

THE REAL NATURE OF K -SPACE

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ABSTRACT. This is an appendix to a talk on interaction between K -space and the reals to be delivered at the meeting “Kantorovich Memorial. Mathematical Economics: Old Problems and New Approaches”; January 8–13, 2004; the Euler International Mathematical Institute at St. Petersburg.

This Appendix briefs the reader on the representation of a K -space in the framework of Boolean valued analysis.¹

1. BOOLEAN VALUED UNIVERSES

The universe of discourse of Boolean valued analysis is a Boolean valued model of ZFC. To sketch its structure, we start with a complete Boolean algebra B . Given an ordinal α , put

$$\mathbb{V}_\alpha^{(B)} := \{x \mid x \text{ is a function} \wedge (\exists \beta)(\beta < \alpha \wedge \text{dom}(x) \subset \mathbb{V}_\beta^{(B)} \wedge \text{im}(x) \subset B)\}.$$

Thus, in more detail we have

$$\begin{aligned} \mathbb{V}_0^{(B)} &:= \emptyset, \\ \mathbb{V}_{\alpha+1}^{(B)} &:= \{x \mid x \text{ is a function with domain in } \mathbb{V}_\alpha^{(B)} \text{ and range in } B\}; \\ \mathbb{V}_\alpha^{(B)} &:= \bigcup_{\beta < \alpha} \mathbb{V}_\beta^{(B)} \quad (\alpha \text{ is a limit ordinal}). \end{aligned}$$

The class

$$\mathbb{V}^{(B)} := \bigcup_{\alpha \in \text{On}} \mathbb{V}_\alpha^{(B)}$$

is a *Boolean valued universe*. An element of the class $\mathbb{V}^{(B)}$ is a *B-valued set*. It is necessary to observe that $\mathbb{V}^{(B)}$ consists only of functions. In particular, \emptyset is the function with domain \emptyset and range \emptyset . Hence, the “lower” levels of $\mathbb{V}^{(B)}$ are organized as follows:

$$\mathbb{V}_0^{(B)} = \emptyset, \quad \mathbb{V}_1^{(B)} = \{\emptyset\}, \quad \mathbb{V}_2^{(B)} = \{\emptyset, (\{\emptyset\}, b) \mid b \in B\}.$$

It is worth stressing that $\alpha \leq \beta \rightarrow \mathbb{V}_\alpha^{(B)} \subset \mathbb{V}_\beta^{(B)}$ for all ordinals α and β . Moreover, the following *induction principle* is valid in $\mathbb{V}^{(B)}$:

$$(\forall x \in \mathbb{V}^{(B)}) ((\forall y \in \text{dom}(x)) \varphi(y) \rightarrow \varphi(x)) \rightarrow (\forall x \in \mathbb{V}^{(B)}) \varphi(x),$$

where φ is a formula of ZFC.

Take an arbitrary formula $\varphi = \varphi(u_1, \dots, u_n)$ of ZFC. If we replace the elements u_1, \dots, u_n by elements $x_1, \dots, x_n \in \mathbb{V}^{(B)}$ then we obtain some statement about the objects x_1, \dots, x_n . It is to this statement that we intend to assign some *truth-value*. Such a value $\llbracket \psi \rrbracket$ must be an element of the algebra B . Moreover,

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¹Extra details may be found in the book: Kusraev A. G. and Kutateladze S. S. *Boolean Valued Analysis*. Kluwer Academic Publishers, Dordrecht, 1996.

it is naturally desired that the theorems of ZFC be true, i.e., attain the greatest truth-value, unity.

We must obviously define the truth-value of a well-formed formula by double induction, on considering the way in which this formula is built up from the atomic formulas $x \in y$ and $x = y$, while assigning truth-values to the latter when x and y range over $\mathbb{V}^{(B)}$ on using the recursive definition of this universe.

It is clear that if φ and ψ are evaluated formulas of ZFC and $\llbracket \varphi \rrbracket \in B$ and $\llbracket \psi \rrbracket \in B$ are their truth-values then we should put

$$\begin{aligned}\llbracket \varphi \wedge \psi \rrbracket &:= \llbracket \varphi \rrbracket \wedge \llbracket \psi \rrbracket, \\ \llbracket \varphi \vee \psi \rrbracket &:= \llbracket \varphi \rrbracket \vee \llbracket \psi \rrbracket, \\ \llbracket \varphi \rightarrow \psi \rrbracket &:= \llbracket \varphi \rrbracket \rightarrow \llbracket \psi \rrbracket, \\ \llbracket \neg \varphi \rrbracket &:= \neg \llbracket \varphi \rrbracket, \\ \llbracket (\forall x) \varphi(x) \rrbracket &:= \bigwedge_{x \in \mathbb{V}^{(B)}} \llbracket \varphi(x) \rrbracket, \\ \llbracket (\exists x) \varphi(x) \rrbracket &:= \bigvee_{x \in \mathbb{V}^{(B)}} \llbracket \varphi(x) \rrbracket,\end{aligned}$$

where the right-hand sides involve the Boolean operations corresponding to the logical connectives and quantifiers on the left-hand sides: \wedge is the meet, \vee is the join, \neg is the complementation, while the implication \rightarrow is introduced as follows: $a \rightarrow b := \neg a \vee b$ for $a, b \in B$. Only these definitions provide the value “unity” for the classical tautologies.

We turn to evaluating the atomic formulas $x \in y$ and $x = y$ for $x, y \in \mathbb{V}^{(B)}$. The intuitive idea consists in the fact that a B -valued set y is a “(lattice-) fuzzy set,” i.e., a “set that contains an element z in $\text{dom}(y)$ with probability $y(z)$.” Keeping this in mind and intending to preserve the logical tautology of $x \in y \leftrightarrow (\exists z \in y) (x = z)$ as well as the axiom of extensionality, we arrive at the following definition by recursion:

$$\begin{aligned}\llbracket x \in y \rrbracket &:= \bigvee_{z \in \text{dom}(y)} y(z) \wedge \llbracket z = x \rrbracket, \\ \llbracket x = y \rrbracket &:= \bigwedge_{z \in \text{dom}(x)} x(z) \rightarrow \llbracket z \in y \rrbracket \wedge \bigwedge_{z \in \text{dom}(y)} y(z) \rightarrow \llbracket z \in x \rrbracket.\end{aligned}$$

Now we can attach some meaning to formal expressions of the form $\varphi(x_1, \dots, x_n)$, where $x_1, \dots, x_n \in \mathbb{V}^{(B)}$ and φ is a formula of ZFC; i.e., we may define the exact sense in which the set-theoretic proposition $\varphi(u_1, \dots, u_n)$ is valid for the assignment of $x_1, \dots, x_n \in \mathbb{V}^{(B)}$.

