Stone–Weierstrass Approximation Revisited

A. G. Kusraev and S. S. Kutateladze

October 27, 2022

Agenda

- Analysis is the technique of differentiation and integration. Differentiation discovers trends, and integration forecasts the future from trends.
- Calculus reduces forecast to numbers, which is scalarization in modern parlance. Some aspects of the latter will be revealed by extending the classical Stone–Weierstrass Approximation Theorem.

Kutateladze, Luxemburg, and Kusraev

Nonstandard Tool Kits

- A model within set theory is *nonstandard* if the membership between the objects of the model differs from that of the originals. In fact the nonstandard tools of today use a couple of set-theoretic models simultaneously. The most popular are *infinitesimal analysis* and *Boolean-valued analysis* .
- Infinitesimal analysis provides us with a novel understanding for the method of indivisibles or monadology, synthesizing the two approaches to calculus which belong to the inventors.
- Boolean valued analysis originated with the famous works by Paul Cohen on the continuum hypothesis and distinguishes itself by the technique of ascending and descending, cyclic envelopes and mixings, and B-sets.

Enter the Reals

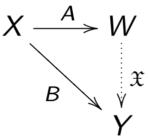
- To operate with preferences, we use group structure. To aggregate and scale, we use linear structure.
- All these are happily provided by the *reals* $\mathbb{R}$, a one-dimensional Dedekind complete vector lattice. A Dedekind complete vector lattice is a *Kantorovich space*.
- Since each number is a measure of quantity, the idea of reducing to numbers is of a universal importance to mathematics. Model theory provides justification of the *Kantorovich heuristic principle* that the members of his spaces are numbers as well.

Environment for Inequality

- Assume that X is a real vector space, Y is a *Kantorovich space*. Let $\mathbb{B} := \mathbb{B}(Y)$ be the *base* of Y, i.e., the complete Boolean algebras of positive projections in Y; and let $m(Y)$ be the universal completion of Y. Denote by $L(X, Y)$ the space of linear operators from X to Y. In case X is furnished with some Y-seminorm on X, by $L^{(m)}(X, Y)$ we mean the *space of dominated operators* from X to Y. As usual, $\{T \leq 0\} := \{x \in X \mid Tx \leq 0\}$; $\ker(T) = T^{-1}(0)$ for $T : X \to Y$. The superscript $^{(m)}$ suggests domination.

Operator Equations

- Find $\mathfrak{X}$ satisfying



- Sard Theorem. $(\exists \mathfrak{X})\ \mathfrak{X}A = B \leftrightarrow \ker(A) \subset \ker(B)$.
- Kantorovich Theorem. *If W is ordered by W_+ and $A(X) - W_+ = W_+ - A(X) = W$, then*

$$(\exists \mathfrak{X} \geq 0)\ \mathfrak{X}A = B \leftrightarrow \{A \leq 0\} \subset \{B \leq 0\}.$$

The Farkas Alternative

- *Let X be a Y-seminormed real vector space, with Y a Kantorovich space. Assume that $A_1, \dots, A_N$ and B belong to $L^{(m)}(X, Y)$. Then one and only one of the following holds:*
  (1) There are $x \in X$ and $b, b' \in \mathbb{B}$ such that $b' \le b$ and
  $$b'Bx > 0, bA_1x \le 0, \dots, bA_Nx \le 0.$$
  (2) There are positive orthomorphisms $\alpha_1, \dots, \alpha_N \in \mathrm{Orth}(m(Y))_+$ such that $B = \sum_{k=1}^{N} \alpha_k A_k$.

