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CONVEXITY AND INEQUALITY

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Appendix: Nomination and Definition

Кутateладзе С. С.

Выпуклость и неравенство

Доклад на Международной конференции «Современные проблемы математики, информатики и биоинформатики», посвященной 100-летию со дня рождения члена-корреспондента АН СССР Алексея Андреевича Ляпунова, 11–14 октября 2011 г., Академгородок, Новосибирск.

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Приложение: Номинация и дефиниция

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CONVEXITY AND INEQUALITY

Linear inequality implies linearity and order. When combined, the two produce an ordered vector space. Each linear inequality in the simplest environment of the sort is some half-space. Simultaneity implies many instances and so leads to the intersections of half-spaces. These yield polyhedra as well as arbitrary convex sets, identifying the theory of linear inequalities with convexity.

Convexity reigns in the federation of geometry, optimization, and functional analysis. Convexity feeds generation, separation, calculus, and approximation. Generation appears as duality; separation, as optimality; calculus, as representation; and approximation, as stability [1].

This talk addresses the origin and the state of the art of the relevant areas with a particular emphasis on the Farkas Lemma [2]. Our aim is to demonstrate how Boolean valued analysis may be applied to simultaneous linear inequalities with operators.

Through Ages

Mathematics resembles linguistics sometimes and pays tribute to etymology, hence, history. Today's convexity is a centenarian, and abstract convexity is much younger.

Convexity traces back to the idea of a solid figure in plane geometry. Book I of Euclid's *Elements* [3] reads:

Definition 13. A boundary is that which is an extremity of anything.

Definition 14. A figure is that which is contained by any boundary or boundaries.

Narrating solid geometry in Book XI, Euclid traveled from solid to surface:

Definition 1. A solid is that which has length, breadth, and depth.

Definition 2. An extremity of a solid is a surface.

Definition 9. Similar solid figures *are those contained by similar planes equal in multitude.*

Definition 10. Equal and similar solid figures *are those contained by similar planes equal in multitude and magnitude.*

Convexity and inequality stem from the remote ages [5]–[7]. But as the acclaimed pioneers who propounded these ideas and anticipated their significance for the future, we must rank the three polymaths:

Joseph-Louis Lagrange (January 25, 1736–April 10, 1813)
Jean-Baptiste Joseph Fourier (March 21, 1768–May 16, 1830)
Hermann Minkowski (June 22, 1864–January 12, 1909)

Joseph Lagrange (1736–1813)

In both research and exposition, he totally reversed the methods of his predecessors. They had proceeded in their exposition from special cases by a species of induction; his eye was always directed to the highest and most general points of view; and it was by his suppression of details and neglect of minor, unimportant considerations that he swept the whole field of analysis with a generality of insight and power never excelled, adding to his originality and profundity a conciseness, elegance, and lucidity which have made him the model of mathematical writers. (Thomas J. McCormack [8])

Joseph Fourier (1768–1830)

He [Fourier] himself was neglected for his work on inequalities, what he called "Analyse indéterminée." Darboux considered that he gave the subject an exaggerated importance and did not publish the papers on this question in his edition of the scientific works of Fourier. Had they been published, linear programming and convex analysis would be included in the heritage of Fourier. (Jean-Pierre Kahane [9])

Hermann Minkowski (1864–1909)

Since my student years Minkowski was my best, most dependable friend who supported me with all the depth and loyalty that was so characteristic of him. Our science, which we loved above all else, brought us together; it seemed to us a garden full of flowers:.. In it, we enjoyed looking for hidden pathways and discovered many a newperspective that appealed to our sense of beauty and when one of us showed it to the other and we marvelled over it together, our joy was complete. He was for me a rare gift from heaven.... and I must be grateful to have possessed that gift for so long. Now death has suddenly torn him from our midst. However, what death cannot take away is his noble image in our hearts and the knowledge that his spirit in us continue to be active. (David Hilbert [10])

Convexity as Abstraction

Stretching a rope taut between two stakes produces a closed straight line segment, the continuum in modern parlance. Rope-stretching raised the problem of measuring the continuum. The continuum hypothesis of set theory is the shadow of the ancient problem of harpedonaptae. Rope-stretching independent of the position of stakes is uniform with respect to direction in space. The mental experiment of uniform rope-stretching yields a compact convex figure.

Convexity has found solid grounds in set theory. The Cantor paradise became an official residence of convexity. Abstraction becomes an axiom of set theory. The abstraction axiom enables us to reincarnate a property, in other words, to collect and to comprehend. The union of convexity and abstraction was inevitable. This yields abstract convexity [11]–[13].

Environment for Convexity

Let $\overline{E}$ be a vector lattice E with the adjoint top $\top := +\infty$ and bottom $\bot := -\infty$. Assume further that H is some subset of E that is by implication a (convex) cone in E, and so the bottom of E lies beyond H. A subset U of H is *convex relative to* H or H*-convex* provided that U is the H*-support set*

$$U_p^H := \{h \in H : h \le p\}$$

of some element p of $\overline{E}$. Limiting finite subsets of H-convex sets yields analogs of polyhedra.