Namely, we say that the *formula* $\varphi(x_1, \dots, x_n)$ *is valid inside* $\mathbb{V}^{(B)}$ or the *elements* x_1, \dots, x_n *possess the property* φ if $\llbracket \varphi(x_1, \dots, x_n) \rrbracket = \mathbf{1}$. In this event, we write $\mathbb{V}^{(B)} \models \varphi(x_1, \dots, x_n)$.

Clearly, the axioms and theorems of the first-order predicate calculus are valid in $\mathbb{V}^{(B)}$. In particular,

- (1) $\llbracket x = x \rrbracket = \mathbf{1}$,
- (2) $\llbracket x = y \rrbracket = \llbracket y = x \rrbracket$,
- (3) $\llbracket x = y \rrbracket \wedge \llbracket y = z \rrbracket \leq \llbracket x = z \rrbracket$,
- (4) $\llbracket x = y \rrbracket \wedge \llbracket z \in x \rrbracket \leq \llbracket z \in y \rrbracket$,
- (5) $\llbracket x = y \rrbracket \wedge \llbracket x \in z \rrbracket \leq \llbracket y \in z \rrbracket$.

It is worth observing that for each formula φ we have

$$\mathbb{V}^{(B)} \models x = y \wedge \varphi(x) \rightarrow \varphi(y),$$

i.e., in a detailed notation

$$(6) \llbracket x = y \rrbracket \wedge \llbracket \varphi(x) \rrbracket \leq \llbracket \varphi(y) \rrbracket.$$

2. PRINCIPLES OF BOOLEAN VALUED ANALYSIS

In a Boolean valued universe $\mathbb{V}^{(B)}$, the relation $\llbracket x = y \rrbracket = \mathbb{1}$ in no way implies that the functions x and y (considered as elements of \mathbb{V}) coincide. For example, the function equal to zero on each level $\mathbb{V}_\alpha^{(B)}$, where $\alpha \geq 1$, plays the role of the empty set in $\mathbb{V}^{(B)}$. This circumstance may complicate some constructions in the sequel.

In this connection, we pass from $\mathbb{V}^{(B)}$ to the *separated Boolean valued universe* $\overline{\mathbb{V}}^{(B)}$ often preserving for it the same symbol $\mathbb{V}^{(B)}$; i.e., we put $\mathbb{V}^{(B)} := \overline{\mathbb{V}}^{(B)}$. Moreover, to define $\overline{\mathbb{V}}^{(B)}$, we consider the relation $\{(x, y) \mid \llbracket x = y \rrbracket = \mathbb{1}\}$ on the class $\mathbb{V}^{(B)}$ which is obviously an equivalence. Choosing an element (a representative of the least rank) in each class of equivalent functions, we arrive at the separated universe $\overline{\mathbb{V}}^{(B)}$. Note that

$$\llbracket x = y \rrbracket = \mathbb{1} \rightarrow \llbracket \varphi(x) \rrbracket = \llbracket \varphi(y) \rrbracket$$

is valid for an arbitrary formula φ of ZFC and elements x and y in $\mathbb{V}^{(B)}$.

Therefore, in the separated universe we can calculate the truth-values of formulas paying no attention to the way of choosing representatives. Furthermore, working with the separated universe, for the sake of convenience we (exercising due caution) often consider some particular representative of an equivalence class rather than the whole class as it is customary, for example, while dealing with function spaces.

The most important properties of a Boolean valued universe $\mathbb{V}^{(B)}$ are stated in the following three propositions:

(1) Transfer Principle. *All theorems of ZFC are true in $\mathbb{V}^{(B)}$; in symbols,*

$$\mathbb{V}^{(B)} \models \text{a theorem of ZFC.}$$

The transfer principle is established by laboriously checking that all axioms of ZFC have truth-value $\mathbb{1}$ and the rules of inference increase the truth-values of formulas. Sometimes, the transfer principle is worded as follows: “ $\mathbb{V}^{(B)}$ is a Boolean valued model of ZFC.”

(2) Maximum Principle. *For each formula φ of ZFC there exists $x_0 \in \mathbb{V}^{(B)}$ for which*

$$\llbracket (\exists x) \varphi(x) \rrbracket = \llbracket \varphi(x_0) \rrbracket.$$

In particular, if it is true in $\mathbb{V}^{(B)}$ that there is an x for which $\varphi(x)$ then there is an element x_0 in $\mathbb{V}^{(B)}$ (in the sense of \mathbb{V}) for which $\llbracket \varphi(x_0) \rrbracket = \mathbb{1}$. In symbols,

$$\mathbb{V}^{(B)} \models (\exists x) \varphi(x) \rightarrow (\exists x_0) \mathbb{V}^{(B)} \models \varphi(x_0).$$

Thus, the *maximum principle* reads:

$$(\exists x_0 \in \mathbb{V}^{(B)}) \llbracket \varphi(x_0) \rrbracket = \bigvee_{x \in \mathbb{V}^{(B)}} \llbracket \varphi(x) \rrbracket$$

for each formula φ of ZFC.

The last equality accounts for the origin of the term “maximum principle.” The proof of the maximum principle is a simple application of the following

(3) Mixing Principle. Let $(b_\xi)_{\xi \in \Xi}$ be a *partition of unity* in B , i.e. a family of elements of a Boolean valued algebra B such that

$$\bigvee_{\xi \in \Xi} b_\xi = \mathbb{1}, \quad (\forall \xi, \eta \in \Xi) (\xi \neq \eta \rightarrow b_\xi \wedge b_\eta = \mathbb{0}).$$

For each family of elements $(x_\xi)_{\xi \in \Xi}$ of the universe $\mathbb{V}^{(B)}$ and each partition of unity $(b_\xi)_{\xi \in \Xi}$ there exists a (unique) mixing of (x_ξ) by (b_ξ) ; i.e. an element x of the separated universe $\mathbb{V}^{(B)}$ such that $b_\xi \leq \llbracket x = x_\xi \rrbracket$ for all $\xi \in \Xi$.

