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LEIBNIZIAN, ROBINSONIAN, AND BOOLEAN VALUED MONADS

НОВОСИБИРСК

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This is an overview of the present-day versions of monadology with some applications to vector lattices and linear inequalities. Two approaches to combining nonstandard set-theoretic models are sketched and illustrated by order convergence, principal projection, and polyhedrality.

Keywords and phrases:
Dedekind complete vector lattice, filter, fragments, principal projection, up-down, descent, ascent, polyhedral Lagrange principle, Boolean valued model, inexact data

Кутателадзе С. С.

Лейбницевы, робинсоновы и булевозначные монады

Обзор современных версий монадологии с некоторыми приложениями к векторным решеткам и линейным неравенствам. Кратко описаны два подхода к комбинированию нестандартных моделей теории множеств и даны иллюстративные приложения к подрядковой сходимости, проектированию на главные компоненты и полиэдральности.

Ключевые слова и фразы:
пространство Канторовича, фильтр, осколки, главная проекция, вверх-вниз, спуск, подъем, полиэдральный принцип Лагранжа, булевозначная модель, неточные данные

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LEIBNIZIAN, ROBINSONIAN, AND BOOLEAN VALUED MONADS

Agenda

The notion of monad is central to external set theory. Justifying the simultaneous use of infinitesimals and the technique of descending and ascending in vector lattice theory requires adaptation of monadology for the implementation of filters in Boolean valued universes. This is still a rather uncharted area of research. The two approaches are available now. One is to apply monadology to the descents of objects. The other consists in applying the standard monadology inside the Boolean valued universe $\mathbb{V}^{(\mathbb{B})}$ over a complete Boolean algebra $\mathbb{B}$, while ascending and descending by the Escher rules.[1] These approaches are sketched and illustrated by tests for order convergence and rules for fragmenting and projecting positive operators in vector lattices. Also, Lagrange's principle is shortly addressed in polyhedral environment with inexact data.

The Origins of Monadology

The concept of monad stems from Ancient Greece. Monadology as a philosophical doctrine is a creation of Leibniz.[2] The general theory of the monads of filters was proposed by Luxemburg[3] within Robinson's nonstandard analysis.[4]

Basics of Monadology

Let $\mathscr{F}$ be a standard filter; ${}^{\circ}\mathscr{F}$, the standard core of $\mathscr{F}$; and ${}^{a}\mathscr{F} := \mathscr{F} \setminus {}^{\circ}\mathscr{F}$, the external set of *remote* elements of $\mathscr{F}$. Note that

$$\mu(\mathscr{F}) := \bigcap {}^{\circ}\mathscr{F} = \bigcup {}^{a}\mathscr{F}$$

is the *monad* of $\mathscr{F}$. Also, $\mathscr{F} = {}^{*}\operatorname{fil}(\{\mu(\mathscr{F})\})$; i.e., $\mathscr{F}$ is the standardization of the collection $\operatorname{fil}(\mu(\mathscr{F}))$ of all supersets of $\mu(\mathscr{F})$.

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[1] Cp. [1] and[2].

[2] Cp. [3], [4], and the Appendix.

[3] Cp. [5].

[4] Cp. [6].

Let $\mathscr{A}$ be a filter on $X \times Y$, and let $\mathscr{B}$ be a filter on $Y \times Z$. Put $\mathscr{B} \circ \mathscr{A} := \mathrm{fil}\{B \circ A \mid A \in \mathscr{A},\ B \in \mathscr{B}\}$, where we may assume all $B \circ A$ nonempty. Then

$$\mu(\mathscr{B} \circ \mathscr{A}) = \mu(\mathscr{B}) \circ \mu(\mathscr{A}).$$

The Granted Horizon Principle

Let X and Y be standard sets. Assume further that $\mathscr{F}$ and $\mathscr{G}$ are standard filters on X and Y respectively satisfying $\mu(\mathscr{F}) \cap {}^{\circ}X \neq \varnothing$. Distinguish a remote set F in ${}^{a}\mathscr{F}$. Given a standard correspondence $f \subset X \times Y$ meeting $\mathscr{F}$, the following are equivalent:

(1) $f(\mu(\mathscr{F}) - F) \subset \mu(\mathscr{G})$;
(2) $(\forall F' \in {}^{a}\mathscr{F})\, f(F' - F) \subset \mu(\mathscr{G})$;
(3) $f(\mu(\mathscr{F})) \subset \mu(\mathscr{G})$.

Boolean Valued Universe

Let $\mathbb{B}$ be a complete Boolean algebra. Given an ordinal α, put

$$V_\alpha^{(\mathbb{B})} := \{x \mid (\exists \beta \in \alpha)\ x : \mathrm{dom}(x) \to \mathbb{B}, \mathrm{dom}(x) \subset V_\beta^{(\mathbb{B})}\}.$$

The *Boolean valued universe* $\mathbb{V}^{(\mathbb{B})}$ is

$$\mathbb{V}^{(\mathbb{B})} := \bigcup_{\alpha \in \mathrm{On}} V_\alpha^{(\mathbb{B})},$$

with On the class of all ordinals.

The truth value $[\![\varphi]\!] \in \mathbb{B}$ is assigned to each formula φ of ZFC relativized to $\mathbb{V}^{(\mathbb{B})}$.

Functional Realization

Let Q be the Stone space of a complete Boolean algebra $\mathbb{B}$. Denote by $\mathfrak{U}$ the (separated) Boolean valued universe $\mathbb{V}^{(\mathbb{B})}$. Given $q \in Q$, put $u \sim_q v \ \leftrightarrow\ q \in [\![u = v]\!]$. Consider the bundle

$$V^Q := \big\{\big(q, \sim_q(u)\big) \mid q \in Q,\ u \in \mathfrak{U}\big\}$$

and denote $\big(q, \sim_q(u)\big)$ by $\widehat{u}(q)$. Hence, $\widehat{u} : q \mapsto \widehat{u}(q)$ is a section of V^Q for every $u \in \mathfrak{U}$. Note that to each $x \in V^Q$ there are $u \in \mathfrak{U}$ and $q \in Q$ satisfying $\widehat{u}(q) = x$. Moreover, we have $\widehat{u}(q) = \widehat{v}(q)$ if and only if $q \in [\![u = v]\!]$.

Make each fiber V^q of V^Q into an algebraic system of signature $\{\in\}$ by letting $V^q \models x \in y \ \leftrightarrow\ q \in [\![u \in v]\!]$, where $u, v \in \mathfrak{U}$ are such that $\widehat{u}(q) = x$ and $\widehat{v}(q) = y$.

The class $\{\widehat{u}(A) \mid u \in \mathfrak{U}\}$, with A a clopen subset of Q, is a base for some topology on V^Q. Thus V^Q as a continuous bundle called a *continuous polyverse*. By a *continuous section* of V^Q we mean a section that is a continuous function. Denote by $\mathfrak{C}$ the class of all continuous sections of V^Q.

The mapping $u \mapsto \widehat{u}$ is a bijection between $\mathfrak{U}$ and $\mathfrak{C}$, yielding a convenient functional realization of the Boolean valued universe $\mathbb{V}^{(\mathbb{B})}$. This universal construction belongs to Gutman and Losenkov.[5]

[5]Cp. [7].