An element $p \in \overline{E}$ is *H-convex* provided that $p = \sup U_p^H$; i.e., p represents the supremum of the H-support set of p. The proper H-convex elements fill the cone $\mathscr{C}(H, \overline{E})$. The *Minkowski duality* $\varphi : p \mapsto U_p^H$ enables us to study convex elements and sets simultaneously.

Lyapunov's Convexity Theorem

The celebrated Lyapunov Convexity Theorem had raised the problem of describing the compact convex sets in finite-dimensional real spaces which serve as the ranges of diffuse measures. These compacta are known in the modern geometrical literature as *zonoids*. Among zonoids we distinguish the Minkowski sums of finitely many straight line segments. These sets, called *zonotopes*, fill a convex cone in the space of compact convex sets, and the cone of zonotopes is dense in the closed cone of all zonoids. The description of the ranges of diffuse vector measures in the Lyapunov Convexity Theorem was firstly found by Chuĭkina practically in the modern terms (see [14]). Soon after that her result was somewhat supplemented and simplified by Glivenko in [15]. The zonotopes of the present epoch were called *parallelohedra* those days.

Zonoids

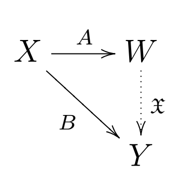
The significant further progress in studying the ranges of diffuse vector measures belong to Reshetnyak and Zalgaller who described zonoids as the results of mixing the linear elements of a rectifiable curve in a finite-dimensional space in 1954 (see [16]). In this same paper they suggested a new prove of the Lyapunov Convexity Theorem and demonstrated that zonotopes are precisely those convex polyhedra whose two-dimensional faces have centers of symmetry. Unfortunately, these results remained practically unnoticed in the West. Analogous results were obtained by Bolker only fifteen years later in 1969 (see [17], [18]).

Environment for Inequality

Assume that X is a real vector space, Y is a *Kantorovich space* also known as a complete vector lattice or a Dedekind complete Riesz space. Let $\mathbb{B} := \mathbb{B}(Y)$ be the *base* of Y, i.e., the complete Boolean algebras of positive projections in Y; and let $m(Y)$ be the universal completion of Y. Denote by $L(X, Y)$ the space of linear operators from X to Y. In case X is furnished with some Y-seminorm on X, by $L^{(m)}(X, Y)$ we mean the *space of dominated operators* from X to Y. As usual, $\{T \le 0\} := \{x \in X \mid Tx \le 0\}$; $\ker(T) = T^{-1}(0)$ for $T : X \to Y$. Also, $P \in \mathrm{Sub}(X, Y)$ means that P is *sublinear*, while $P \in \mathrm{PSub}(X, Y)$ means that P is *polyhedral*, i.e., finitely generated. The superscript $^{(m)}$ suggests domination.

Kantorovich's Theorem

Find $\mathfrak{X}$ satisfying



(1): $(\exists\mathfrak{X})\ \mathfrak{X}A = B \leftrightarrow \ker(A) \subset \ker(B)$.
(2): *If W is ordered by W_+ and $A(X) - W_+ = W_+ - A(X) = W$, then*[1]

$$(\exists\mathfrak{X} \geq 0)\ \mathfrak{X}A = B \leftrightarrow \{A \leq 0\} \subset \{B \leq 0\}.$$

The Farkas Alternative

Let X be a Y-seminormed real vector space, with Y a Kantorovich space. Assume that $A_1, \dots, A_N$ and B belong to $L^{(m)}(X, Y)$.

Then one and only one of the following holds:

(1) *There are $x \in X$ and $b, b' \in \mathbb{B}$ such that $b' \leq b$ and*

$$b'Bx > 0, bA_1x \leq 0, \dots, bA_Nx \leq 0.$$

(2) *There are positive orthomorphisms $\alpha_1, \dots, \alpha_N \in \mathrm{Orth}(m(Y))_+$ such that $B = \sum_{k=1}^{N} \alpha_k A_k$.*

Hidden Dominance

Lemma 1. *Let X be a vector space over some subfield R of the reals $\mathbb{R}$. Assume that f and g are R-linear functionals on X; in symbols, $f, g \in X^{\#} := L(X, \mathbb{R})$.*

For the inclusion

$$\{g \leq 0\} \supset \{f \leq 0\}$$

to hold it is necessary and sufficient that there be $\alpha \in \mathbb{R}_+$ satisfying $g = \alpha f$.

Proof. Sufficiency is obvious.