The *mixing* x of a family (x_ξ) by (b_ξ) is denoted as follows:

$$x = \underset{\xi \in \Xi}{\text{mix}}(b_\xi x_\xi) = \text{mix}\{b_\xi x_\xi \mid \xi \in \Xi\}.$$

3. ASCENDING AND DESCENDING

Boolean valued analysis consists primarily in comparative analysis of a mathematical object or idea simultaneously in two Boolean valued models. This is impossible to achieve without some dialog between the universes \mathbb{V} and $\mathbb{V}^{(B)}$. In other words, we need a smooth mathematical tool-kit for revealing interplay between the interpretations of one and the same fact in the two models \mathbb{V} and $\mathbb{V}^{(B)}$. The relevant technique rests on by the operations of canonical embedding, descent, and ascent to be presented below. We start with the canonical embedding of the von Neumann universe. Given $x \in \mathbb{V}$, we denote by the symbol x^\wedge the *standard name* of x in $\mathbb{V}^{(B)}$; i.e., the element defined by the following recursion schema:

$$\emptyset^\wedge := \emptyset, \quad \text{dom}(x^\wedge) := \{y^\wedge \mid y \in x\}, \quad \text{im}(x^\wedge) := \{\mathbb{1}\}.$$

Observe some properties of the mapping $x \mapsto x^\wedge$ we need in the sequel.

(1) For an arbitrary $x \in \mathbb{V}$ and a formula φ of ZFC we have

$$\llbracket (\exists y \in x^\wedge) \varphi(y) \rrbracket = \bigvee \{ \llbracket \varphi(z^\wedge) \rrbracket : z \in x \},$$

$$\llbracket (\forall y \in x^\wedge) \varphi(y) \rrbracket = \bigwedge \{ \llbracket \varphi(z^\wedge) \rrbracket : z \in x \}.$$

(2) If x and y are elements of \mathbb{V} then, by transfinite induction, we establish

$$x \in y \leftrightarrow \mathbb{V}^{(B)} \models x^\wedge \in y^\wedge,$$

$$x = y \leftrightarrow \mathbb{V}^{(B)} \models x^\wedge = y^\wedge.$$

In other words, the standard name can be considered as an embedding of \mathbb{V} into $\mathbb{V}^{(B)}$. Moreover, it is beyond a doubt that the standard name sends \mathbb{V} onto $\mathbb{V}^{(2)}$, which fact is demonstrated by the next proposition:

(3) The following holds:

$$(\forall u \in \mathbb{V}^{(2)}) (\exists! x \in \mathbb{V}) \mathbb{V}^{(B)} \models u = x^\wedge.$$

(4) A formula is called *bounded* or *restricted* if each bound variable in it is restricted by a bounded quantifier; i.e., a quantifier ranging over a specific set. The latter means that each bound variable x is restricted by a quantifier of the form $(\forall x \in y)$ or $(\exists x \in y)$ for some y .

Restricted Transfer Principle. For each bounded formula φ of ZFC and every collection $x_1, \dots, x_n \in \mathbb{V}$ the following holds:

$$\varphi(x_1, \dots, x_n) \leftrightarrow \mathbb{V}^{(B)} \models \varphi(x_1^\wedge, \dots, x_n^\wedge).$$

Henceforth, working in the separated universe $\overline{\mathbb{V}}^{(B)}$, we agree to preserve the symbol x^\wedge for the distinguished element of the class corresponding to x .

(5) Observe for example that the restricted transfer principle yields:

“ Φ is a correspondence from x to y ”

$$\leftrightarrow \mathbb{V}^{(B)} \models \text{“}\Phi^\wedge \text{ is a correspondence from } x^\wedge \text{ to } y^\wedge\text{”};$$

“ f is a function from x to y ” $\leftrightarrow \mathbb{V}^{(B)} \models \text{“}f^\wedge \text{ is a function from } x^\wedge \text{ to } y^\wedge\text{”}$

(moreover, $f(a)^\wedge = f^\wedge(a^\wedge)$ for all $a \in x$). Thus, the standard name can be considered as a covariant functor of the category of sets (or correspondences) in \mathbb{V} to an appropriate subcategory of $\mathbb{V}^{(2)}$ in the separated universe $\mathbb{V}^{(B)}$.

(6) A set X is *finite* if X coincides with the image of a function on a finite ordinal. In symbols, this is expressed as $\text{Fin}(X)$; hence,

$$\text{Fin}(X) := (\exists n)(\exists f)(n \in \omega \wedge f \text{ is a function} \wedge \text{dom}(f) = n \wedge \text{im}(f) = X)$$

(as usual $\omega := \{0, 1, 2, \dots\}$). Obviously, the above formula is not bounded. Nevertheless there is a simple transformation rule for the class of finite sets under the canonical embedding. Denote by $\mathcal{P}_{\text{Fin}}(X)$ the class of all finite subsets of X :

$$\mathcal{P}_{\text{Fin}}(X) := \{Y \in \mathcal{P}(X) \mid \text{Fin}(Y)\}.$$

For an arbitrary set X the following holds:

$$\mathbb{V}^{(B)} \models \mathcal{P}_{\text{Fin}}(X)^\wedge = \mathcal{P}_{\text{Fin}}(X^\wedge).$$

Given an arbitrary element x of the (separated) Boolean valued universe $\mathbb{V}^{(B)}$, we define the *descent* $x \downarrow$ of x as

$$x \downarrow := \{y \in \mathbb{V}^{(B)} \mid \llbracket y \in x \rrbracket = \mathbb{1}\}.$$

We list the simplest properties of descending:

(1) The class $x \downarrow$ is a set, i.e., $x \downarrow \in \mathbb{V}$ for each $x \in \mathbb{V}^{(B)}$. If $\llbracket x \neq \emptyset \rrbracket = \mathbb{1}$ then $x \downarrow$ is a nonempty set.

(2) Let $z \in \mathbb{V}^{(B)}$ and $\llbracket z \neq \emptyset \rrbracket = \mathbb{1}$. Then for every formula φ of ZFC we have

$$\llbracket (\forall x \in z) \varphi(x) \rrbracket = \bigwedge \{\llbracket \varphi(x) \rrbracket \mid x \in z \downarrow\},$$

$$\llbracket (\exists x \in z) \varphi(x) \rrbracket = \bigvee \{\llbracket \varphi(x) \rrbracket \mid x \in z \downarrow\}.$$

Moreover, there exists $x_0 \in z \downarrow$ such that $\llbracket \varphi(x_0) \rrbracket = \llbracket (\exists x \in z) \varphi(x) \rrbracket$.

