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BOOLEAN VALUED ANALYSIS: SELECTED TOPICS

by

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The book treats Boolean valued analysis. This term signifies the technique of studying properties of an arbitrary mathematical object by means of comparison between its representations in two different set-theoretic models whose construction utilizes principally distinct Boolean algebras. As these models, we usually take the classical Cantorian paradise in the shape of the von Neumann universe and a specially-trimmed Boolean valued universe in which the conventional set-theoretic concepts and propositions acquire bizarre interpretations. Exposition focuses on the fundamental properties of order bounded operators in vector lattices. This volume is intended for the classical analyst seeking new powerful tools and for the model theorist in search of challenging applications of nonstandard models of set theory.

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Монография посвящена булевозначному анализу. Так называют аппарат исследования произвольных математических объектов, основанный на сравнительном изучении их вида в двух моделях теории множеств, конструкции которых основаны на принципиально различных булевых алгебрах. В качестве этих моделей фигурируют классический канторов рай в форме универсума фон Неймана и специально построенный булевозначный универсум, в котором теоретико-множественные понятия и утверждения получают весьма нетрадиционные толкования. Основное внимание уделено фундаментальным свойствам порядково ограниченных операторов в векторных решетках. Книга ориентирована на широкий круг читателей, интересующихся современными теоретико-модельными методами в их приложении к функциональному анализу.

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PREFACE

Humans definitely feel truth but cannot define truth properly. That is what Alfred Tarski explained to us in the 1930s. Mathematics pursues truth by way of proof, as wittily phrased by Saunders Mac Lane. *Boolean valued analysis* is one of the vehicles of the pursuit, resulting from the fusion of analysis and model theory.

Analysis is the technique of differentiation and integration. Differentiation discovers trends, and integration forecasts the future from trends. Analysis opens ways to understanding of the universe.

Model theory evaluates and counts truth and proof. The chase of truth not only leads us close to the truth we pursue but also enables us to nearly catch up with many other instances of truth which we were not aware nor even foresaw at the start of the rally pursuit. That is what we have learned from Boolean valued models of set theory. These models stem from the famous works by Paul Cohen on the continuum hypothesis. They belong to logic and yield a profusion of the surprising and unforeseen visualizations of the ingredients of mathematics. Many promising opportunities are open to modeling the powerful habits of reasoning and verification.

Logic organizes and orders our ways of thinking, manumitting us from conservatism in choosing the objects and methods of research. Logic of today is a fine instrument of pursuing truth and an indispensable institution of mathematical freedom. Logic liberates mathematics, providing nonstandard ways of reasoning.

Some model of set theory is *nonstandard* if the membership between the objects of the model differs from that of the originals. In fact, the nonstandard tools of today use a couple of set-theoretic models simultaneously. Boolean valued models reside within the most popular logical tools.

Boolean valued analysis is a blending of analysis and Boolean valued models which originated and distinguishes itself by ascending and descending, mixing, cycling hulls, etc.

In this book we show how Boolean valued analysis transforms the theory of operators in vector lattices. We focus on the recent results that were not reflected in the monographic literature yet.

In Chapter 1 we collect the Boolean valued prerequisites of the further analysis. Chapter 2 provides the presentation of the reals and complexes within Boolean valued models. In Chapter 3 we give the Boolean valued interpretations of order bounded operators with the emphasis on lattice homomorphisms and disjointness preserving operators. Chapter 4 contains the solution of the Wickstead problem as well as other new results on band preserving operators. Chapter 5 deals with various applications of order continuous operators to injective Banach lattices, Maharam operators, and related topics.

Adaptation of the ideas of Boolean valued models to functional analysis projects among the most important directions of developing the synthetic methods of mathematics. This approach yields the new models of numbers, spaces, and types of equations. The content expands of all available theorems and algorithms. The whole methodology of mathematical research is enriched and renewed, opening up absolutely fantastic opportunities. We can now transform matrices into numbers, embed function spaces into a straight line, yet having still uncharted vast territories of new knowledge.

Quite a long time had passed until the classical functional analysis occupied its present position of the language of continuous mathematics. Now the time has come of the new powerful technologies of model theory in mathematical analysis. Not all theoretical and applied mathematicians have already gained the importance of modern tools and learned how to use them. However, there is no backward traffic in science, and the new methods are doomed to reside in the realm of mathematics for ever and they will shortly become as elementary and omnipresent in analysis as Banach spaces and linear operators.

A. Kusraev
S. Kutateladze

CHAPTER 1

BOOLEAN VALUED REQUISITES

In this chapter we briefly present some prerequisites of the theory of Boolean valued models. All missing details may be found in Bell [43], Jech [184], Kusraev and Kutateladze [248, 249], Takeuti and Zaring [388]. We mainly keep the notation of [248] and [249].

The most important feature of Boolean valued analysis consists in comparative analysis of the standard and nonstandard (Boolean valued) models under consideration which uses the special technique of ascending and descending. Moreover, it is often necessary to carry out some syntactic comparison of formal texts. Therefore, before we launch into the ascending and descending machinery, we have to grasp a clearer idea of the status of mathematical objects in the framework of a formal set theory, the construction of a Boolean valued universe, and the way of assigning the Boolean truth value to each sentence of the language of set theory.

We use several notations for implication: $\Rightarrow$ presents the Boolean operation, $\Longrightarrow$ stands for the logical connective, but often in a set theoretic formula we use $\to$ instead of $\Longrightarrow$ indicating that this formula will be interpreted in some Boolean valued model. We also use $\leftrightarrow$, $\Leftrightarrow$, and $\Longleftrightarrow$ with the similar meaning. The proof-theoretic consequence relation $\vdash$ is applied alongside with the semantic (or model-theoretic) consequence relation $\models$.

Observe that, speaking of a formal set theory, we will freely (because this is in fact unavoidable) adhere to the level of rigor which is current in the mainstream of mathematics and introduce abbreviations by means of the *definor*, i.e. the *assignment operator*, $:=$ without specifying any subtleties.

1.1. Zermelo–Fraenkel Set Theory

At present, the most widespread axiomatic foundation for mathematics is *Zermelo–Fraenkel set theory*. We will briefly recall some of its concepts, outlining the details we need in the sequel.

1.1.1. The alphabet of Zermelo–Fraenkel theory ZF or ZFC, if the presence of choice AC is stressed, comprises the symbols of variables; the parentheses (and); the propositional connectives (i.e., the signs of propositional calculus) $\vee$, $\wedge$, $\to$, $\leftrightarrow$, and $\neg$; the quantifiers $\forall$ and $\exists$; the equality sign $=$; and the symbol of the special binary predicate of *containment* or *membership* $\in$. In general, the domain of the variables of ZF is thought as the world or *universe* of sets. In other words, the universe of ZF contains nothing but sets. We write $x \in y$ rather than $\in (x, y)$ and say that x is an *element* of y.

1.1.2. The formulas of ZF are defined by the routine procedure. In other words, the formulas of ZF are finite texts resulting from the *atomic* formulas $x = y$ and $x \in y$, where x and y are variables of ZF, by reasonably placing parentheses, propositional connectives, and quantifiers

$$\varphi \vee \psi, \quad \varphi \wedge \psi, \quad \neg\,\varphi, \quad \varphi \to \psi, \quad \varphi \leftrightarrow \psi, \quad (\forall\, x)\,\varphi, \quad (\exists\, x)\,\varphi.$$

So, if φ_1 and φ_2 are formulas of ZF and x is a variable then the texts $\varphi_1 \to \varphi_2$ and $(\exists\, x)\,(\varphi_1 \to (\forall\, y)\,\varphi_2) \vee \varphi_1$ are formulas of ZF, whereas $\varphi_1 \exists\, x$ and $\forall\,(x \exists\,\varphi_1 \neg\,\varphi_2$ are not. We attach the natural meaning to the terms *free* and *bound variables* and the term *domain of a quantifier*. For instance, in the formula $(\forall\, x)\,(x \in y)$ the variable x is bound and the variable y is free, whereas in the formula $(\exists\, y)\,(x = y)$ the variable x is free and y is bound (for it is bounded by a quantifier). Henceforth, in order to emphasize that the only free variables in a formula φ are the variables $x_1, \dots, x_n$, we write $\varphi(x_1, \dots, x_n)$. Sometimes such a formula is considered as a "function"; in this event, it is convenient to write $\varphi(\cdot, \dots, \cdot)$ or $\varphi = \varphi(x_1, \dots, x_n)$, implying that $\varphi(y_1, \dots, y_n)$ is a formula of ZF obtained by replacing each free occurrence of x_k by y_k for $k := 1, \dots, n$.

1.1.3. Studying ZF, it is convenient to use some expressive tools absent in the formal language. In particular, in the sequel it is worthwhile employing the concepts of *class* and *definable class* and also the corresponding symbols of *classifiers* like $A_\varphi := A_{\varphi(\cdot)} := \{x : \varphi(x)\}$ and $A_\psi := A_{\psi(\cdot, y)} := \{x : \psi(x, y)\}$, where φ and ψ are formulas of ZF and y is a distinguished collection of variables. If it is desirable to clarify or eliminate the appearing records then we can assume that the use of classes and classifiers is connected only with the conventional agreement on introducing abbreviations. This agreement, sometimes called the *Church schema*, reads:

$$z \in \{x : \varphi(x)\} \leftrightarrow \varphi(z),$$
$$z \in \{x : \psi(x, y)\} \leftrightarrow \psi(z, y).$$

1.1.4. Working within ZF, we will use some notations that are widely spread in mathematics. We start with the most frequent abbreviations:

$$x \neq y := \neg\, x = y, \quad x \notin y := \neg\, x \in y;$$
$$(\forall\, x \in y)\, \varphi(x) := (\forall\, x)\, (x \in y \to \varphi(x));$$
$$(\exists\, !z)\, \varphi(z) := (\exists\, z)\, \varphi(z) \wedge ((\forall\, x)\, (\forall\, y)\, (\varphi(x) \wedge \varphi(y) \to x = y));$$
$$(\exists\, x \in y)\, \varphi(x) := (\exists\, x)\, (x \in y \wedge \varphi(x)).$$

The empty set $\varnothing$, the pair $\{x, y\}$, the singleton $\{x\}$, the ordered pair (x, y), and the ordered n-tuple $(x_1, \dots, x_n)$ are defined as

$$\varnothing := \{x :\ x \neq x\};$$
$$\{x, y\} := \{z :\ z = x \vee z = y\}, \quad \{x\} := \{x, x\},$$
$$(x, y) := \{x, \{x, y\}\};$$
$$(x_1, \dots, x_n) := ((x_1, \dots, x_{n-1}), x_n).$$

The inclusion $\subset$, the union $\bigcup$, the intersection $\bigcap$, the powerset $\mathscr{P}(\cdot)$, and the universe of sets $\mathbb{V}$ are introduced as follows:

$$x \subset y := (\forall\, z)\, (z \in x \to z \in y);$$
$$\bigcup x := \{z : (\exists\, y \in x)\, z \in y\};$$
$$\bigcap x := \{z : (\forall\, y \in x)\, z \in y\};$$
$$\mathscr{P}(x) := \text{“the class of all subsets of } x\text{”} := \{z :\ z \subset x\};$$
$$\mathbb{V} := \text{“the class of all sets”} := \{x :\ x = x\}.$$

Note also that in the sequel we accept more complicated descriptions in which much is presumed:

$$\begin{gathered}\mathrm{Fnc}(f) := \text{“}f\text{ is a function”};\\ \mathrm{dom}(f) := \text{“the domain of }f\text{”};\\ \mathrm{im}(f) := \text{“the range of }f\text{”};\\ \varphi \vdash \psi := \varphi \to \psi := \text{“}\psi\text{ is derivable from }\varphi\text{”};\\ \text{“a class }A\text{ is a set”} := A \in \mathbb{V} := (\exists\, x)\,(\forall\, y)\,(y \in A \leftrightarrow y \in x).\end{gathered}$$

Similar simplifications will be used in rendering more complicated formulas without further stipulation. For instance, instead of some rather involved formulas of ZF we simply write

$$\begin{gathered}f : x \to y \equiv \text{“}f\text{ is a function from }x\text{ to }y\text{”};\\ \text{“}X\text{ is a vector lattice”};\\ U \in L^{\sim}(X, Y) \equiv \text{“}U\text{ is an order bounded linear operator from }X\text{ to }Y\text{.”}\end{gathered}$$

1.1.5. In ZFC, we accept the usual axioms and rules of a first-order theory with equality which fix the standard means of classical reasoning. Recall the equality axioms:

(1) $(\forall\, x)\, x = x$ (*reflexivity*);

(2) $(\forall\, x)\,(\forall\, x)\, x = y \to y = x$ (*symmetry*);

(3) $(\forall\, x)\,(\forall\, y)\,(\forall\, y)\, x = y \wedge y = x \to x = z$ (*transitivity*);

(4) $(\forall\, x)\,(\forall\, y)\,(\forall\, u)\,(\forall\, v)\,\big((x = y \wedge u = v)$
$\to (x \in u \to y \in v)\big)$ (*substitution*).

1.1.6. The classical first-order logic CL has the following axiom schemas (φ, ψ, and ω are arbitrary formulas of CL):

(1) $\varphi \to (\varphi \wedge \varphi)$;

(2) $(\varphi \wedge \psi) \to (\psi \wedge \varphi)$;

(3) $(\varphi \to \psi) \to ((\varphi \wedge \omega) \to (\psi \wedge \omega))$;

(4) $((\varphi \to \psi) \wedge (\psi \to \omega)) \to (\varphi \to \omega)$;

(5) $\psi \to (\varphi \to \psi)$;

(6) $(\varphi \wedge (\varphi \to \psi)) \to \psi$;

(7) $\varphi \to (\varphi \vee \psi)$;

(8) $(\varphi \vee \psi) \to (\psi \vee \varphi)$;

(9) $((\varphi \to \omega) \wedge (\psi \to \omega)) \to ((\varphi \wedge \psi) \to \omega)$;

(10) $\neg\,\varphi \to (\varphi \to \psi)$;

(11) $((\varphi \to \psi) \wedge (\varphi \to \neg\,\psi)) \to \neg\,\varphi$;

(12) $\varphi \vee \neg\,\varphi$.

The only rule of inference in CL is *modus ponens*:

($\mathbb{MP}$) If φ and $\varphi \to \psi$ are provable in CL then so is ψ.

1.1.7. Moreover, the following *special* or *proper* axioms are accepted in ZFC as a correct formalization of the principles of most mathematicians working with sets:

(1) Axiom of Extensionality. *If x and y have the same elements then $x = y$:*

$$(\forall\, x)\,(\forall\, y)\,(x \subset y \wedge y \subset x \to x = y).$$

(2) Axiom of Union. *To each x there exists a set $y = \bigcup x$:*

$$(\forall\, x)\,(\exists\, y)\,\left(y = \bigcup x\right).$$

(3) Axiom of Powerset. *To each x there exists a set $y = \mathscr{P}(x)$:*

$$(\forall\, x)\,(\exists\, y)\,(y = \mathscr{P}(x)).$$

(4) Axiom Schema of Replacement. *If a class A_φ is a function then to each x there exists a set $v = A_\varphi(x) = \{A_\varphi(z) : z \in x\}$:*

$$\begin{gathered}(\forall\, x)\,((\forall\, y)\,(\forall\, z)\,(\forall\, u)\,\varphi(y,z) \wedge \varphi(y,u) \to z = u)\\ \to (\exists\, v)\,(v = \{z : (\exists\, y \in x)\,\varphi(y,z)\}).\end{gathered}$$

(5) Axiom of Foundation. *Each nonempty set has an $\in$-minimal element:*

$$(\forall\, x)\,(x \neq \varnothing \to (\exists\, y \in x)\,(y \cap x = \varnothing)).$$

(6) Axiom of Infinity. *There exists an inductive set:*

$$(\exists\, \omega)\,(\varnothing \in \omega) \wedge (\forall\, x \in \omega)\,(x \cup \{x\} \in \omega).$$

(7) Axiom of Choice. *Each family of nonempty sets has a choice function:*

$$\begin{gathered}(\forall\, F)\,(\forall\, x)\,(\forall\, y)\,((x \neq \varnothing \wedge F : x \to \mathscr{P}(y))\\ \to ((\exists\, f)\, f : x \to y \wedge (\forall\, z \in x)\, f(z) \in F(z)).\end{gathered}$$

1.1.8. Grounding on the above axiomatics, we acquire a clear idea of the class of all sets, the *von Neumann universe* $\mathbb{V}$. As the initial object of all constructions we take the empty set. The elementary step of introducing new sets consists in taking the union of the powersets of the sets already available. Transfinitely repeating these steps, we exhaust the class of all sets. More precisely, we assign $\mathbb{V} := \bigcup_{\alpha \in \mathrm{On}} \mathbb{V}_\alpha$, where On is the class of all ordinals and

$$\mathbb{V}_0 := \varnothing,$$
$$\mathbb{V}_{\alpha+1} := \mathscr{P}(\mathbb{V}_\alpha),$$
$$\mathbb{V}_\beta := \bigcup_{\alpha<\beta} \mathbb{V}_\alpha \quad (\beta \text{ is a limit ordinal}).$$

1.1.9. *The pair* $(\mathbb{V}, \in)$ *is a standard model of* ZFC.

1.2. Boolean Valued Universes

Everywhere below $\mathbb{B}$ is a complete Boolean algebra with supremum (join) $\vee$, meet (infimum) $\wedge$, complement $(\cdot)^*$, unit (top) $\mathbb{1}$, and zero (bottom) $\mathbb{0}$. The necessary information on Boolean algebras can be found in Givant and Halmos [130], Sikorski [365], and Vladimirov [399].

1.2.1. Let $\mathbb{B}$ be a complete Boolean algebra. Given an ordinal α, put

$$\mathbb{V}_\alpha^{(\mathbb{B})} := \Big\{x : \ \mathrm{Funct}(x) \wedge (\exists\, \beta)\, (\beta < \alpha \wedge \mathrm{dom}(x) \subset \mathbb{V}_\beta^{(\mathbb{B})} \wedge \mathrm{im}(x) \subset \mathbb{B})\Big\}.$$

Thus, in more detail we have

$$\mathbb{V}_0^{(\mathbb{B})} := \varnothing,$$
$$\mathbb{V}_{\alpha+1}^{(\mathbb{B})} := \{x : \ x \text{ is a function with domain in } \mathbb{V}_\alpha^{(\mathbb{B})} \text{ and range in } \mathbb{B}\};$$
$$\mathbb{V}_\alpha^{(\mathbb{B})} := \bigcup_{\beta<\alpha} \mathbb{V}_\beta^{(\mathbb{B})} \quad (\beta \text{ is a limit ordinal}).$$

The class

$$\mathbb{V}^{(\mathbb{B})} := \bigcup_{\alpha \in \mathrm{On}} \mathbb{V}_\alpha^{(\mathbb{B})}$$

is a *Boolean valued universe*. An element of $\mathbb{V}^{(\mathbb{B})}$ is a $\mathbb{B}$-*valued set*. Observe that $\mathbb{V}^{(\mathbb{B})}$ consists only of functions. In particular, $\varnothing$ is the function with domain $\varnothing$ and range $\varnothing$. Hence, the "lower" levels of $\mathbb{V}^{(\mathbb{B})}$ are organized as follows:

$$\mathbb{V}_0^{(\mathbb{B})} = \varnothing, \quad \mathbb{V}_1^{(\mathbb{B})} = \{\varnothing\}, \quad \mathbb{V}_2^{(\mathbb{B})} = \big\{\varnothing, \big(\{\varnothing\}, b\big) : b \in \mathbb{B}\big\}.$$

1.2.2. It is worth stressing that $\alpha \leqslant \beta \Longrightarrow \mathbb{V}_\alpha^{(\mathbb{B})} \subset \mathbb{V}_\beta^{(\mathbb{B})}$ is valid for all ordinals α and β. Moreover, we have the *induction principle* for $\mathbb{V}^{(\mathbb{B})}$:

$$\big(\forall\, x \in \mathbb{V}^{(\mathbb{B})}\big) \Big(\big((\forall\, y \in \operatorname{dom}(x))\ \varphi(y) \Longrightarrow \varphi(x)\big) \Longrightarrow \big(\forall\, x \in \mathbb{V}^{(\mathbb{B})}\big)\, \varphi(x)\Big),$$

where φ is a formula of ZFC.

1.2.3. Take an arbitrary formula $\varphi = \varphi(u_1, \dots, u_n)$ of ZFC. If we replace $u_1, \dots, u_n$ by $x_1, \dots, x_n \in \mathbb{V}^{(\mathbb{B})}$ then we obtain some statement about the objects $x_1, \dots, x_n$. It is to this statement that we intend to assign some *Boolean truth value*. Such a truth value $[\![\psi]\!]$ must be an element of $\mathbb{B}$. Moreover, we desire naturally that the theorems of ZFC be true; i.e., they attain the greatest truth value $\mathbb{1} \in \mathbb{B}$, the unity of $\mathbb{B}$.

We must obviously define truth values by double induction, taking into consideration the way in which formulas are built up from atomic formulas and assigning truth values to the atomic formulas $x \in y$ and $x = y$, where $x, y \in \mathbb{V}^{(\mathbb{B})}$ in accord with the way in which $\mathbb{V}^{(\mathbb{B})}$ is constructed.

It is clear that if φ and ψ are evaluated formulas of ZFC and $[\![\varphi]\!] \in \mathbb{B}$ and $[\![\psi]\!] \in \mathbb{B}$ are their truth values then we should put

(1) $[\![\varphi \wedge \psi]\!] := [\![\varphi]\!] \wedge [\![\psi]\!]$;

(2) $[\![\varphi \vee \psi]\!] := [\![\varphi]\!] \vee [\![\psi]\!]$;

(3) $[\![\varphi \to \psi]\!] := [\![\varphi]\!] \Rightarrow [\![\psi]\!]$;

(4) $[\![\neg\, \varphi]\!] := [\![\varphi]\!]^*$;

(5) $[\![(\forall\, x)\, \varphi(x)]\!] := \bigwedge_{x \in \mathbb{V}^{(\mathbb{B})}} [\![\varphi(x)]\!]$;

(6) $[\![(\exists\, x)\, \varphi(x)]\!] := \bigvee_{x \in \mathbb{V}^{(\mathbb{B})}} [\![\varphi(x)]\!]$;

where the right-hand sides involve the Boolean operations that correspond to the logical connectives and quantifiers on the left-hand sides: $\wedge$ is the meet of two elements, $\vee$ is the join of two elements, $*$ is the taking of the complement of an element, and the operation $\Rightarrow$ is introduced as follows: $a \Rightarrow b := a^* \vee b$ $(a, b \in \mathbb{B})$. Moreover, $\bigvee E$ and $\bigwedge E$ stand for the supremum and infimum of a subset $E \subset \mathbb{B}$. Only these definitions provide the value "unit" for the classical tautologies. The elements $x_1 \vee \dots \vee x_n$ and $x_1 \wedge \dots \wedge x_n$ may alternatively be denoted by $\bigvee_{k=1}^n x_k$ and $\bigwedge_{k=1}^n x_k$.

1.2.4. We turn to evaluating the atomic formulas $x \in y$ and $x = y$ for $x, y \in \mathbb{V}^{(\mathbb{B})}$. The intuitive idea consists in the fact that a $\mathbb{B}$-valued set y is a "(lattice) fuzzy set," i.e., a "set that contains an element z in $\mathrm{dom}(y)$ with probability $y(z)$." With this in mind and intending to preserve the logical tautology of $x \in y \leftrightarrow (\exists\, z \in y)\, (x = z)$ as well as the axiom of extensionality, we arrive at the definition by recursion:

$$[\![x \in y]\!] := \bigvee_{z \in \mathrm{dom}(y)} y(z) \wedge [\![z = x]\!],$$

$$[\![x = y]\!] := \bigwedge_{z \in \mathrm{dom}(x)} x(z) \Rightarrow [\![z \in y]\!] \wedge \bigwedge_{z \in \mathrm{dom}(y)} y(z) \Rightarrow [\![z \in x]\!].$$

1.2.5. We are able now to attach some meaning to the formal expressions of the form $\varphi(x_1, \dots, x_n)$, where $x_1, \dots, x_n \in \mathbb{V}^{(\mathbb{B})}$ and φ is a formula of ZFC; i.e., we can define exactly in which sense the set-theoretic proposition $\varphi(u_1, \dots, u_n)$ is valid for $x_1, \dots, x_n \in \mathbb{V}^{(\mathbb{B})}$. Namely, we say that the *formula* $\varphi(x_1, \dots, x_n)$ *is valid within* $\mathbb{V}^{(\mathbb{B})}$ or the *elements* $x_1, \dots, x_n$ *possess the property* φ if $[\![\varphi(x_1, \dots, x_n)]\!] = \mathbb{1}$. In this event we write $\mathbb{V}^{(\mathbb{B})} \vDash \varphi(x_1, \dots, x_n)$.

It is easy to check that the axioms and theorems of the first-order predicate calculus are valid in $\mathbb{V}^{(\mathbb{B})}$. In particular (cp. 1.1.5),

(1) $[\![x = x]\!] = \mathbb{1}$,

(2) $[\![x = y]\!] = [\![y = x]\!]$,

(3) $[\![x = y]\!] \wedge [\![y = z]\!] \leqslant [\![x = z]\!]$,

(4) $[\![x = y]\!] \wedge [\![z \in x]\!] \leqslant [\![z \in y]\!]$,

(5) $[\![x = y]\!] \wedge [\![x \in z]\!] \leqslant [\![y \in z]\!]$.

1.2.6. It is worth observing that for each formula φ we have

$$\mathbb{V}^{(\mathbb{B})} \vDash x = y \wedge \varphi(x) \to \varphi(y),$$

i.e., in terms of Boolean truth values,

$$[\![x = y]\!] \wedge [\![\varphi(x)]\!] \leqslant [\![\varphi(y)]\!].$$

1.2.7. In a Boolean valued universe $\mathbb{V}^{(\mathbb{B})}$, the relation $[\![x = y]\!] = \mathbb{1}$ in no way implies that the functions x and y (considered as elements of $\mathbb{V}$) coincide. For example, the function equal to zero on each layer $\mathbb{V}_{\alpha}^{(\mathbb{B})}$, where $\alpha \geqslant 1$, plays the role of the empty set in $\mathbb{V}^{(\mathbb{B})}$. This circumstance may complicate some constructions in the sequel. In this connection, we pass from $\mathbb{V}^{(\mathbb{B})}$ to the *separated Boolean valued universe* $\overline{\mathbb{V}}^{(\mathbb{B})}$ often preserving for the latter the same symbol $\mathbb{V}^{(\mathbb{B})}$; i.e., we put $\mathbb{V}^{(\mathbb{B})} := \overline{\mathbb{V}}^{(\mathbb{B})}$. Moreover, to define $\overline{\mathbb{V}}^{(\mathbb{B})}$, we consider the relation $\{(x, y) : [\![x = y]\!] = \mathbb{1}\}$ on the class $\mathbb{V}^{(\mathbb{B})}$ which is obviously an equivalence. Choosing an element (a representative of least rank) in each class of equivalent functions, we arrive at the separated universe $\overline{\mathbb{V}}^{(\mathbb{B})}$. Note that

$$[\![x = y]\!] = \mathbb{1} \implies [\![\varphi(x)]\!] = [\![\varphi(y)]\!]$$

is valid for an arbitrary formula φ of ZF and elements x and y in $\mathbb{V}^{(\mathbb{B})}$. Therefore, in the separated universe we can calculate the truth values of formulas paying no attention to the way of choosing representatives. Furthermore, working with the separated universe, for the sake of convenience we often consider (exercising due caution) a concrete representative rather than a class of equivalence as it is customary, for example, while dealing with function spaces.

Concluding the section we state a very useful exhaustion principle for Boolean algebras. A subset of a Boolean algebra is said to be *disjoint* or *antichain* if the meet of its every two elements is $\mathbb{0}$.

1.2.8. Exhaustion Principle. Let $\mathbb{B}$ be a Boolean algebra. To each nonempty set $B \subset \mathbb{B}$ having the least upper bound, there is an antichain $A \subset \mathbb{B}$ such that $\bigvee A = \bigvee B$ and, given $x \in A$, we may find y in B with $x \leqslant y$.

1.3. Transformations of the Boolean Valued Universe

Each homomorphism of a Boolean algebra $\mathbb{B}$ induces a transformation of the Boolean valued universe $\mathbb{V}^{(\mathbb{B})}$. The topic to be discussed in this section is the behavior of these transformations and, in particular, the manner in which they change the Boolean truth values of formulas.

1.3.1. Assume that π is a homomorphism of $\mathbb{B}$ in a complete Boolean algebra $\mathbb{D}$. By recursion on a well-founded relation $y \in \mathrm{dom}(x)$, we define the mapping $\pi^* : \mathbb{V}^{(\mathbb{B})} \to \mathbb{V}^{(\mathbb{D})}$ using the formulas $\mathrm{dom}(\pi^* x) := \{\pi^* y : y \in \mathrm{dom}(x)\}$ and

$$\pi^* x : v \mapsto \bigvee \big\{\pi(x(z)) : \ z \in \mathrm{dom}(x), \ \pi^* z = v\big\}.$$

1.3.2. *If σ is a complete homomorphism of a complete Boolean algebra $\mathbb{A}$ to $\mathbb{B}$ then $(\pi \circ \sigma)^* = \pi^* \circ \sigma^*$. Moreover, $I^*_{\mathbb{B}}$ is the identity mapping on $\mathbb{V}^{(\mathbb{B})}$. If π is injective then π^* is also injective. Moreover,*

$$\pi^* x : \pi^* y \mapsto \pi(x(y)) \quad (y \in \mathrm{dom}(x)).$$

A formula is called *bounded* or *restricted* if each bound variable in it is restricted by a bounded quantifier; i.e., each of its quantifiers occures in the form $(\forall\, x \in y)$ or $(\exists\, x \in y)$ for some y (cp. 1.1.2 and 1.1.4), or it is equivalent in ZFC to a formula of this kind. A formula is of class Σ_1 (or Σ_1-formula) if it is built up from atomic formulas and their negations using only the logical operations $\wedge$, $\vee$, $\forall\, x \in y$, $\exists\, x$, or if it is equivalent in ZFC to such a formula.

1.3.3. *Let π be a complete homomorphism from $\mathbb{B}$ to $\mathbb{D}$, let $\varphi(x_1, \dots, x_n)$ be a formula of* ZFC, *and let $u_1, \dots, u_n \in \mathbb{V}^{(\mathbb{B})}$. Then*

(1) *if φ is a formula of class Σ_1 and π is arbitrary then*

$$\pi([\![\varphi(u_1, \dots, u_n)]\!]^{\mathbb{B}}) \leqslant [\![\varphi(\pi^* u_1, \dots, \pi^* u_n]\!]^{\mathbb{D}};$$

(2) *if φ is a restricted formula and π is arbitrary, or π is an epimorphism and φ is arbitrary; then*

$$\pi([\![\varphi(u_1, \dots, u_n)]\!]^{\mathbb{B}}) = [\![\varphi(\pi^* u_1, \dots, \pi^* u_n)]\!]^{\mathbb{D}}.$$

1.3.4. *Assume that π, φ, and $u_1, \dots, u_n$ are the same as in* 1.3.3 *assume further that one of the following is fulfilled:*

(1) *$\varphi(x_1, \dots, x_n)$ is a formula of class Σ_1, and π is arbitrary;*

(2) *π is an epimorphism and $\varphi(x_1, \dots, x_n)$ is arbitrary.*

Then

$$\mathbb{V}^{(\mathbb{B})} \models \varphi(u_1, \dots, u_n) \Longrightarrow \mathbb{V}^{(\mathbb{D})} \models \varphi(\pi^* u_1, \dots, \pi^* u_n).$$

1.3.5. *Assume that* π, φ, *and* $u_1,\dots,u_n$ *are the same as in* 1.3.3. *Assume further that one of the following is fulfilled:*

(1) φ *is restricted and* π *is a monomorphism;*

(2) π *is an isomorphism and* φ *is arbitrary.*

Then

$$\mathbb{V}^{(\mathbb{B})} \models \varphi(u_1,\dots,u_n) \Longleftrightarrow \mathbb{V}^{(\mathbb{D})} \models \varphi(\pi^*u_1,\dots,\pi^*u_n).$$

We now consider the two important particular cases:

1.3.6. Let $\mathbb{B}_0$ be an order closed subalgebra of a complete Boolean algebra $\mathbb{B}$. Then $\mathbb{B}_0$ is itself a complete Boolean algebra and the least upper bound and the greatest lower bound of every subset of $\mathbb{B}_0$ are the same in $\mathbb{B}_0$ and in $\mathbb{B}$. In these circumstances $\mathbb{V}^{(\mathbb{B}_0)} \subset \mathbb{V}^{(\mathbb{B})}$. Moreover, denoting by $\imath$ the identical embedding of $\mathbb{B}_0$ into $\mathbb{B}$, we then see that $\imath$ is a complete monomorphism and $\imath^*$ is an embedding of $\mathbb{V}^{(\mathbb{B}_0)}$ into $\mathbb{V}^{(\mathbb{B})}$. Thus, the following is immediate from 1.3.5 (1).

If $\varphi(x_1,\dots,x_n)$ *is a restricted formula and* $u_1,\dots,u_n \in \mathbb{V}^{(\mathbb{B}_0)}$ *then*

$$\mathbb{V}^{(\mathbb{B}_0)} \models \varphi(u_1,\dots,u_n) \Longleftrightarrow \mathbb{V}^{(\mathbb{B})} \models \varphi(u_1,\dots,u_n).$$

Since the two-valued algebra $2 := \{\mathbb{0}, \mathbb{1}\}$ may be viewed as a complete subalgebra of the Boolean algebra $\mathbb{B}$, the above is also valid for the universe $\mathbb{V}^{(2)}$. As can easily be seen from 1.4.5 (2,3) below, $\mathbb{V}^{(2)}$ is naturally isomorphic to the von Neumann universe $\mathbb{V}$.

1.3.7. Fix a nonzero $b \in \mathbb{B}$ and consider the relative subalgebra $\bar{\mathbb{B}} := [\mathbb{0}, b] \subset \mathbb{B}$ with unit $\bar{\mathbb{1}} := b$. The mapping $\pi_b : x \mapsto b \wedge x$ $(x \in \mathbb{B})$ is a complete Boolean epimorphism from $\mathbb{B}$ onto $\bar{\mathbb{B}}$. Given $u \in \mathbb{V}^{(\mathbb{B})}$, the element $b \wedge u := \pi_b^*(u) \in \mathbb{V}^{(\bar{\mathbb{B}})}$ is defined by recursion according to 1.3.1:

$$\operatorname{dom}(b \wedge u) := \{b \wedge v : v \in \operatorname{dom}(u)\};$$

$$(b \wedge u)(v) = \bigvee\{\pi(u(z)) : z \in \operatorname{dom}(u),\ b \wedge z = v\}.$$

Applying 1.3.3 (2) we get

$$b \wedge [\![\varphi(u_1,\dots,u_n)]\!]^{\mathbb{B}} = [\![\varphi(b \wedge u_1,\dots,b \wedge u_n)]\!]^{\bar{\mathbb{B}}}.$$

In particular, if $\varphi(x_1,\dots,x_n)$ is a formula of ZFC and $u_1,\dots,u_n \in \mathbb{V}^{(\mathbb{B})}$ then

$$\mathbb{V}^{(\mathbb{B})} \models \varphi(u_1,\dots,u_n) \Longrightarrow \mathbb{V}^{(\bar{\mathbb{B}})} \models \varphi(b \wedge u_1,\dots,b \wedge u_n).$$

1.4. Principles of Boolean Valued Set Theory

The most important properties of a Boolean valued universe $\mathbb{V}^{(\mathbb{B})}$ are stated in the three principles:

1.4.1. Transfer principle. If $\varphi(x_1, \dots, x_n)$ is a theorem of ZFC then

$$(\forall x_1, \dots, x_n \in \mathbb{V}^{(\mathbb{B})})\ \mathbb{V}^{(\mathbb{B})} \models \varphi(x_1, \dots, x_n)$$

is also a theorem of ZFC.

The transfer principle is established by rather laboriously checking that all axioms of ZFC have truth value $\mathbb{1}$ and all applications of the rule of inference increase the truth value of each formula. Sometimes, the transfer principle is worded as follows: "$\mathbb{V}^{(\mathbb{B})}$ is the Boolean valued model of ZFC," or "all theorems of ZFC are true in $\mathbb{V}^{(\mathbb{B})}$," or another simile. Using the transfer principle, we will often simplify the reference and say "by transfer."

1.4.2. Maximum Principle. For each set-theoretic formula $\varphi(u, x_1, \dots, x_n)$ the following is provable in ZFC: for every collection $x_1, \dots, x_n \in \mathbb{V}^{(\mathbb{B})}$ there exists $x_0 \in \mathbb{V}^{(\mathbb{B})}$ such that

$$[\![(\exists\, x)\, \varphi(x)]\!] = [\![\varphi(x_0)]\!].$$

In particular, if it is true in $\mathbb{V}^{(\mathbb{B})}$ that there is x for which $\varphi(x)$ then there is an element x_0 in $\mathbb{V}^{(\mathbb{B})}$ (in the sense of $\mathbb{V}$) for which $[\![\varphi(x_0)]\!] = \mathbb{1}$. In symbols, the following is provable in ZFC:

$$\Big(\mathbb{V}^{(\mathbb{B})} \models (\exists\, x)\, \varphi(x)\Big) \Longrightarrow \Big((\exists\, x_0)\, \mathbb{V}^{(\mathbb{B})} \vDash \varphi(x_0)\Big).$$

In other words, the *maximum principle*

$$(\exists\, x_0 \in \mathbb{V}^{(\mathbb{B})})\, [\![\varphi(x_0)]\!] = \bigvee_{x \in \mathbb{V}^{(\mathbb{B})}} [\![\varphi(x)]\!]$$

is valid for every formula φ of ZFC.

The last equality accounts for the origin of the term *maximum principle*. The proof of the principle is a simple consequence of *mixing*.

A *partition of unity* in a Boolean algebra $\mathbb{B}$ is a family $(b_\xi)_{\xi \in \Xi}$ of elements of $\mathbb{B}$ such that

$$(\forall\, \xi, \eta \in \Xi)\, (\xi \neq \eta \Longrightarrow b_\xi \wedge b_\eta = 0) \quad \text{and} \quad \bigvee\{b_\xi :\ \xi \in \Xi\} = \mathbb{1}.$$

1.4.3. Mixing Principle. Given a family $(x_\xi)_{\xi\in\Xi}$ in $\mathbb{V}^{(\mathbb{B})}$ and a partition of unity $(b_\xi)_{\xi\in\Xi}$ in $\mathbb{B}$, there exists a (unique) *mixture* of (x_ξ) by (b_ξ); i.e., the unique $x \in \mathbb{V}^{(\mathbb{B})}$ such that $b_\xi \leqslant [\![x = x_\xi]\!]$ for all $\xi \in \Xi$.

The *mixture* x of a family (x_ξ) by (b_ξ) is denoted as follows:

$$x = \operatorname{mix}_{\xi\in\Xi}(b_\xi x_\xi) = \operatorname{mix}\{b_\xi x_\xi : \xi \in \Xi\}.$$

A set A of elements of $\mathbb{V}^{(\mathbb{B})}$ is called *cyclic* if the family of elements of A is closed under mixing. The least cyclic set that includes A is the *cyclic hull* of A, and we denote it by $\operatorname{cyc}(A)$.

1.4.4. The comparative analysis, mentioned above, presumes that there is some close interconnection between the universes $\mathbb{V}$ and $\mathbb{V}^{(\mathbb{B})}$. In other words, we need a rigorous mathematical technique that would allow us to reveal the interplay between the interpretations of one and the same fact in the two universes $\mathbb{V}$ and $\mathbb{V}^{(\mathbb{B})}$. The base for the technique is constituted by the operations of canonical embedding, descent, and ascent.

We start with the canonical embedding of the von Neumann universe, while the operations will be presented below. Given $x \in \mathbb{V}$, denote by $x^\wedge$ the *standard name* of x in $\mathbb{V}^{(\mathbb{B})}$, i.e., the element defined by the recursion schema:

$$\varnothing^\wedge := \varnothing, \quad \operatorname{dom}(x^\wedge) := \{y^\wedge : y \in x\}, \quad \operatorname{im}(x^\wedge) := \{\mathbb{1}\}.$$

1.4.5. Observe some simple properties of standard names we need in the sequel. Slightly abusing the language, we will call the passage from a set to its standard name *canonical embedding.*

(1) Given $x \in \mathbb{V}$ and a formula φ of ZF, we have

$$[\![(\exists\, y \in x^\wedge)\, \varphi(y)]\!] = \bigvee\{[\![\varphi(z^\wedge)]\!] : z \in x\},$$
$$[\![(\forall\, y \in x^\wedge)\, \varphi(y)]\!] = \bigwedge\{[\![\varphi(z^\wedge)]\!] : z \in x\}.$$

(2) The canonical embedding is injective. Moreover, for all $x, y \in \mathbb{V}$ we have

$$x \in y \iff \mathbb{V}^{(\mathbb{B})} \models x^\wedge \in y^\wedge,$$
$$x = y \iff \mathbb{V}^{(\mathbb{B})} \models x^\wedge = y^\wedge.$$

(3) The canonical embedding sends $\mathbb{V}$ onto $\mathbb{V}^{(2)}$:

$$(\forall\, u \in \mathbb{V}^{(2)})\, (\exists\, !x \in \mathbb{V})\ \mathbb{V}^{(\mathbb{B})} \models u = x^\wedge.$$

1.4.6. If π is a complete homomorphism from $\mathbb{B}$ to a Boolean algebra $\mathbb{D}$ then $\pi^* x^{\wedge} = x^{\Uparrow}$ for all $x \in \mathbb{V}$, where $(\,\cdot\,)^{\Uparrow}$ is the canonical embedding of $\mathbb{V}$ to $\mathbb{V}^{(\mathbb{D})}$.

1.4.7. Restricted Transfer Principle. For each restricted set-theoretic formula φ the following is provable in ZFC:

$$\varphi(x_1, \dots, x_n) \Longleftrightarrow \mathbb{V}^{(\mathbb{B})} \models \varphi(x_1^{\wedge}, \dots, x_n^{\wedge})$$

for all collection $x_1, \dots, x_n \in \mathbb{V}$.

Henceforth, working in the separated universe $\overline{\mathbb{V}}^{(\mathbb{B})}$, we agree to preserve the symbol $x^{\wedge}$ for the distinguished element of the class corresponding to x.

1.4.8. A *correspondence* from X to Y is a triple (X, Y, F) with $F \subset X \times Y$. The *domain* $\mathrm{dom}(\Phi)$ and the *image* $\mathrm{im}(\Phi)$ of Φ are introduced by

$$\mathrm{dom}(\Phi) := \{x \in X : \ (\exists\, y \in Y)\,(x, y) \in F\};$$
$$\mathrm{im}(\Phi) := \{y \in Y : \ (\exists\, x \in X)\,(x, y) \in F\}.$$

The correspondence Φ is often identified with the point-to-set mapping $x \mapsto \Phi(x) := F(x) := \{y \in Y : (x, y) \in F\}$. Consider another set Z and a correspondence $\Psi := (Y, Z, G)$ from Y to Z. Put

$$F^{-1} := \{(y, x) \in Y \times X : (x, y) \in F\};$$
$$G \circ F := \{(x, z) \in X \times Z : \ (\exists\, y \in Y)(x, y) \in F \wedge (y, z) \in G\}.$$

The correspondences $\Phi^{-1} := (Y, X, F^{-1})$ from Y to X and $\Psi \circ \Phi := (X, Y, G \circ F)$ from X to Z are called the *inverse* Φ and the *composite* of Φ and Ψ. If $A \subset X$ then $\Phi(A) := \bigcup_{x \in A} \Phi(x)$ is the *image* of A under Φ. In particular, $\mathrm{dom}(\Phi) = \Phi^{-1}(Y)$ and $\mathrm{im}(\Phi) = \Phi(X)$. Observe by way of example that the restricted transfer principle yields

$$\text{“}\Phi \text{ is a correspondence from } X \text{ to } Y\text{”}$$
$$\Longleftrightarrow \mathbb{V}^{(\mathbb{B})} \models \text{“}\Phi^{\wedge} \text{ is a correspondence from } X^{\wedge} \text{ to } Y^{\wedge}\text{”};$$
$$\mathbb{V}^{(\mathbb{B})} \models (\Psi \circ \Phi)^{\wedge} = \Psi^{\wedge} \circ \Phi^{\wedge} \wedge (\Phi^{-1})^{\wedge} = (\Phi^{\wedge})^{-1} \wedge \Phi(A)^{\wedge} = \Phi^{\wedge}(A^{\wedge});$$
$$\mathbb{V}^{(\mathbb{B})} \models \mathrm{dom}(\Phi)^{\wedge} = \mathrm{dom}(\Phi^{\wedge}) \wedge \mathrm{im}(\Phi)^{\wedge} = \mathrm{im}(\Phi^{\wedge});$$
$$\text{“}f \text{ is a function from } X \text{ to } Y\text{”}$$
$$\Longleftrightarrow \mathbb{V}^{(\mathbb{B})} \models \text{“}f^{\wedge} \text{ is a function from } X^{\wedge} \text{ to } Y^{\wedge}\text{”}$$
$$\text{(moreover, } f(x)^{\wedge} = f^{\wedge}(x^{\wedge}) \text{ within } \mathbb{V}^{(\mathbb{B})} \text{ for every } x \in X).$$

Thus, the standard name can be considered as a covariant functor of the category of sets (or correspondences) in $\mathbb{V}$ to the appropriate subcategory of $\mathbb{V}^{(2)}$ in the separated universe $\mathbb{V}^{(\mathbb{B})}$.

1.4.9. A set X is *finite* if X coincides with the image of a function on a finite ordinal. In symbols, this is expressed as $\mathrm{fin}(X)$; hence,

$$\mathrm{fin}(X) := (\exists\, n)(\exists\, f)(n \in \omega \wedge \mathrm{Fnc}\,(f) \wedge \mathrm{dom}(f) = n \wedge \mathrm{im}(f) = X).$$

Obviously, the above formula is not bounded. Nevertheless there is a simple transformation rule for the class of finite sets under the canonical embedding. Denote by $\mathscr{P}_{\mathrm{fin}}(X)$ the class of all finite subsets of X:

$$\mathscr{P}_{\mathrm{fin}}(X) := \{Y \in \mathscr{P}(X) : \ \mathrm{fin}(Y)\}.$$

1.4.10. *For an arbitrary set $X \in \mathbb{V}$ we have*

$$\mathbb{V}^{(\mathbb{B})} \models \mathscr{P}_{\mathrm{fin}}(X^{\wedge}) = \mathscr{P}_{\mathrm{fin}}(X)^{\wedge}.$$

1.5. Descents

In this section we define the mapping that assigns to each element $x \in \mathbb{V}^{(\mathbb{B})}$ some subclass of $\mathbb{V}^{(\mathbb{B})}$ which is a set in the sense of $\mathbb{V}$.

1.5.1. Given an arbitrary element x of the (separated) Boolean valued universe $\mathbb{V}^{(\mathbb{B})}$, we define the *descent* $x{\downarrow}$ of x as

$$x{\downarrow} := \big\{y \in \mathbb{V}^{(\mathbb{B})} : [\![y \in x]\!] = \mathbb{1}\big\}.$$

The class $x{\downarrow}$ is a set; i.e., $x{\downarrow} \in \mathbb{V}$ for each $x \in \mathbb{V}^{(\mathbb{B})}$. If $[\![x \neq \varnothing]\!] = \mathbb{1}$ then $x{\downarrow}$ is a nonempty set by the maximum principle. If $[\![a \subset x \wedge b \subset x]\!] = \mathbb{1}$ then $(a \cap b){\downarrow} = a{\downarrow} \cap b{\downarrow}$.

We list the simplest properties of descending:

1.5.2. Let $z \in \mathbb{V}^{(\mathbb{B})}$ and $[\![z \neq \varnothing]\!] = \mathbb{1}$. Then for every formula φ of ZFC we have

$$[\![(\forall\, x \in z)\, \varphi(x)]\!] = \bigwedge\{[\![\varphi(x)]\!] : x \in z{\downarrow}\},$$
$$[\![(\exists\, x \in z)\, \varphi(x)]\!] = \bigvee\{[\![\varphi(x)]\!] : x \in z{\downarrow}\}.$$

Moreover, there exists $x_0 \in z{\downarrow}$ such that $[\![\varphi(x_0)]\!] = [\![(\exists\, x \in z)\, \varphi(x)]\!]$.

1.5.3. Let Φ be a correspondence from X to Y within $\mathbb{V}^{(\mathbb{B})}$. Thus, Φ, X, and Y are elements of $\mathbb{V}^{(\mathbb{B})}$ and, moreover, $[\![\Phi \subset X \times Y]\!] = 1$. There is a unique correspondence $\Phi{\downarrow}$ from $X{\downarrow}$ to $Y{\downarrow}$ such that

$$\Phi{\downarrow}(A{\downarrow}) = \Phi(A){\downarrow}$$

for every nonempty subset A of X within $\mathbb{V}^{(\mathbb{B})}$. The correspondence $\Phi{\downarrow}$ from $X{\downarrow}$ to $Y{\downarrow}$ involved in the above proposition is called the *descent* of the correspondence Φ from X to Y in $\mathbb{V}^{(\mathbb{B})}$.

1.5.4. The correspondence $\Phi{\downarrow}$ is *extensional*; i.e., it satisfies the condition

$$y_1 \in \Phi{\downarrow}(x_1) \Longrightarrow [\![x_1 = x_2]\!] \leqslant \bigvee_{y_2 \in \Phi(x_2)} [\![y_1 = y_2]\!]$$

for all $x_1, x_2 \in \operatorname{dom}(\Phi{\downarrow}) = \operatorname{dom}(\Phi){\downarrow}$.

1.5.5. (1) The descent of the composite of correspondences within $\mathbb{V}^{(\mathbb{B})}$ is the composite of their descents:

$$(\Psi \circ \Phi){\downarrow} = \Psi{\downarrow} \circ \Phi{\downarrow}.$$

(2) The descent of the inverse correspondence within $\mathbb{V}^{(\mathbb{B})}$ is the inverse of its descent:

$$(\Phi^{-1}){\downarrow} = (\Phi{\downarrow})^{-1}.$$

(3) If $I_X \in \mathbb{V}^{(\mathbb{B})}$ is the identity mapping on X within $\mathbb{V}^{(\mathbb{B})}$ then

$$(I_X){\downarrow} = I_{X{\downarrow}}.$$

1.5.6. Suppose that $X, Y, f \in \mathbb{V}^{(\mathbb{B})}$ are such that $[\![X \neq \varnothing]\!] = \mathbb{1}$, $[\![Y \neq \varnothing]\!] = \mathbb{1}$, and $[\![f : X \to Y]\!] = \mathbb{1}$; i.e., f is a mapping from X to Y within $\mathbb{V}^{(\mathbb{B})}$. Then there is a unique mapping $f{\downarrow}$ from $X{\downarrow}$ to $Y{\downarrow}$ for which

$$[\![f{\downarrow}(x) = f(x)]\!] = \mathbb{1} \quad (x \in X{\downarrow}).$$

The descent of a function is *extensional* in the sense that (cp. 1.5.4)

$$[\![x_1 = x_2]\!] \leqslant [\![f{\downarrow}(x_1) = f{\downarrow}(x_2)]\!] \quad (x_1, x_2 \in X{\downarrow}).$$

By 1.5.5 we can consider the descent as a functor from the category of $\mathbb{B}$-valued sets and mappings (correspondences) to the category of the usual (i.e., in the sense of $\mathbb{V}$) sets and mappings (correspondences).

1.5.7. Given $x_1,\dots,x_n \in \mathbb{V}^{(\mathbb{B})}$, denote by $(x_1,\dots,x_n)^{\mathbb{B}}$ the corresponding ordered n-tuple within $\mathbb{V}^{(\mathbb{B})}$. Assume that P is an n-ary relation on X within $\mathbb{V}^{(\mathbb{B})}$; i.e., $X, P \in \mathbb{V}^{(\mathbb{B})}$ and $[\![P \subset X^{n^{\wedge}}]\!] = \mathbb{1}$ $(n \in \omega)$. Then there exists an n-ary relation P' on $X{\downarrow}$ such that

$$(x_1,\dots,x_n) \in P' \Longleftrightarrow [\![(x_1,\dots,x_n)^{\mathbb{B}} \in P]\!] = \mathbb{1}.$$

Slightly abusing notation, we denote the relation P' by the same symbol $P{\downarrow}$ and call it the *descent* of P.

1.5.8. Suppose that $X \in \mathbb{V}$, $X \neq \varnothing$; i.e., X is a nonempty set. Let ι denote the canonical embedding $x \mapsto x^{\wedge}$ $(x \in X)$. Then $\iota(X){\uparrow} = X^{\wedge}$ and $X = \iota^{-1}(X^{\wedge}{\downarrow})$. Using the above relations, we can extend the descent operation to the case in which $[\![\Psi$ is a correspondence from $X^{\wedge}$ to $Y]\!] = \mathbb{1}$, where $Y \in \mathbb{V}^{(\mathbb{B})}$ and $[\![Y \neq \varnothing]\!] = \mathbb{1}$. Namely, we put $\Psi{\downharpoonleft} := \Psi{\downarrow} \circ \iota$. In this case, $\Psi{\downharpoonleft}$ is called the *modified descent* of the correspondence Ψ. (If the context excludes ambiguity then we simply speak of descents using simple arrow.)

It is easy to see that $\Psi{\downharpoonleft}$ is the unique correspondence from X to $Y{\downarrow}$ satisfying the equality

$$\Psi{\downharpoonleft}(x) = \Psi(x^{\wedge}){\downarrow} \quad (x \in X).$$

If $\Psi := g$ is a function then $g{\downharpoonleft}$ is a function from X to $Y{\downarrow}$ uniquely determined by

$$[\![g{\downharpoonleft}(x) = g(x^{\wedge})]\!] = \mathbb{1} \quad (x \in X).$$

1.5.9. Let $[\![X^{\wedge} \to Y]\!]$ stand for the set of all members $g \in \mathbb{V}^{(\mathbb{B})}$ with $[\![g : X^{\wedge} \to Y]\!] = \mathbb{1}$, and $[X \to Y{\downarrow}]$ denote the set of all functions $f : X \to Y{\downarrow}$. The mapping $g \mapsto g{\downharpoonleft}$ is a bijection between $[\![X^{\wedge} \to Y]\!]$ and $[X \to Y{\downarrow}]$. The converse mapping $f \to f{\uparrow}$ is defined in the next section (see 1.6.8).

1.6. Ascents

We now consider some transformation acting in the reverse direction, i.e. sending each subset $x \subset \mathbb{V}^{(\mathbb{B})}$ into an element of $\mathbb{V}^{(\mathbb{B})}$.

1.6.1. Let $x \in \mathbb{V}$ and $x \subset \mathbb{V}^{(\mathbb{B})}$; i.e., let x be some set composed of $\mathbb{B}$-valued sets or, in other words, $x \in \mathscr{P}(\mathbb{V}^{(\mathbb{B})})$. Put $\varnothing{\uparrow} := \varnothing$ and

$$\operatorname{dom}(x{\uparrow}) = x, \quad \operatorname{im}(x{\uparrow}) = \{\mathbb{1}\}$$

if $x \neq \varnothing$. The element $x{\uparrow}$ (of the separated universe $\mathbb{V}^{(\mathbb{B})}$, i.e. the distinguished representative of the class $\{y \in \mathbb{V}^{(\mathbb{B})} : [\![y = x{\uparrow}]\!] = \mathbb{1}\}$) is called the *ascent* of x.

1.6.2. The following hold for every $x \in \mathscr{P}(\mathbb{V}^{(\mathbb{B})})$ and every formula φ:

$$[\![(\forall\, z \in x{\uparrow})\, \varphi(z)]\!] = \bigwedge_{y \in x} [\![\varphi(y)]\!],$$

$$[\![(\exists\, z \in x{\uparrow})\, \varphi(z)]\!] = \bigvee_{y \in x} [\![\varphi(y)]\!].$$

Introducing the ascent of a correspondence $\Phi \subset X \times Y$, we have to bear in mind a possible difference between the domain of departure X and the domain

$$\mathrm{dom}(\Phi) := \{x \in X : \Phi(x) \neq \varnothing\}.$$

This difference is inessential for our further goals; therefore, we assume that, speaking of ascents, we always consider the correspondences Φ that are defined everywhere; i.e., $\mathrm{dom}(\Phi) = X$.

1.6.3. Let $X, Y, \Phi \in \mathbb{V}^{(\mathbb{B})}$, and let Φ be a correspondence from X to Y. There exists a unique correspondence $\Phi{\uparrow}$ from $X{\uparrow}$ to $Y{\uparrow}$ within $\mathbb{V}^{(\mathbb{B})}$ such that

$$\Phi{\uparrow}(A{\uparrow}) = \Phi(A){\uparrow}$$

is valid for every subset A of $\mathrm{dom}(\Phi)$ if and only if Φ is *extensional*; i.e., Φ satisfies the condition

$$y_1 \in \Phi(x_1) \to [\![x_1 = x_2]\!] \leqslant \bigvee_{y_2 \in \Phi(x_2)} [\![y_1 = y_2]\!]$$

for all $x_1, x_2 \in \mathrm{dom}(\Phi)$. In this event, $\Phi{\uparrow} = \Phi'{\uparrow}$, where

$$\Phi' := \{(x, y)^{\mathbb{B}} : \ (x, y) \in \Phi\}.$$

The element $\Phi{\uparrow}$ is called the *ascent* of Φ.

1.6.4. The composite of extensional correspondences is extensional. In addition, the ascent of a composite is equal to the composite of the ascents (within $\mathbb{V}^{(\mathbb{B})}$): Assuming that $\mathrm{dom}(\Psi) \supset \mathrm{im}(\Phi)$ we have

$$\mathbb{V}^{(\mathbb{B})} \models (\Psi \circ \Phi)\!\uparrow = \Psi\!\uparrow \circ\, \Phi\!\uparrow.$$

Note that if Φ and Φ^{-1} are extensional then $(\Phi\!\uparrow)^{-1} = (\Phi^{-1})\!\uparrow$. But in general the extensionality of Φ in no way guarantees the extensionality of Φ^{-1}.

1.6.5. It is worth mentioning that if an extensional correspondence f is a function from X to Y then the ascent $f\!\uparrow$ is a function from $X\!\uparrow$ to $Y\!\uparrow$. Moreover, the extensionality property can be stated as follows:

$$[\![x_1 = x_2]\!] \leqslant [\![f(x_1) = f(x_2)]\!] \quad (x_1, x_2 \in X).$$

It is immediate from the last property that for an extensional function $f : X \to Y$, a family $(x_\xi)_{\xi\in\Xi}$ in X, and a partition of unity $(b_\xi)_{\xi\in\Xi}$ in $\mathbb{B}$ we have

$$f\Big(\operatorname*{mix}_{\xi\in\Xi} b_\xi x_\xi\Big) = \operatorname*{mix}_{\xi\in\Xi} b_\xi f(x_\xi).$$

1.6.6. Given $X \subset \mathbb{V}^{(\mathbb{B})}$, we denote by $\mathrm{mix}(X)$ the set of all mixtures of the form $\mathrm{mix}(b_\xi x_\xi)$, where $(x_\xi) \subset X$ and (b_ξ) is an arbitrary partition of unity. The following are referred to as the *rules for canceling arrows* or the *Escher rules*.

Let X and X' be subsets of $\mathbb{V}^{(\mathbb{B})}$ and let $f : X \to X'$ be an extensional mapping. Suppose that $Y, Y', g \in \mathbb{V}^{(\mathbb{B})}$ are such that $[\![\, Y, Y' \neq \varnothing]\!] = [\![\, g : Y \to Y']\!] = \mathbb{1}$. Then

$$X\!\uparrow\downarrow = \mathrm{mix}(X), \quad Y\!\downarrow\uparrow = Y;$$
$$f = (f\!\uparrow\downarrow)|_X, \quad g = g\!\downarrow\uparrow.$$

Observe that $\mathrm{mix}(X) = \mathrm{cyc}(X)$ (cp. 1.4.3).

1.6.7. Moreover, the mapping $f \mapsto f\!\uparrow$ is a one-to-one embedding of $\mathrm{Ext}(X, Y)$ into $\mathscr{Y}^{\mathscr{X}}\!\downarrow$, where $\mathrm{Ext}(X, Y)$ is the set of all extensional mappings from X to Y and $\mathscr{Y}^{\mathscr{X}}$ is the set of all mappings from $\mathscr{X}$ to $\mathscr{Y}$ within $\mathbb{V}^{(\mathbb{B})}$; i.e., $\mathscr{Y}^{\mathscr{X}}$ is a member of $\mathbb{V}^{(\mathbb{B})}$ defined as

$$\phi \in \mathscr{Y}^{\mathscr{X}} \leftrightarrow \phi : \mathscr{X} \to \mathscr{Y}.$$

This embedding is a bijection whenever $X = \mathrm{mix}(X)$ and $Y = \mathrm{mix}(Y)$.

1.6.8. Let $X \in \mathbb{V}$, $Y \in \mathbb{V}^{(\mathbb{B})}$, and let ι be as in 1.5.8. By analogy with 1.5.8 we can extend the ascent operations to the case that Φ is a correspondence from X to $Y{\downarrow}$. We need only to put $\Phi{\Uparrow} := (\Phi \circ \iota^{-1}){\uparrow}$. In this case, $\Phi{\Uparrow}$ is called the *modified ascent* of Φ. (Again, when there is no ambiguity, we simply speak of ascents and use simple arrows.) Clearly, $\Phi{\Uparrow}$ is the unique correspondence from $X^{\wedge}$ to Y within $\mathbb{V}^{(\mathbb{B})}$ satisfying

$$[\![\Phi{\Uparrow}(x^{\wedge}) = \Phi(x){\uparrow}]\!] = \mathbb{1} \quad (x \in X).$$

Moreover, the correspondence $\Phi \circ \iota^{-1}$ is extensional, and consequently we have $[\![\Phi{\Uparrow}(A^{\wedge}) = \Phi(A){\Uparrow}]\!] = \mathbb{1}$ for every nonempty $A \subset X$. If $\Phi := f$ is a function then $f{\Uparrow}$ is a function from $X^{\wedge}$ to Y within $\mathbb{V}^{(\mathbb{B})}$ uniquely determined by

$$[\![f{\Uparrow}(x^{\wedge}) = f(x)]\!] = \mathbb{1} \quad (x \in X).$$

1.6.9. The following useful fact is immediate from 1.4.10:

$$\mathscr{P}_{\mathrm{fin}}(X{\uparrow}) = \{\theta{\uparrow} : \theta \in \mathscr{P}_{\mathrm{fin}}(X)\}{\uparrow}.$$

1.7. Algebraic $\mathbb{B}$-systems

In this section we describe a category of algebraic systems comprised of descents of Boolean algebraic systems.

1.7.1. A *Boolean set* or a $\mathbb{B}$*-set* is a pair (X, d), where $X \in \mathbb{V}$, $X \neq \varnothing$, and d is a $\mathbb{B}$*-metric* on X; i.e., d is a mapping from $X \times X$ to the Boolean algebra $\mathbb{B}$ which satisfies the following conditions for all $x, y, z \in X$:

(a) $d(x, y) = 0 \Longleftrightarrow x = y$;

(b) $d(x, y) = d(y, x)$;

(c) $d(x, y) \leqslant d(x, z) \vee d(z, y)$.

Each $\varnothing \neq X \subset \mathbb{V}^{(\mathbb{B})}$ gives an example of a $\mathbb{B}$-set if we put

$$d(x, y) := [\![x \neq y]\!] = [\![x = y]\!]^{*} \quad (x, y \in X).$$

Another example is a nonempty $X \in \mathbb{V}$ with the "discrete $\mathbb{B}$-metric" d; i.e., $d(x, y) = \mathbb{1}$ if $x \neq y$ and $d(x, y) = \mathbb{0}$ if $x = y$.

Given $x \in X$, a family (x_{ξ}) in X, and a partition of unity (b_{ξ}) in $\mathbb{B}$, we write $x = \mathrm{mix}(b_{\xi} x_{\xi})$ provided that $b_{\xi} \wedge d(x, x_{\xi}) = \mathbb{0}$ for all ξ. As in 1.4.3, x is called the *mixture* of (x_ξ) by (b_ξ). The mixture, if existent, is unique. A $\mathbb{B}$-set X is called mix-*complete* if $\operatorname{mix}(b_\xi x_\xi)$ exists in X for all families (x_ξ) in X and partitions of unity in $\mathbb{B}$.

1.7.2. Let (X, d) be some $\mathbb{B}$-set. There exist an element $\mathscr{X} \in \mathbb{V}^{(\mathbb{B})}$ and an injection $\iota : X \to X' := \mathscr{X}\!\downarrow$ such that $d(x, y) = [\![\iota x \neq \iota y]\!]$ $(x, y \in X)$ and $X' = \operatorname{mix}(\iota X)$. Thus, every $x' \in X'$ admits the representation $x' = \operatorname{mix}_{\xi\in\Xi}(b_\xi \iota x_\xi)$, where $(x_\xi)_{\xi\in\Xi} \subset X$ and $(b_\xi)_{\xi\in\Xi}$ is a partition of unity in $\mathbb{B}$. The element $\mathscr{X} \in \mathbb{V}^{(\mathbb{B})}$ is referred to as the *Boolean valued representation* of the $\mathbb{B}$-set X. If X is a discrete $\mathbb{B}$-set then $\mathscr{X} = X^{\wedge}$ and $\iota x = x^{\wedge}$ $(x \in X)$. If $X \subset \mathbb{V}^{(\mathbb{B})}$ then $\iota\!\uparrow$ is an injection from $X\!\uparrow$ to $\mathscr{X}$ (within $\mathbb{V}^{(\mathbb{B})}$).

A mapping f from a $\mathbb{B}$-set (X, d) to a $\mathbb{B}$-set (X', d') is said to be *nonexpanding* or *contracting* if $d(x, y) \geqslant d'(f(x), f(y))$ for all $x, y \in X$.

1.7.3. We exhibit some example of a $\mathbb{B}$-set that is important for the sequel. Let X be a vector lattice and $\mathbb{B} := \mathbb{P}(X)$. Put

$$d(x, y) := \{|x - y|\}^{\perp\perp} \quad (x, y \in X).$$

Clearly, d satisfies 1.7.1 (b, c). At the same time, 1.7.1 (a) is valid only provided that X is Archimedean (cp. 2.1.3). Thus, (X, d) is a $\mathbb{B}$-set if and only if the vector lattice X is Archimedean.

1.7.4. Recall that a *signature* is a 3-tuple $\sigma := (F, P, \mathfrak{a})$, where F and P are some (possibly, empty) sets and $\mathfrak{a}$ is a mapping from $F \cup P$ to ω. If the sets F and P are finite then σ is a *finite signature*. In applications we usually deal with algebraic systems of finite signature.

An *n-ary operation* and an *n-ary predicate* on a $\mathbb{B}$-set A are contractive mappings $f : A^n \to A$ and $p : A^n \to \mathbb{B}$, respectively. By definition, f and p *are contractive mappings* provided that

$$d(f(a_0, \dots, a_{n-1}), f(a'_0, \dots, a'_{n-1})) \leqslant \bigvee_{k=0}^{n-1} d(a_k, a'_k),$$

$$d_s\big(p(a_0, \dots, a_{n-1}), p(a'_0, \dots, a'_{n-1})\big) \leqslant \bigvee_{k=0}^{n-1} d(a_k, a'_k)$$

for all a_0, $a'_0, \dots, a_{n-1}$, $a'_{n-1} \in A$, where d is the $\mathbb{B}$-metric of A, and d_s is the *symmetric difference* on $\mathbb{B}$; i.e.,

$$d_s(b_1, b_2) := b_1 \bigtriangleup b_2 := (b_1 \wedge b_2^*) \vee (b_1^* \wedge b_2).$$

Clearly, the above definitions depend on $\mathbb{B}$ and it would be cleaner to speak of $\mathbb{B}$-operations, $\mathbb{B}$-predicates, etc. We adhere to a simpler practice whenever this entails no confusion.

1.7.5. An *algebraic* $\mathbb{B}$-*system* $\mathfrak{A}$ of signature σ is a pair (A, ν), where A is a nonempty $\mathbb{B}$-set, the *underlying set* or *carrier* or *universe* of $\mathfrak{A}$, and ν is a mapping such that

(a) $\operatorname{dom}(\nu) = F \cup P$;

(b) $\nu(f)$ is an $\mathfrak{a}(f)$-ary operation on A for all $f \in F$; and

(c) $\nu(p)$ is an $\mathfrak{a}(p)$-ary predicate on A for every $p \in P$.

It is in common parlance to call ν the *interpretation* of $\mathfrak{A}$, in which case the notations f^ν and p^ν are substitutes for $\nu(f)$ and $\nu(p)$.

The signature of an algebraic $\mathbb{B}$-system $\mathfrak{A} := (A, \nu)$ is often denoted by $\sigma(\mathfrak{A})$; while the universe A of $\mathfrak{A}$, by $|\mathfrak{A}|$. Since $A^0 = \{\varnothing\}$, the nullary operations and predicates on A are mappings from $\{\varnothing\}$ to the set A and to the algebra $\mathbb{B}$ respectively. We agree to identify a mapping $g : \{\varnothing\} \to A \cup \mathbb{B}$ with the element $g(\varnothing)$. Each nullary operation on A thus transforms into the unique member of A. Analogously, the set of all nullary predicates on A turns into the Boolean algebra $\mathbb{B}$. If $F := \{f_1, \dots, f_n\}$ and $P := \{p_1, \dots, p_m\}$ then an algebraic $\mathbb{B}$-system of signature σ is often written down as $(A, \nu(f_1), \dots, \nu(f_n), \nu(p_1), \dots, \nu(p_m))$ or even $(A, f_1, \dots, f_n,\ p_1, \dots, p_m)$. In this event, the expression $\sigma = (f_1, \dots, f_n,\ p_1, \dots, p_m)$ is substituted for $\sigma = (F, P, \mathfrak{a})$.

1.7.6. We now address the $\mathbb{B}$-*valued interpretation* of a first-order language. Consider an algebraic $\mathbb{B}$-system $\mathfrak{A} := (A, \nu)$ of signature $\sigma := \sigma(\mathfrak{A}) := (F, P, \mathfrak{a})$. Let $\varphi(x_0, \dots, x_{n-1})$ be a formula of signature σ with n free variables. Assume given $a_0, \dots, a_{n-1} \in A$. We may readily define the truth value $|\varphi|^{\mathfrak{A}}(a_0, \dots, a_{n-1}) \in \mathbb{B}$ of a formula φ in the system $\mathfrak{A}$ for the given values $a_0, \dots, a_{n-1}$ of the variables $x_0, \dots, x_{n-1}$. The definition proceeds as usual by induction on the complexity of φ: Considering propositional connectives and quantifiers, we put

$$
\begin{aligned}
|\varphi \wedge \psi|^{\mathfrak{A}}(a_0, \dots, a_{n-1}) &:= |\varphi|^{\mathfrak{A}}(a_0, \dots, a_{n-1}) \wedge |\psi|^{\mathfrak{A}}(a_0, \dots, a_{n-1});\\
|\varphi \vee \psi|^{\mathfrak{A}}(a_0, \dots, a_{n-1}) &:= |\varphi|^{\mathfrak{A}}(a_0, \dots, a_{n-1}) \vee |\psi|^{\mathfrak{A}}(a_0, \dots, a_{n-1});\\
|\neg\, \varphi|^{\mathfrak{A}}(a_0, \dots, a_{n-1}) &:= |\varphi|^{\mathfrak{A}}(a_0, \dots, a_{n-1})^*;\\
|(\forall\, x_0)\, \varphi|^{\mathfrak{A}}(a_1, \dots, a_{n-1}) &:= \bigwedge_{a_0 \in A} |\varphi|^{\mathfrak{A}}(a_0, a_1, \dots, a_{n-1});
\end{aligned}
$$

$$|(\exists\, x_0)\, \varphi|^{\mathfrak{A}}\, (a_1, \dots, a_{n-1}) := \bigvee_{a_0 \in A} |\varphi|^{\mathfrak{A}}(a_0, a_1, \dots, a_{n-1}).$$

1.7.7. Now, the case of atomic formulas is in order. Suppose that $p \in P$ symbolizes an m-ary predicate, $q \in P$ is a nullary predicate, and $t_0, \dots, t_{m-1}$ are terms of signature σ assuming values $b_0, \dots, b_{m-1}$ at the given values $a_0, \dots, a_{n-1}$ of the variables $x_0, \dots, x_{n-1}$. By definition, we let

$$|\varphi|^{\mathfrak{A}}(a_0, \dots, a_{n-1}) := \nu(q), \quad \text{if } \varphi := q^{\nu};$$
$$|\varphi|^{\mathfrak{A}}(a_0, \dots, a_{n-1}) := d(b_0, b_1)^{*}, \quad \text{if } \varphi := (t_0 = t_1);$$
$$|\varphi|^{\mathfrak{A}}(a_0, \dots, a_{n-1}) := p^{\nu}(b_0, \dots, b_{m-1}), \quad \text{if } \varphi := p^{\nu}(t_0, \dots, t_{m-1}),$$

where d is a $\mathbb{B}$-metric on A.