Necessity: The case of $f = 0$ is trivial. If $f \neq 0$ then there is some $x \in X$ such that $f(x) \in \mathbb{R}$ and $f(x) > 0$. Denote the image $f(X)$ of X under f by R_0. Put $h := g \circ f^{-1}$, i.e. $h \in R_0^{\#}$ is the only solution for $h \circ f = g$. By hypothesis, h is a positive R-linear functional on R_0. By the Bigard Theorem [24, p. 108] h can be extended to a positive homomorphism $\bar{h} : \mathbb{R} \to \mathbb{R}$, since $R_0 - \mathbb{R}_+ = \mathbb{R}_+ - R_0 = \mathbb{R}$. Each positive automorphism of $\mathbb{R}$ is multiplication by a positive real. As the sought α we may take $\bar{h}(1)$.

The proof of Lemma 1 is complete.

Explicit Dominance

Lemma 2. *Let X be an $\mathbb{R}$-seminormed vector space over some subfield R of $\mathbb{R}$. Assume that $f_1, \dots, f_N$ and g are bounded R-linear functionals on X; in symbols, $f_1, \dots, f_N, g \in X^* := L^{(m)}(X, \mathbb{R})$.*

For the inclusion

$$\{g \leq 0\} \supset \bigcap_{k=1}^{N} \{f_k \leq 0\}$$

to hold it is necessary and sufficient that there be $\alpha_1, \dots, \alpha_N \in \mathbb{R}_+$ satisfying

$$g = \sum_{k=1}^{N} \alpha_k f_k.$$

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[1] Cp. [24, p. 51].

Boolean Modeling

Cohen's final solution of the problem of the cardinality of the continuum within ZFC gave rise to the Boolean valued models by Scott, Solovay, and Vopěnka.[2]

Takeuti coined the term "Boolean valued analysis" for applications of the models to analysis.[3]

Scott forecasted in 1969:[4]

We must ask whether there is any interest in these nonstandard models aside from the independence proof; that is, do they have any mathematical interest? The answer must be yes, but we cannot yet give a really good argument.

In 2009 Scott wrote:[5]

At the time, I was disappointed that no one took up my suggestion. And then I was very surprised much later to see the work of Takeuti and his associates. I think the point is that people have to be trained in Functional Analysis in order to understand these models. I think this is also obvious from your book and its references. Alas, I had no students or collaborators with this kind of background, and so I was not able to generate any progress.

Let $\mathbb{B}$ be a complete Boolean algebra. Given an ordinal α, put

$$V_\alpha^{(\mathbb{B})} := \{x \mid (\exists \beta \in \alpha)\ x : \operatorname{dom}(x) \to \mathbb{B}\ \&\ \operatorname{dom}(x) \subset V_\beta^{(\mathbb{B})}\}.$$

The *Boolean valued universe* $\mathbb{V}^{(\mathbb{B})}$ is

$$\mathbb{V}^{(\mathbb{B})} := \bigcup_{\alpha \in \mathrm{On}} V_\alpha^{(\mathbb{B})},$$

with On the class of all ordinals.

The truth value $[\![\varphi]\!] \in \mathbb{B}$ is assigned to each formula φ of ZFC relativized to $\mathbb{V}^{(\mathbb{B})}$.

Descending and Ascending

Given φ, a formula of ZFC, and y, a member of $\mathbb{V}^{\mathbb{B}}$; put $A_\varphi := A_{\varphi(\cdot,\ y)} := \{x \mid \varphi(x,\ y)\}$.

The *descent* $A_\varphi{\downarrow}$ of a class A_φ is

$$A_\varphi{\downarrow} := \{t \mid t \in \mathbb{V}^{(\mathbb{B})}\ \&\ [\![\varphi(t,\ y)]\!] = \mathbb{1}\}.$$

If $t \in A_\varphi{\downarrow}$, then it is said that t *satisfies* $\varphi(\cdot,\ y)$ *inside* $\mathbb{V}^{(\mathbb{B})}$.

The *descent* $x{\downarrow}$ of $x \in \mathbb{V}^{(\mathbb{B})}$ is defined as

$$x{\downarrow} := \{t \mid t \in \mathbb{V}^{(\mathbb{B})}\ \&\ [\![t \in x]\!] = \mathbb{1}\},$$

i.e. $x{\downarrow} = A_{\cdot \in x}{\downarrow}$. The class $x{\downarrow}$ is a set.

If x is a nonempty set inside $\mathbb{V}^{(\mathbb{B})}$ then

$$(\exists z \in x{\downarrow})[\![(\exists t \in x)\ \varphi(t)]\!] = [\![\varphi(z)]\!].$$

The *ascent* functor acts in the opposite direction.

There is an object $\mathscr{R}$ inside $\mathbb{V}^{(\mathbb{B})}$ modeling $\mathbb{R}$, i.e.,

$$[\![\mathscr{R} \text{ is the reals}]\!] = \mathbb{1}.$$

[2]Cp. [19].
[3]Cp. [20].
[4]Cp. [21].
[5]Letter of April 29, 2009 to S. S. Kutateladze.

Let $\mathscr{R}{\downarrow}$ be the descent of the carrier $|\mathscr{R}|$ of the algebraic system $\mathscr{R} := (|\mathscr{R}|, +, \cdot, 0, 1, \leq)$ inside $\mathbb{V}^{(\mathbb{B})}$.