(3) Let Φ be a correspondence from X to Y in $\mathbb{V}^{(B)}$. Thus, Φ , X , and Y are elements of $\mathbb{V}^{(B)}$ and, moreover, $\llbracket \Phi \subset X \times Y \rrbracket = \mathbb{1}$. There is a unique correspondence $\Phi \downarrow$ from $X \downarrow$ to $Y \downarrow$ such that

$$\Phi \downarrow (A \downarrow) = \Phi(A) \downarrow$$

for every nonempty subset A of the set X inside $\mathbb{V}^{(B)}$. The correspondence $\Phi \downarrow$ from $X \downarrow$ to $Y \downarrow$ of the above proposition is called the *descent* of the correspondence Φ from X to Y in $\mathbb{V}^{(B)}$.

(4) The descent of the composite of correspondences inside $\mathbb{V}^{(B)}$ is the composite of their descents:

$$(\Psi \circ \Phi) \downarrow = \Psi \downarrow \circ \Phi \downarrow.$$

(5) If Φ is a correspondence inside $\mathbb{V}^{(B)}$ then

$$(\Phi^{-1}) \downarrow = (\Phi \downarrow)^{-1}.$$

(6) Let Id_X be the identity mapping inside $\mathbb{V}^{(B)}$ of a set $X \in \mathbb{V}^{(B)}$. Then

$$(\text{Id}_X) \downarrow = \text{Id}_{X \downarrow}.$$

(7) Suppose that $X, Y, f \in \mathbb{V}^{(B)}$ are such that $\llbracket f : X \rightarrow Y \rrbracket = \mathbb{1}$, i.e., f is a mapping from X to Y inside $\mathbb{V}^{(B)}$. Then $f\downarrow$ is a unique mapping from $X\downarrow$ to $Y\downarrow$ for which

$$\llbracket f\downarrow(x) = f(x) \rrbracket = \mathbb{1} \quad (x \in X\downarrow).$$

By virtue of (1)–(7), we can consider the descent operation as a functor from the category of B -valued sets and mappings (correspondences) to the category of the usual sets and mappings (correspondences) (i.e., in the sense of \mathbb{V}).

(8) Given $x_1, \dots, x_n \in \mathbb{V}^{(B)}$, denote by $(x_1, \dots, x_n)^B$ the corresponding ordered n -tuple inside $\mathbb{V}^{(B)}$. Assume that P is an n -ary relation on X inside $\mathbb{V}^{(B)}$; i.e., $X, P \in \mathbb{V}^{(B)}$ and $\llbracket P \subset X^{n\wedge} \rrbracket = \mathbb{1}$, where $n \in \omega$. Then there exists an n -ary relation P' on $X\downarrow$ such that

$$(x_1, \dots, x_n) \in P' \leftrightarrow \llbracket (x_1, \dots, x_n)^B \in P \rrbracket = \mathbb{1}.$$

Slightly abusing notation, we denote the relation P' by the same symbol $P\downarrow$ and call it the *descent* of P .

Let $x \in \mathbb{V}$ and $x \subset \mathbb{V}^{(B)}$; i.e., let x be some set composed of B -valued sets or, in other words, $x \in \mathcal{P}(\mathbb{V}^{(B)})$. Put $\emptyset\uparrow := \emptyset$ and

$$\text{dom}(x\uparrow) = x, \quad \text{im}(x\uparrow) = \{\mathbb{1}\}$$

if $x \neq \emptyset$. The element $x\uparrow$ (of the separated universe $\mathbb{V}^{(B)}$, i.e., the distinguished representative of the class $\{y \in \mathbb{V}^{(B)} \mid \llbracket y = x\uparrow \rrbracket = \mathbb{1}\}$) is called the *ascent* of x .

(1) The following equalities hold for every $x \in \mathcal{P}(\mathbb{V}^{(B)})$ and every formula φ :

$$\llbracket (\forall z \in x\uparrow) \varphi(z) \rrbracket = \bigwedge_{y \in x} \llbracket \varphi(y) \rrbracket,$$

$$\llbracket (\exists z \in x\uparrow) \varphi(z) \rrbracket = \bigvee_{y \in x} \llbracket \varphi(y) \rrbracket.$$

Introducing the ascent of a correspondence $\Phi \subset X \times Y$, we have to bear in mind a possible distinction between the domain of departure X and the domain $\text{dom}(\Phi) := \{x \in X \mid \Phi(x) \neq \emptyset\}$. This circumstance is immaterial for the sequel; therefore, speaking of ascents, we always imply total correspondences; i.e., $\text{dom}(\Phi) = X$.

(2) Let $X, Y, \Phi \in \mathbb{V}^{(B)}$, and let Φ be a correspondence from X to Y . There exists a unique correspondence $\Phi\uparrow$ from $X\uparrow$ to $Y\uparrow$ inside $\mathbb{V}^{(B)}$ such that

$$\Phi\uparrow(A\uparrow) = \Phi(A)\uparrow$$

is valid for every subset A of the set $\text{dom}(\Phi)$ if and only if Φ is *extensional*; i.e., satisfies the condition

$$y_1 \in \Phi(x_1) \rightarrow \llbracket x_1 = x_2 \rrbracket \leq \bigvee_{y_2 \in \Phi(x_2)} \llbracket y_1 = y_2 \rrbracket$$

for $x_1, x_2 \in \text{dom}(\Phi)$. In this event, $\Phi\uparrow = \Phi'\uparrow$, where $\Phi' := \{(x, y)^B \mid (x, y) \in \Phi\}$. The element $\Phi\uparrow$ is called the *ascent* of the initial correspondence Φ .

(3) The composite of extensional correspondences is extensional. Moreover, the ascent of a composite is equal to the composite of the ascents (inside $\mathbb{V}^{(B)}$): On assuming that $\text{dom}(\Psi) \supset \text{im}(\Phi)$ we have

$$\mathbb{V}^{(B)} \models (\Psi \circ \Phi)\uparrow = \Psi\uparrow \circ \Phi\uparrow.$$

Note that if Φ and Φ^{-1} are extensional then $(\Phi\uparrow)^{-1} = (\Phi^{-1})\uparrow$. However, in general, the extensionality of Φ in no way guarantees the extensionality of Φ^{-1} .

