On One Idea by Yu. G. Reshetnyak in Measure Theory

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Abstract. Yu. G. Reshetnyak proposed to consider majorization of measures for studying inequalities over convex surfaces as far back as in 1954. The talk touches the life and state-of-the art of majorization in convex geometry and subdifferential calculus.

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It was long ago in 1954 that Yu. G. Reshetnyak suggested in his Ph. D. thesis [1] to compare (positive) measures on the euclidean unit sphere S_{N-1} as follows.

A measure μ *(linearly) majorizes* or *dominates* a measure ν provided that to each decomposition of S_{N-1} into finitely many disjoint Borel sets $U_1, \dots, U_m$ there are measures $\mu_1, \dots, \mu_m$ whose sum is μ and for which every difference $\mu_k - \nu|_{U_k}$ annihilates all restrictions to S_{N-1} of linear functionals over $\mathbb{R}^N$. In symbols, $\mu \gg_{\mathbb{R}^N} \nu$.

Yu. G. Reshetnyak proved that

$$\int_{S_{N-1}} p\,d\mu \geqslant \int_{S_{N-1}} p\,d\nu$$

for each *sublinear* (= positively homogeneous subadditive) function p on $\mathbb{R}^N$ if $\mu \gg_{\mathbb{R}^N} \nu$.

This gave an important trick for generating positive linear functionals over various classes of convex surfaces and functions.

A similar idea was suggested by L. Loomis [2] in 1962 within Choquet theory:

A measure μ *(affinely) majorizes* a measure ν, both given on a compact convex subset Q of a locally convex space X, provided that to each decomposition of ν into finitely many summands $\nu_1, \dots, \nu_m$ there are measures $\mu_1, \dots, \mu_m$ whose sum is μ and for which every difference $\mu_k - \nu_k$ annihilates all restrictions to Q of affine functionals over X. In symbols, $\mu \gg_{\mathrm{Aff}(Q)} \nu$.

Cartier P., Fell J. M., and Meyer P. A. [3] proved in 1964 that

$$\int_Q f\,d\mu \geqslant \int_Q f\,d\nu$$

for each continuous convex function f on Q if and only if $\mu \gg_{\mathrm{Aff}(Q)} \nu$.

An analogous necessity part for linear majorization was published in 1970, cf. [4, 5].

Majorization in the convex case is famous since the celebrated theorem by I. Schur about diagonal entries and eigenvalues of a hermitian matrix, cf. [6].

This article touches the life of the Reshetnyak idea, inspecting some of its modern versions and applications.

Majorization in Convex Geometry

Let $\mathcal{V}_N$ stand for the set of all *convex figures*, i. e. compact convex sets in the N-dimensional euclidean space $\mathbb{R}^N$. Dealing with convex figures, we usually treat them sociologically as members of the well-known two vector spaces or *parametrizations*, the *Minkowski* and *Blaschke structures*.

The first vector structure rests on the classical *Minkowski duality* which identifies a convex figure $\mathfrak{x}$ that lies in $\mathbb{R}^N$ with its *support function* $\mathfrak{x}(z) := \sup\{(x, z) \mid x \in \mathfrak{x}\}$ for $z \in \mathbb{R}^N$. Considering the members of $\mathbb{R}^N$ as singletons belonging to $\mathcal{V}_N$, note that $\mathbb{R}^N$ lies in $\mathcal{V}_N$. The Minkowski duality induces in $\mathcal{V}_N$

the structure of a cone in the space $C(S_{N-1})$ of continuous functions on the euclidean unit sphere S_{N-1}, the boundary of the unit ball $\mathfrak{z}_N$. This parametrization is the *Minkowski structure.* Addition of the support functions of convex figures amounts to passing to the algebraic sum of the latter, also called the *Minkowski addition.* It is worth observing that the *linear span* $[\mathcal{V}_N]$ of the cone $\mathcal{V}_N$ is dense in $C(S_{N-1})$.