Descending and Ascending

Given φ, a formula of ZFC, and $y \in \mathbb{V}^{\mathbb{B}}$; put $A_\varphi := A_{\varphi(\cdot,\ y)} := \{x \mid \varphi(x,\ y)\}$.

The *descent* $A_\varphi{\downarrow}$ of a class A_φ is

$$A_\varphi{\downarrow} := \{t \mid t \in \mathbb{V}^{(\mathbb{B})}, [\![\varphi(t,\ y)]\!] = \mathbb{1}\}.$$

If $t \in A_\varphi{\downarrow}$, then it is said that t *satisfies* $\varphi(\cdot,\ y)$ *inside* $\mathbb{V}^{(\mathbb{B})}$.

The *descent* $x{\downarrow}$ of $x \in \mathbb{V}^{(\mathbb{B})}$ is defined as

$$x{\downarrow} := \{t \mid t \in \mathbb{V}^{(\mathbb{B})}, [\![t \in x]\!] = \mathbb{1}\},$$

i.e. $x{\downarrow} = A_{\cdot \in x}{\downarrow}$. The class $x{\downarrow}$ is a set.

If x is a nonempty set inside $\mathbb{V}^{(\mathbb{B})}$ then

$$(\exists z \in x{\downarrow})[\![(\exists t \in x)\ \varphi(t)]\!] = [\![\varphi(z)]\!].$$

The *ascent* functor acts in the opposite direction.

The Reals Within

There is an object $\mathscr{R}$ inside $\mathbb{V}^{(\mathbb{B})}$ modeling $\mathbb{R}$, i.e., $[\![\mathscr{R}$ is the reals$]\!] = \mathbb{1}$.

Let $\mathscr{R}{\downarrow}$ be the descent of the carrier $|\mathscr{R}|$ of the algebraic system

$$\mathscr{R} := (|\mathscr{R}|, +,\ \cdot, 0, 1, \le)$$

inside $\mathbb{V}^{(\mathbb{B})}$.

Implement the descent of the structures on $|\mathscr{R}|$ to $\mathscr{R}{\downarrow}$ as follows:

$$x + y = z \leftrightarrow [\![x + y = z]\!] = \mathbb{1};$$
$$xy = z \leftrightarrow [\![xy = z]\!] = \mathbb{1};$$
$$x \le y \leftrightarrow [\![x \le y]\!] = \mathbb{1};$$
$$\lambda x = y \leftrightarrow [\![\lambda^{\wedge} x = y]\!] = \mathbb{1}\ (x, y, z \in \mathscr{R}{\downarrow},\ \lambda \in \mathbb{R}).$$

Gordon Theorem.[6] *$\mathscr{R}{\downarrow}$ with the descended structures is a universally complete vector lattice with base $\mathbb{B}(\mathscr{R}{\downarrow})$ isomorphic to $\mathbb{B}$.*

Filters within $\mathbb{V}^{(\mathbb{B})}$

Let $\mathscr{G}$ be a filterbase on X, with $X \in \mathscr{P}(\mathbb{V}^{(\mathbb{B})})$. Put

$$\mathscr{G}' := \{F \in \mathscr{P}(X{\uparrow}){\downarrow} \mid (\exists G \in \mathscr{G})\, [\![\, F \supset G{\uparrow}\,]\!] = \mathbb{1}\};$$
$$\mathscr{G}'' := \{G{\uparrow} \mid G \in \mathscr{G}\}.$$

Then $\mathscr{G}'{\uparrow}$ and $\mathscr{G}''{\uparrow}$ are bases of the same filter $\mathscr{G}^{\uparrow}$ on $X{\uparrow}$ inside $\mathbb{V}^{(\mathbb{B})}$—the *ascent* of $\mathscr{G}$. If $\mathrm{fil}(\mathscr{G})$ is the set of all mixings of nonempty families of elements of $\mathscr{G}$ and $\mathscr{G}$ consists of cyclic sets; then $\mathrm{fil}(\mathscr{G})$ is a filterbase on X and $\mathscr{G}^{\uparrow} = \mathrm{fil}(\mathscr{G})^{\uparrow}$.

If $\mathscr{F}$ is a filter on X inside $\mathbb{V}^{(\mathbb{B})}$ then put $\mathscr{F}^{\downarrow} := \mathrm{fil}\,(\{F{\downarrow} \mid F \in \mathscr{F}{\downarrow}\})$. The filter $\mathscr{F}^{\downarrow}$ is the *descent* of $\mathscr{F}$. A filterbase $\mathscr{G}$ on $X{\downarrow}$ is *extensional* provided that $\mathrm{fil}\,(\mathscr{G}) = \mathscr{F}$ for some filter $\mathscr{F}$ on X.

The descent of an ultrafilter on X is a *proultrafilter* on $X{\downarrow}$. A filter with a base of cyclic sets is *cyclic*. Proultrafilters are maximal cyclic filters.

[6] Cp. [1, p. 349].

Cyclic Filters and Monads

Fix a standard complete Boolean algebra $\mathbb{B}$ and think of $\mathbb{V}^{(\mathbb{B})}$ to be composed of internal sets. If A is external then the *cyclic hull* $\mathrm{fil}(A)$ of A consists of $x \in \mathbb{V}^{(\mathbb{B})}$ admitting an internal family $(a_\xi)_{\xi\in\Xi}$ of elements of A and an internal partition $(b_\xi)_{\xi\in\Xi}$ of unity in B such that x is the mixing of $(a_\xi)_{\xi\in\Xi}$ by $(b_\xi)_{\xi\in\Xi}$; i.e., $b_\xi x = b_\xi a_\xi$ for $\xi \in \Xi$ or, equivalently, $x = \mathrm{fil}_{\xi\in\Xi}(b_\xi a_\xi)$.

Given a filter $\mathscr{F}$ on $X{\downarrow}$, let

$$\mathscr{F}{\uparrow\downarrow} := \mathrm{fil}\,(\{F{\uparrow\downarrow} \mid F \in \mathscr{F}\}).$$

Then $\mathrm{fil}(\mu(\mathscr{F})) = \mu(\mathscr{F}{\uparrow\downarrow})$ and $\mathscr{F}{\uparrow\downarrow}$ is the greatest cyclic filter coarser than $\mathscr{F}$.

The monad of $\mathscr{F}$ is called *cyclic* if $\mu(\mathscr{F}) = \mathrm{fil}(\mu(\mathscr{F}))$. Unfortunately, the cyclicity of a monad is not completely responsible for extensionality of a filter.

Monad Hulls

The *cyclic monad hull* $\mu_c(U)$ of an external set U is defined as follows:

$$x \in \mu_c(U) \leftrightarrow (\forall^{\mathrm{st}} V = V{\uparrow\downarrow}) V \supset U \to x \in \mu(V).$$

If $\mathbb{B} = 2$, then $\mu_c(U)$ is the monad of the standardization of the external filter of supersets of U, i.e. the (*discrete*) *monad hull* $\mu_d(U)$.

The cyclic monad hull of a set is the cyclic hull of its monad hull $\mu_c(U) = \mathrm{fil}(\mu_d(U))$.

Essential Points

A special role is played by the *essential points* of $X{\downarrow}$ constituting the external set ${}^eX$. By definition, an essential point of ${}^eX$ belongs to the monad of some proultrafilter on $X{\downarrow}$. The collection ${}^eX$ of all essential points of X is usually external.