(4) It is worth mentioning that if an extensional correspondence f is a function from X to Y then its ascent $f\uparrow$ is a function from $X\uparrow$ to $Y\uparrow$. Moreover, the extensionality property can be stated as follows:

$$\llbracket x_1 = x_2 \rrbracket \leq \llbracket f(x_1) = f(x_2) \rrbracket \quad (x_1, x_2 \in X).$$

Given a set $X \subset \mathbb{V}^{(B)}$, we denote by the symbol $\text{mix } X$ the set of all mixings of the form $\text{mix}(b_\xi x_\xi)$, where $(x_\xi) \subset X$ and (b_ξ) is an arbitrary partition of unity. The following propositions are referred to as the *rules for cancelling arrows* or the “*descending-and-ascending*” and “*ascending-and-descending*” rules. It seems smarter to call them *Escher rules*.²

(5) Let X and X' be subsets of $\mathbb{V}^{(B)}$ and $f : X \rightarrow X'$ be an extensional mapping. Suppose that $Y, Y', g \in \mathbb{V}^{(B)}$ are such that $\llbracket Y \neq \emptyset \rrbracket = \llbracket g : Y \rightarrow Y' \rrbracket = \mathbb{1}$. Then

$$X\uparrow\downarrow = \text{mix } X, \quad Y\downarrow\uparrow = Y; \quad f\uparrow\downarrow = f, \quad g\downarrow\uparrow = g.$$

(6) From (6) we easily infer the useful relation:

$$\mathcal{P}_{\text{Fin}}(X\uparrow) = \{\theta\uparrow \mid \theta \in \mathcal{P}_{\text{Fin}}(X)\}\uparrow.$$

Suppose that $X \in \mathbb{V}$, $X \neq \emptyset$; i.e., X is a nonempty set. Let the letter ι denote the standard name embedding $x \mapsto x^\wedge$ ($x \in X$). Then $\iota(X)\uparrow = X^\wedge$ and $X = \iota^{-1}(X^\wedge\downarrow)$. Using the above relations, we may extend the descent and ascent operations to the case in which Φ is a correspondence from X to $Y\downarrow$ and $\llbracket \Psi$ is a correspondence from X^\wedge to $Y \rrbracket = \mathbb{1}$, where $Y \in \mathbb{V}^{(B)}$. Namely, we put $\Phi\uparrow := (\Phi \circ \iota)\uparrow$ and $\Psi\downarrow := \Psi\downarrow \circ \iota$. In this case, $\Phi\uparrow$ is called the *modified ascent* of the correspondence Φ and $\Psi\downarrow$ is called the *modified descent* of the correspondence Ψ . (If the context excludes ambiguity then we briefly speak of ascents and descents using simple arrows.) It is easy to see that $\Psi\uparrow$ is a unique correspondence inside $\mathbb{V}^{(B)}$ satisfying the relation

$$\llbracket \Phi\uparrow(x^\wedge) = \Phi(x)\uparrow \rrbracket = \mathbb{1} \quad (x \in X).$$

Similarly, $\Psi\downarrow$ is a unique correspondence from X to $Y\downarrow$ satisfying the equality

$$\Psi\downarrow(x) = \Psi(x^\wedge)\downarrow \quad (x \in X).$$

If $\Phi := f$ and $\Psi := g$ are functions then these relations take the form

$$\llbracket f\uparrow(x^\wedge) = f(x) \rrbracket = \mathbb{1}, \quad g\downarrow(x) = g(x^\wedge) \quad (x \in X).$$

(1) A *Boolean set* or a *set with B -structure* or just a *B -set* is a pair (X, d) , where $X \in \mathbb{V}$, $X \neq \emptyset$, and d is a mapping from $X \times X$ to the Boolean algebra B such that for all $x, y, z \in X$ the following hold:

- (a) $d(x, y) = \mathbb{0} \leftrightarrow x = y$;
- (b) $d(x, y) = d(y, x)$;
- (c) $d(x, y) \leq d(x, z) \vee d(z, y)$.

An example of a B -set is given by each $\emptyset \neq X \subset \mathbb{V}^{(B)}$ if we put

$$d(x, y) := \llbracket x \neq y \rrbracket = \neg \llbracket x = y \rrbracket \quad (x, y \in X).$$

Another example is a nonempty X with the “discrete B -metric” d ; i.e., $d(x, y) = \mathbb{1}$ if $x \neq y$ and $d(x, y) = \mathbb{0}$ if $x = y$.

(2) Let (X, d) be some B -set. There exist an element $\mathcal{X} \in \mathbb{V}^{(B)}$ and an injection $\iota : X \rightarrow X' := \mathcal{X}\downarrow$ such that $d(x, y) = \llbracket \iota x \neq \iota y \rrbracket$ for all $x, y \in X$ and every element $x' \in X'$ admits the representation $x' = \text{mix}_{\xi \in \Xi} (b_\xi \iota x_\xi)$, where $(x_\xi)_{\xi \in \Xi} \subset X$ and $(b_\xi)_{\xi \in \Xi}$ is a partition of unity in B . The element $\mathcal{X} \in \mathbb{V}^{(B)}$ is referred to as the

²Hofstedter D. R. *Gödel, Escher, Bach: an Eternal Golden Braid. (20th Anniversary Edition)*. Basic Books, New York (1999).

Boolean valued realization of the B -set X . If X is a discrete B -set then $\mathcal{X} = X^\wedge$ and $\iota x = x^\wedge$ for all $x \in X$. If $X \subset \mathbb{V}^{(B)}$ then $\iota \uparrow$ is an injection from $X \uparrow$ to \mathcal{X} (inside $\mathbb{V}^{(B)}$).

A mapping f from a B -set (X, d) to a B -set (X', d') is said to be *nonexpanding* if $d(x, y) \geq d'(f(x), f(y))$ for all $x, y \in X$.

(3) Let X and Y be some B -sets, let \mathcal{X} and \mathcal{Y} be their Boolean valued realizations, and let ι and \varkappa be the corresponding injections $X \rightarrow \mathcal{X} \downarrow$ and $Y \rightarrow \mathcal{Y} \downarrow$. If $f : X \rightarrow Y$ is a nonexpanding mapping then there is a unique element $g \in \mathbb{V}^{(B)}$ such that $\llbracket g : \mathcal{X} \rightarrow \mathcal{Y} \rrbracket = \mathbb{1}$ and $f = \varkappa^{-1} \circ g \downarrow \circ \iota$. We also accept the notations $\mathcal{X} := \mathcal{F}^\sim(X) := X^\sim$ and $g := \mathcal{F}^\sim(f) := f^\sim$.

(4) Moreover, the following are valid:

(1) $\mathbb{V}^{(B)} \models f(A)^\sim = f^\sim(A^\sim)$ for $A \subset X$;

(2) If $g : Y \rightarrow Z$ is a contraction then $g \circ f$ is a contraction and $\mathbb{V}^{(B)} \models (g \circ f)^\sim = g^\sim \circ f^\sim$;

(3) $\mathbb{V}^{(B)} \models$ “ f^\sim is injective” if and only if f is a B -isometry;

(4) $\mathbb{V}^{(B)} \models$ “ f^\sim is surjective” if and only if $\bigvee \{d(f(x), y) \mid x \in X\} = \mathbb{1}$ for all $y \in Y$.