1.7.8. Say that $\varphi(x_0, \dots, x_{n-1})$ is *valid* in $\mathfrak{A}$ at the given values $a_0, \dots, a_{n-1} \in A$ of $x_0, \dots, x_{n-1}$ and write $\mathfrak{A} \models \varphi(a_0, \dots, a_{n-1})$ provided that $|\varphi|^{\mathfrak{A}}(a_0, \dots, a_{n-1}) = \mathbb{1}_{\mathbb{B}}$. The alternative expressions are as follows: $a_0, \dots, a_{n-1} \in A$ *satisfies* $\varphi(x_0, \dots, x_{n-1})$, or $\varphi(a_0, \dots, a_{n-1})$ holds true in $\mathfrak{A}$. In case $\mathbb{B} := \{\mathbb{0}, \mathbb{1}\}$, we arrive at the conventional definition of the validity of a formula in an algebraic system.

Recall that a closed formula φ of signature σ is a *tautology* if φ is valid on every algebraic $\mathbb{2}$-system of signature σ.

1.7.9. Consider algebraic $\mathbb{B}$-systems $\mathfrak{A} := (A, \nu)$ and $\mathfrak{C} := (C, \mu)$ of the same signature σ. The mapping $h : A \to C$ is a *homomorphism* of $\mathfrak{A}$ to $\mathfrak{C}$ provided that, for all $a_0, \dots, a_{n-1} \in A$, the following are valid:

(1) $d_{\mathbb{B}}(h(a_1), h(a_2)) \leqslant d_A(a_1, a_2)$;

(2) $h(f^{\nu}) = f^{\nu}$, $\mathfrak{a}(f) = 0$;

(3) $h(f^{\nu}(a_0, \dots, a_{n-1})) = f^{\nu}(h(a_0), \dots, h(a_{n-1}))$, $0 \neq n := \mathfrak{a}(f)$;

(4) $p^{\nu}(a_0, \dots, a_{n-1}) \leqslant p^{\mu}(h(a_0), \dots, h(a_{n-1}))$, $n := \mathfrak{a}(p)$.

A homomorphism h is called *strong* if

(5) $\mathfrak{a}\ (p) := n \neq 0$ for all $p \in P$, and for all $c_0, \dots, c_{n-1} \in C$ we have

$$\begin{aligned} p^{\mu}&(c_0, \dots, c_{n-1}) \\ &\geqslant \bigvee_{a_0, \dots, a_{n-1} \in A} \Big\{ p^{\nu}(a_0, \dots, a_{n-1}) \wedge d_C(c_0, h(a_0)) \\ &\qquad\qquad \wedge \dots \wedge d_C(c_{n-1}, h(a_{n-1})) \Big\}. \end{aligned}$$

1.7.10. If a homomorphism h is injective and 1.7.9 (1, 4) are fulfilled with equality holding, then h is said to be an *isomorphism from* $\mathfrak{A}$ *to* $\mathfrak{C}$. Undoubtedly, all surjective isomorphisms h and, in particular, the identity mapping $I_A : A \to A$ are strong homomorphisms. The composite of (strong) homomorphisms is a (strong) homomorphism. Clearly, if h is a homomorphism and h^{-1} is a homomorphism too, then h is an isomorphism.

Note again that in the case of the two element Boolean algebra $\mathbb{2} := \{\mathbb{0}, \mathbb{1}\}$ we come to the conventional notions of homomorphism, strong homomorphism, and isomorphism.

1.8. Boolean Valued Algebraic Systems

Before giving the general definition of descent of an algebraic system, consider the descent of a very simple but important algebraic system, the two element Boolean algebra. Choose two arbitrary elements, say $0,\ 1 \in \mathbb{V}^{(\mathbb{B})}$, satisfying $[\![0 \neq 1]\!] = \mathbb{1}_{\mathbb{B}}$. We may for instance assume that $0 := \mathbb{0}_{\mathbb{B}}^{\wedge}$ and $1 := \mathbb{1}_{\mathbb{B}}^{\wedge}$.

1.8.1. *The descent C of the two-element Boolean algebra $\{0,1\}^{\mathbb{B}} \in \mathbb{V}^{(\mathbb{B})}$ is a complete Boolean algebra isomorphic to $\mathbb{B}$. The formulas*

$$[\![\chi(b) = 1]\!] = b, \quad [\![\chi(b) = 0]\!] = b^* \quad (b \in \mathbb{B})$$

yield the isomorphism $\chi : \mathbb{B} \to C$.

1.8.2. Let X and Y be some $\mathbb{B}$-sets, let $\mathscr{X}$ and $\mathscr{Y}$ be their Boolean valued representations, and let ι and $\varkappa$ be the corresponding embeddings $X \to \mathscr{X}{\downarrow}$ and $Y \to \mathscr{Y}{\downarrow}$. If $f : X \to Y$ is a contracting mapping then there is a unique element $g \in \mathbb{V}^{(\mathbb{B})}$ such that $[\![g : \mathscr{X} \to \mathscr{Y}]\!] = \mathbb{1}$ and $f = \varkappa^{-1} \circ g{\downarrow} \circ \iota$. We also accept the denotations $\mathscr{X} := \mathscr{F}^{\sim}(X) := X^{\sim}$ and $g := \mathscr{F}^{\sim}(f) := f^{\sim}$.

1.8.3. The following are valid:

(1) $\mathbb{V}^{(\mathbb{B})} \models f(A)^{\sim} = f^{\sim}(A^{\sim})$ for $A \subset X$.

(2) If $g : Y \to Z$ is a contraction then $g \circ f$ is a contraction and $\mathbb{V}^{(\mathbb{B})} \models (g \circ f)^{\sim} = g^{\sim} \circ f^{\sim}$.

(3) $\mathbb{V}^{(\mathbb{B})} \models$ "$f^{\sim}$ is injective" if and only if f is a $\mathbb{B}$-isometry.

(4) $\mathbb{V}^{(\mathbb{B})} \models$ "$f^{\sim}$ is surjective" if and only if for every $y \in Y$ we have $\bigvee\{d(f(x), y) : x \in X\} = \mathbb{1}$.

1.8.4. Consider an algebraic system $\mathfrak{A}$ of signature $\sigma^\wedge$ within $\mathbb{V}^{(\mathbb{B})}$, and let $[\![\mathfrak{A} = (A, \nu)^{\mathbb{B}}]\!] = \mathbb{1}$ for some A, $\nu \in \mathbb{V}^{(\mathbb{B})}$. The *descent* of $\mathfrak{A}$ is the pair $\mathfrak{A}{\Downarrow} := (A{\downarrow}, \mu)$, where μ is the function determined from the formulas:

$$\mu : f \mapsto (\nu{\downarrow\!\!\!\!\downarrow}(f)){\downarrow} \quad (f \in F),$$
$$\mu : p \mapsto \chi^{-1} \circ (\nu{\downarrow\!\!\!\!\downarrow}(p)){\downarrow} \quad (p \in P).$$

Here χ is the above isomorphism of the Boolean algebras $\mathbb{B}$ and $\{0,1\}^{\mathbb{B}}{\downarrow}$.

In more detail, the modified descent $\nu{\downarrow\!\!\!\!\downarrow}$ is the mapping with domain $\mathrm{dom}(\nu{\downarrow\!\!\!\!\downarrow}) = F \cup P$. Given $p \in P$, observe $[\![\mathfrak{a}\ (p)^\wedge = \mathfrak{a}^\wedge(p^\wedge)]\!] = \mathbb{1}$, $[\![\nu{\downarrow\!\!\!\!\downarrow}(p) = \nu(p^\wedge)]\!] = \mathbb{1}$ and so

$$\mathbb{V}^{(\mathbb{B})} \models \nu{\downarrow\!\!\!\!\downarrow}(p) : A^{\mathfrak{a}\ (f)^\wedge} \to \{0,1\}^{\mathbb{B}}.$$

It is now obvious that $(\nu{\downarrow\!\!\!\!\downarrow}(p)){\downarrow} : (A{\downarrow})^{\mathfrak{a}\ (f)} \to C := \{0,1\}^{\mathbb{B}}{\downarrow}$ and we can put $\mu(p) := \chi^{-1} \circ (\nu{\downarrow\!\!\!\!\downarrow}(p)){\downarrow}$.

1.8.5. Let $\varphi(x_0, \dots, x_{n-1})$ be a fixed formula of signature σ in n free variables. Write down the formula $\Phi(x_0, \dots, x_{n-1}, \mathfrak{A})$ in the language of set theory which formalizes the proposition $\mathfrak{A} \models \varphi(x_0, \dots, x_{n-1})$. Recall that the formula $\mathfrak{A} \models \varphi(x_0, \dots, x_{n-1})$ determines an n-ary predicate on A or, which is the same, a mapping from A^n to $\{0,1\}$. By the maximum and transfer principles, there is a unique element $|\varphi|^{\mathfrak{A}} \in \mathbb{V}^{(\mathbb{B})}$ such that

$$[\![|\varphi|^{\mathfrak{A}} : A^{n^\wedge} \to \{0,1\}^{\mathbb{B}}]\!] = \mathbb{1},$$
$$[\![|\varphi|^{\mathfrak{A}}(a{\uparrow}) = 1]\!] = [\![\Phi(a(0), \dots, a(n-1), \mathfrak{A})]\!] = \mathbb{1}$$

for every function $a : n \to A{\downarrow}$. Instead of $|\varphi|^{\mathfrak{A}}(a{\uparrow})$ we will write $|\varphi|^{\mathfrak{A}}$ $(a_0, \dots, a_{n-1})$, where $a_l := a(l)$. Therefore, the formula

$$\mathbb{V}^{(\mathbb{B})} \models \text{“}\varphi(a_0, \dots, a_{n-1}) \text{ is valid in } \mathfrak{A}\text{”}$$

holds true if and only if $[\![\Phi(a_0, \dots, a_{n-1}, \mathfrak{A})]\!] = \mathbb{1}$.

1.8.6. Let $\mathfrak{A}$ be an algebraic system of signature $\sigma^\wedge$ within $\mathbb{V}^{(\mathbb{B})}$. Then $\mathfrak{A}{\Downarrow}$ is a laterally complete algebraic $\mathbb{B}$-system of signature σ. In this event $\chi \circ |\varphi|^{\mathfrak{A}\Downarrow} = |\varphi|^{\mathfrak{A}}{\downarrow}$ for each formula φ of signature σ. An algebraic system is *laterally* complete whenever its universe is mix-complete.

1.8.7. Let $\mathfrak{A}$ and $\mathfrak{B}$ be algebraic systems of the same signature $\sigma^\wedge$ within $\mathbb{V}^{(\mathbb{B})}$. Put $\mathfrak{A}' := \mathfrak{A}{\downarrow}$ and $\mathfrak{B}' := \mathfrak{B}{\downarrow}$. Then, if h is a homomorphism (strong homomorphism) within $\mathbb{V}^{(\mathbb{B})}$ from $\mathfrak{A}$ to $\mathfrak{B}$ then $h' := h{\downarrow}$ is a homomorphism (strong homomorphism) of the $\mathbb{B}$-systems $\mathfrak{A}'$ and $\mathfrak{B}'$.

Conversely, if $h' : \mathfrak{A}' \to \mathfrak{B}'$ is a homomorphism (strong homomorphism) of algebraic $\mathbb{B}$-systems then $h := h'{\uparrow}$ is a homomorphism (strong homomorphism) from $\mathfrak{A}$ to $\mathfrak{B}$ within $\mathbb{V}^{(\mathbb{B})}$.

1.8.8. *Let $\mathfrak{A} := (A, \nu)$ be an algebraic $\mathbb{B}$-system of signature σ. Then there are $\mathscr{A}$ and $\mu \in \mathbb{V}^{(\mathbb{B})}$ such that the following are fulfilled:*

(1) $\mathbb{V}^{(\mathbb{B})} \models$ *"$(\mathscr{A}, \mu)$ is an algebraic system of signature $\sigma^{\wedge}$".*

(2) *If $\mathfrak{A}' := (A', \nu')$ is the descent of $(\mathscr{A}, \mu)$ then $\mathfrak{A}'$ is a laterally complete algebraic $\mathbb{B}$-system of signature σ.*

(3) *There is an isomorphism $\imath$ from $\mathfrak{A}$ to $\mathfrak{A}'$ such that $A' = \mathrm{mix}(\imath(A))$.*

(4) *For every formula φ of signature σ in n free variables, we have*

$$
\begin{aligned}
|\varphi|^{\mathfrak{A}}(a_0, \dots, a_{n-1}) &= |\varphi|^{\mathfrak{A}'}(\imath(a_0), \dots, \imath(a_{n-1})) \\
&= \chi^{-1} \circ (|\varphi|^{\mathfrak{A}^{\sim}}){\downarrow}(\imath(a_0), \dots, \imath(a_{n-1}))
\end{aligned}
$$

for all $a_0, \dots, a_{n-1} \in A$ and χ the same as in 1.8.1.

1.9. Boolean Valued Ordinals and Cardinals

Now we dwell for a while on the properties of ordinals and cardinals within the Boolean valued universe.

1.9.1. A set x is *transitive* (not to be confused with a transitive relation) if each member of x is also a subset of x:

$$\mathrm{Tr}(x) := (\forall\, y)\,(y \in x \to y \subset x).$$

An *ordinal* is a transitive set well-ordered by membership. The record $\mathrm{Ord}(x)$ means that x is ordinal. The terms *ordinal number* or *transfinite number* are also in common parlance. Denote by On the class of all ordinals. We often let lowercase Greek letters stand for ordinals. Moreover, we use the abbreviations:

$$\alpha < \beta := \alpha \in \beta, \quad \alpha \leqslant \beta := (\alpha \in \beta) \vee (\alpha = \beta), \quad \alpha + 1 := \alpha \cup \{\alpha\}.$$

If $\alpha < \beta$ then we say that α precedes β and β succeeds α.

1.9.2. If $x \subset \mathrm{On}$ is a set then $\bigcup x$ is the least upper bound of x in the class On ordered by the membership relation $\in$. The least upper bound of a set of ordinals x is usually denoted by $\lim(x)$. An ordinal α is a *limit ordinal* if $\alpha \neq \varnothing$ and $\lim(\alpha) = \alpha$.

In other words, α is a limit ordinal provided that α cannot be written down as $\alpha = \beta + 1$ with $\beta \in \mathrm{On}$. The least limit ordinal whose existence is ensured by the axiom of infinity is denoted by ω (or ω_0; see 1.9.4 (2)).

The least ordinal, the zero set $0 := \varnothing$, belongs to ω.

The successor $1 := 0+1 = 0 \cup \{0\} = \{\varnothing\}$ contains the only element 0. Furthermore, $2 := 1 \cup \{1\} = \{0\} \cup \{1\} = \{0,1\} = \{0,\{0\}\}$, $3 := 2 \cup \{2\} = \{0,\{0\},\{\{0,\{0\}\}\}$, etc. Thus,

$$\omega := \{0,\{0\},\{0,\{0\}\},\dots\} = \{0,1,2,\dots\}.$$

The following notation is also used:

$$\mathbb{N} := \omega \setminus \{0\} = \{1,2,\dots\}.$$

The members of ω are *finite ordinals* or *positive integers*. The elements of $\mathbb{N}$ are called *natural numbers* or simply *naturals* by historical reasons. But the whole of ω is called the *naturals* rather often too (since 0 seems very common today).

1.9.3. Two sets are *equipollent*, or *equipotent*, or *of the same cardinality* if there is a bijection of one of them onto the other. An ordinal that is equipotent to no preceding ordinal is a *cardinal*. Each natural is a cardinal.

A cardinal not in ω is an *infinite* cardinal. Therefore, ω is the least infinite cardinal.

Given an ordinal α, we denote by ω_α an infinite cardinal such that the ordered set of all infinite cardinals less than ω_α is similar to α. If such a cardinal exists then it is unique.

1.9.4. Cardinal Comparability Principle. The following are valid:

(1) Infinite cardinals form a well-ordered proper class.

(2) To each ordinal α there is a cardinal ω_α so that the mapping $\alpha \mapsto \omega_\alpha$ is a similarity between the class of ordinals and the class of infinite cardinals.

(3) There is a mapping $|\cdot|$ from the universal class $\mathbb{U}$ onto the class of all cardinals such that the sets x and $|x|$ are equipollent for all $x \in \mathbb{U}$.

1.9.5. Clearly, $\mathrm{Ord}(x)$ is a bounded formula. Since $\lim(\alpha) \leqslant \alpha$ for every ordinal α, the formula $\mathrm{Ord}(x) \wedge x = \lim(x)$ may be rewritten as

$\mathrm{Ord}(x) \wedge (\forall\, t \in x)(\exists\, s \in x)(t \in s)$. Hence, $\mathrm{Ord}(x) \wedge x = \lim(x)$ is a bounded formula as well. Finally, the record

$$\mathrm{Ord}(x) \wedge x = \lim(x) \wedge (\forall\, t \in x)(t = \lim(t) \to t = 0)$$

convinces us that the "least limit ordinal" is a bounded formula too. Hence α is the least limit ordinal if and only if $\mathbb{V}^{(\mathbb{B})} \models$ "$\alpha^{\wedge}$ is the least limit ordinal" by the restricted transfer principle. Since ω is the least limit ordinal in $\mathbb{V}$, we have $\mathbb{V}^{(\mathbb{B})} \models$ "$\omega^{\wedge}$ is the least limit ordinal."

1.9.6. It can be demonstrated that $\mathbb{V}^{(\mathbb{B})} \models$ "$\mathrm{On}^{\wedge}$ is the unique ordinal class that is not an ordinal" (with $\mathrm{On}^{\wedge}$ defined in an appropriate way). Given $x \in \mathbb{V}^{(\mathbb{B})}$, we thus have

$$[\![\mathrm{Ord}(x)]\!] = \bigvee_{\alpha \in \mathrm{On}} [\![x = \alpha^{\wedge}]\!].$$

This yields the convenient formulas for quantification over ordinals:

$$[\![(\forall\, x)\big(\mathrm{Ord}(x) \to \psi(x)\big)]\!] = \bigwedge_{\alpha \in \mathrm{On}} [\![\psi(\alpha^{\wedge})]\!],$$
$$[\![(\exists\, x)\big(\mathrm{Ord}(x) \wedge \psi(x)\big)]\!] = \bigvee_{\alpha \in \mathrm{On}} [\![\psi(\alpha^{\wedge})]\!].$$

1.9.7. *Each ordinal within $\mathbb{V}^{(\mathbb{B})}$ is a mixture of some set of standard ordinals. In other words, given $x \in \mathbb{V}^{(\mathbb{B})}$, we have $\mathbb{V}^{(\mathbb{B})} \models \mathrm{Ord}(x)$ if and only if there are an ordinal $\beta \in \mathrm{On}$ and a partition of unity $(b_\alpha)_{\alpha \in \beta} \subset \mathbb{B}$ such that $x = \mathrm{mix}_{\alpha \in \beta}\, b_\alpha \alpha^{\wedge}$.*

1.9.8. By transfer every Boolean valued model enjoys the classical cardinal comparability principle. In other words, there is a $\mathbb{V}^{(\mathbb{B})}$-class Cn whose elements are only cardinals. Let $\mathrm{Card}(\alpha)$ denote the formula that declares α a cardinal. Within $\mathbb{V}^{(\mathbb{B})}$ we then see that $\alpha \in \mathrm{Cn} \leftrightarrow \mathrm{Card}(\alpha)$. Clearly, the class of ordinals $\mathrm{On}^{\wedge}$ is similar to the class of infinite cardinals, and we denote the similarity from $\mathrm{On}^{\wedge}$ into Cn by $\alpha \mapsto \aleph_\alpha$. In particular, to each standard ordinal $\alpha \in \mathrm{On}$ there is a unique infinite cardinal $\aleph_{\alpha^{\wedge}}$ within $\mathbb{V}^{(\mathbb{B})}$ since $[\![\mathrm{Ord}(\alpha^{\wedge})]\!] = \mathbb{1}$.

1.9.9. Recall that it is customary to refer to the standard names of ordinals and cardinals as *standard ordinals* and *standard cardinals* within $\mathbb{V}^{(\mathbb{B})}$.

(1) The standard name of the least infinite cardinal is the least infinite cardinal:

$$\mathbb{V}^{(\mathbb{B})} \models (\omega_0)^{\wedge} = \aleph_0.$$

(2) Within $\mathbb{V}^{(\mathbb{B})}$ there is a mapping $|\cdot|$ from the universal class $\mathbb{U}_{\mathbb{B}}$ into the class Cn such that x and $|x|$ are equipollent for all x. The standard names of equipollent sets are of the same cardinality:

$$(\forall\, x \in \mathbb{V})\, (\forall\, y \in \mathbb{V})\, \big(|x| = |y| \Longrightarrow [\![|x^{\wedge}| = |y^{\wedge}|]\!] = \mathbb{1}\big).$$

1.9.10. (1) If the standard name of an ordinal α is a cardinal then α is a cardinal too:

$$(\forall\, \alpha \in \mathrm{On})\, \big(\mathbb{V}^{(\mathbb{B})} \models \mathrm{Card}(\alpha^{\wedge})\big) \Longrightarrow \mathrm{Card}(\alpha).$$

(2) The standard name of a finite cardinal is a finite cardinal too:

$$(\forall\, \alpha \in \mathrm{On})\, \big(\alpha < \omega \Longrightarrow \mathbb{V}^{(\mathbb{B})} \models \mathrm{Card}(\alpha^{\wedge}) \wedge \alpha^{\wedge} \in \aleph_0\big).$$

1.9.11. Given $x \in \mathbb{V}^{(\mathbb{B})}$, we have $\mathbb{V}^{(\mathbb{B})} \models \mathrm{Card}(x)$ if and only if there are nonempty set of cardinals Γ and a partition of unity $(b_\gamma)_{\gamma\in\Gamma} \subset \mathbb{B}$ such that $x = \mathrm{mix}_{\gamma\in\Gamma}\, b_\gamma\gamma^{\wedge}$ and $\mathbb{V}^{(\mathbb{B}_\gamma)} \models \mathrm{Card}(\gamma^{\wedge})$ with $\mathbb{B}_\gamma := [\mathbb{0}, b_\gamma]$ for all $\gamma \in \Gamma$. In other words, each Boolean valued cardinal is a mixture of some set of relatively standard cardinals.

1.9.12. A σ-complete Boolean algebra $\mathbb{B}$ is said to be *σ-distributive* if $\mathbb{B}$ satisfies one of the following equivalent conditions (cp. [365, 19.1]):

(1) $\bigwedge_{n\in\mathbb{N}} \bigvee_{m\in\mathbb{N}} b^n_m = \bigvee_{m\in\mathbb{N}^{\mathbb{N}}} \bigwedge_{n\in\mathbb{N}} b^n_{m(n)}$ for all $(b^n_m)_{n,m\in\mathbb{N}}$ in $\mathbb{B}$;

(2) $\bigvee_{n\in\mathbb{N}} \bigwedge_{m\in\mathbb{N}} b^n_m = \bigwedge_{m\in\mathbb{N}^{\mathbb{N}}} \bigvee_{n\in\mathbb{N}} b^n_{m(n)}$ for all $(b^n_m)_{n,m\in\mathbb{N}}$ in $\mathbb{B}$;

(3) $\bigvee_{\varepsilon\in\{1,-1\}^{\mathbb{N}}} \bigwedge_{n\in\mathbb{N}} \varepsilon(n)b_n = \mathbb{1}$ for all $(b_n)_{n\in\mathbb{N}}$ in $\mathbb{B}$.

$\big($Here $1b_n := b_n$ and $(-1)b_n$ is the complement of b_n.$\big)$

It is worth noting that σ-distributive Boolean algebras are often referred to as (ω,ω)-distributive Boolean algebras. This term is related to a more general notion, (α,β)-distributivity, where α and β are arbitrary cardinals.

1.9.13. *If $\mathbb{B}$ is a complete Boolean algebras then the following are equivalent:*

(1) *$\mathbb{B}$ is σ-distributive.*

(2) $\mathbb{V}^{(\mathbb{B})} \models (\aleph_0)^{\aleph_0} = (\omega^\omega)^{\wedge}$.

(3) $\mathbb{V}^{(\mathbb{B})} \models \mathscr{P}(\aleph_0) = \mathscr{P}(\omega)^{\wedge}$.

The latter is the result by Scott on (α, β)-distributive Boolean algebras which was formulated in the case $\alpha = \beta = \omega$ (cp. [43, 2.14]). More details and references are collected in [249].

1.10. Boolean Algebras

In this section we specify the general results of 1.8.6–1.8.8 on algebraic $\mathbb{B}$-systems for Boolean algebras.

1.10.1. Let $\mathbb{B}$ be a complete Boolean algebra and let $\imath$ be a Boolean homomorphism from $\mathbb{B}$ to a Boolean algebra D. Define the mapping $d : D \times D \to \mathbb{B}$ by putting

$$d(x, y) := \bigwedge \{b \in \mathbb{B} : \ \imath(b^*) \wedge x = \imath(b^*) \wedge y\} \quad (x, y \in D).$$

It can easily be seen that d is a *Boolean* (or $\mathbb{B}$-*valued*) *semimetric*; i.e., d satisfies 1.7.1 (b, c) and $d(x, x) = 0$ for all $x \in D$. Moreover, d is a $\mathbb{B}$-metric whenever $\imath$ is a complete homomorphism. The results of Section 1.8 are applicable to D:

If $\imath : \mathbb{B} \to D$ is a complete Boolean homomorphism then D is an algebraic $\mathbb{B}$-system of signature $(\vee, \wedge, *, \mathbb{0}, \mathbb{1})$. This $\mathbb{B}$-system is laterally complete whenever D is complete.

1.10.2. *Let $\mathscr{D}$ be a Boolean algebra within $\mathbb{V}^{(\mathbb{B})}$ and $D := \mathscr{D}{\downarrow}$. Then D is a Boolean algebra and there exists a complete monomorphism $\imath : \mathbb{B} \to D$ such that for all $x, y \in D$ and $b \in \mathbb{B}$ we have*

$$b \leqslant [\![x \leqslant y]\!] \Longleftrightarrow \imath(b) \wedge x \leqslant \imath(b) \wedge y.$$

Moreover, D is order complete if and only if so is $\mathscr{D}$ within $\mathbb{V}^{(\mathbb{B})}$.

$\lhd$ In view of 1.8.6 D is a laterally complete algebraic $\mathbb{B}$-system of signature $(\vee, \wedge, *, \mathbb{0}, \mathbb{1})$. The fact that D is a Boolean algebra follows also from 1.8.6. $\rhd$

1.10.3. *Let $\mathscr{D}_1$ and $\mathscr{D}_2$ be complete Boolean algebras in $\mathbb{V}^{(\mathbb{B})}$. Put $D_k := \mathscr{D}_k{\downarrow}$ and denote by $\imath_k : \mathbb{B} \to D_k$ $(k := 1, 2)$ the monomorphism from 1.10.2. If $h \in \mathbb{V}^{(\mathbb{B})}$ is an internal isomorphism from $\mathscr{D}_1$ to $\mathscr{D}_2$, then $H := h{\downarrow}$ is a Boolean isomorphism from D_1 to D_2 such that the diagram* *commutes:*

$$\begin{tikzcd} & \mathbb{B} \arrow[dl, "\imath_1"'] \arrow[dr, "\imath_2"] & \\ D_1 \arrow[rr, "H"'] & & D_2 \end{tikzcd}$$

Conversely, if $H : D_1 \to D_2$ is an isomorphism of Boolean algebras and the above diagram commutes, then $h := H{\uparrow}$ is a Boolean isomorphism from $\mathscr{D}_1$ to $\mathscr{D}_2$ within $\mathbb{V}^{(\mathbb{B})}$.

$\lhd$ All can be deduced from 1.8.7 and 1.10.2. $\rhd$

1.10.4. *Assume that D is a complete Boolean algebra and $\jmath : \mathbb{B} \to D$ is a complete monomorphism. Then there are a complete Boolean algebra $\mathscr{D}$ within $\mathbb{V}^{(\mathbb{B})}$ and an isomorphism H from D onto $D' := \mathscr{D}{\downarrow}$ such that the diagram commutes:*

$$\begin{tikzcd} & \mathbb{B} \arrow[dl, "\jmath"'] \arrow[dr, "\imath'"] & \\ D \arrow[rr, "H"'] & & D' \end{tikzcd}$$

where $\imath'$ is the monomorphism from $\mathbb{B}$ to D' defined as in 1.10.2.

$\lhd$ According to 1.10.1 D is a laterally complete algebraic $\mathbb{B}$-system of signature $\sigma := \{\vee, \wedge, *, \mathbb{0}, \mathbb{1}\}$. By 1.8.8 we can assume without loss of generality that D coincides with $\mathscr{D}{\downarrow}$ and $\jmath = \imath$ for some algebraic system $\mathscr{D}$ within $\mathbb{V}^{(\mathbb{B})}$ of signature $\sigma^{\wedge}$.

If a formula φ formalizes the axioms of a complete Boolean algebra, then we can check by direct calculation of Boolean truth values that $|\varphi|^D = \mathbb{1}$. From 1.8.8 we deduce $[\![|\varphi|^{\mathscr{D}} = 1]\!] = \mathbb{1}$. Hence, $\mathscr{D}$ is a complete Boolean algebra within $\mathbb{V}^{(\mathbb{B})}$. $\rhd$

1.11. Applications to Boolean Homomorphisms

In this section we demonstrate that some Hahn–Banach type extension results for Boolean homomorphisms can be deduced by Boolean valued interpretation of the properties of filters and ultrafilters.

1.11.1. Let X be a set, and let $\mathbb{B}$ be a complete Boolean algebra. Given $\sigma \in \mathbb{V}^{(\mathbb{B})}$ with $[\![\sigma \subset X^{\wedge}]\!] = \mathbb{1}$, define $h_\sigma : X \to \mathbb{B}$ as

$$h_\sigma(x) := [\![x^{\wedge} \in \sigma]\!] \quad (x \in X).$$

The mapping $\sigma \mapsto h_\sigma$ is a bijection between $\mathscr{P}(X^{\wedge})\downarrow$ and $\mathbb{B}^X$.

$\lhd$ This mapping is clearly injective. Take $h : X \to \mathbb{B}$. Let η stand for the modified ascent of $\chi \circ h : X \to \{0,1\}^{\mathbb{B}}\downarrow$, with χ the same as in 1.8.1. By the maximum principle, we can define $\sigma \in \mathscr{P}(X^{\wedge})$ as $\sigma := \{x \in X^{\wedge} : \eta(x) = 1\}$. Then we derive from 1.8.1 that

$$h(x) = [\![\chi(h(x) = 1]\!] = [\![\eta(x^{\wedge}) = 1]\!] = [\![x^{\wedge} \in \sigma]\!].$$

So, $h = h_\sigma$. $\rhd$

1.11.2. Take another Boolean algebra A. A mapping $p : A \to \mathbb{B}$ is called a *submorphism* (*supermorphism*), provided that $p(\mathbb{1}_A) = \mathbb{1}$ and $p(x \vee y) = p(x) \vee p(y)$ ($p(\mathbb{0}_A) = \mathbb{0}$ and $p(x \wedge y) = p(x) \wedge p(y)$ respectively) for all $x, y \in A$. If $h^* : x \mapsto h(x)^*$ $(x \in A)$ is a Boolean homomorphism then we call $h : A \to \mathbb{B}$ a *Boolean antimorphism.*

The fact that A is a Boolean algebra can be expressed by a restricted formula. Consequently, $\mathbb{V}^{(\mathbb{B})} \models$ "$A^{\wedge}$ is a Boolean algebra."

1.11.3. *Assume that $\sigma \in \mathscr{P}(A^{\wedge})\downarrow$. Then the following hold:*

(1) $\mathbb{V}^{(\mathbb{B})} \models$ *"σ is an ideal" $\Longleftrightarrow$ h_σ^* is a submorphism.*

(2) $\mathbb{V}^{(\mathbb{B})} \models$ *"σ is a filter" $\Longleftrightarrow$ h_σ is a supermorphism.*

(3) $\mathbb{V}^{(\mathbb{B})} \models$ *"σ is an ultrafilter" $\Longleftrightarrow$ h_σ is a Boolean homomorphism.*

(4) $\mathbb{V}^{(\mathbb{B})} \models$ *"σ is a maximal ideal" $\Longleftrightarrow$ h_σ is a Boolean antimorphism.*

$\lhd$ A subset A of a Boolean algebra is a filter (an ideal) if and only if A does not contain $\mathbb{0}$ ($\mathbb{1}$), and the meet (join) of two elements of $\mathbb{B}$ belongs to A if and only if each of the two elements belongs to A.

The same fact holds for the Boolean valued universe by transfer. Therefore, the formulas $\mathbb{V}^{(\mathbb{B})} \models$ "σ is an ideal" and $\mathbb{V}^{(\mathbb{B})} \models$ "ρ is a filter" amount to the two groups of equalities:

$$[\![\mathbb{1}_A^{\wedge} \in \sigma]\!] = \mathbb{0}, \quad [\![x^{\wedge} \vee y^{\wedge} \in \sigma]\!] = [\![x^{\wedge} \in \sigma]\!] \wedge [\![y^{\wedge} \in \sigma]\!];$$

$$[\![\mathbb{0}_A^{\wedge} \in \rho]\!] = \mathbb{0}, \quad [\![x^{\wedge} \wedge y^{\wedge} \in \rho]\!] = [\![x^{\wedge} \in \rho]\!] \wedge [\![y^{\wedge} \in \rho]\!].$$

This yields (1) and (2) by 1.11.1. Furthermore, a filter in a Boolean algebra is an ultrafilter if and only if each element or its Boolean complement belongs to the filter. Interpreting this criterion in the Boolean valued universe we see that $\mathbb{V}^{(\mathbb{B})} \models$ "σ is an ultrafilter" if and only if h_σ is a supermorphism and $[\![x^* \in \sigma]\!] \vee [\![x \in \sigma]\!] = \mathbb{1}$ $(x \in A^\wedge)$, or, equivalently, $h_\sigma(x^*) \vee h_\sigma(x) = \mathbb{1}$ $(x \in A)$. Observe that $h_\sigma(x^*) \wedge h_\sigma(x) = [\![(x^*)^\wedge \in \sigma]\!] \wedge [\![x^\wedge \in \sigma]\!] = [\![0 \in \sigma]\!] = \mathbb{0}$ amounts to the identity $h_\sigma(x^*) = h_\sigma(x)^*$. These arguments prove (3), while (4) is easy from (3). $\triangleright$

1.11.4. Let $\mathrm{Hom}(A, \mathbb{B})$ be the set of all Boolean homomorphisms from A to $\mathbb{B}$. By $\mathfrak{U}(A^\wedge)$ we denote the element of $\mathbb{V}^{(\mathbb{B})}$ such that $[\![\mathfrak{U}(A^\wedge)$ is the set of all ultrafilters in the Boolean algebra $A^\wedge]\!] = \mathbb{1}$.

The mapping $\psi \mapsto h_\psi$ is a bijection between $\mathfrak{U}(A^\wedge){\downarrow}$ and $\mathrm{Hom}(A, \mathbb{B})$.

$\triangleleft$ The claim follows from 1.11.1 and 1.11.3(3). $\triangleright$

1.11.5. Sandwich Theorem. *Let $p, q : A \to \mathbb{B}$ be such that p is a submorphism and q is a supermorphism. Assume that $q(x) \leqslant p(x)$ for all $x \in A$. Then there is $h \in \mathrm{Hom}(A, \mathbb{B})$ satisfying*

$$q(x) \leqslant h(x) \leqslant p(x) \quad (x \in A).$$

$\triangleleft$ By 1.11.1 there are $\rho, \sigma \in \mathscr{P}(A^\wedge){\downarrow}$ such that $q = h_\sigma$ and $p^* = h_\rho$. By 1.11.3 $\mathbb{V}^{(\mathbb{B})} \models$ "σ is a filter" and $\mathbb{V}^{(\mathbb{B})} \models$ "ρ is an ideal." Moreover, $[\![x^\wedge \in \sigma]\!] = q(x) \leqslant p(x) = [\![x^\wedge \notin \rho]\!]$ and so $\mathbb{V}^{(\mathbb{B})} \models$ "$\sigma \cap \rho$ is empty." By the transfer and maximum principles we see that the filters σ and $\rho^* := \{x^* : x \in \rho\}$ lie in some common filter within $\mathbb{V}^{(\mathbb{B})}$. Otherwise, there would exist $x \in \sigma$ and $y \in \rho$ such that $x \wedge y^* = \mathbb{0}$ or, equivalently, $x \leqslant y$. But this would imply that $x \in \rho$, contradicting the condition $\sigma \cap \rho = \varnothing$. We now choose some ultrafilter $\psi \subset A^\wedge$ within $\mathbb{V}^{(\mathbb{B})}$ that includes both σ and ρ^*. Put $h := h_\psi$ and note that h is a Boolean homomorphism by 1.11.3 (3). Clearly, $\sigma \subset \psi$ and $\psi \cap \rho = \varnothing$. Thus $x \in \sigma \to x \in \psi \to x \notin \rho$ for all $x \in A^\wedge$. Calculating the Boolean truth value of the latter formula yields $q(x) \leqslant h(x) \leqslant p(x)$. $\triangleright$

Deriving corollaries to the Sandwich Theorem, we mention two facts about extension of Boolean homomorphisms. The first is analogous to the Hahn–Banach Extension Theorem for linear functionals.

1.11.6. Hahn–Banach Theorem for Boolean Homomorphisms. *Let A_0 be a subalgebra of a Boolean algebra A and let $p : A \to \mathbb{B}$ be a submorphism. Assume that a Boolean homomorphism $h_0 : A_0 \to \mathbb{B}$ satisfies the inequality $h_0(x_0) \leqslant p(x_0)$ for all $x_0 \in A_0$. Then there exists a Boolean homomorphism $h : A \to \mathbb{B}$ such that $h(x) \leqslant p(x)$ $(x \in A)$ and $h(x_0) = h_0(x_0)$ $(x_0 \in A_0)$.*

$\lhd$ Introduce the mapping $q : A \to \mathbb{B}$ by letting

$$q(x) := \bigvee \{h_0(a) : \ a \in A_0, \ a \leqslant x\} \quad (x \in A).$$

Clearly, q is a supermorphism, $q \leqslant p$, and $q|_{A_0} = h_0$. By 1.11.5 there is $h \in \operatorname{Hom}(A, \mathbb{B})$ satisfying $q \leqslant h \leqslant p$. In particular, $h_0|_{A_0} \leqslant h$. Given $x \in A_0$, we hence see that $h(x) = h(x^*)^* \leqslant h_0(x^*)^* = h_0(x)$. Therefore, $h|_{A_0} = h$ and h is a desired homomorphism. $\rhd$

1.11.7. Sikorski Extension Theorem. *Each Boolean homomorphism h_0 from a subalgebra A_0 of an arbitrary Boolean algebra A to a complete Boolean algebra $\mathbb{B}$ admits an extension to a Boolean homomorphism h defined on the whole of A.*

$\lhd$ Let $p(\mathbb{0}_A) = \mathbb{0}$ and $p(x) = \mathbb{1}$ for $\mathbb{0}_A \neq x \in A$. Then p is a submorphism and $h_0 \leqslant p|_{A_0}$. So, the claim follows from 1.11.6.

We may proceed otherwise not appealing to 1.11.6, but recalling 1.11.1 and 1.11.3. Indeed, $[\![A_0^\wedge$ is a subalgebra of the algebra $A^\wedge]\!] = \mathbb{1}$, and by 1.11.1 $h_0 = h_\sigma$ for some $\sigma \in \mathscr{P}(A_0^\wedge){\downarrow}$. By 1.11.3 (3) $[\![\sigma$ is an ultrafilter in $A_0^\wedge]\!] = \mathbb{1}$. The claim follows now from the fact that σ, presenting (within $\mathbb{V}^{(\mathbb{B})}$) a filterbase in $A^\wedge$, admits extension to some ultrafilter $\psi \subset A^\wedge$, so that $h = h_\psi$ is a sought homomorphism. $\rhd$

1.12. Variations on the Theme

The purpose of this section is to present briefly intuitionistic set theory and quantum set theory as counterparts of Boolean valued set theory. This is done by constructing universes based respectively on a complete Heyting algebra and a complete orthomodular lattice, which are reasonable models of set theory. Intuitionistic propositional calculus is based on Heyting algebras and quantum propositional calculus is based on orthomodular lattices, just as classical propositional calculus is based on Boolean algebras.

1.12.A. Heyting Algebras and Orthomodular Lattices

In this section we give a brief overview of the elementary properties of Heyting algebras and quantum logics.

1.12.A.1. Consider some lattice $\mathbb{L}$. The *relative pseudocomplement* of $x \in \mathbb{L}$ with respect to $y \in \mathbb{L}$ is the top of the set $\{z \in \mathbb{L} : x \wedge$

$z \leqslant y\}$. The pseudocomplement of x with respect to y, if existent, is denoted by $x \Rightarrow y$. The following easy property can be viewed as another definition of relative pseudocomplement:

$$z \leqslant x \Rightarrow y \iff x \wedge z \leqslant y.$$

1.12.A.2. A lattice Ω with zero $\mathbb{0}$ and unity $\mathbb{1}$ is called a *Heyting algebra* provided that the relative pseudocomplement $x \Rightarrow y$ exists for every two elements $x, y \in \Omega$. A Heyting algebra is also referred to as a *pseudo-Boolean algebra* or *Brouwer lattice*.

Each distributive Heyting algebra is a distributive lattice.

The lattice $\mathscr{O}(X)$ of all open subsets of a topological space X ordered by inclusion is a complete Heyting algebra. If $A, B, B_\xi \in \mathscr{O}(X)$ then $\bigvee_{\xi\in\Xi} B_\xi = \bigcup_{\xi\in\Xi} B_\xi$ and $A \Rightarrow B$ coincides with the interior of $(X \setminus A) \cup B$.

1.12.A.3. *Given elements x, y, and z of a Heyting algebra, we have*

(1) $x \Rightarrow y = \mathbb{1} \iff x \leqslant y$; $x \Rightarrow \mathbb{1} = \mathbb{1}$; $\mathbb{1} \Rightarrow y = y$.

(2) $(x \Longrightarrow y) \wedge y = y$; $x \wedge (x \Rightarrow y) = x \wedge y$.

(3) $x_1 \leqslant x_2 \implies x_2 \Rightarrow y \leqslant x_1 \Rightarrow y$.

(4) $y_1 \leqslant y_2 \implies x \Rightarrow y_1 \leqslant x \Rightarrow y_2$.

(5) $(x \Rightarrow y) \wedge (x \Rightarrow z) = x \Rightarrow (y \wedge z)$.

(6) $(x \Rightarrow z) \wedge (y \Rightarrow z) = (x \vee y) \Rightarrow z$.

(7) $(x \Rightarrow y) \wedge (y \Rightarrow z) \leqslant (x \Rightarrow z)$.

(8) $(x \Rightarrow y) \leqslant ((x \wedge z) \Rightarrow (y \wedge z))$.

(9) $x \Rightarrow (y \Rightarrow z) = (x \wedge y) \Rightarrow z = y \Rightarrow (x \Rightarrow z)$.

(10) $x \Rightarrow (y \Rightarrow z) \leqslant (x \Rightarrow y) \Rightarrow (x \Rightarrow z)$.

$\triangleleft$ See [344, Theorem I.12.2]. $\triangleright$

1.12.A.4. The *pseudocomplement* of x in a lattice $\mathbb{L}$ with zero is the top of the set $\{y \in \mathbb{L} : x \wedge y = \mathbb{0}\}$. Clearly, if $\mathbb{L}$ is a Heyting algebra then each $x \in \mathbb{L}$ has the pseudocomplement $x^* := x \Rightarrow \mathbb{0}$. Therefore, the properties of pseudocomplements follow from the corresponding properties of relative pseudocomplements.

1.12.A.5. *Given elements x, y, and z of a Heyting algebra, we have*

(1) $x \leqslant y \implies y^* \leqslant x^*$; $x \wedge x^* = \mathbb{0}$.

(2) $x^* = \mathbb{1} \iff x = \mathbb{0}$; $x^* = \mathbb{0} \leftrightarrow x = \mathbb{1}$.

(3) $x \leqslant x^{**}$; $x^* = x^{***}$; $(x \vee x^*)^{**} = \mathbb{1}$.

(4) $(x \vee y)^* = x^* \wedge y^*$; $(x \wedge y)^* \geqslant x^* \vee y^*$.

(5) $x \Rightarrow y^* = y \Rightarrow x^* = (x \wedge y)^*$.

(6) $x \Rightarrow y \leqslant y^* \Rightarrow x^*$; $(x \Rightarrow y) \wedge (x \Rightarrow y^*) = x^*$.

$\lhd$ See [344, Theorem I.12.3]. $\rhd$

1.12.A.6. An element x of a Heyting algebra Ω is *regular* provided that $x^{**} = x$. The set of all regular elements of a Heyting algebra Ω with the order induced from Ω will be denoted by $\mathfrak{R}(\Omega)$. Note that $x \in \Omega$ is regular if and only of $x = y^*$ for some $y \in \Omega$. The following holds (cp. [344, Theorem IV.6.5]):

The ordered set $\mathfrak{R}(\Omega)$ is a Boolean algebra for each Heyting algebra Ω.

1.12.A.7. An *ortholattice* or *orthocomplemented lattice* is a lattice $\mathbb{L}$ with some bottom $\mathbb{0}$ and top $\mathbb{1}$, together with the unary operation $(\cdot)^{\perp} : \mathbb{L} \to \mathbb{L}$, called *orthocomplementation*, such that for all $x, y \in \mathbb{L}$ we have

$$x \wedge x^{\perp} = \mathbb{0}, \quad x \vee x^{\perp} = \mathbb{1}; \quad x^{\perp\perp} := (x^{\perp})^{\perp} = x;$$

$$(x \vee y)^{\perp} = x^{\perp} \wedge y^{\perp}, \quad (x \wedge y)^{\perp} = x^{\perp} \vee y^{\perp}.$$

An ortholattice $\mathbb{L}$ is a Boolean algebra if and only if $\mathbb{L}$ satisfies the distributive law $(x, y, z \in \mathbb{L})$:

$$x \wedge (y \vee z) = (x \vee y) \wedge (x \vee z).$$

We say that some elements x and y of an ortholattice are *orthogonal* and write $x \perp y$ whenever $x \leqslant y^{\perp}$ or, equivalently, $y \leqslant x^{\perp}$.

1.12.A.8. *If $\mathbb{L}$ is an ortholattice then for all $x, y, z \in \mathbb{L}$ the following are equivalent:*

(1) *If $x \leqslant y$ then there exists $u \in \mathbb{L}$ with $x \perp u$ and $y = x \vee u$.*

(2) *$x \leqslant y$ implies $y = x \vee (y \wedge x^{\perp})$.*

(3) *$(x \wedge y) \vee (x^{\perp} \wedge y^{\perp}) = \mathbb{1}$ implies $x = y$.*

(4) *$(x \vee (x^{\perp} \wedge (x \vee y)) = x \vee y$.*

(5) *If $x = (x \wedge y) \vee (x \wedge y^{\perp})$ and $x = (x \wedge z) \vee (x \wedge z^{\perp})$, then $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.*

$\lhd$ See [117, 187, 332]. $\rhd$

1.12.A.9. An ortholattice $\mathbb{L}$ is said to be an *orthomodular lattice* or *quantum logic* if one of (and hence all) the conditions 1.12.A.8 (1–5) is satisfied. An ortholattice is orthomodular if and only if it does not include a subalgebra isomorphic to *hexagon* $\mathbb{O}6$, which is defined as the set $\mathbb{O}6 := \{a,\, b,\, a^\perp,\, b^\perp,\, \mathbb{0},\, \mathbb{1}\}$ with the order $\mathbb{0} < a < b < \mathbb{1}$, $\mathbb{0} < a^\perp < b^\perp < \mathbb{1}$ (cp. [188, p. 22]).

1.12.A.10. A quantum logic will be denoted by $\mathscr{Q}$. Let $H = (H, \langle\cdot,\cdot\rangle)$ be a complex Hilbert space and let $\mathscr{M}$ be a von Neumann algebra on H. Denote by $\mathbb{P}(\mathscr{M})$ the set of all orthogonal projections in $\mathscr{M}$ with the induced order: $P \leqslant Q \iff (\forall\, x \in H)(\langle x, Px\rangle \leqslant \langle x, Qx\rangle)$ $(P, Q \in \mathbb{P}(\mathscr{M}))$. Then $\mathscr{Q} := \mathbb{P}(\mathscr{M})$ is a quantum logic with $P^\perp = I_H - P$ and $P \wedge Q = \lim_{n\to\infty}(P \circ Q)^n$.

1.12.A.11. *For all x and y of a complete orthomodular lattice, the following are equivalent:*

(1) *The sublattice generated by $\{x, x^\perp, y, y^\perp\}$ is distributive.*

(2) $(x \wedge y) \vee (x^\perp \wedge y) \vee (x \wedge y^\perp) \vee (x^\perp \wedge y^\perp) = \mathbb{1}$.

(3) $(x \wedge y) \vee (x^\perp \wedge y) = y$.

(4) $(x \vee y^\perp) \wedge y = x \wedge y$.

$\vartriangleleft$ See [336, Theorems 2.15, 2.17, and 2.19]. $\vartriangleright$

1.12.A.12. Elements x and y of a complete orthomodular lattice $\mathscr{Q}$ are said to be *compatible,* in symbols $x \,\natural\, y$, if one of (and hence all) the equivalent assertions 1.12.A.11 (1–4) is fulfilled. For a subset C of $\mathscr{Q}$ and $x \in \mathscr{Q}$ we put $x \,\natural\, C$, whenever $x \,\natural\, y$ for all $y \in C$. The set of elements compatible with all other elements, called the *center,* of $\mathscr{Q}$ is a complete Boolean algebra.

1.12.A.13. *Let $\mathscr{Q}$ be a complete orthomodular lattice. Assume that $x \in \mathscr{Q}$, $(x_\xi)_{\xi\in\Xi}$ is a family in $\mathscr{Q}$, and $x \,\natural\, x_\xi$ for all $\xi \in \Xi$. Then the following hold:*

(1) $x \wedge \bigvee_{\xi\in\Xi} x_\xi = \bigvee_{\xi\in\Xi} x \wedge x_\xi$.

(2) $x \vee \bigwedge_{\xi\in\Xi} x_\xi = \bigwedge_{\xi\in\Xi} x \vee x_\xi$.

(3) $x \,\natural \bigvee_{\xi\in\Xi} x_\xi$.

(4) $x \,\natural \bigwedge_{\xi\in\Xi} x_\xi$.

$\vartriangleleft$ See [336, Theorems 2.21 and 2.24]. $\vartriangleright$

1.12.A.14. Given a nonempty subset of a complete orthomodular lattice $\mathscr{Q}$, put

$$S(A) = \big\{x \in \mathscr{Q} :\ x \mathrel{\downarrow\!\!\circ} A\ (\forall\, p, q \in A)\, (p \wedge x \mathrel{\downarrow\!\!\circ} q \wedge x)\big\}.$$

The *Boolean domain* $\perp\!\!\!\perp(A)$ of $A \subset \mathscr{Q}$ is defined as $\perp\!\!\!\perp(A) = \bigvee S(A)$. Also, put $\perp\!\!\!\perp(x_1, \dots, x_n) := \perp\!\!\!\perp(A)$ whenever $A = \{x_1, \dots, x_n\}$. It is easily seen from 1.12.A.13 that $\perp\!\!\!\perp(A) \mathrel{\downarrow\!\!\circ} A$ and $p \wedge \perp\!\!\!\perp(A) \mathrel{\downarrow\!\!\circ} q \wedge \perp\!\!\!\perp(A)$ for all $p, q \in A$ (cp. [326, § 2] and [382, Proposition 4 and Corollary 1]).

1.12.B. Intuitionistic Set Theory

In this section we present an intuitionistic set theory $\mathrm{ZF_I}$ based on intuitionistic logic IL.

1.12.B.1. *Intuitionistic predicate calculus* IL is a formal deductive system with a set of logical axioms and a set of rules of deduction. The logical axioms are the same as the classical excluding the axiom scheme 1.1.6 (12). Thus, the logical axioms of IL comprise the axioms schemes 1.1.6 (1–11) and the following axiom schemes: if $\varphi(x)$ is a formula and t is a term then we have $(\forall\, x)\, \varphi \to \varphi(t)$ and $\varphi(t) \to (\exists\, x)\, \varphi$.

We only have the three rules of the predicate calculus: *modus ponens* and the two *quantification laws*:

(MP) If φ and $\varphi \to \psi$ are theorems of CL then ψ is a theorem of CL too.

($\forall$) If x is not free in φ then $\varphi \to \psi$ implies that $\varphi \to (\forall\, x)\, \psi$.

($\exists$) If x is not free in ψ then $\varphi \to \psi$ implies that $(\exists\, x)\, \varphi \to \psi$.

By definition, all theorems of IL are theorems of CL. The converse is obviously false: the CL-theorems $\neg(\neg\, \varphi) \to \varphi$ and $(\neg\, \varphi) \vee \neg(\neg\, \varphi)$ are not theorems of IL. But $\varphi \to \neg\, (\neg\, \varphi)$ and $\neg\neg(\varphi \vee \neg\, \varphi)$ are IL-theorems. Note that neither of the logical connectives $\vee$, $\wedge$, and $\to$ can be expressed through the others in IL.

1.12.B.2. The system $\mathrm{ZF_I}$ of *intuitionistic set theory* is the first order theory with the nonlogical symbols $\in$, $=$, E, where E is a predicate symbol with one argument place and $\mathsf{E}x := x \in \mathsf{E}$ is interpreted as "x exists."

There are two groups of nonlogical axioms: the equality axioms and the ZF type axioms. First, we present the four equality axioms:

$$u = u,$$
$$u = v \to v = u,$$
$$u = v \wedge \varphi(u) \to \varphi(v),$$
$$(\mathsf{E}u \vee \mathsf{E}v \to u = v) \to u = v.$$

1.12.B.3. To formulate the ZF type nonlogical axioms, we use the notation $\dot\forall\, x \dots$ and $\dot\exists\, x \dots$ to abbreviate $\forall\, x\,(\mathsf{E}x \to \dots)$ and $\exists\, x\,(\mathsf{E}x \wedge \dots)$, respectively.

Extension: $\dot\forall\, z\ (z \in u \leftrightarrow z \in v) \wedge (\mathsf{E}u \leftrightarrow \mathsf{E}v) \to u = v$.

Pair: $\dot\exists\, z\ \dot\forall\, x\ (x \in z \leftrightarrow x = u \vee x = v)$.

Union: $\dot\exists\, v\ \dot\forall\, x\ (x \in v \leftrightarrow \dot\exists\, y \in u\ (x \in y))$.

Power: $\dot\exists\, v\ \dot\forall\, x\ (x \in v \leftrightarrow x \subseteq u)$.

Infinity: $\dot\exists\, v\ (\dot\exists\, x \in v \wedge \dot\forall\, x \in v\, \dot\exists\, y \in v\ (x \in y))$.

Separation: $\dot\exists\, v\ \dot\forall\, x\ (x \in v \leftrightarrow x \in u \wedge \varphi(x))$.

Foundation: $\dot\forall\, x\ (\dot\forall\, y \in x \varphi(y) \to \varphi(x)) \to \dot\forall\, x\, \varphi(x)$.

Replacement: $\dot\exists\, v\ (\dot\forall\, x \in u\, \dot\exists\, y\, \varphi(x, y) \to \dot\forall\, x \in u\, \dot\exists\, y \in v\, \varphi(x, y))$.

To the end of this section, we write $\forall\, x$ and $\exists\, x$ instead of $\dot\forall\, x$ and $\dot\exists\, x$, since $\forall\, x$ and $\exists\, x$ always appear in the form $\dot\forall\, x$ and $\dot\exists\, x$.

1.12.B.4. A model of a theory consists of a universe M, a set Ω of truth values, a function $\mathscr{E} : M \to \Omega$, and a function $[\![\,\cdot\,]\!]$ that assigns some truth value $[\![\varphi(x_1, \dots, x_n)]\!] \in \Omega$ to each sentence $\varphi(u_1, \dots, u_n)$ and all $x_1, \dots, x \in M$. We say that $(\Omega, M, \mathscr{E}, [\![\,\cdot\,]\!])$ is a model of $\mathrm{ZF_I}$, if the operations $\wedge$, $\vee$, $\bigwedge$, $\bigvee$, $\Rightarrow$, and $(\cdot)^*$, corresponding to the logical operations $\wedge$, $\vee$, $\forall$, $\exists$, $\to$, and $\neg$, are defined on Ω and satisfy the following conditions for all sentences φ, ψ and an arbitrary formula $\varphi(u)$ with one variable:

(1) $\{[\![\varphi]\!] :\ \varphi \text{ is a sentence}\} = \Omega$;

(2) $[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \wedge [\![\psi]\!]$;

(3) $[\![\varphi \vee \psi]\!] = [\![\varphi]\!] \vee [\![\psi]\!]$;

(4) $[\![\neg\, \varphi]\!] = [\![\psi]\!]^*$;

(5) $[\![\forall\, x\, \varphi(x)]\!] = \bigwedge_{x \in M} (\mathscr{E}x \Rightarrow [\![\varphi(x)]\!])$;

(6) $[\![\exists\, x\, \varphi(x)]\!] = \bigvee_{x \in M} (\mathscr{E}x \wedge [\![\varphi(x)]\!])$;

(7) $[\![\varphi \to \psi]\!] = [\![\varphi]\!] \Rightarrow [\![\psi]\!]$;

(8) if $\vdash \varphi \to \psi$ then $[\![\varphi]\!] = [\![\psi]\!]^*$.

It is easy to see that if $(\Omega, M, \mathscr{E}, [\![\,\cdot\,]\!])$ is a model of $\mathrm{ZF_I}$ then Ω is a Heyting algebra. Conversely, if Ω is a complete Heyting algebra then we can define a universe M and function $[\![\,\cdot\,]\!]$ such that $(\Omega, M, [\![\cdot]\!])$ is a model of $\mathrm{ZF_I}$, as follows.

1.12.B.5. Let $\mathbb{V}$ be a standard universe of ZFC. Define $\mathbb{V}_\alpha^{(\Omega)} \subseteq \mathbb{V}$ for all $\alpha \in \mathrm{Ord}$ by transfinite induction. Assume that $\mathbb{V}_\beta^{(\Omega)}$ is defined already for $\beta < \alpha$ and each element u of $\mathbb{V}_\beta^{(\Omega)}$ is of the form $(D(u), \lfloor u \rfloor, \mathscr{E}u)$, where $D(u) \subset \mathbb{V}_\gamma^{(\Omega)}$ for some $\gamma < \beta$, $\lfloor u \rfloor$ is a function of $D(u)$ into Ω and $\mathscr{E}u \in \Omega$. For convenience we write $u(x)$ instead of $\lfloor u \rfloor(x)$. Now we define $\mathbb{V}_\alpha^{(\Omega)}$ by

$$\mathbb{V}_\alpha^{(\Omega)} = \Big\{ u = (D(u), \lfloor u \rfloor, \mathscr{E}u) : \ (\exists\, \beta < \alpha)\big(D(u) \subset \mathbb{V}_\beta^{(\Omega)}\big)$$
$$\lfloor u \rfloor : \ D(u) \to \Omega \wedge \mathscr{E}u \in \Omega \wedge \forall\, x \in D(u)\, (u(x) \leqslant \mathscr{E}u \wedge \mathscr{E}x) \Big\}.$$

Finally, we define the Heyting valued universe as

$$\mathbb{V}^{(\Omega)} = \bigcup_{\alpha \in \mathrm{On}} \mathbb{V}_\alpha^{(\Omega)}.$$

1.12.B.6. The Heyting truth value $[\![\varphi]\!]$ is defined by induction on the number of logical symbols in φ. An atomic sentence over $\mathbb{V}^{(\Omega)}$ is of the from $u = v$, $u \in v$ or $\mathsf{E}u$, where $u, v \in \mathbb{V}^{(\Omega)}$. Now, $[\![u = v]\!]$ and $[\![u \in v]\!]$ are defined by recursion as follows (cp. 1.1.8 and 1.2.4):

$$[\![u = v]\!]$$
$$= \bigwedge_{x \in D(u)} (u(x) \Rightarrow [\![x \in v]\!]) \wedge \bigwedge_{y \in D(v)} (v(y) \Rightarrow [\![y \in u]\!]) \wedge (\mathscr{E}u \Leftrightarrow \mathscr{E}v),$$
$$[\![u \in v]\!] = \bigvee_{y \in D(v)} (v(y) \wedge [\![u = y]\!]),$$
$$[\![\mathsf{E}u]\!] = \mathscr{E}u.$$

Note that for all $x \in D(u)$ and $y \in D(v)$ we have

$$\max(\mathrm{rank}(x), \mathrm{rank}(y)) < \max(\mathrm{rank}(u), \mathrm{rank}(v)).$$

Hence,

$$[\![x \in v]\!] = \bigvee_{y \in D(v)} \big(v(y) \wedge [\![x = v]\!]\big),$$

$$[\![u = y]\!] = \bigwedge_{x \in D(v)} \big(u(x) \Rightarrow [\![x \in y]\!]\big) \wedge \bigwedge_{t \in D(v)} \big(y(t) \Rightarrow [\![t \in u]\!]\big) \wedge \big(\mathscr{E}u \Leftrightarrow \mathscr{E}y\big)$$

are defined at an earlier stage.

For a sentence with logical symbols $[\![\,\cdot\,]\!]$ is defined as in 1.2.3:

$$[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \wedge [\![\psi]\!],$$

$$[\![\varphi \vee \psi]\!] = [\![\varphi]\!] \vee [\![\psi]\!],$$

$$[\![\varphi \to \psi]\!] = [\![\varphi]\!] \Rightarrow [\![\psi]\!],$$

$$[\![\neg\varphi]\!] = [\![\varphi]\!] \Rightarrow \mathbb{0} = [\![\varphi]\!]^*,$$

$$[\![\exists\, x\varphi(x)]\!] = \bigvee_{x \in \mathbb{V}^{(\Omega)}} \mathscr{E}x \wedge [\![\varphi(x)]\!],$$

$$[\![\forall\, x\varphi(x)]\!] = \bigwedge_{x \in \mathbb{V}^{(\Omega)}} (\mathscr{E}x \Rightarrow [\![\varphi(x)]\!]).$$

1.12.B.7. *If Ω is a complete Heyting algebra then* $(\Omega, \mathbb{V}^{(\Omega)}, \mathscr{E}, [\![\,\cdot\,]\!])$, *defined above, is a model of* $\mathrm{ZF_I}$.

$\lhd$ See [149, 150, 386]. $\rhd$

1.12.C. Quantum Set Theory

1.12.C.1. *Quantum predicate calculus* QL is a formal deductive system, and so it defined as a language consisting of propositions and connectives and the axioms and a rule of inference. Just as classical propositional calculus bases on Boolean algebras, quantum propositional calculus bases on orthomodular lattices. We will avoid going into the details of the quantum propositional calculus. The interested reader is referred to [188, 336].

1.12.C.2. The system $\mathrm{ZF_Q}$ of *quantum set theory* is the first order theory with the nonlogical symbols $\in, =, \underline{\vee}$, where $\underline{\vee} := \underline{\vee}(x_0, \dots, x_n)$ is an n-ary predicate symbol for all $n = 2, 3, \dots$ interpreted as "$x_0, \dots, x_n$ are compatible." The implication can be defined as the *Sasaki hook*: $\varphi \to \psi := \neg\,\varphi \vee (\varphi \wedge \psi)$. Consider the equality axioms:

(1) $u = u$.

(2) $u = v \to v = u$.

(3) $\underline{\vee}\,(u,v,u') \wedge u = u' \wedge u \in v \to u' \in v$.

(4) $\underline{\vee}\,(u,v,u') \wedge u \in v \wedge v = v' \to u \in v'$.

(5) $\underline{\vee}\,(u,v,w) \wedge u = v \wedge v = w \to u = w$.

1.12.C.3. Consider the special axioms of quantum set theory.

(1) *Axiom of Pair*:

$$(\forall\, u,v)\Big(\underline{\vee}\,(u,v) \to (\exists\, x)\,\big(\underline{\vee}\,(u,v,x) \wedge \forall\, y\,(y \in x \leftrightarrow y = u \vee y = v)\big)\Big).$$

(2) *Axiom of Union*:

$$(\forall\, u)\Big(\underline{\vee}\,(u) \to \Big((\exists\, v)\,\underline{\vee}\,(u,v) \wedge (\forall\, x)\,\big(\underline{\vee}\,(x,u) \to (x \in v \leftrightarrow (\exists\, y \in u)\,(x \in y))\big)\Big)\Big).$$

(3) *Axiom of Powerset*:

$$(\forall\, u)\Big(\underline{\vee}\,(u) \to \Big((\exists\, v)\,\underline{\vee}\,(u,v) \wedge (\forall\, t)\,\big(\underline{\vee}\,(u,v,t) \to (t \in v \leftrightarrow (\forall\, x \in t)\,(x \in u))\big)\Big)\Big).$$

(4) *Axiom of Replacement*:

$$(\forall\, u)\Big((\forall\, x \in u)\,(\exists\, y)\,\varphi(x,y) \to (\exists\, v)\,(\forall\, x \in u)\,(\exists\, y \in v)\,\varphi(x,y)\Big).$$

(5) *Axiom of Foundation*:

$$(\forall\, u)\,\Big((\underline{\vee}\,(u) \wedge (\exists\, x \in u)(x \in u)) \to ((\exists\, x \in u)\,(\forall\, y \in x)\neg(y \in u))\Big).$$

(6) *Axiom of Infinity*:

$$(\varnothing \in \omega^{\wedge}) \wedge (\forall\, x \in \omega^{\wedge})\,(x \cup \{x\} \in \omega^{\wedge}).$$

(7) *Axiom of Choice*:

$$(\forall\, u)\Big(\underline{\vee}\,(u) \to (\exists\, v)\,\Big(\underline{\vee}\,(u,v) \wedge (\forall\, x \in u)\,\big((\exists\, y \in x)\,(\exists!z \in u)\,(y \in z) \to (\exists!y \in x)\,(y \in v)\big)\Big)\Big).$$

1.12.C.4. Let $\mathscr{Q}$ be a quantum logic. Given an ordinal α, put

$$\mathbb{V}_{\alpha}^{(\mathscr{Q})} = \Big\{ u : \mathscr{D}(u) \to \mathscr{Q} \text{ and } \mathscr{D}(u) \subseteq \bigcup_{\beta<\alpha} \mathbb{V}_{\beta}^{(\mathscr{Q})} \Big\}.$$

The *$\mathscr{Q}$-valued universe* $\mathbb{V}^{(\mathscr{Q})}$ is defined as

$$\mathbb{V}^{(\mathscr{Q})} = \bigcup_{\alpha\in \mathrm{On}} \mathbb{V}_{\alpha}^{(\mathscr{Q})}.$$

For every $u \in \mathbb{V}^{(\mathscr{Q})}$, the rank of u, denoted by $\mathrm{rank}(u)$, is the least α such that $u \in \mathbb{V}_{\alpha}^{(\mathscr{Q})}$. Clearly, if $u \in \mathscr{D}(v)$ then $\mathrm{rank}(u) < \mathrm{rank}(v)$.

1.12.C.5. Given $u \in \mathbb{V}^{(\mathscr{Q})}$, define the *support* of u, denoted by $\mathbb{L}(u)$, by transfinite recursion on the rank of u:

$$\mathbb{L}(u) := \bigcup_{x\in\mathscr{D}(u)} \mathbb{L}(x) \cup \big\{ u(x) : \ x \in \mathscr{D}(u) \big\}.$$

For $A \subset \mathbb{V}^{(\mathscr{Q})}$ we write $\mathbb{L}(A) := \bigcup_{u\in A} \mathbb{L}(u)$ and define the *Boolean domain* $\underline{\vee}(A)$ of A by the formula $\underline{\vee}(A) := \perp\!\!\perp \mathbb{L}(A)$. We also put

$$\mathbb{L}(u_1, \dots, u_n) := \mathbb{L}(\{u_1, \dots, u_n\}),$$
$$\underline{\vee}(u_1, \dots, u_n) := \underline{\vee}(\{u_1, \dots, u_n\})$$

for all $u_1, \dots, u_n \in \mathbb{V}^{(\mathscr{Q})}$.

Put $x \Rightarrow y := x^{\perp} \vee (x \wedge y)$ for all $x, y \in \mathscr{Q}$. Define the $\mathscr{Q}$-valued truth values for the atomic formulas $[\![u = v]\!] \in \mathscr{Q}$ and $[\![u \in v]\!] \in \mathscr{Q}$ with $u, v \in \mathbb{V}^{(\mathscr{Q})}$ as follows (cp. 1.2.4):

$$[\![u = v]\!] := \bigwedge_{x\in\mathscr{D}(u)} (u(x) \Rightarrow [\![x \in v]\!]) \wedge \bigwedge_{y\in\mathscr{D}(v)} (v(y) \Rightarrow [\![y \in u]\!]),$$

$$[\![u \in v]\!] := \bigvee_{y\in\mathscr{D}(v)} (v(y) \wedge [\![u = y]\!]).$$

1.12.C.7. To each statement φ of ZFC we assign the $\mathscr{Q}$-valued truth value $[\![\varphi]\!]$ just as in 1.2.3 with the only difference that $[\![\neg\,\phi]\!] = [\![\phi]\!]^{\perp}$ is taken instead of $[\![\neg\,\phi]\!] = [\![\phi]\!]^{*}$ and the following additional rule is included

$$[\![\underline{\vee}\,(x_0, \dots, x_n)]\!] = \underline{\vee}\,(u_0, \dots, u_n) \quad (u_0, \dots, u_n \in \mathbb{V}^{(\mathscr{Q})}).$$

1.12.C.8. We say that φ *holds within* $\mathbb{V}^{(\mathscr{Q})}$ and write $\mathbb{V}^{(\mathscr{Q})} \models \varphi$, whenever $[\![\varphi]\!] = \mathbb{1}$. The axioms of equality 1.12.C.2 hold within $\mathbb{V}^{(\mathscr{Q})}$ (cp. [382, Theorem 1] and [326, Theorem 3.3]). At the same time the classical axioms of transitivity 1.1.4 (3) and substitution 1.1.4 (4) fail within $\mathbb{V}^{(\mathscr{Q})}$ (cp. [382, pp. 313, 314]).

1.12.C.9. Given $v \in \mathbb{V}$, define $v^\wedge \in \mathbb{V}^{(2)} \subset \mathbb{V}^{(\mathscr{Q})}$ by putting $\mathscr{D}(v) := \{x^\wedge : x \in v\}$ and $v^\wedge(x^\wedge) = \mathbb{1}$ $(x \in v)$. Then, for every bounded formula $\varphi(x_1, \dots, x_n)$ of ZFC and all $u_1, \dots, u_n \in \mathbb{V}$, we have

$$\mathbb{V} \models \varphi(u_1, \dots, u_n) \Longleftrightarrow \mathbb{V}^{(\mathscr{Q})} \models \varphi(u_1^\wedge, \dots, u_n^\wedge).$$

The following two results were obtained under the assumption that $\mathscr{Q} = \mathbb{P}(\mathscr{M})$, where $\mathscr{M}$ is a von Neumann algebra (cp. 1.12.A.10).

1.12.C.10. *All axioms* 1.12.C.3 (1–7) *are true within the universe* $\mathbb{V}^{(\mathscr{Q})}$*; i.e.,* $\mathbb{V}^{(\mathscr{Q})} \models \mathrm{ZF}_{\mathrm{Q}}$.

$\lhd$ See [382, pp. 315–321]. $\rhd$

1.12.C.11. Transfer Principle. Given a bounded formula $\varphi(x_1, \dots, x_n)$ of ZFC and $u_1, \dots, u_n \in \mathbb{V}^{(\mathscr{Q})}$, the implication holds

$$\mathrm{ZFC} \vdash \varphi(x_1, \dots, x_n) \Longrightarrow \mathbb{V}^{(\mathscr{Q})} \models \underline{\vee}(u_1, \dots, u_n) \to \varphi(u_1, \dots, u_n).$$

$\lhd$ See [326, Theorem 4.6]. $\rhd$

1.13. Comments

1.13.1. (1) The first system of axioms for set theory, existing alongside the Russell type theory and suggested by Zermelo in 1908, coincides essentially with the collection of 1.1.7 (1–4, 6) in which the axiom schema of replacement is replaced by the two of its consequences: the *axiom of separation*—$(\forall\, x)(\exists\, y)(\forall\, z)(z \in y \leftrightarrow (z \in x) \wedge \varphi(z))$ where φ is a formula of ZF and the *axiom of pairing*—$(\forall\, x)(\forall\, y)(\exists\, z)(\forall\, u)(u \in z \leftrightarrow (u = x \vee u = y))$. The axioms of extensionality 1.1.7 (1) and union 1.1.7 (2) had been previously proposed by Frege in 1883 and Cantor in 1899. The idea of the axiom of infinity 1.1.7 (6) stems from Dedekind.