A point $x \in {}^eX$ if and only if x can be separated by a standard set from every standard cyclic set not containing x.

If there is an essential point in the monad of an ultrafilter $\mathscr{F}$ then $\mu(\mathscr{F}) \subset {}^eX$; moreover, $\mathscr{F}{\uparrow\downarrow}$ is a proultrafilter.

A filter $\mathscr{F}$ is extensional if and only if $\mu(\mathscr{F}) = \mu_c({}^e\mu(\mathscr{F}))$. A standard set A is cyclic if and only if A is the cyclic monad hull of ${}^eA$.

Test for the Mixing of Filters

Let $(\mathscr{F}_\xi)_{\xi\in\Xi}$ be a standard family of extensional filters, and let $(b_\xi)_{\xi\in\Xi}$ be a standard partition of unity. The filter $\mathscr{F}$ is the mixing of $(\mathscr{F}_\xi)_{\xi\in\Xi}$ by $(b_\xi)_{\xi\in\Xi}$ if and only if

$$(\forall^{\mathrm{St}} \xi \in \Xi) b_\xi \mu(\mathscr{F}) = b_\xi \mu(\mathscr{F}_\xi).$$

Properties of Essential Points

(1) The image of an essential point under an extensional mapping is an essential point of the image;

(2) Let E be a standard set, and let X be a standard element of $\mathbb{V}^{(\mathbb{B})}$. Consider the product $X^{E^\wedge}$ inside $\mathbb{V}^{(\mathbb{B})}$, where $E^\wedge$ is the standard name of E in $\mathbb{V}^{(\mathbb{B})}$. If x is an essential point of $X^{E^\wedge}{\downarrow}$ then for every standard $e \in E$ the point $x{\downarrow}(e)$ is essential in $X{\downarrow}$;

(3) Let $\mathscr{F}$ be a cyclic filter in $X{\downarrow}$, and let ${}^e\mu(\mathscr{F}) := \mu(\mathscr{F}) \cap {}^eX$ be the set of essential points of its monad. Then ${}^e\mu(\mathscr{F}) = {}^e\mu(\mathscr{F}^{\uparrow\downarrow})$.

Procompactness

Let $(X, \mathscr{U})$ be a uniform space inside $\mathbb{V}^{(\mathbb{B})}$. The descent $(X{\downarrow}, \mathscr{U}^{\downarrow})$ is *procompact* or *cyclically compact* if $(X, \mathscr{U})$ is compact inside $\mathbb{V}^{(\mathbb{B})}$. A similar sense resides in the notion of *pro-total-boundedness* and so on.

Every essential point of $X{\downarrow}$ is nearstandard, i.e., infinitesimally close to a standard point, if and only if $X{\downarrow}$ is procompact.

Existence of many procompact but not compact spaces provides a lot of examples of inessential points.

Test for Proprecompactness

A standard space is the descent of a totally bounded uniform space if and only if its every essential point is prenearstandard, i.e. belongs to the monad of a Cauchy filter.

Vector Lattice Environment

Let Y to be a universally complete vector lattice. By Gordon's Theorem, $Y = \mathscr{R}{\downarrow}$ of the reals $\mathscr{R}$ inside $\mathbb{V}^{(\mathbb{B})}$ over the base $\mathbb{B} := \mathbb{B}(Y)$ of Y.

Denote by $\mathscr{E}$ the filter of order units in Y, i.e. $\mathscr{E} := \{\varepsilon \in Y_+ \mid [\![\, \varepsilon = 0 \,]\!] = \mathbb{0}\}$.

Put $x \approx y \leftrightarrow (\forall^{\rm st} \varepsilon \in \mathscr{E})\, (|x - y| < \varepsilon)$. Given $a, b \in Y$, write $a < b$ if $[\![\, a < b \,]\!] = \mathbb{1}$; in other words, $a > b \leftrightarrow a - b \in \mathscr{E}$. Thus, there is some deviation from the understanding of the theory of ordered vector spaces. Clearly, this is done in order to adhere to the principles of introducing notations while descending and ascending.

Let ${}^{\approx}Y$ be the *nearstandard part* of Y. Given $y \in {}^{\approx}Y$, denote by ${}^{\circ}y$ (or by $\mathrm{st}(y)$) the *standard part* of y, i.e. the unique standard element infinitely close to y.

Order Convergence

For a standard filter $\mathscr{F}$ in Y and a standard $z \in Y$, the following are true:

(1) $\inf_{F \in \mathscr{F}} \sup F \le z \leftrightarrow (\forall y \in {}^{\cdot}\mu(\mathscr{F}{\uparrow}{\downarrow}))\, {}^{\circ}y \le z \leftrightarrow (\forall y \in {}^{e}\mu(\mathscr{F}{\uparrow}{\downarrow}))\, {}^{\circ}y \le z$;

(2) $\sup_{F \in \mathscr{F}} \inf F \ge z \leftrightarrow (\forall y \in {}^{\cdot}\mu(\mathscr{F}{\uparrow}{\downarrow}))\, {}^{\circ}y \ge z \leftrightarrow (\forall y \in {}^{e}\mu(\mathscr{F}{\uparrow}{\downarrow}))\, {}^{\circ}y \ge z$;

(3) $\inf_{F \in \mathscr{F}} \sup F \ge z \leftrightarrow (\exists y \in {}^{\cdot}\mu(\mathscr{F}{\uparrow}{\downarrow}))\, {}^{\circ}y \ge z \leftrightarrow (\exists y \in {}^{e}\mu(\mathscr{F}{\uparrow}{\downarrow}))\, {}^{\circ}y \ge z$;

(4) $\sup_{F \in \mathscr{F}} \inf F \le z \leftrightarrow (\exists y \in {}^{\cdot}\mu(\mathscr{F}{\uparrow}{\downarrow}))\, {}^{\circ}y \le z \leftrightarrow (\exists y \in {}^{e}\mu(\mathscr{F}{\uparrow}{\downarrow})){}^{\circ}y \le z$;

(5) $\mathscr{F} \overset{(o)}{\to} z \leftrightarrow (\forall y \in {}^{e}\mu(\mathscr{F}{\uparrow}{\downarrow}))y \approx z \leftrightarrow (\forall y \in \mu(\mathscr{F}^{\uparrow\downarrow}))y \approx z$.

Here ${}^{\cdot}\mu(\mathscr{F}{\uparrow}{\downarrow}) := \mu(\mathscr{F}{\uparrow}{\downarrow}) \cap {}^{\approx}Y$, and, as usual, ${}^{e}\mu(\mathscr{F}{\uparrow}{\downarrow})$ is the set of essential points of the monad $\mu(\mathscr{F}{\uparrow}{\downarrow})$, i.e. ${}^{e}\mu(\mathscr{F}{\uparrow}{\downarrow}) = \mu(\mathscr{F}{\uparrow}{\downarrow}) \cap {}^{e}\mathscr{R}$.

Infinitesimal Modeling in $\mathbb{V}^{(\mathbb{B})}$

Let us follow the classical approach of Robinson inside $\mathbb{V}^{(\mathbb{B})}$. In other words, the classical and internal universes and the corresponding $*$-map (Robinson's standardization) are understood to be members of $\mathbb{V}^{(\mathbb{B})}$. Moreover, the nonstandard world is supposed to be properly saturated.

