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ALEXANDROV'S APPROACH TO THE
MINKOWSKI PROBLEM

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This article is dedicated to the centenary of the birth of Aleksandr D. Alexandrov (1912–1999). His functional analytical approach to the solving of the Minkowski problem is examined and applied to some new extremal problems of isoperimetric type with conflicting goals.

KEYWORDS AND PHRASES: Blaschke and Minkowski sums, space of convex bodies, decomposition, Urysohn type problems, Pareto optimality

Кутателадзе С. С.

ПОДХОД А. Д. АЛЕКСАНДРОВА К ПРОБЛЕМЕ МИНКОВСКОГО

Работа посвящена столетию со дня рождения Александра Даниловича Александрова (1912–1999). Рассматривается его функционально-аналитический подход к решению проблемы Минковского и некоторые новые применения к экстремальным задачам изопериметрического типа с конфликтующими целями.

КЛЮЧЕВЫЕ СЛОВА И ФРАЗЫ: суммы Бляшке и Минковского, пространство выпуклых тел, мажорация, задачи урысоновского типа, оптимальность по Парето

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The *Mathematics Subject Classification*, produced jointly by the editorial staffs of *Mathematical Reviews* and *Zentralblatt für Mathematik* in 2010, has Section 53C45 “Global surface theory (convex surfaces à la A. D. Aleksandrov).” This article surveys some mathematics of the sort.

Good mathematics starts as a first love. If great, it turns into adult sex and happy marriage. If ordinary, it ends in dumping, cheating or divorce. If awesome, it becomes eternal. Alexandrov’s mathematics is great (see [1]–[3]). To demonstrate, inspect his solution of the Minkowski problem.

Alexandrov’s mathematics is alive, expanding and flourishing for decades. Dido’s problem in the today’s setting is one of the examples.

THE SPACE OF CONVEX BODIES

A *convex figure* is a compact convex set. A *convex body* is a solid convex figure. The *Minkowski duality* identifies a convex figure S in \mathbb{R}^N and its *support function* $S(z) :=$

$\sup\{(x, z) \mid x \in S\}$ for $z \in \mathbb{R}^N$. Considering the members of \mathbb{R}^N as singletons, we assume that \mathbb{R}^N lies in the set \mathcal{V}_N of all compact convex subsets of \mathbb{R}^N .

The Minkowski duality makes \mathcal{V}_N into a cone in the space $C(S_{N-1})$ of continuous functions on the Euclidean unit sphere S_{N-1} , the boundary of the unit ball \mathfrak{z}_N . The *linear span* $[\mathcal{V}_N]$ of \mathcal{V}_N is dense in $C(S_{N-1})$, bears a natural structure of a vector lattice and is usually referred to as the *space of convex sets*.

The study of this space stems from the pioneering breakthrough of Alexandrov in 1937 and the further insights of Radström, Hörmander, and Pinsker.

LINEAR INEQUALITIES OVER CONVEX SURFACES

A measure μ *linearly majorizes* or *dominates* a measure ν on S_{N-1} provided that to each decomposition of S_{N-1} into finitely many disjoint Borel sets U_1, \dots, U_m there are measures μ_1, \dots, μ_m with sum μ such that every difference $\mu_k - \nu|_{U_k}$ annihilates all restrictions to S_{N-1} of linear functionals over \mathbb{R}^N . In symbols, we write $\mu \gg_{\mathbb{R}^N} \nu$.

Reshetnyak proved in 1954 (cp. [4]) that

$$\int_{S_{N-1}} p d\mu \geq \int_{S_{N-1}} p d\nu$$

for each sublinear functional p on \mathbb{R}^N if $\mu \gg_{\mathbb{R}^N} \nu$. This gave an important trick for generating positive linear functionals over various classes of convex surfaces and functions.

CHOQUET'S ORDER

A measure μ *affinely majorizes* or *dominates* a measure ν , both given on a compact convex subset Q of a locally convex space X , provided that to each decomposition of ν into finitely many summands ν_1, \dots, ν_m there are measures μ_1, \dots, μ_m whose sum is μ and for which every difference $\mu_k - \nu_k$ annihilates all restrictions to Q of affine functionals over X . In symbols, $\mu \gg_{\text{Aff}(Q)} \nu$.

Cartier, Fell, and Meyer proved in 1964 (cp. [5]) that

$$\int_Q f d\mu \geq \int_Q f d\nu$$

for each continuous convex function f on Q if and only if $\mu \gg_{\text{Aff}(Q)} \nu$. An analogous necessity part for linear majorization was first published in 1969 (cp. [6]–[8]).