(2) Zermelo set theory appeared in the beginning of the 1920s. It terminated the important stage of formalizing the language of set theory which eliminated the ambiguous descriptions of the tricks for distinguishing sets. But Zermelo's axioms did not make it possible to allow for the heuristic view of Cantor which asserts that the one-to-one image of a set is a set too. This shortcoming was eliminated by Fraenkel in 1922 and Skolem in 1923 who suggested versions of the axiom schema of replacement 1.1.7 (4). These achievements may be considered as the birth of ZFC.

(3) The axiom of foundation 1.1.7 (5) was propounded by Gödel and Bernays in 1941. It replaces the axiom of regularity which was proposed by von Neumann in 1925. The axiom of foundation is independent of the rest of the axioms of ZFC.

(4) The axiom of choice AC 1.1.7 (7) seems to have been used implicitly since long ago (for instance, Cantor used it in 1887 while proving that each infinite set includes a countable subset), whereas it was distinguished by Peano in 1890 and by B. Levi in 1902. The axiom of choice had been propounded by Zermelo in 1904 and remained most disputable and topical for quite a few decades. But the progress of "concrete" mathematics has showed that the possibility of virtual choice is perceived as an obvious and indispensable part of many valuable fragments of modern mathematics. There is no wonder that the axiom of choice is accepted by most of scientists. The discussion of the place and role of the axiom of choice in various areas of mathematics can be found in Gödel [145], Jech [184], Cohen [92], Lévy [279], and Fraenkel, Bar-Hillel, and Lévy [120].

(5) The concept of *continuum* belongs to the most important general tools of science. The mathematical views of the continuum relate to the understanding of a straight line in geometry and time and time-dependent processes in physics. The set-theoretic stance revealed a new enigma of the continuum. Cantor demonstrated that the set of the naturals is not equipollent with the simplest mathematical continuum, the real axis. This gave an immediate rise to the problem of the continuum which consists in determining the cardinalities of the intermediate sets between the naturals and the reals. The continuum hypothesis reads that the intermediate subsets yield no new cardinalities.

1.13.2 **(1)** Boolean valued models were invented for research into the foundations of mathematics. Many delicate properties of the objects of $\mathbb{V}^{(\mathbb{B})}$ depend essentially on the structure of the initial Boolean algebra $\mathbb{B}$. The diversity of opportunities together with a great stock of information on particular Boolean algebras ranks Boolean valued models among the most powerful tools of foundational studies; see Bell [43], Jech [184], and Takeuti and Zaring [388].

(2) Boolean valued analysis stems from the brilliant results of Gödel and Cohen who demonstrated the independence of the continuum hypothesis from the axioms of ZFC. Gödel proved the consistency of the continuum hypothesis with the axioms of ZFC by inventing the universe of constructible sets [145]. Cohen [92] demonstrated the consistency of the negation of the continuum hypothesis with the axioms of ZFC by *forcing*, the new method he invented for changing the properties of available or hypothetical models of set theory. Boolean valued models made Cohen's difficult result simple demonstrating to the working mathematician the independence of the continuum hypothesis with the same visuality as the Poincaré model for non-Euclidean geometry. Those who get acquaintance with this technique are inclined to follow Cohen [92] and view the continuum hypothesis as "obviously false."

(3) The book [344] by Rasiowa and Sikorski is devoted to the basics of Boolean valued models for the predicate calculus. The ideas of using Boolean valued models for simplifying the method of forcing by Cohen had independently been suggested by Solovay [369] and Vopěnka [400, 401] in 1965. Somewhat later Scott and Solovay, as well as Vopěnka in the research of his own, draw the conclusion that the topics of forcing should be addressed within the objects of a Boolean valued universe from the very beginning. The Boolean valued models whose construction was not considered as adverse by the majority of the "traditional" mathematicians have gained much popularity after it was revealed that they allow for deriving the same results as the method of forcing.

(4) The Boolean valued universe $\mathbb{V}^{(\mathbb{B})}$ is used for proving relative consistency of some set theoretic propositions by the following scheme: Assume that the theories $\mathscr{T}$ and $\mathscr{T}'$ are some enrichments of ZF such that the consistency of ZF entails the consistency of $\mathscr{T}'$. Assume further that we can define $\mathbb{B}$ so that $\mathscr{T}' \models$ "$\mathbb{B}$ is a complete Boolean algebra" and $\mathscr{T}' \models [\![\varphi]\!]^{\mathbb{B}} = \mathbb{1}$ for every axiom φ of $\mathscr{T}$. Then the consistency of ZF will imply the consistency $\mathscr{T}$ (cp. Bell [43]).

1.13.3. (1) We now exhibit another interesting Boolean valued model of set theory. Let G be a subgroup of the automorphism group of a complete Boolean algebra $\mathbb{B}$, and let Γ be a filter of subgroups of G; i.e., Γ is a nonempty set of subgroups of G such that $H, K \in \Gamma$ implies $H \cap K \in \Gamma$ while $H \in \Gamma$ and $H \subset K$ imply $K \in \Gamma$ for all subgroups H and K of G. Say that Γ is a *normal filter* if $g \in G$ and $H \in \Gamma$ imply $gHg^{-1} \in \Gamma$. Each $g \in G$ induces the automorphism g^* of $\mathbb{V}^{(\mathbb{B})}$ which is in 1.3.1. The stabilizer $\operatorname{stab}(x)$ of $x \in \mathbb{V}^{(\mathbb{B})}$ is defined as $\operatorname{stab}(x) := \{g \in G : g^*(x) = x\}$. It is easy from 1.3.2 that $\operatorname{stab}(x)$ is a subgroup of G. We define the sets $\mathbb{V}^{(\Gamma)}$ recursively as follows:

$$\mathbb{V}_\alpha^{(\Gamma)} := \{x : \ \mathrm{Funct}(x) \wedge (\exists\, \beta)\, (\beta < \alpha \wedge \operatorname{dom}(x) \subset \mathbb{V}_\beta^{(\mathbb{B})} \\ \wedge \operatorname{im}(x) \subset \mathbb{B}) \wedge \operatorname{stab}(x) \in \Gamma\}.$$

Put $\mathbb{V}^{(\Gamma)} := \{x : (\exists\, \alpha \in \mathrm{On}) x \in \mathbb{V}_\alpha^{(\Gamma)}\}$ and define Boolean truth values by $[\![x \in y]\!]^{(\Gamma)} = [\![x \in y]\!]^{(\mathbb{B})}$, and $[\![x = y]\!]^{(\Gamma)} = [\![x = y]\!]^{(\mathbb{B})}$ for $x, y \in \mathbb{V}^{(\Gamma)}$. Define $[\![\varphi]\!] \in \mathbb{B}$ as in 1.2.3 and 1.2.4.

(2) Scott established that all axioms and so all theorems of ZF are true in $\mathbb{V}^{(\Gamma)}$; see Bell [43, Theorem 3.19]. Scott succeeded in choosing $\mathbb{B}$, G, and Γ in such a way that $\mathbb{V}^{(\Gamma)} \models \neg\mathrm{AC}$. Then it follows that the consistency of ZF implies the consistency of ZF $+\neg$AC. So the model $\mathbb{V}^{(\Gamma)}$ is effective in proving consistency. It seems reasonable to suppose that these models will be useful in Boolean valued analysis, but we are unaware of any applications of the sort yet.

1.13.4. (1) Scott established the maximum and transfer principles of Section 1.4 together with many other properties of Boolean valued models. He also gave the schematic exposition of the models. But the manuscript of 1967 remained unpublished although it was rather widely used by specialists. The literature on Boolean valued models has references to the nonexistent paper by Scott and Solovay which was intended to be an extension of the Scott manuscript. These and other details of the creation and development of the theory of Boolean valued models are disclosed in Scott's introduction to Bell's book [43]

(2) The restricted transfer principle is often referred to as as the Boolean valued version of absoluteness of bounded formulas; see [184, Lemma 14.21]

1.13.5, 1.13.6. (1) Various versions of the tricks of Sections 1.5 and 1.6 are common for studying Boolean valued models. In Kusraev [218, 222] and Kutateladze [267, 268] they appeared as the technique of descents and ascents which proved to be convenient in applications to analysis (cp. Kusraev and Kutateladze [244, 245, 246, 248]). The terminology of "descents and ascents" was suggested by Kutateladze [267, 268] in memory of Escher (whose life and achievements are reflected, for instance, by Locher [285] and Hofstedter [172]). The arrow cancellation rules in 1.6.6 are also called *Escher rules.*

(2) The same symbols $\downarrow$ and $\uparrow$ are used for various operators having the same nature. Therefore, the records like $\mathscr{X}\downarrow$ and $X\uparrow$ can be properly understood only with extra information about the object that is ascended to or descended from a Boolean valued universe. The situation here is pretty similar to that with using the plus sign for recording completely different group operations: addition of numbers, vectors, linear operators, etc. The precise meaning is always reconstructible from the context. We use the symbols $\upharpoonleft$ and $\downharpoonleft$ by analogy.

1.13.7. Boolean valued interpretations have a long history. It seems that the first Boolean valued model (for the theory of types) was suggested by Church in 1951. Since then many authors have considered Boolean valued models for first order propositions and theories; for instance, Halmos, Mostowski, and Tarski. But it was Rasiowa and Sikorski who advanced the technique substantially; cp. [344]. In regard to the theory of algebraic systems look at the definitive monograph by Maltsev [304].

1.13.8. The descent and embedding of an algebraic $\mathbb{B}$-system to a Boolean valued model were accomplished in the articles by Kusraev [218] and Kutateladze [267] on using the method of Solovay and Tennenbaum [370] which they applied to proving Theorem 1.10.4. The descents of various particular algebraic systems were performed by many authors. Part of these results is collected in the books [248, 249] by Kusraev and Kutateladze.

1.13.9. (1) A Boolean algebra is said to satisfy the *countable chain condition* if its every disjoint family of nonzero elements is at most countable. If the complete Boolean algebra $\mathbb{B}$ satisfies the countable chain condition, then cardinals in $\mathbb{V}$ retain their true size in $\mathbb{V}^{(\mathbb{B})}$; i.e., $\alpha \in \mathbb{V}$ is a cardinal if and only if $\alpha^\wedge$ is a cardinal within $\mathbb{V}^{(\mathbb{B})}$ and, consequently $\alpha^\wedge = |\alpha^\wedge|$; see Bell [43, Theorem 1.51]. Clearly, if $\mathbb{B}$ does not satisfy this condition, then it becomes possible for two infinite cardinals $\varkappa < \lambda$ have the coinciding standard names $\varkappa^\wedge$ and $\lambda^\wedge$. More precisely, for infinite cardinals $\varkappa < \lambda$ there exists a complete Boolean algebra $\mathbb{B}$ such that $\mathbb{V}^{(\mathbb{B})} \models |\varkappa^\wedge| = |\lambda^\wedge|$. In this event we say that λ has been *collapsed* to $\varkappa$ in $\mathbb{V}^{(\mathbb{B})}$; see Bell [43, Theorem 5.1, Corollary 5.2, and 5.4].

(2) A Boolean algebra $\mathbb{B}$ is said to be $(\varkappa, \lambda)$-*distributive* if for every family $(b_{\alpha,\beta})_{\alpha<\varkappa,\beta<\lambda}$ in $\mathbb{B}$ we have

$$\bigwedge_{\alpha<\varkappa} \bigvee_{\beta<\lambda} b_{\alpha,\beta} = \bigvee_{\phi\in\lambda^\varkappa} \bigwedge_{\alpha<\varkappa} b_{\alpha,\phi(\alpha)}.$$

It can be shown that $(\varkappa, \lambda)$-distributivity of $\mathbb{B}$ is equivalent to the relation $\mathbb{V}^{(\mathbb{B})} \models (\lambda^{\varkappa})^{\wedge} = (\lambda^{\wedge})^{\varkappa^{\wedge}}$; see Bell [43, 2.14]. The monographs [43] by Bell and [184] by Jech are excellent sources of the facts concerning Boolean valued cardinals.

1.13.10. (1) Let $\mathbb{B}$ and $\mathbb{D}$ be Boolean algebras and let $\mathbb{B} \otimes \mathbb{D}$ be their free product. That is, $\mathbb{B} \otimes \mathbb{D}$ is isomorphic to the Boolean algebra of clopen sets of the Cartesian product of the Stone spaces of $\mathbb{B}$ and $\mathbb{D}$; see Koppelberg [204, Subsection 11.1]. Denote by $\mathbb{B} \widehat{\otimes} \mathbb{D}$ the Dedekind completion of $\mathbb{B} \otimes \mathbb{D}$; see Koppelberg [204, Section 4.3]. Given a Boolean algebra $\mathbb{B}$ and an element $\mathscr{D} \in \mathbb{V}^{(\mathbb{B})}$ satisfying $\mathbb{V}^{(\mathbb{B})} \models$ "$\mathscr{D}$ is the Dedekind completion of the Boolean algebra $\mathbb{D}^{\wedge}$," the Boolean algebras $\mathscr{D}{\downarrow}$ and $\mathbb{B} \widehat{\otimes} \mathbb{D}$ are isomorphic (see Solovay and Tennenbaum [370]).

(2) The results by Solovay and Tennenbaum [370] (Theorems 1.10.2–1.10.4) can serve as a basis for iterating the construction of a Boolean valued model. Assume that $\mathscr{D} \in \mathbb{V}^{(\mathbb{B})}$ and $\mathbb{V}^{(\mathbb{B})} \models$ "$\mathscr{D}$ is a complete Boolean algebra." Using the scheme of Section 1.3 we can construct within $\mathbb{V}^{(\mathbb{B})}$ a few $\mathbb{V}^{(\mathbb{B})}$-classes: the Boolean valued universe $(\mathbb{V}^{(\mathbb{B})})^{(\mathscr{D})}$, the corresponding Boolean truth values $[\![\, \cdot = \cdot \,]\!]^{\mathscr{D}}$ and $[\![\, \cdot \in \cdot \,]\!]^{\mathscr{D}}$ together with the canonical embedding $(\,\cdot\,)^{\wedge}$ of the universal class $\mathbb{U}_{\mathbb{B}}$ to $(\mathbb{V}^{(\mathbb{B})})^{\mathbb{D}}$. Put $\mathbb{D} := \mathscr{D}{\downarrow}$, $\mathbb{W}^{(\mathbb{D})} := (\mathbb{V}^{(\mathbb{B})})^{(\mathscr{D})}{\downarrow}$, $\|\, \cdot = \cdot \,\|^{\mathbb{D}} := ([\![\, \cdot = \cdot \,]\!]^{\mathscr{D}}){\downarrow}$, $\|\, \cdot \in \cdot \,\|^{\mathbb{D}} := ([\![\, \cdot \in \cdot \,]\!]^{\mathscr{D}}){\downarrow}$, and $\jmath := (\,\cdot\,)^{\wedge}{\downarrow}$. Assume that $\imath : \mathbb{B} \to \mathbb{D}$ is the canonical isomorphism, and $\imath^* : \mathbb{V}^{(\mathbb{B})} \to \mathbb{V}^{(\mathbb{D})}$ is the corresponding injection (see Section 1.3). Then there is a unique bijection $h : \mathbb{V}^{(\mathbb{D})} \to \mathbb{W}^{(\mathbb{D})}$ such that $[\![x = y]\!]^{\mathbb{D}} = \|h(x) = h(y)\|^{\mathbb{D}}$ and $[\![x \in y]\!]^{\mathbb{D}} = \|h(x) \in h(y)\|^{\mathbb{D}}$ for all x and $y \in \mathbb{V}^{(\mathbb{B})}$. In this event the diagram commutes:

$$\begin{tikzcd} & \mathbb{V}^{(\mathbb{B})} \arrow[dl, "\imath^*"'] \arrow[dr, "\jmath"] & \\ \mathbb{V}^{(\mathbb{D})} \arrow[rr, "h"'] & & \mathbb{W}^{(\mathbb{D})} \end{tikzcd}$$

See details in Solovay and Tennenbaum [370].

(3) Further iterations of the above construction lead to a transfinite collection of Boolean valued enrichments. This approach leads to the iterated forcing which was used for instance in establishing the relative consistency of the Suslin hypothesis and ZFC which was done in Solovay and Tennenbaum [370].

1.13.11. (1) The Sandwich and Hahn–Banach Theorems for Boolean homomorphisms (i.e. Theorems 1.11.5 and 1.11.6) were obtained by Monteiro [313] who used another method. Some analogous results for distributive lattices were demonstrated by Cignoli [91]. The proofs in Section 1.11 show that the results about extension of Boolean homomorphisms are simply the existence theorem of a nontrivial ultrafilter modulo translation into a Boolean valued model. For instance, Theorem 1.11.4 is a Boolean valued interpretation of the Stone Theorem: *If an ideal I and a filter F are disjoint in a Boolean algebra, then there is a maximal ideal $\mathscr{I}$ including I and disjoint from F as well as there is an ultrafilter $\mathscr{F}$ including F and disjoint from I.*

1.13.12. (1) The system $\mathrm{ZF_I}$ of intuitionistic set theory and construction of a Heyting valued model $\mathbb{V}^{(\Omega)}$ within the theory, as presented in 1.12.C, are due to Grayson [149, 150]. Takeuti and Titani in [386], using Grayson's $\mathrm{ZF_I}$, extended the Solovay and Tennenbaum's results on iterated Cohen extensions in [370] to Heyting valued universes. More precisely, a complete Heyting algebra Ω and the corresponding Heyting valued universe $\mathbb{V}^{(\Omega)}$ are considered in a universe $\mathbb{V}^{(\Omega')}$ with Ω' another complete Heyting algebra. For more detail on complete Heyting algebras, refer to Fourman and Scott [119] in which some related subjects are also discussed.

(2) Let $\mathbb{2} = \{\mathbb{0}, \mathbb{1}\}$ be the complete least subalgebra of a complete Heyting algebra Ω. Just as in 1.4.4 and 1.4.5 the universe $\mathbb{V}$ is equivalent to $\mathbb{V}^{(\mathbb{2})} \subset \mathbb{V}^{(\Omega)}$, so that $\mathbb{V}$ is embedded in $\mathbb{V}^{(\Omega)}$ as a submodel. But the copy $\hat{\mathbb{V}} := \{x^\wedge : x \in \mathbb{V}\}$ of $\mathbb{V}$ is not expressible in the language of $\mathrm{ZF_I}$ on $\mathbb{V}^{(\Omega)}$, since the concepts expressible in the language of $\mathrm{ZF_I}$ on $\mathbb{V}^{(\Omega)}$ are local, whereas $\hat{\mathbb{V}}$ is a global concept. In Takeuti and Titani [387] a modification of $\mathrm{ZF_I}$, the *global intuitionistic set theory* GIZF in which the global concepts are expressible, are presented.

(3) The idea of quantum logic stems from von Neumann's 1932 book on the mathematical foundations of quantum mechanics. In [394, p. 253] he wrote: "As can be seen, the relation between the properties of a physical system on the one hand, and the projections on the other, makes possible a sort of logical calculus with these." A systematic attempt to propose a "propositional calculus" for quantum logic was made in the seminal joint paper [59] by Birkhoff and von Neumann which marked the birth of quantum logic. As regads the history and the main ideas of quantum logic, see Dalla Chiara, Giuntini, and Rédei [102] as well as Foulis, Greechie, Dalla Chiara, and Giuntini [118]. The mathematical and logical investigation of various aspects of quantum mechanics is the topic of the *Handbook of Quantum Logic and Quantum Structures* edited by Engesser, Gabbay, and Lehmann [115]; see also Piron [336] and Ptak and Pulmannova [339].

(4) Quantum set theory was introduced by Takeuti in [382] as the quantum counterpart of Boolean valued set theory. In [382, p. 303] he wrote: "Since quantum logic is an intrinsic logic, i.e. the logic of the quantum world (cp. Birkhoff and von Neumann [59]), it is an important problem to develop mathematics based on quantum logic, more specifically set theory based on quantum logic. It is also a challenging problem for logicians since quantum logic is drastically different from the classical logic or the intuitionistic logic and consequently mathematics based on quantum logic is extremely difficult. On the other hand, mathematics based on quantum logic has a very rich mathematical content."

(5) In [390] Titani presented the lattice valued logic and lattice valued set theory by introducing the basic implication. The completeness of the lattice valued logic was proved in Takano [378]. For an arbitrary complete lattice $\mathbb{L}$, the $\mathbb{L}$-valued universe $\mathbb{V}^{(\mathbb{L})}$ is a model of lattice valued set theory based on the lattice valued logic.

(6) The possibilities are open for defining implication in the quantum logics that satisfy the order known as the Birkhoff–von Neumann requirement (cp. Pavičić and Megill [334]):

$x \Rightarrow y := x^{\perp}(x \wedge y)$ (Sasaki);

$x \Rightarrow y := y \vee (x^{\perp} \wedge y^{\perp})$ (Dishkant);

$x \Rightarrow y := (((x^{\perp} \wedge y) \vee (x^{\perp} \wedge y^{\perp})) \vee (x \wedge (x^{\perp} \vee y)))$ (Kalmbach);

$x \Rightarrow y := (((x \wedge y) \vee (x^p erp \wedge y)) \vee ((x^{\perp}(x \vee y) \wedge y^{\perp})))$ (nontollens);

$x \Rightarrow y := (((x \wedge y) \vee (x^{\perp} \wedge y)) \vee (x^{\perp} \wedge y))$ (relevance).

In a Boolean algebra, all reduce to the classical implication $x \Rightarrow y := (x^{\perp} \vee y)$.

(7) An ortholattice $\mathbb{L}$ is called *weakly orthomodular* provided that $x \equiv y = \mathbb{1} \Longrightarrow x = y$ and $\mathbb{L}$ is called a *weakly distributive ortholattice* whenever $(x \equiv y) \vee (x \equiv y^{\perp}) = (x \wedge y^{\perp}) \vee (x^{\perp} \wedge y) \vee (x^{\perp} \wedge y^{\perp}) = \mathbb{1}$ for all $x, y \in \Omega$, where $x \equiv y := (x \wedge y) \vee (x^{\perp} \wedge y^{\perp})$. There exist weakly distributive ortholattices that are not orthomodular and therefore not distributive, weakly orthomodular ortholattices that are not orthomodular, ortholattices that are neither weakly orthomodular nor weakly distributive, and weakly orthomodular ortholattices that are not weakly distributive (cp. Pavičić and Megill [332, 333]).

(8) Surprisingly, the quantum propositional calculuses and the classical propositional calculuses are noncategorical. Recall that a formal system is called *categorical* and if all its models are isomorphic with one another. More precisely, quantum logic can be modeled by an orthomodular lattice as well as a weakly orthomodular lattice, and the classical logic can be modeled by a Boolean algebra as well as a weakly distributive lattice (cp. Pavičić and Megill [333]).

CHAPTER 2

BOOLEAN VALUED NUMBERS

Boolean valued analysis stems from the fact that the image of the reals in each Boolean valued model presents a universally complete vector lattice. Therefore, the theorems about real numbers can be "externalized" by transfer so as to yield results about universally complete vector lattices. Depending on which Boolean algebra $\mathbb{B}$ (the algebra of measurable sets, regular open sets, or projections in a Hilbert space, etc.) forms the base for constructing the Boolean valued model $\mathbb{V}^{(\mathbb{B})}$, we obtain various vector lattices (the spaces of measurable functions, continuous functions, selfadjoint operators, etc.). Thereby the remarkable opportunity opens up to expand the treasure-trove of knowledge about the reals to a profusion of classical objects of analysis.

In this chapter we show that the most important structure properties of Dedekind complete vector lattices such as representation as function spaces, the Freudenthal Spectral Theorem, functional calculus, etc. are some translations of the properties of the reals in an appropriate Boolean valued model.

As in Chapter 1, to simplify the simultaneous work with two universes, we agree to some extra pedantry in notation. Denoting implication and equivalence in the sequel, we will use $\Longrightarrow$ and $\Longleftrightarrow$ outside $\mathbb{V}^{(\mathbb{B})}$ and $\to$ and $\leftrightarrow$ inside $\mathbb{V}^{(\mathbb{B})}$, while $\Rightarrow$ and $\Leftrightarrow$ we reserve for the Boolean operations: $x \Rightarrow y := x^* \vee y$ and $x \Leftrightarrow y := (x \Rightarrow y) \wedge (y \Rightarrow x)$.

Throughout the sequel $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ symbolize the naturals, the integers, the rationals, the reals, and the complexes.

2.1. Vector Lattices

In this section we give some preliminaries to the theory of vector lattices; a more explicit exposition can be found elsewhere (cp. Akilov and Kutateladze [22], Kusraev [228], Luxemburg and Zaanen [297], Meyer-Nieberg [311], Schaefer [356], Schwarz [361], Vulikh [403], and Zaanen [427]).

2.1.1. Let $\mathbb{F}$ be a linearly ordered field. An *ordered vector space over* $\mathbb{F}$ is a pair $(X, \leqslant)$, where X is a vector space over $\mathbb{F}$ and $\leqslant$ is an order on X satisfying the conditions:

(1) if $x \leqslant y$ and $u \leqslant v$ then $x + u \leqslant y + v$ for all $x, y, u, v \in X$;

(2) if $x \leqslant y$ then $\lambda x \leqslant \lambda y$ for all $x, y \in X$ and $0 \leqslant \lambda \in \mathbb{F}$.

Informally speaking, we can "sum inequalities in X and multiply them by positive members of $\mathbb{F}$." This circumstance is worded as follows: $\leqslant$ is an order compatible with the vector space structure or, briefly, $\leqslant$ is a *vector order*.

2.1.2. The subset $X_+ := \{x \in X : x \geqslant 0\}$ of an ordered vector space X is called the *positive cone* of X. The elements of X_+ are called *positive*. The positive cone X_+ of an ordered vector space X has the properties:

$$X_+ + X_+ \subset X_+, \quad \lambda X_+ \subset X_+ \ (0 \leqslant \lambda \in \mathbb{F}), \quad X_+ \cap -X_+ = \{0\}.$$

Moreover, if X_+ is a subset of a vector space X over $\mathbb{F}$ satisfying the above properties, then X transforms into an ordered vector space over $\mathbb{F}$ by letting

$$x \leqslant y \Longleftrightarrow y - x \in X_+ \quad (x, y \in X).$$

2.1.3. A *vector lattice* is an ordered vector space that is also a lattice. Thereby each finite set $\{x_1, \dots, x_n\} \subset X$ of a vector lattice has the *join* or the least upper bound $\sup\{x_1, \dots, x_n\} := x_1 \vee \dots \vee x_n$ as well as the *meet* or the greatest lower bound $\inf\{x_1, \dots, x_n\} := x_1 \wedge \dots \wedge x_n$. In particular, each element x of a vector lattice has the *positive part* $x^+ := x \vee 0$, the *negative part* $x^- := (-x)^+ := -x \wedge 0$, and the *modulus* $|x| := x \vee (-x)$.

A vector lattice X is called *Archimedean* if for every pair of elements $x, y \in X$ from $(\forall n \in \mathbb{N})\ nx \leqslant y$ it follows that $x \leqslant 0$. In the sequel, all ordered vector spaces are assumed to be Archimedean.

2.1.4. Let X be a vector lattice. If $x, y, z \in X$ then the following hold:

(1) $x = x^+ - x^-,\ |x| = x^+ + x^- = x^+ \vee x^-$.

(2) $x \leqslant y \Longleftrightarrow (x^+ \leqslant y^+ \text{ and } y^- \leqslant x^-)$.

(3) $x \vee y = \frac{1}{2}(x + y + |x - y|),\ x \wedge y = \frac{1}{2}(x + y - |x - y|)$.

(4) $|x| \vee |y| = \frac{1}{2}(|x + y| + |x - y|),\ |x| \wedge |y| = \frac{1}{2}(|x + y| - |x - y|)$.

(5) $x + y = x \vee y + x \wedge y,\ |x - y| = x \vee y - x \wedge y$.

(6) $x + y \vee z = (x + y) \vee (x + z),\ x + y \wedge z = (x + y) \wedge (x + z)$.

(7) $x, y, z \in X_+ \Longrightarrow (x + y) \wedge z \leqslant (x \wedge z) + (y \wedge z)$.

(8) $|x - y| = |x \vee z - y \vee z| + |x \wedge z - x \wedge z|$.

2.1.5. Let (x_α) and (y_α) be families in X for which $\sup(x_\alpha)$ and $\inf(y_\alpha)$ exist. Then the *infinite distributive laws* are valid $(z \in X)$:

(1) $z \wedge \sup_\alpha(x_\alpha) = \sup_\alpha(z \wedge x_\alpha),\ z \vee \inf_\alpha(y_\alpha) = \inf_\alpha(z \vee y_\alpha)$.

Moreover, for every $z \in X$ we have the following:

(2) $z + \sup_\alpha(x_\alpha) = \sup_\alpha(z + x_\alpha)$;

(3) $z + \inf_\alpha(y_\alpha) = \inf_\alpha(z + y_\alpha)$;

(4) $\sup_\alpha(x_\alpha) = -\inf_\alpha(-x_\alpha)$.

2.1.6. An *order interval* in X is a set of the form $[a, b] := \{x \in X : a \leqslant x \leqslant b\}$, where $a, b \in X$. In a vector lattice we have the very useful *Riesz decomposition property:*

(1) $[0, x + y] = [0, x] + [0, y]\ (x, y \in X_+)$.

Note the two corollaries of (1):

(2) $(x_1 + \dots + x_n) \wedge y \leqslant x_1 \wedge y + \dots + x_n \wedge y\ (x_k, y \in X_+)$;

(3) If $x_{k,l} \in X_+$ $(k \leqslant n,\ l \leqslant m)$ and J is the set of all functions from $\{1, \dots, n\}$ to $\{1, \dots, m\}$, then

$$\bigwedge_{k=1}^{n} \sum_{l=1}^{m} x_{k,l} \leqslant \sum_{j \in J} x_{1,j(1)} \wedge \dots \wedge x_{n,j(n)}.$$

2.1.7. Two elements $x, y \in X$ are called *disjoint* if $|x| \wedge |y| = 0$. The disjointness of x and y is denoted by $x \perp y$. Say that two subsets M and N of X are *disjoint* and write $M \perp N$ if $x \perp y$ for all $x \in M$ and $y \in N$. The properties of disjointness are easy from 2.1.4:

$$x \perp y \Longleftrightarrow |x + y| = |x - y| \Longleftrightarrow |x| \vee |y| = |x| + |y|;$$
$$x^+ \perp x^-; \quad (x - x \wedge y) \perp (y - x \wedge y);$$
$$x \perp y \Longrightarrow |x + y| = |x| + |y|,$$
$$(x + y)^+ = x^+ + y^+, \quad (x + y)^- = x^- + y^-.$$

The *disjoint complement* $M^{\perp}$ of $M \subset X$, $M \neq \varnothing$, is defined as

$$M^{\perp} := \{x \in X : (\forall y \in M)\ x \perp y\}.$$

Put $M^{\perp\perp} := (M^{\perp})^{\perp}$. The disjoint complement has the properties:

$$M \subset M^{\perp\perp}, \quad M^{\perp} = M^{\perp\perp\perp}, \quad M^{\perp} \cap M^{\perp\perp} = \{0\}.$$

2.1.8. A nonempty set K in X meeting the identity $K = K^{\perp\perp}$ is called a *band* (a *component* in the Russian literature) of X. Every band of the form $\{x\}^{\perp\perp}$ with $x \in X$ is called *principal.*

The inclusion-ordered set of all bands of X is denoted by $\mathbb{B}(X)$ and presents a complete Boolean algebra. The Boolean operations on $\mathbb{B}(X)$ take the shape:

$$L \wedge K = L \cap K, \quad L \vee K = (L \cup K)^{\perp\perp}, \quad L^{*} = L^{\perp} \quad (L, K \in \mathbb{B}(X)).$$

Let $u \in X_{+}$ and $e \wedge (u - e) = 0$ for some $0 \leqslant e \in X$. Then e is said to be a *component* or a *fragment* of u.

The set $\mathbb{C}(u)$ of all components of u with the order induced by X is a Boolean algebra. The lattice operations in $\mathbb{C}(u)$ are taken from X, while the Boolean complement has the form $e^{*} := u - e$ $(e \in \mathbb{C}(u))$. If an order unit $\mathbb{1}$ is fixed in X then the notation $\mathbb{C}(X) := \mathbb{C}(X, \mathbb{1}) := \mathbb{C}(\mathbb{1})$ is also in use.

2.1.9. A band B in a vector lattice X is said to be a *projection band* if $X = B \oplus B^{\perp}$. It can easily be seen that B is a projection band if and only if B is an order ideal and for every $x \in X_{+}$ the supremum of $B_{+} \cap [0, x]$ exists in X and belongs to B. The projection from X onto B along $B^{\perp}$ is called a *band projection* or an *order projection* and denoted by $[B]$ or P_B. A linear operator $P : X \to X$ is a band projection if and only if $P^2 = P$ and $0 \leqslant Px \leqslant x$ for all $x \in X_{+}$. Moreover,

$$[B]x := \sup(B_{+} \cap [0, x]) \quad (x \in X_{+}),$$
$$[B]x := [B]x^{+} - [B]x^{-} \quad (x \in X).$$

(1) *The set $\mathbb{P}(X)$ of all band projections ordered by $\pi \leqslant \rho \iff \pi \circ \rho = \pi$ is a Boolean algebra. The Boolean operations on $\mathbb{P}(X)$ take the shape*

$$\pi \wedge \rho = \pi \circ \rho, \quad \pi \vee \rho = \pi + \rho - \pi \circ \rho, \quad \pi^{*} = I_X - \pi \quad (\pi, \rho \in \mathbb{P}(X)).$$

The band projection onto a principal band is called *principal.*

(2) *The principal projection* $\pi_u := [u] := [u^{\perp\perp}]$, *where* $0 \leqslant u \in X$, *can be calculated by the rule simpler than that above:*

$$\pi_u x = \sup\{x \wedge (nu) : n \in \mathbb{N}\}.$$

A vector lattice X is said to have the *projection property* (*principal projection property*) if each band (each principal band) in $\mathbb{B}(X)$ is a projection band.

2.1.10. (1) A linear subspace J of a vector lattice X is called an *order ideal* or *o-ideal* (or, finally, just an *ideal,* when it is clear from the context what is meant) if the inequality $|x| \leqslant |y|$ implies $x \in J$ for all $x \in X$ and $y \in J$. Each order ideal of a vector lattice is a vector lattice. If an ideal J possesses the additional property $J^{\perp\perp} = X$ (or, which is the same, $J^{\perp} = \{0\}$) then J is referred to as an *order dense ideal* of X (the term "foundation" is also used in the Russian literature).

(2) A *vector sublattice* is a vector subspace $X_0 \subset X$ such that $x \wedge y$, $x \vee y \in X_0$ for all $x, y \in X_0$. We say that a sublattice X_0 is *order dense* or *minorizing* if, for every $0 \neq x \in X_+$, there exists $x_0 \in X_0$ satisfying $0 < x_0 \leqslant x$. We say that X_0 is a *majorizing* or *massive* sublattice if, for every $x \in X$, there exists $x_0 \in X_0$ such that $x \leqslant x_0$. Thus, X_0 is a minorizing or a majorizing sublattice if and only if $X_+ \setminus \{0\} = X_+ + X_{0+} \setminus \{0\}$ or $X = X_+ + X_0$, respectively.

(3) A set in X is called (*order*) *bounded* (or *o-bounded*) if it is included in some order interval. The *o*-ideal generated by the element $0 \leqslant u \in X$ is the set $X(u) := \bigcup_{n=1}^{\infty}[-nu, nu]$; clearly, $X(u)$ is the least *o*-ideal in X containing u.

If $X(u) = X$ then we say that u is a *strong unit* or *strong order unit*. If $X(u)^{\perp\perp} = X$ then we say that u is an *order unit* or *weak order unit.* It is evident that an element $u \in X_+$ is an order unit if and only if $\{u\}^{\perp\perp} = X$; i.e., if there is X no nonzero element in X disjoint from u. Recall that the term *unit* is often replaced with *unity* by tradition. This leads to no confusion.

(4) An element $x \geqslant 0$ of a vector lattice is called *discrete* if $[0, x] = [0, 1]x$; i.e., if $0 \leqslant y \leqslant x$ implies $y = \lambda x$ for some $0 \leqslant \lambda \leqslant 1$. A vector lattice X is called *discrete* or *atomic* if, for every $0 \neq y \in X_+$, there exists a discrete element $x \in X$ such that $0 < x \leqslant y$. If X lacks nonzero discrete elements then X is said to be *continuous* or *diffuse.*

2.1.11. A vector lattice X is said to be *Dedekind complete* or *order complete* whenever each nonempty upper bounded subset of X has the least upper bound (or, equivalently, whenever each nonempty lower bounded subset of X has the greatest lower bound).

If, in a vector lattice X, the least upper bounds (or equivalently the greatest lower bounds) exist only for countable bounded sets, then X is called *Dedekind σ-complete* or *order σ-complete*. A Dedekind complete vector lattice and a Dedekind σ-complete vector lattice are frequently referred to in the Russian literature as *K-space* (= *Kantorovich space*) and a *K_σ-space*, respectively.

2.1.12. *Let X be a Dedekind complete vector lattice. Then X has the projection property and the mapping $B \mapsto [B]$ is an isomorphism of the Boolean algebras $\mathbb{B}(X)$ and $\mathbb{P}(X)$. If there is an order unit $\mathbb{1}$ in X then the mappings $P \mapsto P(\mathbb{1})$ from $\mathbb{P}(X)$ in $\mathbb{C}(X)$ and $e \mapsto \{e\}^{\perp\perp}$ from $\mathbb{C}(X)$ in $\mathbb{B}(X)$ are isomorphisms of Boolean algebras, too.*

2.1.13. Recall that a vector lattice is called *laterally complete* whenever each disjoint set positive vectors in it has a supremum. A vector lattice that is at the same time laterally complete and Dedekind complete is referred to as a *universally complete* vector lattice.

A linear operator $T : X \to Y$ is a *lattice homomorphism* provided that T preserves the joins and meets of nonempty finite sets.

If X is an Archimedean vector lattice then there exists a unique (up to lattice isomorphism) universally complete vector lattice $X^{\mathfrak{u}}$ (called the *universal completion* of X) such that X is lattice isomorphic to an order dense sublattice of $X^{\mathfrak{u}}$. Identifying X with its copy in $X^{\mathfrak{u}}$, we have the vector sublattice inclusion $X \subset X^{\mathfrak{u}}$ with X order dense in $X^{\mathfrak{u}}$. In particular, the Dedekind completion X^δ of X can be identified with the ideal generated by X in $X^{\mathfrak{u}}$, and so we have the lattice isomorphisms $X \subset X^\delta \subset X^{\mathfrak{u}}$ with X^δ order dense in $X^{\mathfrak{u}}$.

2.2. Gordon's Theorem

In this section we will demonstrate that the externalization of the Boolean valued reals is a universally complete vector lattice.

2.2.1. By a *field of reals* we mean an algebraic system that satisfies the axioms of an Archimedean ordered field (with distinct zero and unity) and the axiom of completeness. Recall the two well-known assertions:

(1) *There exists a field of reals $\mathbb{R}$ that is unique up to isomorphism.*

(2) *If $\mathbb{P}$ is an Archimedean ordered field then there is an isomorphic embedding h of the field $\mathbb{P}$ into $\mathbb{R}$ such that the image $h(\mathbb{P})$ is a subfield of $\mathbb{R}$ containing a subfield of rationals. In particular, $h(\mathbb{P})$ is dense in $\mathbb{R}$.*

2.2.2. Successively applying the transfer and maximum principles of Boolean valued analysis to 2.2.1(1), we find an element $\mathscr{R} \in \mathbb{V}^{(\mathbb{B})}$ for which $[\![\mathscr{R}$ is a field of reals$]\!] = \mathbb{1}$. Moreover, if an arbitrary $\mathscr{R}' \in \mathbb{V}^{(\mathbb{B})}$ satisfies the condition $[\![\mathscr{R}'$ is a field of reals$]\!] = \mathbb{1}$ then it also satisfies $[\![$the ordered fields $\mathscr{R}$ and $\mathscr{R}'$ are isomorphic$]\!] = \mathbb{1}$. In other words, there exists a field of reals $\mathscr{R}$ in $\mathbb{V}^{(\mathbb{B})}$ and such a field is unique up to isomorphism. We call $\mathscr{R}$ the *reals* in $\mathbb{V}^{(\mathbb{B})}$.

Note also that $\varphi(x)$, formally presenting the expressions of the axioms of an Archimedean ordered field x, is restricted; therefore, $[\![\varphi(\mathbb{R}^{\wedge})]\!] = \mathbb{1}$; i.e., $[\![\mathbb{R}^{\wedge}$ is an Archimedean ordered field$]\!] = \mathbb{1}$. "Pulling" 2.2.1(2) by transfer we conclude that $[\![\mathbb{R}^{\wedge}$ is isomorphic to a dense subfield of the field $\mathscr{R}]\!] = \mathbb{1}$. In this regard, we further assume that $\mathscr{R}$ is the reals in $\mathbb{V}^{(\mathbb{B})}$ and $\mathbb{R}^{\wedge}$ is a dense subfield of $\mathscr{R}$. It is easy to note that the elements $0 := 0^{\wedge}$ and $1 := 1^{\wedge}$ are the zero and unit of $\mathscr{R}$.

2.2.3. Let us consider the descent $\mathscr{R}{\downarrow}$ of the algebraic system

$$\mathscr{R} := (|\mathscr{R}|, \oplus, \odot, \preccurlyeq, 0, 1)$$

(see 1.5.6, 1.5.7, and 1.8.4). In other words, we consider the descent of the universe $|\mathscr{R}|$ of the system $\mathscr{R}$ together with the descended operations $+ := \oplus{\downarrow}$, $\cdot := \odot{\downarrow}$ and order $\leqslant := \preccurlyeq{\downarrow}$. In more detail, we introduce addition, multiplication, and order on $\mathscr{R}{\downarrow}$ by the formulas

$$\begin{gathered} z = x + y \Longleftrightarrow [\![z = x \oplus y]\!] = \mathbb{1}, \\ z = x \cdot y \Longleftrightarrow [\![z = x \odot y]\!] = \mathbb{1}, \\ x \leqslant y \Longleftrightarrow [\![x \preccurlyeq y]\!] = \mathbb{1} \\ (x, y, z \in |\mathscr{R}|{\downarrow}). \end{gathered}$$

Also, we can introduce multiplication by the standard reals in $\mathscr{R}{\downarrow}$ by

$$y = \lambda x \Longleftrightarrow [\![y = \lambda^{\wedge} \odot x]\!] = \mathbb{1} \quad (\lambda \in \mathbb{R},\ x, y \in |\mathscr{R}|{\downarrow}).$$

For simplicity, in the sequel we identify $\mathscr{R}$ and $|\mathscr{R}|$ and denote the operations and order on $\mathscr{R}$ and $\mathscr{R}{\downarrow}$ by the same symbols $+$, $\cdot$, and $\leqslant$.

2.2.4. Gordon Theorem. *Let $\mathscr{R}$ be the reals in $\mathbb{V}^{(\mathbb{B})}$. Then $\mathscr{R}{\downarrow}$ (with the descended operations and order) is a universally complete vector lattice with a weak order unit $\mathbb{1} := 1^{\wedge}$. Moreover, there exists a Boolean isomorphism χ of $\mathbb{B}$ onto the Boolean algebra of band projections $\mathbb{P}(\mathscr{R}{\downarrow})$ (or components of the unit $\mathbb{C}(\mathbb{1})$) such that*

$$\begin{gathered}\chi(b)x = \chi(b)y \Longleftrightarrow b \leqslant [\![\, x = y \,]\!], \\ \chi(b)x \leqslant \chi(b)y \Longleftrightarrow b \leqslant [\![\, x \leqslant y \,]\!]\end{gathered} \tag{$\mathbb{G}$}$$

for all $x, y \in \mathscr{R}{\downarrow}$ and $b \in \mathbb{B}$.

$\lhd$ We omit an elementary verification of the fact that $\mathscr{R}{\downarrow}$ is a vector space over $\mathbb{R}$ and an ordered set. The remaining part is given in 2.2.5 and 2.2.6 below. $\rhd$

2.2.5. *The algebraic system $\mathscr{R}{\downarrow}$ is a universally complete vector lattice with weak order unit $\mathbb{1} := 1^{\wedge}$.*

$\lhd$ Show that the operations and order agree on $\mathscr{R}{\downarrow}$. Take elements $x, y \in \mathscr{R}{\downarrow}$ such that $x \leqslant y$. This means that

$$\mathbb{V}^{(\mathbb{B})} \models \text{“}x \text{ and } y \text{ are reals and } x \leqslant y.\text{”}$$

Let $u := x + z$, $v := y + z$, $x' := \lambda x$, and $y' := \lambda y$, where $z \in \mathscr{R}{\downarrow}$ and $\lambda \in \mathbb{R}$, $\lambda \geqslant 0$. By the definition of operations and order on $\mathscr{R}{\downarrow}$, we have $\mathbb{V}^{(\mathbb{B})} \models$ “x', y', u, and v are reals; moreover, $u = x + z$, $v = y + z$, $x' = \lambda^{\wedge} x$, and $y' = \lambda^{\wedge} y$.” The inequality $\lambda \geqslant 0$ implies $\mathbb{V}^{(\mathbb{B})} \models \lambda^{\wedge} \geqslant 0^{\wedge} = 0$. Using the requested properties of real numbers within $\mathbb{V}^{(\mathbb{B})}$, we obtain $\mathbb{V}^{(\mathbb{B})} \models$ “$u \leqslant v$ and $x' \leqslant y'$.” Thereby $u \leqslant v$ and $x' \leqslant y'$ by 2.2.3.

Let us show that the supremum of a nonempty bounded set A exists. Suppose that $A \subset \mathscr{R}{\downarrow}$ is bounded above by $y \in \mathscr{R}{\downarrow}$. By definition, $[\![\, x \leqslant y \,]\!] = \mathbb{1}$ for every $x \in A$. Then $\mathbb{V}^{(\mathbb{B})} \models$ “$A{\uparrow}$ is a set of reals bounded above by y” or, in view of 1.6.2,

$$[\![\, (\forall\, x \in A{\uparrow})\,(x \leqslant y) \,]\!] = \bigwedge_{x \in A} [\![\, x \leqslant y \,]\!] = \mathbb{1}.$$

The completeness of $\mathscr{R}$ yields

$$[\![\, (\exists a \in \mathscr{R})(a = \sup(A{\uparrow})) \,]\!] = \mathbb{1}.$$

By the maximum principle we find $a \in \mathbb{V}^{(\mathbb{B})}$ such that $[\![\, a \in \mathscr{R} \,]\!] = [\![\, a = \sup(A{\uparrow}) \,]\!] = \mathbb{1}$. Thereby $a \in \mathscr{R}{\downarrow}$ and if $z \in \mathscr{R}{\downarrow}$ is an upper bound of A then, as was already shown, $[\![\, z \text{ is an upper bound of } A{\uparrow} \,]\!] = \mathbb{1}$; therefore, $[\![\, a \leqslant z \,]\!] = \mathbb{1}$ or $a \leqslant z$. Consequently, a is the supremum of A in $\mathscr{R}{\downarrow}$. Incidentally, we have established that $a = \sup(A)$ if and only if $[\![\, a = \sup(A{\uparrow}) \,]\!] = \mathbb{1}$. In particular, given arbitrary $x, x_1, x_2 \in \mathscr{R}{\downarrow}$, we have $x = x_1 \vee x_2$ if and only if

$$[\![\, x = \sup\{x_1, x_2\} = x_1 \vee x_2 \,]\!] = \mathbb{1},$$

since $[\![\, \{x_1, x_2\}{\uparrow} = \{x_1, x_2\} \,]\!] = \mathbb{1}$. Of course, an analogous assertion is valid for meets.

Finally, take an arbitrary disjoint set $A \subset \mathscr{R}{\downarrow}$ of positive elements. We may see from the above remarks and 1.5.2 that

$$[\![\, (\forall\, x_1 \in A{\uparrow})(\forall\, x_2 \in A{\uparrow}) x_1 \wedge x_2 = 0 \,]\!] = \bigwedge_{x_1, x_2 \in A} [\![\, x_1 \wedge x_2 = 0 \,]\!] = \mathbb{1}.$$

Hence, the numerical set $A{\uparrow}$ (within $\mathbb{V}^{(\mathbb{B})}$) consists of pairwise disjoint positive elements. For such a set we have only the two possibilities open: either $[\![\, A{\uparrow} = \{0\} \,]\!] = \mathbb{1}$ and then $A \subset A{\uparrow}{\downarrow} = \{0\}$, or $[\![\, A{\uparrow} = \{0, a\} \,]\!] = \mathbb{1}$ for some $0 < a \in \mathscr{R}{\downarrow}$ (by the maximum principle) and then $[\![\, \sup(A{\uparrow}) = a \,]\!] = \mathbb{1}$. As was mentioned above, the latter is equivalent to the equality $a = \sup A$. We may conclude now that $\mathscr{R}{\downarrow}$ is a universally complete vector lattice. Recalling that $1 := 1^{\wedge}$ is the unity of the field $\mathscr{R}$ within $\mathbb{V}^{(\mathbb{B})}$ and using the formulas of 1.5.2, we find

$$\mathbb{1} = [\![\, (\forall\, x \in \mathscr{R})(x \wedge 1 = 0 \to x = 0) \,]\!] = \bigwedge_{x \in \mathscr{R}\downarrow} [\![\, x \wedge 1 = 0 \,]\!] \Rightarrow [\![\, x = 0 \,]\!].$$

Hence, we see that $[\![\, x \wedge 1 = 0 \,]\!] \leqslant [\![\, x = 0 \,]\!]$ for each $x \in \mathscr{R}{\downarrow}$. If $x \wedge 1 = 0$ then $[\![\, x \wedge 1 = 0 \,]\!] = \mathbb{1}$ and so $[\![\, x = 0 \,]\!] = \mathbb{1}$; i.e., $x = 0$. Thereby 1 is a weak order unit of the vector lattice $\mathscr{R}{\downarrow}$. $\triangleright$

2.2.6. *There exists an isomorphism χ of the Boolean algebra $\mathbb{B}$ onto $\mathbb{P}(\mathscr{R}{\downarrow})$ such that for all $x, y \in \mathscr{R}{\downarrow}$ and $b \in \mathbb{B}$ the equivalences of* 2.2.4($\mathbb{G}$) *hold.*

$\triangleleft$ Let us introduce some mapping $\chi : \mathbb{B} \to \mathbb{P}(\mathscr{R}{\downarrow})$. Take an arbitrary element $b \in \mathbb{B}$ and put $\chi(b)x := \operatorname{mix}\{bx, b^*0\}$ for $x \in \mathscr{R}{\downarrow}$. In other words, the element $\chi(b)x \in \mathscr{R}{\downarrow}$ is uniquely determined by the relations (cp. 1.4.3):

$$b \leqslant [\![\, \chi(b)x = x \,]\!], \quad b^* \leqslant [\![\, \chi(b)x = 0 \,]\!].$$

This implies that $\pi := \chi(b) : \mathscr{R}{\downarrow} \to \mathscr{R}{\downarrow}$ is an extensional mapping. Indeed, given $x, y \in \mathscr{R}{\downarrow}$ we have (cp. 1.2.5 (3)):

$$[\![x = y]\!] \wedge b \leqslant [\![x = y]\!] \wedge [\![x = \pi x]\!] \wedge [\![y = \pi y]\!] \leqslant [\![\pi x = \pi y]\!],$$
$$[\![x = y]\!] \wedge b^* \leqslant [\![x = y]\!] \wedge [\![\pi x = 0]\!] \wedge [\![\pi y = 0]\!] \leqslant [\![\pi x = \pi y]\!].$$

If $\rho := \pi{\uparrow}$ then $[\![\rho : \mathscr{R} \to \mathscr{R}]\!] = \mathbb{1}$ by 1.6.5 and $\rho = \operatorname{mix}\{bI_{\mathscr{R}}, b^*0\}$ by definition. Since 0 and $I_{\mathscr{R}}$ are idempotent positive linear mappings from $\mathscr{R}$ to $\mathscr{R}$, so is π. Moreover, $[\![(\forall x \in \mathscr{R}_+) \rho x \leqslant x]\!] = \mathbb{1}$; therefore, $\pi x \leqslant x$ for all $x \in \mathscr{R}{\downarrow}_+$. Thus, $\pi = \chi(b)$ is a band projection. Since ρ is positive, we have $[\![x \leqslant y \to \rho x \leqslant \rho y]\!] = \mathbb{1}$ for $x, y \in \mathscr{R}{\downarrow}$ and so

$$[\![x \leqslant y]\!] \leqslant [\![\rho x \leqslant \rho y]\!] = [\![\pi x \leqslant \pi y]\!].$$

Assume that $\pi x \leqslant \pi y$. Then $[\![\pi x = \pi y]\!] = \mathbb{1}$ and by 1.2.6

$$b \leqslant [\![\pi x \leqslant \pi y]\!] \wedge [\![\pi x = x]\!] \wedge [\![\pi y = y]\!] \leqslant [\![x \leqslant y]\!].$$

Conversely, if we assume that $b \leqslant [\![x \leqslant y]\!]$ then $b \leqslant [\![\pi x \leqslant \pi y]\!]$ by the above observation. Moreover,

$$b^* \leqslant [\![\pi x = 0]\!] \wedge [\![\pi y = 0]\!] \wedge [\![0 \leqslant 0]\!] \leqslant [\![\pi x \leqslant \pi y]\!];$$

consequently, $[\![\pi x \leqslant \pi y]\!] = \mathbb{1}$ or $\pi x \leqslant \pi y$.

Thereby we have established the second of the required equivalences 2.2.4 ($\mathbb{G}$). The first ensues from that by the formula $u = v \iff u \leqslant v \wedge v = u$.

It remains to demonstrate that the mapping χ is an isomorphism between the Boolean algebras $\mathbb{B}$ and $\mathbb{P}(\mathscr{R}{\downarrow})$. Take an arbitrary band projection $\pi \in \mathbb{P}(\mathscr{R}{\downarrow})$ and put $b := [\![\pi{\uparrow} = I_{\mathscr{R}}]\!]$. The fact that a band projection is extensional (and so the ascent $\pi{\uparrow}$ of π is well-defined) follows from 2.2.4 ($\mathbb{G}$), because

$$\begin{aligned} c = [\![x = y]\!] &\implies \chi(c)x = \chi(c)y \implies \pi\chi(c)x = \pi\chi(c)y \\ &\implies \chi(c)\pi x = \chi(c)\pi y \implies c \leqslant [\![\pi x = \pi y]\!]. \end{aligned}$$

Since π is idempotent, $\pi{\uparrow}$ as well is an idempotent mapping in $\mathscr{R}$; i.e., either $\pi{\uparrow} = I_{\mathscr{R}}$ or $\pi{\uparrow} = 0$. Hence, we derive $b^* = [\![\pi{\uparrow} \neq I_{\mathscr{R}}]\!] = [\![\pi{\uparrow} = 0]\!]$ and thereby $\pi{\uparrow} = \operatorname{mix}\{bI_{\mathscr{R}}, b^*(\mathbb{0})\}$. The mixture is unique; therefore, $\pi{\uparrow} = \chi(b){\uparrow}$; i.e., $\pi = \chi(b)$. Thus, χ is a bijection between $\mathbb{B}$ and $\mathbb{P}(\mathscr{R}{\downarrow})$.

Take arbitrary $b_1, b_2 \in \mathbb{B}$ and put $\rho_k := \chi(b_k)\!\uparrow$ $(k := 1, 2)$. Recalling that $\rho_k = \operatorname{mix}\{b_k I_{\mathscr{R}}, b_k^* \mathbb{0}\}$, we derive

$$[\![\chi(b_1 \wedge b_2)\!\uparrow = I_{\mathscr{R}}]\!] = b_1 \wedge b_2 = [\![\rho_1 = I_{\mathscr{R}} \wedge \rho_2 = I_{\mathscr{R}}]\!] \leqslant [\![\rho_1 \circ \rho_2 = I_{\mathscr{R}}]\!],$$
$$[\![\chi(b_1 \wedge b_2)\!\uparrow = 0]\!] = (b_1 \wedge b_2)^* = [\![\rho_1 = \mathbb{0} \vee \rho_2 = \mathbb{0}]\!] \leqslant [\![\rho_1 \circ \rho_2 = \mathbb{0}]\!].$$

From this, using 1.2.5 (3), we obtain

$$b_1 \wedge b_2 \leqslant [\![\chi(b_1 \wedge b_2)\!\uparrow = I_{\mathscr{R}}]\!] \wedge [\![\rho_1 \circ \rho_2 = I_{\mathscr{R}}]\!]$$
$$\leqslant [\![\chi(b_1 \wedge b_2)\!\uparrow = \rho_1 \circ \rho_2]\!],$$

$$(b_1 \wedge b_2)^* \leqslant [\![\chi(b_1 \wedge b_2)\!\uparrow = 0]\!] \wedge [\![\rho_1 \circ \rho_2 = 0]\!]$$
$$\leqslant [\![\chi(b_1 \wedge b_2)\!\uparrow = \rho_1 \circ \rho_2]\!].$$

Thus, $[\![\chi(b_1 \wedge b_2)\!\uparrow = \rho_1 \circ \rho_2]\!] = \mathbb{1}$ and taking into account the identity $[\![\rho_1 \circ \rho_2 = \chi(b_1)\!\uparrow \circ \chi(b_2)\!\uparrow = (\chi(b_1) \wedge \chi(b_2))\!\uparrow]\!] = \mathbb{1}$ (see 1.6.4) we arrive at the desired property $[\![\chi(b_1 \wedge b_2)\!\uparrow = \rho_1 \circ \rho_2 = (\chi(b_1) \wedge \chi(b_2))\!\uparrow]\!] = \mathbb{1}$ or, equivalently, $\chi(b_1 \wedge b_2) = \chi(b_1) \wedge \chi(b_2)$.

In particular, $0 = \chi(b) \wedge \chi(b^*)$ for $\chi(0) = 0$. Given elements $\rho := \chi(b)\!\uparrow$ and $\rho' := \chi(b^*)$, we have $[\![\rho, \rho' \in \{\mathbb{0}, I_{\mathscr{R}}\}; \rho = \mathbb{0}$ or $\rho' = \mathbb{0}$; and ρ and ρ' do not vanish simultaneously $]\!] = \mathbb{1}$. Hence, we see that $[\![\rho + \rho' = I_{\mathscr{R}}]\!] = \mathbb{1}$ and thereby $\chi(b) + \chi(b^*) = I_{\mathscr{R}\downarrow}$. Summarizing, we conclude that χ preserves meets and complements; i.e., χ is an isomorphism. $\triangleright$

2.3. Gordon's Theorem Revisited

In this section we examine some additional properties of Boolean valued reals: multiplicative structure, complexification, and some absoluteness.

2.3.1. An *ordered algebra* over an ordered field $\mathbb{F}$ is an ordered vector space X over $\mathbb{F}$ which is simultaneously an algebra over the same field and satisfies the condition: if $x \geqslant 0$ and $y \geqslant 0$ then $xy \geqslant 0$ for all $x, y \in X$. To characterize the positive cone X_+ of an ordered algebra X, we must add to what was said in 2.1.2 the property $X_+ \cdot X_+ \subset X_+$. We say that X is a *lattice ordered algebra* if X is a vector lattice and an ordered algebra simultaneously. A lattice-ordered algebra is an *f-algebra* if, for all $a, x, y \in X_+$, the condition $x \perp y = 0$ implies that $(ax) \perp y$

and $(xa) \perp y$. The multiplication on every (Archimedean) f-algebra is commutative and associative. An f-algebra is called *semiprime* if $xy = 0$ implies $x \perp y$ for all x and y. Clearly, an f-algebra is semiprime if and only if it do not contain nonzero nilpotent elements. The semiprimness of an f-algebra is equivalent to saying that there is no strictly positive element with zero square in it. A multiplicative unit vector of an Archimedean f-algebra is a weak order unit. Moreover, an f-algebra with unit is semiprime.

2.3.2. Theorem. *The universally complete vector lattice $\mathscr{R}{\downarrow}$ with the descended multiplications is a semiprime f-algebra with ring unit $\mathbb{1} := 1^\wedge$. Moreover, for every $b \in \mathbb{B}$ the band projection $\chi(b)$ acts as multiplication by $\chi(b)\mathbb{1}$.*

$\lhd$ The multiplicative structure on $\mathscr{R}{\downarrow}$ was defined in 2.2.3. As in the Gordon Theorem, we establish that $\mathscr{R}{\downarrow}$ is a semiprime f-algebra. Take $x \in \mathscr{R}{\downarrow}$ and $b \in \mathbb{B}$. By the definition of $\chi(b)$, we have $b \leqslant [\![\chi(b)x = x]\!]$ and $b^* \leqslant [\![\chi(b^*)x = 0]\!]$. Applying these to $x := 1^\wedge$ and appealing to the definition of multiplication on $\mathscr{R}{\downarrow}$, we obtain $b \leqslant [\![x = x \cdot 1^\wedge = x \cdot \chi(b)1^\wedge]\!]$ and $b^* \leqslant [[\, 0 = x \cdot 0 = x \cdot \chi(b)1^\wedge]\!]$. Thereby

$$[\![\chi(b)x = x \cdot \chi(b)1^\wedge]\!] \geqslant [\![\chi(b)x = x]\!] \wedge [\![x = x \cdot \chi(b)1^\wedge]\!] \geqslant b.$$

In a similar way, $b^* \leqslant [\![\chi(b)x = \chi(b)1^\wedge \cdot x]\!]$. Hence, $[\![\chi(b)x = x \cdot \chi(b)1^\wedge]\!] = \mathbb{1}$. $\rhd$

We see from the above that the mapping $b \mapsto \chi(b)1^\wedge$ $(b \in \mathbb{B})$ is a Boolean isomorphism between $\mathbb{B}$ and the algebra $\mathbb{C}(\mathscr{R}{\downarrow}) := \mathbb{C}(1^\wedge)$ of the components of the weak order unit $1^\wedge$. This isomorphism is denoted by the same letter χ. Thus, depending on the context, $x \mapsto \chi(b)x$ is either the appropriate band projection or the operator of multiplication by $\chi(b)\,\mathbb{1}$.

2.3.3. A *complex vector lattice* is defined to be the complexification $X_{\mathbb{C}} := X \oplus iX$ (with i standing for the imaginary unit) of a real vector lattice X; i.e., the additive group of $X \times X$ is endowed additionally with the scalar multiplication $(\alpha + i\beta)(x, y) = (\alpha x - \beta y, \alpha y + \beta x)$ for all $\alpha, \beta \in \mathbb{R}$ and $x, y \in X$. Identifying $x \in X$ with $(x, 0) \in X$ and iy with $(0, y)$, we will write $x + iy$ instead of (x, y). Often it is additionally required that the modulus

$$|z| := \sup\{(\cos\theta)\, x + (\sin\theta)\, y : \ 0 \leqslant \theta < 2\pi\}$$

exists for every element $z := x + iy \in X \oplus iX$. This requirement is automatically satisfied in a uniformly complete vector lattice, so that a Dedekind complete complex vector lattice is the complexification of a Dedekind complete real vector lattice. The Riesz decomposition property remains valid in every complex vector lattice: For all $z, z_1, z_2 \in X_{\mathbb{C}}$ with $|z| \leqslant |z_1| + |z_2|$ there exist $v_1, v_2 \in X_{\mathbb{C}}$ satisfying $z = v_1 + v_2$ and $|v_k| \leqslant |z_k|$ $(k = 1, 2)$.

Speaking about the order properties of the complex vector lattice $X_{\mathbb{C}}$, we mean its real part X. The concepts of sublattice, ideal, band, projection, etc. are naturally translated to the case of a complex vector lattice by appropriate complexification. For example, a subset A from $X_{\mathbb{C}}$ is said to be order bounded if the set $\{|z| :\ z \in A\}$ is order bounded in X. The the disjointness relation $\perp$ in $X_{\mathbb{C}}$ is defined as $z_1 \perp z_2 \iff |z_1| \wedge |z_2| = 0$, etc.

A *complex f-algebra* is defined as the complexification $A \oplus iA$ of a real f-algebra A and is denoted by $A_{\mathbb{C}}$. The multiplication on $A_{\mathbb{C}}$ is given by $(x + iy)(u + iv) = (xu - yv) + i(xv + yu)$ for all $x, y, u, v \in A$. The modulus $|z|$ of $z = x + iy$ is introduced by the above formula. Then $|z_1 z_2| = |z_1||z_2|$ and $z_1 \perp z_2$ implies $wz_1 \perp z_2$ for all $z_1, z_2, w \in A_{\mathbb{C}}$. Evidently, $A_{\mathbb{C}}$ has the same identity element as A.

2.3.4. Theorem. *Let $\mathscr{C}$ be the complexes in $\mathbb{V}^{(\mathbb{B})}$. Then the algebraic system $\mathscr{C}{\downarrow}$ is a universally complete complex f-algebra. Moreover, $\mathscr{C}{\downarrow}$ the complexification of the universally complete real f-algebra $\mathscr{R}{\downarrow}$; i.e., $\mathscr{C}{\downarrow} = \mathscr{R}{\downarrow} \oplus i\mathscr{R}{\downarrow}$.*

$\lhd$ Since $\mathbb{C} = \mathbb{R} \oplus i\mathbb{R}$ symbolizes a bounded formula, we have $[\![\, \mathbb{C}^{\wedge} = \mathbb{R}^{\wedge} \oplus \mathbb{R}^{\wedge} \,]\!] = \mathbb{1}$ (cp. 1.4.7), where i is the imaginary unit and the element $i^{\wedge}$ is denoted by the same letter i. From 2.2.2 we see that $[\![\, \mathbb{C}^{\wedge}$ is a dense subfield of the field $\mathscr{C}\,]\!] = \mathbb{1}$ and, in particular, $[\![\, i$ is the imaginary unit of the field $\mathscr{C}\,]\!] = \mathbb{1}$. If $z \in \mathscr{C}{\downarrow}$ then z is a complex number within $\mathbb{V}^{(\mathbb{B})}$; therefore,

$$[\![\, (\exists\, !x \in \mathscr{R})\, (\exists\, !y \in \mathscr{R})\, z = x + iy \,]\!] = \mathbb{1}.$$

The maximum principle implies that we have the unique pair of elements $x, y \in \mathbb{V}^{(\mathbb{B})}$ such that

$$[\![\, x, y \in \mathscr{R} \,]\!] = [\![\, z = x + iy \,]\!] = \mathbb{1}.$$

Hence, we obtain $x, y \in \mathscr{R}{\downarrow}$ such that $z = x + iy$, and so $\mathscr{C}{\downarrow} = \mathscr{R}{\downarrow} \oplus i\mathscr{R}{\downarrow}$. Appealing to the Gordon Theorem and 2.3.2 completes the proof. $\rhd$

Consider complete Boolean algebras $\mathbb{B}$ and $\mathbb{D}$ and a complete epimorphism $h : \mathbb{B} \to \mathbb{D}$. Denote by $h^* : \mathbb{V}^{(\mathbb{B})} \to \mathbb{V}^{(\mathbb{D})}$ the corresponding epimorphism of Boolean valued universes (cp. Section 1.3).

2.3.5. *Let $\mathscr{R}$ be the reals within $\mathbb{V}^{(\mathbb{B})}$. Then $h^*(\mathscr{R})$ is the reals within $\mathbb{V}^{(\mathbb{D})}$ and the mapping $h^* : x \mapsto h^*(x)$ is an order continuous lattice homomorphism from $\mathscr{R}{\downarrow}$ onto $h^*(\mathscr{R}){\downarrow}$. Moreover, if $\chi : \mathbb{B} \to \mathbb{P}(\mathscr{R}{\downarrow})$ and $\bar{\chi} : \mathbb{D} \to \mathbb{P}(h^*(\mathscr{R}){\downarrow})$ are Boolean isomorphisms from the Gordon Theorem then the diagram*

$$\begin{array}{ccc} \mathbb{B} & \xrightarrow{\ \chi\ } & \mathbb{P}(\mathscr{R}{\downarrow}) \\ {\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle h^*} \\ \mathbb{D} & \xrightarrow{\ \bar{\chi}\ } & \mathbb{P}(h^*(\mathscr{R}){\downarrow}) \end{array}$$

commutes. In particular, if $\ker(h) = [\mathbb{0}, b]$ for some $b \in \mathbb{B}$ and $\pi_b := \chi(b)$ then $\ker(h^*) = \ker(\pi_b)$, so that $h^*(\mathscr{R}){\downarrow}$ can naturally be identified with the band $\pi_b(\mathscr{R}{\downarrow})$ in the universally complete vector lattice $\mathscr{R}{\downarrow}$.

$\lhd$ It follows from 1.3.3 (2) that $h^*(\mathscr{R})$ is the reals within $\mathbb{V}^{\mathbb{D}}$ and h^* is a lattice homomorphism from $\mathscr{R}{\downarrow}$ onto $h^*(\mathscr{R}){\downarrow}$. If $x \in \mathscr{R}{\downarrow}$, $\bar{\mathbb{1}} := \mathbb{1}_{\mathbb{D}}$ is the unit of $\mathbb{D}$ and $\pi_b := \chi(b)$ then by 1.3.3 (2) and 1.4.6 we have

$$\begin{aligned} h^*(x) = 0 \iff [\![h^*(x) = 0^{\hat{\wedge}}]\!] = \bar{\mathbb{1}} \iff h([\![x = 0^\wedge]\!]) = h(b^*) \\ \iff b \leqslant [\![x = 0^\wedge]\!] \iff \pi_b x = 0 \end{aligned}$$

and so $\ker(h^*) = \ker(\chi(b))$. To prove the commutativity of the above diagram it is enough to estimate Boolean truth values for all $a \in \mathbb{B}$:

$$\begin{gathered} [\![\bar{\chi}(h(a)) = h^*(\chi(a))]\!]^{\mathbb{D}} \geqslant [\![\bar{\chi}(h(a)) = 1^{\hat{\wedge}}]\!]^{\mathbb{D}} \wedge [\![h^*(\chi(a)) = h^*(1^\wedge)]\!]^{\mathbb{D}} \\ = b \wedge a \wedge b \wedge [\![\chi(a) = 1^\wedge]\!]^{\mathbb{B}} = b \wedge a, \quad h(a) \wedge h([\![\chi(a) = 1^\wedge]\!]^{\mathbb{B}}) = h(a), \\ [\![\bar{\chi}(h(a)) = h^*(\chi(a))]\!]^{\mathbb{D}} \geqslant [\![\bar{\chi}(h(a)) = 0^{\hat{\wedge}}]\!]^{\mathbb{D}} \wedge [\![h^*(\chi(a)) = h^*(0^\wedge)]\!]^{\mathbb{D}} \\ = b \wedge a^* \wedge b \wedge [\![\chi(a) = 0^\wedge]\!]^{\mathbb{B}} = b \wedge a^*, \quad h(a^*) \wedge h([\![\chi(a) = 0^\wedge]\!]^{\mathbb{B}}) = h(a^*), \end{gathered}$$

whence $[\![\bar{\chi}(h(a)) = h^*(\chi(a))]\!]^{\mathbb{D}} \geqslant h(a) \vee h(a^*) = \bar{\mathbb{1}}$. It follows that $\bar{\chi}(h(a)) = h^*(\chi(a))$ for all $a \in \mathbb{D}$ and the proof is complete. $\rhd$

2.3.6. Consider the relative Boolean algebra $\bar{\mathbb{B}} := [\mathbb{0}, b] \subset \mathbb{B}$ with $0 \neq b \in \mathbb{B}$. Then $b \wedge \mathscr{R}$ is the reals within $\mathbb{V}^{\bar{\mathbb{B}}}$ and the mapping $h_b : x \mapsto b \wedge x$ is an order continuous lattice homomorphism from $\mathscr{R}{\downarrow}$ onto $(b \wedge \mathscr{R}){\downarrow}$. Moreover, if $\chi : \mathbb{B} \to \mathbb{P}(\mathscr{R}{\downarrow})$ and $\bar{\chi} : \bar{\mathbb{B}} \to \mathbb{P}((b \wedge \mathscr{R}){\downarrow})$ are Boolean isomorphisms from the Gordon Theorem then $\bar{\chi}(b \wedge a) = b \wedge \chi(a)$ for all $a \in \mathbb{B}$. In particular, $\ker(h_b) = \ker(\pi_b)$, $\pi_b := \chi(b)$, so that $(b \wedge \mathscr{R}){\downarrow}$ can naturally be identified with the band $\pi_b(\mathscr{R}{\downarrow})$.

