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This book presents some new results on vector lattices and operators between them that base on the powerful logical technique of Boolean valued analysis. The book is intended for the classical analyst seeking new powerful tools and for the model theorist in search of challenging applications of nonstandard models of set theory.

BOOLEAN VALUED ANALYSIS:
SELECTED TOPICS

A. G. Kusraev
S. S. Kutateladze

MM

TRENDS IN SCIENCE
THE SOUTH OF RUSSIA

MATHEMATICAL MONOGRAPH

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Kusraev A. G., Kutateladze S. S.

Boolean Valued Analysis: Selected Topics / Ed. A. E. Gutman.—Vladikavkaz: SMI VSC RAS, 2014.—iv+364 p.—(Trends in Science: The South of Russia. A Mathematical Monograph. Issue 6).

The book treats Boolean valued analysis. This term signifies the technique of studying properties of an arbitrary mathematical object by means of comparison between its representations in two different set-theoretic models whose construction utilizes principally distinct Boolean algebras. As these models, we usually take the classical Cantorian paradise in the shape of the von Neumann universe and a specially-trimmed Boolean valued universe in which the conventional set-theoretic concepts and propositions acquire bizarre interpretations. Exposition focuses on the fundamental properties of order bounded operators in vector lattices. This volume is intended for the classical analyst seeking new powerful tools and for the model theorist in search of challenging applications of nonstandard models of set theory.

Кусраев А. Г., Кутателадзе С. С.

Булевозначный анализ: Избранные темы / отв. ред. А. Е. Гутман.—Владикавказ: ЮМИ ВНЦ РАН и РСО-А, 2014.—iv+364 с.—(Итоги науки. Юг России. Математическая монография. Вып. 6).

Монография посвящена булевозначному анализу. Так называют аппарат исследования произвольных математических объектов, основанный на сравнительном изучении их вида в двух моделях теории множеств, конструкции которых основаны на принципиально различных булевых алгебрах. В качестве этих моделей фигурируют классический канторов рай в форме универсума фон Неймана и специально построенный булевозначный универсум, в котором теоретико-множественные понятия и утверждения получают весьма нетрадиционные толкования. Основное внимание уделено фундаментальным свойствам порядково ограниченных операторов в векторных решетках. Книга ориентирована на широкий круг читателей, интересующихся современными теоретико-модельными методами в их приложении к функциональному анализу.

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PREFACE

Humans definitely feel truth but cannot define truth properly. That is what Alfred Tarski explained to us in the 1930s. Mathematics pursues truth by way of proof, as wittily phrased by Saunders Mac Lane. *Boolean valued analysis* is one of the vehicles of the pursuit, resulting from the fusion of analysis and model theory.

Analysis is the technique of differentiation and integration. Differentiation discovers trends, and integration forecasts the future from trends. Analysis opens ways to understanding of the universe.

Model theory evaluates and counts truth and proof. The chase of truth not only leads us close to the truth we pursue but also enables us to nearly catch up with many other instances of truth which we were not aware nor even foresaw at the start of the rally pursuit. That is what we have learned from Boolean valued models of set theory. These models stem from the famous works by Paul Cohen on the continuum hypothesis. They belong to logic and yield a profusion of the surprising and unforeseen visualizations of the ingredients of mathematics. Many promising opportunities are open to modeling the powerful habits of reasoning and verification.

Logic organizes and orders our ways of thinking, manumitting us from conservatism in choosing the objects and methods of research. Logic of today is a fine instrument of pursuing truth and an indispensable institution of mathematical freedom. Logic liberates mathematics, providing nonstandard ways of reasoning.

Some model of set theory is *nonstandard* if the membership between the objects of the model differs from that of the originals. In fact, the nonstandard tools of today use a couple of set-theoretic models simultaneously. Boolean valued models reside within the most popular logical tools.

Boolean valued analysis is a blending of analysis and Boolean valued models which originated and distinguishes itself by ascending and descending, mixing, cycling hulls, etc.

In this book we show how Boolean valued analysis transforms the theory of operators in vector lattices. We focus on the recent results that were not reflected in the monographic literature yet.

In Chapter 1 we collect the Boolean valued prerequisites of the further analysis. Chapter 2 provides the presentation of the reals and complexes within Boolean valued models. In Chapter 3 we give the Boolean valued interpretations of order bounded operators with the emphasis on lattice homomorphisms and disjointness preserving operators. Chapter 4 contains the solution of the Wickstead problem as well as other new results on band preserving operators. Chapter 5 deals with various applications of order continuous operators to injective Banach lattices, Maharam operators, and related topics.

Adaptation of the ideas of Boolean valued models to functional analysis projects among the most important directions of developing the synthetic methods of mathematics. This approach yields the new models of numbers, spaces, and types of equations. The content expands of all available theorems and algorithms. The whole methodology of mathematical research is enriched and renewed, opening up absolutely fantastic opportunities. We can now transform matrices into numbers, embed function spaces into a straight line, yet having still uncharted vast territories of new knowledge. The chase of truth not only leads us close to the truth we pursue but also enables us to nearly catch up with many other instances of truth which we were unaware nor even foresaw at the start of the rally pursuit. That is what we have learned from Boolean valued analysis.

Quite a long time had passed until the classical functional analysis occupied its present position of the language of continuous mathematics. Now the time has come of the new powerful technologies of model theory in mathematical analysis. Not all theoretical and applied mathematicians have already gained the importance of modern tools and learned how to use them. However, there is no backward traffic in science, and the new methods are doomed to reside in the realm of mathematics for ever and they will shortly become as elementary and omnipresent in analysis as Banach spaces and linear operators.

A. Kusraev
S. Kutateladze

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