Recall that a *signature* is a 3-tuple $\sigma := (F, P, \mathfrak{a})$, where F and P are some (possibly, empty) sets and \mathfrak{a} is a mapping from $F \cup P$ to ω . If the sets F and P are finite then σ is a *finite signature*. In applications we usually deal with algebraic systems of finite signature.

An *n-ary operation* and an *n-ary predicate* on a B -set A are contractive mappings $f : A^n \rightarrow A$ and $p : A^n \rightarrow B$ respectively. By definition, f and p are *contractive* provided that

$$d(f(a_0, \dots, a_{n-1}), f(a'_0, \dots, a'_{n-1})) \leq \bigvee_{k=0}^{n-1} d(a_k, a'_k),$$

$$d_s(p(a_0, \dots, a_{n-1}), p(a'_0, \dots, a'_{n-1})) \leq \bigvee_{k=0}^{n-1} d(a_k, a'_k)$$

for all $a_0, a'_0, \dots, a_{n-1}, a'_{n-1} \in A$, where d is the B -metric of A , and d_s is the *symmetric difference* on B ; i.e., $d_s(b_1, b_2) := (b_1 x \wedge \neg b_2) \vee (\neg b_1 \wedge b_2)$.

Clearly, the above definitions depend on B and it would be cleaner to speak of B -operations, B -predicates, etc. We adhere to a simpler practice whenever this entails no confusion.

An *algebraic B-system* \mathfrak{A} of signature σ is a pair (A, ν) , where A is a nonempty B -set, the *underlying set* or *carrier* or *universe* of \mathfrak{A} , and ν is a mapping such that (a) $\text{dom}(\nu) = F \cup P$; (b) $\nu(f)$ is an $\mathfrak{a}(f)$ -ary operation on A for all $f \in F$; and (c) $\nu(p)$ is an $\mathfrak{a}(p)$ -ary predicate on A for all $p \in P$.

It is in common parlance to call ν the *interpretation* of \mathfrak{A} in which case the notation f^ν and p^ν are common substitutes for $\nu(f)$ and $\nu(p)$.

The signature of an algebraic B -system $\mathfrak{A} := (A, \nu)$ is often denoted by $\sigma(\mathfrak{A})$; while the carrier A of \mathfrak{A} , by $|\mathfrak{A}|$. Since $A^0 = \{\emptyset\}$, the nullary operations and predicates on A are mappings from $\{\emptyset\}$ to the set A and to the algebra B respectively. We agree to identify a mapping $g : \{\emptyset\} \rightarrow A \cup B$ with the element $g(\emptyset)$. Each nullary operation on A thus transforms into a unique member of A . Analogously, the set of all nullary predicates on A turns into the Boolean algebra B . If $F := \{f_1, \dots, f_n\}$ and $P := \{p_1, \dots, p_m\}$ then an algebraic B -system of signature σ is often written down as $(A, \nu(f_1), \dots, \nu(f_n), \nu(p_1), \dots, \nu(p_m))$ or even

$(A, f_1, \dots, f_n, p_1, \dots, p_m)$. In this event, the expression $\sigma = (f_1, \dots, f_n, p_1, \dots, p_m)$ is substituted for $\sigma = (F, P, \mathfrak{a})$.

We now address the B -valued interpretation of a first-order language. Consider an algebraic B -system $\mathfrak{A} := (A, \nu)$ of signature $\sigma := \sigma(\mathfrak{A}) := (F, P, \mathfrak{a})$.

Let $\varphi(x_0, \dots, x_{n-1})$ be a formula of signature σ with n free variables. Assume given $a_0, \dots, a_{n-1} \in A$. We are ready to define the truth-value $|\varphi|^{\mathfrak{A}}(a_0, \dots, a_{n-1}) \in B$ of a formula φ in the system \mathfrak{A} for the given values a_0, \dots, a_{n-1} of the variables x_0, \dots, x_{n-1} . The definition proceeds as usual by induction on the length of φ : Considering propositional connectives and quantifiers, we put

$$\begin{aligned} |\varphi \wedge \psi|^{\mathfrak{A}}(a_0, \dots, a_{n-1}) &:= |\varphi|^{\mathfrak{A}}(a_0, \dots, a_{n-1}) \wedge |\psi|^{\mathfrak{A}}(a_0, \dots, a_{n-1}); \\ |\varphi \vee \psi|^{\mathfrak{A}}(a_0, \dots, a_{n-1}) &:= |\varphi|^{\mathfrak{A}}(a_0, \dots, a_{n-1}) \vee |\psi|^{\mathfrak{A}}(a_0, \dots, a_{n-1}); \\ |\neg\varphi|^{\mathfrak{A}}(a_0, \dots, a_{n-1}) &:= \neg|\varphi|^{\mathfrak{A}}(a_0, \dots, a_{n-1}); \\ |(\forall x_0)\varphi|^{\mathfrak{A}}(a_1, \dots, a_{n-1}) &:= \bigwedge_{a_0 \in A} |\varphi|^{\mathfrak{A}}(a_0, \dots, a_{n-1}); \\ |(\exists x_0)\varphi|^{\mathfrak{A}}(a_1, \dots, a_{n-1}) &:= \bigvee_{a_0 \in A} |\varphi|^{\mathfrak{A}}(a_0, \dots, a_{n-1}). \end{aligned}$$

Now, the case of atomic formulas is in order. Suppose that $p \in P$ symbolizes an m -ary predicate, $q \in P$ is a nullary predicate, and t_0, \dots, t_{m-1} are terms of signature σ assuming values b_0, \dots, b_{m-1} at the given values a_0, \dots, a_{n-1} of the variables x_0, \dots, x_{n-1} . By definition, we let

$$\begin{aligned} |\varphi|^{\mathfrak{A}}(a_0, \dots, a_{n-1}) &:= \nu(q), \quad \text{if } \varphi = q^\nu; \\ |\varphi|^{\mathfrak{A}}(a_0, \dots, a_{n-1}) &:= \neg d(b_0, b_1), \quad \text{if } \varphi = (t_0 = t_1); \\ |\varphi|^{\mathfrak{A}}(a_0, \dots, a_{n-1}) &:= p^\nu(b_0, \dots, b_{m-1}), \quad \text{if } \varphi = p^\nu(t_0, \dots, t_{m-1}), \end{aligned}$$

where d is a B -metric on A .