DECOMPOSITION THEOREM

Majorization is a vast subject (cp. [9]). The general form for many cones is as follows (cp. [10]):

Assume that H_1, \dots, H_N are cones in a vector lattice X . Assume further that f and g are positive linear functionals on X . The inequality

$$f(h_1 \vee \dots \vee h_N) \geq g(h_1 \vee \dots \vee h_N)$$

holds for all $h_k \in H_k$ ($k := 1, \dots, N$) if and only if to each decomposition of g into a sum of N positive terms $g = g_1 + \dots + g_N$ there is a decomposition of f into a sum of N positive terms $f = f_1 + \dots + f_N$ such that

$$f_k(h_k) \geq g_k(h_k) \quad (h_k \in H_k; k := 1, \dots, N).$$

ROTATIONAL SYMMETRY

Assume that a plane convex figure $\mathfrak{x}_0 \in \mathcal{V}_2$ has the symmetry axis $A_{\bar{z}}$ with generator \bar{z} . Assume further that \mathfrak{x}_{00} is the result of rotating \mathfrak{x}_0 around the symmetry axis $A_{\bar{z}}$ in \mathbb{R}^3 . Consider the problem:

$$\begin{aligned} & \mathfrak{x} \in \mathcal{V}_3; \\ & \mathfrak{x} \text{ is a convex body of rotation around } A_{\bar{z}}; \\ & \mathfrak{x} \supset \mathfrak{x}_{00}; \langle \mathfrak{z}_N, \mathfrak{x} \rangle \geq \langle \mathfrak{z}_N, \bar{\mathfrak{x}} \rangle; \\ & (-p(\mathfrak{x}), b_{\bar{z}}(\mathfrak{x})) \rightarrow \inf. \end{aligned}$$

Each Pareto-optimal solution is the result of rotating around the symmetry axis a Pareto-optimal solution of the plane internal Urysohn problem with flattening in the direction of the axis.

SOAP BUBBLES

Little is known about the analogous problems in arbitrary dimensions. An especial place is occupied by the result of Porogelov who demonstrated that the ‘‘soap bubble’’ in a tetrahedron has the form of the result of the rolling of a ball over a solution of the internal Urysohn problem, i.e. the weighted Blaschke sum of a tetrahedron and a ball.

THE EXTERNAL URYSOHN PROBLEM WITH FLATTENING

Given are some convex body $\mathfrak{x}_0 \in \mathcal{V}_N$ and flattening direction $\bar{z} \in S_{N-1}$. Considering $\mathfrak{x} \supset \mathfrak{x}_0$ of fixed integral breadth, maximize volume and minimize breadth in the flattening direction: $\mathfrak{x} \in \mathcal{V}_N$; $\mathfrak{x} \supset \mathfrak{x}_0$; $\langle \mathfrak{x}, \mathfrak{z}_N \rangle \geq \langle \bar{\mathfrak{x}}, \mathfrak{z}_N \rangle$; $(-p(\mathfrak{x}), b_{\bar{z}}(\mathfrak{x})) \rightarrow \inf$.

For a feasible convex body $\bar{\mathfrak{x}}$ to be a Pareto-optimal solution of the external Urysohn problem with flattening it is necessary and sufficient that there be positive reals α, β , and a convex figure \mathfrak{x} satisfying

$$\begin{aligned} & \mu(\bar{\mathfrak{x}}) + \mu(\mathfrak{x}) \gg_{\mathbb{R}^N} \alpha \mu(\mathfrak{z}_N) + \beta(\varepsilon_{\bar{z}} + \varepsilon_{-\bar{z}}); \\ & V(\bar{\mathfrak{x}}) + V_1(\mathfrak{x}, \bar{\mathfrak{x}}) = \alpha V_1(\mathfrak{z}_N, \bar{\mathfrak{x}}) + 2N\beta b_{\bar{z}}(\bar{\mathfrak{x}}); \\ & \bar{\mathfrak{x}}(z) = \mathfrak{x}_0(z) \quad (z \in \text{spt}(\mu(\mathfrak{x}))). \end{aligned}$$

OPTIMAL CONVEX HULLS

Given $\mathfrak{v}_1, \dots, \mathfrak{v}_m$ in \mathbb{R}^N , place \mathfrak{x}_k within \mathfrak{v}_k , for $k := 1, \dots, m$, maximizing the volume of each of the $\mathfrak{x}_1, \dots, \mathfrak{x}_m$ and minimizing the integral breadth of their convex hull:

$$\mathfrak{x}_k \subset \mathfrak{v}_k; (-p(\mathfrak{x}_1), \dots, -p(\mathfrak{x}_m), \langle \text{co}\{\mathfrak{x}_1, \dots, \mathfrak{x}_m\}, \mathfrak{z}_N \rangle) \rightarrow \inf.$$