The second parametrization, *Blaschke structure*, results from identifying the coset of translates $\{z+\mathfrak{x} \mid z \in \mathbb{R}^N\}$ of a *convex body* $\mathfrak{x}$, which is by definition a convex figure with nonempty interior, and the corresponding measure on the unit sphere which we call the *surface area measure* of the coset of $\mathfrak{x}$ and denote by $\mu(\mathfrak{x})$. The soundness of this parametrization rests on the celebrated Alexandrov Theorem of recovering a convex surface from its surface area measure. Each surface area measure is an *Alexandrov measure.* So we call a positive measure on the unit sphere which is supported by no great hypersphere and which annihilates singletons. The last property of a measure is referred to as translation invariance in the theory of convex surfaces. Thus, each Alexandrov measure is a translation-invariant linear functional over the cone $\mathcal{V}_N$. The cone of positive translation-invariant measures in the dual $C'(S_{N-1})$ of $C(S_{N-1})$ is denoted by $\mathcal{A}_N$. We now agree on some preliminaries.

Given $\mathfrak{x}, \mathfrak{y} \in \mathcal{V}_N$, let the record $\mathfrak{x} =_{\mathbb{R}^N} \mathfrak{y}$ mean that $\mathfrak{x}$ and $\mathfrak{y}$ are equal up to translation or, in other words, are translates of one another. Note that $=_{\mathbb{R}^N}$ is the equivalence associated with the preorder $\geqslant_{\mathbb{R}^N}$ on $\mathcal{V}_N$ symbolizing the possibility of inserting one figure into the other by translation. Arrange the factor set $\mathcal{V}_N/\mathbb{R}^N$ which consists of the cosets of translates of the members of $\mathcal{V}_N$. Clearly, $\mathcal{V}_N/\mathbb{R}^N$ is a cone in the factor space $[\mathcal{V}_N]/\mathbb{R}^N$ of the vector space $[\mathcal{V}_N]$ by the subspace $\mathbb{R}^N$.

Some natural bijection exists between $\mathcal{V}_N/\mathbb{R}^N$ and $\mathcal{A}_N$. Namely, we identify the coset of singletons with the zero measure. To the straight line segment with endpoints x and y, we assign the measure

$$|x-y|(\varepsilon_{(x-y)/|x-y|} + \varepsilon_{(y-x)/|x-y|}),$$

where $|\cdot|$ stands for the euclidean norm and the symbol ε_z for $z \in S_{N-1}$ stands for the *Dirac measure* supported at z. If the dimension of the affine span $\operatorname{Aff}(\mathfrak{x})$ of a representative $\mathfrak{x}$ of a coset in $\mathcal{V}_N/\mathbb{R}^N$ is greater than unity, then we assume that $\operatorname{Aff}(\mathfrak{x})$ is a subspace of $\mathbb{R}^N$ and identify this class with the surface area measure of $\mathfrak{x}$ in $\operatorname{Aff}(\mathfrak{x})$ which is some measure on $S_{N-1} \cap \operatorname{Aff}(\mathfrak{x})$ in this event. Extending the measure by zero to a measure on S_{N-1}, we obtain the member of $\mathcal{A}_N$ that we assign to the coset of all translates of $\mathfrak{x}$. The fact that this correspondence is one-to-one follows easily from the Alexandrov Theorem.

The vector space structure on the set of regular Borel measures induces in $\mathcal{A}_N$ and, hence, in $\mathcal{V}_N/\mathbb{R}^N$ the structure of a cone or, strictly speaking, the structure of a commutative $\mathbb{R}_+$-operator semigroup with cancellation. This structure on $\mathcal{V}_N/\mathbb{R}^N$ is called the *Blaschke structure.* Note that the sum of the surface area measures of $\mathfrak{x}$ and $\mathfrak{y}$ generates a unique class $\mathfrak{x}\#\mathfrak{y}$ which is referred to as the *Blaschke sum* of $\mathfrak{x}$ and $\mathfrak{y}$.