Implement the descent of the structures on $|\mathscr{R}|$ to $\mathscr{R}{\downarrow}$ as follows:

$$x + y = z \leftrightarrow [\![x + y = z]\!] = \mathbb{1};$$
$$xy = z \leftrightarrow [\![xy = z]\!] = \mathbb{1};$$
$$x \leq y \leftrightarrow [\![x \leq y]\!] = \mathbb{1};$$
$$\lambda x = y \leftrightarrow [\![\lambda^{\wedge} x = y]\!] = \mathbb{1}\ (x, y, z \in \mathscr{R}{\downarrow},\ \lambda \in \mathbb{R}).$$

Gordon Theorem.[6] *$\mathscr{R}{\downarrow}$ with the descended structures is a universally complete vector lattice with base $\mathbb{B}(\mathscr{R}{\downarrow})$ isomorphic to $\mathbb{B}$.*

We will proceed by Boolean valued analysis:

Theorem 1. *Assume that $A_1, \dots, A_N$ and B belong to $L^{(m)}(X, Y)$.*

The following are equivalent:

(1) *Given $b \in \mathbb{B}$, the operator inequality $bBx \leq 0$ is a consequence of the simultaneous linear operator inequalities $bA_1x \leq 0, \dots, bA_Nx \leq 0$, i.e.,*

$$\{bB \leq 0\} \supset \{bA_1 \leq 0\} \cap \dots \cap \{bA_N \leq 0\}.$$

(2) *There are positive orthomorphisms $\alpha_1, \dots, \alpha_N \in \operatorname{Orth}(m(Y))$ such that*

$$B = \sum_{k=1}^{N} \alpha_k A_k;$$

i.e., B lies in the operator convex conic hull of $A_1, \dots, A_N$.

Proof. (2) $\to$ (1): If $B = \sum_{k=1}^{N} \alpha_k A_k$ for some positive $\alpha_1, \dots, \alpha_N$ in $\operatorname{Orth}(m(Y))$ while $bA_kx \leq 0$ for $b \in \mathbb{B}$ and $x \in X$, then

$$bBx = b\sum_{k=1}^{N} \alpha_k A_k x = \sum_{k=1}^{N} \alpha_k bA_k x \leq 0$$

since orthomorphisms commute and projections are orthomorphisms of $m(Y)$.

(1) $\to$ (2): Consider the separated Boolean valued universe $\mathbb{V}^{(\mathbb{B})}$ over the base $\mathbb{B}$ of Y. By the Gordon Theorem the ascent $Y{\uparrow}$ of Y is $\mathscr{R}$, the reals inside $\mathbb{V}^{(\mathbb{B})}$.

Using the canonical embedding, we see that $X^{\wedge}$ is an $\mathscr{R}$-seminormed vector space over the standard name $\mathbb{R}^{\wedge}$ of the reals $\mathbb{R}$.

Moreover, $\mathbb{R}^{\wedge}$ is a subfield and sublattice of $\mathscr{R} = Y{\uparrow}$ inside $\mathbb{V}^{(\mathbb{B})}$.

(1) $\to$ (2):

Put $f_k := A_k{\uparrow}$ for all $k := 1, \dots, N$ and $g := B{\uparrow}$. Clearly, all $f_1, \dots, f_N, g$ belong to $(X^{\wedge})^*$ inside $\mathbb{V}^{\mathbb{B}}$.

Define the finite sequence

$$f : \{1, \dots, N\}^{\wedge} \to (X^{\wedge})^*$$

as the ascent of $(f_1, \dots, f_N)$. In other words, the truth values are as follows:

$$[\![f_{k^{\wedge}}(x^{\wedge}) = A_k x]\!] = \mathbb{1}, \quad [\![g(x^{\wedge}) = Bx]\!] = \mathbb{1}$$

for all $x \in X$ and $k := 1, \dots, N$.

(1) $\to$ (2):

[6] Cp. [19, p. 349].

Put

$$b := [\![A_1x \leq 0^\wedge]\!] \wedge \dots \wedge [\![A_Nx \leq 0^\wedge]\!].$$

Then $bA_kx \leq 0$ for all $k := 1, \dots, N$ and $bBx \leq 0$ by (1).

Therefore,

$$[\![A_1x \leq 0^\wedge]\!] \wedge \dots \wedge [\![A_Nx \leq 0^\wedge]\!] \leq [\![Bx \leq 0^\wedge]\!].$$

In other words,

$$[\![(\forall k := 1^\wedge, \dots, N^\wedge) f_k(x^\wedge) \leq 0^\wedge]\!] = \bigwedge_{k:=1,\dots,N} [\![f_{k^\wedge}(x^\wedge) \leq 0^\wedge]\!] \leq [\![g(x^\wedge) \leq 0^\wedge]\!].$$

(1) → (2):

By Lemma 2 inside $\mathbb{V}^{(\mathbb{B})}$ and the maximum principle of Boolean valued analysis, there is a finite sequence $\alpha : \{1^\wedge, \dots, N^\wedge\} \to \mathscr{R}_+$ inside $\mathbb{V}^{(\mathbb{B})}$ satisfying

$$[\![(\forall x \in X^\wedge)\ g(x) = \sum_{k=1^\wedge}^{N^\wedge} \alpha(k) f_k(x)]\!] = \mathbb{1}.$$

Put $\alpha_k := \alpha(k^\wedge) \in \mathscr{R}_+{\downarrow}$ for $k := 1, \dots, N$.