Descent Standardization

The descent of the $*$-map is referred to as *descent standardization.* Alongside the term "descent standardization" the expressions like "B-standardization," "prostandardization," etc. are in common parlance. Furthermore, denote the Robinson standardization of a B-set A by ${}^*A$.

The *descent standardization* of a set A with B-structure, i.e. a subset of $\mathbb{V}^{(\mathbb{B})}$, is defined as $({}^*(A{\uparrow})){\downarrow}$ and is denoted by ${}_*A$ (it is meant here that $A{\uparrow}$ is an element of the standard universe located inside $\mathbb{V}^{(\mathbb{B})}$).

Thus, ${}^*a \in {}_*A \leftrightarrow a \in A{\uparrow}{\downarrow}$. The *descent standardization* ${}_*\Phi$ of an *extensional correspondence* Φ is also defined in a natural way.

Considering the descent standardizations of the standard names of elements of the von Neumann universe $\mathbb{V}$, use the abbreviations ${}^*x := {}^*(x^\wedge)$ and ${}_*x := ({}^*x){\downarrow}$ for $x \in \mathbb{V}$. The rules of placing and omitting asterisks (by default) in descent standardization are also assumed as liberal as those for the Robinson $*$-map.

Transfer

Let $\varphi = \varphi(x, y)$ be a formula of ZFC without any free variables other than x and y. Then

$$(\exists x \in {}_*F)\,[\![\, \varphi(x, {}^*z) \,]\!] = \mathbb{1} \leftrightarrow (\exists x \in F{\downarrow})\,[\![\, \varphi(x, z) \,]\!] = \mathbb{1};$$

$$(\forall x \in {}_*F)\,[\![\, \varphi(x, {}^*z) \,]\!] = \mathbb{1} \leftrightarrow (\forall x \in F{\downarrow})\,[\![\, \varphi(x, z) \,]\!] = \mathbb{1}$$

for a nonempty element F in $\mathbb{V}^{(\mathbb{B})}$ and for every z.

Idealization

Let $X{\uparrow}$ and Y be classical elements of $\mathbb{V}^{(\mathbb{B})}$, and let $\varphi = \varphi(x, y, z)$ be a formula of ZFC. Then

$$(\forall^{\mathrm{fin}} A \subset X)\,(\exists y \in {}_*Y)\,(\forall x \in A)\,[\![\, \varphi({}^*x, y, z) \,]\!] = \mathbb{1}$$
$$\leftrightarrow (\exists y \in {}_*Y)\,(\forall x \in X)\,[\![\, \varphi({}^*x, y, z) \,]\!] = \mathbb{1}$$

for an internal element z in $\mathbb{V}^{(\mathbb{B})}$.

Descending Monads

Given a filter $\mathscr{F}$ of sets with B-structure, define the *descent monad* $m(\mathscr{F})$ of $\mathscr{F}$ as

$$m(\mathscr{F}) := \bigcap_{F \in \mathscr{F}} {}_*F.$$

Let $\mathscr{E}$ be a set of filters, and let $\mathscr{E}^{\uparrow} := \{\mathscr{F}^{\uparrow} \mid \mathscr{F} \in \mathscr{E}\}$ be its ascent to $\mathbb{V}^{(\mathbb{B})}$. The following are equivalent:

(1) *the set of cyclic hulls of $\mathscr{E}$, i.e. $\mathscr{E}{\uparrow}{\downarrow} := \{\mathscr{F}{\uparrow}{\downarrow} \mid \mathscr{F} \in \mathscr{E}\}$, is bounded above;*
(2) *$\mathscr{E}^{\uparrow}$ is bounded above inside $\mathbb{V}^{(\mathbb{B})}$;*
(3) *$\bigcap\{m(\mathscr{F}) \mid \mathscr{F} \in \mathscr{E}\} \neq \varnothing$.*

Moreover, in this event

$$m(\sup \mathscr{E}{\uparrow}{\downarrow}) = \bigcap\{m(\mathscr{F}) \mid \mathscr{F} \in \mathscr{E}\}; \quad \sup \mathscr{E}^{\uparrow} = (\sup \mathscr{E})^{\uparrow}.$$

It is worth noting that for an infinite set of descent monads, its union, and even the cyclic hull of this union, is not a descent monad in general. The situation here is the same as for ordinary monads.

Nonstandard Tests for a Proultrafilter

The following are equivalent:
(1) $\mathscr{U}$ *is a proultrafilter;*
(2) $\mathscr{U}$ *is an extensional filter with inclusion-minimal descent monad;*
(3) *the representation* $\mathscr{U} = (x)^{\downarrow} := \mathrm{fil}\,(\{U{\uparrow}{\downarrow} \mid x \in {}_*A\})$ *holds for each point* x *of the descent monad* $m(\mathscr{U})$;
(4) $\mathscr{U}$ *is an extensional filter whose descent monad is easily caught by a cyclic set; i.e. either* $m(\mathscr{U}) \subset {}_*U$ *or* $m(\mathscr{U}) \subset {}_*(X \setminus U)$ *for every* $U = U{\uparrow}{\downarrow}$;
(5) $\mathscr{U}$ *is a cyclic filter satisfying the condition: for every cyclic* U, *if* ${}_*U \cap m(\mathscr{A}) \neq \varnothing$ *then* $U \in \mathscr{U}$.

Nonstandard Test for the Mixing of Filters

Let $(\mathscr{F}_\xi)_{\xi\in\Xi}$ *be a family of filters, let* $(b_\xi)_{\xi\in\Xi}$ *be a partition of unity, and let* $\mathscr{F} = \mathrm{fil}_{\xi\in\Xi}(b_\xi \mathscr{F}_\xi^{\uparrow})$ *be the mixing of* $\mathscr{F}_\xi^{\uparrow}$ *by* b_ξ. *Then*

$$m(\mathscr{F}^{\downarrow}) = \mathrm{fil}_{\xi\in\Xi}(b_\xi m(\mathscr{F}_\xi)).$$

Nonstandard Test for Procompactness

A point y of ${}_*X$ is called *descent-nearstandard* or simply *nearstandard* if there is no danger of misunderstanding whenever ${}^*x \approx y$ for some $x \in X{\downarrow}$; i.e., $(x, y) \in m(\mathscr{U}^{\downarrow})$, with $\mathscr{U}$ the uniformity on X.