Let $C(S_{N-1})/\mathbb{R}^N$ stand for the factor space of $C(S_{N-1})$ by the subspace of all restrictions of linear functional on $\mathbb{R}^N$ to S_{N-1}. Denote by $[\mathcal{A}_N]$ the space $\mathcal{A}_N - \mathcal{A}_N$ of translation-invariant measures. It is easy to see that $[\mathcal{A}_N]$ is also the linear span of the set of Alexandrov measures. The spaces $C(S_{N-1})/\mathbb{R}^N$ and $[\mathcal{A}_N]$ are set in duality by the canonical bilinear form

$$\langle f, \mu \rangle = \frac{1}{N} \int_{S_{N-1}} f d\mu \quad (f \in C(S_{N-1})/\mathbb{R}^N,\ \mu \in [\mathcal{A}_N]).$$

For $\mathfrak{x} \in \mathcal{V}_N/\mathbb{R}^N$ and $\mathfrak{y} \in \mathcal{A}_N$, the quantity $\langle \mathfrak{x}, \mathfrak{y} \rangle$ coincides with the *mixed volume* $V_1(\mathfrak{y}, \mathfrak{x})$. The space $[\mathcal{A}_N]$ is usually furnished with the weak topology induced by the above indicated duality with $C(S_{N-1})/\mathbb{R}^N$.

By the *dual* K^* of a given cone K in a vector space X in duality with another vector space Y, we mean the set of all positive linear functionals on K; i. e., $K^* := \{y \in Y \mid (\forall x \in K)\langle x, y \rangle \geqslant 0\}$. Recall also that to a convex subset U of X and a point $\bar{x}$ in U there corresponds the cone

$$U_{\bar{x}} := \operatorname{Fd}(U, \bar{x}) := \{h \in X \mid (\exists \alpha \geqslant 0)\ \bar{x} + \alpha h \in U\}$$

which is called the *cone of feasible directions* to U at $\bar{x}$. Fortunately, description is available for all dual cones we need.

1.1. *The dual $\mathcal{A}_N^*$ of $\mathcal{A}_N$ is the positive cone of $C(S_{N-1})/\mathbb{R}^N$.*

1.2. *Let $\bar{\mathfrak{x}} \in \mathcal{A}_N$. Then the dual $\mathcal{A}_{n,\bar{\mathfrak{x}}}^*$ of the cone of feasible directions at $\bar{\mathfrak{x}}$ may be represented as follows*

$$\mathcal{A}_{n,\bar{\mathfrak{x}}}^* = \{f \in \mathcal{A}_N^* \mid \langle \bar{\mathfrak{x}}, f \rangle = 0\}.$$

1.3. *Let* $\mathfrak{x}$ *and* $\mathfrak{y}$ *be convex figures. Then*
(1) $\mu(\mathfrak{x}) - \mu(\mathfrak{y}) \in \mathcal{V}_N^* \leftrightarrow \mu(\mathfrak{x}) \gg_{\mathbb{R}^N} \mu(\mathfrak{y})$;
(2) *If* $\mathfrak{x} \geqslant_{\mathbb{R}^N} \mathfrak{y}$ *then* $\mu(\mathfrak{x}) \gg_{\mathbb{R}^N} \mu(\mathfrak{y})$;
(3) $\mathfrak{x} \geqslant_{\mathbb{R}^2} \mathfrak{y} \leftrightarrow \mu(\mathfrak{x}) \gg_{\mathbb{R}^2} \mu(\mathfrak{y})$.

1.4. *Let* $\bar{\mathfrak{x}}$ *and* $\mathfrak{y}$ *be convex figures. Then*
(1) *If* $\mathfrak{y} - \bar{\mathfrak{x}} \in \mathcal{A}^*_{N,\bar{\mathfrak{x}}}$ *then* $\mathfrak{y} =_{\mathbb{R}^N} \bar{\mathfrak{x}}$;
(2) *If* $\mu(\mathfrak{y}) - \mu(\bar{\mathfrak{x}}) \in \mathcal{V}^*_{N,\bar{\mathfrak{x}}}$ *then* $\mathfrak{y} =_{\mathbb{R}^N} \bar{\mathfrak{x}}$.

In the sequel we never distinguish between a convex figure, the respective coset of translates in $\mathcal{V}_N/\mathbb{R}^N$, and the corresponding measure in $\mathcal{A}_N$.

It is worth of noting that the volume $V(\mathfrak{x}) := \langle \mathfrak{x}, \mathfrak{x} \rangle$ of a convex figure $\mathfrak{x}$ is a homogeneous polynomial of degree N with respect to the Minkowski structure. That is why to calculate the subdifferential of $V(\cdot)$ is an easy matter. The particular feature of the Minkowski structure is a complicated construction of the dual of the cone of compact convex sets whose description bases on the relation $\gg_{\mathbb{R}^N}$ in the space of measures $[\mathcal{A}_N]$. If we use the Blaschke addition in the space of dimension $N \geqslant 3$ then the dual of the cone of convex surfaces is rather simple whereas volume fails to be a homogeneous polynomial, which complicates analysis.