Multiplication by an element in $\mathscr{R}{\downarrow}$ is an orthomorphism of $m(Y)$. Moreover,

$$B = \sum_{k=1}^{N} \alpha_k A_k,$$

which completes the proof.

Absence of Dominance

Lemma 1, describing the consequences of a single inequality, does not restrict the class of functionals under consideration.

The analogous version of the Farkas Lemma simply fails for two simultaneous inequalities in general.

The inclusion $\{f = 0\} \subset \{g \leq 0\}$ equivalent to the inclusion $\{f = 0\} \subset \{g = 0\}$ does not imply that f and g are proportional in the case of an arbitrary subfield of $\mathbb{R}$. It suffices to look at $\mathbb{R}$ over the rationals $\mathbb{Q}$, take some discontinuous $\mathbb{Q}$-linear functional on $\mathbb{Q}$ and the identity automorphism of $\mathbb{Q}$.

Theorem 2. *Take A and B in $L(X, Y)$. The following are equivalent:*

(1) $(\exists \alpha \in \operatorname{Orth}(m(Y)))\ B = \alpha A$*;*

(2) *There is a projection $\varkappa \in \mathbb{B}$ such that*

$$\{\varkappa bB \leq 0\} \supset \{\varkappa bA \leq 0\}; \quad \{\neg\varkappa bB \leq 0\} \supset \{\neg\varkappa bA \geq 0\}$$

for all $b \in \mathbb{B}$.[7]

Proof. Boolean valued analysis reduces the claim to the scalar case. Applying Lemma 1 twice and writing down the truth values, complete the proof.

[7]As usual, $\neg\varkappa := \mathbb{1} - \varkappa$.

Inexact Data

Let X be a vector lattice. An *interval operator* $\mathbf{T}$ from X to Y is an order interval $[\underline{T}, \overline{T}]$ in $L^{(r)}(X, Y)$, with $\underline{T} \le \overline{T}$.[8]

The interval equation $\mathbf{B} = \mathfrak{X}\mathbf{A}$ has a *weak interval solution* provided that $(\exists \mathfrak{X})(\exists A \in \mathbf{A})(\exists \mathbf{B} \in \mathbf{B})\ \mathbf{B} = \mathfrak{X}\mathbf{A}$.

Given an interval operator $\mathbf{T}$ and $x \in X$, put

$$P_{\mathbf{T}}(x) = \overline{T}x_+ - \underline{T}x_-.$$

Call $\mathbf{T}$ *adapted* in case $\overline{T} - \underline{T}$ is the sum of finitely many disjoint addends.

Put $\sim(x) := -x$ for all $x \in X$.

Theorem 3. *Let X be a vector lattice, and let Y be a Kantorovich space. Assume that $\mathbf{A}_1, \dots, \mathbf{A}_N$ are adapted interval operators and $\mathbf{B}$ is an arbitrary interval operator in the space of order bounded operators $L^{(r)}(X, Y)$.*

The following are equivalent:

(1) *The interval equation*

$$\mathbf{B} = \sum_{k=1}^{N} \alpha_k \mathbf{A}_k$$

has a weak interval solution $\alpha_1, \dots, \alpha_N \in \operatorname{Orth}(Y)_+$.

(2) *For all $b \in \mathbb{B}$ we have*

$$\{b\mathfrak{B} \ge 0\} \supset \{b\mathfrak{A}_1^{\sim} \le 0\} \cap \dots \cap \{b\mathfrak{A}_N^{\sim} \le 0\},$$

where $\mathfrak{A}_k^{\sim} := P_{\mathbf{A}_k} \circ \sim$ for $k := 1, \dots, N$ and $\mathfrak{B} := P_{\mathbf{B}}$.