A set $A{\uparrow}{\downarrow}$ *is procompact if and only if every point of* ${}_*A$ *is descent-nearstandard.*

Rules of Descent Standardization

Let $\varphi = \varphi(x)$ be a formula of ZFC. The truth value of φ is constant on the descent monad of every proultrafilter $\mathscr{A}$; i.e.,

$$(\forall x, y \in m(\mathscr{A}))\, [\![\, \varphi(x) \,]\!] = [\![\, \varphi(y) \,]\!].$$

Let $\varphi = \varphi(x, y, z)$ *be a formula of* ZFC, *and let* $\mathscr{F}$ *and* $\mathscr{G}$ *be filters of sets with* B-*structure.*

The following quantification rules are valid (for internal y *and* z *in* $\mathbb{V}^{(\mathbb{B})}$*):*
(1) $(\exists x \in m(\mathscr{F}))\, [\![\, \varphi(x, y, z) \,]\!] = \mathbb{1} \leftrightarrow (\forall F \in \mathscr{F})\, (\exists x \in {}^*F)\, [\![\, \varphi(x, y, z) \,]\!] = \mathbb{1}$;
(2) $(\forall x \in m(\mathscr{F}))\, [\![\, \varphi(x, y, z) \,]\!] = \mathbb{1} \leftrightarrow (\exists F \in \mathscr{F}^{\uparrow\downarrow})(\forall x \in {}_*F)\, [\![\, \varphi(x, y, z) \,]\!] = \mathbb{1}$;
(3) $(\forall x \in m(\mathscr{F}))\, (\exists y \in m(\mathscr{G}))[\![\, \varphi(x, y, z) \,]\!] = \mathbb{1}$
$\leftrightarrow (\forall G \in \mathscr{G})\, (\exists F \in \mathscr{F}^{\uparrow\downarrow})\, (\forall x \in {}^*F)\, (\exists y \in {}^*G)\, [\![\, \varphi(x, y, z) \,]\!] = \mathbb{1}$;
(4) $(\exists x \in m(\mathscr{F}))\, (\forall y \in m(\mathscr{G}))\, [\![\, \varphi(x, y, z) \,]\!] = \mathbb{1}$
$\leftrightarrow (\exists G \in \mathscr{G}^{\uparrow\downarrow})\, (\forall F \in \mathscr{F})\, (\exists x \in {}^*F)\, (\forall y \in {}^*G)\, [\![\, \varphi(x, y, z) \,]\!] = \mathbb{1}$.

The Case of Standardized Free Variables

(1) $(\exists x \in m(\mathscr{F}))[\![\, \varphi(x, {}^*y, {}^*z) \,]\!] = \mathbb{1} \leftrightarrow (\forall F \in \mathscr{F})(\exists x \in F{\uparrow}{\downarrow})[\![\, \varphi(x, y, z) \,]\!] = \mathbb{1}$;
(2) $(\forall x \in m(\mathscr{F}))[\![\, \varphi(x, {}^*y, {}^*z) \,]\!] = \mathbb{1} \leftrightarrow (\exists F \in \mathscr{F}^{\uparrow\downarrow})(\forall x \in F)[\![\, \varphi(x, y, z) \,]\!] = \mathbb{1}$;
(3) $(\forall x \in m(\mathscr{F}))(\exists y \in m(\mathscr{G}))[\![\, \varphi(x, y, {}^*z) \,]\!] = \mathbb{1}$
$\leftrightarrow (\forall G \in \mathscr{G})(\exists F \in \mathscr{F}^{\uparrow\downarrow})(\forall x \in F)(\exists y \in G{\uparrow}{\downarrow})[\![\, \varphi(x, y, z) \,]\!] = \mathbb{1}$;
(4) $(\exists x \in m(\mathscr{F}))(\forall y \in m(\mathscr{G}))[\![\, \varphi(x, y, {}^*z) \,]\!] = \mathbb{1}$
$\leftrightarrow (\exists G \in \mathscr{G}^{\uparrow\downarrow})(\forall F \in \mathscr{F})(\exists x \in F{\uparrow}{\downarrow})(\forall y \in G)[\![\, \varphi(x, y, z) \,]\!] = \mathbb{1}$.

Again in Vector Lattices

The fact that E is a vector lattice is a bounded formula, say, $\varphi(E,\mathbb{R})$. Hence, recalling the bounded transfer principle, we come to the equality $[\![\,\varphi(E^{\wedge},\mathbb{R}^{\wedge})\,]\!]=\mathbb{1}$; i.e., $E^{\wedge}$ is a vector lattice over the ordered field $\mathbb{R}^{\wedge}$ inside $\mathbb{V}^{(\mathbb{B})}$.

Let $E^{\wedge\sim}$ be the space of regular $\mathbb{R}^{\wedge}$-linear functionals from $E^{\wedge}$ to $\mathscr{R}$. It is easy that $E^{\wedge\sim}:=L^{\sim}(E^{\wedge},\mathscr{R})$ is a K-space, i.e. a Dedekind complete vector lattice, inside $\mathbb{V}^{(\mathbb{B})}$. Since $E^{\wedge\sim}$ is a K-space, the descent $E^{\wedge\sim}{\downarrow}$ of $E^{\wedge\sim}$ is a K-space too.

Turn to the universally complete vector lattice $F:=\mathscr{R}{\downarrow}$. For every operator $T\in L^{\sim}(E,F)$ the ascent $T{\uparrow}$ is defined by the equality $[\![\,Tx=T{\uparrow}(x^{\wedge})\,]\!]=\mathbb{1}$ for all $x\in E$. If $\tau\in E^{\wedge\sim}$, then $[\![\,\tau:E^{\wedge}\to\mathscr{R}\,]\!]=\mathbb{1}$; hence, the operator $\tau{\downarrow}:E\to F$ is available. Moreover, $\tau{\downarrow}{\uparrow}=\tau$. On the other hand, $T{\uparrow}{\downarrow}=T$.

For every $T\in L^{\sim}(E,F)$ the ascent $T{\uparrow}$ is a regular $\mathbb{R}^{\wedge}$-functional on $E^{\wedge}$ inside $\mathbb{V}^{(\mathbb{B})}$; i.e., $[\![\,T{\uparrow}\in E^{\wedge\sim}\,]\!]=\mathbb{1}$. The mapping $T\mapsto T{\uparrow}$ is a linear and lattice isomorphism between $L^{\sim}(E,F)$ and $E^{\wedge\sim}{\downarrow}$.

A Few Classes of Operators

An operator $S\in L^{\sim}(E,F)$ is a *fragment* of $0\le T\in L^{\sim}(E,F)$ if $S\wedge(T-S)=0$. Say that T is *F-discrete* whenever $[0,T]=[0,I_F]\circ T$; i.e., for every $0\le S\le T$ there is an operator $0\le\alpha\le I_F$ satisfying $S=\alpha\circ T$. Let $L^{\sim}_a(E,F)$ be the band of $L^{\sim}(E,F)$ generated by F-discrete operators, and write $L^{\sim}_d(E,F):=L^{\sim}_a(E,F)^{\perp}$. The bands $(E^{\wedge\sim})_a$ and $(E^{\wedge\sim})_d$ are introduced similarly. The elements of $L^{\sim}_d(E,F)$ are usually referred to as *F-diffuse* operators. The $\mathbb{R}$-discrete or $\mathbb{R}$-diffuse operators are called for the sake of brevity *discrete* or *diffuse* functionals.

Applying the Escher Rules

Consider $S,T\in L^{\sim}(E,F)$ and put $\tau:=T{\uparrow}$; $\sigma:=S{\uparrow}$. The following are true:

(1) $T\ge 0\ \leftrightarrow\ [\![\,\tau\ge 0\,]\!]=\mathbb{1}$;

(2) S is a fragment of $T\ \leftrightarrow\ [\![\,\sigma$ is a fragment of $\tau\,]\!]=\mathbb{1}$;

(3) T is F-discrete$\leftrightarrow [\![\,\tau$ is discrete$\,]\!]=\mathbb{1}$;

(4) $T\in L^{\sim}_a(E,F)\ \leftrightarrow\ [\![\,\tau\in(E^{\wedge\sim})_a\,]\!]=\mathbb{1}$;

(5) $T\in L^{\sim}_d(E,F)\ \leftrightarrow\ [\![\,\tau\in(E^{\wedge\sim})_d\,]\!]=\mathbb{1}$.