2.3.7. Let $h : \mathbb{B} \to \mathbb{D}$ be an isomorphism of Boolean algebras. Then $h^*(\mathscr{R})$ is the reals within $\mathbb{V}^{(\mathbb{D})}$ and the mapping $h^* : x \mapsto h^*(x)$ is a lattice isomorphism of $\mathscr{R}{\downarrow}$ onto $h^*(\mathscr{R}){\downarrow}$. Moreover, if χ and $\bar{\chi}$ are the same as in 2.3.5 then $h^* \circ \chi = \bar{\chi} \circ h$.

2.4. Boolean Valued Reals Translated

Henceforth, $\mathscr{R}$ denotes the reals within $\mathbb{V}^{(\mathbb{B})}$. We will clarify the meaning of the least upper and greatest lower bounds, order limits, carriers, and spectral systems in the vector lattice $\mathscr{R}{\downarrow}$.

2.4.1. First, we will introduce a few definitions we need. The order on a vector lattice generates various types of convergence. Let $(\mathrm{A}, \leqslant)$ be an upward directed set. A net $(x_\alpha) := (x_\alpha)_{\alpha \in \mathrm{A}}$ in X is called *increasing* (*decreasing*) if $x_\alpha \leqslant x_\beta$ $(x_\beta \leqslant x_\alpha)$ for $\alpha \leqslant \beta$ $(\alpha, \beta \in \mathrm{A})$.

We say that a net (x_α) in a vector lattice X *o-converges* to $x \in X$ if there exists a decreasing net $(e_\beta)_{\beta \in \mathrm{B}}$ in X such that $\inf\{e_\beta : \beta \in \mathrm{B}\} = 0$ and for each $\beta \in \mathrm{B}$ there is $\alpha(\beta) \in \mathrm{A}$ with $|x_\alpha - x| \leqslant e_\beta$ for all $\alpha \geqslant \alpha(\beta)$. In this event, we call x the *o-limit* of the net (x_α) and write $x = o\text{-}\lim x_\alpha$ or $x_\alpha \xrightarrow{(o)} x$.

If a net (e_β) in this definition is replaced by a sequence $(\lambda_n e)_{n \in \mathbb{N}}$, where $0 \leqslant v \in X_+$ and $(\lambda_n)_{n \in \mathbb{N}}$ is a numerical sequence with $\lim_{n \to \infty} \lambda_n = 0$, then we say that a net $(x_\alpha)_{\alpha \in \mathrm{A}}$ *converges relatively uniformly* or more precisely *e-uniformly* to $x \in X$. The elements e and x are called the *regulator of convergence* and the *r-limit* of (x_α), respectively. The notations $x = r\text{-}\lim_{\alpha \in \mathrm{A}} x_\alpha$ and $x_\alpha \xrightarrow{(r)} x$ are also frequent.

A net $(x_\alpha)_{\alpha \in \mathrm{A}}$ is called o-fundamental (r-fundamental with regulator e) provided that the net $(x_\alpha - x_\beta)_{(\alpha,\beta) \in \mathrm{A} \times \mathrm{A}}$ o-converges (respectively, r-converges with regulator e) to zero. A vector lattice is said to be (*relatively*) *uniformly complete* if every r-fundamental sequence in it is r-convergent.

2.4.2. Define the sum of an infinite family $(x_\xi)_{\xi \in \Xi}$ in a vector lattice X. Given $\theta := \{\xi_1, \dots, \xi_n\} \in \mathscr{P}_{\mathrm{fin}}(\Xi)$, put $y_\theta := x_{\xi_1} + \dots + x_{\xi_n}$. So, we arrive at the net $(y_\theta)_{\theta \in \Theta}$, where $\Theta := \mathscr{P}_{\mathrm{fin}}(\Xi)$ is ordered by inclusion. Assuming that there is some x satisfying $x = o\text{-}\lim_{\theta \in \Theta} y_\theta$, we call the family (x_ξ) *summable in order* or *order summable* or *o-summable*. The element x is the *o-sum* of (x_ξ) and we write $x = o\text{-}\sum_{\xi \in \Xi} x_\xi$. Obviously, if $x_\xi \geqslant 0$ $(\xi \in \Xi)$ then for the o-sum of the family (x_ξ) to exist it is necessary and sufficient that the net $(y_\theta)_{\theta\in\Theta}$ has join, in which case $o\text{-}\sum_{\xi\in\Xi} x_\xi = \sup_{\theta\in\Theta} y_\theta$. If (x_ξ) is a disjoint family then

$$o\text{-}\sum_{\xi\in\Xi} x_\xi = \sup_{\xi\in\Xi} x_\xi^+ - \sup_{\xi\in\Xi} x_\xi^-.$$

2.4.3. *Let $(b_\xi)_{\xi\in\Xi}$ be a partition of unity in $\mathbb{B}$ and let $(x_\xi)_{\xi\in\Xi}$ be a family in $\mathscr{R}{\downarrow}$. Then*

$$\operatorname*{mix}_{\xi\in\Xi}(b_\xi x_\xi) = o\text{-}\sum_{\xi\in\Xi} \chi(b_\xi)x_\xi.$$

$\lhd$ If $x := \operatorname{mix}_{\xi\in\Xi}(b_\xi x_\xi)$ then $b_\xi \leqslant [\![\, x = x_\xi \,]\!]$ $(\xi \in \Xi)$ (cp. 1.4.3). According to 2.2.4 ($\mathbb{G}$), $\chi(b_\xi)x_\xi = \chi(b_\xi)x$ for all $\xi \in \Xi$. Summing the last equalities over ξ, we arrive at what was required. $\rhd$

2.4.4. *For a nonempty set $A \subset \mathscr{R}{\downarrow}$ and all $a \in \mathscr{R}$ and $b \in \mathbb{B}$ the equivalences hold:*

$$b \leqslant [\![\, a = \sup(A{\uparrow}) \,]\!] \iff \chi(b)a \ = \sup \chi(b)(A);$$
$$b \leqslant [\![\, a = \inf(A{\uparrow}) \,]\!] \iff \chi(b)a = \inf \chi(b)(A).$$

$\lhd$ We will prove only the first equivalence. In view of 2.2.4 ($\mathbb{G}$), the equality

$$\chi(b)\, a = \sup\{\chi(b)\, x : \ x \in A\}$$

holds if and only if $b \leqslant [\![\, x \leqslant a \,]\!]$ for all $x \in A$ and the formula $(\forall\, x \in A)(b \leqslant [\![\, x \leqslant y \,]\!])$ implies $b \leqslant [\![\, a \leqslant y \,]\!]$ for each $y \in \mathscr{R}{\downarrow}$.

Using the rules for calculating the truth values for quantifiers (cp. 1.2.3), we can represent the conditions under consideration in equivalent form:

$$b \leqslant [\![\, (\forall\, x \in A{\uparrow})\, x \leqslant a \,]\!],$$
$$b \leqslant [\![\, (\forall\, y \in \mathscr{R})\, (A{\uparrow} \leqslant y \to a \leqslant y) \,]\!].$$

This system of inequalities is equivalent to $b \leqslant [\![\, a = \sup(A{\uparrow}) \,]\!]$. $\rhd$

2.4.5. *Let A be an upward directed set and let $s : \mathrm{A} \to \mathscr{R}{\downarrow}$ be a net in $\mathscr{R}{\downarrow}$. Then $\mathrm{A}^{\wedge}$ is directed upward and $\sigma := s{\uparrow} : \mathrm{A}^{\wedge} \to \mathscr{R}$ is a net in $\mathscr{R}$ (within $\mathbb{V}^{(\mathbb{B})}$). Moreover,*

$$b \leqslant [\![\, x = \lim \sigma \,]\!] \iff \chi(b)x = o\text{-}\lim \chi(b) \circ s$$

for arbitrary $x \in \mathscr{R}{\downarrow}$ *and* $b \in \mathbb{B}$.

$\lhd$ The assertion "A is an upward directed set" is a bounded formula. By 1.4.7, we have $\mathbb{V}^{(\mathbb{B})} \models$ "$\mathrm{A}^{\wedge}$ is an upward directed set." The equality $\chi(b)x = o\text{-}\lim \chi(b) \circ s$ means that there exists a net $d : \mathrm{A} \to \mathscr{R}{\downarrow}$ for which the following system of conditions is compatible:

$$\alpha \leqslant \beta \to d(\alpha) \leqslant d(\beta) \quad (\alpha, \beta \in \mathrm{A}), \quad \inf_{\alpha \in \mathrm{A}} d(\alpha) = 0,$$
$$|\chi(b)x - \chi(b)s(\alpha)| \leqslant d(\alpha) \quad (\alpha \in \mathrm{A}).$$

Taking account of the easy formula $[\![\, s(\mathrm{A}){\uparrow} = \sigma(\mathrm{A}^{\wedge}) \,]\!] = \mathbb{1}$ (see 1.6.9) and putting $\delta := d{\uparrow}$, we see that the system of conditions is equivalent to the simultaneous inequalities:

$$b \leqslant [\![\inf \sigma(\mathrm{A}^{\wedge}) = 0]\!],$$
$$b \leqslant [\![(\forall\, \alpha, \beta \in \mathrm{A}^{\wedge})\, (\alpha \leqslant \beta \to \sigma(\alpha) \leqslant \sigma(\beta))]\!],$$
$$b \leqslant [\![(\forall\, \alpha \in \mathrm{A}^{\wedge})\, (|x - \sigma(\alpha)| < \delta(\alpha))]\!],$$

whose short form is just as follows: $b \leqslant [\![\, x = \lim \sigma \,]\!]$. $\rhd$

2.4.6. *Suppose that* A *and* $\sigma \in \mathbb{V}^{(\mathbb{B})}$ *are such that* $[\![$ A *is directed upward and* $\sigma : \mathrm{A} \to \mathscr{R} \,]\!] = \mathbb{1}$. *Then* $\mathrm{A}{\downarrow}$ *is an upward directed set and so the mapping* $s := \sigma{\downarrow} : \mathrm{A}{\downarrow} \to \mathscr{R}{\downarrow}$ *is a net in* $\mathscr{R}{\downarrow}$. *Moreover,*

$$b \leqslant [\![\, x = \lim \sigma \,]\!] \iff \chi(b)x = o\text{-}\lim \chi(b) \circ s$$

for all $x \in \mathscr{R}{\downarrow}$ *and* $b \in \mathbb{B}$.

$\lhd$ The proof is similar to that of 2.4.5. $\rhd$

2.4.7. *Let* f *be a mapping from a nonempty set* Ξ *to* $\mathscr{R}{\downarrow}$ *and* $g := f{\uparrow}$. *Then*

$$b \leqslant \left[\!\!\left[\, x = \sum_{\xi \in \Xi^{\wedge}} g(\xi) \,\right]\!\!\right] \iff \chi(b)x = o\text{-}\sum_{\xi \in \Xi} \chi(b) f(\xi)$$

for all $x \in \mathscr{R}{\downarrow}$ *and* $b \in \mathbb{B}$.

$\lhd$ First of all observe that the required equivalence holds for a finite set $\Xi_0 \subset \Xi$. Afterwards, apply 2.4.5 to the net $s : \mathscr{P}_{\mathrm{fin}}(\Xi) \to \mathscr{R}{\downarrow}$, where $\mathscr{P}_{\mathrm{fin}}(\Xi)$ is the set of finite subsets of Ξ and $s(\theta) := \sum_{\xi \in \theta} f(\xi)$, and use the formula $[\![\, \mathscr{P}_{\mathrm{fin}}(\Xi)^{\wedge} = \mathscr{P}_{\mathrm{fin}}(\Xi^{\wedge}) \,]\!] = \mathbb{1}$ (cp. 1.4.10). $\rhd$

2.4.8. Let X be a vector lattice with the principal projection property and a weak order unit $\mathbb{1}$. We call the projection of $\mathbb{1}$ to the band $\{x\}^{\perp\perp}$

the *trace* of x and denoted it by e_x. Thus, $e_x := \sup\{\mathbb{1} \wedge (n|x|) : n \in \mathbb{N}\}$. Clearly, the trace e_x serves both as a weak order unit of $\{x\}^{\perp\perp}$ and a component of $\mathbb{1}$.

Given a real λ, denote the trace of the positive part of $\lambda\mathbb{1} - x$ by e_λ^x:

$$e_\lambda^x := e_{(\lambda\mathbb{1}-x)^+} = \sup\{\mathbb{1} \wedge (n(\lambda\mathbb{1} - x)^+) :\ n \in \mathbb{N}\}.$$

The function $\lambda \mapsto e_\lambda^x$ $(\lambda \in \mathbb{R})$ arising in this case is called the *spectral system* or *characteristic* of x.

2.4.9. *The following hold for every $x \in \mathscr{R}{\downarrow}$:*

$$e_x := \chi([\![\, x \neq 0 \,]\!]), \quad e_\lambda^x = \chi([\![\, x < \lambda^\wedge \,]\!]) \quad (\lambda \in \mathbb{R}).$$

$\lhd$ A real t is distinct from zero if and only if the join of the set $\{1 \wedge (n|t|) : n \in \omega\}$ is equal to 1. Consequently, for $x \in \mathscr{R}{\downarrow}$ the transfer principle yields $b := [\![\, x \neq 0 \,]\!] = [\![\, 1^\wedge = \sup A \,]\!]$, where $A \in \mathbb{V}^{(\mathbb{B})}$ is determined by the formula $A := \{1^\wedge \wedge (n|x|) : n \in \omega^\wedge\}$. If $C := \{1^\wedge \wedge (n|x|) : n \in \omega\}$ then we prove that $[\![\, C{\uparrow} = A \,]\!] = \mathbb{1}$ using the second formula of 1.6.2 and the representation $\omega^\wedge = (\iota\omega){\uparrow}$ of 1.5.8. Hence, $[\![\, \sup(A) = \sup(C{\uparrow}) \,]\!] = \mathbb{1}$. Using 2.4.4, we derive

$$b = [\![\, \sup(C{\uparrow}) = 1^\wedge \,]\!] = [\![\, \sup(C) = 1^\wedge \,]\!] = [\![\, e_x = 1^\wedge \,]\!].$$

On the other hand, $[\![\, e_x = 0 \,]\!] = [\![\, e_x = 1^\wedge \,]\!]^* = b^*$. By 2.2.4 ($\mathbb{G}$), we can write down

$$\chi(b)e_x = \chi(b)1^\wedge = \chi(b), \quad \chi(b^*)e_x = 0 \Longrightarrow \chi(b)e_x = e_x.$$

Finally, $\chi(b) = e_x$.

Take $\lambda \in \mathbb{R}$ and put $y := (\lambda\mathbb{1} - x)^+$. Since $[\![\, \lambda^\wedge = \lambda\mathbb{1} \,]\!] = \mathbb{1}$, we have $[\![\, y = (\lambda^\wedge - x)^+ \,]\!] = \mathbb{1}$. Consequently, $e_\lambda^x = e_y = \chi([\![\, y \neq 0 \,]\!])$. It remains to observe that within $\mathbb{V}^{(\mathbb{B})}$ the number $y = (\lambda^\wedge - x) \vee 0$ is distinct from zero if and only if $\lambda^\wedge - x > 0$; i.e., $[\![\, y \neq 0 \,]\!] = [\![\, x < \lambda^\wedge \,]\!]$. $\rhd$

2.5. Vector Lattices Within Boolean Valued Reals

The aim of this section is to demonstrate that an Archimedean vector lattice is represented as a vector sublattice of the internal reals $\mathscr{R}$ in an appropriate Boolean valued universe considered as a vector lattice over the field of standard reals.

2.5.1. Representation Theorem. *Let X be an Archimedean vector lattice, let $\mathscr{R}$ be the reals within $\mathbb{V}^{(\mathbb{B})}$, and let $\jmath$ be an isomorphism of $\mathbb{B}$ onto the Boolean algebra $\mathbb{B}(X)$. Then there exists an element $\mathscr{X} \in \mathbb{V}^{(\mathbb{B})}$ satisfying the conditions:*

(1) *$\mathbb{V}^{(\mathbb{B})} \models$ "$\mathscr{X}$ is a vector sublattice of the field $\mathscr{R}$ considered as a vector lattice over the subfielf $\mathbb{R}^\wedge$".*

(2) *$X' := \mathscr{X}{\downarrow}$ is a laterally complete vector sublattice of $\mathscr{R}{\downarrow}$ which is majorizing and invariant under each band projection $\chi(b)$ $(b \in \mathbb{B})$.*

(3) *$\iota(X)$ is an order dense sublattice in $\mathscr{R}{\downarrow}$ for some o-continuous lattice isomorphism $\iota : X \to X'$.*

(4) *For every $b \in \mathbb{B}$ the band projection in $\mathscr{R}{\downarrow}$ onto $\{\iota(\jmath(b))\}^{\perp\perp}$ coincides with $\chi(b)$.*

$\lhd$ Put $d(x,y) := \jmath^{-1}(\{|x-y|\}^{\perp\perp})$. Let $\mathscr{X}$ be the Boolean valued representation of the $\mathbb{B}$-set (X,d) and $X' := \mathscr{X}{\downarrow}$ (cp. 1.7.1 and 1.7.2). By 1.7.2, without loss of generality we can assume that $X \subset X'$, $d(x,y) = [\![\, x \neq y \,]\!]$ $(x,y \in X)$, and $X' = \operatorname{mix}(X)$. Further, furnish X' with a vector lattice structure. To this end, take $\lambda \in \mathbb{R}$ and $x,y \in X'$ of the form $x := \operatorname{mix}(b_\xi x_\xi)$ and $y := \operatorname{mix}(b_\xi y_\xi)$, where $(x_\xi) \subset X$, $(y_\xi) \subset X$, and (b_ξ) is a partition of unity in $\mathbb{B}$. Put

$$x + y := \operatorname{mix}(b_\xi(x_\xi + y_\xi)),$$
$$\lambda x := \operatorname{mix}(b_\xi(\lambda x_\xi)),$$
$$x \leqslant y \Longleftrightarrow x = \operatorname{mix}(b_\xi(x_\xi \wedge y_\xi)).$$

Within $\mathbb{V}^{(\mathbb{B})}$, we define the addition $\oplus$, multiplication $\odot$, and order $\preccurlyeq$ on $\mathscr{X}$ as the ascents of the corresponding objects on X'. More precisely, the operations $\oplus : \mathscr{X} \times \mathscr{X} \to \mathscr{X}$ and $\odot : \mathbb{R}^\wedge \times \mathscr{X} \to \mathscr{X}$ and the predicate $\preccurlyeq \subset \mathscr{X} \times \mathscr{X}$ are determined from the formulas:

$$[\![\, x \oplus y = x + y \,]\!] = \mathbb{1},$$
$$[\![\, \lambda^\wedge \odot x = \lambda x \,]\!] = \mathbb{1} \quad (x,y \in X',\ \lambda \in \mathbb{R}),$$
$$[\![\, x \preccurlyeq y \,]\!] = \bigvee \{ [\![\, x = x' \,]\!] \wedge [\![\, y = y' \,]\!] : x', y' \in X',\ x' \leqslant y' \}.$$

Thus, we can claim that $\mathscr{X}$ is a vector lattice over the field $\mathbb{R}^\wedge$ and, in particular, a lattice ordered group within $\mathbb{V}^{(\mathbb{B})}$. Also, it is clear that the Archimedean axiom is valid on $\mathscr{X}$, since X' is an Archimedean lattice.

Note that if $x \in X_+$ then $\{x\}^{\perp\perp} = d(x,0) = [\![\, x \neq 0 \,]\!]$; i.e., $\{x\}^\perp = [\![\, x = 0 \,]\!]$. Consequently, we have

$$[\![\, x = 0 \,]\!] \vee [\![\, y = 0 \,]\!] = \{x\}^\perp \vee \{y\}^\perp = \mathbb{1}_B$$

for every pair of disjoint $x, y \in X$. Hence, we easily derive that $[\![\, \mathscr{X}$ is linearly ordered$\,]\!] = \mathbb{1}$, for

$$[\![\, (\forall\, x \in \mathscr{X})(\forall\, y \in \mathscr{X})\ (|x| \wedge |y| = 0 \to x = 0 \vee y = 0)\,]\!] = \mathbb{1}.$$

It is well known that an Archimedean linearly ordered group is isomorphic to an additive subgroup of the reals. Applying this assertion to $\mathscr{X}$ within $\mathbb{V}^{(\mathbb{B})}$, without loss of generality we can assume that $\mathscr{X}$ is an additive subgroup of $\mathscr{R}$. Furthermore, we presume that $1^\wedge \in \mathscr{X}$, since otherwise $\mathscr{X}$ could be replaced by the isomorphic group $e^{-1}\mathscr{X}$ with $0 < e \in \mathscr{X}$. The multiplication $\odot$ represents a continuous $\mathbb{R}^\wedge$-bilinear mapping from $\mathbb{R}^\wedge \times \mathscr{X}$ to $\mathscr{X}$. Let $\beta : \mathscr{R} \times \mathscr{R} \to \mathscr{R}$ be its extension by continuity. Then β is $\mathscr{R}$-bilinear and $\beta(1^\wedge, 1^\wedge) = 1^\wedge \odot 1^\wedge = 1^\wedge$. Consequently, β coincides with the usual multiplication on $\mathscr{R}$; i.e., $\mathscr{X}$ is a vector sublattice of the field $\mathscr{R}$ considered as a vector lattice over $\mathbb{R}^\wedge$. Thereby $X' \subset \mathscr{R}{\downarrow}$.

The fact that X' is majorizing in $\mathscr{R}{\downarrow}$ ensues obviously, since $[\![\, \mathscr{X}$ is dense in $\mathscr{R}\,]\!] = \mathbb{1}$ and by the maximum principle for each $y \in \mathscr{R}{\downarrow}$ there exists $x \in \mathscr{X}{\downarrow}$ with $[\![y \leqslant x]\!] = \mathbb{1}$, whence $y \leqslant x$. Prove that X is minorizing in X'.

It follows from the properties of the isomorphism χ (cp. 2.2.6) that

$$\chi(b)\iota x = 0 \iff \jmath(b) \leqslant \{x\}^{\perp} \iff x \in \jmath(b^{\perp}),$$

whatever $b \in \mathbb{B}$ and $x \in X_+$ might be. Hence, $\chi(b)$ is the band projection onto the band in $\mathscr{R}{\downarrow}$ generated by $\iota(\jmath(b))$. Moreover, if $\chi(b)x = 0$ for all $x \in X_+$ then $b = \{0\}$. Thus, for every $b \in \mathbb{B}$ we can find a positive element $y \in X$ for which $y = \chi(b)y$. Take $0 < z \in X'$. The representation $z = o\text{-}\sum_{\xi \in \Xi} \chi(b_\xi)x_\xi$ is valid, where (b_ξ) is a partition of unity in $\mathbb{B}$ and $(x_\xi) \subset X_+$. We see that $\chi(b_\xi)x_\xi \neq 0$ at least for one index ξ. Let $\pi := \chi(b_\xi) \circ \chi([\![\, x_\xi \neq 0\,]\!])$ and let y be a strictly positive element in X such that $y = \pi y$. Then for $x_0 := y \wedge x_\xi$ we have $0 < x_0 \leqslant \pi x_\xi \leqslant \chi(b_\xi)x_\xi \leqslant z$ and $x_0 \in X$. Thereby X is minorizing in X'. $\triangleright$

Observe some corollaries to Theorems 2.2.4 and 2.5.1 with the same denotations $\mathbb{B}$, X, X', $\mathscr{X}$, ι, and $\mathscr{R}$.

2.5.2. A few additional remarks are in order.

(1) *For every $x' \in X'$ there exist a family $(x_\xi) \subset X$ and a partition of unity (π_ξ) in $\mathbb{P}(\mathscr{R}{\downarrow})$ such that*

$$x' = o\text{-}\sum_{\xi \in \Xi} \pi_\xi \iota x_\xi.$$

$\lhd$ By 1.7.2 $X' = \mathrm{mix}(\imath(X))$ and 2.4.3 yields the result. $\rhd$

(2) *For every $x \in \mathscr{R}{\downarrow}$ and $\varepsilon > 0$ there is $x_\varepsilon \in X'$ such that $|x - x_\varepsilon| \leqslant \varepsilon\mathbb{1}$.*

$\lhd$ This is a consequence of the fact that $[\![\, \mathscr{X}$ is dense in $\mathscr{R}\,]\!] = \mathbb{1}$. $\rhd$

(3) *X is laterally complete if and only if $X = X'$.*

$\lhd$ The sufficiency is obvious. If X is laterally complete then X has the projection property (see Veksler and Geyler [398]) and the claim follows from (1). $\rhd$

2.5.3. The element $\mathscr{X} \in \mathbb{V}^{(\mathbb{B})}$ arising in 2.5.1 is called the *Boolean valued representation* of X. Thus, the Boolean valued representations of Archimedean vector lattices are vector sublattices of the reals $\mathscr{R}$ considered as a vector lattice over the field $\mathbb{R}^\wedge$.

The vector lattice $X' = \mathscr{X}{\downarrow}$ is called the *lateral completion* of X and denoted by X^λ. We identify X and its lattice isomorphic image $\iota(X)$ in X^λ, so that we consider X as a sublattice of its lateral completion X^λ.

Given a vector sublattice L of a laterally complete vector lattice X', denote by $\lambda(L)$ the laterally complete sublattice in X' generated by L, i.e. the smallest laterally complete sublattice in X' including L. It is easy to check that $\lambda(L) = L{\uparrow}{\downarrow}$ and hence, by 1.6.6 and 2.4.3, $\lambda(L)$ comprises all $x \in X$ of the form $x = o\text{-}\sum \pi_\xi y_\xi$ with an arbitrary family (x_ξ) in L and partition of unity (π_ξ) in $\mathbb{P}(X')$.

The lateral completion of a vector lattice is essentially unique: If Z is a laterally complete vector lattice and X is lattice isomorphic to an order dense sublattice Y of Z, then X^λ is lattice isomorphic to the sublattice Y' of Z consisting of all $y \in Z$ representable as $y = \sum \pi_\xi y_\xi$ with an arbitrary family (y_ξ) in Y and a partition of unity in $\mathbb{P}(Y)$.

2.5.4. *If $h : X \to \mathscr{R}{\downarrow}$ is a lattice isomorphism and for every $b \in \mathbb{B}$ the band projection onto the band in $\mathscr{R}{\downarrow}$ generated by $h(\jmath(b))$ coincides with $\chi(b)$ then there exists $a \in \mathscr{R}{\downarrow}$ such that $hx = a \cdot \iota(x)$ $(x \in X)$. If there exists a weak order unit $\mathbb{1}$ in X then the isomorphism ι is uniquely determined by the requirement $\iota\mathbb{1} = 1$.*

$\lhd$ Indeed, if $X_0 := \mathrm{im}\,\iota$ and $h_0 := h \circ \iota^{-1}$ then the isomorphism $h_0 : X_0 \to \mathscr{R}{\downarrow}$ is extensional; therefore, for $\tau := h_0{\uparrow}$ we have $[\![$ the mapping $\tau : \mathscr{X} \to \mathscr{R}$ is isotonic, injective, and additive $]\!] = \mathbb{1}$. Consequently, h_0 is continuous and has the form $\tau(\alpha) = a \cdot \alpha$ $(\alpha \in \mathscr{R})$, where a is a fixed element in $\mathscr{R}{\downarrow}$. Hence, we derive that $h_0(y) = a \cdot y$ $(y \in X_0)$ or $h(x) = a \cdot \iota(x)$ $(x \in X)$. $\rhd$

2.5.5. *If X is a Dedekind complete vector lattice then $\mathscr{X} = \mathscr{R}$, $X' = \mathscr{R}{\downarrow}$, and $\iota(X)$ is an order dense ideal in $\mathscr{R}{\downarrow}$. Moreover, $\iota^{-1} \circ \chi(b) \circ \iota$ is the band projection onto $\jmath(b)$ for every $b \in \mathbb{B}$.*

$\lhd$ If X is order complete then so is X'. From 2.4.4 we see that the order completeness of X' is equivalent to the axiom of existence of suprema (infima) for bounded sets in $\mathscr{X}$. By 2.2.1 (1), $\mathscr{X} = \mathscr{R}$ and $X' = \mathscr{R}{\downarrow}$. Let $e \in X_+$, $y \in \mathscr{R}{\downarrow}$, and $|y| \leqslant \iota e$. Since $\iota(X)$ is an order dense sublattice in $\mathscr{R}{\downarrow}$, we have $y^+ = \sup \iota(A)$, where $A := \{x \in X_+ : \iota x \leqslant y^+\}$. But the set A is bounded in X by e; therefore, $\sup A \in X$ and $y^+ = \iota(\sup A) \in \iota X$. Similarly, $y^- \in \iota(X)$ and, finally, $y \in \iota(X)$. $\rhd$

2.5.6. *The image $\iota(X)$ coincides with the whole $\mathscr{R}{\downarrow}$ if and only if X is a universally complete vector lattice.*

$\lhd$ If X is a Dedekind complete vector lattice then $\mathscr{X} = \mathscr{R}$ by 2.5.5 and, hence, $\mathscr{R}{\downarrow} = \mathscr{X}{\downarrow} = \operatorname{mix} \iota(X)$. But for the universally complete vector lattice X we have $\operatorname{mix} \iota(X) = \iota(X)$. The converse is obvious. $\rhd$

2.5.7. *Universally complete vector latices are isomorphic if and only if so are their bases.*

$\lhd$ If X and Y are universally complete vector lattices and the Boolean algebras $\mathbb{B}(X)$ and $\mathbb{B}(Y)$ are isomorphic then by 2.5.6 X and Y are isomorphic to the same vector lattice $\mathscr{R}{\downarrow}$ with $\mathscr{R} \in \mathbb{V}^{(\mathbb{B})}$ and $\mathbb{B} \simeq \mathbb{B}(X) \simeq \mathbb{B}(Y)$. On the other hand, if h is an isomorphism from X onto Y then the mapping $K \mapsto h(K)$ $(K \in \mathbb{B}(X))$ is an isomorphism of the Boolean algebras $\mathbb{B}(X)$ and $\mathbb{B}(Y)$. $\rhd$

2.5.8. *Let X be a universally complete vector lattice with a weak order unit $\mathbb{1}$. Then we can uniquely define the multiplication on X so as to make X into a semiprime f-algebra and $\mathbb{1}$, into a ring unit.*

$\lhd$ By 2.5.4 and 2.5.6, we can assume that $X = \mathscr{R}{\downarrow}$ and $\mathbb{1} = 1^{\wedge}$. The existence of the required multiplication on X follows from 2.3.2. Assume that there is another multiplication $\odot : X \times X \to X$ on X and $(X, +, \odot, \leqslant)$ is a semiprime f-algebra with unity $\mathbb{1}$. The semiprimeness of the f-algebra implies that $\odot$ is an extensional mapping. But then the ascent $\times := \odot{\uparrow}$ is a multiplication on $\mathscr{R}$. By uniqueness of the multiplicative structure on $\mathscr{R}$, we conclude that $\times = \cdot$. Hence, we derive that $\odot$ coincides with the original multiplication on X (cp. 2.3.2). $\rhd$

2.6. Order Convergence

Interpreting the concept of convergent numerical net within $\mathbb{V}^{(\mathbb{B})}$ and using 2.4.5 and 2.5.5, we obtain some useful tests for o-convergence in a Dedekind complete vector lattice. Recall that $[x]$ stands for the band projection onto the band generated by x.

2.6.1. *Let $(x_\alpha)_{\alpha\in\mathrm{A}}$ be an order bounded net in a Dedekind complete vector lattice X and $x \in X$. The following are equivalent:*

(1) *(x_α) o-converges to x.*

(2) *o-$\lim_{\alpha\in\mathrm{A}}[e]\big[(|x_\alpha - x| - e)^+\big] = 0$ in $\mathbb{P}(X)$ for all positive $e \in X$.*

(3) *For every $e \in X_+$ there exists a partition $(\pi_\alpha)_{\alpha\in\mathrm{A}}$ of $[e]$ in the Boolean algebra $\mathbb{P}(X)$ such that*

$$\pi_\alpha |x - x_\beta| \leqslant e \quad (\alpha, \beta \in \mathrm{A},\ \beta \geqslant \alpha).$$

(4) *For every $e \in X_+$ there exists an increasing net $(\rho_\alpha)_{\alpha\in\mathrm{A}}$ in the Boolean algebra $\mathbb{P}(X)$ such that*

$$\rho_\alpha |x - x_\beta| \leqslant e \quad (\alpha, \beta \in \mathrm{A},\ \beta \geqslant \alpha).$$

$\lhd$ Without loss of generality we can assume that X is an order dense ideal of the universally complete vector lattice $\mathscr{R}{\downarrow}$ (cp. 2.5.5).

(1) $\Longleftrightarrow$ (2): Let σ be the modified ascent of the mapping $s : \alpha \to x_\alpha$. Then $[\![\sigma \text{ is a net in } \mathscr{R}]\!] = \mathbb{1}$. By 2.4.5, o-$\lim s = x$ if and only the identity if $[\![\lim \sigma = x]\!] = \mathbb{1}$ holds. We may rewrite this identity as

$$\mathbb{1} = [\![(\forall\, \varepsilon \in \mathscr{R}_+)(\varepsilon > 0 \to (\exists\, \alpha \in \mathrm{A}^\wedge)(\forall\, \beta \in \mathrm{A}^\wedge)$$
$$(\beta \geqslant \alpha \to |x - x_\beta| \leqslant \varepsilon))]\!]. \quad (*)$$

Calculating the Boolean truth values for the quantifiers, we find the following equivalent form

$$(\forall\, \varepsilon \in X_+)\, [\![\varepsilon \neq 0]\!] \leqslant \bigvee_{\alpha\in\mathrm{A}} \bigwedge_{\substack{\beta\in\mathrm{A}\\ \beta\geqslant\alpha}} [\![|x - x_\beta| - \varepsilon \leqslant 0]\!]$$

which in turn amounts to the formula

$$(\forall\, \varepsilon \in X_+)\left([\![\varepsilon \neq 0]\!] = \bigvee_{\alpha\in\mathrm{A}} \bigwedge_{\substack{\beta\in\mathrm{A}\\ \beta\geqslant\alpha}} [\![\varepsilon \neq 0]\!] \wedge [\![(|x - x_\beta| - \varepsilon)^+ = 0]\!]\right).$$

Since $\chi([\![(|x-x_\beta|-\varepsilon)^+]\!]) = \big[(|x_\alpha - x| - \varepsilon)^+\big]$ and $\chi([\![\varepsilon \neq 0]\!]) = [\varepsilon]$ (cp. 2.4.9), we see from the above that $x_\alpha \overset{(o)}{\to} x$ if and only if

$$\begin{aligned}
0 =& [\![\varepsilon \neq 0]\!] \wedge [\![\varepsilon = 0]\!] \\
=& [\![\varepsilon \neq 0]\!] \wedge \Bigg(\bigvee_{\alpha\in \mathrm{A}} \bigwedge_{\substack{\beta\in \mathrm{A}\\ \beta\geqslant\alpha}} [\![\varepsilon \neq 0]\!] \wedge [\![(|x-x_\beta|-\varepsilon)^+ = 0]\!] \Bigg)^* \\
=& [\![\varepsilon \neq 0]\!] \wedge \bigwedge_{\alpha\in \mathrm{A}} \bigvee_{\substack{\beta\in \mathrm{A}\\ \beta\geqslant\alpha}} [\![\varepsilon = 0]\!] \vee [\![(|x-x_\beta|-\varepsilon)^+ \neq 0]\!] \\
\geqslant& \bigwedge_{\alpha\in \mathrm{A}} \bigvee_{\substack{\beta\in \mathrm{A}\\ \beta\geqslant\alpha}} [\![\varepsilon \neq 0]\!] \wedge [\![(|x-x_\beta|-\varepsilon)^+ \neq 0]\!] \\
=& \, o\text{-}\lim_{\alpha\in \mathrm{A}} [\varepsilon]\big[(|x_\alpha - x| - \varepsilon)^+\big].
\end{aligned}$$

(1) $\iff$ (3): Arguing as in (1) $\iff$ (2) and putting $b := \chi^{-1}([\varepsilon])$ and $c_\alpha := \big\{ \bigwedge [\![|x-x_\beta| \leqslant \varepsilon]\!] : \beta \in \mathrm{A}, \beta \geqslant \alpha \big\}$, we find that the equality $o\text{-}\lim x_\alpha = x$ is equivalent to the formula

$$\begin{aligned}
(\forall\, \varepsilon \in X_+)\big(\exists\, (c_\alpha)_{\alpha\in\mathrm{A}} \subset \mathbb{B}\big)\Bigg(& \bigvee_{\alpha\in\mathrm{A}} c_\alpha = b \\
& \wedge (\forall\, \beta \in \mathrm{A})\Big(\beta \geqslant \alpha \Rightarrow c_\alpha \leqslant [\![|x_\alpha - x| \leqslant \varepsilon]\!]\Big)\Bigg).
\end{aligned}$$

By the exhaustion principle for Boolean algebras, there exist a partition of unity $(d_\xi)_{\xi\in\Xi}$ in $\mathbb{B}$ and a mapping $\delta : \Xi \to \mathrm{A}$ such that $d_\xi \leqslant c_{\delta(\xi)}$ $(\xi \in \Xi)$. Put $b_\alpha := b \wedge \bigvee\{d_\xi : \alpha = \delta(\xi)\}$ if $\alpha \in \delta(\Xi)$ and $b_\alpha = 0$ if $\alpha \notin \delta(\Xi)$. We see that $(b_\alpha)_{\alpha\in\mathrm{A}}$ is a partition of b and $b_\alpha \leqslant c_\alpha$ $(\alpha \in \mathrm{A})$. Thus, if $x_\alpha \to x$ then for every $\varepsilon \in X_+$ there is a partition of unity (b_α) such that

$$b_\alpha \leqslant [\![|x-x_\beta| \leqslant \varepsilon]\!] \quad (\alpha, \beta \in \mathrm{A},\ \beta \geqslant \alpha).$$

As follows from 2.2.4 ($\mathbb{G}$), the latter means that

$$\pi_\alpha |x - x_\beta| \leqslant \pi_\alpha \varepsilon \leqslant \varepsilon \quad (\alpha, \beta \in \mathrm{A},\ \beta \geqslant \alpha),$$

where $\pi_\alpha := \chi(b_\alpha)$. Since (π_α) is a partition of $[\varepsilon]$ in $\mathbb{P}(X)$, the necessity is proven.

To prove the sufficiency, observe that if the conditions are satisfied and $a := \limsup |x_\alpha - x|$ then

$$\pi_\alpha a \leqslant \bigvee_{\beta \geqslant \alpha} |x_\beta - x| \leqslant \pi_\alpha \varepsilon$$

for all $\alpha \in \mathrm{A}$. Consequently,

$$0 \leqslant a = \sum \pi_\alpha a \leqslant \sum \pi_\alpha \varepsilon = \varepsilon.$$

Since $\varepsilon \in X_+$ is arbitrary, we have $a = 0$ and $o\text{-}\lim_\alpha x_\alpha = x$.

(3) $\iff$ (4): We only have to put $\rho_\alpha := \bigvee\{\pi_\beta : \beta \in \mathrm{A}, \alpha \leqslant \beta\}$ in (3). $\rhd$

2.6.2. Corollary. *Assume that X has a weak order unit $\mathbb{1}$, $(x_\alpha)_{\alpha \in \mathrm{A}}$ is an order bounded net in X, and $x \in X$. The following are equivalent:*

(1) *The net (x_α) o-converges to x.*

(2) *For every $0 < \varepsilon \in \mathbb{R}$ the equality $o\text{-}\lim_{\alpha \in \mathrm{A}} e_\varepsilon^{y(\alpha)} = \mathbb{1}$ with $y(\alpha) := |x - x_\alpha|$ holds in $\mathbb{P}(X)$.*

(3) *For every $n \in \mathbb{N}$ the equality $o\text{-}\lim_{\alpha \in \mathrm{A}} \big[(|x - x_\alpha| - \mathbb{1}/n)^+\big] = 0$ holds in $\mathbb{P}(X)$.*

(4) *For every $0 < \varepsilon \in \mathbb{R}$ there exists a partition of unity $(\pi_\alpha)_{\alpha \in \mathrm{A}}$ in $\mathbb{P}(X)$ such that $\pi_\alpha |x - x_\beta| \leqslant \varepsilon \mathbb{1}$ for all $\alpha, \beta \in \mathrm{A}$, $\beta \geqslant \alpha$.*

(5) *For every $0 < \varepsilon \in \mathbb{R}$ there exists an increasing net $(\rho_\alpha)_{\alpha \in \mathrm{A}}$ in $\mathbb{P}(X)$ such that $\rho_\alpha |x - x_\beta| \leqslant \varepsilon \mathbb{1}$ for all $\alpha, \beta \in \mathrm{A}$, $\beta \geqslant \alpha$.*

$\lhd$ The proof proceeds along the same lines as before. Since $[\![\mathbb{R}^\wedge$ is dense in $\mathscr{R}]\!] = \mathbb{1}$, we can rewrite $(*)$ in equivalent form:

$$\mathbb{1} = [\![(\forall\, \varepsilon \in \mathbb{R}^\wedge)(\varepsilon > 0 \to (\exists\, \alpha \in \mathrm{A}^\wedge)(\forall\, \beta \in \mathrm{A}^\wedge)\,(\beta \geqslant \alpha \to x_\beta < \varepsilon))]\!].$$

In further arguments we should replace $[\![|x_\alpha - x| \leqslant \varepsilon]\!]$ by $[\![|x_\alpha - x| \leqslant \varepsilon^\wedge]\!]$ and take it into account that $[\![(\varepsilon 1)^\wedge = \varepsilon^\wedge 1^\wedge = \varepsilon^\wedge \mathbb{1}]\!] = \mathbb{1}$ and $\chi\big([\![x_\beta < \varepsilon^\wedge]\!]\big) = e_\varepsilon^{x_\beta} = \chi^{-1}\big(\big[(|x_\alpha - x| - \varepsilon)^+\big]\big)$ (cp. 2.4.9). $\rhd$

2.6.3. Corollary. *Suppose that A is an order bounded set in a Dedekind complete vector lattice X. Then the following hold:*

(1) *$x = \inf(A)$ if and only if for every $\varepsilon \in X_+$ there exists a partition $(\pi_a)_{a \in A}$ of the band projection $[\varepsilon]$ in $\mathbb{P}(X)$ such that*

$$\pi_a(a - x) \leqslant \varepsilon \quad (a \in A).$$

(2) *$x = \sup(A)$ if and only if for every $\varepsilon \in X_+$ there exists a partition $(\pi_a)_{a\in A}$ of the band projection $[\varepsilon]$ in $\mathbb{P}(X)$ such that*

$$\pi_a(x - a) \leqslant \varepsilon \quad (a \in A).$$

$\lhd$ Suffice it to interpret the definitions of the least upper bound and the greatest lower bound of a bounded set of reals within $\mathbb{V}^{(\mathbb{B})}$ with $\mathbb{B} := \mathbb{P}(X)$. $\rhd$

2.7. Freudenthal Spectral Theorem

In the present section we will show that the properties of a spectral system can be deduced from the properties of reals. We start with several useful remarks to be applied below without further specification.

2.7.1. Take a Dedekind σ-complete vector lattice X. By 2.5.1, we can assume that X is a vector sublattice of the universally complete vector lattice $\mathscr{R}{\downarrow}$, where, as usual, $\mathscr{R}$ is the reals within $\mathbb{V}^{(\mathbb{B})}$ and $\mathbb{B} := \mathbb{B}(X)$. Moreover, the ideal $\widehat{X} := I(X)$ generated by X in $\mathscr{R}{\downarrow}$ is an order dense ideal of $\mathscr{R}{\downarrow}$ and an o-completion of X. Each weak order unit in X is also a weak order unit in $\mathscr{R}{\downarrow}$. The countable joins and meets in X are inherited from $\mathscr{R}{\downarrow}$. In more detail, if the least upper (greatest lower) bound x of a sequence $(x_n) \subset X$ exists in $\mathscr{R}{\downarrow}$ then x is also the least upper (greatest lower) bound in X, provided that $x \in X$.

So, it does not matter whether the o-limit (o-sum) of a sequence in X is calculated in X or $\mathscr{R}{\downarrow}$, provided the result belongs to X. The same is true for the r-limit and r-sums. In particular, $\mathbb{C}(X)$ is a σ-subalgebra of a complete Boolean algebra $\mathbb{C}(\mathscr{R}{\downarrow})$, while the trace e_x and the spectral system $\mathbb{R} \ni \lambda \mapsto e^x_\lambda$ of an element $x \in X$ calculated in $\mathscr{R}{\downarrow}$ are an element of $\mathbb{C}(X)$ and a mapping from $\mathbb{R}$ to $\mathbb{C}(X)$ respectively.

By an easy application of the Boolean valued approach we prove the properties of a spectral system. According to the above remarks, we lose no generality in assuming that the Dedekind σ-complete vector lattice under consideration coincides with $\mathscr{R}{\downarrow}$. But then the claims can easily be derived from the elementary properties of reals with the help of 2.4.9. In 2.7.2–2.7.5, X is an arbitrary Dedekind σ-complete vector lattice with a weak order unit $\mathbb{1}$ and $\mathbb{P}$ is a dense subfield of $\mathbb{R}$.

2.7.2. *The spectral system $\lambda \mapsto e^x_\lambda$ $(\lambda \in \mathbb{R})$ of $x \in X$ has the properties:*

(1) $(\forall\, \lambda, \mu \in \mathbb{R})\ (\lambda \leqslant \mu \Longrightarrow e^x_\lambda \leqslant e^x_\mu)$.

(2) $e^x_{+\infty} := \bigvee_{\mu \in \mathbb{P}} e^x_\mu = \mathbb{1}$ *and* $e^x_{-\infty} := \bigwedge_{\mu \in \mathbb{P}} e^x_\mu = \mathbb{0}$.

(3) $e^x_\lambda = \bigvee \{ e^x_\mu :\ \mu \in \mathbb{P},\ \mu < \lambda \}$ $(\lambda \in \mathbb{R})$.

$\lhd$ Observe first that $\mathbb{P}^\wedge$ is a dense subfield of $\mathscr{R}$ within $\mathbb{V}^{(\mathbb{B})}$.

(1): If $\lambda, \mu \in \mathbb{P}$, $\lambda \leqslant \mu$, and $x \in \mathbb{R}$; then, obviously, trivially $x < \lambda \to x < \mu$. By transfer $[\![x < \lambda^\wedge]\!] \Rightarrow [\![x < \mu^\wedge]\!] = \mathbb{1}$ or equivalently $[\![x < \lambda^\wedge]\!] \leqslant [\![x < \mu^\wedge]\!]$, and the result follows from 2.4.9.

(2): Take $x \in \mathscr{R}{\downarrow}$ and consider the two formulas $\varphi(x, \mathbb{P}) := (\exists\, t \in \mathbb{P})\,(x < t)$ and $\psi(x, \mathbb{P}) := (\forall\, t \in \mathbb{P})\,(x < t)$. For a real x the formula $\varphi(x, \mathbb{P})$ is true and $\psi(x, \mathbb{P})$ is false. Consequently, by transfer $[\![\varphi(x, \mathbb{P}^\wedge)]\!] = \mathbb{1}$ and $[\![\psi(x, \mathbb{P}^\wedge)]\!] = \mathbb{0}$. Calculating the Boolean truth values for the quantifiers by the rules of 1.4.5 (1) yields

$$\bigvee_{t \in \mathbb{P}} [\![x < t^\wedge]\!] = \mathbb{1}, \qquad \bigwedge_{t \in \mathbb{P}} [\![x < t^\wedge]\!] = \mathbb{0},$$

which is equivalent to (2) by 2.4.9.

(3): Applying the transfer principle to $x < \lambda \leftrightarrow (\exists\, \mu \in \mathbb{P})\, x < \mu < \lambda$ and taking it into account that by 1.4.7 $[\![\mu^\wedge < \lambda^\wedge]\!] = \mathbb{1}$ whenever $\mu < \lambda$ and $[\![\mu^\wedge < \lambda^\wedge]\!] = \mathbb{0}$ otherwise, we deduce

$$\begin{aligned} [\![x < \lambda^\wedge]\!] &= \bigvee_{\mu \in \mathbb{P}} [\![x < \mu^\wedge]\!] \wedge [\![\mu^\wedge < \lambda^\wedge]\!] \\ &= \bigvee \{ [\![x < \mu^\wedge]\!] :\ \mu \in \mathbb{P},\ \mu < \lambda \}. \end{aligned}$$

It remains to appeal to 2.4.9. $\rhd$

2.7.3. *Given $x, y \in X$, we have*

(1) $e^{x+y}_\lambda = \bigvee \{ e^x_\mu \wedge e^y_\nu :\ \mu, \nu \in \mathbb{P},\ \mu + \nu = \lambda \}$.

(2) $e^{x \cdot y}_\lambda = \bigvee \{ e^x_\mu \wedge e^y_\nu : 0 \leqslant \mu,\ \nu \in \mathbb{P},\ \mu\nu = \lambda \} \quad (x \geqslant 0,\ y \geqslant 0)$.

$\lhd$ We confine demonstration to (2). Take positive elements $x, y \in \mathscr{R}{\downarrow}$ and $0 < t \in \mathbb{P}$. Then x, y, and $t^\wedge$ are reals in $\mathbb{V}^{(\mathbb{B})}$. For reals we have

$$x \geqslant 0 \wedge y \geqslant 0 \to (xy < t^\wedge \leftrightarrow (\exists\, r, s \in \mathbb{P}^\wedge_+)(x < r \wedge y < s \wedge rs = t)).$$

By transfer and the rules of 1.4.5 (1) for Boolean truth values, we obtain

$$[\![xy < t^\wedge]\!] = \bigvee_{\substack{0\leqslant r,s\in\mathbb{P}\\ rs=t}} [\![x < r^\wedge]\!] \wedge [\![y < s^\wedge]\!].$$

Hence, (2) ensues if we apply χ to both sides of the preceding equality (cp. 2.4.9). $\triangleright$

2.7.4. *If $x, y \in X$ and $\varnothing \neq A \subset X$ then the assertions hold:*

(1) $x \leqslant y \iff (\forall\,\lambda \in \mathbb{P})\ (e_\lambda^y \leqslant e_\lambda^x)$.

(2) $e_\lambda^{x\vee y} = e_\lambda^x \wedge e_\lambda^y$ *for all* $\lambda \in \mathbb{R}$.

(3) $e_\lambda^{x\wedge y} = e_\lambda^x \vee e_\lambda^y$ *for all* $\lambda \in \mathbb{R}$.

(4) $x = \inf(A) \iff (\forall\,\lambda \in \mathbb{P})\,(e_\lambda^x = \bigvee\{e_\lambda^a : a \in A\})$.

$\triangleleft$ Clearly, (2) is immediate from the equivalence

$$(\forall x, y, \lambda \in \mathscr{R})\ x \vee y < \lambda \leftrightarrow (x < \lambda) \wedge (y < \lambda)$$

and (3) is a particular case of (4). Prove (1) and (4).

(1): Observe first that $x \leqslant y \leftrightarrow (\forall\,t \in \mathbb{P})(y < t \to x < t)$. By transfer and the properties of Boolean truth values

$$[\![x \leqslant y]\!] = \bigwedge_{\lambda\in\mathbb{P}} [\![y < \lambda^\wedge]\!] \Rightarrow [\![x < \lambda^\wedge]\!]$$

for all $x, y \in X$. Since the formulas $x \leqslant y$ and $[\![x \leqslant y]\!] = \mathbb{1}$ are equivalent, $x \leqslant y$ is fulfilled if and only if $[\![y < \lambda^\wedge]\!] \leqslant [\![x < \lambda^\wedge]\!]$ or equivalently $e_\lambda^y \leqslant e_\lambda^x$ for all $\lambda \in \mathbb{P}$.

(4): If A is a nonempty subset of X then $A{\uparrow}$ is a set of reals within $\mathbb{V}^{(\mathbb{B})}$ and $\inf(A) < t \leftrightarrow (\exists\,a \in A{\uparrow})(a < t)$. By 1.6.2 and 2.4.4, we obtain the chain of equivalent formulas:

$$\begin{aligned} x = \inf(A) &\iff [\![x = \inf(A{\uparrow})]\!] = \mathbb{1} \\ &\iff [\![(\forall\,t \in \mathbb{P}^\wedge)(x < t \leftrightarrow \inf(A{\uparrow}) < t)]\!] = \mathbb{1} \\ &\iff (\forall\,t \in \mathbb{P})[\![x < t^\wedge]\!] = [\![(\exists\,a \in A{\uparrow})(a < t^\wedge)]\!] \\ &\iff (\forall\,t \in \mathbb{P})[\![x < t^\wedge]\!] = \bigvee_{a\in A} [\![a < t^\wedge]\!]. \end{aligned}$$

Appealing to 2.4.9 completes the proof of (4). $\triangleright$

2.7.5. *Given $x, y \in X$, $\alpha \in \mathbb{R}$, and $c \in \mathbb{C}(\mathbb{1})$, for all $\lambda \in \mathbb{R}$ the following are valid:*

(1) $e_\lambda^{\alpha x} = e_{\lambda/\alpha}^x$ $(\alpha > 0)$, $e_\lambda^{\alpha x} = e_{-\lambda/\alpha}^{-x}$ $(\alpha < 0)$.

(2) $e_\lambda^{-x} = \bigvee\{\mathbb{1} - e_{-\mu}^x : \mu \in \mathbb{P},\, \mu < \lambda\} = (\mathbb{1} - e_{-\lambda}^x) \cdot e_{(x+\lambda\mathbb{1})}$.

(3) $e_\lambda^{|x|} = e_\lambda^x \wedge (\mathbb{1} - e_{-\lambda}^x) \wedge e_{x+\lambda\mathbb{1}}$ $(\lambda \geqslant 0)$, $e_\lambda^{|x|} = \mathbb{0}$ $(\lambda < 0)$.

(4) $e_\lambda^{cx} = c \wedge e_\lambda^x + c^*$ $(\lambda > 0)$, $\quad e_\lambda^{cx} = c \wedge e_\lambda^x$ $(\lambda \leqslant 0)$.

$\lhd$ Note that (1) is easily seen from $(\lambda/\alpha)^\wedge = \lambda^\wedge/\alpha^\wedge$ and (3) is immediate from (2) and 2.7.4 (2). Turn to proving (2) and (4).

(2): The inequality $-x < \lambda$ can be written in the two equivalent forms:

$$-x < \lambda \iff (\neg(x < -\lambda)) \wedge (x + \lambda \neq 0),$$
$$-x < \lambda \iff (\exists\, \mu \in \mathbb{P})(\neg(x < -\mu)) \wedge (\mu < \lambda).$$

Applying transfer and using the equivalence $[\![\lambda^\wedge < \mu^\wedge]\!] = \mathbb{1} \iff \lambda < \mu$ (cp. 1.4.7), we get

$$[\![-x < \lambda^\wedge]\!] = [\![x < -\lambda^\wedge]\!]^* \wedge [\![x + \lambda^\wedge \mathbb{1} \neq 0]\!];$$
$$[\![-x < \lambda^\wedge]\!] = \bigvee\{[\![x < -\mu^\wedge]\!]^* : \ \mu \in \mathbb{P},\, \mu < \lambda\}.$$

The desired result follows from 2.4.9.

(4): Take $c \in \mathbb{C}(X)$ and choose $b \in \mathbb{B}$ with $c = \chi(b)$. If $a \in \{0, 1\}$ then for all $x \in \mathbb{R}$ and $\lambda \in \mathbb{P}$ we evidently have

$$ax < \lambda \leftrightarrow (a = 1 \wedge x < \lambda) \vee (a = 0 \wedge 0 < \lambda).$$

Since $[\![c \in \{0^\wedge, 1^\wedge\}^{\mathbb{B}}]\!] = \mathbb{1}$, the transfer principle together with 1.2.3 (1, 2, 4) and 2.2.6 yields

$$\begin{aligned}[\![cx < \lambda^\wedge]\!] &= ([\![c = 1^\wedge]\!] \wedge [\![x < \lambda^\wedge]\!]) \vee ([\![c = 0^\wedge]\!] \wedge [\![0^\wedge < \lambda^\wedge]\!]) \\ &= (b \wedge [\![x < \lambda^\wedge]\!]) \vee (b^* \wedge [\![0^\wedge < \lambda^\wedge]\!].\end{aligned}$$

If $\lambda > 0$ then $[\![0^\wedge < \lambda^\wedge]\!] = \mathbb{1}$ and $e_\lambda^{cx} = \chi([\![cx < \lambda^\wedge]\!]) = c \wedge e_\lambda^x + c^*$; if $\lambda < 0$ then $[\![0^\wedge < \lambda^\wedge]\!] = \mathbb{0}$ and $e_\lambda^{cx} = \chi([\![cx < \lambda^\wedge]\!]) = c \wedge e_\lambda^x$. $\rhd$

2.7.6. Sometimes it is important to have an estimate rather than knowing the exact values of the spectral system. For example, if

$x = \sup(A)$ then e_λ^x is calculated by a more complicated formula than 2.7.4 (4):

$$x = \sup(A) \iff (\forall\,\lambda \in \mathbb{P})\left(e_\lambda^x = \bigvee_{\nu<\lambda} \bigwedge\{e_\nu^a : a \in A\}\right).$$

At the same time for every $0 < \varepsilon \in \mathbb{R}$ the following hold:

(1) $\bigwedge_{a\in A} e_{\lambda-\varepsilon}^a \leqslant e_\lambda^{\sup(A)} \leqslant \bigwedge_{a\in A} e_\lambda^a$.

(2) $\mathbb{1} - e_{\varepsilon-\lambda}^x \leqslant e_\lambda^{-x} \leqslant \mathbb{1} - e_{-\lambda}^x$.

(3) $e_\lambda^x \wedge (\mathbb{1} - e_{\varepsilon-\lambda}^x) \leqslant e_\lambda^{|x|} \leqslant e_\lambda^x \wedge (\mathbb{1} - e_{-\lambda}^x)$ $\ (\lambda \in \mathbb{R}_+)$.

(4) $\mathbb{1} - e_{\varepsilon+1/\lambda}^x \leqslant e_\lambda^{x^{-1}} \leqslant \mathbb{1} - e_{1/\lambda}^x$ $\ (x \in X_+,\ 0 < \lambda \in \mathbb{R})$.

Applying 2.7.3, 2.7.4 (1–3), and 2.7.5 (1) to the inequalities $2(x\wedge y) \leqslant x + y \leqslant 2(x \vee y)$ and $(x \wedge y)^2 \leqslant xy \leqslant (x \vee y)^2$ yields the estimates:

(5) $e_{\lambda/2}^x \wedge e_{\lambda/2}^y \leqslant e_\lambda^{x+y} \leqslant e_{\lambda/2}^x \vee e_{\lambda/2}^y$ $\ (x, y \in X;\ \lambda \in \mathbb{R})$.

(6) $e_{\sqrt{\lambda}}^x \wedge e_{\sqrt{\lambda}}^y \leqslant e_\lambda^{xy} \leqslant e_{\sqrt{\lambda}}^x \vee e_{\sqrt{\lambda}}^y$ $\ (x, y \in X_+;\ \lambda \in \mathbb{R}_+)$.

2.7.7. Freudenthal Spectral Theorem. *Let X be an arbitrary Dedekind σ-complete vector lattice with order unit $\mathbb{1}$. Every element $x \in X$ admits the representation*

$$x = \int_{-\infty}^{\infty} \lambda\, de_\lambda^x,$$

where the integral is understood to be the $\mathbb{1}$-uniform limit of the integral sums

$$x(\beta) := \sum_{n\in\mathbb{Z}} \tau_n (e_{t_{n+1}}^x - e_{t_n}^x), \quad t_n \leqslant \tau_n \leqslant t_{n+1},$$

as $\delta(\beta) := \sup_{n\in\mathbb{Z}} (t_{n+1} - t_n) \to 0$, with $\beta := (t_n)_{n\in\mathbb{Z}}$ being a partition of the real line.

$\lhd$ We may assume that $\mathscr{R}{\downarrow}$ is a universal completion of X and $X \subset \mathscr{R}{\downarrow}$. Let $x \in X$, $\beta := (t_n)_{n\in\mathbb{Z}}$ be a partition of $\mathbb{R}$, and $t_n < \tau_n < t_{n+1}$ $(n \in \mathbb{Z})$. Put $b_n := e_{t_{n+1}}^x - e_{t_n}^x$. Then

$$\begin{aligned} b_n &= [\![t_n^\wedge \leqslant x < t_{n+1}^\wedge]\!] \wedge [\![t_n^\wedge \leqslant \tau_n^\wedge < t_{n+1}^\wedge]\!] \wedge [\![t_{n+1}^\wedge - t_n^\wedge \leqslant \delta(\beta)^\wedge]\!] \\ &\leqslant [\![|x - \tau_n^\wedge| \leqslant \delta(\beta)^\wedge]\!]. \end{aligned}$$

Since $x(\beta) := \operatorname{mix}_{n\in\mathbb{Z}}(b_n\tau_n^\wedge)$, we derive

$$[\![|x - x(\beta)| \leqslant \delta(\beta)^\wedge]\!] = \mathbb{1} \quad \text{or} \quad |x - x(\beta)| \leqslant \delta(\beta)\mathbb{1}.$$

It remains to recall the remarks of 2.7.1. ▷

2.7.8. In particular, the Freudenthal Spectral Theorem states that if X is a Dedekind σ-complete vector lattice and $e \in X_+$ then every $x \in X(e)$ can be e-uniformly approximated by the linear combinations of components of e; i.e., by the elements of the form $\sum_{k=1}^n \lambda_k e_k$, where $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ and $e_1, \dots, e_n \in \mathbb{C}(e)$. In the case when the latter holds in a vector lattice X we say that X possesses the *weak Freudenthal property*. It may happen that every $x \in X(e)$ can be e-uniformly approximated by linear combinations $\sum_{k=1}^n \lambda_k \pi_k e$, where $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ and $\pi_1, \dots, \pi_n \in \mathbb{P}(X)$. Then a vector lattice X is said to possesses the *strong Freudenthal property*. Clearly, a vector lattice with the principal projection property possesses the strong Freudenthal property. The converse is false.

2.8. Representation of Vector Lattices

By the Freudenthal Spectral Theorem, the mapping that assigns to each element of a Dedekind σ-complete vector lattice with weak order unit its spectral system is one-to-one and transforms the vector lattice structure in a definite way. This circumstance suggests that an arbitrary Dedekind σ-complete vector lattice with weak order unit can be represented as some space of “abstract spectral system.” We will expatiate upon this.

2.8.1. A *spectral system* or *resolution of the identity* in a Boolean algebra $\mathbb{B}$ is defined as a mapping $e : \mathbb{R} \to \mathbb{B}$ satisfying the conditions

(1) $s \leqslant t \to e(s) \leqslant e(t) \quad (s, t \in \mathbb{R})$;

(2) $\bigvee_{t\in\mathbb{R}} e(t) = \mathbb{1}$, $\bigwedge_{t\in\mathbb{R}} e(t) = \mathbb{0}$;

(3) $\bigvee_{s\in\mathbb{R}, s<t} e(s) = e(t)$ $(t \in \mathbb{R})$.

Let $\mathfrak{S}(\mathbb{B})$ be the set of all spectral systems in $\mathbb{B}$. Introduce some order by the formula

$$e' \leqslant e'' \iff (\forall\, t \in \mathbb{R})(e''(t) \leqslant e'(t)) \quad (e', e'' \in \mathfrak{S}(\mathbb{B})).$$

2.8.2. Further, suppose that $\mathbb{B}$ is a σ-algebra and choose some countable dense subfield $\mathbb{P}$ of $\mathbb{R}$. By 2.8.1 (3), every spectral function is uniquely determined by its values on $\mathbb{P}$.

Given $e', e'' \in \mathfrak{S}(\mathbb{B})$, we can define the mapping

$$e : t \mapsto \bigvee\{e'(r) \wedge e''(s) : r, s \in \mathbb{P},\ r + s = t\} \quad (t \in \mathbb{P}),$$
$$e : t \mapsto \bigvee\{e(s) : s \in \mathbb{P},\ s < t\} \quad (t \in \mathbb{R})$$

which is obviously a spectral function in $\mathbb{B}$. Putting $e' + e'' := e$, we obtain the structure of a commutative group in $\mathfrak{S}(\mathbb{B})$. In particular, the zero element $\bar{\mathbb{0}}$ is defined as $\bar{\mathbb{0}}(t) := \mathbb{1}$ if $t > 0$ and $\bar{\mathbb{0}}(t) := \mathbb{0}$ if $t \leqslant 0$, while $-e(t) = \bigvee\{\mathbb{1} - e(-s) : s \in \mathbb{P},\ s < t\}$. Set $\bar{\mathbb{1}}(t) := \mathbb{1}$ if $t > 1$ and $\bar{\mathbb{1}}(t) := \mathbb{0}$ if $t \leqslant 1$. Finally, the scalar multiplication $(\alpha, e) \mapsto \alpha e$ $(\alpha \in \mathbb{R},\ e \in \mathfrak{S}(\mathbb{B}))$ is defined as

$$(\alpha e)(t) := e(t/\alpha) \quad (\alpha > 0,\ t \in \mathbb{R}),$$
$$(\alpha e)(t) := (-e)(-t/\alpha) \quad (\alpha < 0,\ t \in \mathbb{R}).$$

To each element $b \in \mathbb{B}$ we assign the spectral system $\bar{b}$ that is defined as

$$\bar{b}(t) := \begin{cases} \mathbb{1}, & \text{for } t > 1, \\ b^* := \mathbb{1} - b, & \text{for } 0 < t \leqslant 1, \\ \mathbb{0}, & \text{for } t \leqslant 0. \end{cases}$$

2.8.3. Theorem. *Let $\mathbb{B}$ be a complete Boolean algebra. The set $\mathfrak{S}(\mathbb{B})$ with the above operations and order is a universally complete vector lattice with a weak order unit $\bar{\mathbb{1}}$. The mapping h assigning to each $x \in \mathscr{R}{\downarrow}$ the spectral system $t \mapsto [\![x < t^\wedge]\!]$ $(t \in \mathbb{R})$ is a lattice isomorphism from $\mathscr{R}{\downarrow}$ onto $\mathfrak{S}(\mathbb{B})$. The mapping $b \mapsto \bar{b}$ is a Boolean isomorphism of $\mathbb{B}$ onto $\mathbb{C}(\bar{\mathbb{1}})$.*

$\lhd$ By 2.4.9 and 2.7.2 $h(x) \in \mathfrak{S}(\mathbb{B})$. According to 2.7.3, and 2.7.4 h preserves addition, multiplication, and lattice operations. Moreover, h is one-to-one, since the equality $h(x) = h(y)$ means

$$[\![x < t^\wedge]\!] = [\![y < t^\wedge]\!] \quad (t \in \mathbb{R})$$

or equivalently (cp. 1.4.5 (1))

$$[\![(\forall\, t \in \mathbb{R}^\wedge)\, (x < t \leftrightarrow y < t)]\!] = \mathbb{1}$$

and the latter amounts to the coincidence of x and y within $\mathbb{V}^{(\mathbb{B})}$. By Gordon's Theorem, it remains to establish that h is surjective. Take an arbitrary spectral system $e : \mathbb{R} \to \mathbb{B}$. Let $\beta := (t_n)_{n \in \mathbb{Z}}$ be a partition of the real axis; i.e., $t_n < t_{n+1}$ $(n \in \mathbb{Z})$, $\lim_{n\to\infty} t_n = \infty$, and $\lim_{n\to-\infty} t_n = -\infty$. The disjoint sum

$$\bar{x}(\beta) := \sum_{n\in\mathbb{Z}} t_{n+1}(\chi(e(t_{n+1})) - \chi(e(t_n)))$$

exists in the universally complete vector lattice $\mathscr{R}{\downarrow}$; here χ is the isomorphism of $\mathbb{B}$ onto $\mathbb{C}(\mathscr{R}{\downarrow})$ (cp. 2.2.4 and 2.3.2). Denote by A the set of all elements $\bar{x}(\beta)$. Each element of the form

$$\underline{x}(\beta) := \sum_{n\in\mathbb{Z}} t_n(\chi(e(t_{n+1})) - \chi(e(t_n)))$$

is a lower bound of A. Therefore, there exists $x := \inf(A) := \inf\{\bar{x}(\beta)\}$. It is easy to observe that

$$e_\lambda^{\bar{x}(\beta)} = \bigvee\{\chi(e(t_n)) :\ t_n < \lambda\}.$$

Hence, from 2.7.4 (4) we infer

$$e_\lambda^x = \bigvee_{a\in A} e_\lambda^a = \bigvee_{t\in\mathbb{R}, t<\lambda} \chi(e(t)) = \chi(e(\lambda)) \quad (\lambda \in \mathbb{R}).$$

Using 2.4.9, we conclude that $h(x)(t) = \chi^{-1}([\![x < t^\wedge]\!]) = e(t)$. $\triangleright$

Let us derive several important corollaries to 2.8.3.

2.8.4. Corollary. *A universally complete vector lattice X with unit $\mathbb{1}$ is isomorphic to the Dedekind complete vector lattice $\mathfrak{S}(\mathbb{B})$, where $\mathbb{B} := \mathbb{C}(\mathbb{1})$. The isomorphism is established by the mapping that assigns to each $x \in X$ the spectral system $\lambda \mapsto e_\lambda^x$ $(\lambda \in \mathbb{R})$.*

$\triangleleft$ It suffices to compare 2.5.6 and 2.8.3. $\triangleright$

2.8.5. Corollary. *For an arbitrary σ-algebra $\mathbb{B}$, the set $\mathfrak{S}(\mathbb{B})$ (with the structure defined as in 2.8.2) is a universally σ-complete vector lattice with order unit. Conversely, every universally σ-complete vector lattice X with order unit is isomorphic to $\mathfrak{S}(\mathbb{B})$, where $\mathbb{B} := \mathbb{C}(X)$.*

$\triangleleft$ Let $\widehat{\mathbb{B}}$ be an o-completion of the σ-algebra $\mathbb{B}$. According to 2.8.3, $\mathfrak{S}(\widehat{\mathbb{B}})$ is a universally complete vector lattice. The set $\mathfrak{S}(\mathbb{B})$ lies in $\mathfrak{S}(\widehat{\mathbb{B}})$. Moreover, it is easily seen from 2.7.3–2.7.5 and 2.8.4 that $\mathfrak{S}(\mathbb{B})$ is a vector subspace of $\mathfrak{S}(\widehat{\mathbb{B}})$ and the countable suprema and infima in $\mathfrak{S}(\mathbb{B})$ are inherited from $\mathfrak{S}(\widehat{\mathbb{B}})$. Consequently, $\mathfrak{S}(\mathbb{B})$ is a Dedekind σ-complete vector lattice with order unit. The same arguments imply that every countable disjoint set of elements in $\mathfrak{S}(\mathbb{B})$ is bounded.

Take an arbitrary Dedekind σ-complete vector lattice X with order unit and denote by $\widehat{X}$ its universal completion. If $\mathbb{B} = \mathbb{C}(X)$ and $\widehat{\mathbb{B}} := \mathbb{C}(\widehat{X})$ then $\widehat{\mathbb{B}}$ is an o-completion of $\mathbb{B}$. By 2.8.4, the spaces $\widehat{X}$ and $\mathfrak{S}(\widehat{\mathbb{B}})$ are isomorphic; moreover, $\mathfrak{S}(\mathbb{B})$ is the image of X by 2.7.6. $\triangleright$

2.8.6. We proceed now to the functional representation of vector lattices. Some additional definitions and facts are needed for this purpose.

Let Q be a topological space. Recall that a topological space Q is called *extremally* (*quasiextremally*) *disconnected* or simply *extremal* (*quasiextremal*) if the closure of an arbitrary open set (open F_σ-set) in Q is open or, which is equivalent, the interior of an arbitrary closed set (closed G_δ-set) is closed. Clearly, an extremal (quasiextremal) space is totally disconnected. If Q is Hausdorff compact then the respective terms *Stonean* and *quasi-Stonean* are in common parlance as well.

Let $\Lambda \subset \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\}$. Given a function $f : Q \to \overline{\mathbb{R}}$ and $\lambda \in \overline{\mathbb{R}}$, put

$$\{f < \lambda\} := \{q \in Q : f(q) < \lambda\}, \quad \{f \leqslant \lambda\} := \{q \in Q : f(q) \leqslant \lambda\}.$$

Consider a mapping $\lambda \mapsto U_\lambda \subset Q$ that is assumed to be increasing: $\lambda \leqslant \mu$ implies $U_\lambda \subset U_\mu$. This mapping is said to be *strictly increasing* if and only if $\mathrm{cl}(U_\lambda) \subset \mathrm{int}(U_\mu)$ for all $\lambda, \mu \in \Lambda$ with $\lambda < \mu$. Say that a function $f : Q \to \overline{\mathbb{R}}$ *calibrates* the mapping $\lambda \mapsto U_\lambda$ whenever $\{f < \lambda\} \subset U_\lambda \subset \{f \leqslant \lambda\}$ for all $\lambda \in \Lambda$.