For some feasible $\bar{\mathfrak{x}}_1, \dots, \bar{\mathfrak{x}}_m$ to have a Pareto-optimal convex hull it is necessary and sufficient that there be $\alpha_1, \dots, \alpha_m \in \mathbb{R}_+$ not vanishing simultaneously and positive Borel measures μ_1, \dots, μ_m and ν_1, \dots, ν_m on S_{N-1} such that

$$\begin{aligned} & \nu_1 + \dots + \nu_m = \mu(\mathfrak{z}_N); \\ & \bar{\mathfrak{x}}_k(z) = \mathfrak{v}_k(z) \quad (z \in \text{spt}(\mu_k)); \\ & \alpha_k \mu(\bar{\mathfrak{x}}_k) = \mu_k + \nu_k \quad (k := 1, \dots, m). \end{aligned}$$

rule. But this idea fails since the interior of \mathcal{V}_N is empty. The fact that DC-functions are dense in $C(S_{N-1})$ is not helpful at all.

Alexandrov extended the volume to the positive cone of $C(S_{N-1})$ by the formula $V(f) := \langle f, \mu(\text{co}(f)) \rangle$ with $\text{co}(f)$ the envelope of support functions below f . He also observed that $V(f) = V(\text{co}(f))$. The ingenious trick settled all for the Minkowski problem. This was done in 1938 but still is one of the summits of convexity.

In fact, Alexandrov suggested a functional analytical approach to extremal problems for convex surfaces. To follow it directly in the general setting is impossible without the above description of the polar cones. The obvious limitations of the Lagrange multiplier rule are immaterial in the case of convex programs. It should be emphasized that the classical isoperimetric problem is not a Minkowski convex program in dimensions greater than 2. The convex counterpart is the Urysohn problem of maximizing volume given integral breadth [11]. The constraints of inclusion type are convex in the Minkowski structure, which opens way to complete solution of new classes of Urysohn-type problems (cp. [12]).

THE EXTERNAL URYSOHN PROBLEM

Among the convex figures, circumscribing \mathfrak{x}_0 and having integral breadth fixed, find a convex body of greatest volume.

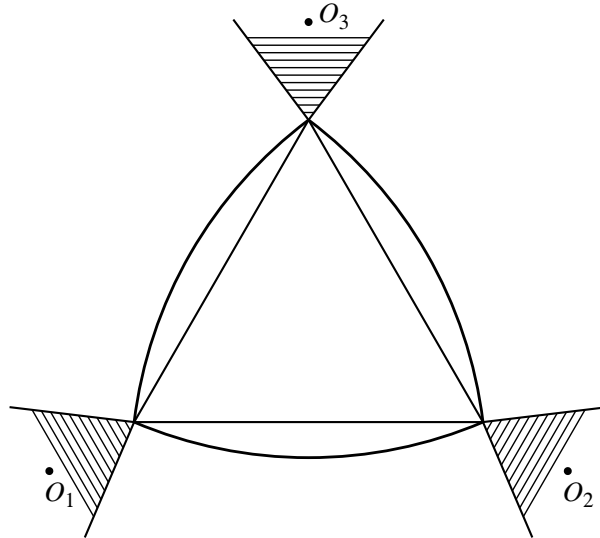
A feasible convex body $\bar{\mathfrak{x}}$ is a solution to the external Urysohn problem if and only if there are a positive measure μ and a positive real $\bar{\alpha} \in \mathbb{R}_+$ satisfying

- (1) $\bar{\alpha}\mu(\mathfrak{z}_N) \gg_{\mathbb{R}^N} \mu(\bar{\mathfrak{x}}) + \mu$;
- (2) $V(\bar{\mathfrak{x}}) + \frac{1}{N} \int_{S_{N-1}} \bar{\mathfrak{x}} d\mu = \bar{\alpha}V_1(\mathfrak{z}_N, \bar{\mathfrak{x}})$;
- (3) $\bar{\mathfrak{x}}(z) = \mathfrak{x}_0(z)$ for all z in the support of μ .

SOLUTIONS

If $\mathfrak{x}_0 = \mathfrak{z}_{N-1}$ then $\bar{\mathfrak{x}}$ is a *spherical lens* and μ is the restriction of the surface area function of the ball of radius $\bar{\alpha}^{1/(N-1)}$ to the complement of the support of the lens to S_{N-1} .