(6) T is a lattice homomorphism $\leftrightarrow\ [\![\,\tau$ is a lattice homomorphism$\,]\!]=\mathbb{1}$.

Generating Sets of Projections

Let E stand for a vector lattice and F, for a K-space. A set $\mathscr{P}$ of band projections in $L^{\sim}(E,F)$ *generates the fragments of* T, $0\le T\in L^{\sim}(E,F)$, provided that $Tx^{+}=\sup\{pTx\mid p\in\mathscr{P}\}$ for all $x\in E$. If this happens for all $0\le T\in L^{\sim}(E,F)$, then $\mathscr{P}$ is a *generating set*.

Put $F:=\mathscr{R}{\downarrow}$ and let p be a band projection in $L^{\sim}(E,F)$. Then there is a unique element $p{\uparrow}\in\mathbb{V}^{(\mathbb{B})}$ such that $[\![\,p{\uparrow}$ is a band projection in $E^{\wedge\sim}\,]\!]=\mathbb{1}$ and $(pT){\uparrow}=p{\uparrow}\,T{\uparrow}$ for all $T\in L^{\sim}(E,F)$.

Scalarizing Fragments

Consider some set $\mathscr{P}$ of band projections in $L^{\sim}(E,F)$ and a positive operator $T \in L^{\sim}(E,F)$. Put $\tau := T{\uparrow}$ and $\mathscr{P}{\uparrow} := \{p{\uparrow} \mid p \in \mathscr{P}\}{\uparrow}$. Then $[\![\, \mathscr{P}{\uparrow}$ is a set of band projections in $E^{\wedge\sim} \,]\!] = \mathbb{1}$ and the following are true:

(1) $\mathscr{P}$ generates the fragments of $T \leftrightarrow$
$[\![\, \mathscr{P}{\uparrow}$ generates the fragments of $\tau \,]\!] = \mathbb{1}$;

(2) $\mathscr{P}$ is a generating set $\leftrightarrow$ $[\![\, \mathscr{P}{\uparrow}$ is a generating set $]\!] = \mathbb{1}$.

Up-Down

Given a set A in a K-space, denote by $A^{\vee}$ the result of adjoining to A suprema of every nonempty finite subset of A. Let $A^{\uparrow}$ stand for the result of adjoining to A suprema of nonempty increasing nets of elements of A. The symbols $A^{\uparrow\downarrow}$ and $A^{\uparrow\downarrow\uparrow}$ are understood naturally.[7]

Put $\mathscr{P}(f) := \{pf \mid p \in \mathscr{P}\}$ and note that E will for a time being stand for a vector lattice over a dense subfield of $\mathbb{R}$ while $\mathscr{P}$ is a set of band projections in $E^{\sim}$. Let $\mathfrak{E}(f)$ be the set of all fragments of f.

Generating Scalar Fragments

Theorem. *The following are equivalent:*

(1) $\mathscr{P}(f)^{\vee(\uparrow\downarrow\uparrow)} = \mathfrak{E}(f)$;

(2) *$\mathscr{P}$ generates the fragments of f;*

(3) $(\forall x \in {}^{\circ}E)(\exists p \in \mathscr{P})pf(x) \approx f(x^{+})$;

(4) *a functional g in $[0,f]$ is a fragment of f if and only if*

$$\inf_{p\in\mathscr{P}} (p^{\perp} g(x) + p(f-g)(x)) = 0$$

for every $0 \le x \in E$;

(5) $(\forall g \in {}^{\circ}\mathfrak{E}(f))(\forall x \in {}^{\circ}E_{+})(\exists p \in \mathscr{P})|pf - g|(x) \approx 0$;

(6) $\inf\{|pf-g|(x) \mid p \in \mathscr{P}\} = 0$ *for all fragments $g \in \mathfrak{E}(f)$ and $x \ge 0$;*

(7) *for $x \in E_{+}$ and $g \in \mathscr{E}(f)$ there is an element $p \in \mathscr{P}(f)^{\vee(\uparrow\downarrow\uparrow)}$, satisfying*

$$|pf - g|(x) = 0.$$

Proof. The implications (1) → (2) → (3) are obvious.

(3) → (4): We will work within the *standard entourage*; i.e., we presume that all free variables are standard. Note first that validity of the sought equality for all functionals g and f satisfying $0 \le g \le f$ amounts to the existence of $p \in \mathscr{P}$, given a standard $x \ge 0$, such that $p^{\perp} g(x) \approx 0$ and $p(f-g)(x) \approx 0$. As usual, $p^{\perp}$ is the *complementary band projection* to p. Thus, ${}^{\circ}p(g \wedge (f-g))(x) \le {}^{\circ}p(f-g)(x) = 0$ and ${}^{\circ}p^{\perp}((f-g)\wedge g)(x) \le {}^{\circ}p^{\perp} g(x) = 0$, i.e. $g \wedge (f-g) = 0$.

Prove now that, on assuming (3), the sought equality ensues from the conventional criterion for disjointness:

$$\inf\{g(x_1) + (f-g)(x_2) \mid x_1 \ge 0,\ x_2 \ge 0,\ x_1 + x_2 = x\} = 0.$$

Given a standard x, find internal positive x_1 and x_2 such that $x = x_1 + x_2$ and, moreover, $g(x_1) \approx 0$ and $f(x_2) \approx g(x_2)$. By (3), it follows from the Kreĭn–Milman Theorem that the

[7] Cp. [8]–[10].

fragment g belongs to the weak closure of $\mathscr{P}(f)$. In particular, there is an element $p \in \mathscr{P}$ satisfying $g(x_1) \approx pf(x_1)$ and $g(x_2) \approx pf(x_2)$. Thus, $p^\perp g(x_2) \approx 0$, because $p^\perp g \leq p^\perp f$. Finally, $p^\perp g(x) \approx 0$. Hence,

$$p(f-g)(x) = pf(x_2) + pf(x_1) - pg(x)$$
$$\approx g(x_2) + g(x_1) - pg(x) \approx p^\perp g(x) \approx 0.$$

This yields the claim.

(4) $\to$ (5): Using the equality $|pf-g|(x) = p^\perp g(x) + p(f-g)(x)$, we may find $p \in \mathscr{P}$ so that $p^\perp g(x) \approx 0$ and $p(f-g)(x) \approx 0$. This justifies the claim.

The equivalence (5) $\leftrightarrow$ (6) is clear.

The implications (5) $\to$ (7) $\to$ (1) are standard. The proof is complete.

Principal Bands in the Scalar Case

For positive functionals f and g and for a generating set of band projections $\mathscr{P}$, the following are equivalent:

(1) $g \in \{f\}^{\perp\perp}$;

(2) If x is a *limited* element of E, i.e. $x \in {}^{\mathrm{fin}}E := \{x \in E \mid (\exists \overline{x} \in {}^\circ E)|x| \leq \overline{x}\}$, then $pg(x) \approx 0$ whenever $pf(x) \approx 0$ for $p \in \mathscr{P}$;

(3) $(\forall x \in E_+)(\forall \varepsilon > 0)(\exists \delta > 0)(\forall p \in \mathscr{P}) pf(x) \leq \delta \to pg(x) \leq \varepsilon$.