In the sequel we use the following notations:

$$p : \mathfrak{x} \mapsto V^{1/N}(\mathfrak{x}) \quad (\mathfrak{x} \in \mathcal{V}_N/\mathbb{R}^N); \quad \widehat{p} : \mathfrak{x} \mapsto V^{(N-1)/N}(\mathfrak{x}) \quad (\mathfrak{x} \in \mathcal{A}_N).$$

The *Minkowski inequality* is thus paraphrased as $\langle \mathfrak{x}, \mathfrak{y} \rangle \geqslant p(\mathfrak{x})\widehat{p}(\mathfrak{y})$.

1.5. Brunn–Minkowski Theorem. *The functional* p *is superlinear on the cone* $\mathcal{V}_N$.

The following important proposition was most likely known to H. Minkowski.

1.6. *The functional* $\widehat{p}$ *is superlinear on the cone* $\mathcal{A}_N$.

1.7. Herglotz Theorem. *The function* p *is concave on the convex set* $\mathcal{A}_N$.

We now illustrate majorization with isoperimetric problems. For simplicity, let us start in two dimensions.

1.8. Internal Isoperimetric Problem. *Among the convex figures lying in a fixed convex body* $\mathfrak{x}_0$ *and having perimeter equal to* $S(\bar{\mathfrak{x}})$*, find a figure of maximal area.*

A convex body is sometimes identified with its boundary and so referred to as a (closed) *convex surface*. Existence is easy for Problem 1.8 and its analogs on using the Blaschke Choice Theorem which proclaims that the set is compact of convex figures lying in a fixed convex figure. Uniqueness of a solution to within translation rests on the strict convexity of volume which amounts to the conditions of equality holding in the isoperimetric inequality.

1.9. Optimality Criterion. *A feasible convex body* $\bar{\mathfrak{x}}$ *is a solution to the internal isoperimetric problem if and only if there are a convex figure* $\mathfrak{x} \in \mathcal{V}_2$ *and a real* $\bar{\alpha} \in \mathbb{R}_+$ *satisfying*
(1) $\bar{\mathfrak{x}} =_{\mathbb{R}^2} \mathfrak{x} + \bar{\alpha}\mathfrak{z}_2$;
(2) $\bar{\mathfrak{x}}(z) = \mathfrak{x}_0(z)$ *for all* z *in* $\operatorname{supp}(\mathfrak{x})$.

Abstracting Problem 1.8, we may replace the condition on the perimeter calculated with respect to the classical euclidean metric by a constraint on the perimeter in an arbitrary Minkowski geometry defined by a possibly asymmetric conical segment. Such a perimeter is simply the *mixed area* with an appropriate convex figure. A solution to the so-modified problem remains to be some "parallel" set to a *critical figure* x. In the general case a solution is the sum of a critical figure and a scaled polar of the unit disk of the original Minkowski geometry.

1.10. External Isoperimetric Problem. *Among the convex figures that include a fixed convex figure* $\mathfrak{x}_0$ *and has perimeter equal to* $S(\bar{\mathfrak{x}})$*, find a convex figure of greatest area.*

An equivalent convex program is stated as follows:
(1) $\mathfrak{x} \in \mathcal{V}$;
(2) $-\mathfrak{x} \leqslant -\mathfrak{x}_0$;
(3) $S(\mathfrak{x}) \leqslant S(\bar{\mathfrak{x}})$;

(4) $p(\mathfrak{x}) \to \max$.

The presence of the minus sign in (2) complicates the optimality criterion rather than its derivation since the dual $\mathcal{V}^*_{2,\bar{\mathfrak{x}}}$ of the cone of feasible directions at an irregular convex figure $\bar{\mathfrak{x}}$ differs from zero in general. To obviate this obstacle by appealing to 1.4(2) is impossible in contrast to the case of Problem 1.8, since easy examples show that not all elements of $\mathcal{V}^*_{2,\bar{\mathfrak{x}}}$ are of the shape $\mathfrak{y} - \bar{\mathfrak{x}}$.