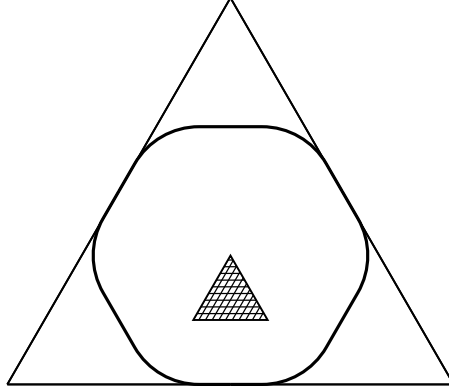
If \mathfrak{x}_0 is an equilateral triangle then the solution $\bar{\mathfrak{x}}$ looks as follows:



$\bar{\mathfrak{x}}$ is the union of \mathfrak{x}_0 and three congruent slices of a circle of radius $\bar{\alpha}$ and centers O_1 – O_3 , while μ is the restriction of $\mu(\mathfrak{z}_2)$ to the subset of S_1 comprising the endpoints of the unit vectors of the shaded zone.

SYMMETRIC SOLUTIONS

This is the general solution of the internal Urysohn problem inside a triangle in the class of centrally symmetric convex figures:



CURRENT HYPERPLANES

Find two convex figures $\bar{\mathfrak{x}}$ and $\bar{\mathfrak{y}}$ lying in a given convex body \mathfrak{x}_0 , separated by a hyperplane with the unit outer normal z_0 , and having the greatest total volume of $\bar{\mathfrak{x}}$ and $\bar{\mathfrak{y}}$ given the sum of their integral breadths.

A feasible pair of convex bodies $\bar{\mathfrak{x}}$ and $\bar{\mathfrak{y}}$ solves the internal Urysohn problem with a current hyperplane if and only if there are convex figures \mathfrak{x} and \mathfrak{y} and positive reals $\bar{\alpha}$ and $\bar{\beta}$ satisfying

- (1) $\bar{\mathfrak{x}} = \mathfrak{x} \# \bar{\alpha} \mathfrak{z}_N$;
- (2) $\bar{\mathfrak{y}} = \mathfrak{y} \# \bar{\alpha} \mathfrak{z}_N$;
- (3) $\mu(\mathfrak{x}) \geq \bar{\beta} \varepsilon_{z_0}$, $\mu(\mathfrak{y}) \geq \bar{\beta} \varepsilon_{-z_0}$;
- (4) $\bar{\mathfrak{x}}(z) = \mathfrak{x}_0(z)$ for all $z \in \text{spt}(\mathfrak{x}) \setminus \{z_0\}$;
- (5) $\bar{\mathfrak{y}}(z) = \mathfrak{x}_0(z)$ for all $z \in \text{spt}(\mathfrak{y}) \setminus \{-z_0\}$, with $\text{spt}(\mathfrak{x})$ standing for the support of \mathfrak{x} , i.e. the support of the surface area measure $\mu(\mathfrak{x})$ of \mathfrak{x} .

IS DIDO'S PROBLEM SOLVED?

From a utilitarian standpoint, the answer is definitely in the affirmative. There is no evidence that Dido experienced any difficulties, showed indecisiveness, and procrastinated the choice of the tract of land. Practically speaking, the situation in which Dido made her decision was not as primitive as it seems at the first glance.

Assume that Dido had known the isoperimetric property of the circle and had been aware of the symmetrization processes that were elaborated in the nineteenth century. Would this knowledge be sufficient for Dido to choose the tract of land? Definitely, it would not. The real coastline may be rather ragged and craggy. The photo snaps of coastlines are exhibited as the most visual examples of fractality. From a theoretical standpoint, the free boundary in Dido's plane problem may be nonrectifiable, and so the concept of area as the quantity to be optimized is itself rather ambiguous. Practically speaking, the situation in which Dido made her decision was not as primitive as it seems at the first glance. Choosing the tract of land, Dido had no right to trespass the territory under the control of the local sovereign. She had to choose the tract so as to encompass the camps of her subjects and satisfy some fortification requirements. Clearly, this generality is unavailable in the mathematical models known as the classical isoperimetric problem.

Nowadays there is much research aiming at the problems with conflicting goals (cp., for instance, [13]). One of the simplest and most popular approach is based on the concept of Pareto-optimum.

PARETO OPTIMALITY

Consider a bunch of economic agents each of which intends to maximize his own income. The *Pareto efficiency principle* asserts that as an effective agreement of the conflicting goals it is reasonable to take any state in which nobody can increase his income in any way other than diminishing the income of at least one of the other fellow members. Formally speaking, this implies the search of the maximal elements of the set comprising the tuples of incomes of the agents at every state; i.e., some vectors of a finite-dimensional arithmetic space endowed with the coordinatewise order. Clearly, the concept of Pareto optimality was already abstracted to arbitrary ordered vector spaces.