Principal Projections in the Scalar Case

Let f and g be positive functionals on E, and let x be a positive element of E. The following representations of the band projection π_f onto the band $\{f\}^{\perp\perp}$ are valid:

(1) $\pi_f g(x) \risingdotseq \inf{}^*\{{}^\circ pg(x) \mid p^\perp f(x) \approx 0,\, p \in \mathscr{P}\}$ (the symbol $\risingdotseq$ means that the formula is *exact*, i.e., the equality is attained);

(2) $\pi_f g(x) = \sup_{\varepsilon>0} \inf\{pg(x) \mid p^\perp f(x) \leq \varepsilon,\, p \in \mathscr{P}\}$;

(3) $\pi_f g(x) \risingdotseq \inf{}^*\{{}^\circ g(y) \mid f(x-y) \approx 0,\, 0 \leq y \leq x\}$;

(4) $(\forall \varepsilon > 0)\,(\exists \delta > 0)\,(\forall p \in \mathscr{P})\, pf(x) < \delta \to \pi_f g(x) \leq p^\perp g(x) + \varepsilon$;
$(\forall \varepsilon > 0)\,(\forall \delta > 0)\,(\exists p \in \mathscr{P})\, pf(x) < \delta \wedge p^\perp g(x) \leq \pi_f g(x) + \varepsilon$;

(5) $(\forall \varepsilon > 0)\,(\exists \delta > 0)\,(\forall 0 \leq y \leq x)\, f(x-y) \leq \delta \to \pi_f g(x) \leq g(y) + \varepsilon$;
$(\forall \varepsilon > 0)\,(\forall \delta > 0)\,(\exists 0 \leq y \leq x)\, f(x-y) \leq \delta \wedge g(y) \leq \pi_f g(x) + \varepsilon$.

Up-Down for Operators

For a set of band projections $\mathscr{P}$ in $L^\sim(E,F)$ and $0 \leq S \in L^\sim(E,F)$ the following are equivalent:

(1) $\mathscr{P}(S)^{\vee(\uparrow\downarrow\uparrow)} = \mathfrak{E}(S)$;

(2) $\mathscr{P}$ generates the fragments of S;

(3) an operator $T \in [0,S]$ is a fragment of S if and only if

$$\inf_{p \in \mathscr{P}} (p^\perp Tx + p(S-T)x) = 0$$

for all $0 \leq x \in E$;

(4) $(\forall x \in {}^\circ E)\,(\exists p \in \mathscr{P}{\uparrow\downarrow})\, pSx \approx Sx^+$.

Principal Bands in the Operator Case

For positive operators S and T and a generating set $\mathscr{P}$ of band projections in $L^{\sim}(E,F)$, the following are equivalent:

(1) $T \in \{S\}^{\perp\perp}$;

(2) $(\forall x \in {}^{\mathrm{fin}}E)\,(\forall p \in \mathscr{P})\,(\forall \pi \in \mathbb{B})\,\pi p S x \approx 0 \to \pi p T x \approx 0$;

(3) $(\forall x \in {}^{\mathrm{fin}}E)\,(\forall \pi \in \mathbb{B})\,\pi S x \approx 0 \to \pi T x \approx 0$;

(4) $(\forall x \geq 0)\,(\forall \varepsilon \in \mathscr{E})\,(\exists \delta \in \mathscr{E})\,(\forall p \in \mathscr{P})\,(\forall \pi \in \mathbb{B})\,\pi p S x \leq \delta \to \pi p T x \leq \varepsilon$;

(5) $(\forall x \geq 0)\,(\forall \varepsilon \in \mathscr{E})\,(\exists \delta \in \mathscr{E})\,(\forall \pi \in \mathbb{B})\pi S x \leq \delta \to \pi T x \leq \varepsilon$.

Principal Projections in the Operator Case

Let E be a vector lattice, and let F be a K-space having the filter of order units $\mathscr{E}$ and the base $\mathbb{B}$. Suppose that S and T are positive operators in $L^{\sim}(E,F)$ and R is the band projection of T to the band $\{S\}^{\perp\perp}$. For a positive $x \in E$, the following are valid:

(1) $Rx = \sup_{\varepsilon\in\mathscr{E}} \inf\{\pi T y + \pi^{\perp} S x \mid 0 \leq y \leq x, \pi \in \mathbb{B},\ \pi S(x-y) \leq \varepsilon\}$;

(2) $Rx = \sup_{\varepsilon\in\mathscr{E}} \inf\{(\pi p)^{\perp} T x \mid \pi p S x \leq \varepsilon,\ p \in \mathscr{P}, \pi \in \mathbb{B}\}$,

where $\mathscr{P}$ is a generating set of band projections in F.

The Polyhedral Lagrange Principle

Turn to the revisited Farkas Lemma.[8] Let X be a Y-seminormed real vector space, with Y a K-space. Given are some dominated polyhedral sublinear operators $P_1,\dots,P_N$ from X to Y and a dominated sublinear operator $P : X \to Y$.

The finite value of the constrained problem

$$P_1(x) \leq u_1,\dots,P_N(x) \leq u_N, \quad P(x) \to \inf$$

is the value of the unconstrained problem for an appropriate Lagrangian without any constraint qualification but polyhedrality.

Interval Operator Inequalities

Polyhedrality is omnipresent and so finds applications in inexact data processing.[9] Let X be a Y-seminormed real space, with Y a K-space. Assume given a dominated polyhedral sublinear operator $P : X \to Y$, a dominated sublinear operator $Q : X \to Y$, and $u, v \in Y$. Assume further that $\{P \leq u\} \neq \varnothing$.

The following are equivalent:

(1) *for all $b \in \mathbb{B}$, with $\mathbb{B}$ the base of Y, the sublinear operator inequality $bQ\circ \sim (x) \geq -bv$ is a consequence of the polyhedral sublinear operator inequality $bP(x) \leq bu$, i.e., $\{bP \leq bu\} \subset \{bQ\circ \sim \geq -bv\}$, with $\sim (x) := -x$ for all $x \in X$;*

(2) *there are $A \in \partial(P)$, $B \in \partial(Q)$, and a positive orthomorphism $\alpha \in \mathrm{Orth}(m(Y))$ on the universal completion $m(Y)$ of Y satisfying $B = \alpha A$, $\alpha u \leq v$.*

[8] Cp. [11]–[13].

[9] Cp. [14].

References

[1] Kusraev A. G. and Kutateladze S. S., *Introduction to Boolean Valued Analysis.* Moscow: Nauka, 2005.

[2] Kusraev A. G. and Kutateladze S. S., "Boolean methods in positivity," J. Appl. Indust. Math., **2**:1, 81–99 (2008).

[3] Leibniz G. W., "Monadology," In: Collected Works, Vol. 1 [in Russian], Mysl′, Moscow, 1982, pp. 143–429.

[4] Kutateladze S. S., "The mathematical background of Lomonosov's contribution," J. Appl. Indust. Math., **5**:2, 155–162 (2011).

[5] Luxemburg W. A. J., "A general theory of monads," In: *Applications of Model Theory to Algebra, Analysis and Probability.* Holt, Rinehart and Minston, New York, 1966, pp. 18–86.