1.11. Optimality Criterion. *A feasible convex body $\bar{\mathfrak{x}}$ is a solution to the external isoperimetric problem if and only if there are a critical figure $\mathfrak{x}$ and a positive real $\bar{\alpha}$ satisfying*

(1) $\bar{\alpha}\mathfrak{z}_2 \geqslant_{\mathbb{R}^2} \mathfrak{x} + \bar{\mathfrak{x}}$;

(2) $\bar{\mathfrak{x}}(z) + \mathfrak{x}(z) = \bar{\alpha}\mathfrak{z}_2(z)$ *for all* $z \in \operatorname{supp}(\bar{\mathfrak{x}})$;

(3) $\bar{\mathfrak{x}}(z) = \mathfrak{x}_0(z)$ *for all* $z \in \operatorname{supp}(\mathfrak{x})$.

Duality analysis of isoperimetric-type problems in $\mathbb{R}^N$ for $N \geqslant 3$ has a few particularities which may be summarized as follows:

Replace support functions with surface area measures in many dimensions.

We now illustrate this motto by example.

1.12. External Urysohn Problem. *Among the convex figures, including $\mathfrak{x}_0$ and having integral width fixed, find a convex body of greatest volume.*

1.13. Optimality Criterion. *A feasible convex body $\bar{\mathfrak{x}}$ is a solution to Problem 1.13 if and only if there are a positive critical measure μ and a positive real $\bar{\alpha} \in \mathbb{R}_+$ satisfying*

(1) $\bar{\alpha}\mu(\mathfrak{z}_N) \gg_{\mathbb{R}^N} \mu(\bar{\mathfrak{x}}) + \mu$;

(2) $V(\bar{\mathfrak{x}}) + \frac{1}{N}\int_{S_{N-1}} \bar{\mathfrak{x}}\,d\mu = \bar{\alpha}V_1(\mathfrak{z}_N, \bar{\mathfrak{x}})$;

(3) $\bar{\mathfrak{x}}(z) = \mathfrak{x}_0(z)$ *for all z in the support of μ.*

If, in particular, $\mathfrak{x}_0 = \mathfrak{z}_{N-1}$ then the sought body is a *spherical lens*, that is, the intersection of two balls of the same radius; while the critical measure is the restriction of the surface area measure of the ball of radius $\bar{\alpha}^{1/(N-1)}$ to the complement of the support of the lens to S_{N-1}. If $\mathfrak{x}_0 = \mathfrak{z}_1$ and $N = 3$ then our result implies that we should seek a solution in the class of the so-called spindle-shaped constant-width surfaces of revolution.

See [7] for more details on majorization and isoperimetric-type problems in convex geometry.

Majorization in Convex Analysis

The following chant, *to each decomposition of ν into m summands there is a decomposition of μ such that the kth difference of the fragments is positive on some cone H_k for $k = 1, \dots, m$*, clearly amounts to the inequality

$$\mu(h_1 \vee \cdots \vee h_m) \geqslant \nu(h_1 \vee \cdots \vee h_m)$$

for all $h_k \in H_k$ and $k = 1, \dots, m$. Analytically, we see two composites with the sublinear operator of taking a supremum and inclusion of their subdifferentials. It turns fruitful to abstract the situation as follows:

Consider a *K-space* E, i. e. an order complete vector lattice, and an arbitrary nonempty set $\mathfrak{A}$. Denote by $l_\infty(\mathfrak{A}, E)$ the set of all (order) bounded mappings from $\mathfrak{A}$ into E; i.e., $f \in l_\infty(\mathfrak{A}, E)$ if and only if $f : \mathfrak{A} \to E$ and the set $\{f(\alpha) : \alpha \in \mathfrak{A}\}$ is order bounded in E. It is easy to verify that $l_\infty(\mathfrak{A}, E)$, endowed with the coordinatewise algebraic operations and order, is a K-space. The operator $\varepsilon_{\mathfrak{A},E}$ acting from $l_\infty(\mathfrak{A}, E)$ into E by the rule

$$\varepsilon_{\mathfrak{A},E} : f \mapsto \sup\{f(\alpha) : \alpha \in \mathfrak{A}\} \quad (f \in l_\infty(\mathfrak{A}, E))$$

is called the *canonical sublinear operator* given $\mathfrak{A}$ and E. We often write $\varepsilon_{\mathfrak{A}}$ instead of $\varepsilon_{\mathfrak{A},E}$ when it is clear from the context what K-space is meant. The notation ε_m is used when the cardinality of the set $\mathfrak{A}$ equals m and the operator ε_m is called *finitely-generated*.