Say that $\varphi(x_0, \dots, x_{n-1})$ is *valid* in \mathfrak{A} at the given values $a_0, \dots, a_{n-1} \in A$ of x_0, \dots, x_{n-1} and write $\mathfrak{A} \models \varphi(a_0, \dots, a_{n-1})$ provided that $|\varphi|^{\mathfrak{A}}(a_0, \dots, a_{n-1}) = \mathbb{1}_B$. Alternative expressions are as follows: $a_0, \dots, a_{n-1} \in A$ *satisfy* $\varphi(x_0, \dots, x_{n-1})$; or $\varphi(a_0, \dots, a_{n-1})$ *holds true* in \mathfrak{A} . In the case of the two-element Boolean algebra $B := \mathbb{2} := \{0, \mathbb{1}\}$, we arrive at the conventional definition of the validity of a formula in an algebraic system.

Recall that a closed formula φ of signature σ is a *tautology* if φ is valid on every algebraic $\mathbb{2}$ -system of signature σ .

Consider algebraic B -systems $\mathfrak{A} := (A, \nu)$ and $\mathfrak{D} := (D, \mu)$ of the same signature σ . The mapping $h : A \rightarrow D$ is a *homomorphism* of \mathfrak{A} to \mathfrak{D} provided that, for all $a_0, \dots, a_{n-1} \in A$, the following are valid:

- (1) $d_B(h(a_1), h(a_2)) \leq d_A(a_1, a_2)$;
- (2) $h(f^\nu) = f^\mu$ if $\mathfrak{a}(f) = 0$;
- (3) $h(f^\nu(a_0, \dots, a_{n-1})) = f^\mu(h(a_0), \dots, h(a_{n-1}))$ if $0 \neq n := \mathfrak{a}(f)$;
- (4) $p^\nu(a_0, \dots, a_{n-1}) \leq p^\mu(h(a_0), \dots, h(a_{n-1}))$, with $n := \mathfrak{a}(p)$.

A homomorphism h is called *strong* if

(5) $\mathfrak{a}(p) := n \neq 0$ for $p \in P$, and, for all $d_0, \dots, d_{n-1} \in D$ the following inequality holds:

$$\begin{aligned} & p^\mu(d_0, \dots, d_{n-1}) \\ & \geq \bigvee_{a_0, \dots, a_{n-1} \in A} \{p^\nu(a_0, \dots, a_{n-1}) \wedge d_D(d_0, h(a_0)) \wedge \dots \wedge d_D(d_{n-1}, h(a_{n-1}))\}. \end{aligned}$$

If a homomorphism h is injective and (1) and (4) are fulfilled with equality holding; then h is said to be a *isomorphism from \mathfrak{A} to \mathfrak{D}* . Undoubtedly, each surjective isomorphism h and, in particular, the identity mapping $\text{Id}_A : A \rightarrow A$ are strong homomorphisms. The composite of (strong) homomorphisms is a (strong) homomorphism. Clearly, if h is a homomorphism and h^{-1} is a homomorphism too, then h is an isomorphism.

Note again that in the case of the two-element Boolean algebra we come to the conventional concepts of homomorphism, strong homomorphism, and isomorphism.

Before giving a general definition of the descent of an algebraic system, consider the descent of the two-element Boolean algebra. Choose two arbitrary elements, $0, 1 \in \mathbb{V}^{(B)}$, satisfying $\llbracket 0 \neq 1 \rrbracket = \mathbb{1}_B$. We may for instance assume that $0 := 0_B^\wedge$ and $1 := \mathbb{1}_B^\wedge$.

(1) *The descent D of the two-element Boolean algebra $\{0, 1\}^B \in \mathbb{V}^{(B)}$ is a complete Boolean algebra isomorphic to B . The formulas*

$$\llbracket \chi(b) = 1 \rrbracket = b, \quad \llbracket \chi(b) = 0 \rrbracket = \neg b \quad (b \in B)$$

define an isomorphism $\chi : B \rightarrow D$.

Consider now an algebraic system \mathfrak{A} of signature σ^\wedge inside $\mathbb{V}^{(B)}$, and let $\llbracket \mathfrak{A} = (A, \nu)^B \rrbracket = \mathbb{1}$ for some $A, \nu \in \mathbb{V}^{(B)}$. The *descent* of \mathfrak{A} is the pair $\mathfrak{A} \downarrow := (A \downarrow, \mu)$, where μ is the function determined from the formulas:

$$\begin{aligned} \mu : f &\mapsto (\nu \downarrow (f)) \downarrow \quad (f \in F), \\ \mu : p &\mapsto \chi^{-1} \circ (\nu \downarrow (p)) \downarrow \quad (p \in P). \end{aligned}$$

Here χ is the above isomorphism of the Boolean algebras B and $\{0, 1\}^B \downarrow$.

In more detail, the modified descent $\nu \downarrow$ is the mapping with domain $\text{dom}(\nu \downarrow) = F \cup P$. Given $p \in P$, observe $\llbracket \mathfrak{A}(p)^\wedge = \mathfrak{A}^\wedge(p^\wedge) \rrbracket = \mathbb{1}$, $\llbracket \nu \downarrow (p) = \nu(p^\wedge) \rrbracket = \mathbb{1}$ and so

$$\mathbb{V}^{(B)} \models \nu \downarrow (p) : A^{a(f)^\wedge} \rightarrow \{0, 1\}^B.$$

It is now obvious that $(\nu \downarrow (p)) \downarrow : (A \downarrow)^{a(f)} \rightarrow D := \{0, 1\}^B \downarrow$ and we may put $\mu(p) := \chi^{-1} \circ (\nu \downarrow (p)) \downarrow$.

Let $\varphi(x_0, \dots, x_{n-1})$ be a fixed formula of signature σ in n free variables. Write down the formula $\Phi(x_0, \dots, x_{n-1}, \mathfrak{A})$ in the language of set theory which formalizes the proposition $\mathfrak{A} \models \varphi(x_0, \dots, x_{n-1})$. Recall that the formula $\mathfrak{A} \models \varphi(x_0, \dots, x_{n-1})$ determines an n -ary predicate on A or, which is the same, a mapping from A^n to $\{0, 1\}$. By the maximum and transfer principles, there is a unique element $|\varphi|^\mathfrak{A} \in \mathbb{V}^{(B)}$ such that

$$\llbracket |\varphi|^\mathfrak{A} : A^{n^\wedge} \rightarrow \{0, 1\}^B \rrbracket = \mathbb{1},$$

$$\llbracket |\varphi|^\mathfrak{A}(a \uparrow) = 1 \rrbracket = \llbracket \Phi(a(0), \dots, a(n-1), \mathfrak{A}) \rrbracket = \mathbb{1}$$

for all $a : n \rightarrow A \downarrow$. Instead of $|\varphi|^\mathfrak{A}(a \uparrow)$ we will write $|\varphi|^\mathfrak{A}(a_0, \dots, a_{n-1})$, where $a_l := a(l)$. Therefore, the formula

$$\mathbb{V}^{(B)} \models \text{“}\varphi(a_0, \dots, a_{n-1}) \text{ is valid in } \mathfrak{A}\text{”}$$

holds true if and only if $\llbracket \Phi(a_0, \dots, a_{n-1}, \mathfrak{A}) \rrbracket = \mathbb{1}$.