Inhomogeneous Inequalities

- **Theorem.** *Let X be a Y-seminormed real vector space, with Y a Kantorovich space. Assume given some dominated operators $A_1, \dots, A_N, B \in L^{(m)}(X, Y)$ and elements $u_1, \dots, u_N, v \in Y$. The following are equivalent:*
  (1) *For all $b \in \mathbb{B}$ the inhomogeneous operator inequality $bBx \le bv$ is a consequence of the consistent simultaneous inhomogeneous operator inequalities $bA_1x \le bu_1, \dots, bA_Nx \le bu_N$, i.e.,*

$$\{bB \le bv\} \supset \{bA_1 \le bu_1\} \cap \dots \cap \{bA_N \le bu_N\}.$$

  (2) *There are positive orthomorphisms $\alpha_1, \dots, \alpha_N \in \mathrm{Orth}(m(Y))$ satisfying*

$$B = \sum_{k=1}^{N} \alpha_k A_k; \quad v \ge \sum_{k=1}^{N} \alpha_k u_k.$$

Boolean Modeling

- Cohen's final solution of the problem of the cardinality of the continuum within ZFC gave rise to the Boolean valued models by Scott, Solovay, and Vopěnka.
- Takeuti coined the term "Boolean valued analysis" for applications of the models to analysis.

Scott's Comments

- Scott forecasted in 1969:

  We must ask whether there is any interest in these nonstandard models aside from the independence proof; that is, do they have any mathematical interest? The answer must be yes, but we cannot yet give a really good argument.

- In 2009 Scott wrote:[1]

  At the time, I was disappointed that no one took up my suggestion. And then I was very surprised much later to see the work of Takeuti and his associates. I think the point is that people have to be trained in Functional Analysis in order to understand these models. I think this is also obvious from your book and its references. Alas, I had no students or collaborators with this kind of background, and so I was not able to generate any progress.

[1] Letter of April 29, 2009 to S. S. Kutateladze.

Boolean Valued Universe

- Let $\mathbb{B}$ be a complete Boolean algebra. Given an ordinal α, put

$$V_\alpha^{(\mathbb{B})} := \{x \mid (\exists \beta \in \alpha)\ x : \mathrm{dom}(x) \to \mathbb{B}\ \&\ \mathrm{dom}(x) \subset V_\beta^{(\mathbb{B})}\}.$$

- The *Boolean valued universe* $\mathbb{V}^{(\mathbb{B})}$ is

$$\mathbb{V}^{(\mathbb{B})} := \bigcup_{\alpha \in \mathrm{On}} V_\alpha^{(\mathbb{B})},$$

  with On the class of all ordinals.
- The truth value $[\![\varphi]\!] \in \mathbb{B}$ is assigned to each formula φ of ZFC relativized to $\mathbb{V}^{(\mathbb{B})}$.

Descending and Ascending

- Given φ, a formula of ZFC, and y, a member of $\mathbb{V}^{\mathbb{B}}$; put $A_\varphi := A_{\varphi(\cdot,\ y)} := \{x \mid \varphi(x,\ y)\}$.
- The *descent* $A_\varphi{\downarrow}$ of a class A_φ is

$$A_\varphi{\downarrow} := \{t \mid t \in \mathbb{V}^{(\mathbb{B})} \ \&\ [\![\varphi(t,\ y)]\!] = \mathbb{1}\}.$$

- If $t \in A_\varphi{\downarrow}$, then it is said that t *satisfies* $\varphi(\cdot,\ y)$ *inside* $\mathbb{V}^{(\mathbb{B})}$.
- The *descent* $x{\downarrow}$ of $x \in \mathbb{V}^{(\mathbb{B})}$ is defined as

$$x{\downarrow} := \{t \mid t \in \mathbb{V}^{(\mathbb{B})} \ \&\ [\![t \in x]\!] = \mathbb{1}\},$$

  i.e. $x{\downarrow} = A_{\cdot \in x}{\downarrow}$. The class $x{\downarrow}$ is a set.
- If x is a nonempty set inside $\mathbb{V}^{(\mathbb{B})}$ then

$$(\exists z \in x{\downarrow})[\![(\exists t \in x)\ \varphi(t)]\!] = [\![\varphi(z)]\!].$$

- The *ascent* functor acts in the opposite direction.