[6] Dauben J. W., *The Creation of Nonstandard Analysis. A Personal and Mathematical Odyssey.* Princeton: Princeton University Press, 1995.

[7] Gutman A. E. and Losenkov G. A., "Functional representation of a Boolean valued universe," In: *Nonstandard Analysis and Vector Lattices* (Ed. S. S. Kutateladze), Dordrecht, Kluwer Academic Publishers, 2000, pp. 81–104.

[8] Pagter B. de, "The components of a positive operator," Indag. Math., **45**, No. 2, 229–241 (1983).

[9] Kusraev A. G. and Kutateladze S. S., "Nonstandard methods and Kantorovich spaces," In: *Nonstandard Analysis and Vector Lattices* (Ed. S. S. Kutateladze), Dordrecht, Kluwer Academic Publishers, 2000, pp. 1–79.

[10] Kusraev A. G. and Kutateladze S. S., "On the calculus of order bounded operators," Positivity, **9**:3, 327–339 (2005).

[11] Kutateladze S. S., "The Farkas lemma revisited," Siberian Math. J., **51**:1, 78–87 (2010).

[12] Kutateladze S. S., "Boolean trends in linear inequalities," J. Appl. Indust. Math., **4**:3, 340–348 (2010).

[13] Kutateladze S. S., "The polyhedral Lagrange principle," Siberian Math. J., **52**:3, 484–486 (2011).

[14] Fiedler M. et al., *Linear Optimization Problems with Inexact Data.* New York: Springer, 2006.

Appendix

Leibniz's Monadology

The outlook of Leibniz, proliferating with his works, occupies a unique place in human culture. We can hardly find in the philosophical treatises of his predecessors and later thinkers something comparable with the phantasmagoric conceptions of monads, the special and stunning constructs of the world and mind which precede, comprise, and incorporate all infinite advents of the eternity.

We must always bear in mind that the seventeenth century was enchanted by the microscope. It was already in the 1610s that microscopes were mass-produced in many European countries. From the 1660s Europe enjoyed Antony van Leeuwenhoek's microscope. *Gulliver's Travels* by Jonathan Swift, published in 1726, exhibits most vivid examples of the depth of penetration of the ideology of interplay between large and small quantities into the cultural media of the age of the Enlightenment.

Impressed by the new discoveries, Leibniz wrote (cp. [1, p. 42]):

> Whence we see that there is a world of created things, of living beings, of animals, of entelechies, of souls, in the minutest particle of matter.... Every portion of matter may be conceived as like a garden full of plants and like a pond full of fish. But every branch of a plant, every member of an animal, and every drop of the fluids within it, is also such a garden or such a pond.

It is worth emphasizing that mathematics was the true source of the philosophical ideas of Leibniz who believed that "science is necessary for true happiness." Note the observation of Child who translated into English and commented the early mathematical papers of Leibniz (cp. [2, Preface]):

> The main ideas of his philosophy are to be attributed to his mathematical work, and not *vice versa.*

The *Monadology* (or *La Monadologie*) [3, pp. 413–428] is usually dated as of 1714. This article was never published during Leibniz's life. Moreover, it is generally accepted that the very term "monad" had appeared in his writings since 1690 when he was already an established and prominent scholar.

The special attention to the origin of the term "monad" and the particular investigation into the date of its first appearance in the works by Leibniz are in fact the present-day products. There are now a few if any cultivated persons who never got acquaintance with the basics of planimetry and heard nothing of Euclid. But nobody has ever met the concept of "monad" on the school bench. Neither the contemporary translations of Euclid's *Elements* nor the popular school text-books contain this seemingly exotic term. The concept of "monad" is, however, fundamental not only for Euclidean geometry but also for the whole science of the Ancient Hellas.

By Definition I of Book VII of Euclid's *Elements* [4], a monad is "that by virtue of which each of the things that exist is called one." Euclid proceeds with Definition II: "A number is a multitude composed of monads." Note that the present-day translations of the Euclid treatise substitute "unit" for "monad."

A contemporary reader can hardly understand why Sextus Empiricus, an outstanding scepticist of the second century, wrote when presenting the mathematical views of his predecessors as follows [5]: "Pythagoras said that the origin of the things that exist is a monad by virtue of which each of the things that exist is called one." And furthermore: "A point is structured as a monad; indeed, a monad is a certain origin of numbers and likewise a point is a certain origin of lines." Now some place is in order for the excerpt which can easily be misconceived as a citation from the *Monadology*: "A whole as such is indivisible and a monad, since it is a monad, is not divisible. Or, if it splits into many pieces it becomes a union of many monads rather than a [simple] monad."

It is worth observing that the ancients sharply perceived an exceptional status of the start of counting. In order to count, one should firstly particularize the entities to count and only then to proceed with putting these entities into correspondence with some symbolic series of numerals. We begin counting with making "each of the things one." The especial role of the start of counting is reflected in the almost millennium-long dispute about whether or not the unit (read, monad) is a natural number. We feel today that it is excessive to distinguish the key role of the unit or monad which signifies the start of counting. But this was not always so.

From the times of Euclid, all serious scientists knew about existence of the two basic concepts of mathematics: a point and a monad. By Definition I of Book I of Euclid's *Elements*: "A point is that which has no parts." Clearly this definition differs drastically from the definition of monad as that which makes one from many. The cornerstone of geometry is other than that of arithmetic. Without clear understanding of this circumstance it is impossible to comprehend the essence of the views of Leibniz. By the way, the modern set theory refers to "that which has no parts" as the empty set, the starting cardinal of the von Neumann universe. The present-day mathematics seems to have no concept that is vocalized as "that which many makes into one." We will return to the modern mathematical definition of monad shortly.

As a top mathematician of his epoch, Leibniz was in full command of Euclidean geometry. Therefore, rather bewildering is Item 1 of the *Monadology*, where Leibniz gave the first idea of what his monad actually is:

> The Monad, of which we shall here speak, is nothing but a simple substance, which enters into compounds. By "simple" is meant "without parts."

This definition of monad as a "simple" substance without parts coincides with the Euclidean definition of point. At the same time the reference to the compounds consisting of monads reminds us the structure of the definition of number which belongs to Euclid.

The synthesis of both primary definitions of Euclid in the Leibnizian monad is not accidental. As a mathematician by belief, from his earliest childhood, Leibniz dreamed of "some sort of calculus" that operates in the "alphabet of human thoughts" and possesses the same beauty, strength, and integrity as mathematics in solving arithmetical and geometrical problems. Leibniz devoted many articles to invention of this universal logical calculus. He remarked that his general methodological views are grounded on the "studying of the ways of analysis in mathematics to which I was subjected with such an ardency that I do not know whether there are many to be found today who invested much more toil into it than me."

References

[1] Mates B., *The Philosophy of Leibniz: Metaphysics and Language*. New York and Oxford: Oxford Univ. Press (1986).

[2] Child J. M., *The Early Mathematical Manuscripts of Leibniz* Chicago: The Open Court Publishing Company, 1920; Mineola: Dover Publ., 2005.

[3] Leibniz G. W., *Collected Works*, Vol. 1. Moscow: Mysl′, 1982.

[4] Euclid, *Elements*, In Three Volumes Moscow and Leningrad: Gostekhizdat, 1949.

[5] Sextus Empiricus, *Collected Works*, Vol. 1. Moscow: Mysl′, 1976.

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