2.8.7. *Assume that Q is a topological space and Λ is a dense subset of $\overline{\mathbb{R}}$. A mapping $U : \lambda \mapsto U_\lambda$ from Λ to $\mathscr{P}(Q)$ is strictly increasing if and only if there is a unique continuous function $f : Q \to \overline{\mathbb{R}}$ that calibrates U.*

$\triangleleft$ See [228, 1.4.1 (1)]. $\triangleright$

2.8.8. *Let Q be a quasiextremal compact space. Assume that Q_0 is an open dense F_σ-set in Q and $f : Q_0 \to \mathbb{R}$ is a continuous function. Then there is a unique continuous function $\bar{f} : Q \to \overline{\mathbb{R}}$ such that $f(t) = \bar{f}(t)$ $(t \in Q_0)$.*

$\triangleleft$ Indeed, if $U_\mu := \mathrm{cl}(\{f < \mu\})$ then the mapping $\mu \mapsto U_\mu$ $(\mu \in \mathbb{R})$ is strictly increasing. Therefore, by 2.8.7, there is a unique function $\bar{f} : Q \to \overline{\mathbb{R}}$ satisfying

$$\{\bar{f} < \mu\} \subset U_\mu \subset \{\bar{f} \leqslant \mu\} \quad (\mu \in \mathbb{R}).$$

Obviously, the restriction of $\bar{f}$ to Q_0 coincides with f. ▷

2.8.9. Let Q be a quasiextremal compact space. Denote by $C_\infty(Q)$ the set of all continuous functions $x : Q \to \overline{\mathbb{R}}$ assuming the values $\pm\infty$ possibly on a rare set. The order on $C_\infty(Q)$ is defined by putting $x \leqslant y$ whenever $x(t) \leqslant y(t)$ for all $t \in Q$. Take $x, y \in C_\infty(Q)$ and put $Q_0 := \{|x| < +\infty\} \cap \{|y| < +\infty\}$. In this case Q_0 is open and dense in Q. According to 2.8.8, we have the unique continuous functions $u, v : Q \to \overline{\mathbb{R}}$ such that $u(t) = x(t) + y(t)$ and $v(t) = x(t) \cdot y(t)$ for $t \in Q_0$. So we can define addition and multiplication on $C_\infty(Q)$ by putting $x + y := u$ and $xy := v$. The identically one function $\mathbb{1}$ is an order and ring unit in $C_\infty(Q)$. The scalar multiplication on $C_\infty(Q)$ is defined as $\lambda x := (\lambda\mathbb{1})x$.

The space $C_\infty(Q)$ with the above algebraic operations and order is vector lattice and a semiprime f-algebra. The following result tells us that $C_\infty(Q)$ is universally σ-complete.

2.8.10. *Let Q be the Stone space of a σ-algebra $\mathbb{B}$. The vector lattices $C_\infty(Q)$ and $\mathfrak{S}(\mathbb{B})$ are lattice isomorphic. In particular, $C_\infty(Q)$ is a universally σ-complete vector lattice with unit for every quasiextremal compact space Q.*

◁ Take $e \in \mathfrak{S}(\mathbb{B})$. Let G_t be a clopen set in Q corresponding to the element $e(t) \in \mathbb{B}$. The mapping $t \mapsto G_t$ $(t \in \mathbb{R})$ is strictly increasing, so that by 2.8.6, there exists a unique continuous function $\hat{e} : Q \to \overline{\mathbb{R}}$ such that

$$\{\hat{e} < t\} \subset G_t \subset \{\hat{e} \leqslant t\} \quad (t \in \mathbb{R}).$$

It follows from 2.7.2 (2, 3) that the closed set $\bigcap\{G_t : t \in \mathbb{R}\}$ has empty interior and the open set $\bigcup\{G_t : t \in \mathbb{R}\}$ is dense in Q. Hence, the function $\hat{e}$ is finite everywhere, except possibly the points of a nowhere dense set; therefore, $\hat{e} \in C_\infty(Q)$. It is easy to check that the mapping $e \mapsto \hat{e}$ is the sought lattice isomorphism. ▷

2.8.11. Theorem. *Let Q be the Stone space of a complete Boolean algebra $\mathbb{B}$, and let $\mathscr{R}$ be the reals within $\mathbb{V}^{(\mathbb{B})}$. The vector lattice $C_\infty(Q)$ is isomorphic to the universally complete vector lattice $\mathscr{R}{\downarrow}$. The isomorphism is established by assigning to an element $x \in \mathscr{R}{\downarrow}$ the function $\hat{x} : Q \to \overline{\mathbb{R}}$ by the formula*

$$\hat{x}(q) := \inf\{t \in \mathbb{R} : [\![x < t^{\wedge}]\!] \in q\} \quad (q \in Q).$$

◁ The proof is immediate from 2.8.10 and 2.8.3. ▷

2.8.12. *Let X be an Archimedean vector lattice and let Q be the Stone space of the Boolean algebra $\mathbb{B}(X)$. Then X is isomorphic to a minorizing sublattice $X_0 \subset C_\infty(Q)$. Moreover, X is an order dense ideal of $C_\infty(Q)$ (coincides with $C_\infty(Q)$) if and only if X is a Dedekind complete vector lattice (a universally complete vector lattice).*

$\triangleleft$ See 2.8.10, 2.5.1, 2.5.5, and 2.5.6. $\triangleright$

2.8.13. Theorem. *Let X be a universally σ-complete vector lattice with an order unit $\mathbb{1}$ and let Q be the Stone space of the Boolean algebra $\mathbb{C}(X, \mathbb{1})$. Then X is lattice isomorphic to $C_\infty(Q)$. Moreover, X can uniquely be equipped with an f-algebra multiplication with $\mathbb{1}$ as ring unit; in this event X and $C_\infty(Q)$ are f-algebra isomorphic.*

$\triangleleft$ Immediate from Corollary 2.8.5 and 2.8.10. $\triangleright$

2.9. Spectral Measure and Integral

In the sequel, we need the concept of integral with respect to a spectral measure.

2.9.1. Suppose that (Ω, Σ) is a measure space; i.e., Ω is a nonempty set and Σ is a fixed σ-algebra of subsets of Ω. A *spectral measure* is defined to be a σ-continuous Boolean homomorphism μ from Σ into the Boolean σ-algebra $\mathbb{B}$. More precisely, a mapping $\mu : \Sigma \to \mathbb{B}$ is a spectral measure if $\mu(\Omega \setminus A) = \mathbb{1} - \mu(A)$ $(A \in \Sigma)$ and

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \bigvee_{n=1}^{\infty} \mu(A_n)$$

for each sequence (A_n) of elements of Σ.

Let $\mathbb{B} := \mathbb{C}(X, \mathbb{1})$ be the Boolean algebra of components of $\mathbb{1}$ in a Dedekind σ-complete vector lattice X with a fixed unit $\mathbb{1}$. Take a measurable function $f : \Omega \to \mathbb{R}$. Given an arbitrary partition of the real axis

$$\beta := (\lambda_k)_{k\in\mathbb{Z}}, \quad \lambda_k < \lambda_{k+1}\ (k \in \mathbb{Z}), \quad \lim_{n\to\pm\infty} \lambda_n = \pm\infty,$$

put $A_k := f^{-1}([\lambda_k, \lambda_{k+1}))$ and compose the integral sums

$$\underline{\sigma}(f, \beta) := \sum_{-\infty}^{\infty} \lambda_k \mu(A_k), \quad \overline{\sigma}(f, \beta) := \sum_{-\infty}^{\infty} \lambda_{k+1} \mu(A_k),$$

where the sums are calculated in X. It is clear that

$$\underline{\sigma}(f,\beta) \leqslant \sum_{-\infty}^{\infty} f(t_k)\mu(A_k) \leqslant \overline{\sigma}(f,\beta)$$

for every choice of $t_k \in A_k$ ($k \in \mathbb{Z}$). Also, it is evident that $\underline{\sigma}(f,\beta)$ increases and $\overline{\sigma}(f,\beta)$ decreases as we refine the partition β. If there exists an element $x \in X$ such that $\sup\{\underline{\sigma}(f,\beta)\} = x = \inf\{\overline{\sigma}(f,\beta)\}$, where the suprema and infima are calculated over all partitions $\beta := (\lambda_k)_{k\in\mathbb{Z}}$ of the real axis, then we say that f is *integrable with respect to* μ or the *spectral integral* $I_\mu(f)$ exists; in this event we write

$$I_\mu(f) := \int_T f\,d\mu := \int_T f(t)\,d\mu(t) := x.$$

2.9.2. *The spectral integral $I_\mu(f)$ exists for every bounded measurable function f. If X is a universally σ-complete vector lattice then every almost everywhere finite measurable function is integrable with respect to each spectral measure.*

$\lhd$ Note that $A_k \cap A_l = \varnothing$ ($k \neq l$) and $\bigcup_{k\in\mathbb{Z}} A_k = \Omega$; therefore, $(\mu(A_k))_{k\in\mathbb{Z}}$ is a partition of unity in the Boolean algebra $\mathbb{B}$. Putting $\delta := \sup_{k\in\mathbb{Z}}\{\lambda_{k+1} - \lambda_k\}$, we can write down

$$0 \leqslant \overline{\sigma}(f,\beta) - \underline{\sigma}(f,\beta) \leqslant \sum_{k\in\mathbb{Z}} \delta\mu(A_k) = \delta\mathbb{1}.$$

Consequently, a measurable function f is integrable with respect to μ if and only if $\overline{\sigma}(f,\beta)$ and $\underline{\sigma}(f,\beta)$ exist at least for one partition β. If f is bounded then the sums $\overline{\sigma}(f,\beta)$ and $\underline{\sigma}(f,\beta)$ contain at most finitely many nonzero summands. If X is a universally σ-complete vector lattice and a measurable function f is arbitrary then the sums also make sense, since in this case they involve at most countably many pairwise disjoint elements. $\rhd$

2.9.3. Theorem. *Let $X := \mathscr{R}{\downarrow}$ and let μ be a spectral measure with values in $\mathbb{B} := \mathbb{C}(X) := \mathbb{C}(1^\wedge)$. Then for each measurable function f the integral $I_\mu(f)$ is the unique element of the Dedekind complete vector lattice X satisfying the condition*

$$[\![I_\mu(f) < \lambda^\wedge]\!] = \mu(\{f < \lambda\}) \quad (\lambda \in \mathbb{R}).$$

◁ Take an arbitrary real $\lambda \in \mathbb{R}$ and a partition of the real axis $\beta := (\lambda_k)_{k\in\mathbb{Z}}$ such that $\lambda_0 = \lambda$. If $b := [\![I_\mu(f) < \lambda^\wedge]\!]$ then

$$b = [\![(\exists\, t \in \mathbb{R}^\wedge)\, (I_\mu(f) < t \wedge t < \lambda^\wedge)]\!].$$

By the exhaustion principle, there exist a partition $(b_\xi)_{\xi\in\Xi}$ of b and a family $(t_\xi)_{\xi\in\Xi} \subset \mathbb{R}$ such that $t_\xi < \lambda$ and $b_\xi \leqslant [\![I_\mu(f) \leqslant t_\xi^\wedge]\!]$ for all ξ. Hence, applying 2.2.4 ($\mathbb{G}$), we derive

$$b_\xi \underline{\sigma}(f,\beta) \leqslant t_\xi b_\xi < \lambda b_\xi \quad (\xi \in \Xi)$$

and further

$$\lambda_k b_\xi \mu(A_k) \leqslant t_\xi b_\xi \mu(A_k) < \lambda b_\xi \mu(A_k) \quad (\xi \in \Xi,\ k \in \mathbb{Z}).$$

For $k \geqslant 1$ we have $\lambda_k > \lambda$; therefore, $b_\xi \mu(A_k) = 0$. So,

$$b = \bigvee_{\xi\in\Xi} b_\xi \leqslant \bigwedge_{k=1}^{\infty} \mu(A_k)^* = \mu\left(\Omega - \bigcup_{k=1}^{\infty} A_k\right) = \mu(\{f < \lambda\}).$$

On the other hand, $b^* = [\![I_\mu(f) \geqslant \lambda^\wedge]\!]$ and, by 2.2.4 ($\mathbb{G}$), we again infer that $\lambda b^* \leqslant b^* I_\mu(f) \leqslant b^* \overline{\sigma}(f,\beta)$ or

$$\lambda b^* \mu(A_k) \leqslant b^* \lambda_k \mu(A_k) \quad (k \in \mathbb{Z}).$$

For $k < 0$ we have $\lambda_k < \lambda$; therefore, $b^* \mu(A_k) = 0$. Consequently,

$$b^* \leqslant \bigwedge_{k=-1}^{-\infty} \mu(A_k)^* = \mu\left(\Omega - \bigcup_{k=-1}^{-\infty} A_k\right) = \mu(\{f \geqslant \lambda\}).$$

This implies $b \geqslant \mu(\{f < \lambda\})$ and we finally obtain $b = \mu(\{f < \lambda\})$.

Assume that

$$[\![x < \lambda^\wedge]\!] = \mu(\{f < \lambda\}) \quad (\lambda \in \mathbb{R})$$

for some $x \in \mathscr{R}{\downarrow}$. Then by what was established above we have $[\![x < \lambda^\wedge]\!] = [\![I_\mu(f) < \lambda^\wedge]\!]$ for all $\lambda \in \mathbb{R}$. This is equivalent to

$$[\![(\forall\, \lambda \in \mathbb{R}^\wedge)\, (x < \lambda \leftrightarrow I_\mu(f) < \lambda)]\!] = \mathbb{1}.$$

Hence, recalling that $\mathbb{R}^\wedge$ is dense in $\mathscr{R}$, we get the equality $[\![x = I_\mu(f)]\!] = \mathbb{1}$ or $x = I_\mu(f)$. ▷

2.9.4. *Take a measurable function $f : \Omega \to \mathbb{R}$ and a spectral measure $\mu : \Sigma \to \mathbb{B} := \mathbb{C}(X)$, where X is a Dedekind complete vector lattice. If the integral $I_\mu(f) \in X$ exists then $\lambda \mapsto \mu(\{f < \lambda\})$ $(\lambda \in \mathbb{R})$ coincides with the spectral system of $I_\mu(f)$.*

$\triangleleft$ Suffice it to compare 2.4.9 with 2.9.3. $\triangleright$

2.9.5. Theorem. *Let X be a universally σ-complete vector lattice with an order unit $\mathbb{1}$ and let $\mathscr{M}(\Omega, \Sigma)$ stand for the unital f-algebra of measurable real functions on Ω. Given a spectral measure $\mu : \Sigma \to \mathbb{B}_0$, $\mathbb{B}_0 := \mathbb{C}(X, \mathbb{1})$, the spectral integral $I_\mu(\cdot)$ is a sequentially o-continuous f-algebra homomorphism from $\mathscr{M}(\Omega, \Sigma)$ to X.*

$\triangleleft$ Without loss of generality we can assume that $X \subset \mathscr{R}{\downarrow}$ and $\mathscr{R}{\downarrow}$ is a universal completion of X (cp. 2.5.1 (3)). Here $\mathscr{R}$ is the field of the reals in $\mathbb{V}^{(\mathbb{B})}$, where $\mathbb{B}$ is a completion of the algebra $\mathbb{B}_0$. It is obvious that the operator I_μ is linear and positive. Prove its sequential o-continuity. Take a decreasing sequence $(f_n)_{n\in\mathbb{N}}$ of measurable functions such that $\lim_{n\to\infty} f_n(t) = 0$ for all $t \in \Omega$, and let $x_n := I_\mu(f_n)$ $(n \in \mathbb{N})$ and $0 < \varepsilon \in \mathbb{R}$. If we assign $A_n := \{t \in \Omega : f_n(t) < \varepsilon\}$ then $\Omega = \bigcup_{n=1}^\infty A_n$. By 2.9.4, we can write down

$$o\text{-}\lim_{n\to\infty} e_\varepsilon^{x_n} = o\text{-}\lim_{n\to\infty} \mu(A_n) = \bigvee_{n=1}^{\infty} \mu(A_n) = \mathbb{1}.$$

Appealing to the test for o-convergence 2.6.2 (2), we obtain $o\text{-}\lim_{n\to\infty} x_n = 0$. Further, given arbitrary measurable functions $f, g : \Omega \to \mathbb{R}$, we derive from 2.7.4 (2) and 2.9.4 that

$$\begin{gathered} e_\lambda^{I(f\vee g)} = \mu(\{f \vee g < \lambda\}) \\ = \mu(\{f < \lambda\}) \wedge \mu(\{g < \lambda\}) = e_\lambda^{I(f)} \wedge e_\lambda^{I(g)} = e_\lambda^{I(f)\vee I(g)} \end{gathered}$$

(with $I := I_\mu$); consequently, $I(f \vee g) = I(f) \vee I(g)$. It means that I_μ is a lattice homomorphism. In a similar way, for $f \geqslant 0$ and $g \geqslant 0$ it follows from 2.7.3 (2) and 2.9.4 that

$$\begin{aligned} e_\lambda^{I(f\cdot g)} &= \mu(\{f \cdot g < \lambda\}) = \mu\Bigg(\bigcup_{\substack{r,s\in\mathbb{Q}_+ \\ rs=\lambda}} \{f < r\} \cap \{g < s\}\Bigg) \\ &= \bigvee_{\substack{r,s\in\mathbb{Q}_+ \\ rs=\lambda}} \mu(\{f < r\}) \wedge \mu(\{g < s\}) = \bigvee_{\substack{r,s\in\mathbb{Q}_+ \\ rs=\lambda}} e_r^{I(f)} \wedge e_s^{I(g)} = e_\lambda^{I(f)\cdot I(g)} \end{aligned}$$

for $0 < \lambda \in \mathbb{Q}$, with $\mathbb{Q}$ the *rationals*. Thus, $I(f \cdot g) = I(f) \cdot I(g)$. The validity of the latter equality for arbitrary functions f and g ensues from the above-established properties of the spectral integral:

$$\begin{aligned} I(f \cdot g) &= I(f^+g^+) + I(f^-g^-) - I(f^+g^-) - I(f^-g^+) \\ &= I(f)^+I(g)^+ + I(f)^-I(g)^- - I(f)^+I(g)^- - I(f)^-I(g)^+ \\ &= I(f) \cdot I(g). \ \triangleright \end{aligned}$$

2.10. Functional Calculus

In a universally σ-complete vector lattice X with an order unit we can define $\phi(x_1, \dots, x_n) \in X$, given a finite collection $x_1, \dots, x_N \in X$ and a Borel measurable function $\phi : \mathbb{R}^N \to \mathbb{R}$. To this end, we need the auxiliary result:

2.10.1. Loomis–Sikorski Theorem. *Let Q be the Stone space of a Boolean σ-algebra $\mathbb{B}$. Denote by $\mathrm{Clop}_\sigma(Q)$ the σ-algebra of subsets of Q generated by the collection $\mathrm{Clop}(Q)$ of all clopen subsets of Q. Let Δ stand for the σ-ideal of $\mathrm{Clop}_\sigma(Q)$ comprising all meager sets. If $\imath$ is an isomorphism of $\mathbb{B}$ onto $\mathrm{Clop}(Q)$ and φ is the quotient mapping of $\mathrm{Clop}_\sigma(Q)$ onto the quotient algebra $\mathrm{Clop}_\sigma(Q)/\Delta$ then the mapping $h := \varphi \circ \imath$ is an isomorphism of $\mathbb{B}$ onto $\mathrm{Clop}_\sigma(Q)/\Delta$.*

$\triangleleft$ Observe that h is a homomorphism as the composite of two homomorphisms. If $h(b) = 0$ then $\imath(b) \in \Delta$ and $\imath(b) = \varnothing$, since no nonempty clopen set is meager. Thus h is injective. To prove that h is surjective put

$$\mathscr{F} := \{A \in \mathrm{Clop}_\sigma(Q) : \ (\exists\, b \in \mathbb{B})\, \varphi(A) = h(b)\}.$$

Since $\mathrm{Clop}(Q) \subset \mathscr{F} \subset \mathrm{Clop}_\sigma(Q)$, it suffices to observe that $\mathscr{F}$ is a σ-algebra. If $A \in \mathscr{F}$ with $\varphi(A) = h(b)$ then $\varphi(Q \setminus A) = h(b^*)$, so that $Q \setminus A \in \mathscr{F}$. Consider a sequence (A_n) of $\mathscr{F}$ and choose a sequence (b_n) of $\mathbb{B}$ such that $\varphi(A_n) = h(b_n)$. Put $A := \bigcup_{n=1}^\infty \imath(b_n)$ and $A_0 := \mathrm{cl}(A) \setminus A$. Since Q is quasiextremal, $\mathrm{cl}(A)$ is clopen and A_0 is rare. Thus, we have the representation $\imath\left(\bigvee_{n=1}^\infty b_n\right) = A_0 \cup A$ from which we easily deduce

$$\begin{aligned} \varphi\left(\bigcup_{n=1}^\infty A_n\right) &= \varphi\left(A_0 \cup \bigcup_{n=1}^\infty A_n\right) = \varphi\left(A_0 \cup \bigcup_{n=1}^\infty \imath(b_n)\right) \\ &= \varphi\left(\imath\left(\bigvee_{n=1}^\infty b_n\right)\right) = h\left(\bigvee_{n=1}^\infty b_n\right) \end{aligned}$$

and the result follows. $\triangleright$

2.10.2. *Let $e_1, \dots, e_N : \mathbb{R} \to \mathbb{B}$ be a finite collection of spectral systems with values in a σ-algebra $\mathbb{B}$. Then there exists a unique $\mathbb{B}$-valued spectral measure μ defined on the Borel σ-algebra $\mathscr{B}or(\mathbb{R}^N)$ of the space $\mathbb{R}^N$ such that*

$$\mu\left(\prod_{k=1}^{N}(-\infty, \lambda_k)\right) = \bigwedge_{k=1}^{N} e_k(\lambda_k)$$

for all $\lambda_1, \dots, \lambda_N \in \mathbb{R}$.

$\vartriangleleft$ Without loss of generality we can assume that $\mathbb{B} = \mathrm{Clop}(Q)$, where Q is the Stone space of $\mathbb{B}$. According to 2.8.7, there are continuous functions $x_k : Q \to \overline{\mathbb{R}}$ such that $e_k(\lambda) = \mathrm{cl}\{x_k < \lambda\}$ for all $\lambda \in \mathbb{R}$ and $k := 1, \dots, N$. Put $f(t) := (x_1(t), \dots, x_N(t))$ if all $x_k(t)$ are finite and $f(t) := \infty$ if $x_k(t) = \pm\infty$ at least for one index k. Thereby we have defined some continuous mapping $f : Q \to \mathbb{R}^N \cup \{\infty\}$ (the neighborhood filterbase of the point ∞ is composed of the complements to various balls with center the origin). It is clear that f is measurable with respect to the Borel algebras $\mathscr{B}or(Q)$ and $\mathscr{B}or(\mathbb{R}^N)$. Let $\mathrm{Clop}_\sigma(Q)$ and φ be the same as in 2.10.1.

Define the mapping $\mu : \mathscr{B}or(\mathbb{R}^N) \to \mathbb{B}$ by the formula

$$\mu(A) := \varphi\big(f^{-1}(A)\big) \quad (A \in \mathscr{B}or(\mathbb{R}^N)).$$

It is obvious that μ is a spectral measure. If $A := \prod_{k=1}^{N}(-\infty, \lambda_k)$ then

$$f^{-1}(A) = \bigcap_{k=1}^{N}\{x_k < \lambda_k\},$$

and so $\mu(A) = e_1(\lambda_1) \wedge \dots \wedge e_N(\lambda_N)$. If ν is another spectral measure with the same properties as μ then the set $\mathscr{B} := \{A \in \mathscr{B}or(\mathbb{R}^N) : \nu(A) = \mu(A)\}$ is a σ-algebra containing all sets of the form

$$\prod_{k=1}^{N}(-\infty, \lambda_k) \quad (\lambda_1, \dots, \lambda_N \in \mathbb{R}).$$

Hence, $\mathscr{B} = \mathscr{B}or(\mathbb{R}^N)$. $\vartriangleright$

2.10.3. Let us take an ordered collection of elements $x_1, \dots, x_N$ in a Dedekind σ-complete vector lattice X with unity $\mathbb{1}$. Let $e^{x_k} : \mathbb{R} \to$

$\mathbb{B} := \mathbb{C}(\mathbb{1})$ denote the spectral system of the element x_k. By 2.10.2, there exists a spectral measure $\mu : \mathscr{B}or(\mathbb{R}^N) \to \mathbb{B}$ such that

$$\mu\left(\prod_{k=1}^{N}(-\infty, \lambda_k)\right) = \bigwedge_{k=1}^{N} e^{x_k}(\lambda_k).$$

We may see that the measure μ is uniquely determined by the ordered collection $\mathfrak{x} := (x_1, \dots, x_N) \in X^N$. For this reason, we write $\mu_{\mathfrak{x}} := \mu$ and say that $\mu_{\mathfrak{x}}$ is the spectral measure of the collection $\mathfrak{x}$. The following denotations are accepted for the integral of a measurable function $f : \mathbb{R}^N \to \mathbb{R}$ with respect to the spectral measure $\mu_{\mathfrak{x}}$:

$$\hat{\mathfrak{x}}(f) := f(\mathfrak{x}) := f(x_1, \dots, x_N) := I_\mu(f).$$

If $\mathfrak{x} = (x)$ then we also write $\hat{x}(f) := f(x) := I_\mu(f)$ and call $\mu_x := \mu$ the *spectral measure of an element* of x. Recall that the space $\mathscr{B}(\mathbb{R}^N, \mathbb{R})$ of all Borel functions in $\mathbb{R}^N$ is a universally complete Dedekind σ-complete vector lattice and a semiprime f-algebra.

2.10.4. Theorem. *The spectral measures of the element $f(\mathfrak{x})$ and a collection $\mathfrak{x} := (x_1, \dots, x_N)$ satisfy the equality*

$$\mu_{f(\mathfrak{x})} = \mu_{\mathfrak{x}} \circ f^{\urcorner},$$

where $f^{\urcorner} : \mathscr{B}or(\mathbb{R}) \to \mathscr{B}or(\mathbb{R}^N)$ is the homomorphism acting by the rule $A \mapsto f^{-1}(A)$. In particular,

$$(f \circ g)(\mathfrak{x}) = g(f(\mathfrak{x}))$$

for $f \in \mathscr{B}(\mathbb{R}^N, \mathbb{R})$ and $g \in \mathscr{B}(\mathbb{R}, \mathbb{R})$ whenever $f(\mathfrak{x})$ and $g(f(\mathfrak{x}))$ exist.

$\triangleleft$ By 2.9.4, we have

$$\mu_{f(\mathfrak{x})}(-\infty, t) = e_t^{f(\mathfrak{x})} = [\![f(\mathfrak{x}) < t]\!] = \mu_{\mathfrak{x}} \circ f^{-1}(-\infty, t)$$

for every $t \in \mathbb{R}$. Hence, the spectral measures $\mu_f(\mathfrak{x})$ and $\mu_{\mathfrak{x}} \circ f^{\urcorner}$ on $\mathscr{B}or(\mathbb{R})$ coincide on the intervals of the form $(-\infty, t)$. Reasoning in a standard manner, we then conclude that the measures coincide everywhere. To complete the proof, it suffices to observe that $(g \circ f)^{\urcorner} = f^{\urcorner} \circ g^{\urcorner}$ and apply what was established above twice. $\triangleright$

2.10.5. Theorem. *For every ordered collection $\mathfrak{x} := (x_1, \dots, x_N)$ of elements of a universally σ-complete vector lattice X with unit $\mathbb{1}$, the* *mapping* $\hat{\mathfrak{x}} : f \mapsto \hat{\mathfrak{x}}(f)$ *is the unique sequentially* o*-continuous* f*-algebra homomorphism from* $\mathscr{B}(\mathbb{R}^N, \mathbb{R})$ *to* X *satisfying the conditions*

$$\hat{\mathfrak{x}}(\mathbf{1}) = \mathbb{1}, \quad \hat{\mathfrak{x}}(dt_k) = x_k \quad (k := 1, \dots, N),$$

where $\mathbf{1}$ *is the identically one function on* $\mathbb{R}^N$ *and* $dt_k : (t_1, \dots, t_N) \mapsto t_k$ *stands for the* k*th coordinate function on* $\mathbb{R}^N$*.*

$\lhd$ As was established in 2.9.5, the mapping $f \mapsto \hat{\mathfrak{x}}(f)$ is a sequentially o-continuous homomorphism of f-algebras. From 2.10.4 we have

$$\mu_{dt_k(\mathfrak{x})} = \mu_{\mathfrak{x}} \circ (dt_k)^{\leftarrow} = \mu_{x_k}.$$

Consequently, the elements $\hat{\mathfrak{x}}(dt_k) = dt_k(\mathfrak{x})$ and x_k coincide, for they have the same spectral function. If $h : \mathscr{B}(\mathbb{R}^N, \mathbb{R}) \to X$ is another homomorphism of f-algebras with the same properties as $\hat{\mathfrak{x}}(\cdot)$, then h and $\hat{\mathfrak{x}}(\cdot)$ coincide on all polynomials. Afterwards, we infer that h and $\hat{\mathfrak{x}}(\cdot)$ coincide on the whole $\mathscr{B}(\mathbb{R}^N, \mathbb{R})$ due to o-continuity. $\rhd$

2.11. Boolean Valued Vector Lattices

In this section we will show that a vector lattice arises as the Boolean valued interpretation of a vector lattice if and only if the latter admits the structure of an f-module.

2.11.1. Let A be an f-algebra. Recall that every f-algebra is commutative. A vector lattice X is said to be an f*-module over* A if the following are satisfied:

(1) X is a module over A (with respect to the multiplication $A \times X \ni (a, x) \mapsto ax \in X$).

(2) $ax \geqslant 0$ for all $a \in A_+$ and $x \in X_+$.

(3) $x \perp y$ implies $ax \perp y$ for all $a \in A_+$ and $x, y \in X$.

A vector lattice X has the natural f-module structure over $\operatorname{Orth}(X)$:

$$\pi x := \pi(x) \quad (x \in X,\, \pi \in \operatorname{Orth}(X)).$$

Clearly, X is an f-module over an arbitrary f-submodule $A \subset \operatorname{Orth}(X)$ and, in particular, over $\mathscr{Z}(X)$. If a Dedekind complete vector lattice Y

is an f-module over an f-algebra A then the space $L^{\sim}(X,Y)$ of regular operators from X to Y also has the natural f-module structure:

$$(aT) : x \mapsto a(Tx) \quad (x \in X) \quad (a \in A,\, T \in L^{\sim}(X,Y)).$$

2.11.2. *Let X be an f-module over an f-algebra A. Then the following hold:*

(1) *$(a \vee b)x = (ax) \vee (vx)$ and $(a \wedge b)x = (ax) \wedge (vx)$ for all $a, b \in A$ and $x \in X_+$.*

(2) *$|ax| = |a||x|$ for all $a \in A$ and $x \in X$.*

(3) *$a \perp b$ implies $ax \perp by$ for all $a, b \in A$ and $x, y \in X$.*

It is clear that if $a \in A$ and the operator π_a in X is defined as $\pi_a x := ax$ then $\pi_a \in \mathrm{Orth}(X)$. Moreover, the mapping $h : a \mapsto \pi_a$ is a positive algebra homomorphism from A to $\mathrm{Orth}(X)$ and so h is a lattice homomorphism, since $\mathrm{Orth}(X)$ is a semiprime f-algebra. Conversely, if X is a vector lattice and h is an f-algebra homomorphism from an f-algebra A to $\mathrm{Orth}(X)$, then h induces an f-algebra structure over A on X by putting $ax := h(a)x$ $(x \in X)$.

If the f-algebra A has a unit element $e \in A$ then π_e is a band projection in X. An f-module X is called *unital* if $\pi_e = I_X$.

2.11.3. Let X be a vector lattice, $\mathbb{B}$ a complete Boolean algebra and $\jmath$ a complete homomorphism from $\mathbb{B}$ into $\mathbb{B}(X)$. Say that X is a vector $\mathbb{B}$-*lattice* if $\jmath(b)$ is a projection band for all $b \in \mathbb{B}$. In this case we identify $\jmath(b)$ with the corresponding band projection $[\jmath(b)]$ and write $\mathbb{B} \subset \mathbb{P}(X)$. A vector $\mathbb{B}$-lattice X is said to be $\mathbb{B}$-*complete* if for every family $(x_\xi)_{\xi\in\Xi}$ in X and every partition of unity $(b_\xi)_{\xi\in\Xi}$ in $\mathbb{B}$ there exist $x \in X$ such that $b_\xi x = b_\xi x_\xi$ for all $\xi \in \Xi$. This element x is called the *mixture* of $(x_\xi)_{\xi\in\Xi}$ by $(b_\xi)_{\xi\in\Xi}$ and we write $x := \mathrm{mix}_{\xi\in\Xi}\, b_\xi x_\xi$ (cp. 1.4.3).

Denote by $\mathrm{St}_0(\mathbb{B})$ the subspace of $\mathrm{Orth}(X)$ consisting of the operators $\sum_{k=1}^n \lambda_k \pi_k$ where $\pi_1, \dots, \pi_n$ are pairwise disjoint members of $\mathbb{B}$ and $\lambda_1, \dots, \lambda_n \in \mathbb{R}$. Clearly, $A_0 := \mathrm{St}_0(\mathbb{B})$ is an f-subalgebra of $\mathrm{Orth}(X)$ and X is an f-module over A_0.

If X is $\mathbb{B}$-complete then for every partition of unity $(\pi_\xi)_{\xi\in\Xi}$ in $\mathbb{B}$ and every family of reals $(\lambda_\xi)_{\xi\in\Xi}$ there exist an orthomorphism $T := \sum_{\xi\in\Xi} \lambda_\xi \pi_\xi \in \mathrm{Orth}(X)$ such that $\pi_\xi T = \lambda_\xi \pi_\xi$ for all $\xi \in \Xi$. Let $\mathrm{St}(\mathbb{B})$ stand for the set of all orthomorphisms of this form. Then $A := \mathrm{St}(\mathbb{B})$ is an f-subalgebra of $\mathrm{Orth}(X)$ and X is an f-module over A.

2.11.4. Theorem. *If $\mathscr{X} \in \mathbb{V}^{(\mathbb{B})}$ is an internal vector lattice over the internal field $\mathbb{R}^\wedge$, then $\mathscr{X}{\downarrow}$ is a $\mathbb{B}$-complete vector lattice over $\mathbb{R}$ and there exists a complete Boolean monomorphism $\jmath : \mathbb{B} \to \mathbb{P}(\mathscr{X}{\downarrow})$ with*

$$b \leqslant [\![x \leqslant y]\!] \iff \jmath(b)x \leqslant \jmath(b)y \quad (x, y \in \mathscr{X}{\downarrow};\ b \in \mathbb{B}).$$

Moreover, there exists a Boolean isomorphism $\varkappa$ from $\mathbb{P}(\mathscr{X}{\downarrow})$ onto $\mathbb{P}(\mathscr{X}){\downarrow}$ such that the diagram commutes

$$\begin{tikzcd}
 & \mathbb{B} \arrow[dl, "\jmath"'] \arrow[dr, "\imath"] & \\
\mathbb{P}(\mathscr{X}{\downarrow}) \arrow[rr, "\varkappa"'] & & \mathbb{P}(\mathscr{X}){\downarrow}
\end{tikzcd}$$

where ι is defined as in 1.10.2.

$\lhd$ The proof can be given along the lines of the proof of the Gordon Theorem with obvious modifications. Alternatively we can use 1.8.6. In the latter case $\mathscr{X}$ is considered as an algebraic system with the universe $|\mathscr{X}|$, the nullary operation 0, the 1-ary operations $\{+\} \cup \mathbb{R}^\wedge$, and the predicates $\{=, \leqslant\}$. The symbol $\lambda \in \mathbb{R}^\wedge$ is identified with the operation $x \mapsto \lambda x$ $(x \in \mathscr{X})$. Then it should be observed that for each $\pi \in \mathbb{P}(X)$ the set $c_\pi := \{(x, y) \in X \times X : \pi x = \pi y\}$ is a congruence of the algebraic system $\mathscr{X} = (|\mathscr{X}|, 0, +, (\lambda)_{\lambda \in \mathbb{R}^\wedge}, \leqslant)$ and the mapping $\pi \mapsto c_\pi$ is an isomorphism of $\mathbb{P}(X)$ onto a complete Boolean algebra of congruences of $\mathscr{X}$ (cp. 1.7.5 and 1.7.9). $\rhd$

2.11.5. *For every vector $\mathbb{B}$-lattice X there exists $\mathscr{X}, \mathscr{X}^\delta \in \mathbb{V}^{(\mathbb{B})}$ such that the following hold:*

(1) *$[\![\mathscr{X}$ is a vector lattice over $\mathbb{R}^\wedge]\!] = \mathbb{1}$.*

(2) *there is a lattice isomorphism h from X to $\mathscr{X}{\downarrow}$ satisfying $\mathscr{X}{\downarrow} = \mathrm{mix}(h(X))$ and $b = h^{-1} \circ \jmath(b) \circ h$ for all $b \in \mathbb{B}$.*

(3) *$[\![\mathscr{X}^\delta$ is a Dedekind completion of $\mathscr{X}]\!] = \mathbb{1}$ and $\mathscr{X}^\delta{\downarrow}$ is a Dedekind completion of $\mathscr{X}{\downarrow}$; i.e., $(\mathscr{X}{\downarrow})^\delta = (\mathscr{X}^\delta){\downarrow}$.*

$\lhd$ Let X be a vector $\mathbb{B}$-lattice. Define $d, P_{\leqslant} : X \times X \to \mathbb{B}$ by putting

$$d(x, y) := \bigwedge \{b \in \mathbb{B} :\ b^* x = b^* y\};$$

$$P_{\leqslant}(x, y) := \bigwedge \{b \in \mathbb{B} :\ b^* x \leqslant b^* y\}$$

for $x, y \in X$. It is immediate from the definitions that d is a $\mathbb{B}$-metric on X and for all $x, y, u, v \in X$ and $\lambda \in \mathbb{R}$ the inequalities hold:

$$d(x+u, y+v) \leqslant d(x,y) \vee d(u,v),$$
$$d(\lambda x, \lambda y) \leqslant d(x,y),$$
$$P_{\leqslant}(x,y) \bigtriangleup P_{\leqslant}(u,v) \leqslant d(x,u) \vee d(y,v).$$

It follows that $\mathfrak{A} = (X, +, (\lambda)_{\lambda\in\mathbb{R}}, P_{\leqslant})$ is an algebraic $\mathbb{B}$-system of signature $\{+, P_{\leqslant}\} \cup \mathbb{R}$. By 1.8.8 within $\mathbb{V}^{(\mathbb{B})}$ there exists an algebraic system $\mathscr{X} := \mathscr{A}$ of signature $\{+, P_{\leqslant}\}^{\wedge} \cup \mathbb{R}^{\wedge}$ with the properties 1.8.8 (2–4). Direct calculation of truth values on using 1.7.6 shows that $|\varphi|^{\mathscr{X}} = \mathbb{1}$ with φ the formula of signature $\{+, P_{\leqslant}\} \cup \mathbb{R}$ formalizing the sentence "X is a vector lattice over $\mathbb{R}$." It follows from 1.8.8 (4) that $\mathscr{X}$ is a vector lattice over $\mathbb{R}^{\wedge}$ within $\mathbb{V}^{(\mathbb{B})}$. Observe also that (2) follows from 1.8.8 (3).

The claim (3) amounts to saying that $[\![\mathscr{X}$ is an order dense majorizing sublattice of $\mathscr{X}^{\delta}]\!] = \mathbb{1}$ if and only if $\mathscr{X}{\downarrow}$ is an order dense majorizing sublattice of $\mathscr{X}^{\delta}{\downarrow}$ (cp. Luxemburg and Zaanen [297, Theorem 32.7]). An easy verification of the latter is left to the reader. $\rhd$

2.11.6. Theorem. *If $\mathscr{X}$ is a universally complete vector lattice in $\mathbb{V}^{(\mathbb{B})}$, then $\mathscr{X}{\downarrow}$ is a universally complete vector lattice and there exists a lattice isomorphism $\jmath$ from $\mathscr{R}{\downarrow}$ onto an order closed sublattice $X_0 \subset X$ with $\mathbb{1} := \jmath(1^{\wedge}) \in X_0$ a weak order unit of X. Moreover, there is a Boolean isomorphism $\varkappa$ from $\mathbb{P}(\mathscr{X}){\downarrow}$ onto $\mathbb{P}(\mathscr{X}{\downarrow})$ such that the diagram*

$$\begin{array}{ccc} \mathbb{B} & \xrightarrow{\ \chi\ } & \mathbb{P}(\mathscr{R}{\downarrow}) \\ {\scriptstyle\iota}\big\downarrow & & \big\downarrow{\scriptstyle\jmath} \\ \mathbb{P}(\mathscr{X}){\downarrow} & \xrightarrow{\ \varkappa\ } & \mathbb{P}(\mathscr{X}{\downarrow}) \end{array}$$

commutes, where ι and χ are defined respectively as in 1.10.1 *and* 2.2.4 *and $\varkappa$ is induced by $\jmath$ with $\jmath : \pi \mapsto [\jmath(\pi 1^{\wedge})]$ $(\pi \in \mathbb{P}(\mathscr{R}{\downarrow}))$.*

$\lhd$ Working within $\mathbb{V}^{(\mathbb{B})}$ and using transfer, we put $\mathscr{D} := \mathbb{P}(\mathscr{X})$ and observe that $\mathscr{X}$ is lattice isomorphic to $\mathfrak{S}(\mathscr{D})$ by 2.8.4. At the same time, by 1.10.1 $\mathbb{D} := \mathscr{D}{\downarrow}$ is a complete Boolean algebra and there is a Boolean isomorphism from $\mathbb{B}$ onto an order closed subalgebra $\mathbb{D}_0 \subset \mathbb{D}$. Moreover, according to 2.8.3 and 2.8.5 $\mathfrak{S}(\mathbb{D})$ is a universally complete vector lattice and $\mathfrak{S}(\mathbb{D}_0)$ is an order closed sublattice in $\mathfrak{S}(\mathbb{D})$ isomorphic to $\mathscr{R}{\downarrow}$. Prove that $\mathfrak{S}(\mathscr{D}){\downarrow}$ and $\mathfrak{S}(\mathbb{D})$ are isomorphic algebraic systems. If $\epsilon \in \mathfrak{S}(\mathscr{D}){\downarrow}$

then $[\![\epsilon : \mathscr{R} \to \mathscr{D}$ is a spectral function$]\!] = \mathbb{1}$. Since spectral functions as well as operations and order on $\mathfrak{S}(\mathscr{D})$ are uniquely determined by the values on a dense subfield in $\mathscr{R}$ and since $\mathbb{R}^\wedge$ is a dense subfield of $\mathscr{R}$, we can replace ϵ by its restriction onto $\mathbb{R}^\wedge$. Denote by e the modified descent of $\epsilon|_{\mathbb{R}^\wedge}$; i.e., $e : \mathbb{R} \to \mathbb{D}$ is defined by $[\![e(t) = \epsilon(t^\wedge)]\!] = \mathbb{1}$ $(t \in \mathbb{R})$. An easy calculation ensures that e is a spectral function. $\triangleright$

2.11.7. Let $\mathbb{F}$ be a dense subfield of $\mathbb{R}$ and let X be a vector lattice over $\mathbb{F}$. A Dedekind complete vector lattice X^δ over $\mathbb{R}$ is said to be a *Dedekind completion* of X whenever X is lattice isomorphic to a majorizing order dense sublattice X^δ (which is identified with X).

If X is Archimedean then X has a Dedekind completion X^δ unique up to lattice isomorphism. This fact can be proved by the method of cuts just as the classical result (cp. [297, Theorems 32.3 and 32.5] and [403, Theorems II.3.2 and IV.11.1]).

It follows also that if X is Archimedean then X has a *universal completion* $X^{\mathfrak{u}}$ unique up to a lattice isomorphism; i.e., X^δ is a universally complete vector lattice and X is lattice isomorphic to an order dense vector subspace of X.

2.11.8. *Let X be a vector $\mathbb{B}$-lattice. Then there exist $\mathscr{X}, \mathscr{X}^{\mathfrak{u}} \in \mathbb{V}^{(\mathbb{B})}$ such that*

(1) *$[\![\mathscr{X}$ is a vector lattice over $\mathbb{R}^\wedge$ and $\mathscr{X}^{\mathfrak{u}}$ is a universal completion of $\mathscr{X}]\!] = \mathbb{1}$.*

(2) *There is a lattice isomorphism h from X into $\mathscr{X}^{\mathfrak{u}}\!\downarrow$ such that $(\mathscr{X}^{\mathfrak{u}}\!\downarrow, h)$ is a universal completion of X and $b = h^{-1} \circ \jmath(b) \circ h$ for all $b \in \mathbb{B}$, where $\jmath$ is defined as in* 2.11.4.

$\triangleleft$ Put $\mathbb{D} := \mathbb{P}(X)$ and $\mathbb{B} := \mathbb{P}(X_0)$ and let $\jmath$ be the embedding of $\mathbb{B}$ to $\mathbb{D}$. Then $\mathfrak{S}(\mathbb{D})$ and $\mathfrak{S}(\mathbb{B})$ are universally complete vector lattices isomorphic to X and X_0, respectively. By 1.9.3 there exists a complete Boolean algebra $\mathscr{D}$ in $\mathbb{V}^{(\mathbb{B})}$ and we have a Boolean isomorphism h from $\mathbb{D}$ onto $\mathscr{D}\!\downarrow$ and a Boolean isomorphism ι from $\mathbb{B}$ onto an order closed subalgebra in $\mathscr{D}\!\downarrow$ such that $\iota = \jmath \circ h$. By transfer $\mathscr{X} := \mathfrak{S}(\mathscr{D})$ is a universally complete vector lattice in $\mathbb{V}^{(\mathbb{B})}$ and $\mathbb{P}(\mathscr{X})$ is isomorphic to $\mathscr{D}$. It remains to appeal to 2.11.3. $\triangleright$

2.11.9. *Let X be an f-module over $\mathscr{L}(Y)$ with Y a Dedekind complete vector lattice and $\mathbb{B} = \mathbb{P}(Y)$. Then there exist $\mathscr{X}, \mathscr{X}^\delta, \mathscr{X}^{\mathfrak{u}} \in \mathbb{V}^{(\mathbb{B})}$ such that*

(1) *$[\![\mathscr{X}$ is a vector lattice over $\mathscr{R}]\!] = \mathbb{1}$, $\mathscr{X}\!\downarrow$ is an f-module over $A^{\mathfrak{u}}$,* *and there is an f-module isomorphism h from X to $\mathscr{X}{\downarrow}$ satisfying $\mathscr{X}{\downarrow} = \operatorname{mix}(h(X))$;*

(2) *$[\![\mathscr{X}^{\delta}$ is a Dedekind completion of $\mathscr{X}]\!] = \mathbb{1}$, $\mathscr{X}^{\delta}{\downarrow}$ is an f-module over $A^{\mathfrak{u}}$, and $\mathscr{X}{\downarrow}$ is f-module isomorphic to an order dense f-submodule in $(\mathscr{X}{\downarrow})^{\delta}$.*

(3) *$[\![\mathscr{X}^{\mathfrak{u}}$ is a universal completion of $\mathscr{X}]\!] = \mathbb{1}$, $\mathscr{X}^{\mathfrak{u}}{\downarrow}$ is an f-module over $A^{\mathfrak{u}}$, and $\mathscr{X}{\downarrow}$ is f-module isomorphic to an order dense f-submodule in $(\mathscr{X}{\downarrow})^{\mathfrak{u}}$.*

2.12. Variations on the Theme

In this section we raise the following question: Which uniformities are generated by the metrics that take values in some vector lattices? It is clear that if (X,ρ) is a metric space, Λ is a vector lattice and $0 < e \in \Lambda$, then the Λ-valued metric $(x,y) \mapsto \rho(x,y)\cdot e$ $(x,y \in X)$ determines the same uniformity as ρ. Consequently, the question raised becomes nontrivial only if we additionally require that the Λ-valued metric uses a substantial part of Λ rather than just its one-dimensional subspace spanned by e. This extra assumption, for instance, provides decomposability. Thus, we are to clarify necessary and sufficient conditions for a uniformity $\mathscr{F}$ on X to be generated by a decomposable metric with values in a universally complete vector lattice.

2.12.A. Vector Lattice Valued Metrics

We introduce the main definitions and notation that are dealt with henceforth.

2.12.A.1. Consider a nonempty set X and a vector lattice Λ. A mapping $\rho : X \times X \to \Lambda$ is called a (*vector*, or Λ-*valued*) *semimetric on* X if for all $x,y,z \in X$ the axioms are valid:

(1) $\rho(x,y) \geqslant 0$, $\rho(x,x) = 0$,

(2) $\rho(x,y) = \rho(y,x)$,

(3) $\rho(x,y) \leqslant \rho(x,z) + \rho(z,y)$.

A semimetric ρ is said to be a *metric* if from $\rho(x,y) = 0$ it follows that $x = y$ for all x and y in X.

Much of the sequel is valid for general semimetrics, but we confine exposition to the case of vector metrics. A pair (X,ρ) is said to be a Λ-*metric space*, if X is a nonempty set and ρ is a metric on X with values in some vector lattice Λ.

2.12.A.2. Given an arbitrary net $(x_\alpha)_{\alpha\in A}$ in a Λ-metric space (X,ρ), we say that

(a) (x_α) *ρo-converges* (*ρr-converges*) to an element $x\in X$ if

$$o\text{-}\lim_{\alpha\in A}\rho(x,x_\alpha)=0 \quad \Big(r\text{-}\lim_{\alpha\in A}\rho(x,x_\alpha)=0\Big);$$

(b) (x_α) is *ρo-fundamental* (*ρr-fundamental*), if

$$o\text{-}\lim_{\beta,\alpha\in A}\rho(x_\alpha,x_\beta)=0 \quad \Big(r\text{-}\lim_{\alpha,\beta\in A}\rho(x_\alpha,x_\beta)=0\Big);$$

(c) the space (X,ρ) is *ρo-complete* (*ρr-complete*) if each ρo-fundamental (ρr-fundamental) net in X has a ρo-limit (ρr-limit);

(d) a subspace $X_0\subset X$ is *ρo-closed* (*ρr-closed*) if X_0 contains the ρo-limits (ρr-limits) of all ρo-converging (ρr-converging) nets in X_0.

A vector metric enables us to provide the underlying set both with a Boolean metric and a uniformity.

2.12.A.3. We assume henceforth that the vector lattice Λ under consideration is Dedekind complete. Take a Λ-metric space X with a Λ-valued metric ρ. Let $\jmath$ be an isomorphism of a complete Boolean algebra $\mathbb{B}$ onto the base $\mathbb{P}(\Lambda)$ of Λ. Define the mapping $d:X\times X\to\Lambda$ by putting $d(x,y):=\jmath^{-1}([\rho(x,y)])$ for all $x,y\in X$. Recall that $[u]$ stands for the band projection onto $u^{\perp\perp}$. We can easily check that d is a Boolean metric on X. Thus, each Λ-metric space transforms canonically into a $\mathbb{B}$-set. This fact allows us to use Boolean valued models for the study of Λ-metric spaces.

Take an arbitrary family $(x_\xi)_{\xi\in\Xi}$ in X and a partition of unity $(b_\xi)_{\xi\in\Xi}$ in $\mathbb{B}$. An element $x\in X$ coincides with the mixture $\operatorname{mix}(b_\xi x_\xi)$ (relative to the canonical $\mathbb{B}$-metric d) if and only if $\pi_\xi d(x_\xi,x)=0$ for all $\xi\in\Xi$, where $\pi_\xi=\jmath(b_\xi)$. For convenience, from now on assume that $\mathbb{B}=\mathbb{P}(\Lambda)$.

Recall that mixing in a $\mathbb{B}$-set is not always possible. A Λ-metric space X as well as its metric ρ is called mix-*complete* or *laterally complete* if there exist mixtures of all families (x_ξ) in X by all partitions of unity (b_ξ) in $\mathbb{B}$. We say that (X,ρ) is *decomposable* if there exist mixtures of all finite collections by all finite partitions of unity.

2.12.A.4. Let Λ be a universally complete vector lattice. Let $\mathscr{E}$ denote the filter of all order units in Λ; i.e., $e\in\mathscr{E}$ means that $0\leqslant e\in\Lambda$ and $\{e\}^{\perp\perp}=\Lambda$. The collection of sets $\{[-e,e],e\in\mathscr{E}\}$ constitutes a base of zero neighborhoods for the unique topology τ making (Λ, τ) a complete separated topological group (but not a topological vector space). Given $e \in \mathscr{E}$, put $U(\rho, e) := \{(x, y) \in X^2 : \rho(x, y) \leqslant e\} = \rho^{-1}([-e, e])$. Clearly, the sets $U(\rho, e)$ $(e \in \mathscr{E})$ form a fundamental system of entourages for the unique uniformity on X. This uniformity will be referred to as the *uniformity generated by the* Λ-*metric* ρ. Below, while talking about the uniform structure of a Λ-metric space (X, ρ), we bear in mind the uniformity.

2.12.A.5. The inclusion-ordered set of all uniformities $\mathscr{U}(X)$ on a set X forms a complete lattice. The bottom $\mathbb{0}$ of the lattice is the uniformity having the single entourage X^2. Consider the interval $[\mathbb{0}, \mathscr{F}] := \{\mathscr{F}' \in \mathscr{U}(X) : \mathbb{0} \leqslant \mathscr{F}' \leqslant \mathscr{F}\}$. The *complete Boolean algebra of components* of $\mathscr{F}$ is defined as the complete Boolean algebra $\mathscr{B} \subset [\mathbb{0}, \mathscr{F}]$ in which suprema are inherited from the lattice $\mathscr{U}(X)$ and $\mathscr{F}$ serves as the order unit of $\mathscr{B}$. It is easily seen that if $\mathbb{B}$ is a complete Boolean algebra and there is an injective mapping $\chi : \mathbb{B} \to [\mathbb{0}, \mathscr{F}]$ preserving the suprema of all sets and having $\chi(\mathbb{1}) = \mathscr{F}$, then $\chi(\mathbb{B})$ is a complete Boolean algebra of components of $\mathscr{F}$.

2.12.A.6. Now we give the main definition of the current section: Fix the complete Boolean algebra $\mathscr{B}$ of components of a uniformity $\mathscr{F}$. Take a family $(x_\xi)_{\xi \in \Xi}$ in a space X and a partition of unity $(\mathscr{F}_\xi)_{\xi \in \Xi}$ in $\mathscr{B}$. We assume that there is $x := \bigvee_{\xi \in \Xi} \mathscr{F}_\xi(x_\xi)$, where the supremum is taken in the complete lattice of all filters on X and $\mathscr{F}_\xi(x_\xi)$ denotes the set of subsets $V(x_\xi) := \{y \in X : (x_\xi, y) \in V\}$ $(V \in \mathscr{F}_\xi)$. We call x the *mixture* of (x_ξ) by $(\mathscr{F}_\xi)$ and denote x by $\mathscr{B}\text{-mix}_{\xi \in \Xi}(\mathscr{F}_\xi x_\xi)$. The space X is called $\mathscr{B}$-*decomposable* ($\mathscr{B}$-*complete*) if there exist $\mathscr{B}$-mixtures of all finite (arbitrary) families in X by all finite (arbitrary) partitions of unity in $\mathscr{B}$.

2.12.A.7. A filterbase $\mathscr{F}_0$ is called a $\mathscr{B}$-*cyclic base* of a uniformity $\mathscr{F}$ provided that

(1) every entourage $V_0 \in \mathscr{F}_0$ is closed under $\mathscr{B}$-mixing; i.e., if $((x_\xi, y_\xi))_{\xi \in \Xi}$ lies in V_0, while $(\mathscr{F}_\xi)_{\xi \in \Xi}$ is a partition of unity in $\mathscr{B}$ such that there exist $x := \mathscr{B}\text{-mix}_{\xi \in \Xi}(\mathscr{F}_\xi x_\xi)$ and $y := \mathscr{B}\text{-mix}_{\xi \in \Xi} \mathscr{F}_\xi y_\xi)$, then $(x, y) \in V_0$;

(2) each set $V \in \mathscr{F}$ includes a subset of the form

$$V_0 := \mathscr{B}\text{-}\underset{\xi \in \Xi}{\mathrm{mix}}(\mathscr{F}_\xi V_\xi) := \big\{\mathscr{B}\text{-}\underset{\xi \in \Xi}{\mathrm{mix}}(\mathscr{F}_\xi x_\xi) : x_\xi \in V_\xi \ (\xi \in \Xi)\big\},$$

where (V_ξ) is a family in $\mathscr{F}_0$ and $(\mathscr{F}_\xi)$ is a partition of unity in $\mathscr{B}$.

2.12.A.8. We say that a uniform space is *completely metrizable* by a Dedekind complete vector lattice Λ if its uniformity is determined by a decomposable Λ-valued metric as in 2.12.A.4.

Consider the three examples of mix-complete metric spaces.

2.12.A.9. Take a metric space $(\mathscr{X}, \rho)$ in the Boolean-valued model $\mathbb{V}^{(\mathbb{B})}$, with a fixed complete Boolean algebra $\mathbb{B}$. Let $\mathscr{R}$ be the reals within $\mathbb{V}^{(\mathbb{B})}$ and $\Lambda := \mathscr{R}{\downarrow}$.

If $X = \mathscr{X}{\downarrow}$ and $\bar{\rho} := \rho{\downarrow}$ then $\bar{\rho} : X^2 \to \Lambda$ is a metric and the space $(X, \bar{\rho})$ is mix-complete. The latter space is ρo-complete if and only if the internal metric space $(\mathscr{X}, \rho)$ is complete within $\mathbb{V}^{(\mathbb{B})}$.

2.12.A.10. Let Q be an extremally disconnected compact space and let (X, ρ) be a metric space. Denote by $C_\infty(Q, X)$ the set of cosets of continuous mappings from comeager subsets of Q into X. To put it in more detail, an element $z \in C_\infty(Q, X)$ is uniquely determined by the conditions: (a) for every $u \in z$, there is a comeager subset $Q(u) \subset Q$ (i.e. the complement of a meager subset) such that u is a continuous mapping from $Q(u)$ into X; (b) if $u, v \in z$ then $u(t) = v(t)$ for all $t \in Q(u) \cap Q(v)$. Take arbitrary elements $y, z \in C_\infty(Q, X)$. Let $u \in y$ and $v \in z$. Then the function $t \mapsto \rho(u(t), v(t))$ $(t \in Q(u) \cap Q(v))$ is defined on a comeager set and is continuous. Consequently, it determines the unique element w of $C_\infty(Q) := C_\infty(Q, \mathbb{R})$; the element w is independent of the choice of $u \in y$ and $v \in z$. We set $\bar{\rho}(y, z) := w$ by definition. Clearly, that $\bar{\rho}$ is a vector metric on $C_\infty(Q, X)$ with values in $C_\infty(Q)$.

If (X, ρ) is a metric space then $(C_\infty(Q, X), \bar{\rho})$ is a mix-complete Λ-metric space with $\Lambda := C_\infty(Q)$. Moreover, $C_\infty(Q, X)$ is $\bar{\rho}o$-complete if and only if X is complete.

2.12.A.11. Consider a metric space (X, ρ). Let τ be the topology on X determined by the metric ρ. A mapping $\varphi : \tau \to \mathbb{B}$ is called a *Cauchy $\mathbb{B}$-filter* if φ satisfies the conditions:

(1) $\varphi(\varnothing) = \mathbb{0}$;

(2) $\varphi(U \cap V) = \varphi(U) \wedge \varphi(V)$ for all $U, V \in \tau$;

(3) $\bigvee\{\varphi(V) : V \times V \subset \{\rho < \varepsilon\}\} = \mathbb{1}$ for every $0 < \varepsilon \in \mathbb{R}$.

Say that a Cauchy $\mathbb{B}$-filter φ is *minimal* of φ has the uniform regularity property:

$$\varphi(U) = \bigvee\{\varphi(V) : V \in \tau,\ U_{\varepsilon,\rho}(V) \subset U,\ 0 < \varepsilon \in \mathbb{R}\} \quad (U \in \tau),$$

where $U_{\varepsilon,\rho}(V) = \{x \in X : (\exists v \in V)\ \rho(v, x) < \varepsilon\}$. Denote the set of minimal Cauchy $\mathbb{B}$-filters by $\mathbb{B}^X$. Let Λ be the universally complete vector lattice of all spectral systems in $\mathbb{B}$ (see Theorem 2.8.3). We will determine a Λ-valued metric on $\mathbb{B}^X$. Given $\varphi, \psi \in \mathbb{B}^X$ we put

$$e_\lambda := \bigvee\{\varphi(U) \wedge \psi(V) : U, V \in \tau;\ U \times V \subset \{\rho < \lambda\}\},$$

if $0 < \lambda \in \mathbb{R}$ and $e_\lambda := 0$ if $\lambda \leqslant 0$. It can be verified that the mapping $e : \lambda \to e_\lambda$ $(\lambda \in \mathbb{R})$ is a spectral system in $\mathbb{B}$. We put $r(\varphi, \psi) := e$.

The mapping $r : \mathbb{B}^X \times \mathbb{B}^X \to \Lambda$ *is a metric and* $(\mathbb{B}^X, r)$ *is a mix-complete* Λ-metric space.

2.12.B. Metrization by Vector Lattices

The aim of this section is to prove the following metrization result.

2.12.B.1. Theorem. *A separated uniform space* $(X, \mathscr{F})$ *is completely metrizable by a Dedekind complete vector lattice* Λ *with unit if and only if* X *is* $\mathscr{B}$*-decomposable and the uniformity* $\mathscr{F}$ *possesses a countable* $\mathscr{B}$*-cyclic base with respect to some complete Boolean algebra* $\mathscr{B}$ *of components of* $\mathscr{F}$ *which is isomorphic to* $\mathbb{P}(\Lambda)$.

2.12.B.2. *Let* $\rho : X \times X \to \Lambda$ *be a decomposable metric generating the uniformity* $\mathscr{F}$ *as in* 2.12.A.4. *Then* $\mathscr{F}$ *has a countable* $\mathscr{B}$*-cyclic base with respect to some complete Boolean algebra* $\mathscr{B}$ *of components of* $\mathscr{F}$ *which is isomorphic to* $\mathbb{P}(\Lambda)$.

$\lhd$ Associate to each projection $b \in \mathbb{B} := \mathbb{B}(\Lambda)$ the uniformity $\mathscr{F}^b$ on X that is determined by the fundamental system of entourages

$$U(b\rho, e) := \{(x, y) \in X^2 : b\rho(x, y) < e\},$$

where e ranges over the filter of order units $\mathscr{E}$ in Λ. Clearly, the mapping $b \mapsto \mathscr{F}^b$ $(b \in \mathbb{B})$ is injective, preserves suprema, and associates $\mathscr{F}$ with the unit of the algebra $\mathbb{B}$ by our assumption of metrizability. Consequently, the mapping gives an isomorphism of $\mathbb{B}$ onto the Boolean algebra of components of $\mathscr{F}$. As for the conditions required, we, for example, will prove $\mathscr{B}$-decomposability, where $\mathscr{B}$ is the image of $\mathbb{B}$ under the isomorphism indicated.

Take $x, y \in X$ and $b \in \mathbb{B}$ and put $z := \rho\text{-mix}\{bx, b^*y\}$. This means that $z \in X$ and $b\rho(x, z) = b^*\rho(y, z) = 0$. Let some neighborhoods of x and y in the uniform topology corresponding to $\mathscr{F}^b$ and $\mathscr{F}^{b^*}$ look like $U := \{u \in X : b\rho(x, u) \leqslant e\}$ and $V := \{v \in X : b^*\rho(y, v) \leqslant e\}$, where $e \in \mathscr{E}$. If $W := U \cap V$ then for every $w \in W$ we have

$$b\rho(z, w) \leqslant b(\rho(z, x) + \rho(x, w)) = b\rho(x, w) \leqslant e,$$
$$b^*\rho(z, w) \leqslant b^*(\rho(z, y) + \rho(y, w)) = b^*\rho(y, w) \leqslant e;$$

i.e., $\rho(z,w) \leqslant e$. Therefore, $W \in \mathscr{F}(z)$. Thus, $z = \lim(\mathscr{F}^b(x) \vee \mathscr{F}^{b^*}(y))$. The similar reasoning demonstrates that the sequence $\{\rho \leqslant n^{-1}\mathbb{1}\}$ $(n \in \mathbb{N})$ forms a $\mathscr{B}$-cyclic filterbase of the entourages of $\mathscr{F}$. $\triangleright$

2.12.B.3. *Let $(X,\mathscr{F})$ be a $\mathscr{B}$-decomposable separated uniform space with $\mathscr{B}$ the complete Boolean algebra of components of $\mathscr{F}$ isomorphic to $\mathbb{B}$. Then X is a decomposable $\mathbb{B}$-set.*

$\triangleleft$ Assume that there is an isomorphism $b \mapsto \mathscr{F}^b$ of a complete Boolean algebra $\mathbb{B}$ onto the Boolean algebra $\mathscr{B}$ of components of $\mathscr{F}$. Given a pair of elements $x, y \in X$, put

$$d(x,y) := \bigwedge \left\{ b \in \mathbb{B} :\ (x,y) \in \bigcap \mathscr{F}^{b^*} \right\}.$$

It is obvious that $d(x,y) = d(y,x)$ and $d(x,x) = \mathbb{0}$ for all $x, y \in X$. Assume that $d(x,y) = \mathbb{0}$. Then

$$\mathscr{F} = \mathscr{F}^{\mathbb{1}} = \bigvee \left\{ \mathscr{F}^b :\ (x,y) \in \bigcap \mathscr{F}^b \right\};$$

consequently, $(x,y) \in \bigcap \mathscr{F} = \Delta(X^2)$ or $x = y$. Take $b, c \in \mathbb{B}$ such that $(x,y) \in \bigcap \mathscr{F}^b$ and $(z,y) \in \bigcap \mathscr{F}^c$. Since $\mathscr{F}^{b\wedge c}$ is a uniformity on X, the set $V := \bigcap \mathscr{F}^{c\wedge b}$ is symmetric. Moreover, $(x,z), (z,y) \in V$; hence, $(x,y) \in V$. From the definition of d we have $d(x,y) \leqslant (b \wedge c)^* = b^* \vee c^*$. By taking infima over b^* and c^*, we arrive to the triangle inequality for d.

Thus, (X,d) is a $\mathbb{B}$-set; let us prove that X is decomposable. To this end, take arbitrary $x, y \in X$ and $b \in \mathbb{B}$. By assumption $z = \mathscr{B}$-$\mathrm{mix}\{bx, b^*y\}$ exists. From the containment $z \in \bigcap(\mathscr{F}^b(x) \vee \mathscr{F}^{b^*}(y))$ it is clear that $(x,z) \in \bigcap \mathscr{F}^b$ and $(y,z) \in \bigcap \mathscr{F}^{b^*}$. Hence, from the definition of d we obtain $d(x,z) \leqslant b^*$ and $d(x,y) \leqslant b$, or, which is the same, $b \wedge d(x,z) = 0$ and $b^* \wedge d(x,y) = 0$. This means that $z = d$-$\mathrm{mix}(bx, b^*y)$, where d-mix denotes the mixing operation on the $\mathbb{B}$-set (X,d). So the decomposability of (X,d) is corroborated. The same reasoning shows that the mixtures $\mathscr{B}$-$\mathrm{mix}_{\xi\in\Xi}(\mathscr{F}^b\xi x_\xi)$ and $z = d$-$\mathrm{mix}_{\xi\in\Xi}(b_\xi x_\xi)$ coincide for all $(x_\xi) \subset X$ and $(b_\xi) \subset \mathbb{B}$. Therefore, in what follows we will simply write mix, while denoting the two mixing operations. $\triangleright$

2.12.B.4. We are able now to prove 2.12.B.1.

$\triangleleft$ Let $\mathscr{X} \in \mathbb{V}^{(\mathbb{B})}$ be a Boolean valued representation of a $\mathbb{B}$-set (X,d). Without loss of generality, we can assume that $X \subset X' := \mathscr{X}{\downarrow} \subset \mathbb{V}^{(\mathbb{B})}$ and that $[\![x \neq y]\!] = d(x,y)$ for $x, y \in X$ (cp. 1.7.2). Let $\mathscr{F}_0$ be a countable $\mathscr{B}$-cyclic filterbase of $\mathscr{F}$. Put $\mathfrak{F} := \{V{\uparrow} : V \in \mathscr{F}\}{\uparrow}$, $\mathfrak{F}_0 := \{V{\uparrow} : V \in$

$\mathscr{F}_0\}{\uparrow}$. Show that $(\mathscr{X}, \mathfrak{F})$ is a uniform space and that $\mathfrak{F}_0$ is a countable filterbase of $\mathfrak{F}$. The fact that $\mathfrak{F}$ is a filterbase within $\mathbb{V}^{(\mathbb{B})}$ is plain from the calculation:

$$[\![(\forall\, A, B \in \mathfrak{F})\, A \cap B \in \mathfrak{F}]\!] = \bigwedge_{A,B\in\mathscr{F}} [\![A{\uparrow} \cap B{\uparrow} \in \mathfrak{F}]\!]$$
$$\geqslant \bigwedge_{A,B\in\mathscr{F}} [\![(A\cap B){\uparrow} \in \mathfrak{F}]\!] = \bigwedge_{C\in\mathscr{F}} [\![C{\uparrow} \in \mathfrak{F}]\!] = \mathbb{1}.$$

Assuming that $A \in \mathscr{F}$, $B \in \mathbb{V}^{(\mathbb{B})}$, and $[\![A{\uparrow} \subset B \subset \mathscr{X}^2]\!] = \mathbb{1}$, and putting $V_B := B{\downarrow} \cap X^2$ we see that $V_B \in \mathscr{F}$ and $[\![B \in \mathfrak{F}]\!] = \mathbb{1}$, since $A \subset A{\uparrow}{\downarrow} \cap X^2 \subset V_B$ and $[\![B \in \mathfrak{F}]\!] = \bigvee_{V\in\mathscr{F}} [\![V{\uparrow} = B]\!] \geqslant [\![V_B{\uparrow} = B]\!] = \mathbb{1}$. Now it is easy to estimate

$$[\![(\forall\, A \in \mathfrak{F})(\forall\, B \subset \mathscr{X}^2)(A \subset B \to B \in \mathfrak{F}]\!]$$
$$= \bigwedge_{A\in\mathscr{F}} \bigwedge \big\{[\![B \in \mathfrak{F}]\!] :\ [\![A{\uparrow} \subset B \subset \mathscr{X}^2]\!] = \mathbb{1}\big\} = \mathbb{1},$$

so that $\mathfrak{F}$ is a filter within $\mathbb{V}^{(\mathbb{B})}$.

Demonstrate that $\mathfrak{F}_0$ is a filterbase of $\mathfrak{F}$. Take an arbitrary entourage $A \in \mathscr{F}$. We will establish that there is an element $B \in \mathbb{V}^{(\mathbb{B})}$ for which $[\![B \in \mathfrak{F}_0]\!] = \mathbb{1}$ and $[\![A{\uparrow} \subset B]\!] = \mathbb{1}$. The last equalities are equivalent to $B \in \mathfrak{F}_0{\downarrow} = \mathrm{mix}\{V{\uparrow} : V \in \mathscr{F}_0\}$ and $\mathrm{mix}(A) \subset B{\downarrow}$. The latter are fulfilled since $\mathscr{F}_0$ is a $\mathscr{B}$-cyclic base (cp. 2.12.A.7 (2)).

Observe that every set $V \in \mathscr{F}_0$ is cyclic (cp. 2.12.A.7 (1)); consequently, $(V \circ V){\uparrow} = V{\uparrow} \circ V{\uparrow}$. Also, the equality $(V^{-1}){\uparrow} = (V{\uparrow})^{-1}$ is true for every $V \subset X$. From here we see that

$$[\![(\forall\, U \in \mathfrak{F})(\exists\, V \in \mathfrak{F}_0) V \circ V \subset U]\!] = \mathbb{1},$$
$$[\![(\forall\, U \in \mathfrak{F})(\exists\, V \in \mathfrak{F})(V^{-1} = V \wedge V \subset U)]\!] = \mathbb{1}.$$

If $[\![(x, y) \in \bigcap(\mathfrak{F}_0)]\!] = \mathbb{1}$ then $(x, y) \in A{\uparrow}{\downarrow} = \mathrm{mix}(A) = A$ for every $A \in \mathscr{F}_0$ (cp. 2.12.A.7 (1)). Since $\mathscr{F}$ is a Hausdorff uniformity, it follows that $x = y$. Thus, $[\![\bigcap \mathfrak{F}^0 = I_{\mathscr{X}}]\!] = \mathbb{1}$; i.e., $\mathfrak{F}$ is a Hausdorff uniformity within $\mathbb{V}^{(\mathbb{B})}$.