Let X and E be ordered vector spaces. An operator $p : X \to E$ is called *increasing* or *isotonic* if for all $x_1, x_2 \in X$ from $x_1 \leqslant x_2$ it follows that $p(x_1) \leqslant p(x_2)$. An increasing linear operator is also called *positive*. As usual, the collection of all positive linear operators in the space $L(X, E)$ of all linear operators is denoted by $L^+(X, E)$. Obviously, the positivity of a linear operator T amounts to the inclusion $T(X^+) \subset E^+$, where $X^+ := \{x \in X : x \geqslant 0\}$ and $E^+ := \{e \in E : e \geqslant 0\}$ are the *positive cones* in X and E respectively.

Recall that the set $\partial p := \partial p(0) = \{T \in L(X,\ E) : (\forall x \in X)\ Tx \leqslant p(x)\}$ is the *subdifferential* (at zero) or the *support set* of a sublinear operator p.

2.1. *A sublinear operator p from an ordered vector space X into a K-space E is increasing if and only if its support set ∂p consists of positive operators, i.e. $\partial p \subset L^+(X,\ E)$.*

2.2. *A canonical operator is increasing and sublinear. A finitely-generated canonical operator is order continuous.*

Consider a set $\mathfrak{A}$ of linear operators acting from a vector space X into a K-space E. The set $\mathfrak{A}$ is *weakly* (*order*) *bounded* if the set $\{\alpha x : \alpha \in \mathfrak{A}\}$ is order bounded for every $x \in X$. We denote by $\langle\mathfrak{A}\rangle x$ the mapping that assigns the element $\alpha x \in E$ to each $\alpha \in \mathfrak{A}$, i.e. $\langle\mathfrak{A}\rangle x : \alpha \mapsto \alpha x$. If $\mathfrak{A}$ is weakly order bounded, then $\langle\mathfrak{A}\rangle x \in l_\infty(\mathfrak{A},\ E)$ for every fixed $x \in X$. Consequently, we obtain the linear operator $\langle\mathfrak{A}\rangle : X \to l_\infty(\mathfrak{A},\ E)$ that acts as $\langle\mathfrak{A}\rangle : x \mapsto \langle\mathfrak{A}\rangle x$. Associate with $\mathfrak{A}$ one more operator

$$p_{\mathfrak{A}} : x \mapsto \sup\{\alpha x : \alpha \in \mathfrak{A}\} \quad (x \in X).$$

The operator $p_{\mathfrak{A}}$ is sublinear. The support set $\partial p_{\mathfrak{A}}$ is denoted by $\operatorname{cop}(\mathfrak{A})$ and is called the *support hull* of $\mathfrak{A}$. These definitions entail the following statement:

2.3. *If p is a sublinear operator with $\partial p = \operatorname{cop}(\mathfrak{A})$, then the representation holds*

$$p = \varepsilon_{\mathfrak{A}} \circ \langle\mathfrak{A}\rangle.$$

Clearly, $\partial p = \operatorname{cop}(\partial p)$. Consequently, every sublinear operator $p : X \to E$ admits the above representation with $\mathfrak{A} := \partial p$. Due to this fact the canonical sublinear operator is very useful in various problems connected with sublinear operators and, particularly, in calculating support sets and support hulls.

Let $\text{´}_{\mathfrak{A}} := \text{´}_{\mathfrak{A},E}$ be the embedding of E into $l_\infty(\mathfrak{A},\ E)$ which assigns the constant mapping $\alpha \mapsto e$ $(\alpha \in \mathfrak{A})$ to every element $e \in E$ so that $(\text{´}_{\mathfrak{A}} e)(\alpha) = e$ for all $\alpha \in \mathfrak{A}$.