(2) Let \mathfrak{A} be an algebraic system of signature σ^\wedge inside $\mathbb{V}^{(B)}$. Then $\mathfrak{A} \downarrow$ is a universally complete algebraic B -system of signature σ . In this event,

$$\chi \circ |\varphi|^\mathfrak{A} \downarrow = |\varphi|^\mathfrak{A} \downarrow$$

for each formula φ of signature σ .

(3) Let \mathfrak{A} and \mathfrak{B} be algebraic systems of the same signature σ^\wedge inside $\mathbb{V}^{(B)}$. Put $\mathfrak{A}' := \mathfrak{A} \downarrow$ and $\mathfrak{B}' := \mathfrak{B} \downarrow$. Then, if h is a homomorphism (strong homomorphism)

inside $\mathbb{V}^{(B)}$ from \mathfrak{A} to \mathfrak{B} then $h' := h \downarrow$ is a homomorphism (strong homomorphism) of the B -systems \mathfrak{A}' and \mathfrak{B}' .

Conversely, if $h' : \mathfrak{A}' \rightarrow \mathfrak{B}'$ is a homomorphism (strong homomorphism) of algebraic B -systems then $h := h' \uparrow$ is a homomorphism (strong homomorphism) from \mathfrak{A} to \mathfrak{B} inside $\mathbb{V}^{(B)}$.

Let $\mathfrak{A} := (A, \nu)$ be an algebraic B -system of signature σ . Then there are \mathcal{A} and $\mu \in \mathbb{V}^{(B)}$ such that the following are fulfilled:

- (1) $\mathbb{V}^{(B)} \models “(\mathcal{A}, \mu) \text{ is an algebraic system of signature } \sigma^\wedge”$;
- (2) If $\mathfrak{A}' := (A', \nu')$ is the descent of (\mathcal{A}, μ) then \mathfrak{A}' is a universally complete algebraic B -system of signature σ ;
- (3) There is an isomorphism ι from \mathfrak{A} to \mathfrak{A}' such that $A' = \text{mix}(\iota(A))$;
- (4) For every formula φ of signature σ in n free variables, the equalities hold

$$\begin{aligned} |\varphi|^{\mathfrak{A}}(a_0, \dots, a_{n-1}) &= |\varphi|^{\mathfrak{A}'}(\iota(a_0), \dots, \iota(a_{n-1})) \\ &= \chi^{-1} \circ (|\varphi|^{\mathfrak{A}'}) \downarrow (\iota(a_0), \dots, \iota(a_{n-1})) \end{aligned}$$

for all $a_0, \dots, a_{n-1} \in A$ and χ the same as above.

4. THE REALS DESCEND INTO A K -SPACE

We are now ready to apply the technique of Boolean valued analysis to the algebraic system of paramount importance for mathematics, the field of real numbers.

By the transfer and maximum principles, there is an element $\mathcal{R} \in \mathbb{V}^{(B)}$ such that $\mathbb{V}^{(B)} \models “\mathcal{R} \text{ is an ordered field of the reals.}”$ It is obvious that inside $\mathbb{V}^{(B)}$ the field \mathcal{R} is unique up to isomorphism; i.e., if \mathcal{R}' is another field of the reals inside $\mathbb{V}^{(B)}$ then $\mathbb{V}^{(B)} \models “\mathcal{R} \text{ and } \mathcal{R}' \text{ are isomorphic.}”$

It is an easy matter to show that \mathbb{R}^\wedge is an Archimedean ordered field inside $\mathbb{V}^{(B)}$ and so we may assume that $\mathbb{V}^{(B)} \models “\mathbb{R}^\wedge \subset \mathcal{R} \text{ and } \mathcal{R} \text{ is the (metric) completion of } \mathbb{R}^\wedge.”$ Regarding the unity 1 of \mathbb{R} , notice that $\mathbb{V}^{(B)} \models “1 := 1^\wedge \text{ is an order unit of } \mathcal{R}.”$

Consider the descent $\mathcal{R} \downarrow$ of the algebraic system $\mathcal{R} := (|\mathcal{R}|, +, \cdot, 0, 1, \leq)$. By implication, we equip the descent of the underlying set of \mathcal{R} with the descended operations and order of \mathcal{R} . In more detail, the addition, multiplication, and order on $\mathcal{R} \downarrow$ appear in accord with the following rules:

$$x + y = z \leftrightarrow \llbracket x + y = z \rrbracket = \mathbb{1}, \quad xy = z \leftrightarrow \llbracket xy = z \rrbracket = \mathbb{1},$$

$$x \leq y \leftrightarrow \llbracket x \leq y \rrbracket = \mathbb{1}, \quad \lambda x = y \leftrightarrow \llbracket \lambda^\wedge x = y \rrbracket = \mathbb{1}$$

$$(x, y, z \in \mathcal{R} \downarrow, \lambda \in \mathbb{R}).$$

Gordon Theorem. *Let \mathcal{R} be the reals in $\mathbb{V}^{(B)}$. Assume further that $\mathcal{R} \downarrow$ stands for the descent $|\mathcal{R} \downarrow|$ of the underlying set of \mathcal{R} equipped with the descended operations and order. Then the algebraic system \mathcal{R} is a universally complete K -space.*

Moreover, there is a (canonical) isomorphism χ from the Boolean algebra B onto the base $H_{\mathcal{R} \downarrow}$ of $\mathcal{R} \downarrow$ such that the following hold:

$$\chi(b)x = \chi(b)y \leftrightarrow b \leq \llbracket x = y \rrbracket,$$

$$\chi(b)x \leq \chi(b)y \leftrightarrow b \leq \llbracket x \leq y \rrbracket$$

for all $x, y \in \mathcal{R}_\downarrow$ and $b \in B$.

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