Inhomogeneous Inequalities

Theorem 4. *Let X be a Y-seminormed real vector space, with Y a Kantorovich space. Assume given some dominated operators $A_1, \dots, A_N, B \in L^{(m)}(X, Y)$ and elements $u_1, \dots, u_N, v \in Y$. The following are equivalent:*

(1) *For all $b \in \mathbb{B}$ the inhomogeneous operator inequality $bBx \le bv$ is a consequence of the consistent simultaneous inhomogeneous operator inequalities $bA_1x \le bu_1, \dots, bA_Nx \le bu_N$, i.e.,*

$$\{bB \le bv\} \supset \{bA_1 \le bu_1\} \cap \dots \cap \{bA_N \le bu_N\}.$$

(2) *There are positive orthomorphisms $\alpha_1, \dots, \alpha_N \in \operatorname{Orth}(m(Y))$ satisfying*

$$B = \sum_{k=1}^{N} \alpha_k A_k; \quad v \ge \sum_{k=1}^{N} \alpha_k u_k.$$

Theorem 5.[9] *Let X be a Y-seminormed real vector space, with Y a Kantorovich space. Assume that $A \in L^{(m)}(X, Y^s)$, $B \in L^{(m)}(X, Y^t)$, $u \in Y^s$, and $v \in Y^t$, where s and t are some naturals.*

The following are equivalent:

(1) *For all $b \in \mathbb{B}$ the inhomogeneous operator inequality $bBx \le bv$ is a consequence of the consistent inhomogeneous inequality $bAx \le bu$, i.e., $\{bB \le bv\} \supset \{bA \le bu\}$.*

[8] Cp. [22].

[9] Cp. [23].

(2) *There is some $s \times t$ matrix with entries positive orthomorphisms of $m(Y)$ such that $B = \mathfrak{X}A$ and $\mathfrak{X}u \le v$ for the corresponding linear operator $\mathfrak{X} \in L_+(Y^s, Y^t)$.*

Theorem 6. *Let X be a Y-seminormed complex vector space, with Y a Kantorovich space. Assume given some $u_1, \dots, u_N, v \in Y$ and dominated operators $A_1, \dots, A_N, B \in L^{(m)}(X, Y_{\mathbb{C}})$ from X into the complexification $Y_{\mathbb{C}} := Y \otimes iY$ of Y.*[10] *Assume further that the inhomogeneous simultaneous inequalities $|A_1 x| \le u_1, \dots, |A_N x| \le u_N$ are consistent. Then the following are equivalent:*

(1) $\{b|B(\cdot)| \le bv\} \supset \{b|A_1(\cdot)| \le bu_1\} \cap \dots \cap \{b|A_N(\cdot)| \le bu_N\}$ *for all* $b \in \mathbb{B}$.

(2) *There are complex orthomorphisms $c_1, \dots, c_N \in \mathrm{Orth}(m(Y)_{\mathbb{C}})$ satisfying*

$$B = \sum_{k=1}^{N} c_k A_k; \quad v \ge \sum_{k=1}^{N} |c_k| u_k.$$

Lemma 3. *Let X be a real vector space. Assume that $p_1, \dots, p_N \in \mathrm{PSub}(X) := \mathrm{PSub}(X, \mathbb{R})$ and $p \in \mathrm{Sub}(X)$. Assume further that $v, u_1, \dots, u_N \in \mathbb{R}$ make consistent the simultaneous sublinear inequalities $p_k(x) \le u_k$, with $k := 1, \dots, N$.*

The following are equivalent:

(1) $\{p \ge v\} \supset \bigcap\limits_{k=1}^{N} \{p_k \le u_k\}$;

(2) *there are $\alpha_1, \dots, \alpha_N \in \mathbb{R}_+$ satisfying*

$$(\forall x \in X)\ p(x) + \sum_{k=1}^{N} \alpha_k p_k(x) \ge 0, \quad \sum_{k=1}^{N} \alpha_k u_k \le -v.$$

Proof. (2) $\to$ (1): If x is a solution to the simultaneous inhomogeneous inequalities $p_k(x) \le u_k$ with $k := 1, \dots, N$, then

$$0 \le p(x) + \sum_{k=1}^{N} \alpha_k p_k(x) \le p(x) + \sum_{k=1}^{N} \alpha_k u_k(x) \le p(x) - v.$$

(1) $\to$ (2): Given $(x,t) \in X \times \mathbb{R}$, put $\bar{p}_k(x,t) := p_k(x) - tu_k$, $\bar{p}(x,t) := p(x) - tv$ and $\tau(x,t) := -t$. Clearly, $\tau, \bar{p}_1, \dots, \bar{p}_N \in \mathrm{PSub}(X \times \mathbb{R})$ and $\bar{p} \in \mathrm{Sub}(X \times \mathbb{R})$. Take

$$(x,t) \in \{\tau \le 0\} \cap \bigcap_{k=1}^{N} \{\bar{p}_k \le 0\}.$$

If, moreover, $t > 0$; then $u_k \ge p_k(x/t)$ for $k := 1, \dots, N$ and so $p(x/t) \le v$ by hypothesis. In other words $(x,t) \in \{\bar{p} \le 0\}$. If $t = 0$ then take some solution $\bar{x}$ of the simultaneous inhomogeneous polyhedral inequalities under study.