2.4. *The following relations are true*:

$$\varepsilon_{\mathfrak{A},E} \circ \text{´}_{\mathfrak{A},E} = I_E, \quad \text{´}_{\mathfrak{A},E} \circ \varepsilon_{\mathfrak{A},E}(f) \geqslant f \quad (f \in l_\infty(\mathfrak{A},\ E)),$$

where I_E is the identity mapping in E.

2.5. *Let F be another K-space and $p : E \to F$ be an increasing sublinear operator. Then*

$$\partial\left(p \circ \varepsilon_{\mathfrak{A},E}\right) = \left\{T \in L^+(l_\infty(\mathfrak{A},\ E),\ F) : T \circ \text{´}_{\mathfrak{A}} \in \partial p\right\}.$$

This reveals the nature of decompositions we use in majorization. Moreover, we are now in a position to describe the subdifferential of a composite sublinear operator by means of decompositions.

2.6. *For the support set of a canonical sublinear operator the following representation holds*:

$$\partial\varepsilon_{\mathfrak{A},E} = \left\{\alpha \in L^+\left(l_\infty(\mathfrak{A},\ E),\ E\right) : \alpha \circ \text{´}_{\mathfrak{A},E} = I_E\right\}.$$

2.7. *For each weakly order bounded set $\mathfrak{A}$ of linear operators the equality holds*:

$$\operatorname{cop}(\mathfrak{A}) = \partial\varepsilon_{\mathfrak{A}} \circ \langle\mathfrak{A}\rangle.$$

The time has come to demonstrate how the canonical operator technique applies to subdifferential calculus. For instance, we calculate the support set of a composite sublinear operator.

2.8. Theorem. *Let $p_1 : X \to E$ be a sublinear operator and let $p_2 : E \to F$ be an increasing sublinear operator. Then*

$$\partial(p_2 \circ p_1) = \left\{T \circ \langle\partial p_1\rangle : T \in L^+(l_\infty(\partial p_1,\ E),\ F)\ \&\ T \circ \text{´}_{\partial p_1} \in \partial p_2\right\}.$$

Furthermore, if $\partial p_1 = \operatorname{cop}(\mathfrak{A}_1)$ and $\partial p_2 = \operatorname{cop}(\mathfrak{A}_2)$, then

$$\partial(p_2 \circ p_1) = \left\{T \circ \langle\mathfrak{A}_1\rangle : T \in L^+(l_\infty(\mathfrak{A}_1,\ E),\ F)\ \&\ \left(\exists\alpha \in \partial\varepsilon_{\mathfrak{A}_2}\right)\ T \circ \text{´}_{\mathfrak{A}_1} = \alpha \circ \langle\mathfrak{A}_2\rangle\right\}.$$

See [8] for more details on subdifferential calculus.

Conclusion

The above glimpses at the developments arising from the Reshetnyak "offhand" idea prove its vitality, so enjoying the happy author.

References

[1] Reshetnyak Yu. G., *On the Length and Swerve of a Curve and the Area of a Surface.* (Ph. D. Thesis, unpublished), Leningrad State University, 1954. (Russian)
[2] Loomis L., *Unique direct integral decomposition on convex sets*, Amer. Math. J. **84** n. 3 (1962), 509–526.
[3] Cartier P., Fell J. M., and Meyer P. A., *Comparaison des mesures poertÂes par un ensemble convexe compact*, Bull. Soc. Math. France **94** (1964), 435–445. (French)
[4] Kutateladze S. S., *Positive Minkowski-linear functionals over convex surfaces*, Soviet Math. Dokl. **11** n. 3 (1970), 767-769.
[5] Kutateladze S. S., *Choquet boundaries in K-spaces*, Russian Math. Surv. **30** n. 4 (1975), 115–155.
[6] Marshall A. W. and Olkin I., *Inequalities: Theory of Majorization and Its Applications*, Academic Press, New York etc., 1979.
[7] Kutateladze S. S., *Parametrization of isoperimetric-type problems in convex geometry*, Siberian Advances in Mathematics **9** n. 3 (1999), 132-143.
[8] Kusraev A. G. and Kutateladze S. S., *Subdifferentials*: *Theory and Applications*, Kluwer, Dordrecht, 1995.

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