributions of a given, but arbitrary, finite order m: as the *continuous linear functionals* on the space C^m_{comp} of compactly supported functions of class C^m. He keeps the integer m fixed; he never considers the intersection C^∞_{comp} of the spaces C^m_{comp} for all m. This is all the more surprising, since he proves that C^{m+1}_{comp}C is dense in C^m_{comp} by the Wiener procedure of convolving functions $f \in C^m_{\mathrm{comp}}$ with a sequence of functions belonging to C^∞_{comp}! In connection with this apparent blindness to the possible role of mentioned to Henri Cartan his inclination to use the elements of C^∞_{comp} as test functions, Cartan tried to dissuade him: "They are too freakish (*trop monstrueuses*)."
>
> Using transposition, Sobolev defines the multiplication of the functionals belonging to C^m_{comp} by the functions belonging to C^m and the differentiation of those functionals: d/dx maps $(C^m_{\mathrm{comp}})^*$ into $(C^{m+1}_{\mathrm{comp}})^*$. But again there is no mention of Dirac $\delta(x)$ nor of convolution, and no link is made with the Fourier transform. He limits himself to applying his new approach to reformulating and solving the Cauchy problem for linear hyperbolic equations. And he does not try to build on his remarkable discoveries. Only after the war does he invent the Sobolev spaces H^m and then only for integers $m \geq 0$. Needless to say, Schwartz had not read Sobolev's articles, what with military service and a world war (and Western mathematicians' ignorance of the works of their Soviet colleagues). There is no doubt that knowing those articles would have spared him months of anxious uncertainty.

F. Tréves should be honored for drifting aside from the practice of evaluating publications from what they lack when he wrote somewhat later about that which made the name of Schwartz immortal:[10]

> Granted that Schwartz might have been replaceable as the inventor of distributions, what can still be regarded as his greatest contributions to their theory? This writer can mention at least two that will endure: (1) deciding that the Schwartz space $\mathscr{S}$ of rapidly decaying functions at infinity and its dual $\mathscr{S}'$ are the "right" framework for Fourier analysis, (2) the Schwartz kernel theorem.

The Tréves opinion coincides practically verbatim with the narration of Schwartz in his autobiography published firstly in 1997. Moreover, Schwartz had even remarked there about Sobolev that[11]

> he did not develop his theory in view of general applications, but with a precise goal: he wanted to define the generalized solution of a partial differential equation with a second term and initial conditions. He includes the initial conditions in the second term in the form of functionals on the boundary and obtains in this way a remarkable theorem on second order hyperbolic partial differential equations. Even today this remains one of the most beautiful applications of the theory of distributions, and he found it in a rigorous manner. The astounding thing is that he stopped at this point. His 1936 article, written in French, is entitled "Nouvelle méthode à résoudre de problème de Cauchy pour les équations linéares hyperboliques normales." After this article, he did nothing further in this fertile direction. In other words, Sobolev himself did not fully understand the importance of his discovery.

[9]These are references to the articles in *Sbornik* [2, 3].

[10]Ibid., p. 1077.

[11]Cp. [11, p. 222].

It is impossible to agree with these opinions. Rather strange is to read about the absence of any mention of the Dirac delta-function among the generalized functions of Sobolev, since δ obviously belongs to each of the spaces $(C^m_{\mathrm{comp}})^*$.

Disappointing is the total neglect of the classical treatise of Sobolev [6] which was a desk of many specialists in functional analysis and partial differential equations for decades.[12] Finally, Schwartz was not recruited in 1997 and did not participate in a world war. Therefore, there were some other reasons for him to overlook the Sobolev book [8] which contains the principally new appications of distributions to computational mathematics. Sobolev based his pioneering results in numerical integration on developing the theory of the Fourier transform of distributions which was created by Schwartz.

Prudent in the appraisals, exceptionally tactful, and modest in his ripe years, Sobolev always abstained from any bit of details of the history of distribution theory neither in private conversations nor in his numerous writings. The opinion that he decided worthy to be left to the future generations about this matter transpires in his concise comments on the origins of distribution theory in his book [8, Ch. 8] which was printed in 1974:

> The generalized functions are "ideal elements" that complete the classical function spaces in much the same way as the real numbers complete the set of rationals. In this chapter we concisely present the theory of these functions which we need in the sequel. We will follow the way of presentation close to that which was firstly used by the author in 1935 in [16].[13] The theory of generalized functions was further developed by L. Schwartz [21] who has in particular considered and studied the Fourier transform of a generalized function.[14] Historically, the generalized function had appeared explicitly in the studies in theoretical physics as well as in the works of J. Hadamard, M. Riesz, S. Bochner, et al.

Therefore, we can agree only in part with the following statement by Schwartz [11, p. 236]:

> Sobolev and I and all the others who came before us were influenced by out time, our environment and our own previous work. It makes it less glorious, but since we were both ignorant of the work of many other people, we still had to develop plenty of originality.

Most mathematicians agree that Israel Gelfand could be ranked as the best arbiter in distribution theory. The series *Generalized Functions* written by him and his students was started in the mid 1950s and remains one of the heights of the world mathematical literature, the encyclopedia of distribution theory. In the preface to the first edition of the first volume of this series, Gelfand wrote:[15]

> It was S. L. Sobolev who introduced generalized functions in explicit and now generally accepted form in 1936.... The monograph of Schwartz *Théorie des Distributiones* appeared in 1950–1951. In this book Schwartz systemized the theory of generalized functions, interconnected all previous approaches, laid the theory of topological linear spaces in the foundations of the theory of generalized functions, and obtain a number of essential and far-reaching results. After the publication of *Théorie des Distributiones*, the generalized functions won exceptionally swift and wide popularity just in two or three years.

This is an accurate and just statement. We may agree with it.

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[12]Published in 1950 by Leningrad State University, reprinted in 1962 by the Siberian Division of the Academy of Sciences of the USSR in Novosibirsk, and translated into English by the American Mathematical Society in 1963. The third Russian edition was printed by the Nauka Publishers in 1988.

[13]This is a reference to the article of 1936 in *Sbornik* [3].

[14]Cp. [25, p. 355]. This is a curious misprint: the correct reference to Schwartz's two-volume set should be [47].

[15]Cp. [16].

5. Classicism and Romanticism

Pondering over the fates of Sobolev and Schwartz, it is impossible to obviate the problem of polarization of the opinions about the mathematical discovery tied with these persons. The hope is naive that this problem will ever received a simple and definitive answer that satisfies and convinces everyone. It suffices to consider the available experience that concerns other famous pairs of mathematicians whose fates and contributions raise the quandaries that sometimes lasted for centuries and resulted in the fierce clashes of opinions up to the present day. The sources of these phenomena seem of a rather universal provenance that is not concealed in the particular traits of the persons in question but resides most probably in the nature of mathematical creativity.

Using quite a risky analogy, we may say that mathematics has some features that are associated with the trends of artistry which are referred to traditionally as classicism and romanticism. It is hard to fail discerning the classic lineaments of the Hellenistic tradition in the writings of Euclid, Newton, Bolyai, Hilbert, and Bourbaki. It is impossible to fail to respond to the accords of the romantic anthem of the human genius which sound in the pages of the writings of Diophant, Leibniz, Lobachevskiĭ, Poincaré, and Arnold.

The magnificent examples of mathematical classicism and romanticism glare from the creative contributions of Sobolev and Schwartz. These scholars and their achievements will remain with us for ever.

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