Take a mapping φ from the naturals ω onto $\mathscr{F}_0$. Put $\psi(n) := \varphi(n){\uparrow}(n \in \omega)$. Then $\psi{\uparrow}$ is a mapping from $\omega^{\wedge}$ onto $\mathfrak{F}_0$ within $\mathbb{V}^{(\mathbb{B})}$. Since $\psi{\uparrow}(\omega^{\wedge}) = \psi(\omega){\uparrow}$, we have $\mathrm{im}\,(\psi{\uparrow}) = \mathfrak{F}_0$ by 1.6.8 and hence $\mathfrak{F}_0$ is a countable set within $\mathbb{V}^{(\mathbb{B})}$.

Thus, $\mathbb{V}^{(\mathbb{B})} \models [\![(X, \mathfrak{F})$ is a Hausdorff uniform space with a countable base of the uniformity$]\!]$. By the well-known metrization theorem from general topology, the uniformity $\mathfrak{F}$ is generated by some metric p. Put $X := \mathscr{X}{\downarrow}$ and $\rho' := p{\downarrow}$. By 2.12.A.9, (X', ρ') is a Λ-metric space, where $\Lambda = \mathscr{R}{\downarrow}$. It is easy that $U(p, \varepsilon){\downarrow} = U(\rho', \varepsilon)$ for $\varepsilon \in \Lambda^+$, $[\![\varepsilon > 0]\!] = \mathbb{1}$. If ρ is the restriction of the metric ρ' to X then $U(\rho, \varepsilon) = U(\rho', \varepsilon) \cap X^2$; consequently, ρ is the required metric on X. $\triangleright$

2.12.C. Boolean Compactness

In this section we present the notion of a cyclically compact (or mix-compact) set arising as a Boolean valued interpretation of compactness.

2.12.C.1. Suppose that (X, ρ) is a Λ-metric space, $(x_n)_{n\in\mathbb{N}} \subset X$, and $x \in X$. Say that a *sequence* $(x_n)_{n\in\mathbb{N}}$ *approximates* x if $\inf_{n\geqslant k} \rho(x_n, x) = 0$ for all $k \in \mathbb{N}$. Call a set $K \subset X$ mix-*compact* if K is mix-complete and for every sequence $(x_n)_{n\in\mathbb{N}} \subset K$ there is $x \in K$ such that $(x_n)_{n\in\mathbb{N}}$ approximates x. It is clear that in case $\Lambda = \mathbb{R}$ mix-compactness is equivalent to compactness in the metric topology.

2.12.C.2. As is easily seen, mix-compactness is an absolute concept in the following sense: *If X and Y are Λ-metric spaces, X is a decomposable subspace of Y, and $K \subset X$ then the* mix-*compactness of K in X is equivalent to that in Y.*

2.12.C.3. *Suppose that $\mathscr{X}$ is a metric space within $\mathbb{V}^{(\mathbb{B})}$.*

(1) *A subset $K \subset \mathscr{X}{\downarrow}$ is* mix-*compact if and only if K is* mix-*complete and $\mathbb{V}^{(\mathbb{B})} \models$ "$K{\uparrow}$ is a compact subset of $\mathscr{X}$."*

(2) $\mathbb{V}^{(\mathbb{B})} \models$ *"$\mathscr{K}$ is a compact subset of $\mathscr{X}$" if and only if $\mathscr{K}{\downarrow}$ is a* mix-*compact subset of $\mathscr{X}{\downarrow}$.*

$\triangleleft$ (1): The compactness of $K{\uparrow}$ within $\mathbb{V}^{(\mathbb{B})}$ is equivalent to

$$\mathbb{V}^{(\mathbb{B})} \models (\forall\, \sigma : \mathbb{N}^{\wedge} \to K{\uparrow})(\exists\, x \in K{\uparrow})(\forall\, k \in \mathbb{N}^{\wedge}) \\ \inf\left\{p(\sigma(n), x) : \, k \leqslant n \in \mathbb{N}^{\wedge}\right\} = 0.$$

Taking account of 1.6.8 and recalling the equality $\mathrm{cyc}(K) = K$ (cp. 1.6.6), we conclude that the above formula amounts to $(\forall\, s : \mathbb{N} \to K)$ $(\exists\, x \in K)(\forall\, k \in \mathbb{N})\varphi$, where $\varphi := (\mathbb{V}^{(\mathbb{B})} \models \inf\{p(s{\uparrow}(n), x) : \, n \geqslant k^{\wedge}\} = 0)$.

It remains to observe that

$$\varphi \iff \mathbb{V}^{(\mathbb{B})} \models \text{“}\big(\forall\, e \in \mathscr{R}_+\big)\big((\forall\, n \geqslant k^{\wedge})(e \leqslant p(s{\uparrow}(n),x)) \to e = 0\big)\text{”}$$
$$\iff \big(\forall\, e \in \Lambda_+\big)\big((\forall\, n \geqslant k)\big(e \leqslant \rho(s(n),x)\big) \implies e = 0\big)$$
$$\iff \inf\big\{\rho(s(n),x) :\, n \geqslant k\big\} = 0.$$

(2): Put $K := \mathscr{K}{\downarrow}$. If $\mathbb{V}^{(\mathbb{B})} \models$ “$\mathscr{K}$ is a compact subset of $\mathscr{X}$” then, using the obvious mix-completeness of K and applying (1), we conclude that K is a mix-compact subset of $\mathscr{X}{\downarrow}$. Conversely, if K is a mix-compact subset of $\mathscr{X}{\downarrow}$ then, as $K{\uparrow} = \mathscr{K}$, we have $\mathbb{V}^{(\mathbb{B})} \models$ “$\mathscr{K}$ is a compact subset of $\mathscr{X}$” due to (1). $\triangleright$

2.12.C.4. Denote by $\mathrm{Prt}_{\mathbb{N}}(\mathbb{B})$ the set of sequences $\nu : \mathbb{N} \to \mathbb{B}$ that are partitions of unity of the Boolean algebra $\mathbb{B}$. For $\nu_1, \nu_2 \in \mathrm{Prt}_{\mathbb{N}}(\mathbb{B})$, the formula $\nu_1 \ll \nu_2$ abbreviates the following: If $m, n \in \mathbb{N}$ and $\nu_1(m) \wedge \nu_2(n) \neq 0_{\mathbb{B}}$ then $m < n$.

Let (X, ρ) be a mix-complete Λ-metric space. Given a mix-complete subset $K \subset X$, a sequence $s : \mathbb{N} \to K$, and a partition $\nu \in \mathrm{Prt}_{\mathbb{N}}(\mathbb{B})$, put $s_\nu := \mathrm{mix}_{n\in\mathbb{N}}\, \nu(n)s(n)$. A *cyclic subsequence* of $s : \mathbb{N} \to K$ is each sequence of the form $(s_{\nu_k})_{k\in\mathbb{N}}$, where $(\nu_k)_{k\in\mathbb{N}} \subset \mathrm{Prt}_{\mathbb{N}}(\mathbb{B})$ and $\nu_k \ll \nu_{k+1}$ for all $k \in \mathbb{N}$.

A subset $K \subset X$ is called *cyclically compact* if K is mix-complete and each sequence of elements in K admits a cyclic subsequence convergent to an element of K in the metric ρ.

2.12.C.5. *Let X be a* mix-*complete Λ-metric space. A subset $K \subset X$ is cyclically compact if and only if K is* mix-*compact.*

$\triangleleft \implies$: Let K be a cyclically compact subset of X. Consider an arbitrary sequence $s : \mathbb{N} \to K$. By the definition of cyclic compactness there exist a sequence $(\nu_k)_{k\in N} \subset \mathrm{Prt}_{\mathbb{N}}(\mathbb{B})$ and an element $x \in K$ such that $(\forall\, k \in \mathbb{N})(\nu_k \ll \nu_{k+1})$ and $o\text{-}\lim_{k\to 0} \rho(s_{\nu_k}, x) = 0$. The inspection of the latter formulas shows that

$$\inf\big\{\rho(\varkappa, x) :\, \varkappa \in \mathrm{mix}\{s(n) :\, n \geqslant k\}\big\} = 0$$

for all $k \in \mathbb{N}$ and so the sequence s approximates $x \in K$, since for each $\varkappa = \mathrm{mix}_{n\geqslant k}\, \pi_n s(n)$, with $(\pi_n)_{n\geqslant k} \in \mathrm{Prt}_{\mathbb{N}}(\mathbb{B})$, we have

$$\pi_m\Big(\inf_{n\geqslant k} \rho(s(n),x)\Big) \leqslant \pi_m \rho(s(m),x)$$
$$\leqslant \pi_m\rho(s(m),\varkappa) + \pi_m\rho(\varkappa,x) = \pi_m\rho(\varkappa,x) \leqslant \rho(\varkappa,x)$$

for all $m \geqslant k$ and, consequently,

$$\inf_{n \geqslant k} \rho(s(n), x) = \sup_{m \geqslant k} \pi_m \Big(\inf_{n \geqslant k} \rho(s(n), x) \Big) \leqslant \rho(\varkappa, x).$$

$\Longleftarrow$: Suppose now that K is a mix-compact subset of X and let $s : \mathbb{N} \to K$. According to 2.12.B.3 we can assume that X is a decomposable subset of $\mathscr{X}{\downarrow}$, where $\mathbb{V}^{(\mathbb{B})} \models$ "$(\mathscr{X}, p)$ is a metric space." Put $\sigma := s{\uparrow}$. Then $\mathbb{V}^{(\mathbb{B})} \models \sigma : \mathbb{N} \to K{\uparrow}$. Moreover, 2.12.C.2 and 2.12.C.3 (1) imply $\mathbb{V}^{(\mathbb{B})} \models$ "$K{\uparrow}$ is a compact subset of $\mathscr{X}$." Applying the classical compactness criterion within $\mathbb{V}^{(\mathbb{B})}$, consider $x \in K$ and $\mathscr{N} \in \mathbb{V}^{(\mathbb{B})}$ such that

$$\mathbb{V}^{(\mathbb{B})} \models \text{“}\mathscr{N} : \mathbb{N} \to \mathbb{N},\ \mathscr{N}(k) < \mathscr{N}(k+1),$$
$$p(\sigma(\mathscr{N}(k)), x) \leqslant \frac{1}{k} \ \text{ for each } \ k \in \mathbb{N}.$$

Put $\nu_k(n) := [\![\mathscr{N}(k^\wedge) = n^\wedge]\!]$ for all $k, n \in \mathbb{N}$. A routine verification shows that $\nu_k \in \mathrm{Prt}_{\mathbb{N}}(\mathbb{B})$ and $(\forall\, k \in \mathbb{N})\ \nu_k \ll \nu_{k+1}$. Moreover, for each $k \in \mathbb{N}$ we have $\mathbb{V}^{(\mathbb{B})} \models s_{\nu_k} = \sigma(\mathscr{N}(k^\wedge))$ and, consequently, $\rho(s_{\nu_k}, x) \leqslant \frac{1}{k} e$. $\triangleright$

2.12.C.6. Let (X, ρ), Λ, and $\mathscr{E}$ be the same as in 2.11.A.4. Take a projection $\pi \in \mathbb{B}(X)$ and an order unit $e \in \mathscr{E}$. The set $\Theta \subset X$ will be called a (ρ, e, π)-*net* in X if, for every $x \in X$, there is an element $y \in \Theta$ such that $\pi\rho(x, y) \leqslant e$. The next fact is an interpretation of the Hausdorff compactness criterion in a Boolean valued model.

2.12.C.7. *For a decomposable Λ-metric space (X, ρ), the following are equivalent:*

(1) *X is cyclically compact.*

(2) *X is ρo-complete and, for every $e \in \mathscr{E}$, there exist a sequence $(\Theta_n)_{n \in \mathbb{N}}$ of finite subsets $\Theta_n \subset X$ and a countable partition of unity $(\pi_n)_{n \in \mathbb{N}}$ in $\mathbb{P}(\Lambda)$ such that* $\mathrm{mix}(\Theta_n)$ *is a (ρ, e, π_n)-net in X for all $n \in \mathbb{N}$.*

2.13. Comments

2.13.1. (1) In the history of functional analysis, the rise of the theory of ordered vector spaces is commonly ascribed to the contributions of Birkhoff, Freudenthal, Kantorovich, Nakano, Riesz, et al. At present, the theory of ordered vector spaces and its applications constitute a noble branch of mathematics, representing one of the main sections of contemporary functional analysis. The various aspects of vector lattices and positive operators are presented in the monographs by Abramovich and Aliprantis [5]; Akilov and Kutateladze [22]; Aliprantis and Burkinshaw [26, 28]; Kusraev [222, 228]; Kantorovich, Vulikh, and Pinsker [196]; Lacey [275]; Lindenstrauss and Tzafriri [281]; Luxemburg and Zaanen [297]; Meyer-Nieberg [311]; Schaefer [356]; Schwarz [361]; Vulikh [403]; and Zaanen [427]).

(2) The credit for finding the most important instance of ordered vector spaces, a Dedekind complete vector lattice or a Kantorovich space, belongs to Kantorovich. This notion appeared in Kantorovich's first article on this topic [191] where he stated an important methodological principle, the *heuristic transfer principle* for Kantorovich spaces: "the elements of a Dedekind complete vector lattice are generalized numbers."

(3) At the very beginning of the development of the theory, many attempts were made at formalizing the above heuristic principle. These led to the so-called theorems of *relation preservation* which claimed that if some proposition involving finitely many functional relations is proven for the reals then an analogous fact remains valid automatically for the elements of every Dedekind complete vector lattice (cp. [196, 403]). The depth and universality of Kantorovich's principle were demonstrated within Boolean valued analysis.

2.13.2. (1) The Boolean valued status of the concept of Kantorovich space (= Dedekind complete vector lattice) is established by Gordon's Theorem obtained in Gordon [133]. This fact can be interpreted as follows: *A universally complete vector lattice is the interpretation of the reals in an appropriate Boolean valued model.* Moreover, it turns out that each theorem on the reals (in the framework of ZFC) has an analog for the corresponding Dedekind complete vector lattice. The theorems are transferred by means of the precisely-defined procedures: *ascent, descent,* and *canonical embedding,* that is, algorithmically as a matter of fact. Descending the basic scalar fields opens a turnpike to the intensive application of Boolean valued models in functional analysis. The technique of Boolean valued analysis demonstrates its efficiency in studying Banach spaces and algebras as well as lattice normed spaces and modules. The corresponding results are collected and elaborated in Kusraev and Kutateladze [249, Chapters 10–12].

(2) If if μ is a Maharam measure and $\mathbb{B}$ in Theorem 2.2.4 is the algebra of all μ-measurable sets modulo sets of μ-measure zero, then $\mathscr{R}{\downarrow}$

is isomorphic to the universally complete vector lattice $L^0(\mu)$ of (cosets) measurable functions. This fact (for the Lebesgue measure on an interval) was already known to Scott and Solovay [368]. If $\mathbb{B}$ is a complete Boolean algebra of projections in a Hilbert space H then $\mathscr{R}{\downarrow}$ is isomorphic to the space of all selfadjoint operators A on H admitting spectral resolution $A = \int \lambda\, dE_\lambda$ with $E_\lambda \in \mathbb{B}$ for all $\lambda \in \mathbb{R}$. The two indicated particular cases of Gordon's Theorem were intensively and fruitfully exploited by Takeuti [379, 380, 381].

(3) The object $\mathscr{R}{\downarrow}$ for general Boolean algebras was also studied by Jech [180, 181, 182] who in fact rediscovered Gordon's Theorem. The difference is that in [184] a universally complete (complex) vector lattice with unit element is defined by another system of axioms and is referred to as a complete Stone algebra. The contemporary forms of the above mentioned relation preservation theorems, basing on Boolean valued models, may be found in Gordon [135] and Jech [181] (cp. also [248]).

(4) The forcing method splits naturally into the two parts: one is general and the other, special. The general part comprises the apparatus of Boolean valued models of set theory; i.e., the construction of a Boolean valued universe $\mathbb{V}^{(\mathbb{B})}$ and interpretation of the set-theoretic propositions in $\mathbb{V}^{(\mathbb{B})}$. Here, a complete Boolean algebra $\mathbb{B}$ is arbitrary. The special part consists in constructing particular Boolean algebras $\mathbb{B}$ providing some special (usually, pathological and exotic) properties of the objects (for example, Dedekind complete vector lattices) obtained from $\mathbb{V}^{(\mathbb{B})}$. Both parts are of interest in their own right, but the most impressive results stem from their combination. In this book, like in the most part of research in Boolean valued analysis, we primarily use the general part of the forcing method, using in some places cardinal collapsing phenomena. The special part is widely employed for proving independence or consistency (cp. Bell [43], Dales and Woodin [101], Jech [184], Rosser [350], Takeuti and Zaring [388]).

2.13.3. (1) Theorems 2.3.2 and 2.3.4 are companions for Gordon's Theorem from the very beginning of Boolean values analysis. The complex structure of $\mathscr{C}{\downarrow}$ was intensively employed by Takeuti [380]–[384] and multiplication on $\mathscr{R}{\downarrow}$ was examined by Gordon [134, 136, 137]. Sometimes it is useful to consider another companions of the Gordon Theorem treating *quaternions* and *octonions*.

(2) Let $\mathbb{H}$ be the quaternion algebra and let $\mathbb{O}$ be the Cayley algebra. Recall that the Cayley algebra is an 8-dimensional algebra over $\mathbb{R}$ which is noncommutative and nonassociative, and the elements of $\mathbb{O}$ are

Cayley numbers or *octonions*. Then $[\![\, \mathbb{H}^\wedge$ and $\mathbb{O}^\wedge$ are normed algebras over the field $\mathbb{R}^\wedge]\!] = \mathbb{1}$. Let $\mathscr{H}$ and $\mathscr{O}$ stand for the norm completions respectively of $\mathbb{H}^\wedge$ and $\mathbb{O}^\wedge$ within $\mathbb{V}^{(\mathbb{B})}$. It is easy to show (using, for example, the Hurwitz Theorem) that $[\![\, \mathscr{H}$ is the quaternion algebra $]\!] = \mathbb{1}$ and $[\![\mathscr{O}$ is the Cayley algebra $]\!] = \mathbb{1}$. If Q denotes the Stone space of $\mathbb{B}$ then the descents (restricted descents) of $\mathscr{H}$ and $\mathscr{O}$ can be described as $C_\infty(Q, \mathbb{H})$ and $C_\infty(Q, \mathbb{O})$ ($C(Q, \mathbb{H})$ and $C(Q, \mathbb{O})$), respectively. These objects occur in classification and representation theory of Jordan operator algebras; see Ajupov [15, 16]; Hanshe-Olsen and Störmer [165].

(3) Subsections 2.3.5–2.3.7 can be considered as analytical versions of Shoenfield type absoluteness theorems (see Takeuti [380, Lemma 2.7]): *Let $\mathbb{B}_0$ be a complete subalgebra of $\mathbb{B}$ and let $\mathscr{R}$ be the reals within $\mathbb{V}^{(\mathbb{B}_0)}$. If $u_1, \dots, u_n \in \mathscr{R}{\downarrow}$ and φ is a* ZFC*-formula of the class Σ_2^1 or Π_2^1 then* $[\![\varphi(u_1, \dots, u_n)]\!]^{\mathbb{B}} = [\![\varphi(u_1, \dots, u_n)]\!]^{\mathbb{B}}$.

2.13.4. The interconnections between the properties of numerical objects in $\mathscr{R}$ and the corresponding objects in the universally complete vector lattice $\mathscr{R}{\downarrow}$, indicated in 2.4.3–2.4.7 were actually obtained by Gordon [135, Section 3] for countable sets and sequences. The general case in Section 2.3 is treated by repeating essentially the same argument. Proposition 2.4.9, allowing us to translate into the internal language of $\mathscr{R}$ the claims about traces and characteristics in $\mathscr{R}{\downarrow}$, was also established by Gordon; see [133, Theorem 3] and [135, Theorem 3]. These results underlie the technology of translating the knowledge about numbers to theorems on the elements, sequences, and subsets of universally complete vector lattices.

2.13.5. (1) The Representation Theorem 2.5.1 is due to Kusraev [223]. A close result (in other terms) is presented in Jech's article [182], where the Boolean valued interpretation is developed for the theory of linearly ordered sets. Corollaries 2.5.7 and 2.5.8 are well known (cp. Kantorovich, Vulikh, and Pinsker [196] and Vulikh [403]). Some subsystems of the reals $\mathscr{R}$ appear not only as the Boolean valued representation of Archimedean vector lattices.

(2) For instance, the following assertions were stated in Kusraev [223] and proved in Kusraev and Kutateladze [244, 248]: (a) a Boolean valued representation of an Archimedean lattice-ordered group is a subgroup of the additive group of $\mathscr{R}$; (b) an Archimedean f-ring includes two complementary bands one of which has the zero multiplication and is realized as in (a) and the other, as a subring of $\mathscr{R}$; (c) an Archimedean

f-algebra contains two complementary bands one of which is realized as in 2.5.1 and the other, as a sublattice and subalgebra of $\mathscr{R}$ considered as a lattice-ordered algebra over $\mathbb{R}^\wedge$ (also see Jech [182]).

2.13.6. The tests of 2.6.2 (2, 5) for o-convergence (in the case of sequences) were obtained by Kantorovich and Vulikh (cp. [196]). As is seen from 2.6.1, these tests are merely the interpretation of convergence properties of numerical nets (sequences).

2.13.7. (1) The Spectral Theorem 3.7.7 was proved by Freudenthal [127]. It remains true for vector lattices with the principal projection property (see Veksler [395]; Luxemburg and Zaanen [297]). Then Veksler introduced slightly different concepts of weak and strong Freudenthal properties for general vector lattices and characterized them by the corresponding projection properties; see [395, Theorems 2.3, 2.5, 2.8–2.10].

(2) The weak and strong Freudenthal properties in the sense of 2.7.8 were introduced and studied by Lavrič [277]. To characterize the spaces with strong Freudenthal property we need the definition. Two elements of a vector lattice X are *completely disjoint* if they lie in two disjoint projection bands of X. The following was proved in Lavrič [277]: *A vector lattice X has the strong Freudenthal property if and only if every two disjoint elements in X are completely disjoint. A vector lattice X has the weak Freudenthal property if and only if for every two elements $e, d \in X$ there are disjoint elements $e_0 \in [0, e]$ and $d_0 \in [0, d]$ such that $X(e + d) = X(e_0 + d_0)$.*

(3) The Boolean valued proofs of the Freudenthal Spectral Theorem, as well as the properties 2.7.2 and 2.7.3 (1) were given by Gordon [135], while 2.7.3 (2), 2.7.4, and 2.7.5 are collected in Kusraev and Kutateladze [248, 249]. Of course, these formulas as well as the estimates in 2.7.6 were mostly known and employed by various authors; see for example the papers of Nakano [318], Vulikh [403], Luxemburg and de Pagter [294].

(4) The formulas for e_x similar to those of 5.7.3 (1, 2), 5.7.4 (2, 3), and 5.7.5 (1–3) are trivial: $e_x = e_{|x|} = e_{\alpha x}$ $(0 \neq \alpha \in \mathbb{R})$; $e_{x\wedge y} = e_{xy} = e_x \wedge e_y$ and $e_{x\vee y} = e_{x+y} = e_x \vee e_y$ $(x \geqslant 0,\ y \geqslant 0)$; $e_{xy} = e_x \wedge e_y$ $(x, y \in X$ arbitrary). To ensure the latter we need only to interpret within $\mathbb{V}^{(\mathbb{B})}$ the simple proposition $(\forall\, s, t \in \mathscr{R})(st \neq 0 \leftrightarrow s \neq 0 \wedge t \neq 0)$ and apply 5.4.9.

2.13.8. The fact that for a complete Boolean algebra $\mathbb{B}$ the set $\mathfrak{S}(\mathbb{B})$ of spectral functions is a universally complete vector lattice with the

Boolean algebra of bands isomorphic to $\mathbb{B}$ (cp. 2.8.3) is due to Kantorovich [196]. The claim of 2.8.4 was obtained by Pinsker (cp. [196]). The representation of an arbitrary Dedekind complete vector lattice as an order dense ideal in $C_\infty(Q)$ was established independently from one another by Vulikh and Ogasawara (cp. [196, 403]).

2.13.9. (1) The starting point of the theory of spectral measures was von Neumann's classical theorem: *Each normal operator on a Hilbert space admits a spectral resolution with commutable orthogonal projections.* By the classical definition, a spectral measure is a Boolean homomorphism of a Boolean algebra of sets to the Boolean algebra of projections (cp. Dunford and Schwartz [112]). If need be, the countable additivity condition or some regularity requirements are added. Motivated by spectral theory, much effort has been made to extend the spectral theory of hermitian operators on a Hilbert space to Banach spaces. The third part of the Dunford and Schwartz treatise [112] is devoted to the corresponding theory of spectral operators. Recall that an operator T is called *spectral* if there is a spectral measure P on the Borel sets of the complex plane such that $P(A)T = TP(A)$ for all $A \in \mathscr{B}or(\mathbb{C})$ and the spectrum of $P(A)T|_{P(A)}$ lies in the closure of A.

(2) The Bade Reflexivity Theorem tells us that a bounded linear operator T on a Banach space X belongs to the strongly closed algebra generated by a σ-complete Boolean algebra $\mathbb{B}$ of projections on X if and only if T keeps invariant every $\mathbb{B}$-invariant subspace of X (cp. Bade [39]). Schaefer [354] discovered the key role that is played by order in abstracting the method of spectral measures and Bade's reflexivity results to locally convex spaces. This article has started the systematic study of the operator algebras generated by Boolean algebras of projections within the theory of Riesz spaces; see Dodds and de Pagter [106, 107]; Dodds and Ricker [109]; and Dodds, de Pagter, and Ricker [108].

2.13.10. (1) The Borel functions of an element of an arbitrary Dedekind complete vector lattice with unit seem to be first considered by Sobolev (see [367] and also [403]). Theorems 2.10.4 and 2.10.5 in the above generality were obtained by Kusraev and Malyugin [252, 254]. Some Boolean valued proof of Theorem 2.10.4 is also given by Jech in [180]. Further details are available in the books by Kusraev and Kutateladze [244, 248].

(2) Kusraev and Malyugin constructed in [254] the Borel functional calculus of countable and uncountable collections of elements of

Dedekind complete vector lattices. The following was proved in particular: *Let X be a universally complete vector lattice with unit $\mathbb{1}$ and let $\mathbf{x} := (x_k)_{k\in\mathbb{N}}$ be an arbitrary sequence in X. There exists a unique sequentially order continuous f-algebra homomorphism $\widehat{\mathbf{x}}$ from $\mathscr{B}(\mathbb{R}^{\mathbb{N}}, \mathbb{R})$ to X such that $\mathbf{x}(dt_k) = x_k$ for all $k \in \mathbb{N}$.*

(3) From 2.10.2 it follows that: *For every resolution of the identity $(e_\alpha)_{\alpha\in\mathbb{R}}$ with range in a σ-algebra $\mathbb{B}$ there is a unique spectral measure $\mu : \mathscr{B}or(\mathbb{R}) \to B$ satisfying $\mu((-\infty, \alpha)) = e_\alpha$ $(\alpha \in \mathbb{R})$.* This fact was firstly revealed by Sobolev in [367]. But the extension method that led to 2.10.2 differs significantly from the Carathéodory extension and bases on the Loomis–Sikorski representation of Boolean σ-algebras (co. 2.10.1).

(4) Under the assumptions of Theorem 2.10.5, for each $e \in \mathbb{C}(\mathbb{1})$ we have $ef(x_1, \dots, x_N) = f(ex_1, \dots, ex_N) + e^*f(0, \dots, 0)$ (cp. Kantorovich, Vulikh, and Pinsker [196, Proposition 3.14]). Indeed, if $e\mathfrak{x} := (ex_1, \dots, ex_N)$, $\mu = \mu_{\mathfrak{x}}$, $\bar\mu := \mu_{e\mathfrak{x}}$, and μ_0 is the $\{0,1\}$-valued measure on $\mathscr{B}(\mathbb{R}^N, \mathbb{R})$ with support $\{0\}$ then $e\mu = \bar\mu + e^*\mu_0$ by the definition of $\mu_{\mathfrak{x}}$ in 2.10.3 and 2.7.5 (4); therefore, $f(x_1, \dots, x_N) = I_\mu(f) = I_{\bar\mu}(f) + I_{\mu_0}(f) = f(ex_1, \dots, ex_N) + e^*f(0, \dots, 0)$.

(5) If the function f in Theorem 2.10.5 is positive homogeneous ($f(\lambda t) = \lambda f(t)$ for $\lambda \in \mathbb{R}_+$ and $t \in \mathbb{R}^N$) then $f(x_1, \dots, x_N)$ do not depend on the choice of an order unit $\mathbb{1} \in X$. This fact was first observed by Riesz [347] and Vulikh [196, Theorema 3.54]. Homogeneous functional calculus in uniformly complete vector lattices stems from Lozanovskiĭ [290] and Krivine [210]; see further development in Buskes, de Pagter, and van Rooij [79], Lindenstrauss and Tzafriri [281], Szulga [374]. Concerning various generalizations of the homogeneous functional calculus see Haydon, Levy, and Raynaud [170], Kusraev [237], Kusraev and Kutateladze [250], Tasoev [389].

2.13.11. (1) Vector lattices within Boolean valued models were first considered by Gordon [134]; Theorems 2.11.4 and 2.11.6 are essentially contained in [134, Theorem 1].

(2) The f-module structure is inevitable in the theory of order bounded operators, since $L^\sim(X, Y)$ is always an f-module over the f-algebra $\mathrm{Orth}(Y)$. Nevertheless, f-modules were introduced and studied in Luxemburg and de Pagter [293] more than sixty years after Kantorovich had started a systematic study of order bounded operators [191, 193].

2.13.12. (1) In 1934 Kurepa introduced the so-called *espaces pseudodistanciès*, i.e. the spaces metrized by means of an ordered vector space

[212]. Soon after that Kantorovich developed the theory of abstract normed spaces; i.e., the vector spaces with a norm that takes values in an order complete vector lattice [193]. These objects turned out very useful in the study of functional equations by successive approximations (cp. Kantorovich [192, 194]) and in the related areas of analysis (cp. Kantorovich, Vulikh, and Pinsker [196], Kollatz [203], and Kusraev [222]). Lattice normed spaces (Kusraev [228]) and randomly normed spaces (Haydon, Levy, and Raynaud [170]) are special examples of spaces with lattice valued metric. At the same time their structural properties never gained adequate research. Metrization by means of a semifield (= a kind of a vector lattice) was studied also by several authors in a series of articles by Uzbekistan mathematicians (see Antonovskiĭ, Boltjanskiĭ, and Sarymsakov [32]).

(2) The claim of 2.12.B.1 was justified in Kusraev [225]. The proof employs the following simple idea: By Gordon's Theorem, metrization by a Dedekind complete vector lattice is nothing else but the usual metrization (i.e. by means of the reals) in the corresponding Boolean valued model. Successive implementation of this idea results in the notion of Boolean algebra of components of uniformity which reflects the main structural peculiarity of the uniformities metrizable by order complete vector lattices.

(3) The concept of cyclical compactness was first studied by Kusraev [216, 222]. Section 8.5 in Kusraev [228] deals with the cyclically compact linear operators on $\mathbb{B}$-cyclic Banach spaces and Kaplansky–Hilbert modules. Recently Gönüllü [146]–[148] undertook the study of Schatten type classes of operators (which are cyclically compact) on Kaplansky–Hilbert modules.

(4) The notion of mix-compact subset of lattice normed space was introduced in Gutman and Lisovskaya [152]. Basing on Boolean valued analysis, they prove the analogs of the three classical theorems for arbitrary lattice normed spaces over universally complete Riesz spaces, namely, the boundedness principle, the Banach–Steinhaus Theorem, and the uniform boundedness principle for a compact convex set; see [152, Theorems 2.4, 2.6, 3.3]. These theorems generalize the analogous results by Ganiev and Kudajbergenov [128] which were established for Banach–Kantorovich spaces over the lattice of measurable functions by the specific technique of the theory of measurable Banach bundles with lifting (see Gutman [158, 160] and Kusraev [228]).

(5) Take an arbitrary metric space (X, p). Then, $(X^\wedge, p^\wedge)$ is a metric space within $\mathbb{V}^{(\mathbb{B})}$. If τ is the topology on X generated by the metric p then $[\![\tau^\wedge$ is a base of the topology on $X^\wedge$ generated by the metric $\rho^\wedge]\!] = \mathbb{1}$. Let $(\mathscr{X}, \rho)$ denote the completion of the metric space $(X^\wedge, \rho^\wedge)$ within $\mathbb{V}^{(\mathbb{B})}$. The elements of $\mathscr{X}$ are the minimal Cauchy filters identifiable with the mappings $\tau^\wedge : \mathscr{X} \to \{0, 1\}$. Thus, with every $\varphi \in \mathscr{X}{\downarrow}$ we uniquely associate the Cauchy $\mathbb{B}$-filter $\bar{\varphi}$ by the formula $\bar{\varphi}(V) := [\![V^\wedge \in \varphi]\!] = [\![\varphi(V^\wedge) = 1]\!]$ $(V \in \tau)$.

The mapping $\varphi \to \bar{\varphi}$ $(\varphi \in \mathscr{X}{\downarrow})$ is an isometric bijection from $\mathscr{X}{\downarrow}$ onto $\mathbb{B}^X$. If (X, ρ) is complete then the mapping that sends a $\mathbb{B}$-filter $\varphi \in \mathbb{B}^X$ to the function $f : q \to \lim \varphi^{-1}(q)$ $(q \in Q)$, $\varphi^{-1}(q) := \{\varphi^{-1}(V) : V \in q\}$ determines an isometric bijection of $\mathbb{B}^X$ onto $C_\infty(Q, X)$. Isometry is understood in the sense of Λ-valued metrics (see 2.12.A.10 and 2.12.A.11).

(6) The Boolean extensions $\mathbb{B}^X$ of general uniform structures was studied in a series of articles by Gordon and Lyubetskiĭ (see [134, 139, 140]). Boolean extensions of locally compact abelian groups as well as the corresponding harmonic analysis were developed by Takeuti [380, 381]. Other interesting results on the structure of Boolean extensions can also be found in the above articles.

CHAPTER 3

Order Bounded Operators

The aim of the three subsequent chapters is to apply Boolean valued analysis to order bounded operators and establish some variants of the *Boolean valued transfer principle* from functionals to operators. The presentation below is rather transparent as we use the well-developed technique of "nonstandard scalarization." This technique implements the Kantorovich heuristic principle and reduces operator problems to the case of functionals. The principal scheme works as follows:

First, we establish that some class of operators $\mathbb{T}$ admits a Boolean valued representation $\mathscr{T}$ which turns out to be a Boolean valued class of functionals. More precisely, we prove that each operator $T \in \mathbb{T}$ embeds into an appropriate Boolean valued model $\mathbb{V}^{(\mathbb{B})}$, becoming a functional $\tau \in \mathscr{T}$ within $\mathbb{V}^{(\mathbb{B})}$. Then the Boolean valued transfer principle tells us that each theorem about τ within Zermelo–Fraenkel set theory has its counterpart for the original operator T interpreted as the Boolean valued functional τ. Translation of theorems from $\tau \in \mathbb{V}^{(\mathbb{B})}$ to $T \in \mathbb{V}$ is carried out by the Boolean valued ascending–descending machinery together with principles of Boolean valued analysis.

This chapter focuses on the structure of disjointness preserving operators and some related concepts. To save room, using the facts of vector lattice theory we will accept the terminology and notation of Aliprantis and Burkinshow [28] and Meyer-Nieberg [311].

3.1. Positive Operators

This section collects some basic facts on positive operators that we need in what follows.

3.1.1. Let X and Y be vector lattices. A linear mapping T from X to Y is called a *positive operator* if T carries positive vectors to positive vectors; in symbols, $0 \leqslant x \in X \Longrightarrow 0 \leqslant Tx \in Y$ or $T(X_+) \subset Y_+$. An operator T is said to be *regular* if T can be written as a difference of two positive operators and *order bounded* or shortly *o-bounded* provided that T sends each order bounded subset of X to an order bounded subset of Y. We will often omit the indication to linearity if this is implied by context.

Let $L(X,Y)$ stand for the space of all linear operators from X to Y. The sets of all regular, order bounded, and positive operators from X to Y are denoted by $L^r(X,Y)$, $L^\sim(X,Y)$, and $L_+(X,Y) := L^\sim(X,Y)_+$, respectively. Clearly, $L^r(X,Y)$ and $L^\sim(X,Y)$ are vector subspaces of $L(X,Y)$ and $L_+(X,Y)$ is a convex cone in $L(X,Y)$.

The order on the spaces of regular and order bounded operators is induced from the cone of positive operators $L_+(X,Y)$; i.e.,

$$T \geqslant 0 \iff T \in L_+(X,Y), \quad S \geqslant T \iff S - T \geqslant 0.$$

3.1.2. A linear operator $T \in L(X,Y)$ is said to be *dominated* by a positive operator $S \in L(X,Y)$ provided that $|Tx| \leqslant S(|x|)$ for all $x \in X$. In this event S is called a *dominant* or *majorant* of T. A positive operator T is dominated by itself; i.e., $|Tx| \leqslant T(|x|)$ for all $x \in X$.

(1) *A linear operator T is dominated if and only if T is regular.*

$\lhd$ Indeed, if S is a dominant of T then $T = S-(S-T)$, while $(S-T)$ and S are positive. If $T = S - R$ for some positive $S, R \in L(X,Y)$ then $Tx \leqslant |Sx| + |Rx| \leqslant (S+R)(|x|)$; i.e., $S+R$ is a dominant of T. $\rhd$

(2) *Let $T : X \to Y$ be a regular operator and let S be a dominant of T. If a net (x_α) converges to x in X with regulator $e \in X_+$ then (Tx_α) converges to Tx with regulator Se. In particular, every regular operator is r-continuous.*

$\lhd$ Assume that $|x_\alpha - x| \leqslant \lambda_n e$ for $\alpha \geqslant \alpha(n)$, where $e \in X_+$ and $\lim_{n\to\infty} \lambda_n = 0$. Then for each dominant S of T we have

$$|Tx_\alpha - Tx| \leqslant S(|x_\alpha - x|) \leqslant \lambda_n Se \qquad \big(\alpha \geqslant \alpha(n)\big),$$

which implies the convergence of (Tx_α) to Tx with regulator Se. $\rhd$

(3) Kantorovich Lemma. *Let X be a vector lattice, and let Y be an arbitrary real vector space. Assume that U is an additive and positive homogeneous mapping from X_+ to Y; i.e., $U : X_+ \to Y$ satisfies the conditions:*

$$U(x+y) = Ux + Uy, \quad U(\lambda x) = \lambda Ux \quad (0 \leqslant \lambda \in \mathbb{R};\ x, y \in X_+).$$

Then U has the unique linear extension T on the whole lattice X. Moreover, if Y is a vector lattice and $U(X_+) \subset Y_+$ then T is positive.

$\lhd$ Define T *by differences*: $Tx := Ux^+ - Ux^-$ $(x \in X)$. Then T is a sought extension whose uniqueness is obvious from the representation $x = x^+ - x^-$ (cp. 3.1.2 (1)). $\rhd$

We now formulate the celebrated Riesz–Kantorovich Theorem.

3.1.3. Riesz–Kantorovich Theorem. *Let X and Y be vector lattices with Y Dedekind complete. The set $L^\sim(X,Y)$ of all order bounded linear operators from X to Y, ordered by the cone of positive operators $L^\sim(X,Y)_+$, is a Dedekind complete vector lattice. In particular,*

$$L^\sim(X,Y) = L^r(X,Y).$$

Definitions 3.1.1 make it clear that every positive operator is order bounded. Consequently, so is the difference of positive operators. In other words, every regular operator is order bounded. The converse fails in general, holding in case that Y is Dedekind complete, as follows from the Riesz–Kantorovich Theorem.

3.1.4. The proof of the Riesz–Kantorovich Theorem yields the formulas for presenting the lattice operations on $L^\sim(X,Y)$ by pointwise calculations. The collection of these formulas is usually called the *calculus of order bounded operators* or shortly *order calculus*. We will exhibit the main formulas of order calculus below.

Let X and Y be the same as above. For all $S,T \in L^\sim(X,Y)$ and $x \in X_+$ the following hold:

(1) $(S \vee T)x = \sup\{Sx_1 + Tx_2 :\ x_1, x_2 \geqslant 0,\ x = x_1 + x_2\}$.

(2) $(S \wedge T)x = \inf\{Sx_1 + Tx_2 :\ x_1, x_2 \geqslant 0,\ x = x_1 + x_2\}$.

(3) $S^+x = \sup\{Sy :\ 0 \leqslant y \leqslant x\}$.

(4) $S^-x = \sup\big\{Sy :\ -x \leqslant y \leqslant 0\big\} = -\inf\{Sy :\ 0 \leqslant y \leqslant x\}$.

(5) $|S|x = \sup\{|Sy| :\ |y| \leqslant x\}$.

(6) $|S|x = \sup\big\{\sum_{k=1}^n |Sx_k| :\ x_1, \dots, x_n \geqslant 0,\ x = \sum_{k=1}^n x_k,\ n \in \mathbb{N}\big\}$.

(7) $|Sx| \leqslant |S|(|x|)$ $(x \in X)$.

3.1.5. An operator $T : X \to Y$ between vector lattices is said to be *order continuous,* provided that, for every net (x_α) in X order convergent to $x \in X$, the net (Tx_α) order converges to Tx in Y. Say that T is *sequentially order continuous*, if for every sequence (x_n) in X with order limit $x \in X$, the sequence (Tx_n) is order convergent to Tx in Y. It is useful to note that a positive operator T is (sequential) order continuous if and only if $\inf(T(A)) = 0$ in Y for an arbitrary (countable) downward directed set $A \subset X$ with $\inf(A) = 0$.

The collections of all order bounded order continuous operators and order bounded sequentially order continuous operators from X to Y will be denoted by $L_n^{\sim}(X, Y)$ and $L_c^{\sim}(X, Y)$, respectively. (Note that every order continuous operator is order bounded; see Aliprantis and Burkinshaw [28, Lemma 1.54].)

3.1.6. Ogasawara Theorem. *If X and Y are vector lattices with Y Dedekind complete then $L_n^{\sim}(X, Y)$ and $L_c^{\sim}(X, Y)$ are bands of $L^{\sim}(X, Y)$.*

$\lhd$ See Aliprantis and Burkinshaw [28, Theorem 1.57]. $\rhd$

The following results tells us that the classical Hahn–Banach Theorem remains valid for operators if we take Dedekind complete vector lattice as a range space. An operator p from a (real) vector space V to an ordered vector space Y is called *sublinear* whenever $p(u+v) \leqslant p(u)+p(v)$ and $p(\lambda v) = \lambda p(v)$ for all $u, v \in V$ and all $0 \leqslant \lambda \in \mathbb{R}$.

3.1.7. Hahn–Banach–Kantorovich Theorem. *Let V be a (real) vector space, let Y be a Dedekind complete vector lattice, and let $p : V \to Y$ be a sublinear operator. If U is a vector subspace of V and $S : U \to Y$ is a linear operator satisfying $S(u) \leqslant p(u)$ for all $u \in U$, then there exists some linear operator $T : V \to Y$ such that $Tu = Su$ $(u \in U)$ (i.e., T is a linear extension of S to all of V) and $T(v) \leqslant p(v)$ for all $v \in V$.*

Now we present several extension results for positive operators which will be needed in what follows.

3.1.8. Theorem. *Let X_0, X, and Y be vector lattices with Y Dedekind complete and X_0 a vector sublattice in X. Assume that $S_0 : X_0 \to Y$ and $T : X \to Y$ are positive operators and $S_0 x \leqslant Tx$ for all $0 \leqslant x \in X_0$. Then there exists a positive operator $S : X \to Y$ extending S_0 and satisfying $S \leqslant T$.*

$\lhd$ If $p(x) := T(x^+)$ $(x \in X)$ then p is a sublinear operator from X to Y and $S_0 x \leqslant p(x)$ for all $x \in X_0$. By the Hahn–Banach–Kantorovich Theorem there exists a linear operator $S : X \to Y$ extending S_0 and satisfying $Sx \leqslant p(x)$ for all $x \in X$. The latter implies that $0 \leqslant S \leqslant T$. $\rhd$

3.1.9. Theorem. *Let X_0, $\widehat{X}$, and Y be vector lattices with Y Dedekind complete and X_0 an order ideal in $\widehat{X}$. Assume that $T_0 : X_0 \to Y$ is a positive operator and define X as the set of all $x \in \widehat{X}$ for which $T_0([0,|x|] \cap X_0)$ is order bounded in Y. Then X is an order ideal in $\widehat{X}$ including X_0 and there exists a positive extension T of T_0 to all of X such that $T \leqslant S$ for every positive extension S of T_0 to X. Moreover, T is order continuous if so is T_0.*

$\lhd$ Define the operator $T : X_+ \to Y$ as $Tx := \sup\big\{T_0 u : \ u \in X_0,\ 0 \leqslant u \leqslant x\big\}$ for all $x \in X_+$. Then T is additive and positive homogeneous, so it can be extended to X by differences (cp. 3.1.2 (3)). The resulting operator satisfies the desired conditions and is called the *least extension* of T_0 (cp. Aliprantis and Burkinshaw [28, Theorem 1.30] and Kusraev [228, Propositions 3.1.3 (1, 2)]). $\rhd$

3.1.10. Theorem. *Let X_0 be an order dense majorizing vector sublattice of a vector lattice X, and let Y be a Dedekind complete vector lattice. If $T_0 : X_0 \to Y$ is a positive order continuous operator then there exists a unique order continuous linear extension $T : X \to Y$ of T_0 to all of X.*

$\lhd$ The required extension $T : X \to Y$ is defined first on X_+ as

$$T(x) := \sup\{T(x_0) : \ x_0 \in X_0 \text{ and } 0 \leqslant x_0 \leqslant x\} \quad (x \in X_+)$$

and then T is extended to the whole of X by differences; see 3.1.2 (3). More details can be found in Aliprantis and Burkinshaw [28, Theorem 1.65]. $\rhd$

3.1.11. Let X and Y be vector lattices. For a linear operator T from X to Y the following are equivalent:

(1) $T(x \vee y) = Tx \vee Ty$ $(x, y \in X)$.

(2) $T(x \wedge y) = Tx \wedge Ty$ $(x, y \in X)$.

(3) $x \wedge y = 0 \implies Tx \wedge Ty = 0$ $(x, y \in X)$.

(4) $T(x^+) = (Tx)^+$ $(x \in X)$.

(5) $T(|x|) = |Tx|$ $(x \in X)$.

(6) $[0, T] = [0, I_Y] \circ T$.

3.1.12. Note that 3.1.11 (1) means by definition that T is a lattice homomorphism. So T is a lattice isomorphism whenever T enjoys one (and hence all) of the properties listed in 3.1.11.

An injective lattice homomorphism from X to Y is called a *lattice* (more exactly *order*) *monomorphism*, or an *isomorphic embedding*, or even a *lattice isomorphism* from X to Y.

If a lattice homomorphism $T : X \to Y$ is one-to-one then X and Y are called *lattice* (or *order*) *isomorphic*. The same is worded also as follows: T is an *order isomorphism* between X and Y. The set of all lattice homomorphisms from X to Y is denoted by $\mathrm{Hom}(X, Y)$.

3.1.13. Theorem. *Let X and Y be vector lattices with Y Dedekind complete. If X_0 is a majorizing vector sublattice of X and $T_0 : X_0 \to Y$ is a lattice homomorphism, then there exists a lattice homomorphism $T : X \to Y$ extending T_0.*

$\lhd$ See Aliprantis and Burkinshaw [28, Theorem 2.29] and Kusraev [228, Proposition 3.3.11 (2)]. $\rhd$

3.1.14. Consider a vector lattice X. An order bounded linear operator $\pi : X \to X$ is an *orthomorphism* in X whenever for all $x, y \in X$ from $x \perp y$ it follows that $Tx \perp y$. The set of all orthomorphisms in X is denoted by $\mathrm{Orth}(X)$.

Clearly, $\mathrm{Orth}(X)$ is a vector subspace of $L^\sim(X)$ which we will consider with the order induced from $L^\sim(X)$. In case that X is a Dedekind complete vector lattice, $\mathrm{Orth}(X)$ coincides with the band of $L^\sim(X)$ which is generated by the identity operator I_X.

For more details on orthomorphisms see de Pagter [327] and Zaanen [427]. Some special properties of orthomorphisms will be addressed in Chapter 4.

3.2. Bilinear Operators

In this section we introduce the classes of bilinear operators on products of vector lattices. The main purpose is to agree on notation and terminology and give a brief outline of some useful facts. For an extended discussion of this subject see the survey paper Bu, Buskes, and Kusraev [72].

3.2.1. Let X, Y, and Z be vector lattices. A bilinear operator B from $X \times Y$ to Z is called *positive* if $B(x, y) \geqslant 0$ for all $0 \leqslant x \in X$ and $0 \leqslant y \in Y$. This amounts to saying that the linear operators

$$
\begin{aligned}
B(u, \cdot) &: y \mapsto B(u, y) \quad (y \in Y),\\
B(\cdot, v) &: x \mapsto B(x, v) \quad (x \in X)
\end{aligned}
$$

are positive for all $0 \leqslant u \in X$ and $0 \leqslant v \in Y$. Given a positive bilinear operator B, we have

$$|B(x,y)| \leqslant B(|x|,|y|) \quad (x \in X,\ y \in Y).$$

A bilinear operator is called *order bounded* if it sends order bounded sets in $X \times Y$ to order bounded sets in Z, and *regular* if it can be represented as the difference of two positive bilinear operators. Denote by $BL^r(X,Y;Z)$ and $BL_+(X,Y;Z)$ the sets of all regular and positive bilinear operators from $X \times Y$ to Z.

3.2.2. A bilinear operator $B : X \times Y \to Z$ is said to be *of order bounded variation* if

$$\Sigma B[x;y] := \Bigg\{ \sum_{k=1}^{n} \sum_{l=1}^{m} |B(x_k, y_l)| :\ 0 \leqslant x_k \in X\ (1 \leqslant k \leqslant n \in \mathbb{N}),$$
$$0 \leqslant y_l \in X\ (1 \leqslant l \leqslant m \in \mathbb{N}),\ x = \sum_{k=1}^{n} x_k,\ y = \sum_{l=1}^{m} y_l \Bigg\}$$

is order bounded in Z for all $0 \leqslant x \in X$ and $0 \leqslant y \in Y$. The set of all bilinear operators B from $X \times Y$ to Z of order bounded variation (order bounded) is denoted by $BL^{bv}(X,Y;Z)$ $\big(BL^{\sim}(X,Y;Z)\big)$ and forms an ordered vector space with the positive cone $BL_+(X,Y;Z)$. Obviously, $BL^r(X,Y;Z) \subset BL^{bv}(X,Y;Z) \subset BL^{\sim}(X,Y;Z)$ and $BL^r(X,Y;Z)$ has the induced order. The converse inclusion may be false.

3.2.3. *If Z is Dedekind complete then $BL^r(X,Y;Z) = BL^{bv}(X,Y;Z)$ and this space is a Dedekind complete vector lattice. In particular, every regular bilinear operator $B \in BL^r(X,Y;Z)$ has the modulus $|B|$ and*

$$|B|(x,y) = \sup \Sigma B[x;y] \quad (0 \leqslant x \in X,\ 0 \leqslant y \in Y),$$
$$|B(x,y)| \leqslant |B|(|x|,|y|) \quad (x \in X,\ y \in Y).$$

3.2.4. *For a bilinear operator $B : X \times Y \to Z$ the following are equivalent:*

(1) *$B(u,\cdot)$ and $B(\cdot,v)$ are lattice homomorphisms for all $u \in X_+$ and $v \in Y_+$.*

(2) *$|B(x,y)| = B(|x|,|y|)$ for all $x \in X$ and $y \in Y$.*

(3) *$B(x,y)^+ = B(x^+,y^+) + B(x^-,y^-)$ for all $x \in X$ and $y \in Y$.*

(4) *$B(x,y) \wedge B(u,v) = 0$, whenever $x,u \in X_+$ and $y,v \in Y_+$ are such that either $x \wedge u = 0$ or $y \wedge v = 0$.*

3.2.5. A bilinear operator $B : X \times Y \to Z$ is said to be a *lattice bimorphism* if B satisfies one of (and then all) the conditions of 3.2.4.

The lattice bimorphisms are simple in structure modulo lattice homomorphisms, as will be shown below in 3.12.A.3: *Each lattice bimorphism $B : X \times Y \to Z$ admits the representation*

$$B(x,y) = S(x)T(y) \quad (x \in X,\ y \in Y),$$

where $S : X \to Z^{\mathfrak{u}}$ and $T : Y \to Z^{\mathfrak{u}}$ are lattice homomorphisms with values in the universal completion $Z^{\mathfrak{u}}$ of Z and $Z^{\mathfrak{u}}$ is equipped with an f-algebra multiplication uniquely determined by a choice of an order unit in $Z^{\mathfrak{u}}$.

3.2.6. Theorem. *Let X and Y be vector lattices. Then there exist a unique up to isomorphism vector lattice $X \bar{\otimes} Y$ and a bimorphism $\phi : X \times Y \to X \bar{\otimes} Y$ such that the following are satisfied:*

(1) *Whenever Z is a vector lattice and $\psi : X \times Y \to Z$ is a lattice bimorphism, there is a unique lattice homomorphism $T : X \bar{\otimes} Y \to Z$ with $T \circ \phi = \psi$.*

(2) *The bimorphism ϕ induces an embedding of the algebraic tensor product $X \otimes Y$ into $X \bar{\otimes} Y$.*

(3) *$X \otimes Y$ is dense in $X \bar{\otimes} Y$ in the sense that for every $v \in X \bar{\otimes} Y$ there exist $x_0 \in X$ and $y_0 \in Y$ such that for every $\varepsilon > 0$ there is an element $u \in X \otimes Y$ with $|v - u| \leqslant \varepsilon x_0 \otimes y_0$.*

(4) *If $0 < v \in X \bar{\otimes} Y$, then here exist $x \in X_+$ and $y \in Y_+$ with $0 < x \otimes y \leqslant v$.*

$\lhd$ This fact was established in Fremlin [121, Theorem 4.2]. See another proof in Grobler and Labuschagne [152]. $\rhd$

3.2.7. The vector lattice $X \bar{\otimes} Y$ is called the *Fremlin tensor product* of vector lattices X and Y. The lattice bimorphism ϕ is conventionally denoted by $\otimes$ and the algebraic tensor product $X \otimes Y$ is regarded as already embedded into $X \bar{\otimes} Y$. Some additional remarks are in order here.

(1) Let ψ and T be the same as in Theorem 3.2.6 (1). Suppose that for all $x \in X_+$ and $y \in X_+$ the equality $\psi(x,y) = 0$ implies $x = 0$ or

$y = 0$. In this case T is injective and so sends $X \bar{\otimes} Y$ onto a vector sublattice of X generated by $\operatorname{im} \psi := \psi(X \times Y)$.

(2) In particular, if X_0 and Y_0 are vector sublattices in X and Y, respectively, then the tensor product $X_0 \bar{\otimes} Y_0$ is isomorphic to the vector sublattice in $X \bar{\otimes} Y$ generated by $X_0 \otimes Y_0$. Therefore, $X_0 \bar{\otimes} Y_0$ is regarded as a vector sublattice of $X \bar{\otimes} Y$; see Fremlin [121, Corollaries 4.4 and 4.5].

3.2.8. Theorem. *Let X, Y, and Z be vector lattices with Z uniformly complete. Then for every positive bilinear operator B from $X \times Y$ to Z there exists a unique positive linear operator $T : X \bar{\otimes} Y \to Z$ such that $B = T\otimes$.*

$\lhd$ See Fremlin [121, Theorem 4.2]. See another proof in Grobler and Labuschagne [152]. $\rhd$

3.2.9. Thus, the Fremlin tensor product possesses the following universal property: the set of positive bilinear operators on the Cartesian product of two Archimedean vector lattices with values in a uniformly complete vector lattice is in a one-to-one correspondence with the set of positive linear operators on the Fremlin tensor product of given vector lattices. More precisely, if X, Y, and Z are vector lattices with Z relatively uniformly complete, then the mapping $T \mapsto T\otimes$ constitutes an isomorphism of the two pairs of ordered vector spaces:

(1) $L^r(X \bar{\otimes} Y, Z)$ and $BL^r(X, Y; Z)$;

(2) $L^\sim(X \bar{\otimes} Y, Z)$ and $BL^{bv}(X, Y; Z)$.

The first assertion is immediate from Theorem 3.2.8 and the second was established in Buskes and van Rooij [83].

3.2.10. A bilinear operator $B : X \times X \to Z$ is called *orthosymmetric* if $x \perp y$ implies $B(x, y) = 0$ for arbitrary $x, y \in X$. The difference of two positive orthosymmetric bilinear operators is called *orthoregular*. Denote by $BL^{or}(X, Z)$ the space of all orthoregular bilinear operators from $X \times X$ to Z and order $BL^{or}(X, Z)$ by the cone of positive orthosymmetric operators. Recall also that a bilinear operator $B : X \times X \to G$ is said to be *symmetric* if $B(x, y) = B(y, x)$ for all $x, y \in X$ and *positive semidefinite* if $B(x, x) \geqslant 0$ for every $x \in X$. It is not difficult to see that a lattice bimorphism $B : X \times X \to Y$ is orthosymmetric, symmetric, and positive semidefinite simultaneously (cp. Buskes and Kusraev [78, Proposition 1.7]).

3.2.11. Let X be a vector lattice. A pair $(X^{\odot}, \odot)$ is called a *square* of X provided that the following hold:

(1) $X^{\odot}$ is a vector lattice;

(2) $\odot : X \times X \to X^{\odot}$ is an orthosymmetric bimorphism;

(3) for every vector lattice Y, whenever B is an orthosymmetric bimorphism from $X \times X$ to Y, there exists a unique lattice homomorphism $\hat{B} : X^{\odot} \to Y$ such that $B = \hat{B}\odot$.

3.2.12. Theorem. *The square of an Archimedean vector lattice exists and is essentially unique; i.e., if for some vector lattice $X^{\circledcirc}$ and symmetric lattice bimorphism $\circledcirc : X \times X \to X^{\circledcirc}$ the pair $(X^{\circledcirc}, \circledcirc)$ obeys the universal property* 3.2.11 (3), *then there exists a lattice isomorphism i from $X^{\odot}$ onto $E^{\circledcirc}$ such that $i\odot = \circledcirc$ (and, of course, $i^{-1}\circledcirc = \odot$).*

$\lhd$ Denote by J the least relatively uniformly closed order ideal in the Fremlin tensor product $X \bar{\otimes} X$ which includes $\{x \otimes y : x, y \in X,\ x \perp y\}$. Put $X^{\odot} := X \bar{\otimes} X / J$. Let $\phi : X \bar{\otimes} X \to E^{\odot}$ be the quotient homomorphism and put $\odot := \phi \otimes$. Then $X^{\odot}$ is an Archimedean vector lattice and $\odot$ is a lattice bimorphism. Observe that $\odot$ is orthosymmetric. Indeed, if $x \perp y$ then $x \otimes y \in J = \ker(\phi)$, whence $x \odot y = \phi(x \otimes y) = 0$. Moreover, the pair $(X^{\odot}, \odot)$ meets the universal property 3.2.11 (3). $\rhd$

3.2.13. Theorem. *Let X and Y be vector lattices with Y uniformly complete. Then for every bilinear orthoregular operator $B : X \times X \to Y$ there exists a unique linear regular operator $\hat{B} : X^{\odot} \to Y$ such that*

$$B(x, y) = \hat{B}(x \odot y) \quad (x, y \in X).$$

The correspondence $B \mapsto \hat{B}$ is an isomorphism of the ordered vector spaces $BL^{or}(X, Y)$ and $L^{r}(X^{\odot}, Y)$.

Thus, the role of the square of vector lattices in the theory of positive orthosymmetric bilinear operators is similar to that of the Fremlin tensor product of vector lattices in the general theory of positive bilinear operators.

3.3. Boolean Valued Positive Functionals

We will demonstrate in this section how Boolean valued analysis translates some results from order bounded functionals to operators.

Below X and Y stand for vector lattices with Y an order dense sublattice of $\mathscr{R}{\downarrow}$, while $\mathscr{R}$ is the reals within $\mathbb{V}^{\mathbb{B}}$ and $\mathbb{B} = \mathbb{P}(Y)$.

3.3.1. The fact that X is a vector lattice over the ordered field $\mathbb{R}$ may be rewritten as a restricted formula, say, $\varphi(X, \mathbb{R})$. Hence, recalling the restricted transfer principle, we come to the identity $[\![\, \varphi(X^{\wedge}, \mathbb{R}^{\wedge}) \,]\!] = \mathbb{1}$ which amounts to saying that $X^{\wedge}$ is a vector lattice over the ordered field $\mathbb{R}^{\wedge}$ within $\mathbb{V}^{(\mathbb{B})}$. The positive cone X_+ is defined by the restricted formula $\varphi(X, X_+) \equiv (\forall\, x \in X_+)(x \in X) \wedge (\forall\, x \in X)(x \in X_+ \leftrightarrow x \geqslant 0)$. Hence $(X^{\wedge})_+ = (X_+)^{\wedge}$ by restricted transfer. By the same reason

$$|x^{\wedge}| = |x|^{\wedge}, \quad (x \vee y)^{\wedge} = x^{\wedge} \vee y^{\wedge}, \quad (x \wedge y)^{\wedge} = x^{\wedge} \wedge y^{\wedge}$$

for all $x, y \in X$, since the lattice operations $\vee$, $\wedge$, and $|\cdot|$ in X are defined by restricted formulas. In particular,

$$x \perp y \Longleftrightarrow [\![x^{\wedge} \perp y^{\wedge}]\!] = \mathbb{1} \quad (x, y \in X).$$

3.3.2. Let $X^{\wedge\sim} := L^{\sim}_{\mathbb{R}^{\wedge}}(X^{\wedge}, \mathscr{R})$ be the space of regular $\mathbb{R}^{\wedge}$-linear functionals from $X^{\wedge}$ to $\mathscr{R}$. More precisely, $\mathscr{R}$ is considered as a vector space over the field $\mathbb{R}^{\wedge}$ and by the maximum principle there exists $X^{\wedge\sim} \in \mathbb{V}^{(\mathbb{B})}$ such that $[\![X^{\wedge\sim}$ is a vector space over $\mathscr{R}$ of order bounded $\mathbb{R}^{\wedge}$-linear functionals from $X^{\wedge}$ to $\mathscr{R}$ which is ordered by the cone of positive functionals$\,]\!] = \mathbb{1}$. A functional $\tau \in X^{\wedge\sim}$ is positive if $[\![(\forall\, x \in X^{\wedge})\tau(x) \geqslant 0]\!] = \mathbb{1}$.

It can easily be seen that the Riesz–Kantorovich Theorem remains true if X is a vector lattice over a dense subfield $\mathbb{P} \subset \mathbb{R}$, while Y is a Dedekind complete vector lattice (over $\mathbb{R}$), and $L^{\sim}(X, Y)$ is replaced by $L^{\sim}_{\mathbb{P}}(X, Y)$, the real vector space of all order bounded $\mathbb{P}$-linear operators from X to Y which is ordered by the cone of positive operators; i.e., $L^{\sim}_{\mathbb{P}}(X, Y)$ is a Dedekind complete vector lattice. Moreover, the formulas of order calculus of 3.1.4 are preserved.

According to this observation $X^{\wedge\sim}$ is a Dedekind complete vector lattice within $\mathbb{V}^{(\mathbb{B})}$ and for all $\sigma, \tau \in X^{\wedge\sim}$ and $x \in (X^{\wedge})_+$ we have

$$(\sigma \vee \tau)x = \sup\{\sigma x_1 + \tau x_2 : \ x_1, x_2 \in X^{\wedge}_+, \ x = x_1 + x_2\},$$
$$(\sigma \wedge \tau)x = \inf\{\sigma x_1 + \tau x_2 : \ x_1, x_2 \in X^{\wedge}_+, \ x = x_1 + x_2\}.$$

Thus, the descent $X^{\wedge\sim}{\downarrow}$ of $X^{\wedge\sim}$ is a Dedekind complete vector lattice.

3.3.3. Theorem. *Let X and Y be vector lattices with Y universally complete and represented as $Y = \mathscr{R}{\downarrow}$. Given $T \in L^{\sim}(X, Y)$, the modified ascent $T{\uparrow}$ is an order bounded $\mathbb{R}^{\wedge}$-linear functional on $X^{\wedge}$ within*

$\mathbb{V}^{(\mathbb{B})}$*; i.e.,* $[\![T\!\uparrow \in X^{\wedge\sim}]\!] = \mathbb{1}$. *The mapping* $T \mapsto T\!\uparrow$ *is a lattice isomorphism between the Dedekind complete vector lattices* $L^{\sim}(X,Y)$ *and* $X^{\wedge\sim}\!\downarrow$.

$\lhd$ Observe first that $T \mapsto T\!\uparrow$ is a bijection from Y^X to $\mathscr{R}^{X^\wedge}\!\downarrow$. To this end, recall that for every $T \in L^{\sim}(X,Y)$ the modified ascent $T\!\uparrow$ is defined by the relation $[\![Tx = T\!\uparrow\!(x^\wedge)]\!] = \mathbb{1}$ $(x \in X)$, while for every $\tau \in X^{\wedge\sim}\!\downarrow$ we have $[\![\tau : X^\wedge \to \mathscr{R}]\!] = \mathbb{1}$ and so the modified descent $\tau\!\downarrow : X \to Y$ is available (cp. 1.6.8 and 1.5.8). Moreover, by the Escher rules 1.6.6 we have $\tau\!\downarrow\uparrow = \tau$ and $T\!\uparrow\downarrow = T$. Assuming that T is linear and putting $\tau = T\!\uparrow$, for all $x, y \in X$ and $\lambda \in \mathbb{R}$ we deduce within $\mathbb{V}^{(\mathbb{B})}$:

$$\tau(x^\wedge + y^\wedge) = \tau((x+y)^\wedge) = T(x+y) = Tx + Ty = \tau(x^\wedge) + \tau(y^\wedge),$$
$$\tau(\lambda^\wedge x^\wedge) = \tau((\lambda x)^\wedge) = T(\lambda x) = \lambda Tx = \lambda^\wedge \tau(x^\wedge).$$

The $\mathbb{R}^\wedge$-linearity of τ within $\mathbb{V}^{(\mathbb{B})}$ follows from the calculations:

$$\begin{aligned} &[\![(\forall\, x \in X^\wedge)(\forall\, y \in X^\wedge)(\tau(x+y) = \tau(x) + \tau(y))]\!] \\ &\quad = \bigwedge_{x,y \in X} [\![\tau(x^\wedge + y^\wedge) = \tau(x^\wedge) + \tau(y^\wedge)]\!] = \mathbb{1}; \\ &\quad [\![(\forall\, \lambda \in \mathbb{R}^\wedge)(\forall\, x \in X^\wedge)(\tau(\lambda x) = \lambda \tau(x))]\!] \\ &\quad\quad = \bigwedge_{\lambda \in \mathbb{R}} \bigwedge_{x \in X} [\![\tau(\lambda^\wedge x^\wedge) = \lambda^\wedge \tau(x^\wedge)]\!] = \mathbb{1}. \end{aligned}$$

Suppose that T is order bounded and put $\bar u = \sup\{|Tx| : |x| \leqslant u\}$ for $u \in X$. Denote by $\varphi(u,v)$ the formula $(\forall\, x \in X^\wedge)(|x| \leqslant u \to |\tau(x)| \leqslant v)$ and observe that

$$\begin{aligned} [\![\varphi(u^\wedge, \bar u)]\!] &= \bigwedge_{x \in X} [\![|x^\wedge| \leqslant u^\wedge]\!] \Rightarrow [\![|\tau(x^\wedge)| \leqslant \bar u]\!] \\ &= \bigwedge_{x \in X} (\mathbb{1} \Rightarrow [\![|Tx| \leqslant \bar u]\!]) = \mathbb{1}. \end{aligned}$$

It follows that τ is order bounded within $\mathbb{V}^{(\mathbb{B})}$:

$$\begin{aligned} &[\![\tau \text{ is order bounded}]\!] = [\![(\forall\, u \in X^\wedge)(\exists\, v \in \mathscr{R}) \varphi(u,v)]\!] \\ &\quad = \bigwedge_{u \in X} [\![(\exists\, v \in \mathscr{R}) \varphi(u^\wedge, v)]\!] \geqslant \bigwedge_{u \in X} [\![(\forall\, u \in X^\wedge) \varphi(u^\wedge, \bar u)]\!] = \mathbb{1}. \end{aligned}$$

Thus, $T \in L^{\sim}(X,Y)$ implies $[\![\tau \in X^{\wedge\sim}]\!]$. The converse can be handled in a similar way. Consequently, $T \mapsto T{\uparrow}$ is a linear bijection from $L^{\sim}(X,Y)$ into $[\![X^{\wedge\sim}]\!]$. It follows from

$$[\![\tau \geqslant 0]\!] = \bigwedge_{x \in X_+} [\![\tau(x^{\wedge}) \geqslant 0]\!] = \bigwedge_{x \in X_+} [\![Tx \geqslant 0]\!]$$

that T is positive if and only if $[\![\tau \text{ is positive}]\!] = \mathbb{1}$, so that $T \mapsto T{\uparrow}$ is the desired lattice isomorphism. $\triangleright$

3.3.4. We now formulate a few corollaries to 3.3.3. First, we introduce necessary definitions. An operator $T \in L^{\sim}(X,Y)$ is said to be *disjointness preserving* if $x \perp y$ implies $Tx \perp Ty$ for all $x, y \in X$. Let $L^{\sim}_{dp}(X,Y)$ stand for the set of all order bounded disjointness preserving operators from X to Y.

Let $L^{\sim}_{a}(X,Y)$ be the band generated by $\operatorname{Hom}(X,Y)$ in $L^{\sim}(X,Y)$ and $L^{\sim}_{d}(X,Y)$ be the disjoint complement of $\operatorname{Hom}(X,Y)$:

$$L^{\sim}_{a}(X,Y) := \operatorname{Hom}(X,Y)^{\perp\perp}, \quad L^{\sim}_{d}(X,Y) := \operatorname{Hom}(X,Y)^{\perp}.$$

If Y is Dedekind complete then $L^{\sim}(X,Y) = L^{\sim}_{a}(X,Y) \oplus L^{\sim}_{d}(X,Y)$ holds, and so every $T \in L^{\sim}(X,Y)$ has the unique decomposition $T = T_a + T_d$, where $T_a \in L^{\sim}_{a}(X,Y)$ and $T_d \in L^{\sim}_{d}(X,Y)$. The elements of $L^{\sim}_{d}(X,Y)$ are usually referred to as *diffuse* operators, while the elements of $L^{\sim}_{a}(X,Y)$ are called *pseudoembedding* operators or pseudoembeddings. Also, define within $\mathbb{V}^{(\mathbb{B})}$ the band of order bounded $\mathbb{R}^{\wedge}$-linear atomic, disjointness preserving, and diffuse functionals: namely $(X^{\wedge})^{\sim}_{a} := \operatorname{Hom}_{\mathbb{R}^{\wedge}}(X^{\wedge}, \mathscr{R})^{\perp\perp}$, $(X^{\wedge})^{\sim}_{dp} := L^{\sim}_{dp}(X^{\wedge}, \mathscr{R})$, and $(X^{\wedge})^{\sim}_{d} := \operatorname{Hom}_{\mathbb{R}^{\wedge}}(X^{\wedge}, \mathscr{R})^{\perp}$.

Recall that $S \in L^{\sim}(X,Y)$ is a *component* or a *fragment* of $0 \leqslant T \in L^{\sim}(X,Y)$ if $S \wedge (T - S) = 0$. *Boolean valued representation of a band preserving operators* obtained in Theorem 4.3.4 reduces some properties of band preserving operators to Boolean valued interpretations of the properties of solutions to Cauchy functional equation.

3.3.5. Corollary. *Consider $R, S, T \in L^{\sim}(X, \mathscr{R}{\downarrow})$ and $b \in \mathbb{B}$. Put $\rho := R{\uparrow}$, $\sigma := S{\uparrow}$, $\tau := T{\uparrow}$, and $\pi := \chi(b)$. The following are true:*

(1) $b \leqslant [\![\, \sigma \leqslant \tau \,]\!] \iff \pi S \leqslant \pi T$.

(2) $b \leqslant [\![\, \sigma = |\tau| \,]\!] \iff \pi S = \pi |T|$.

(3) $b \leqslant [\![\, \rho = \sigma \vee \tau \,]\!] \iff \pi R = \pi S \vee \pi T$.

(4) $b \leqslant [\![\, \rho = \sigma \wedge \tau \,]\!] \iff \pi R = \pi S \wedge \pi T$.