Since $x \in K := \bigcap_{k=1}^{N} \{p_k \le 0\}$; therefore, $p_k(\bar{x} + x) \le p(x) + p_k(x) \le u_k$ for all $k := 1, \dots, N$. Hence, $p(\bar{x} + x) \ge v$ by hypothesis. So the sublinear functional p is bounded below on the cone K. Consequently, p assumes only positive values on K. In other words, $(x,0) \in \{\bar{p} \le 0\}$. Thus $\{\bar{p} \ge 0\} \supset \bigcap_{k=1}^{N} \{\bar{p}_k \le 0\}$ and by Lemma 2.2 of [1] there are positive reals $\alpha_1, \dots, \alpha_N, \beta$ such that for all $(x,t) \in X \times \mathbb{R}$ we have $\bar{g}(x) + \beta\tau(x) + \sum_{k=1}^{N} \alpha_k \bar{p}_k(x) \ge 0$. Clearly, the so-obtained parameters $\alpha_1, \dots, \alpha_N$ are what we sought for. The proof of Lemma 3 is complete.

[10] Cp. [19, p. 338].

Theorem 7. *Let X be a Y-seminormed real vector space, with Y a Kantorovich space. Given are some dominated polyhedral sublinear operators $P_1, \dots, P_N \in \mathrm{PSub}^{(m)}(X, Y)$ and a dominated sublinear operator $P \in \mathrm{Sub}^{(m)}(X, Y)$. Assume further that $u_1, \dots, u_N, v \in Y$ make consistent the simultaneous inhomogeneous inequalities*

$$P_1(x) \le u_1, \dots, P_N(x) \le u_N.$$

The following are equivalent:

(1) *for all $b \in \mathbb{B}$ the inhomogeneous sublinear operator inequality $bP(x) \ge bv$ is a consequence of the simultaneous inhomogeneous sublinear operator inequalities $bP_1(x) \le bu_1, \dots, bP_N(x) \le bu_N$, i.e.,*

$$\{bP \ge bv\} \supset \{bP_1 \le bu_1\} \cap \dots \cap \{bP_N \le bu_N\};$$

(2) *there are positive $\alpha_1, \dots, \alpha_N \in \mathrm{Orth}(m(Y))$ satisfying*

$$(\forall x \in X)\ P(x) + \sum_{k=1}^{N} \alpha_k P_k(x) \ge 0, \quad \sum_{k=1}^{N} \alpha_k u_k \le -v.$$

Lagrange's Principle. *The finite value of the constrained problem*

$$P_1(x) \le u_1, \dots, P_N(x) \le u_N, \quad P(x) \to \inf$$

is the value of the unconstrained problem for an appropriate Lagrangian without any constraint qualification other that polyhedrality.

The Slater condition allows us to eliminate polyhedrality as well as considering a unique target space. This is available in a practically unrestricted generality [24].

About the new trends relevant to the Farkas Lemma see [25]–[29].

Freedom and Inequality

Convexity is the theory of linear inequalities in disguise, tailored by set theory with a plentitude of bizarre visualizations of the figments of intuition.

Abstraction is the freedom of generalization. Freedom is the loftiest ideal and idea of man, but it is demanding, limited, and vexing. So is abstraction. So are its instances in convexity, hence, in simultaneous inequalities. Convexity and inequality supersede linearity because there are inequalities other than interpretations of simultaneous equalities.

Inequality is the first and foremost phenomenon of being. Equality is second historically, linguistics notwithstanding.

Freedom presumes liberty and equality. Inequality paves way to freedom.

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Appendix

NOMINATION AND DEFINITION

Nomination is a principal ingredient of education and transfer of knowledge. Nomination differs from definition. The latter implies the description of something new with the already available notions. Nomination is the calling of something, which is the starting point of any definition. Of course, the frontiers between nomination and definition are misty and indefinite rather than rigid and clear-cut.

Any child, meeting an unknown animal in the Zoo, will ask: "What is this?" The answer: "An opossum" will suit him usually. The sounds of the word "opossum" are quite familiar to the young person. The name is identified with a new image and this knowledge suffices. The definition of opossum for an adult will contain something of the sort: "*Didelphimorphia* is the order of common opossums of the Western Hemisphere." There are no grounds to assume that the approach of a child is less reasonable that the attitude of an adult.

The process of acquaintance or introduction of strangers reminds us of a meeting of a child and opossum. The phase "Meet Joe Blake" carried no information about Brad Pitt prior to the famous remake by Martin Brest. The name of a person we meet for the first time is important but little informative, enabling us to determine just a collection of namesakes. Only the number of a social security card in the USA or the requisites of a passport in Russia may serve as a unique identification of a taxpayer from the point of view of an agent of the Internal Revenue Service. However, it is highly improbable to find a pedant so meticulous that he will recite the social security number of an old crony he introduced.

Science is impossible without concepts to be clarified and explicated in definitions. The functioning of science is in a sense the development of concepts. There are no grounds to believe that science uses the laws of nomination and definition which are completely incomprehensible in the simplest example of the meeting of a child and opossum.

Definition is rational, whereas nomination is universal. It is not by chance that nomination plays a key role in the versatile manifestations of mysticism, occultism, and religion. The lexicon of science consists of concepts. The evolution of concepts is a historical witness revealing the features of bygone times. The traces of any epoch are reflected in its most abstract concepts. Neglecting the historical background makes it impossible to understand correctly not only the generally accepted concepts but also the brand-new terms like nanotechnology or quantum logic.