The Reals Within or Kantorovich's Scalars

- There is an object $\mathscr{R}$ inside $\mathbb{V}^{(\mathbb{B})}$ modeling $\mathbb{R}$, i.e.,

$$[\![\mathscr{R} \text{ is the reals}]\!] = \mathbb{1}.$$

- Let $\mathscr{R}{\downarrow}$ be the descent of the carrier $|\mathscr{R}|$ of the algebraic system $\mathscr{R} := (|\mathscr{R}|, +, \cdot, 0, 1, \leq)$ inside $\mathbb{V}^{(\mathbb{B})}$.
- Implement the descent of the structures on $|\mathscr{R}|$ to $\mathscr{R}{\downarrow}$ as follows:

$$\begin{gathered} x + y = z \leftrightarrow [\![x + y = z]\!] = \mathbb{1}; \\ xy = z \leftrightarrow [\![xy = z]\!] = \mathbb{1}; \\ x \leq y \leftrightarrow [\![x \leq y]\!] = \mathbb{1}; \\ \lambda x = y \leftrightarrow [\![\lambda^{\wedge} x = y]\!] = \mathbb{1}\ (x, y, z \in \mathscr{R}{\downarrow},\ \lambda \in \mathbb{R}). \end{gathered}$$

- **Gordon Theorem.** *$\mathscr{R}{\downarrow}$ with the descended structures is a universally complete vector lattice with base $\mathbb{B}(\mathscr{R}{\downarrow})$ isomorphic to $\mathbb{B}$.*

Two-Point Relations

- The kernel $\ker(T)$ of each *two-point relation* $T := T_1 - T_2$ with T_1 and T_2 lattice homomorphisms is evidently a sublattice of X, since it is determined by the equation $\ker(T) = \{x \in X : \ T_1x = T_2x\}$.
- **The Meyer Theorem.** *Each order bounded disjointness preserving operator between vector lattices is a two-point relation.*
- Thus, each stratum bT of an order bounded disjointness preserving operator $T : X \to Y$ is a two-point relation on X and so the kernel of bT is a vector sublattice of X. In fact, the converse is valid too.
  Theorem. *An order bounded linear operator from a vector lattice to a Dedekind complete vector lattice is a two-point relation if and only if the kernel of its every stratum is a vector sublattice of the ambient vector lattice.*

Grothendieck subspaces

- **Definition.** A subspace H of a vector lattice X is a *Grothendieck subspace* (G-subspace, for short) provided that

$$(\forall x, y \in H)\,(x \vee y \vee 0 + x \wedge y \wedge 0 \in H).$$

- **Theorem.** *The modulus of an order bounded operator $T : X \to Y$ is the sum of a pair of lattice homomorphisms if and only if* $\ker(bT)$ *is a G-subspace of X for every $b \in \mathbb{B}$.*
- $[\![\ker(\tau)$ *is a Grothendieck subspace of* $X^{\wedge}]\!] = \mathbb{1}$
$\iff$ $(\forall b \in \mathbb{B})\,(\ker(bT)$ *is a Grothendieck subspace of* $X)$.

Stone–Weierstrass Approximation

- **Theorem.** *Let $\mathbb{B}$ be the base of a Dedekind complete vector lattice Y with order convergence. Assume that Q is a $\mathbb{B}$-cyclically compact set Q and $X = C(Q, Y)$ is the space of continuous vector-functions from X to Y. Then the following hold:*
- *A sublattice L of X is such that* $\mathrm{mix}(L)$ *does not coincide with X if and only if there are lattice homomorphisms $S, T : X \to Y$ satisfying $L \subset ker(S - T)$.*
- *A Grothendieck subspace L of X is such that* $\mathrm{mix}(L)$ *does not coincide with X if and only if there are lattice homomorphisms $S, T : X \to Y$ satisfying $L \subset ker(S + T)$.*

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