(5) $b \leqslant [\![\, \sigma \perp \tau \,]\!] \iff \pi S \perp \pi T.$

(6) $b \leqslant [\![\, \sigma \in \mathbb{C}(\tau) \,]\!] \iff \pi S \in \mathbb{C}(\pi T).$

$\lhd$ According to 2.3.6 for each $b \in \mathbb{B}$ we have $\mathbb{V}^{([\mathbb{0},b])} \models b \wedge \tau \in (X^{\wedge})^{\sim}$ and $(b \wedge \tau)\!\downarrow = \chi(b) \circ T$. Thus, it suffices to observe that the vector lattices $L^{\sim}(X, (b \wedge \mathscr{R})\!\downarrow)$ and $b \wedge (X^{\wedge})^{\sim}_{+}$ are lattice isomorphic in view of 1.3.7 and Theorem 3.3.3. $\rhd$

3.3.6. Corollary. *Consider $S, T \in L^{\sim}(X,Y)$ and put $\tau := T\!\uparrow$, $\sigma := S\!\uparrow$. The following are true:*

(1) $T \in \mathrm{Hom}(X,Y) \iff [\![\, \tau \in \mathrm{Hom}(X^{\wedge}, \mathscr{R}) \,]\!] = \mathbb{1}.$

(2) $T \in L^{\sim}_{dp}(X,Y) \iff [\![\, \tau \in (X^{\wedge})^{\sim}_{dp} \,]\!] = \mathbb{1}.$

(3) $T \in L^{\sim}_{a}(X,Y) \iff [\![\, \tau \in (X^{\wedge})^{\sim}_{a} \,]\!] = \mathbb{1}.$

(4) $T \in L^{\sim}_{d}(X,Y) \iff [\![\, \tau \in (X^{\wedge})^{\sim}_{d} \,]\!] = \mathbb{1}.$

$\lhd$ This is immediate from 3.3.5. $\rhd$

3.3.7. *Let X be a vector lattice and $Y := \mathscr{R}\!\downarrow$. Given $T \in L^{\sim}(X,Y)$ and a family $(T_\xi)_{\xi\in\Xi}$ in $L^{\sim}(X,Y)$, put $\tau := T\!\uparrow$ and $\tau_\xi := T_\xi\!\uparrow$. Then for each partition of unity $(b_\xi)_{\xi\in\Xi}$ in $\mathbb{B}$ we have $\tau = \mathrm{mix}_{\xi\in\Xi}\, b_\xi \tau_\xi$ if and only if the representation holds*

$$Tx = o\text{-}\sum_{\xi\in\Xi} \chi(b_\xi) T_\xi x \quad (x \in X).$$

$\lhd$ It follows from 3.3.5 (1) that $b_\xi \leqslant [\![\tau = \tau_\xi]\!]$ if and only if $\chi(b_\xi)T = \chi(b_\xi)T_\xi$. Summing the last identity over all $\xi \in \Xi$ we see that the desired representation of T is equivalent to the relation $\tau = \mathrm{mix}_{\xi\in\Xi}\, b_\xi \tau_\xi$. $\rhd$

3.4. Disjointness Preserving Operators

We intend here to demonstrate that some properties of disjointness preserving operators are just Boolean valued interpretations of elementary properties of disjointness preserving functionals.

3.4.1. Theorem. *For an order bounded linear functional $f : X \to \mathbb{R}$ the following are equivalent:*

(1) *f preserves disjointness.*

(2) *$|f|$ is a lattice homomorphism.*

(3) *If* $g \in X^\sim$ *and* $0 \leqslant g \leqslant |f|$ *then* $g = \lambda|f|$ *for some* $\lambda \in [0, 1]$.

(4) *If* g *is a component of* $|f|$ *then either* $g = 0$ *or* $g = |f|$.

(5) *Either* f *or* $-f$ *is a lattice homomorphism.*

(6) $|f|(|x|) = |f(|x|)| = |f(x)|$ *for all* $x \in X$.

(7) $\ker(f) := f^{-1}(0)$ *is an order ideal in* X.

$\lhd$ (1) $\iff$ (2): Assume that f preserves disjointness and $x, y \in X_+$ are disjoint, while $|f|(x) \wedge |f|(y) > 0$. Then by formula 3.1.4 (5) there exist $x', y' \in X$ with $|x'| \leqslant x$, $|y'| \leqslant y$, $|f(x')| > 0$, and $|f(y')| > 0$. At the same time $x' \perp y'$ and we should have $|f(x')| \wedge |f(y')| = 0$ by hypothesis; a contradiction. Thus, (1) $\Longrightarrow$ (2) and the converse follows from 3.1.4 (7).

(2) $\Longrightarrow$ (3): Put $h := |f|$ and observe that $\ker(h) \subset \ker(g)$, since $|g(x)| \leqslant g(|x|) \leqslant h(|x|) = |h(x)|$. Thus $g = \lambda h \leqslant h$ for some $0 \leqslant \lambda \leqslant 1$.

(3) $\Longrightarrow$ (4): According to (3), each component g of $|f|$ is of the form $g = \lambda|f|$. It follows that $0 = g \wedge (|f| - g) = \min\{\lambda, (1 - \lambda)\}|f|$, so that either $\lambda = 0$ or $\lambda = 1$.

(4) $\Longrightarrow$ (5): Since $|f|$ is the sum of disjoint components f^+ and f^-, either $f^- = 0$, in which case $|f| = f$, or $f^+ = 0$, in which case $|f| = -f$. Moreover, $|f|$ is a lattice homomorphism. Otherwise there is a pair of disjoint elements $x, y \in X$ with $|f|(x) > 0$ and $|f|(y) > 0$. So, there exists a component g of f such that $g(x) = f(x)$ and $g(y) = 0$. Thus neither $g = 0$ nor $g = f$, which is a contradiction.

(5) $\Longrightarrow$ (6): In both cases of (5) the needed relation is trivial.

(6) $\Longrightarrow$ (7): If $|y| \leqslant |x|$ and $x \in \ker(f)$ then from (6) we have $|f(y)| \leqslant |f|(|y|) \leqslant |f|(|x|) = |f(x)| = 0$, and so $y \in \ker(f)$.

(7) $\Longrightarrow$ (1): We have only to note that for every pair of disjoint elements $x, y \in X$ either $x \in \ker(f)$ or $y \in \ker(f)$. Assuming the contrary, we can choose nonzero $s, t \in \mathbb{R}$ with $sx + ty \in \ker(f)$, since nonzero disjoint elements are linearly independent and $\ker(f)$ is a hyperplane. It follows that $|x| \leqslant (|x| + (|t|/|s|)|y|) = |x + (t/s)y| \in \ker(f)$ and $x \in \ker(f)$; a contradiction. $\rhd$

3.4.2. Theorem. *Assume that* Y *has the projection property. An order bounded linear operator* $T : X \to Y$ *is disjointness preserving if and only if* $\ker(bT)$ *is an order ideal in* X *for every* $b \in \mathbb{P}(Y)$.

$\lhd$ The necessity is obvious, so only the sufficiency will be proved. Suppose that $\ker(bT)$ is an order ideal in X for every $b \in \mathbb{P}(Y)$. Let $|y| \leqslant |x|$ and $b := [\![Tx = 0]\!]$. Then $bTx = 0$ by 2.2.4 ($\mathbb{G}$) and $bTy = 0$ by the hypothesis. Recalling 2.2.4 ($\mathbb{G}$) once again, we see that $b \leqslant [\![Ty = 0]\!]$. Thus $[\![Tx = 0]\!] \leqslant [\![Ty = 0]\!]$ or, which is the same, $[\![Tx = 0]\!] \Rightarrow [\![Ty = 0]\!] = \mathbb{1}$. Put $\tau := T\!\uparrow$ and ensure that $\ker(\tau)$ is an order ideal in $X^\wedge$ within $\mathbb{V}^{(\mathbb{B})}$. Since $|x| \leqslant |y|$ if and only if $[\![|x^\wedge| \leqslant |y^\wedge|]\!] = \mathbb{1}$, we deduce:

$$
\begin{aligned}
&[\![\ker(\tau) \text{ is an order ideal in } X^\wedge]\!] \\
&= [\![(\forall\, x, y \in X^\wedge)\,(\tau(x) = 0 \wedge |y| \leqslant |x| \,\rightarrow\, \tau(y) = 0)]\!] \\
&= \bigwedge_{x,y \in X} [\![(\tau(x^\wedge) = 0]\!] \wedge [\![|y^\wedge| \leqslant |x^\wedge|]\!] \,\Rightarrow\, [\![\tau(y^\wedge) = 0]\!] \\
&= \bigwedge \Big\{ [\![T(x) = 0]\!] \,\Rightarrow\, [\![T(y) = 0]\!] :\ x, y \in X,\ |y| \leqslant |x| \Big\} = \mathbb{1}.
\end{aligned}
$$

According to 3.4.1 (7) τ is a disjointness preserving functional within $\mathbb{V}^{(\mathbb{B})}$ and so T is also disjointness preserving by 3.3.5 (1). $\triangleright$

As is well known each order bounded disjointness preserving operator between vector lattices has a modulus. This is obvious in the special situation of functionals to which the general case is reduced by means of Boolean valued interpretation.

3.4.3. Meyer Theorem. *For every order bounded disjointness preserving linear operator $T : X \to Y$ between vector lattices the modulus $|T|$, positive part T^+, and negative part T^- exist and are lattice homomorphisms. Moreover,*

$$
|T|x = |Tx|, \quad T^+x = (Tx)^+, \quad T^-x = (Tx)^- \quad (x \in X_+).
$$

In particular, an order bounded disjointness preserving operator is regular.

$\triangleleft$ By the Gordon Theorem we can assume that Y is an order dense sublattice in $\mathscr{R}\!\downarrow$. Again, put $\tau := T\!\uparrow$ and note that $\tau \in (X^{\wedge\sim})_{dp}$ by 3.3.6 (2) and $|\tau|$ exists within $\mathbb{V}^{(\mathbb{B})}$. By 3.3.5 (2) $|\tau|\!\downarrow$ is the modulus of T in $L^\sim(X, \mathscr{R}\!\downarrow)$. Moreover, $|\tau|$ and $|T|$ both are lattice homomorphisms in view of 3.4.1 (2) and 3.3.6 (1). But $[\![|\tau|(x) = |\tau x|$ for all $x \in (X^\wedge)_+]\!] = \mathbb{1}$ according to 3.4.1 (6). Putting this fact into $\mathbb{V}^{(\mathbb{B})}$ and recalling 3.3.5 (2), we obtain $|T|(x) = |Tx|$ for all $x \in X_+$. It follows that $|T|(u) = |T|(u^+) - |T|(u^-) = |T(u^+)| - |T(u^-)| \in Y$ for all $u \in X$, so that $|T|$ exists in $L^\sim(X, Y)$. Other properties of T^+ and T^- can easily be deduced from above by using the formulas $T^+ = (|T| + T)/2$ and $T^- = (|T| - T)/2$. $\triangleright$

3.4.4. Theorem. *Let Y have the projection property. For an order bounded disjointness preserving linear operator $T \in L^{\sim}(X,Y)$ there exists a band projection $\pi \in \mathbb{P}(Y)$ such that $T^+ = \pi|T|$ and $T^- = \pi^{\perp}|T|$. In particular, $T = (\pi - \pi^{\perp})|T|$ and $|T| = (\pi - \pi^{\perp})T$.*

$\lhd$ Once again we reduce the problem to the case of functionals by putting $\tau := T\!\uparrow$. As before, $\tau \in (X^{\wedge\sim})_{dp}$ and by, 3.4.1 (5), either $\tau^- = 0$ or $\tau^+ = 0$ within $\mathbb{V}^{(\mathbb{B})}$. Put $\pi := [\![\tau^- = 0]\!]$ and observe that $\pi = [\![|\tau| = \tau^+]\!]$ and $\pi^{\perp} = [\![\tau^- \neq 0]\!] \leqslant [\![\tau^+ = 0]\!]$, since $[\![\tau^- \neq 0 \to \tau^+ = 0]\!] = \mathbb{1}$. By Corollary 3.3.5 (2, 3) we obtain $\pi|T| = \pi T^+$ and $\pi^{\perp}T^+ = 0$. Putting together the last two relations we arrive at the first of the desired identities $\pi|T| = \pi T^+ + \pi^{\perp}T^+ = T^+$. The second is immediate from the first: $\pi^{\perp}|T| = |T| - \pi|T| = |T| - T^+ = T^-$. $\rhd$

3.4.5. Corollary. *Let X and Y be vector lattices and let $T \in L^{\sim}(X,Y)$ be disjointness preserving. Then $(Tx)^+ \perp (Ty)^-$ for all $x, y \in X_+$.*

$\lhd$ Given $x, y \in X_+$ we can write $(Tx)^+ = (Tx) \vee 0 \leqslant T^+x = \pi|T|x$. Similarly, $(Ty)^- \leqslant \pi^{\perp}|T|y$, and so $(Tx)^+ \wedge (Ty)^- = 0$. $\rhd$

3.4.6. Theorem. *Let X and Y be vector lattices with Y Dedekind complete, and let $S, T : X \to Y$ be order bounded disjointness preserving operators. The following are equivalent:*

(1) $T \in \{S\}^{\perp\perp}$.

(2) $Tx \in \{Sx\}^{\perp\perp}$ *for all* $x \in X$.

(3) $\pi Sx = 0 \implies \pi Tx = 0$ *for all* $x \in X$ *and* $\pi \in \mathbb{P}(X)$.

(4) *There exists* $\pi \in \mathrm{Orth}(J(T(X)), Y)$ *such that* $T = \pi \circ S$, *where* $J(Y_0)$ *is an order ideal generated by* Y_0 *in* Y.

$\lhd$ Assume that S and T are positive, since otherwise we can replace them by their modules. Again, put $\tau := T\!\uparrow$ and $\sigma := S\!\uparrow$ and observe that for $k = 1, 2, 3, 4$ we have $(k) \iff [\![(k^{\circ})]\!] = \mathbb{1}$, where

$$(1^{\circ}) := \tau \in \{\sigma\}^{\perp\perp},$$
$$(2^{\circ}) := \tau(x) \in \{\sigma(x)\}^{\perp\perp} \ \text{ for all } \ x \in X^{\wedge},$$
$$(3^{\circ}) := \sigma(x) = 0 \to \tau(x) = 0 \ \text{ for all } \ x \in X^{\wedge},$$
$$(4^{\circ}) := \tau = \alpha\sigma \ \text{ for some } \ \alpha \in \mathscr{R}.$$

Now working within $\mathbb{V}^{(\mathbb{B})}$ we see that $(2^{\circ}) \to (3^{\circ})$ and $(4^{\circ}) \to (1^{\circ})$ are trivial, (3°) implies that $\ker(\sigma) \subset \ker(\tau)$ and so $\tau = \alpha\sigma$ for some $\alpha \in \mathscr{R}$, whence (4°). Finally, if (1°) holds then $\tau = \sup_{n\in\mathbb{N}}(n\sigma)\wedge\tau$ and $|\tau(x)| = \tau(|x|) \in \{\sigma(|x|)\}^{\perp\perp}$, because $((n\sigma)\wedge\tau)(|x|) \leqslant n\sigma(|x|) \in \{\sigma(|x|)\}^{\perp\perp}$. Thus, (1°) → (2°) and the proof is complete. ▷

3.4.7. Corollary. *For a positive linear operator $T : X \to Y$ the following are equivalent:*

(1) *T is a lattice homomorphism.*

(2) *If $S \in L^{\sim}(X, Y)$ and $0 \leqslant S \leqslant T$ then there exists an orthomorphism $\pi \in \mathrm{Orth}(Y)$ such that $0 \leqslant \pi \leqslant I_Y$ and $S = \pi \circ T$.*

(3) *If $S \in \mathbb{C}(T)$ then there exists $\pi \in \mathbb{P}(Y)$ such that $S = \pi \circ T$.*

◁ The proof goes along similar lines using 3.4.1 (3, 4) and 3.3.5 (1, 2, 6). ▷

3.4.8. Theorem. *Let X and Y be vector lattices with Y Dedekind complete. For a pair of disjointness preserving operators T_1 and T_2 from X to Y there exist a band projection $\pi \in \mathbb{P}(Y)$, a lattice homomorphism $T \in \mathrm{Hom}(X, Y)$, and orthomorphisms $S_1, S_2 \in \mathrm{Orth}(Y)$ such that*

$$|S_1| + |S_2| = \pi, \quad \pi T_1 = S_1 T, \quad \pi T_2 = S_2 T,$$
$$\mathrm{im}(\pi^{\perp} T_1)^{\perp\perp} = \mathrm{im}(\pi^{\perp} T_2)^{\perp\perp} = \pi^{\perp}(Y), \quad \pi^{\perp} T_1 \perp \pi^{\perp} T_2.$$

◁ As usual, there is no loss of generality in assuming that $Y = \mathscr{R}{\downarrow}$. Put $\tau_1 := T_1{\uparrow}$ and $\tau_2 := T_2{\uparrow}$. The desired result is a Boolean valued interpretation of the following fact: If the disjointness preserving functionals τ_1 and τ_2 are not proportional then they are both nonzero and disjoint. Indeed, if $\tau := |\tau_1| \wedge |\tau_2| \neq 0$ then both $|\tau_1|$ and $|\tau_2|$ are positive multiples of τ by 3.4.1 (3); therefore, τ_1 and τ_2 are proportional. Put $b := [\![\tau_1 \text{ and } \tau_2 \text{ are proportional}]\!]$ and $\pi := \chi(b)$. Then within $\mathbb{V}^{([\mathbb{0},b])}$ there exist a lattice homomorphism $\tau : X^{\wedge} \to \mathscr{R}$ and reals $\sigma_1, \sigma_2 \in \mathscr{R}$ such that $\tau_i = \sigma_i \tau$. If the function $\bar\sigma_i$ is defined as $\bar\sigma_i : \lambda \mapsto \sigma_i \lambda$ $(\lambda \in \mathscr{R})$, then the operators $S_1 := \bar\sigma_1{\downarrow}$, $S_2 := \bar\sigma_2{\downarrow}$, and $T := \tau{\downarrow}$ (with the modified descents taken from $\mathbb{V}^{([\mathbb{0},b])}$; see 1.3.7) satisfy the first line of required conditions. Moreover, $\pi^{\perp} = \chi(b^*)$ and by transfer we have $b^* = [\![\tau_1 \neq 0]\!] \wedge [\![\tau_2 \neq 0]\!] \wedge [\![|\tau_1| \wedge |\tau_2| = 0]\!]$, so that the second line of required conditions is also satisfied by 3.3.5 (5) and 3.8.4. ▷

3.4.9. Theorem. *Let X and Y be vector lattices with Y Dedekind complete. The sum $T_1 + T_2$ of two disjointness preserving operators $T_1, T_2 : X \to Y$ is disjointness preserving if and only if there exist pairwise disjoint band projections $\pi, \pi_1, \pi_2 \in \mathbb{P}(Y)$, orthomorphisms*

$S_1, S_2 \in \mathrm{Orth}(Y)$ and a lattice homomorphism $T \in \mathrm{Hom}(X, Y)$ such that the following system of relations is consistent

$$\begin{aligned} \pi + \pi_1 + \pi_2 = I_Y, &\quad |S_1| + |S_2| = \pi, \\ T(X)^{\perp\perp} = \pi(Y), &\quad \pi_1 T_2 = \pi_2 T_1 = 0, \\ \pi T_1 = S_1 T, &\quad \pi T_2 = S_2 T. \end{aligned}$$

Consequently, in this case $T_1 + T_2 = \pi_1 T_1 + \pi_2 T_2 + (S_1 + S_2)T$.

$\lhd$ The sufficiency is obvious. To prove the necessity we apply Theorem 3.4.8 and note that only the claim concerning π_1 and π_2 is needed to check. Using the same notation put $b_1 := [\![\tau_2 = 0]\!]$, $b_2 := [\![\tau_1 = 0]\!]$ and $b := [\![$ both τ_1 and τ_2 are nonzero$]\!]$. Observe that the sum of two disjoint functionals that preserve disjointness is disjointness preserving if and only if at least one of them is zero, since otherwise each of them is proportional to their sum; a contradiction. Thus, in view of the transfer principle $b^* = b_1 \vee b_2$ or $b_0 \vee b_1 \vee b_2 = \mathbb{1}$. Moreover, we can assume by replacing b_1 with $b_1 \wedge b_2^*$, if necessary, that b_0, b_1, b_2 are pairwise disjoint. Putting $\pi_i := \chi(b_i)$ $(i = 1, 2)$, we see that $\pi_1\pi_2 = \pi_1\pi = \pi_2\pi = 0$ and $\pi_1 + \pi_2 + \pi = I_Y$. Using 2.2.4 ($\mathbb{G}$), we conclude that $b_1 \leqslant [\![\tau_2 = 0]\!]$ implies $\pi_1 T_2 = 0$ and $b_2 = [\![\tau_1 = 0]\!]$ implies $\pi_2 T_1 = 0$. $\rhd$

3.4.10. Corollary. *The sum $T_1 + T_2$ of two disjointness preserving operators $T_1, T_2 : X \to Y$ is disjointness preserving if and only if $T_1(x_1) \perp T_2(x_2)$ for all $x_1, x_2 \in X$ with $x_1 \perp x_2$.*

$\lhd$ The necessity is immediate from Theorem 3.4.9, since $T_1 = \pi_1 T_1 + S_1 T$ and $T_2 = \pi_2 T_2 + S_2 T$. To see the sufficiency, observe that if T_1 and T_2 meet the above condition then $T_k x_1 \perp T_l x_2$ $(k, l := 1, 2)$ and so $(T_1 + T_2)(x_1) \perp (T_1 + T_2)(x_2)$ for every pair of disjoint elements $x_1, x_2 \in X$. $\rhd$

3.5. Differences of Lattice Homomorphisms

This section answers the following question: Which closed hyperplane in a Banach lattice is a vector sublattice? It turns out that each hyperspace with this property is exactly the kernel of the difference of some lattice homomorphisms on the initial vector lattice. The starting point of this question is the celebrated Stone Theorem about the structure of vector sublattices in the Banach lattice $C(Q, \mathbb{R})$ of continuous real functions on a compact space Q. This theorem may be rephrased in the above terms as follows:

3.5.1. Stone Theorem. *Each closed vector sublattice of $C(Q,\mathbb{R})$ is the intersection of the kernels of some differences of lattice homomorphisms on $C(Q,\mathbb{R})$.*

3.5.2. In view of this theorem it is reasonable to refer to a difference of lattice homomorphisms on a vector lattice X as a two-point relation on X. We are not obliged to assume here that the lattice homomorphisms under study act into the reals $\mathbb{R}$. Thus a linear operator $T : X \to Y$ between vector lattices is said to be a *two-point relation* on X whenever it is written as a difference of two lattice homomorphisms. An operator $bT := b \circ T$ with $b \in \mathbb{B} := \mathbb{P}(Y)$ is called a *stratum* of T.

3.5.3. The kernel $\ker(T)$ of every two-point relation $T := T_1 - T_2$ with $T_1, T_2 \in \operatorname{Hom}(X,Y)$ is evidently a sublattice of X, since it is determined by an equation $\ker(T) = \{x \in X : T_1 x = T_2 x\}$. On using of the above terminology, the Meyer Theorem 3.4.3 reads as follows: *Each order bounded disjointness preserving operator between vector lattices is a two-point relation.* Thus, each stratum bT of an order bounded disjointness preserving operator $T : X \to Y$ is a two-point relation on X and so its kernel is a vector sublattice of X. In fact, the converse is valid too.

3.5.4. Theorem. *An order bounded linear operator from a vector lattice to a Dedekind complete vector lattice is a two-point relation if and only if the kernel of its every stratum is a vector sublattice of the ambient vector lattice.*

$\lhd$ The proof presented below in 3.5.9 and 3.5.10 follows along the general lines: Using the canonical embedding and ascending to the Boolean valued universe $\mathbb{V}^{(\mathbb{B})}$, we reduce the matter to characterizing scalar two-point relations on vector lattices over dense subfields of the reals $\mathbb{R}$. To solve the resulting scalar problem, we use one of the formulas of subdifferential calculus, namely the Moreau–Rockafellar Formula. $\rhd$

3.5.5. We need some additional concepts. Recall that $p : X \to \mathbb{R} \cup \{+\infty\}$ is called a *sublinear functional* if $p(0) = 0$, $p(x+y) \leqslant p(x) + p(y)$, and $p(\lambda x) = \lambda p(x)$ for all $x, y \in X$ and $0 < \lambda \in \mathbb{R}$. The *subdifferential* (at zero) ∂p of a sublinear functional p is defined as

$$\partial p := \{l : X \to \mathbb{R} :\ l \text{ is linear and } lx \leqslant p(x) \text{ for all } x \in X\}.$$

The *effective domain* $\operatorname{dom}(p) := \{x \in X : p(x) < +\infty\}$ of a sublinear functional p is a cone. By a *cone* K we always mean a convex cone which is a subset of X with the properties $K + K = K$ and $\lambda K \subset K$ $(\lambda \in \mathbb{R}_+)$. Evidently, $p \leqslant q$ implies $\partial p \subset \partial q$; the converse is also true whenever $\operatorname{dom}(p) = \operatorname{dom}(q) = X$.

3.5.6. Assume now that X is a vector lattice. If p is *increasing* (i.e., $x_1 \leqslant x_2 \Longrightarrow p(x_1) \leqslant p(x_2)$) then ∂p consists of positive functionals. It can easily be seen from the Hahn–Banach Theorem that the converse is also true whenever $\operatorname{dom}(p) = X$. Indeed, we can pick a linear functional $f \in \partial p$ with $f(x_1) = p(x_1)$ and, assuming $\partial p \subset X^\sim_+$ and $x_1 \leqslant x_2$, we get $p(x_1) = f(x_1) \leqslant f(x_2) \leqslant p(x_2)$.

Take a positive functional f on X. The representation $f = f_1 + \dots + f_N$ will be called a *positive* $(N$-$)$*decomposition* of f whenever $f_1, \dots, f_N$ are positive functionals on X. In this event we say also that $(f_1, \dots, f_N)$ is a positive decomposition of f.

Given a positive functional $f \in X^\sim$ define the function $p_f : X^N \to \mathbb{R}$ as $p_f(x_1, \dots, x_N) = f(x_1 \vee \dots \vee x_N)$. Then p is sublinear and increasing and ∂p_f consists of all positive decompositions of f; i.e., the representation

$$\partial p_f = \Big\{(f_1, \dots, f_N) : \ 0 \leqslant f_k \in X^\sim, \ f = \sum_{k=1}^{N} f_k\Big\}$$

holds. Indeed, $(f_1, \dots, f_N) \in \partial p_f$ means that for all $x_1, \dots, x_N$ we have

$$f_1(x_1) + \dots + f_N(x_N) \leqslant f(x_1 \vee \dots \vee x_N).$$

Taking x_j to be zero for all $j \neq i$ and $x_i \leqslant 0$ yields $f_i \geqslant 0$ and putting $x_1 = \dots = x_N$ gives $f = f_1 + \dots + f_N$. This proves the inclusion $\subset$, while the converse inclusion is trivial.

3.5.7. Moreau–Rockafellar Formula. *Assume that X is a real vector space and $p, q : X \to \mathbb{R} \cup \{+\infty\}$ are sublinear functionals. If* $\operatorname{dom}(p) - \operatorname{dom}(q) = \operatorname{dom}(q) - \operatorname{dom}(p)$ *then*

$$\partial(p + q) = \partial p + \partial q.$$

$\triangleleft$ The inclusion $\partial p + \partial q \subset \partial(p + q)$ is trivial. To see the converse inclusion take $l \in \partial r$ with $r := p + q$ and construct a sublinear functional $P : X \times X \to \mathbb{R}$ such that $(f, g) \in \partial P$ implies $f \in \partial p$, $g \in \partial q$, and $l = f + g$. By hypothesis $X_0 := \operatorname{dom}(p) - \operatorname{dom}(q)$ is a subspace of X and therefore the set $H(x,y) := \{h \in X : x+h \in \operatorname{dom}(p),\ y+h \in \operatorname{dom}(q)\}$ is nonempty for all $x, y \in X_0$. Define $P_0 : X_0 \times X_0 \to \mathbb{R}$ as

$$P_0(x,y) := \inf\{p(x+h) + q(y+h) - l(h) :\ h \in H(x,y)\}.$$

If π is a linear projection of X onto X_0 then $P = P_0 \circ \pi$ is the desired sublinear functional. It remains to observe that $\partial P \neq \varnothing$ by the Hahn–Banach Theorem. $\triangleright$

3.5.8. Decomposition Theorem. *Assume that $H_1, \dots, H_N$ are cones in a vector lattice X. Assume further that f and g are positive functionals on X. The inequality*

$$f(h_1 \vee \dots \vee h_N) \geqslant g(h_1 \vee \dots \vee h_N)$$

holds for all $h_k \in H_k$ $(k := 1, \dots, N)$ if and only if to each positive decomposition $(g_1, \dots, g_N)$ of g there is a positive decomposition $(f_1, \dots, f_N)$ of f such that

$$f_k(h_k) \geqslant g_k(h_k) \quad (h_k \in H_k;\ k := 1, \dots, N).$$

$\triangleleft$ Note that the left-hand side of the first inequality of the claim is the sublinear functional p_f of N variables. The right-hand side is the sublinear functional p_g of the same variables. Put $H := H_1 \times \dots \times H_N$ and define $q_H : X^N \to \mathbb{R} \cup \{+\infty\}$ by letting $q_H(u) = 0$ if $u \in H$ and $q_H(u) = +\infty$ if $u \notin H$. Evidently, q_H is sublinear and ∂q_H consists of all N-tuples $(g_1, \dots, g_N)$ such that $g_k : X \to \mathbb{R}$ is linear and $g_k|_{H_k} \leqslant 0$ for all $k := 1, \dots, N$. Note that $p_g \leqslant p_f + q_H$. Using 3.5.6 and the Moreau–Rockafellar formula 3.5.7 we obtain

$$\partial p_g \subset \partial(p_f + p_H) = \partial p_f + \partial q_H.$$

Consequently for every positive decomposition $g = g_1 + \dots + g_N$ we have $(g_1, \dots, g_N) \in \partial p_g$ and so $(g_1, \dots, g_N) = (f_1, \dots, f_N) + (\bar f_1, \dots, \bar f_N)$ with $(f_1, \dots, f_N) \in \partial p_f$ and $(\bar f_1, \dots, \bar f_N) \in \partial q_H$. It follows from the above remarks that $(f_1, \dots, f_N)$ is a positive decomposition of f and $g_k|_{H_k} = f_k|_{H_k} + \bar f_k|_{H_k} \leqslant f_k|_{H_k}$. $\triangleright$

3.5.9. *An order bounded functional on a vector lattice is a two-point relation if and only if its kernel is a vector sublattice of the ambient vector lattice.*

◁ Let l be an order bounded functional on a vector lattice X. We may present l as the difference of the two disjoint functionals $l = l^+ - l^-$, where l^+ and l^- are the positive and negative parts of l. For convenience, we put $f := l^+$, $g := l^-$, and $H := \ker(l)$. It suffices to demonstrate only that g is a lattice homomorphism; i.e., $[0, g] = [0, 1]g$ (cp. 3.4.1 (3)).

So, we take $0 \leqslant g_1 \leqslant g$ and put $g_2 := g - g_1$. We may assume that $g_1 \neq 0$ and $g_1 \neq g$, since otherwise there is nothing left to prove. By hypothesis, for all $h_1, h_2 \in \ker(l)$ we have the inequality

$$f(h_1 \vee h_2) \geqslant g(h_1 \vee h_2).$$

By the Decomposition Theorem there is a decomposition of f into the sum of some positive terms $f = f_1 + f_2$ such that $f_1(h) - g_1(h) = 0$ and $f_2(h) - g_2(h) = 0$ for all $h \in H$. Since H is the zero hyperplane of $l = f - g$, we see that there are reals α and β satisfying $f_1 - g_1 = \alpha(f - g)$ and $f_2 - g_2 = \beta(f - g)$. Clearly, $\alpha + \beta = 1$ (for otherwise $f = g$ and $l = 0$). Therefore, one of the reals α and β is strictly positive. If $\alpha > 0$ then we have $g_1 = \alpha g$ for f and g are disjoint. Since g_1 is not equal to zero, it follows that $0 \leqslant \alpha \leqslant 1$ and $g_1 \in [0, 1]g$. If $\beta > 0$ then, arguing similarly, we see that $g_2 = \beta g$. Since $g_1 \neq g$; therefore, $g_2 \neq 0$. Hence, $0 \leqslant \beta \leqslant 1$ and we again see that $g_1 \in [0, 1]g$. ▷

3.5.10. Proof of Theorem 3.5.4.

◁ We ought to demonstrate only the sufficiency part of the claim. So, let T be an order bounded operator from X to Y and the kernel $\ker(bT) := (bT)^{-1}(0)$ of each stratum of T is a vector sublattice of X.

We reduce the problem to 3.5.9 by means of Boolean valued "scalarization." Without loss of generality, we can assume that Y is a nonzero space embedded as an order dense ideal in the universally complete vector lattice $\mathscr{R}{\downarrow}$.

Denote by $\tau := T{\uparrow}$ the modified ascent of T to $\mathbb{V}^{(\mathbb{B})}$. Then by 1.6.8

$$[\![\tau : X^\wedge \to \mathscr{R}]\!] = \mathbb{1}, \quad (\forall\, x \in X)\ [\![\tau(x^\wedge) = Tx]\!] = \mathbb{1}.$$

Straightforward calculations of truth values show that $T^+{\uparrow} = \tau^+$ and $T^-{\uparrow} = \tau^-$ within $\mathbb{V}^{(\mathbb{B})}$; see 3.3.5. Moreover, $[\![\ker(\tau)$ is a vector sublattice of $X^\wedge]\!] = \mathbb{1}$. Indeed, given $x, y \in X$, put

$$b := [\![Tx = 0^\wedge]\!] \wedge [\![Ty = 0^\wedge]\!].$$

This means that $x, y \in \ker(bT)$ by 2.2.4 ($\mathbb{G}$). Hence, we see by hypothesis that $bT(x \vee y) = 0$, whence $b \leqslant [\![T(x \vee y) = 0^\wedge]\!]$ again by 2.2.4 ($\mathbb{G}$).

Replacing T by τ yields

$$[\![\tau(x^\wedge) = 0^\wedge \wedge \tau(y^\wedge) = 0^\wedge]\!] \leqslant [\![\tau(x \vee y)^\wedge = 0^\wedge]\!].$$

A straightforward calculation of Boolean truth values with use of the last estimate

$$\begin{aligned}
&[\![\ker(\tau) \text{ is a vector sublattice of } X^\wedge]\!] \\
&= [\![(\forall\, x, y \in X^\wedge)(\tau(x) = 0^\wedge \wedge \tau(y) = 0^\wedge \to \tau(x \vee y) = 0^\wedge)]\!] \\
&= \bigwedge_{x,y \in X} [\![\tau(x^\wedge) = 0^\wedge \wedge \tau(y^\wedge) = 0^\wedge]\!] \Rightarrow [\![\tau((x \vee y)^\wedge) = 0^\wedge]\!] = \mathbb{1}.
\end{aligned}$$

completes the proof. $\triangleright$

3.6. Sums of Lattice Homomorphisms

In this section we will give a description for an order bounded operator T whose modulus may be presented as the sum of two lattice homomorphisms in terms of the properties of the kernels of the strata of T.

3.6.1. Recall that a subspace H of a vector lattice is a *G-space* or *Grothendieck subspace* provided that H enjoys the property:

$$(\forall\, x, y \in H)\ (x \vee y \vee 0 + x \wedge y \wedge 0 \in H).$$

3.6.2. Theorem. *Let X and Y be vector lattices with Y Dedekind complete. The modulus of an order bounded operator $T : X \to Y$ is the sum of some pair of lattice homomorphisms if and only if the kernel of each stratum bT of T with $b \in \mathbb{B} := \mathbb{P}(Y)$ is a Grothendieck subspace of the ambient vector lattice X.*

We argue further as follows: Using the functors of canonical embedding and ascent to the Boolean valued universe $\mathbb{V}^{(\mathbb{B})}$, we reduce the matter to characterizing a Grothendieck hyperspace that serves as the kernel of an order bounded functional over a dense subring of the reals $\mathbb{R}$. The scalar case is settled by the following four auxiliary propositions.

3.6.3. *A functional l is the sum of some pair of lattice homomorphisms if and only if l is a positive functional with kernel a Grothendieck subspace.*

$\triangleleft$ *Necessity* is almost evident. Indeed, assume that $l = f + g$ with f and g lattice homomorphisms. Take h_k such that $f(h_k) + g(h_k) = 0$ for $k := 1, 2$. Then

$$f(h_1 \vee h_2 \vee 0) = f(h_1) \vee f(h_2) \vee 0$$
$$= (-g(h_1)) \vee (-g(h_2)) \vee 0 = -g(h_1) \wedge g(h_2) \wedge 0.$$

Similarly, $g(h_1 \vee h_2 \vee 0) = -f(h_1 \wedge h_2 \wedge 0)$. Finally, these yield

$$l(h_1 \vee h_2 \vee 0 + h_1 \wedge h_2 \wedge 0) = (f + g)(h_1 \vee h_2 \vee 0 + h_1 \wedge h_2 \wedge 0) = 0.$$

Hence, $\ker(l)$ is a Grothendieck subspace.

Sufficiency: Take $l \geqslant 0$ and assume that $\ker(l)$ is a Grothendieck subspace. If l has no components other than 0 and l then l is a lattice homomorphism and we are done. Recall that component of f is an extreme point of the order interval $[0, f]$.

Let f be a component of l such that $0 \neq f$ and $f \neq l$. Denote by g the component of l disjoint from f; i.e., $g := l - f$. Clearly, $g \neq 0$ and $g \neq f$. Check that $[0, f] = [0, 1]f$. To this end, take a functional f_1 such that $0 \leqslant f_1 \leqslant f$, $f_1 \neq 0$, and $f_1 \neq f$. Put $f_2 := f - f_1$.

Since H is a Grothendieck subspace; therefore,

$$h_1 \vee h_2 \vee h_3 + h_1 \wedge h_2 \wedge h_3$$
$$= (h_1 - h_3) \vee (h_2 - h_3) \vee 0 + (h_1 - h_3) \wedge (h_2 - h_3) \wedge 0 + 2h_3 \in H$$

for all $h_1, h_2, h_3 \in H$. Consequently,

$$(\forall\, h_1, h_2, h_3 \in H)\ l(h_1 \vee h_2 \vee h_3) \geqslant l((-h_1) \vee (-h_2) \vee (-h_3)).$$

The decomposition of f into the sum $f = f_1 + f_2$ yields the decomposition $l = f_1 + f_2 + g$ of l into a sum of positive terms. By the Decomposition Theorem 3.5.8, there is a decomposition of l into a sum of positive terms $l = l_1 + l_2 + l_3$ such that for all $h \in H$ we have

$$l_1(h) \geqslant f_1(-h), \quad l_2(h) \geqslant f_2(-h), \quad l_3(h) \geqslant g(-h).$$

Since H is the hyperplane of l, there are reals $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ such that

$$f_1 + l_1 = \alpha_1(f + g), \quad f_2 + l_2 = \alpha_2(f + g), \quad g + l_3 = \alpha_3(f + g).$$

Summing up all these equalities and recalling that $l \neq 0$, we see that $\alpha_1 + \alpha_2 + \alpha_3 = 2$.

Summing up the first two equalities, we arrive at the following: $f + l_1 + l_2 = (\alpha_1 + \alpha_2)(f + g)$. We thus obtain $(\alpha_1 + \alpha_2 - 1)f + (\alpha_1 + \alpha_2)g \geqslant 0$ and $\alpha_1 + \alpha_2 \geqslant 1$ since f and g are disjoint. Similarly, $(\alpha_3 - 1)g + \alpha_3 f \geqslant 0$ and $(\alpha_3 - 1)g \geqslant 0$. Since $g \neq 0$; therefore, $\alpha_3 \geqslant 1$. Finally, $\alpha_1 + \alpha_2 = 1$ and $\alpha_3 = 1$. We thus conclude that $l_3 = f$ and $l_1 + l_2 = g$. Moreover, $f_1 - \alpha_1 f = \alpha_1 g - l_1$. Since $0 \leqslant f_1 \leqslant f$ and $0 \leqslant l_1 \leqslant g$; therefore, $|f_1 - \alpha_1 f| \leqslant (1 + |\alpha_1|)f$ and $|\alpha_1 g - l_1| \leqslant (1 + |\alpha_1|)g$. Since f and g are disjoint, we have $f_1 = \alpha_1 f$. Since $f \geqslant f_1 \neq 0$, we see that $1 > \alpha_1 > 0$ and the proof is complete. $\triangleright$

3.6.4. *Given an order bounded functional l on a vector lattice X, assume that $l_+ \neq 0$ and $l_- \neq 0$. The kernel* $\ker(l)$ *is a Grothendieck subspace of X if and only if l_+ and l_- are lattice homomorphisms on X (or, which is the same,* $\ker(l)$ *is a vector sublattice).*

$\triangleleft$ *Sufficiency* is obvious.

Necessity: Put $f := l_+ \neq 0$, $g := l_- \neq 0$, and $H := \ker(f - g)$. Assume further that $0 \leqslant \overline{f} \leqslant f$, $0 \leqslant \overline{g} \leqslant g$, and $h_1, h_2, h_3 \in H$. Clearly,

$$
\begin{aligned}
&f(h_1 \vee h_2 \vee \ h_3) + g((-h_1) \vee (-h_2) \vee (-h_3)) \\
\geqslant\ &f((-h_1) \vee (-h_2 \vee (-h_3)) + g(h_1 \vee h_2 \vee h_3) \geqslant \overline{f}(-h_1) + \overline{g}(h_2).
\end{aligned} \tag{1}
$$

Given $x_1, x_2, x_3 \in X$, put

$$
\begin{gathered}
\varepsilon_3(x, y, z) =: x_1 \vee x_2 \vee x_3;\ \sigma(x) := -x; \\
m(x_1, x_2, x_3) := \overline{f}(-x_1) + \overline{g}(x_2); \quad p := f \circ \varepsilon_3; \quad q := g \circ \varepsilon_3 \circ \sigma.
\end{gathered}
$$

Using subdifferential calculus and the new notation, we can rephrase (1) as follows:

$$
m \in \partial(p + q + \delta(H^3)) = \partial(p) + \partial(q) + \partial(\delta((H^3)),
$$

where $\delta(U)$ is the indicator of a set U (i.e., $\delta(U)x = 0$ for $x \in U$ and $\delta(U)x = +\infty$ for $x \notin U$) and $\partial(s)$ is the subdifferential of a sublinear functional s.

The Decomposition Theorem yields some decompositions of f and g in the sums of positive terms $f = f_1 + f_2 + f_3$ and $g = g_1 + g_2 + g_3$ such that

$$
\begin{gathered}
f_1 + g_1 \circ \sigma - \overline{f} \circ \sigma \in \partial(\delta(H)); \quad f_2 + g_2 \circ \sigma - \overline{g} \in \partial(\delta(H)); \\
f_3 + g_3 \circ \sigma \in \partial(\delta(H)).
\end{gathered}
$$

Since H is the hyperplane of l; therefore, $\partial(\delta(H)) = \{tl : t \in \mathbb{R}\}$. Hence, there are reals $\alpha, \beta, \gamma \in \mathbb{R}$ satisfying

$$f_1 - g_1 + \overline{f} = \alpha(f - g); \quad f_2 - g_2 - \overline{g} = \beta(f - g); \quad f_3 - g_3 = \gamma(f - g).$$

Put $t := \alpha + \beta + \gamma - 1$. Addition yields $tf - \overline{f} = tg - \overline{g}$. Hence,

$$0 \leqslant |tf - \overline{f}| = |tg - \overline{g}| = |tf - \overline{f}| \wedge |tg - \overline{g}| \leqslant (1 + |t|)(f \wedge g) = 0.$$

Consequently, $\overline{f} = tf$ and $\overline{g} = tg$. By hypotheses, $0 \leqslant t \leqslant 1$. Therefore, $[0, f] = [0, 1]f$ and $[0, g] = [0, 1]g$, and so l is a difference of lattice homomorphisms. We conclude that $\ker(l)$ is a vector sublattice and the proof is complete. $\triangleright$

3.6.5. *The kernel of an order bounded functional is a Grothendieck subspace if and only if so is the kernel of the modulus of this functional.*

$\triangleleft$ *Sufficiency:* Let $l : X \to \mathbb{R}$ be an order bounded functional. If $\ker(|l|)$ is a Grothendieck subspace then $|l| = l_1 + l_2$ by 3.6.4, where l_1 and l_2 are lattice homomorphisms. Two lattice homomorphisms are either disjoint or proportional. Without loss of generality, we can assume that l_1 and l_2 are disjoint components of $|l|$ each distinct from 0 and $|l|$ (otherwise, l would be a lattice homomorphism). The order interval $[0, |l|]$ lies in some plane since $[0, 1]l_1 + [0, 1]l_2 = [0, |l|]$. Consequently, every extreme point of $[0, |l|]$ belongs to the set $\{0, l_1, l_2, |l|\}$. Since l_+ and l_- are also disjoint components of $|l|$, we see that $l = l_1 + l_2$ or $l = l_1 - l_2$ or $l = l_2 - l_1$. In the first case $\ker(l)$ is a Grothendieck subspace by 3.6.4; and in the remaining two cases it is a vector sublattice (thus, a Grothendieck subspace) of X.

Necessity: Assume that $\ker(l)$ is a Grothendieck subspace. If either of the functionals l_+ and l_- equals zero then $\ker(|l|) = \ker(l)$ and we are done.

If $l_+ \neq 0$ and $l_- \neq 0$ then l_+ and l_- are lattice homomorphisms by 3.6.4. Thus, $|l|$ is the sum of a pair of lattice homomorphisms. By 3.6.3, $\ker(|l|)$ is a Grothendieck subspace. $\triangleright$

3.6.6. *The kernel of an order bounded functional is a Grothendieck subspace if and only if the modulus of this functional is the sum of a pair of lattice homomorphisms.*

$\triangleleft$ Let l be an order bounded functional. By 3.6.5 the kernel $\ker(l)$ of l is a Grothendieck subspace if and only if so is the subspace $\ker(|l|)$. Since $|l|$ is a positive functional, we are done by 3.6.3. $\triangleright$

3.6.7. PROOF OF THEOREM 3.6.2.

$\lhd$ We start with "scalarizing" the problem. Without loss of generality, we can assume that Y is a nonzero space embedded as an order dense ideal in the universally complete vector lattice $\mathscr{R}{\downarrow}$ which is the descent of the reals $\mathscr{R}$ within $V^{(\mathbb{B})}$.

We further let $X^\wedge$ stand for the standard name of X in $\mathbb{V}^{(\mathbb{B})}$. Clearly, $X^\wedge$ is a vector lattice over $\mathbb{R}^\wedge$ within the Boolean valued universe $\mathbb{V}^{(\mathbb{B})}$. Denote by $l := T{\Uparrow}$ the modified ascent of T to $\mathbb{V}^{(\mathbb{B})}$. By Theorem 3.3.3 $[\![l \in (X^\wedge)^\sim]\!] = \mathbb{1}$ and $[\![l(x^\wedge) = Tx]\!] = \mathbb{1}$ for all $x \in X$. Working within $\mathbb{V}^{(\mathbb{B})}$, we see

$$[\![\ker(l) \text{ is a Grothendieck subspace of } X^\wedge]\!]$$

$$= [\![(\forall\, x, y \in X^\wedge)(l(x) = 0^\wedge \wedge l(y) = 0^\wedge \to l(x \vee y \vee 0 + x \wedge y \wedge 0) = 0^\wedge)]\!] \quad (2)$$

$$= \bigwedge_{x,y \in X} [\![l(x^\wedge) = 0^\wedge \wedge l(y^\wedge) = 0^\wedge \to l((x \vee y \vee 0 + x \wedge y \wedge 0)^\wedge) = 0^\wedge]\!].$$

Sufficiency: Take $x, y \in X$ and put $b := [\![Tx = 0^\wedge]\!] \wedge [\![Ty = 0^\wedge]\!]$. In view of 2.2.4 ($\mathbb{G}$) this means that $x, y \in \ker(bT)$. By hypothesis the kernel of each stratum bT is a Grothendieck subspace. Hence, $bT(x \vee y \vee 0 + x \wedge y \wedge 0) = 0$. In other words,

$$[\![Tx = 0^\wedge]\!] \wedge [\![Ty = 0^\wedge]\!] \leqslant [\![T(x \vee y \vee 0 + x \wedge y \wedge 0) = 0^\wedge]\!].$$

By (2) in follows that

$$[\![\ker(l) \ \text{ is a Grothendieck subspace of } X^\wedge]\!] = \mathbb{1}.$$

Applying 3.6.6 to the order bounded functional l within $\mathbb{V}^{(\mathbb{B})}$ and using the maximum principle, we see that l is the sum of two lattice homomorphisms l_1 and l_2 within $\mathbb{V}^{(\mathbb{B})}$. Define the operators $T_1, T_2 : X \to \mathscr{R}{\downarrow}$ as $T_1 := l_1{\Downarrow}$ and $T_2 := l_2{\Downarrow}$. According to 3.3.6 (1) T_1 and T_2 are lattice homomorphisms satisfying $T_1 + T_2 = T$. Since Y is an ideal of $\mathscr{R}{\downarrow}$, the ranges of T_1 and T_2 lie in Y.

Necessity: Assume that $|T|$ is the sum of a pair of lattice homomorphisms. By Theorem 3.3.3, the ascent of the sum of some summands is obviously the sum of the ascents of the summands, and so l is an order bounded functional within $\mathbb{V}^{(\mathbb{B})}$ whose modulus is the sum of some pair of lattice homomorphisms. From 3.6.6 it follows that

$$[\![\ker(l) \ \text{ is a Grothendieck subspace of } X^\wedge]\!] = \mathbb{1}.$$

Considering (2), we infer that the kernel of each stratum bT of T is a Grothendieck subspace of X. Indeed, for $x, y \in X$ it follows by (2) that

$$[\![l(x^\wedge) = 0^\wedge \wedge l(y^\wedge) = 0^\wedge \to l((x \vee y \vee 0 + x \wedge y \wedge 0)^\wedge) = 0^\wedge]\!] = \mathbb{1}.$$

Therefore,

$$[\![l(x^\wedge) = 0^\wedge]\!] \wedge [\![l(y^\wedge) = 0^\wedge]\!] \leqslant [\![l((x \vee y \vee 0 + x \wedge y \wedge 0)^\wedge) = 0^\wedge]\!].$$

Consequently, if $b \in \mathbb{B}$ and $bTx = bTy = 0$ then $b \leqslant [\![Tx = 0]\!] \wedge [\![Ty = 0]\!]$ by 2.2.4 ($\mathbb{G}$) or, taking into account the definition of $l = T\!\uparrow$, $b \leqslant [\![lx^\wedge = 0^\wedge]\!] \wedge [\![ly^\wedge = 0^\wedge]\!]$ and we obtain

$$[\![l((x \vee y \vee 0 + x \wedge y \wedge 0)^\wedge) = 0^\wedge]\!] \geqslant b.$$

Finally, one more use of 2.2.4 ($\mathbb{G}$) gives $bT(x \vee y \vee 0 + x \wedge y \wedge 0) = 0$.

The proof of the theorem is complete. ▷

3.7. Polydisjoint Operators

The aim of the present section is to describe the order ideal in the space of order bounded operators which is generated by the order bounded disjointness preserving operators in terms of n-disjoint operators.

3.7.1. Let X and Y be vector lattices and let n be a positive integer. A linear operator $T : X \to Y$ is *n-disjoint* if, for every disjoint collection of $n + 1$ elements $x_0, \dots, x_n \in X$, the meet of $\big\{|Tx_k| : k := 0, 1, \dots, n\big\}$ equals zero; symbolically:

$$(\forall\, x_0, x_1 \dots, x_n \in X)\ x_k \perp x_l\ (k \neq l) \implies |Tx_0| \wedge \dots \wedge |Tx_n| = 0.$$

An operator is called *polydisjoint* if it is n-disjoint for some $n \in \mathbb{N}$.

Evidently, if an operator is n-disjoint then it is m-disjoint for all $m \geqslant n$.

A 1-disjoint operator is just a disjointness preserving operator. Theorem 3.6.2 tells us that 2-disjoint operators are just those satisfying the condition: the kernel of every stratum is a Grothendieck subspace.

Consider some simple properties of n-disjoint operators.

3.7.2. *Let X and Y be vector lattices with Y Dedekind complete. An operator $T \in L^\sim(X,Y)$ is n-disjoint if and only if its modulus $|T|$ is n-disjoint.*

$\lhd$ Sufficiency is obvious from the inequality $|T(x)| \leqslant |T|(|x|)$ $(x \in X)$. Suppose that the operator T is n-disjoint. Take pairwise disjoint elements $e_0, \dots, e_n \in X_+$. Observe that if $|x_k| \leqslant e_k$ then $x_k \perp x_l$ $(k \neq l)$; therefore, $|Tx_0| \wedge \dots \wedge |Tx_n| = 0$. Passing to the supremum over $x_0, \dots, x_n$ in the last equality, we obtain $|T|e_0 \wedge \dots \wedge |T|e_n = 0$ by 3.1.4 (5). $\rhd$

3.7.3. *Let $T_1, \dots, T_n$ be order bounded disjointness preserving operators from X to Y. Then $T := T_1 + \dots + T_n$ is n-disjoint.*

$\lhd$ Take $x_0, x_1, \dots, x_n \in X$ with $x_k \perp x_l$ for all $k \neq l$. Then, from 2.1.6 (3) we have

$$\bigwedge_{k=1}^{n} |Tx_k| \leqslant \sum_{\iota \in I} |T_{\iota(0)}x_0| \wedge \dots \wedge |T_{\iota(n)}x_n|,$$

where I is the set of all mappings from $\{0, 1, \dots, n\}$ to $\{1, \dots, n\}$. Evidently, each summand on the right-hand side contains at least two identical indices. Assuming that $m := \iota(k) = \iota(l)$ for some $k \neq l$ we deduce

$$\begin{aligned} |T_{\iota(0)}x_0| \wedge |T_{\iota(1)}x_1| \wedge \dots \wedge |T_{\iota(n)}x_n| &\leqslant |T_m x_k| \wedge |T_m x_l| \\ &\leqslant |T_m|(|x_k|) \wedge |T_m|(|x_l|) = |T_m|(|x_k| \wedge |x_l|) = 0. \end{aligned}$$

It follows $\bigwedge_{k=1}^{n} |Tx_k| = 0$ and the proof is complete. $\rhd$

In the next two propositions we will characterize n-disjoint order bounded functionals; i.e., "scalarize" the problem.

3.7.4. *Assume that $f \in C(Q)'$ is n-disjoint for some $n \in \mathbb{N}$. Then there exist $q_1, \dots, q_n \in Q$ and $a_1, \dots, a_n \in \mathbb{R}$ such that*

$$f = \sum_{k=1}^{n} a_k \delta_{q_k}$$

where $\delta_q \in C(Q)'$ is the Dirac delta measure $x \mapsto x(q)$ $(x \in C(Q))$ at $q \in Q$.

$\lhd$ Assume that f is an n-disjoint functional. According to 3.7.2 there is no loss of generality in assuming that f is positive. Prove that the corresponding Radon measure μ is a linear combination of n Dirac delta measures. This is equivalent to saying that the support of μ contains at most n points. If there are $n+1$ points $q_0, q_1, \dots, q_n \in Q$ in the support of μ then we can choose pairwise disjoint compact neighborhoods $U_0, U_1, \dots, U_n$ of these points and next take pairwise disjoint open sets $V_k \subset Q$ with $\mu(U_k) > 0$ and $U_k \subset V_k$ $(k = 0, 1, \dots, n)$. Using the Tietze–Urysohn Theorem, construct a continuous function x_k on Q which vanishes on $Q \setminus V_k$ and is identically one on U_k. Then $x_0 \wedge x_1 \wedge \dots \wedge x_n = 0$ but none of $f(x_0), f(x_1), \dots, f(x_n)$ is equal to zero, since $f(x_k) \geqslant \mu(U_k) > 0$ for all $k := 0, 1, \dots, n$. This contradiction shows that the support of μ contains at most n points. $\triangleright$

3.7.5. *An order bounded functional on a vector lattice is n-disjoin if and only if it is representable as a disjoint sum of n order bounded disjointness preserving functionals. This representation is unique up to permutation.*

$\triangleleft$ Let f be an n-disjoint functional on a vector lattice X and denote by m the least natural for which f is m-disjoint. Then there exists a disjoint collection $x_1, \dots, x_m \in X$ such that none of the reals $f(x_1), \dots, f(x_m)$ is equal to zero. Let $J(e)$ stand for the order ideal in X generated by $e := |x_1| + \dots + |x_m|$. By the Kakutani–Kreĭns Representation Theorem we can consider $J(e)$ as a norm dense vector sublattice of $C(Q)$ for some Hausdorff compact topological space Q. Clearly, $f|_{J(e)}$ admits the unique extension f^e by continuity to the whole of $C(Q)$; moreover, f^e is m-disjoint. By 3.7.4 $f|_{J(e)} = f^e|_{J(e)}$ is representable as a sum of m nonzero order bounded disjointness preserving functionals $f^e_1, \dots, f^e_m$. Given $x \in X$, we can choose m nonzero order bounded disjointness preserving functionals $f_1^{e(x)}, \dots, f_m^{e(x)}$ on $J(e(x))$ with $e(x) = |x| + e$ such that $f|_{J(e(x))} = f_1^{e(x)} + \dots + f_m^{e(x)}$. Finally, the functional f_k defined on X by letting $f_k(x) := f_k^{e(x)}(x)$ is order bounded and disjointness preserving, while $f = f_1 + \dots + f_m$. The functionals f_k and f_l with $k \neq l$ are disjoint. Indeed, for every $x \in X_+$ the functionals $f_k^{e(x)}$ and $f_l^{e(x)}$ are disjoint and so

$$
\begin{aligned}
(f_k \wedge f_l)(x) &= \inf \left\{ f_k(x_1) + f_l(x_2) :\ x_1, x_2 \in X_+,\ x_1 + x_2 = x \right\} \\
&= \inf \left\{ f_k^{e(x)}(x_1) + f_l^{e(x)}(x_2) :\ x_1, x_2 \in J(e(x))_+,\ x_1 + x_2 = x \right\} = 0.
\end{aligned}
$$

If $m < n$ then some zero terms should be added. $\triangleright$

3.7.6. *Let $T : X \to \mathscr{R}{\downarrow}$ be an order bounded linear operator and $\tau := T{\uparrow}$. Then τ is $n^\wedge$-disjoint if and only if T is n-disjoint.*

$\lhd$ Put $b := [\![\tau \text{ is } n^\wedge\text{-disjoint}]\!]$ and ensure that if T is n-disjoint then $b = \mathbb{1}$. Identifying $n^\wedge$ with $\{0,\dots,n-1\}^\wedge$ and using 1.5.2, we deduce

$$b = \Big[\!\!\Big[(\forall\,\nu : n^\wedge \to X^\wedge)\big((\forall\,k,l \leqslant n^\wedge)(k \neq l \to \nu(k) \perp \nu(l)) \\ \to \inf_{k\in n^\wedge} |\tau(\nu(k))| = 0\big)\Big]\!\!\Big]$$

$$= \bigwedge\Big\{[\![\inf\{|\tau(\nu(\imath))| :\ \imath \in n^\wedge\} = 0]\!] :\ \nu \in [\![n^\wedge \to X^\wedge]\!], \\ [\![(\forall\,k,l \in n^\wedge)(k \neq l \to \nu(k) \perp \nu(l))]\!] = \mathbb{1}\Big\}$$

$$= \bigwedge\Big\{[\![\inf\{|\tau(\mathrm{im}(\nu))| = 0]\!] :\ \nu{\downarrow} \in [n \to X^\wedge{\downarrow}], \\ (\forall\,k \neq l)[\![\nu{\downarrow}(k) \perp \nu{\downarrow}(l)]\!] = \mathbb{1}\Big\}.$$

Since $X^\wedge{\downarrow} = \mathrm{mix}\{x^\wedge : x \in X\}$, we can choose a partition of unity $(b_\xi)_{\xi\in\Xi}$ and a finite collection of families $(x_{\xi,k})_{\xi\in\Xi}$ $(k := 0,1,\dots,n)$ in X such that $\nu{\downarrow}(k) = \mathrm{mix}_{\xi\in\Xi}(b_\xi x^\wedge_{\xi,k})$. It follows from $[\![\nu{\downarrow}(k) \perp \nu{\downarrow}(l)]\!] = \mathbb{1}$ that $x_{\xi,k} \perp x_{\xi,l}$ whenever $b_\xi \neq 0$ and $k \neq l$. Putting $A_\xi := \{x_{\xi,0},\dots,x_{\xi,n}\}$, we can easily check that $b_\xi \leqslant [\![\mathrm{im}(\nu) = A^\wedge_\xi]\!]$. Working within the relative universe $\mathbb{V}^{(\mathbb{B}_\xi)}$ with $\mathbb{B}_\xi := [\mathbb{0}, b_\xi]$ and using 1.6.8 and 2.2.4 ($\mathbb{G}$), we deduce $\tau(A^\wedge_\xi) = T(A_\xi){\uparrow}$ and $\inf|T(A_\xi){\uparrow}| = \inf|T(A_\xi)|$, so that

$$\mathbb{V}^{(\mathbb{B}_\xi)} \models \inf|\tau(\mathrm{im}(\nu))| = \inf|\tau(A^\wedge_\xi)| = \inf|T(A_\xi){\uparrow}| = \inf|T(A_\xi)|.$$

Thus, $\inf|T(A_\xi)| = |Tx_{\xi,0}| \wedge \dots \wedge |Tx_{\xi,n}| = 0$ or, equivalently, $[\![\inf|T(A_\xi)| = 0]\!] = \mathbb{1}$ for all ξ, since T is n-disjoint. Using 1.2.5 (3) we deduce

$$b_\xi \leqslant [\![\inf|\tau(\mathrm{im}(\nu))| = \inf|T(A_\xi)|]\!] \wedge [\![\inf|T(A_\xi)| = 0]\!] \\ \leqslant [\![\inf|\tau(\mathrm{im}(\nu))| = 0]\!],$$

so that $b = \mathbb{1}$. The converse is demonstrated similarly. $\rhd$

3.7.7. Theorem. *An order bounded operator from a vector lattice to a Dedekind complete vector lattice is n-disjoint for some $n \in \mathbb{N}$ if and only if it is representable as a disjoint sum of n order bounded disjointness preserving operators.*

◁ Assume that X and Y are vector lattices with Y Dedekind complete. Assume further that $T \in L(X, Y)$ is order bounded and n-disjoint. Let $\tau \in \mathbb{V}^{(\mathbb{B})}$ be an internal linear functional on $X^\wedge$ with $[\![Tx = \tau(x^\wedge)]\!] = \mathbb{1}$ for all $x \in X$. Then τ is order bounded n-disjoint functional by 3.3.3 and 3.7.6. According to the transfer principle, applying 3.7.5 to τ yields some pairwise disjoint order bounded disjointness preserving functionals $\tau_1, \dots, \tau_n$ on $X^\wedge$ with $\tau = \tau_1 + \dots + \tau_n$. It remains to observe that by Theorem 3.3.3 $T_k = \tau_k{\downarrow}$ is an order bounded disjointness preserving operator from X to Y and $T = T_1 + \dots + T_n$. Moreover, if $k \neq l$ then T_k and T_l are disjoint by Corollary 3.3.5 (5). ▷

3.7.8. *The representation of an order bounded n-disjoint operator in Theorem 3.7.7 is unique up to mixing: if $T = T_1 + \dots + T_n = S_1 + \dots + S_m$ for two disjoint collections $\{T_1, \dots, T_n\}$ and $\{S_1, \dots, S_m\}$ of order bounded disjointness preserving operators then for every $\jmath = 1, \dots, m$ there exists a disjoint collection of band projections $\pi_{1\jmath}, \dots, \pi_{n\jmath} \in \mathbb{P}(Y)$ such that*

$$S_\jmath = \pi_{1\jmath} T_1 + \dots + \pi_{n\jmath} T_n$$

for all $\jmath := 1, \dots, m$.

◁ Let T_k, τ, and τ_k be the same as in the proof of 3.7.7 and $\sigma_k := S_k{\uparrow}$. Then

$$[\![\tau = \tau_1 + \dots + \tau_{n^\wedge} = \sigma_1 + \dots + \sigma_{m^\wedge}]\!] = \mathbb{1}.$$

It follows from the uniqueness of the representation in 3.7.5 that $[\![(\forall\, \jmath \leqslant m^\wedge)\,(\exists\, \imath \leqslant n^\wedge)\,(\sigma_\imath = \tau_\jmath)]\!] = \mathbb{1}$. Evaluating the Boolean truth values for quantifiers according to 1.2.3 yields $\mathbb{1} = \bigvee_{\imath=1}^{n} [\![\sigma_\jmath = \tau_\imath]\!]$ for every $\jmath \leqslant m$. For every $\jmath \leqslant m$ we can take a partition of unity $(b_{\imath\jmath})_{\imath=1}^{n}$ such that $[\![\sigma_\jmath = \tau_\imath]\!] \geqslant b_{\imath\jmath}$ or, equivalently, $\sigma_\jmath = \operatorname{mix}_{\imath \leqslant n} b_{\imath\jmath}\tau_\imath$. It follows that $S_\jmath = \pi_{1\jmath} T_1 + \dots + \pi_{n\jmath} T_n$ for all $\jmath := 1, \dots, m$, where $\pi_{\imath\jmath} = \chi(b_{\imath\jmath})$ by 3.3.7. ▷

3.7.9. Corollary. *A positive linear operator from a vector lattice to a Dedekind complete vector lattice is n-disjoint if and only if it is the disjoint sum of n lattice homomorphisms.*

3.7.10. Corollary. *The set of polydisjoint operators from a vector lattice to a Dedekind complete vector lattice coincides with the order ideal in the vector lattice of order bounded linear operators generated by lattice homomorphisms or, equivalently, by disjointness preserving operators.*

3.8. Sums of Disjointness Preserving Operators

In this section we examine the problem of finding conditions for the sum of a finite collection of order bounded disjointness preserving operators to be n-disjoint. We will start with the case of functionals.

3.8.1. *For a finite collection of order bounded disjointness preserving functionals $f_1, \dots, f_N$ on X the following are equivalent:*

(1) *$f_\imath + f_\jmath$ is disjointness preserving for all $1 \leqslant \imath, \jmath \leqslant N$.*

(2) *$|f_1| + \dots + |f_N|$ is a lattice homomorphism.*

(3) *There exists a lattice homomorphism $h : X \to \mathbb{R}$ such that $f_\imath = \lambda_\imath h$ $(\imath := 1, \dots, N)$ for some $\lambda_1, \dots, \lambda_N \in [0, 1] \subset \mathbb{R}$.*

(4) *If $f_\imath \neq 0$ and $f_\jmath \neq 0$, then $|f_\imath| \wedge |f_\jmath| \neq 0$ for all $1 \leqslant \imath, \jmath \leqslant N$.*

(5) *$f_\imath \neq 0$ and $f_\jmath \neq 0$ imply $\ker(f_\imath) = \ker(f_\jmath)$ for all $1 \leqslant \imath, \jmath \leqslant N$.*

$\lhd$ (1) $\Longrightarrow$ (2): Assume that (1) holds, while $h := |f_1| + \dots + |f_N|$ is not a lattice homomorphism. Find naturals $1 \leqslant \imath, \jmath \leqslant N$ with $f_\imath \neq 0$, $f_\jmath \neq 0$, and $f_\imath \perp f_\jmath$. It follows that $|f_\imath + f_\jmath| = |f_\imath| + |f_\jmath|$ is not a lattice homomorphism, whereas $f_\imath + f_\jmath$ is disjointness preserving; a contradiction.

(2) $\Longrightarrow$ (3): This is immediate from 3.4.1 (3) with $h := |f_1| + \dots + |f_N|$.

(5) $\Longrightarrow$ (3): If $f_1 = \dots = f_N = 0$, there is nothing to prove. Otherwise, choose a natural $\jmath \leqslant N$ with $f_\jmath \neq 0$. Then for each nonzero $f_\imath$ we have $\ker(f_\jmath) = \ker(f_\imath)$ and so $f_\imath = \lambda_i f_\jmath$ for some nonzero $\lambda_\imath \in \mathbb{R}$. Put $\lambda_\imath = 0$ whenever $f_\imath = 0$. It remains to put $h := |f_1| \vee \dots \vee |f_N|$.

The remaining implications (3) $\Longrightarrow$ (4) $\Longrightarrow$ (5) and (3) $\Longrightarrow$ (1) are obvious. $\rhd$

3.8.2. *Assume that $n, N \in \mathbb{N}$ and $n < N$. For a finite collection of order bounded disjointness preserving functionals $f_1, \dots, f_N$ on X the following are equivalent:*

(1) *The sum $g_1 + \dots + g_{n+1}$ is n-disjoint for an arbitrary permutation $(g_1, \dots, g_N)$ of $(f_1, \dots, f_N)$.*

(2) *$|f_1| + \dots + |f_N|$ is n-disjoint.*

(3) *There is a permutation $(g_1, \dots, g_N)$ of $(f_1, \dots, f_N)$ such that $g_1, \dots, g_n$ are pairwise disjoint and, for $\imath := n + 1, \dots, N$, the representation $g_\imath = \lambda_\imath g_{\kappa(\imath)}$ holds with some $\kappa(\imath) \in \{1, \dots, n\}$ and $\lambda \in \mathbb{R}$, $|\lambda| \leqslant 1$.*

$\lhd$ Simple arguments similar to those in 3.8.1 will do the trick. $\rhd$

3.8.3. Having settled the scalar case, let us discuss the conditions under which the sum of order bounded linear operators is disjointness preserving or n-disjoint with $n \geqslant 1$. The following definition is motivated by 3.7.8.

Given two collections $\mathscr{T} := (T_1, \dots, T_N)$ and $\mathscr{S} := (S_1, \dots, S_N)$ of linear operators from X to Y, say that $\mathscr{S}$ is a $\mathbb{P}(Y)$-*permutation* of $\mathscr{T}$ whenever there exists an $N \times N$ matrix $(\pi_{\imath,l})$ with entries from $\mathbb{P}(Y)$, whose rows and columns are partitions of unity in $\mathbb{P}(Y)$ such that $S_\imath = \sum_{l=1}^N \pi_{\imath,l} T_l$ for all $\imath := 1, \dots, N$ (and so $T_l = \sum_{\imath=1}^N \pi_{\imath,l} S_\imath$ for all $l := 1, \dots, N$).

The *range projection* R_T of an operator $T : X \to Y$ is the least band projection in Y with $T = R_T \circ T$ or, equivalently, R_T is a band projection onto the band $T(X)^{\perp\perp}$ in Y.

3.8.4. *Let T be an order bounded linear operator from X to $Y := \mathscr{R}{\downarrow}$ and $\tau := T{\uparrow}$. Then $\chi([\![\tau \neq 0]\!])$ coincides with the range projection R_T.*

$\lhd$ It follows from the Gordon Theorem that, given $y \in Y$, the band projection $[y]$ onto $\{y\}^{\perp\perp}$ coincides with $\chi([\![y \neq 0]\!])$. Therefore, we can calculate

$$
\begin{aligned}
b &:= [\![\tau \neq 0]\!] = [\![(\exists\, x \in X^{\wedge})\tau(x) \neq 0]\!] \\
&= \bigvee_{x \in X} [\![\tau(x^{\wedge}) \neq 0]\!] = \bigvee_{x \in X} [\![T(x) \neq 0]\!].
\end{aligned}
$$

It remains to observe that $\chi(b) = \bigvee_{x \in X} \chi([\![T(x) \neq 0]\!]) = \bigvee_{x \in X} [T(x)] = R_T$. $\rhd$

3.8.5. *Given $\tau, \sigma \in \mathbb{V}^{(\mathbb{B})}$ with $[\![\tau, \sigma : \{1, \dots, N\}^{\wedge} \to (X^{\wedge})^{\sim}]\!] = \mathbb{1}$. For $l \in \{1, \dots, N\}$ put $\tau_l := \tau{\downarrow}(l)$, $\sigma_l := \sigma{\downarrow}(l)$, $T_l := \tau_l{\downarrow}$, and $S_l := \sigma_l{\downarrow}$. Denote $(\tau_1, \dots, \tau_{N^{\wedge}}) := \operatorname{im}(\tau)$ and $(\sigma_1, \dots, \sigma_{N^{\wedge}}) := \operatorname{im}(\sigma)$. Then $(\sigma_1, \dots, \sigma_{N^{\wedge}})$ is a permutation of $(\tau_1, \dots, \tau_{N^{\wedge}})$ within $\mathbb{V}^{(\mathbb{B})}$ if and only if $(S_1, \dots, S_N)$ is a $\mathbb{P}(Y)$-permutation of $(T_1, \dots, T_N)$.*

$\lhd$ Assume that $(\sigma_1, \dots, \sigma_{N^{\wedge}})$ is a permutation of $(\tau_1, \dots, \tau_{N^{\wedge}})$. Take some permutation $\nu : \{1, \dots, N\}^{\wedge} \to \{1, \dots, N\}^{\wedge}$ such that $\sigma_\imath = \tau_{\nu(\imath)}$ $(\imath \in \{1, \dots, N\}^{\wedge})$. By 1.5.8 $\nu{\downarrow}$ is a function from $\{1, \dots, N\}$ to $(\{1, \dots, N\}^{\wedge}){\downarrow} = \operatorname{mix}(\{1^{\wedge}, \dots, N^{\wedge}\})$. Thus, for each $\imath \in \{1, \dots, N\}$ there exists a partition of unity $(b_{\imath,l})_{l=1}^N$ such that $\nu{\downarrow}(\imath) = \operatorname{mix}_{l \leqslant N}(b_{\imath,l} l^{\wedge})$.