Mental continuity is a priceless gift enabling us to preserve the experience of our ancestors. The first transfinite act of mankind is the birth of the idea of the collection of all natural numbers. The concept of actual infinity remains a challenge for the intellectual efforts of the scientists of all eras and states from *Metaphysics* by Aristotle and *Psammites* by Archimedes. The monads of Leibniz alongside the fluxions and fluents of Newton are product of the heroic epoch of telescope and microscope. The von Neumann universe of the mid-twentieth century implements the Pythagorean thesis that "all is number." Measuring infinity by number is the crux of the revealing research of the genius Cantor. So long was the crooked way of the mysteries and nominata of reason from Paleolith to this day.

We find the origins of the modern science in Ancient Greece. The books by Euclid are the most important source of scientific traditions. From the antic times, geometry deals with the quantitative and qualitative properties of spacial forms and relations. The criteria for equality of triangles provide instances of qualitative geometric knowledge. Finding lengths, areas, and volumes exemplifies quantitative research. The incommensurability of the side and diagonal of a square became an outstanding discovery of Euclidean geometry. It was the first time when science has confronted the problem of counting the continuum.

When our ancestors had demonstrated the absence of any common measure of the side and diagonal of a square, they understood that rational numbers are scarce for practical purposes. It is worth recalling that the set of rational numbers is equipollent with the collection of natural numbers. This means that all rational numbers comprise a countable set, thus serving as an instance of the cardinal number that we use to express the size of the imaginary collection of all entries of the natural series. The discovery that the side and diagonal of a square are incommensurable is the height of mathematics as awesome and ethereal as the independence of the fifth postulate, the axiom of choice, and the continuum hypothesis.

The definitions of Euclid's *Elements*, the greatest scientific treatise in the history of the mankind, reflect the geometric vision of the world of his epoch. Geometry is part of the culture of the ancient world which was invented to meet various human needs. Its mystic, explorative, and economic sources coexisted in the common cultural environment of the man of the pre-Bible times. The strongest quest of geometry stemmed from the cadastral surveying aimed at regular taxation. The famous harpedonaptae of Egypt were tax agents who used ropes for measuring the tracts of land. The tricks and techniques of harpedonaptae were used in construction. Pyramids were erected long before the abstract definition of the geometrical form of a pyramid. It is impossible to ignore the fact that the rope stretching taut between two stakes is a mental icon of the concept of a straight line segment which is the continuum of the modern mathematics. The problem of the continuum, the greatest puzzle for the first mathematical minds of the twentieth century is a shadow of the practical task of commensuration of diverse fragments of lines.

Bewildering is the history of the abstract geometric concepts of point, monad, figure, and solid which came from the remote ages. We are rarely aware of the fact the secondary school arithmetic and geometry are the finest gems of the intellectual legacy of our forefathers. There is no literate who fails to recognize a triangle. However, just a few know an appropriate formal definition. This is not by chance at all, since the definition of triangle is absent in the *Elements*. Euclid spoke about three-lateral figures, emphasizing that "a figure is that which is contained by any boundary or boundaries." Clearly, his definitions remind us of the technology of cadastral surveying of his times. It is worth observing that the institution of property is much older than the art and science of geometry. To measure a tract of land from outside is legitimate whereas trespassing the borders is forbidden. The ancient rope stretchers had similar restrictions for measuring the constructions like pyramids. Clearly, the surveyors of the Kheops pyramid would mum every single word about the interiors of this building.

In the modern parlance, we say that Euclid considered convex figures and solid bodies. The concept of convexity seems quite elementary today. Part of a plane or space is called *convex* provided that no straight line segment between any two points of this part lies within the object under consideration. If we drive three stakes in a tract of land and stretch a lasso whose loop surrounds the stakes, we will single out a triangle. The harpedonaptae did exactly the same, but the interior of the tract to be measured might be inaccessible to the surveyors without permission of the owner. Nowadays we also measure property and levy taxes but any unauthorized attempt to stretch a rope within somebody's property is still a felony of trespassing on land. The definitions of Euclid are listed among the immortal witnesses of the ancient economic relations.

Mathematics is the first science of the knowing man. *Homo sapiens* perceives the reality and himself in the external world by the physiological methods that enable him to discern items and distinguish their forms. The abstract forms and relations of the human mind are the starting points of man's scientific nomination and definition. Euclidean geometry is an exemplar of rational creativity for two and a half millennia. The modern science possesses many hundreds of new theories, while nominating and defining hundreds of thousands of the new objects and concepts unknown to Euclid. However, the method of scientific research remains practically the same. Euclid could learn the fundamentals of any appealing present-day scientific discipline with the same ease as the children of any race or nationality throughout the world master the basics of Euclidean geometry.

Continuity of generations safeguards the immortality of science.

Кутателадзе Семён Самсонович

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