By way of example, consider a few multiple criteria problems of isoperimetric type. For more detail, see [14].

VECTOR ISOPERIMETRIC PROBLEM

Given are some convex bodies η_1, \dots, η_M . Find a convex body \mathfrak{x} encompassing a given volume and minimizing each of the mixed volumes $V_1(\mathfrak{x}, \eta_1), \dots, V_1(\mathfrak{x}, \eta_M)$. In symbols,

$$\mathfrak{x} \in \mathcal{A}_N; \widehat{p}(\mathfrak{x}) \geq \widehat{p}(\bar{\mathfrak{x}}); (\langle \eta_1, \mathfrak{x} \rangle, \dots, \langle \eta_M, \mathfrak{x} \rangle) \rightarrow \inf.$$

Clearly, this is a Slater regular convex program in the Blaschke structure.

Each Pareto-optimal solution $\bar{\mathfrak{x}}$ of the vector isoperimetric problem has the form

$$\bar{\mathfrak{x}} = \alpha_1 \eta_1 + \dots + \alpha_m \eta_m,$$

where $\alpha_1, \dots, \alpha_m$ are positive reals.

THE LEIDENFROST PROBLEM

Given the volume of a three-dimensional convex figure, minimize its surface area and vertical breadth.

By symmetry everything reduces to an analogous plane two-objective problem, whose every Pareto-optimal solution is by 2 a *stadium*, a weighted Minkowski sum of a disk and a horizontal straight line segment.

A plane spheroid, a Pareto-optimal solution of the Leidenfrost problem, is the result of rotation of a stadium around the vertical axis through the center of the stadium.

INTERNAL URYSOHN PROBLEM WITH FLATTENING

Given are some convex body $\mathfrak{x}_0 \in \mathcal{V}_N$ and some flattening direction $\bar{z} \in S_{N-1}$. Considering $\mathfrak{x} \subset \mathfrak{x}_0$ of fixed integral breadth, maximize the volume of \mathfrak{x} and minimize the breadth of \mathfrak{x} in the flattening direction: $\mathfrak{x} \in \mathcal{V}_N; \mathfrak{x} \subset \mathfrak{x}_0; \langle \mathfrak{x}, \mathfrak{z}_N \rangle \geq \langle \bar{\mathfrak{x}}, \mathfrak{z}_N \rangle; (-p(\mathfrak{x}), b_{\bar{z}}(\mathfrak{x})) \rightarrow \inf$.

For a feasible convex body $\bar{\mathfrak{x}}$ to be Pareto-optimal in the internal Urysohn problem with the flattening direction \bar{z} it is necessary and sufficient that there be positive reals α, β and a convex figure \mathfrak{x} satisfying

$$\begin{aligned} \mu(\bar{\mathfrak{x}}) &= \mu(\mathfrak{x}) + \alpha \mu(\mathfrak{z}_N) + \beta(\varepsilon_{\bar{z}} + \varepsilon_{-\bar{z}}); \\ \bar{\mathfrak{x}}(z) &= \mathfrak{x}_0(z) \quad (z \in \text{spt}(\mu(\mathfrak{x}))). \end{aligned}$$

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OPTIMAL CONVEX HULLS

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$$\mathfrak{x}_k \subset \mathfrak{v}_k; (-p(\mathfrak{x}_1), \dots, -p(\mathfrak{x}_m), \langle \text{co}\{\mathfrak{x}_1, \dots, \mathfrak{x}_m\}, \mathfrak{z}_N \rangle) \rightarrow \inf.$$

For some feasible $\bar{\mathfrak{x}}_1, \dots, \bar{\mathfrak{x}}_m$ to have a Pareto-optimal convex hull it is necessary and sufficient that there be $\alpha_1, \dots, \alpha_m \in \mathbb{R}_+$ not vanishing simultaneously and positive Borel measures μ_1, \dots, μ_m and ν_1, \dots, ν_m on S_{N-1} such that

$$\begin{aligned} & \nu_1 + \dots + \nu_m = \mu(\mathfrak{z}_N); \\ & \bar{\mathfrak{x}}_k(z) = \mathfrak{v}_k(z) \quad (z \in \text{spt}(\mu_k)); \\ & \alpha_k \mu(\bar{\mathfrak{x}}_k) = \mu_k + \nu_k \quad (k := 1, \dots, m). \end{aligned}$$

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