Since ν is injective, we have

$$\begin{aligned}\mathbb{1} &= [\![(\forall\,\imath,\jmath \in \{1,\dots,N\}^\wedge)(\nu(\imath)=\nu(\jmath)\to \imath=\jmath)]\!]\\ &= \bigwedge_{\imath,\jmath=1}^{N} [\![\nu(\imath^\wedge)=\nu(\jmath^\wedge)\to \imath^\wedge=\jmath^\wedge]\!]\\ &= \bigwedge_{\imath,\jmath=1}^{N} [\![\nu{\downarrow}(\imath)=\nu{\downarrow}(\jmath)]\!] \Rightarrow [\![\imath^\wedge=\jmath^\wedge]\!],\end{aligned}$$

and so $[\![\nu{\downarrow}(\imath)=\nu{\downarrow}(\jmath)]\!] \leqslant [\![\imath^\wedge=\jmath^\wedge]\!]$ for all $\imath,\jmath \leqslant N$. Taking this inequality and the definition of $\nu{\downarrow}$ into account yields

$$b_{\imath,l}\wedge b_{\jmath,l} \leqslant [\![\nu{\downarrow}(\imath)=l^\wedge]\!]\wedge[\![\nu{\downarrow}(\jmath)=l^\wedge]\!] \leqslant [\![\nu{\downarrow}(\imath)=\nu{\downarrow}(\jmath)]\!] \leqslant [\![\imath^\wedge=\jmath^\wedge]\!],$$

so that $\imath\neq\jmath$ implies $b_{\imath,l}\wedge b_{\jmath,l}=0$ (because $\imath\neq\jmath \iff [\![\imath^\wedge=\jmath^\wedge]\!]=\mathbb{0}$ by 1.4.5 (2)). At the same time, the surjectivity of ν implies

$$\begin{aligned}\mathbb{1} &= [\![(\forall\, l\in\{1,\dots,N\}^\wedge)(\exists\,\imath\in\{1,\dots,N\}^\wedge) l=\nu(\imath)]\!]\\ &= \bigwedge_{l=1}^{N}\bigvee_{\imath=1}^{N}[\![l^\wedge=\nu{\downarrow}(\imath)]\!] = \bigwedge_{l=1}^{N}\bigvee_{\imath=1}^{N} b_{\imath,l}.\end{aligned}$$

Hence, $(b_{\imath,l})_{\imath=1}^{N}$ is a partition of unity in $\mathbb{B}$ for all $l=1,\dots,N$. By the choice of ν it follows that $b_{\imath,l}\leqslant[\![\sigma_\imath=\tau_l]\!]$, because of the estimations

$$\begin{aligned}b_{\imath,l} &\leqslant [\![\sigma(\imath^\wedge)=\tau(\nu(\imath^\wedge))]\!]\wedge[\![\nu(\imath^\wedge)=l^\wedge]\!]\\ &\leqslant [\![\sigma(\imath^\wedge)=\tau(l^\wedge)]\!]=[\![\sigma_\imath=\tau_l]\!].\end{aligned}$$

Put $\pi_{\imath,l}:=\chi(b_{\imath,l})$ and observe that $b_{\imath,l}\leqslant[\![\sigma_\imath(x^\wedge)=\tau_l(x^\wedge)]\!]\leqslant[\![S_i x = T_l x]\!]$ for all $x\in X$ and $1\leqslant\imath,\jmath\leqslant N$. Using 2.2.4 ($\mathbb{G}$), we obtain $\pi_{\imath,l}S_\imath = \pi_{\imath,l}T_l$ and so $S_\imath=\sum_{l=1}^{N}\pi_{\imath,l}T_l$ for all $1\leqslant\imath\leqslant N$. Clearly, $(\pi_{\imath,l})$ is the $N\times N$ matrix as required in Definition 3.8.3. The sufficiency can be seen by the same reasoning in the reverse direction. $\triangleright$

3.8.6. Theorem. *For a finite collection of order bounded disjointness preserving linear operators $T_1,\dots,T_N$ from X to Y the following are equivalent:*

(1) *$T_\imath+T_\jmath$ is disjointness preserving for all $1\leqslant\imath,\jmath\leqslant N$.*

(2) *$|T_1|+\dots+|T_N|$ is a lattice homomorphism.*

(3) *There exist a lattice homomorphism $T: X\to Y$ and orthomor-*

phisms $\varrho_1, \dots, \varrho_\jmath \in \mathscr{Z}(Y)$ *such that* $T_\imath = \varrho_\imath T$ $(\imath := 1, \dots, N)$.

(4) *If* $R_{T_\imath} \circ R_{T_\jmath} \leqslant R_{|T_\imath| \wedge |T_\jmath|}$ *for all* $1 \leqslant \imath, \jmath \leqslant N$.

(5) *For* $\pi \in \mathbb{P}(Y)$ *and* $1 \leqslant \imath, \jmath \leqslant N$ *the inequality* $\pi \leqslant R_{T_\imath} \circ R_{T_\jmath}$ *implies* $\ker(\pi T_\imath) = \ker(\pi T_\jmath)$.

$\lhd$ We may assume without loss of generality that $Y = \mathscr{R}{\downarrow}$. Put $f_\imath := T_\imath{\uparrow}$ $(\imath := 1, \dots, N)$. Note that 3.8.1 is valid within $\mathbb{V}^{(\mathbb{B})}$ in view of the transfer principle, so that it suffices to ensure that 3.8.6 (k) is equivalent to the interpretation of 3.8.1 (k) within $\mathbb{V}^{(\mathbb{B})}$ for all $k = 1, \dots, 5$. For $k = 1, 2$ the equivalences are obvious. Putting $h := T{\uparrow}$ and using 3.3.6 (1) yields $[\![3.8.1\,(3)]\!] = \mathbb{1} \iff$ 3.8.6 (3). Furthermore, 3.8.6 (4) may be symbolized as $\Phi \equiv (\forall\, \imath, \jmath \in \{1, \dots, N\}^\wedge)(f_\imath \neq 0 \wedge f_\jmath \neq 0 \to |f_\imath| \wedge |f_\jmath| \neq 0)$, so that

$$[\![\Phi]\!] = \bigwedge_{\imath, \jmath \leqslant N} [\![f_\imath \neq 0]\!] \wedge [\![f_\jmath \neq 0]\!] \Rightarrow [\![|f_\imath| \wedge |f_\jmath| \neq 0]\!].$$

Thus, $[\![3.8.1\,(4)]\!] = \mathbb{1}$ if and only if $[\![f_\imath \neq 0]\!] \wedge [\![f_\jmath \neq 0]\!] \leqslant [\![|f_\imath| \wedge |f_\jmath| \neq 0]\!]$ for all $\imath, \jmath \leqslant N$. The latter is equivalent to 3.8.6 (4) by 3.8.4. The remaining equivalence of $[\![3.8.1\,(5)]\!] = \mathbb{1}$ and 3.8.6 (5) is verified by combining the above arguments with the proof of 3.4.2. $\rhd$

3.8.7. Theorem. *Let* $n, N \in \mathbb{N}$ *and* $n < N$. *For a collection of order bounded disjointness preserving linear operators* $T_1, \dots, T_N$ *from* X *to* Y *the following are equivalent:*

(1) *For an arbitrary* $\mathbb{P}(Y)$*-permutation* $S_1, \dots, S_N$ *of* $T_1, \dots, T_N$ *the sum* $S_1 + \dots + S_{n+1}$ *is* n*-disjoint.*

(2) $|T_1| + \dots + |T_N|$ *is* n*-disjoint.*

(3) *There exists a* $\mathbb{P}(Y)$*-permutation* $S_1, \dots, S_N$ *of* $T_1, \dots, T_N$ *such that* $S_1, \dots, S_n$ *are pairwise disjoint and each of* $S_{n+1}, \dots, S_N$ *is representable as* $S_\jmath = \sum_{\imath=1}^{n} \alpha_{\imath,\jmath} S_\imath$ *for some pairwise disjoint* $\alpha_{1,\jmath}, \dots, \alpha_n \in \mathscr{Z}(Y)$ $(\jmath := n+1, \dots, N)$.

$\lhd$ We can assume that $Y = \mathscr{R}{\downarrow}$ and put $\tau_l := T_l{\uparrow}$. The equivalence (1) $\iff$ (2) is immediate and we need only check (2) $\iff$ (3). Moreover, there is no loss of generality in assuming that $T_1, \dots, T_N$ are lattice homomorphisms so that $\tau_1, \dots, \tau_N$ are also assumed to be lattice homomorphisms within $\mathbb{V}^{(\mathbb{B})}$.

(2) $\Longrightarrow$ (3): Supposing (2) and working within $\mathbb{V}^{(\mathbb{B})}$, observe that $\tau_1 + \dots + \tau_N$ is $n^\wedge$-disjoint and so there exists a permutation $\nu : \{1, \dots, N\}^\wedge \to \{1, \dots, N\}^\wedge$ such that $\tau_{\nu(1)}, \dots, \tau_{\nu(n)}$ are pairwise disjoint lattice homomorphisms, while each of the homomorphisms

$\tau_{\nu(n+1)},\dots,\tau_{\nu(N)}$ is proportional to some of $\tau_{\nu(1)},\dots,\tau_{\nu(n)}$ with a constant of modulus $\leqslant 1$. The latter is formalized as follows:

$$\Phi \equiv (\forall\, \imath \in \{n+1,\dots,N\}^{\wedge})(\exists\, \jmath \in \{1,\dots,n\}^{\wedge}) \\ (\exists\, \beta \in \mathscr{R})(|\beta| \leqslant 1 \wedge \tau_{\nu(\imath)} = \beta\tau_{\nu(\jmath)}).$$

Put $S_\imath := \tau_{\nu(\imath^{\wedge})}\!\downarrow$ $(\imath := 1,\dots,N)$. Then $(S_1,\dots,S_N)$ is a $\mathbb{P}(Y)$-permutation of $(T_1,\dots,T_N)$ and $(S_1,\dots,S_n)$ are pairwise disjoint by 3.3.5 (5). Moreover, $[\![\Phi]\!] = \mathbb{1}$ by transfer. Hence,

$$\mathbb{1} = \bigwedge_{\imath=n+1}^{N} \bigvee_{\jmath=1}^{n} [\![(\exists\, \beta)(\beta \in \mathscr{R})(|\beta| \leqslant 1 \wedge \tau_{\nu(\imath^{\wedge})} = \beta\tau_{\nu(\jmath^{\wedge})})]\!].$$

It follows that for each $n+1 \leqslant \imath \leqslant N$ there is a partition of unity $\{b_{\imath,1},\dots,b_{\imath,n}\}$ in $\mathbb{B}$ such that $b_{\imath,\jmath} \leqslant [\![(\exists\, \beta)(\beta \in \mathscr{R})(|\beta| \leqslant 1 \wedge \tau_{\nu(\imath^{\wedge})} = \beta\tau_{\nu(\jmath^{\wedge})})]\!]$. According to the maximum principle there exists $\beta_{\imath,\jmath} \in \mathscr{R}\!\downarrow$ with $b_{\imath,\jmath} \leqslant [\![|\beta_{\imath,\jmath}| \leqslant 1]\!] \wedge [\![\tau_{\nu(\imath^{\wedge})} = \beta_{\imath,\jmath}\tau_{\nu(\jmath^{\wedge})}]\!]$. Observe that for each $x \in X$ we have

$$b_{\imath,\jmath} \leqslant [\![\tau_{\nu(\imath^{\wedge})} = \beta_{\imath,\jmath}\tau_{\nu(\jmath^{\wedge})})]\!] \leqslant [\![\tau_{\nu(\imath^{\wedge})}(x^{\wedge}) = \beta_{\imath,\jmath}\tau_{\nu(\jmath^{\wedge})}(x^{\wedge})]\!] \\ \wedge [\![\tau_{\nu(\imath^{\wedge})}(x^{\wedge}) = S_\imath x]\!] \wedge [\![\tau_{\nu(\jmath^{\wedge})}(x^{\wedge}) = S_\jmath x]\!] \leqslant [\![S_\imath x = \beta_{\imath,\jmath} S_\jmath x]\!].$$

Putting $\pi_{\imath,\jmath} := \chi(b_{\imath,\jmath})$ and $\alpha_{\imath,\jmath} := \pi_{\imath,\jmath}\beta_{\imath,\jmath}$ and using the Gordon Theorem, we see that $\pi_{\imath,\jmath} S_\imath x = \alpha_{\imath,\jmath} S_\jmath x$, whence $S_\imath = \sum_{\jmath=1}^{n} \alpha_{\imath,\jmath} S_\jmath$ as required.

(3) $\Longrightarrow$ (2): This is demonstrated along the above lines making use of 3.7.6, 3.8.2, and 3.8.5. $\triangleright$

3.9. Representation of Disjointness Preserving Operators

The main result of the present section is representation of an arbitrary order bounded disjointness preserving operator as a strongly disjoint sum of operators admitting some weight-shift-weight factorization.

3.9.1. Let $\mathbb{B}$ be a complete Boolean algebra and let φ be a $\mathbb{2}$-valued Boolean homomorphism on $\mathbb{B}$ with $\mathbb{2} := \{0, 1\} \subset \mathbb{R}$. Define $\mathscr{D}(\varphi)$ as the set of all spectral systems $x \in \mathfrak{S}(\mathbb{B})$ satisfying $\varphi(x(s)) = 0$ and

$\varphi(x(t)) = 1$ for some $s, t \in \mathbb{R}$. For $x \in \mathscr{D}(\varphi)$ we can choose $t = -s > 0$, since the function $t \mapsto \varphi(x(t))$ is increasing, so that

$$\mathscr{D}(\varphi) := \{x \in \mathfrak{S}(\mathbb{B}) : \ (\exists s \in \mathbb{R}_+) \ \varphi(x(s)) = 1, \ \varphi(x(-s)) = 0\}.$$

Moreover, for every finite collection $x_1, \dots, x_n \in \mathfrak{S}(\mathbb{B})$ there is $0 < s \in \mathbb{R}$ such that $\varphi(x_\imath(s)) = 1$ and $\varphi(x_\imath(-s)) = 0$ for all $\imath = 1, \dots, n$.

Recall that $\mathfrak{S}(\mathbb{B})$ is a universally complete vector lattice with zero element $\bar{\mathbb{0}}$ and weak order unit $\bar{\mathbb{1}}$ defined as $\bar{\mathbb{0}}(t) := \mathbb{1}$ if $t > 0$ and $\bar{\mathbb{0}}(t) := \mathbb{0}$ if $t \leqslant 0$, $\bar{\mathbb{1}}(t) := \mathbb{1}$ if $t > 1$ and $\bar{\mathbb{1}}(t) := \mathbb{0}$ if $t \leqslant 1$ (cp. 2.8.2 and 2.8.3). Moreover, $b \mapsto \bar{b}$ in 2.8.2 is a Boolean isomorphism of $\mathbb{B}$ onto $\mathbb{C}(\bar{\mathbb{1}})$ and we will identify these two Boolean algebras. Denote by $J(\mathbb{1})$ the order ideal in $\mathfrak{S}(\mathbb{B})$ generated by $\bar{\mathbb{1}}$.

3.9.2. *$\mathscr{D}(\varphi)$ is simultaneously an order dense ideal in $\mathfrak{S}(\mathbb{B})$ and an f-subalgebra with unit $\bar{\mathbb{1}}$.*

$\lhd$ Take arbitrary $x, y \in \mathscr{D}(\varphi)$, $0 < \alpha \in \mathbb{R}$, and $z \in \mathfrak{S}(\mathbb{B})$ with $0 \leqslant z \leqslant |x|$, and choose $s \in \mathbb{R}_+$ such that $\varphi(x(s)) = \varphi(y(s)) = 1$ and $\varphi(x(\varepsilon - s)) = \varphi(x(-s)) = \varphi(y(-s)) = 0$ for some $0 < \varepsilon \in \mathbb{R}$. Then $\varphi((x+y)(2s)) = 1$ and $\varphi((x+y)(-2s)) = 0$ by 2.7.6 (5), $\varphi((\alpha x)(\alpha s)) = 1$ and $\varphi((\alpha x)(-\alpha s)) = 0$ by 2.7.5 (1), $\varphi(|x|(s)) = 1$ by 2.7.6 (3) and $\varphi(|x|(-s)) = 0$ by 2.7.4 (1), since $|x| \geqslant \bar{\mathbb{0}}$. Moreover, $\varphi(z(s)) = 1$ and $\varphi(z(-s)) = 0$ by 2.7.4 (1) and the above proved property of $|x|$. It follows that $x + y$, αx, $|x|$, and z lie in $\mathscr{D}(\varphi)$ and so $\mathscr{D}(\varphi)$ is an order ideal. It remains to observe that for the same x and y we have $\varphi(|xy|(s^2)) = 1$ and $\varphi(|xy|(-s^2)) = 0$ by 2.7.6 (6), so that $xy \in \mathscr{D}(\varphi)$ and hence $\mathscr{D}(\varphi)$ is an f-subalgebra of $\mathfrak{S}(\mathbb{B})$ containing $\bar{\mathbb{1}}$. $\rhd$

3.9.3. *A spectral system $x \in \mathfrak{S}(\mathbb{B})$ is contained in $\mathscr{D}(\varphi)$ if and only if there exists a countable partition of unity (b_n) in $\mathbb{B}$ such that $\varphi(b_m) = 1$ for some $m \in \mathbb{N}$ and $b_n x \in J(\mathbb{1})$ for all $n \in \mathbb{N}$.*

$\lhd$ By 3.9.2 we can assume that x is positive. Take a partition $(t_n)_{n=0}^{\infty}$ of the real half-line $\mathbb{R}_+$ and put $b_n := x(t_n) - x(t_{n-1})$ for all $n \in \mathbb{N}$. Clearly, (b_n) is a partition of unity in $\mathbb{B}$. If $x \in \mathscr{D}(\varphi)$ then $\varphi(x(t_m)) = 1$ for some $m \in \mathbb{N}$ and we can choose the first natural m with this property. Then $\varphi(b_m) = \varphi(x(t_m)) - \varphi(x(t_{m-1})) = 1$. At the same time $x(t) \geqslant x(t_n) \geqslant b_n$ whenever $t \geqslant t_n$ and so $(b_n x)(t) = b_n \wedge x(t) + b_n^* = \mathbb{1}$ by 2.7.5 (4). It follows from 2.7.4 (1) that $0 \leqslant b_n x \leqslant t_n \mathbb{1}$.

Conversely, if a partition of unity (b_n) satisfy the above condition then $\varphi(b_m^*) = 0$ and $\varphi((b_m x)(t_0)) = 1$ for some $t_0 \geqslant t_m$. In view of

2.7.5 (4) we have $\varphi(x(t_0)) = \varphi(b_m \wedge x(t_0) + b_m^*) = \varphi((b_m x)(t_0)) = 1$, so that $x \in \mathscr{D}(\varphi)$. $\triangleright$

3.9.4. *The order ideal $J(\mathbb{1})$ is uniformly dense in $\mathscr{D}(\varphi)$.*

$\triangleleft$ By 3.9.3 $x \in \mathscr{D}(\varphi)$ can be written as $x = o\text{-}\sum_{n=1}^{\infty} b_n x_n$, where $x_n \in J(\mathbb{1})$ for all $n \in \mathbb{N}$ and (b_n) is a partition of unity in $\mathbb{B}$ with $\varphi(b_m) = 1$ for some $m \in \mathbb{N}$. Put $y_n = \sum_{k=1}^{n} b_k x_k$ and $e = o\text{-}\sum_{n=1}^{\infty} n b_n x_n$. Clearly, e exists in $\mathfrak{S}(\mathbb{B})$ and $e \in \mathscr{D}(\varphi)$ by 3.9.3. Moreover, $y_n \in J(\mathbb{1})$, and $|x - y_n| \leqslant (1/n)e$ $(n \in \mathbb{N})$. $\triangleright$

3.9.5. *Let $\mathbb{B}$ be a complete Boolean algebra and let φ be a 2-valued Boolean homomorphism on $\mathbb{B}$. Then there exists a unique lattice homomorphism $\widehat{\varphi} : \mathscr{D}(\varphi) \to \mathbb{R}$ with $\widehat{\varphi}|_{\mathbb{B}} = \varphi$. Moreover,*

$$\begin{aligned}\widehat{\varphi}(x) &= \sup\{t \in \mathbb{R} :\ \varphi(x(t)) = 0\}\\ &= \inf\{t \in \mathbb{R} :\ \varphi(x(t)) = 1\} \quad (x \in \mathscr{D}(\varphi)).\end{aligned}$$

$\triangleleft$ The above formula correctly defines some function $\widehat{\varphi} : \mathscr{D}(\varphi) \to \mathbb{R}$, since for every $x \in \mathscr{D}(\varphi)$ the two sets $A = (\varphi \circ x)^{-1}(0)$ and $B = (\varphi \circ x)^{-1}(1)$ form a disjoint partition of the real line with $s < t$ for all $s \in A$ and $t \in B$. It is immediate from 2.7.5 (1) and the definition of $\widehat{\varphi}$ that $\widehat{\varphi}(\alpha x) = \alpha\widehat{\varphi}(x)$ for all $\alpha \in \mathbb{R}_+$. From 2.7.6 (2) we see that $\varphi(-x(t)) = 1$ implies $\varphi(x(\varepsilon - t)) = 0$ $(\varepsilon > 0)$ and $\varphi(x(-t)) = 1$ implies $\varphi(-x(t)) = 0$. Consequently, for every $0 < \varepsilon \in \mathbb{R}$, making use of 2.7.6 (2) we deduce

$$\begin{aligned}\widehat{\varphi}(-x) &= \inf\{t \in \mathbb{R} :\ \varphi((-x)(t)) = 1\}\\ &\leqslant \inf\{t \in \mathbb{R} :\ \varphi(x(\varepsilon - t)) = 0\}\\ &= -\sup\{t - \varepsilon \in \mathbb{R} :\ \varphi(x(t)) = 0\}\\ &= -\widehat{\varphi}(x) + \varepsilon = -\inf\{t \in \mathbb{R} :\ \varphi(x(t)) = 1\} + \varepsilon\\ &= \sup\{t \in \mathbb{R} :\ \varphi(x(-t)) = 1\} + \varepsilon\\ &\leqslant \sup\{t \in \mathbb{R} :\ \varphi(-x(t)) = 0\} + \varepsilon\\ &= \widehat{\varphi}(-x) + \varepsilon.\end{aligned}$$

It follows that $\widehat{\varphi}(-x) = -\widehat{\varphi}(x)$. Observe now that if $\varphi((x + y)(r)) = 0$ and $r = s + t$, then either $\varphi(x(s)) = 0$ or $\varphi(y(t)) = 0$. Using this fact we deduce

$$\begin{aligned}\widehat{\varphi}(x + y) &= \sup\{r \in \mathbb{R} :\ \varphi((x + y)(r)) = 0\}\\ &\leqslant \inf\{t \in \mathbb{R} :\ \varphi(x(s)) = 0 \text{ or } \varphi(y(t)) = 0\}\\ &= \sup\{s \in \mathbb{R} :\ \varphi(x(t)) = 0\} + \sup\{t \in \mathbb{R} :\ \varphi(y(t)) = 0\}\\ &= \widehat{\varphi}(x) + \widehat{\varphi}(y).\end{aligned}$$

Replacing x by $-x$ and y by $-y$ and applying the identity $\widehat{\varphi}(-x) = -\widehat{\varphi}(x)$ just proved we obtain $\widehat{\varphi}(x+y) = \widehat{\varphi}(x) + \widehat{\varphi}(y)$. Thus, $\hat{\varphi}$ is a linear functional. Moreover $\widehat{\varphi}$ is a lattice homomorphism, since the identity $\widehat{\varphi}(x \vee y) = \widehat{\varphi}(x) \vee \widehat{\varphi}(y)$ is immediate from 2.7.4 (2).

The uniqueness of $\widehat{\varphi}$ follows from 3.9.4 in view of 3.1.2 (2). $\triangleright$

3.9.6. *Let X be a vector lattice with the projection property and $\mathbb{B} := \mathbb{B}(X)$. If $f : X \to \mathbb{R}$ is a nonzero disjointness preserving functional then there exists a unique Boolean homomorphism $\varphi : \mathbb{B} \to \{0, 1\}$ such that*

$$\operatorname{im}(f|_K) = \varphi(K)\mathbb{R} \quad (K \in \mathbb{B}).$$

$\triangleleft$ Define $\varphi : \mathbb{B} \to \{0, 1\}$ by putting $\varphi(K) := 0$ if $K \subset \ker(f)$ and $\varphi(K) := 1$ otherwise. Assume that $K \in \mathbb{B}$ is not contained in $\ker(f)$. Then $f(x) \neq 0$ for some $x \in K$ and $K^\perp \subset \{x\}^\perp \subset \ker(f)$. Thus, for every $K \in \mathbb{B}$ either $K \subset \ker(f)$ or $K^\perp \subset \ker(f)$. Using this simple properties one can easily ensure that $\varphi(K \wedge L) = \varphi(K) \wedge \varphi(L)$ for all $K, L \in \mathbb{B}$. Now use the projection property and observe that either $\varphi(K) = 1$ or $\varphi(K^\perp) = 1$, since $X = K + K^\perp$ and $\varphi(X) = 1$. It follows that $\varphi(K)^\perp = \varphi(K^\perp)$ and φ is a Boolean homomorphism from $\mathbb{B}$ to $\{0, 1\}$. Moreover, by the definition of φ we have $\operatorname{im}(f|_K) = \{0\} = \varphi(K)\mathbb{R}$ whenever $K \subset \ker(f)$ and $\operatorname{im}(f|_K) = \mathbb{R} = \varphi(K)\mathbb{R}$ otherwise. $\triangleright$

The Boolean homomorphism φ constructed from f is called the *shadow* of f. It is immediate from the definition that f and $|f|$ have the same shadow. The Boolean homomorphism φ induces a homomorphism from $\mathbb{P}(X)$ to $\mathbb{2}$ defined as $\pi \mapsto \varphi(\pi(X))$ and denoted by the same letter φ. From the definition of the shadow it is also clear that $\varphi(\pi) f \circ \pi = f \circ \pi$ and $\varphi(\pi) f \circ \pi^\perp = 0$. Therefore, $f \circ \pi = \varphi(\pi) f$ for all $\pi \in \mathbb{P}(X)$.

3.9.7. In the rest of this section, X and Y are vector lattices considered as order dense sublattices in their universal completions $X^{\mathfrak{u}}$ and $Y^{\mathfrak{u}}$. Moreover, we assume that Y is Dedekind complete. We fix the weak order units $\mathbb{1}$ and $\hat{\mathbb{1}}$ in $X^{\mathfrak{u}}$ and $Y^{\mathfrak{u}}$, respectively, so that $X^{\mathfrak{u}}$ and $Y^{\mathfrak{u}}$ are also f-algebras with the multiplicative units $\mathbb{1}$ and $\hat{\mathbb{1}}$. Recall that orthomorphisms in $X^{\mathfrak{u}}$ and $Y^{\mathfrak{u}}$ are multiplication operators and we identify them with the corresponding multipliers. Note that some notions in this section depend on a specific choice of the unities $\mathbb{1}$ and $\hat{\mathbb{1}}$.

For every $e \in X^{\mathfrak{u}}$, there exists a unique element $1/e \in X^{\mathfrak{u}}$ such that $e(1/e) = [e]\mathbb{1}$. The product $x(1/e)$ is denoted by x/e for brevity. Put $X/e := \{x/e \in X^{\mathfrak{u}} : x \in X\}$. Then X/e is a vector sublattice of $X^{\mathfrak{u}}$ and

$h : x \mapsto x/e$ is an order bounded band preserving operator from X onto X/e with $h(e) = [e]\mathbb{1}$. If e is invertible in the f-algebra $X^{\mathfrak{u}}$ then h is a lattice isomorphism of X onto X/e and $h(e) = \mathbb{1}$.

3.9.8. Theorem. *Let X be a vector lattice over an ordered field $\mathbb{P}$ with $\mathbb{Q} \subset \mathbb{P} \subset \mathbb{R}$ and $\mathbb{B} := \mathbb{B}(X)$. Let $f : X \to \mathbb{R}$ be a nonzero order bounded band preserving $\mathbb{P}$-linear functional. Then there exist a $\mathbb{2}$-valued Boolean homomorphism φ on $\mathbb{B}$ and $\alpha \in \mathbb{R}$ such that $X/e \subset \mathscr{D}(\varphi)$ and*

$$f(x) = \alpha\widehat{\varphi}(x/e) \quad (x \in X). \tag{1.1}$$

$\lhd$ Because f is nonzero, there exists $e \in X_+$ with $0 < p := |f|(e) \in \mathbb{P}$. There is no loss of generality in assuming that e is a weak order unit, since f preserves disjointness and $\{e\}^\perp \subset \ker(f)$. Denote by $g : X \to \mathbb{R}$ a lattice homomorphism which is an extension of $|f|$ to the Dedekind completion X^δ. (Assume further that $X \subset X^\delta$ and X^δ is an order dense ideal in $\mathfrak{S}(\mathbb{B})$.) Such extension exists by Theorem 3.1.13. We can also identify $\mathbb{B}(X)$ and $\mathbb{B}(X^\delta)$, since $B \mapsto B \cap X$ is an isomorphism of $\mathbb{B}(X^\delta)$ onto $\mathbb{B}(X)$. As was mentioned in 3.9.7, $h : x \mapsto x/e$ is a lattice isomorphism from X^δ onto X^δ/e. By 3.9.6 there exists a unique Boolean homomorphism $\varphi : \mathbb{B} \to \{0,1\}$ such that

$$g([K]x) = \varphi(K)g(x) \quad (x \in X^\delta,\ K \in \mathbb{B}).$$

Observe now that if $X^\delta/e \subset \mathscr{D}(\varphi)$ then $g_1 := g \circ h^{-1}$ and $g_2 := \widehat{\varphi}|_{X^\delta/e}$ are real lattice homomorphisms on X^δ/e with $g_1(\mathbb{1}) = p$ and $g_2(\mathbb{1}) = 1$. As can be seen using 3.4.1, two lattice homomorphisms are either disjoint or proportional. But the first case is impossible, since

$$\begin{aligned}(g_1 \wedge g_2)(\mathbb{1}) &= \inf\{g(\pi e) + \widehat{\varphi}(\pi^\perp \mathbb{1}) :\ \pi \in \mathbb{P}(X^\delta)\}\\ &= \inf\{\varphi(K)p + \varphi(K^\perp) :\ K \in \mathbb{B}(X^\delta)\} \geqslant \min\{p, 1\} > 0,\end{aligned}$$

where $K = \pi(X^\delta)$. Thus, g_1 and g_2 are proportional and, since $g|_X = \pm f$, we have $g = \alpha\widehat{\varphi} \circ h$ for some $\alpha \in \mathbb{R}$.

It remains to show that $X^\delta/e \subset \mathscr{D}(\varphi)$. Observe first that e is a weak order unit in X^δ, whence $h(e) = \mathbb{1} = \bar{\mathbb{1}}$. Moreover, $b(t) = e_t^b$ $(t \in \mathbb{R})$ for all $b \in \mathbb{B}$ (cp. 2.8.4). Take arbitrary $0 \leqslant x \in X^\delta/e$ and $n \in \mathbb{N}$ and note that $e_\lambda^{x/n} = e_{n\lambda}^x \leqslant e_n^x = e_\lambda^{\mathbb{1} - e_n^x}$, whenever $0 \leqslant \lambda < 1$. If $\lambda \geqslant 1$ then $e_\lambda^{x/n} \leqslant \mathbb{1} = e_\lambda^{\mathbb{1} - e_n^x}$ and $e_\lambda^{x/n} = \mathbb{0}$ for $\lambda < 0$. Thus, $e_\lambda^{x/n} \leqslant e_\lambda^{\mathbb{1} - e_n^x}$ for all

$\lambda \in \mathbb{R}$, whence $\mathbb{1} - e_n^x \leqslant x/n$ by $2.7.4\dot{(}1)$. Now we can estimate

$$|g_1(\mathbb{1})|(1 - \varphi(e_n^x)) = |g_1(\mathbb{1}) - \varphi(e_n^x)g_1(\mathbb{1})|$$
$$= |g_1(\mathbb{1} - e_n^x)| \leqslant |g_1(\mathbb{1})|/n \to 0.$$

It follows that $\varphi(e_n^x) = 1$ for some $n \in \mathbb{N}$, so that $x \in \mathscr{D}(\varphi)$. $\triangleright$

3.9.9. Theorem. *Let X and Y be vector lattices and let $T : X \to Y$ be an order bounded disjointness preserving operator such that $\{T(e)\}^{\perp\perp} = Y$ for some $e \in X_+$. Then there exit an order dense sublattice Y_0 in $Y^{\mathfrak{u}}$, an order dense ideal $\mathscr{D}(\Phi)$ in $X^{\mathfrak{u}}$, a lattice homomorphism $\Phi : \mathscr{D}(\Phi) \to Y_0$, and an orthomorphism W from Y_0 to Y such that $x \mapsto x/e$ is an orthomorphism from X into $\mathscr{D}(\Phi)$, $\mathbb{1} \in \mathscr{D}(\Phi)$, $\Phi(\mathbb{1}) = \hat{\mathbb{1}}$, and*

$$T(x) = W\widehat{\Phi}(x/e) \quad (x \in X). \tag{1.2}$$

$\triangleleft$ There is no loss of generality in assuming that e and $|T|e$ are weak order units in X and Y, respectively, since T vanishes on $\{e\}^\perp$. In accordance with the Gordon Theorem, we can assume that Y is an order dense sublattice of $\mathscr{R}{\downarrow}$ and $T = \tau{\downarrow}$ for an internal order bounded disjointness preserving functional $\tau : X^\wedge \to \mathscr{R}$ with $\mathscr{R}, \tau \in \mathbb{V}^{(\mathbb{B})}$ and $\mathbb{B} = \mathbb{B}(Y)$. Then $[\![\tau(e^\wedge) = Te \neq 0]\!] = \mathbb{1}$ and so $[\![\tau(e^\wedge) \neq 0]\!] = \mathbb{1}$. Working within $\mathbb{V}^{(\mathbb{B})}$ we can apply Theorem 3.9.8 and pick a Boolean homomorphism $\varphi : \mathbb{B}(X^\wedge) \to \{0, 1\}$ and $\alpha \in \mathscr{R}$ such that $X^\wedge/e^\wedge \subset \mathscr{D}(\varphi) \subset (X^\wedge)^{\mathfrak{u}}$, $\widehat{\varphi} : \mathscr{D}(\varphi) \to \mathscr{R}$ is a lattice homomorphism with $\widehat{\varphi}(\mathbb{1}^\wedge) = 1$ and $\tau(x) = \alpha\widehat{\varphi}(x/e)$ for all $x \in X^\wedge$.

Clarify some details of such a representation. Recall that $X^\wedge$ is an order dense sublattice in $(X^{\mathfrak{u}})^\wedge$ and $(X^{\mathfrak{u}})^\wedge$ is an order dense sublattice in $(X^\wedge)^{\mathfrak{u}}$. It follows that $\mathbb{1}^\wedge \in (X^{\mathfrak{u}})^\wedge$ is a weak order unit in $(X^\wedge)^{\mathfrak{u}}$. Moreover $(X^\wedge)^{\mathfrak{u}}$ is an f-algebra with the multiplicative unit $\mathbb{1}^\wedge$ and $\mathscr{D}(\varphi)$ is an order dense ideal and an f-subalgebra in $(X^\wedge)^{\mathfrak{u}}$ containing $\mathbb{1}^\wedge$. Thus, the multiplication operator $\mu_e : x \mapsto x/e^\wedge$ on $(X^\wedge)^{\mathfrak{u}}$ induces a lattice isomorphism of $X^\wedge$ into $\mathscr{D}(\varphi)$. If m_α denotes a linear function $y \mapsto \alpha y$ on $\mathscr{R}$, then $\tau = m_\alpha \circ \widehat{\varphi} \circ \mu_e$.

Now we examine the descent of this representation. By Theorem 2.11.6 $Z := (X^\wedge)^{\mathfrak{u}}{\downarrow}$ is a universally complete vector lattice. Moreover, $Z_0 := \mathscr{D}(\varphi){\downarrow}$ is an order dense ideal in Z containing $\mathbb{1}^\wedge$ and $\widehat{\varphi}{\downarrow}$ is a lattice homomorphism from Z_0 to $\mathscr{R}{\downarrow}$. Note that $x \mapsto x^\wedge$ is a lattice homomorphism from $X^{\mathfrak{u}}$ into $(X^\wedge)^{\mathfrak{u}}{\downarrow}$, so that we can identify $X^{\mathfrak{u}}$ with a sublattice in $(X^\wedge)^{\mathfrak{u}}{\downarrow}$ (and so $\mathbb{1}$ with $\mathbb{1}^\wedge$). Keeping this in mind, denote by Φ and w the restrictions of $\widehat{\varphi}{\downarrow}$ onto $Z_0 \cap X^{\mathfrak{u}}$ and $\mu{\downarrow}$ onto X, respectively. Since the multiplication on $(X^{\wedge})^{\mathfrak{u}}{\downarrow}$ is the descent of the multiplication on $(X^{\wedge})^{\mathfrak{u}}$, we conclude that $\mu{\downarrow}$ is the multiplication operator with the same multiplier e, that is $\mu{\downarrow} : x \mapsto x/e$. By the same reasons $m_\alpha{\downarrow}$ is the multiplication operator on $\mathscr{R}{\downarrow}$ with the multiplier $\alpha \in \mathscr{R}{\downarrow}$. Put $Y_0 = \Phi(X/e)$ and denote by W the restriction of $m_\alpha{\downarrow}$ onto Y_0. Then Y_0 is a sublattice of $Y^{\mathfrak{u}}$, and W is an orthomorphism from Y_0 to Y. From 1.5.5 (1) we have

$$Tx = \tau{\downarrow}(x) = (m_\alpha{\downarrow} \circ \widehat{\varphi}{\downarrow} \circ \mu_e{\downarrow})x = (W \circ \Phi \circ w)x$$

for all $x \in X$ and the proof is complete. $\rhd$

3.9.10. The above representation is called the *weight-shift-weight factorization* of T. The operator Φ, whose existence is asserted in 3.9.9, is called the *shift* of T. We say that an operator $S : \mathscr{D}(S) \to Y_0$ is a *shift operator*, if $\mathscr{D}(S)$ and Y_0 are order dense ideals in $X^{\mathfrak{u}}$ and $Y^{\mathfrak{u}}$, respectively, and S is the shift of some order bounded disjointness preserving operator $T : X \to Y$. The operator Φ of the representation can be defined for an arbitrary order bounded disjointness preserving operator T just as in the proof of Theorem 3.9.9, but there is not enough room in $X^{\mathfrak{u}}$ to provide the weight-shift-weight factorization of T.

A *weight system* is a family of pairs $w := \big((b_\xi, e_\xi)\big)_{\xi\in\Xi}$ such that $(b_\xi)_{\xi\in\Xi}$ is a partition of unity in $\mathbb{P}(Y^{\mathfrak{u}})$ and $(e_\xi)_{\xi\in\Xi}$ is a family of positive elements in X and the representation $b_\xi T = W_\xi \circ \Phi \circ (\cdot/e_\xi)$ holds for all $\xi \in \Xi$. In this case, $o\text{-}\sum_{\xi\in\Xi} b_\xi \Phi(\mathbb{1}/e_\xi) = \hat{\mathbb{1}}$ and, putting $W := o\text{-}\sum_{\xi\in\Xi} b_\xi T e_\xi$, we obtain the representation

$$T = o\text{-}\sum_{\xi\in\Xi} b_\xi W \circ \Phi \circ (\cdot/e_\xi),$$

which will be written shortened as follows: $T = W \circ \Phi \circ w$.

3.9.11. Theorem. *Let X and Y be vector lattices and let $T : X \to Y$ be an order bounded disjointness preserving operator. Then there exist Φ and W as in Theorem 3.9.9 and a weight system $w \in \mathscr{W}(X, \mathbb{B})$ such that*

$$Tx = W \circ \widehat{\Phi} \circ w.$$

$\lhd$ Let τ be the same as in 3.9.10. If $[\![\tau = 0]\!] = \mathbb{1}$ then $T = 0$ and there is nothing to prove. If $[\![\tau \neq 0]\!] = \mathbb{1}$ then, in view of the ZFC-theorem

$\psi := \tau \neq 0 \to (\exists p \in \mathscr{R})(\exists\, e \in X^{\wedge})(0 < p \wedge p = \tau(e))$, we have by transfer $[\![\psi]\!] = \mathbb{1}$ and according to maximum principle there exists $p \in Y$ such that

$$\mathbb{1} = \bigvee_{e \in X} [\![p = \tau(e^{\wedge}) \wedge p > 0]\!] = [\![0 < p]\!] \wedge \bigvee_{e \in X} [\![p = T(e)]\!].$$

Thus, p is a weak order unit in Y and there exists a partition of the unit $(b_\xi)_{\xi \in \Xi}$ in $\mathbb{B}$, and a family $(e_\xi)_{\xi \in \Xi}$ in X such that $b_\xi \leqslant [\![Te_\xi = p]\!]$ for all ξ. It follows that $b_\xi T e_\xi = b_\xi p$. By Theorem 3.9.9, for every $\xi \in \Xi$, we have $X/e_\xi \subset \mathscr{D}(\Phi)$, and there exist a vector sublattice $Y_\xi \subset b_\xi Y^{\mathfrak{u}}$ and an orthomorphism $W_\xi : Y_\xi \to Y$ such that $(\cdot/e_\xi)$ is an orthomorphism from X to X_ξ and $\pi_\xi \circ T = W_\xi \circ \pi_\xi \circ \Phi \circ (\cdot/e_\xi)$. Clearly, $w = (b_\xi, e_\xi)_{\xi \in \Xi}$ is a weight system for Φ. If Y_0 is a sublattice in $Y^{\mathfrak{u}}$ generated by $\bigcup_{\xi \in \Xi} Y_\xi$ and W is the restriction of the sum $o\text{-}\sum_{\xi \in \Xi} b_\xi \circ W_\xi$ to Y_0, then

$$Tx = o\text{-}\sum_{\xi \in \Xi} b_\xi \circ Tx = o\text{-}\sum_{\xi \in \Xi} (W_\xi \circ b_\xi \circ \widehat{\Phi})(x/e_\xi) = (W \circ \widehat{\Phi} \circ w)x$$

for all $x \in X$. The proof is complete. $\triangleright$

3.10. Pseudoembedding Operators

In this section we will give a description of the band generated by disjointness preserving operators in the vector lattice of order bounded operators. First we examine the scalar case.

3.10.1. *For an arbitrary vector lattice X there exist a unique cardinal γ and a disjoint family $(\varphi_\alpha)_{\alpha<\gamma}$ of nonzero lattice homomorphisms $\varphi_\alpha : X \to \mathbb{R}$ such that every $f \in X^{\sim}$ admits the unique representation*

$$f = f_d + o\text{-}\sum_{\alpha<\gamma} \lambda_\alpha \varphi_\alpha$$

where $f_d \in X^{\sim}_d$ and $(\lambda_\alpha)_{\alpha<\gamma} \subset \mathbb{R}$. The family $(\varphi_\alpha)_{\alpha<\gamma}$ is unique up to permutation and positive scalar multiplication.

$\triangleleft$ The Dedekind complete vector lattice $X^{\sim}$ splits into the direct sum of the atomic band $X^{\sim}_a$ and the diffuse band $X^{\sim}_d$; therefore, each functional $f \in E^{\sim}$ admits the unique representation $f = f_a + f_d$ with

$f_a \in X_a^\sim$ and $f_d \in X_d^\sim$. Let γ be the cardinality of the set $\mathscr{K}$ of one-dimensional bands in $X_a^\sim$ (= atoms in $\mathbb{B}(X^\sim)$). Then there exists a family of lattice homomorphisms $(\varphi_\alpha : X \to \mathbb{R})_{\alpha<\gamma}$ such that $\mathscr{K} = \{\varphi_\alpha^{\perp\perp} : \alpha < \gamma\}$. It remains to observe that the mapping sending a family of reals $(\lambda_\alpha)_{\alpha<\gamma}$ to the functional $x \mapsto \sum_{\alpha<\gamma} \lambda_\alpha\varphi_\alpha(x)$ implements a lattice isomorphism between $X_a^\sim$ and some ideal in the vector lattice $\mathbb{R}^\gamma$.

If $(\psi_\alpha)_{\alpha<\gamma}$ is a disjoint family of nonzero real lattice homomorphisms on X with $X_a^\sim = \{\psi_\alpha : \alpha < \gamma\}^{\perp\perp}$, then for all $\alpha, \beta < \gamma$ the functionals φ_α and ψ_β are either disjoint or proportional with a strictly positive coefficient, so that there exist a permutation $(\omega_\beta)_{\beta<\gamma}$ of $(\varphi_\alpha)_{\alpha<\gamma}$ and a unique family $(\mu_\beta)_{\beta<\gamma}$ in $\mathbb{R}_+$ such that $\psi_\beta = \mu_\beta\omega_\beta$ for all $\beta < \gamma$. $\triangleright$

3.10.2. Given two families $(S_\alpha)_{\alpha\in\mathrm{A}}$ and $(T_\beta)_{\beta\in\mathrm{B}}$ in $L^\sim(X,Y)$, say that $(S_\alpha)_{\alpha\in\mathrm{A}}$ is a $\mathbb{P}(Y)$-*permutation* of $(T_\beta)_{\beta\in\mathrm{B}}$ whenever there exists a double family $(\pi_{\alpha,\beta})_{\alpha\in\mathrm{A},\,\beta\in\mathrm{B}}$ in $\mathbb{P}(Y)$ such that

$$S_\alpha = \sum_{\beta\in\mathrm{B}} \pi_{\alpha,\beta}T_\beta$$

for all $\alpha \in \mathrm{A}$, while $(\pi_{\alpha,\bar\beta})_{\alpha\in\mathrm{A}}$ and $(\pi_{\bar\alpha,\beta})_{\beta\in\mathrm{B}}$ are partitions of unity in $\mathbb{B}(Y)$ for all $\bar\alpha \in \mathrm{A}$ and $\bar\beta \in \mathrm{B}$. It is easily seem that in case $Y = \mathbb{R}$ this amounts to saying that there is a bijection $\nu : \mathrm{A} \to \mathrm{B}$ with $S_\alpha = T_{\nu(\alpha)}$ for all $\alpha \in \mathrm{A}$; i.e., $(S_\alpha)_{\alpha\in\mathrm{A}}$ is a permutation of $(T_\beta)_{\beta\in\mathrm{B}}$. We also say that $(S_\alpha)_{\alpha\in\mathrm{A}}$ is $\mathrm{Orth}(Y)$-multiple of $(T_\alpha)_{\alpha\in\mathrm{A}}$ whenever there exists a family of orthomorphisms $(\pi_\alpha)_{\alpha\in\mathrm{A}}$ in $\mathrm{Orth}(Y)$ such that $S_\alpha = \pi_\alpha T_\alpha$ for all $\alpha \in \mathrm{A}$. In case $Y = \mathbb{R}$ we evidently get that S_α is a scalar multiple of T_α for all $\alpha \in \mathrm{A}$.

Using above notation define the two mappings $\mathscr{S} : \mathrm{A} \to X^{\wedge\sim}{\downarrow}$ and $\mathscr{T} : \mathrm{B} \to X^{\wedge\sim}{\downarrow}$ within $\mathbb{V}^{(\mathbb{B})}$ by putting $\mathscr{S}(\alpha) := S_\alpha{\uparrow}$ $(\alpha \in \mathrm{A})$ and $\mathscr{T}(\beta) := T_\beta{\uparrow}$ $(\beta \in \mathrm{B})$.

3.10.3. *Define the internal mappings $\tau, \sigma \in \mathbb{V}^{(\mathbb{B})}$ as $\sigma := \mathscr{S}{\uparrow}$ and $\tau := \mathscr{T}{\uparrow}$. Then $(\sigma(\alpha))_{\alpha\in\mathrm{A}^\wedge}$ is a permutation of $(\tau(\beta))_{\beta\in\mathrm{B}^\wedge}$ within $\mathbb{V}^{(\mathbb{B})}$ if and only if $(S_\alpha)_{\alpha\in\mathrm{A}}$ is a $\mathbb{P}(Y)$-permutation of $(T_\beta)_{\beta\in\mathrm{B}}$.*

$\triangleleft$ Assume that $(\sigma(\alpha))_{\alpha\in\mathrm{A}^\wedge}$ is a permutation of $(\tau(\beta))_{\beta\in\mathrm{B}^\wedge}$ within $\mathbb{V}^{(\mathbb{B})}$. Then there is a bijection $\nu : \mathrm{B}^\wedge \to \mathrm{A}^\wedge$ such that $\sigma(\alpha) = \tau(\nu(\alpha))$ for all $\alpha \in \mathrm{A}^\wedge$. By 1.5.8 $\nu{\downarrow}$ is a function from A to $(\mathrm{B}^\wedge){\downarrow} = \mathrm{mix}(\{\beta^\wedge : \beta \in \mathrm{B}\})$. Thus, for each $\alpha \in \mathrm{A}$ there exists a partition of unity $(b_{\alpha,\beta})_{\beta\in\mathrm{B}}$

such that $\nu{\downarrow}(\alpha) = \mathrm{mix}_{\beta \in \mathrm{B}}(b_{\alpha,\beta}\beta^{\wedge})$. Since $\nu{\downarrow}$ is injective, we have

$$
\begin{aligned}
\mathbb{1} &= [\![(\forall \alpha_1, \alpha_2 \in \mathrm{A}^{\wedge})(\nu(\alpha_1) = \nu(\alpha_2) \to \alpha_1 = \alpha_2)]\!] \\
&= \bigwedge_{\alpha_1,\alpha_2 \in \mathrm{A}} [\![\nu(\alpha_1^{\wedge}) = \nu(\alpha_2^{\wedge}) \to \alpha_1^{\wedge} = \alpha_2^{\wedge}]\!] \\
&= \bigwedge_{\alpha_1,\alpha_2} [\![\nu{\downarrow}(\alpha_1) = \nu{\downarrow}(\alpha_2)]\!] \Rightarrow [\![\alpha_1^{\wedge} = \alpha_2^{\wedge}]\!],
\end{aligned}
$$

and so $[\![\nu{\downarrow}(\alpha_1) = \nu{\downarrow}(\alpha_2)]\!] \leqslant [\![\alpha_1^{\wedge} = \alpha_2^{\wedge}]\!]$ for all $\alpha_1, \alpha_2 \in \mathrm{A}$. Taking this inequality and the definition of $\nu{\downarrow}$ into account yields

$$
\begin{aligned}
b_{\alpha_1,\beta} \wedge b_{\alpha_2,\beta} &\leqslant [\![\nu{\downarrow}(\alpha_1) = \beta^{\wedge}]\!] \wedge [\![\nu{\downarrow}(\alpha_2) = \beta^{\wedge}]\!] \\
&\leqslant [\![\nu{\downarrow}(\alpha_1) = \nu{\downarrow}(\alpha_2)]\!] \leqslant [\![\alpha_1^{\wedge} = \alpha_2^{\wedge}]\!],
\end{aligned}
$$

so that $\alpha_1 \neq \alpha_2$ implies $b_{\alpha_1,\beta} \wedge b_{\alpha_2,\beta} = \mathbb{0}$ (because $\alpha_1 \neq \alpha_2 \Longleftrightarrow [\![\alpha_1^{\wedge} = \alpha_2^{\wedge}]\!] = \mathbb{0}$ by 1.4.5 (2)). At the same time, surjectivity of ν implies

$$
\begin{aligned}
\mathbb{1} &= [\![(\forall\, \beta \in \mathrm{B}^{\wedge})(\exists\, \alpha \in \mathrm{A}^{\wedge})\beta = \nu(\alpha)]\!] \\
&= \bigwedge_{\beta \in \mathrm{B}} \bigvee_{\alpha \in \mathrm{A}} [\![\beta^{\wedge} = \nu{\downarrow}(\alpha)]\!] = \bigwedge_{\beta \in \mathrm{B}} \bigvee_{\alpha \in \mathrm{A}} b_{\alpha,\beta}.
\end{aligned}
$$

It follows that $(b_{\alpha,\beta})_{\alpha \in \mathrm{A}}$ is a partition of unity in $\mathbb{B}$ for all $\beta \in \mathrm{B}$. By the choice of ν it follows that $b_{\alpha,\beta} \leqslant [\![\sigma(\alpha^{\wedge}) = \tau(\beta^{\wedge})]\!]$, because of the estimations

$$
\begin{aligned}
b_{\alpha,\beta} &\leqslant [\![\sigma(\alpha^{\wedge}) = \tau(\nu(\alpha^{\wedge}))]\!] \wedge [\![\nu(\alpha^{\wedge}) = \beta^{\wedge}]\!] \\
&\leqslant [\![\sigma(\alpha^{\wedge}) = \tau(\beta^{\wedge})]\!] = [\![\mathscr{S}(\alpha) = \mathscr{T}(\beta)]\!].
\end{aligned}
$$

Put now $\pi_{\alpha,\beta} := \chi(b_{\alpha,\beta})$ and observe that $b_{\alpha,\beta} \leqslant [\![\mathscr{S}(\alpha)x^{\wedge} = \mathscr{T}(\beta)x^{\wedge}]\!] \leqslant [\![S_\alpha x = T_\beta x]\!]$ for all $\alpha \in \mathrm{A}$, $\beta \in \mathrm{B}$, and $x \in X$. Using 2.2.4 ($\mathbb{G}$), we obtain $\pi_{\alpha,\beta} S_\alpha = \pi_{\alpha,\beta} T_\beta$ and so $S_\alpha = \sum_{\beta \in \mathrm{B}} \pi_{\alpha,\beta} T_\beta$ for all $\alpha \in \mathrm{A}$. Clearly, $(\pi_{\alpha,\beta})$ is the family as required in Definition 3.10.2. The sufficiency is shown by the same reasoning in the reverse direction. $\triangleright$

3.10.4. Recall that the elements of the band $L_d^{\sim}(X,Y) := \mathrm{Hom}(X,Y)^{\perp}$ are referred to as *diffuse operators*; see 3.3.4. An order bounded operator $T : X \to Y$ is said to be *pseudoembedding* if T belongs to the complimentary band $L_a^{\sim}(X,Y) := \mathrm{Hom}(X,Y)^{\perp\perp}$, the band generated by all disjointness preserving operators.

A nonempty set $\mathscr{D}$ of positive operators from X to Y is called *strongly generating* if $\mathscr{D}$ is disjoint and $S(X)^{\perp\perp} = Y$ for all $S \in \mathscr{D}$. If, in addition, $\mathscr{D}^{\perp\perp} = B$, then we say also that $\mathscr{D}$ *strongly generates* the band $B \subset L^\sim(X, Y)$ or B is strongly generated by $\mathscr{D}$. In case $Y = \mathbb{R}$, the strongly generating sets in $X^\sim = L^\sim(X, \mathbb{R})$ are precisely those that consist of pairwise disjoint nonzero positive functionals.

Given a cardinal γ and a universally complete vector lattice Y, say that a vector lattice X is (γ, Y)-*homogeneous* if the band $L_a^\sim(X, Y)$ is strongly generated by a set of lattice homomorphisms of cardinality γ and for every nonzero projection $\pi \in \mathbb{P}(Y)$ and every strongly generating set $\mathscr{D}$ in $L_a^\sim(X, \pi Y)$ we have $\operatorname{card}(\mathscr{D}) \geqslant \gamma$. We say also that X is (γ, π)-homogeneous if $\pi \in \mathbb{P}(Y)$ and X is $(\gamma, \pi Y)$-homogeneous. Evidently, the $(\gamma, \mathbb{R})$-homogeneity of a vector lattice X amounts just to saying that the band $X_a^\sim$ is generated in $X^\sim$ by a disjoint set of nonzero lattice homomorphisms of cardinality γ or, equivalently, the cardinality of the set of atoms in $\mathbb{P}(X^\sim)$ equals γ.

Take $\mathscr{D} \subset L^\sim(X, \mathscr{R}{\downarrow})$ and $\Delta \in \mathbb{V}^{(\mathbb{B})}$ with $[\![\Delta \subset (X^\wedge)^\sim]\!] = \mathbb{1}$. Put $\mathscr{D}_\uparrow := \{T{\uparrow} : T \in \mathscr{D}\}{\uparrow}$ and $\Delta^\downarrow := \{\tau{\downarrow} : \tau \in \Delta{\downarrow}\}$. Let $\operatorname{mix}(\mathscr{D})$ stand for the set of all $T \in L^\sim(X, \mathscr{R}{\downarrow})$ representable as $Tx = o\text{-}\sum_{\xi\in\Xi} \pi_\xi T_\xi x$ $(x \in X)$ with arbitrary partition of unity $(\pi_\xi)_{\xi\in\Xi}$ in $\mathbb{P}(\mathscr{R}{\downarrow})$ and family $(T_\xi)_{\xi\in\Xi}$ in $\mathscr{D}$.

3.10.5. *Let $\Delta \subset (X^\wedge)^\sim$ is a disjoint set of nonzero positive functionals of cardinality $\gamma^\wedge$ within $\mathbb{V}^{(\mathbb{B})}$. Then there exists a strongly generating set of positive operators $\mathscr{D}$ from X to $\mathscr{R}{\downarrow}$ of cardinality γ such that $\Delta = \mathscr{D}_\uparrow$ and $\Delta^\downarrow = \operatorname{mix}(\mathscr{D})$.*

$\lhd$ If Δ obeys the conditions then there is $\phi \in \mathbb{V}^{(\mathbb{B})}$ such that $[\![\phi : \gamma^\wedge \to \Delta \text{ is a bijection}]\!] = \mathbb{1}$. Note that $\phi{\downarrow}$ sends γ into $\Delta{\downarrow} \subset (X^\wedge)^\sim{\downarrow}$ by 1.5.8. By Theorem 3.3.3, we can define the mapping $\alpha \mapsto \Phi(\alpha)$ from γ to $L^\sim(X, \mathscr{R}{\downarrow})$ by putting $\Phi(\alpha) := (\phi{\downarrow}(\alpha)){\downarrow}$. Put $\mathscr{D} := \{\Phi(\alpha) : \alpha \in \gamma\}$ and note that $\mathscr{D} \subset \Delta^\downarrow$. By 1.6.6 and surjectivity of ϕ we have $\Delta{\downarrow} = \varphi(\gamma^\wedge){\downarrow} = \operatorname{mix}\{\phi{\downarrow}(\alpha)) : \alpha \in \gamma\}$ and combining this with 3.3.7 we get $\Delta = \mathscr{D}_\uparrow$ and $\Delta^\downarrow = \operatorname{mix}(\mathscr{D})$.

The injectivity of ϕ implies that to $[\![(\forall \alpha, \beta \in \gamma^\wedge)(\alpha \neq \beta \to \phi(\alpha) \neq \phi(\beta)]\!] = \mathbb{1}$. Replacing the universal quantifier by the supremum over $\alpha, \beta \in \gamma^\wedge$, from 1.4.5 (1) and 1.4.5 (2) we deduce that

$$\mathbb{1} = \bigwedge_{\alpha,\beta\in\gamma} [\![\alpha^\wedge \neq \beta^\wedge]\!] \Rightarrow [\![\varphi(\alpha^\wedge) \neq \phi(\beta)^\wedge]\!] = \bigwedge_{\substack{\alpha,\beta\in\gamma \\ \alpha\neq\beta}} [\![\Phi(\alpha) \neq \Phi(\beta)]\!],$$

and so $\alpha \neq \beta$ implies $\Phi(\alpha) \neq \Phi(\beta)$ for all $\alpha, \beta \in \gamma$. Thus Φ is injective and the cardinality of $\mathscr{D}$ is γ. The fact that $\mathscr{D}$ is strongly generating follows from 3.3.5 (5) and 3.8.4. $\triangleright$

3.10.6. *If $\mathscr{D}$ is a strongly generating set of positive operators from X to $\mathscr{R}{\downarrow}$ of cardinality γ then $\Delta = \mathscr{D}_{\uparrow} \subset (X^{\wedge})^{\sim}$ is a disjoint set of nonzero positive functionals of cardinality $|\gamma^{\wedge}|$ within $\mathbb{V}^{(\mathbb{B})}$.*

$\triangleleft$ Assume that $\mathscr{D} \subset L(X, \mathscr{R}{\downarrow})$ is a strongly generating set of cardinality γ. Then there is a bijection $f : \gamma \to \mathscr{D}{\uparrow}$. Moreover, $\alpha \neq \beta$ implies $[\![f(\alpha) \perp f(\beta)]\!] = \mathbb{1}$ by 3.3.5 (5) and $[\![f(\alpha) \neq 0]\!] = \mathbb{1}$ by 3.8.4. Interpreting in $\mathbb{V}^{(\mathbb{B})}$ the ZFC-theorem

$$(\forall f, g \in X^{\sim})(f \neq 0 \wedge g \neq 0 \wedge f \perp g \to f \neq g)$$

yields $[\![f(\alpha) \neq f(\beta)]\!] = \mathbb{1}$ for all $\alpha, \beta \in \gamma$, $\alpha \neq \beta$. It follows that $\phi := f{\uparrow}$ is a bijection from $\gamma^{\wedge}$ onto $\Delta = (\mathscr{D}{\uparrow}){\uparrow}$, so that the cardinality of Δ is $|\gamma^{\wedge}|$. The proof is completed by the arguments similar to those in 3.10.5. $\triangleright$

3.10.7. *A vector lattice X is $(\gamma, \mathscr{R}{\downarrow})$-homogeneous for some cardinal γ if and only if $[\![\gamma^{\wedge}$ is a cardinal and $X^{\wedge}$ is $(\gamma^{\wedge}, \mathscr{R})$-homogeneous $]\!] = \mathbb{1}$.*

$\triangleleft$ *Sufficiency:* Assume that $\gamma^{\wedge}$ is a cardinal and $X^{\wedge}$ is $(\gamma^{\wedge}, \mathscr{R})$-homogeneous within $\mathbb{V}^{(\mathbb{B})}$. The latter means that $(X^{\wedge})^{\sim}_{a}$ is generated by a disjoint set of nonzero lattice homomorphisms $\Delta \subset (X^{\wedge})^{\sim}$ of cardinality $\gamma^{\wedge}$ within $\mathbb{V}^{(\mathbb{B})}$. By 3.10.5 there exists a strongly generating set $\mathscr{D}$ in $L^{\sim}_{a}(X, \mathscr{R}{\downarrow})$ of cardinality $\gamma^{\wedge}$ such that $\Delta = \mathscr{D}_{\uparrow}$. Take nonzero $\pi \in \mathbb{P}(\mathscr{R}{\downarrow})$ and put $b := \chi^{-1}(\pi)$. Recall that we can identify $L^{\sim}(X, \pi(\mathscr{R}{\downarrow}))$ and $L^{\sim}(X, (b \wedge \mathscr{R}){\downarrow})$. If $\mathscr{D}'$ is a strongly generating set in $L^{\sim}_{a}(X, \pi(\mathscr{R}{\downarrow}))$ of cardinality β then $\mathscr{D}'_{\uparrow}$ strongly generates $(X^{\wedge})^{\sim}_{a}$ and has cardinality $|\beta^{\wedge}|$ within the relative universe $\mathbb{V}^{([\mathbb{0}, b])}$. By 1.3.7 $\gamma^{\wedge} = |\beta^{\wedge}| \leqslant \beta^{\wedge}$ and so $\gamma \leqslant \beta$.

Necessity: Assume now that X is $(\gamma, \mathscr{R}{\downarrow})$-homogeneous and a set lattice homomorphisms $\mathscr{D}$ of cardinality γ generates strongly the band $L^{\sim}_{a}(X, \mathscr{R}{\downarrow})$. Then $\Delta = \mathscr{D}_{\uparrow}$ generates the band $(X^{\wedge})^{\sim}_{a}$ and the cardinalities of Δ and $\gamma^{\wedge}$ coincide; i.e., $|\Delta| = |\gamma^{\wedge}|$. By 1.9.11 the cardinal $|\gamma^{\wedge}|$ has the representation $|\gamma^{\wedge}| = \operatorname{mix}_{\alpha \leqslant \gamma} b_{\alpha} \alpha^{\wedge}$, where $(b_{\alpha})_{\alpha \leqslant \gamma}$ is a partition of unity in $\mathbb{B}$. It follows that $b_{\alpha} \leqslant [\![\Delta$ is a generating set in $(X^{\wedge})^{\sim}_{a}$ of cardinality $\alpha^{\wedge}]\!] = \mathbb{1}$. If $b_{\alpha} \neq \mathbb{0}$ then $b_{\alpha} \wedge \Delta$ is a generating set in $(X^{\wedge})^{\sim}_{a}$ of cardinality $|\gamma^{\wedge}| = \alpha^{\wedge} \leqslant \gamma^{\wedge}$ in the relative universe $\mathbb{V}^{[\mathbb{0}, b_{\alpha}]}$. Put $\pi_{\alpha} = \chi(b_{\alpha})$ and $\pi_{\alpha} \circ \mathscr{D} := \{\pi_{\alpha} \circ T : T \in \mathscr{D}\}$. Clearly, $b_{\alpha} \wedge \Delta = (\pi_{\alpha} \mathscr{D})_{\uparrow}$ and so $\pi_{\alpha} \circ \mathscr{D}$ strongly generates the band $L^{\sim}_{a}(X, \mathscr{R}{\downarrow})$. By hypothesis $\mathscr{D}$ is $(\gamma, \mathscr{R}{\downarrow})$-homogeneous, consequently, $\alpha \geqslant \gamma$, so that $\alpha = \gamma$, since

$\alpha \leqslant \gamma$ if and only if $\alpha^\wedge \leqslant \gamma^\wedge$. Thus, $|\gamma^\wedge| = \gamma^\wedge$ whenever $b_\alpha \neq \mathbb{0}$ and $\gamma^\wedge$ is a cardinal within $\mathbb{V}^{(\mathbb{B})}$. $\triangleright$

3.10.8. *Let X be a (γ, Y)-homogeneous vector lattice for some universally complete vector lattice Y and a nonzero cardinal γ. Then there exists a strongly generating family of lattice homomorphisms $(\Phi_\alpha)_{\alpha<\gamma}$ from X to Y such that each operator $T \in L_a^\sim(X, Y)$ admits the unique representation $T = o\text{-}\sum_{\alpha<\gamma} \sigma_\alpha \circ \Phi_{\gamma,\alpha}$, where $(\sigma_\alpha)_{\alpha<\gamma}$ is a family of orthomorphisms in* $\operatorname{Orth}(Y)$.

$\triangleleft$ This is immediate from the definitions in 3.10.4. $\triangleright$

3.10.9. Theorem. *Let X and Y be vector lattices with Y universally complete. Then there are a nonempty set of cardinals Γ and a partition of unity $(Y_\gamma)_{\gamma\in\Gamma}$ in $\mathbb{B}(Y)$ such that X is (γ, Y_γ)-homogeneous for every $\gamma \in \Gamma$.*

$\triangleleft$ We may assume without loss of generality that $Y = \mathscr{R}{\downarrow}$. The transfer principle tells us that according to 3.10.1 there exists a cardinal $\varkappa$ within $\mathbb{V}^{(\mathbb{B})}$ such that $(X^\wedge)_a^\sim$ is generated by a disjoint set $\mathscr{H}$ of nonzero $\mathbb{R}^\wedge$-linear lattice homomorphisms of cardinality $\varkappa$ or, equivalently, $[\![X^\wedge \text{ is } (\varkappa, \mathscr{R})\text{-homogeneous}]\!] = \mathbb{1}$. By 1.9.11 there is a nonempty set of cardinals Γ and a partition of unity $(b_\gamma)_{\gamma\in\Gamma}$ in $\mathbb{B}$ such that $\varkappa = \operatorname{mix}_{\gamma\in\Gamma} b_\gamma \gamma^\wedge$. It follows that $b_\gamma \leqslant [\![X^\wedge \text{ is } (\gamma^\wedge, \mathscr{R})\text{-homogeneous}]\!]$ for all $\gamma \in \Gamma$. Passing to the relative subalgebra $\mathbb{B}_\gamma := [\mathbb{0}, b_\gamma]$ and considering 1.3.7 we conclude that $\mathbb{V}^{(\mathbb{B}_\gamma)} \models$ "$X^\wedge$ is $(\gamma^\wedge, b_\gamma \wedge \mathscr{R})$-homogeneous", so that X is $(\gamma, (b_\gamma \wedge \mathscr{R}){\downarrow})$-homogeneous by 3.10.7. In view of 2.3.6 $(b_\gamma \wedge \mathscr{R}){\downarrow}$ is lattice isomorphic to Y_γ, so the desired result follows. $\triangleright$

3.10.10. Theorem. *Let X and Y be vector lattices with Y universally complete. Then there is a nonempty set of cardinals Γ, a partition of unity $(Y_\gamma)_{\gamma\in\Gamma}$ in $\mathbb{B}(Y)$, and to each cardinal $\gamma \in \Gamma$ there is a disjoint family of lattice homomorphisms $(\Phi_{\gamma,\alpha})_{\alpha<\gamma}$ from X to Y_γ such that*

(1) $\Phi_{\gamma,\alpha}(X)^{\perp\perp} = Y_\gamma \neq \{0\}$ *for all $\gamma \in \Gamma$ and $\alpha < \gamma$.*

(2) *X is (γ, Y_γ)-homogeneous for all $\gamma \in \Gamma$.*

(3) *For each order dense sublattice $Y_0 \subset Y$ each $T \in L^\sim(X, Y_0)$ admits the unique representation*

$$T = T_d + o\text{-}\sum_{\gamma\in\Gamma} o\text{-}\sum_{\alpha<\gamma} \sigma_{\gamma,\alpha} \circ \Phi_{\gamma,\alpha},$$

with $T_d \in L_d^\sim(X, Y)$ and $\sigma_{\gamma,\alpha} \in \operatorname{Orth}(\Phi_{\gamma,\alpha}, Y_0)$.

For every $\gamma \in \Gamma$ the family $(\Phi_{\gamma,\alpha})_{\alpha<\gamma}$ is unique up to $\mathbb{P}(Y)$-permutation and $\operatorname{Orth}(Y_\gamma)_+$*-multiplication.*

$\triangleleft$ The existence of $(Y_\gamma)_{\gamma\in\Gamma}$ and $(\Phi_{\gamma,\alpha})_{\gamma\in\Gamma,\,\alpha<\gamma}$ with the required properties is immediate from 3.10.8 and 3.10.9. The uniqueness follows from 3.10.1 and 3.10.3. $\triangleright$

3.11. Diffuse operators

In this section, we give necessary and sufficient conditions under which an order bounded linear operator is diffuse. We first handle the case of functionals and then obtain a general result by means of Boolean valued interpretation of the scalar result.

3.11.1. We need a property of additive measures on Boolean algebras. Consider a Boolean algebra $\mathscr{B}$. A function $\mu : \mathscr{B} \to \mathbb{R}$ is called *additive* if $\mu(a \vee b) = \mu(a) + \mu(b)$ for all $a, b \in \mathscr{B}$ with $a \wedge b = 0$ and *completely additive* whenever $\mu(\bigvee D) = \sum_{d\in D} \mu(d)$ for every disjoint subset $D \subset \mathscr{B}$. A positive (that is, $(\forall\, b \in \mathscr{B})\mu(b) \geqslant 0$) additive function μ is completely additive if and only if it is *order continuous*; i.e., $\lim_\alpha \mu(b_\alpha) = 0$ for every decreasing net (b_α) in $\mathscr{B}$ with $0 = \inf_\alpha b_\alpha$.

Say that $b_0 \in \mathscr{B}$ is a *μ-atom* if $\mu(b_0) \neq 0$ and for every $b \in \mathscr{B}$ with $b \leqslant b_0$ either $\mu(b) = 0$ or $\mu(b - b_0) = 0$. An additive function μ is said to be *nonatomic* on $\mathscr{B}$ if there are no μ-atoms in $\mathscr{B}$ or, equivalently, for each $b \in \mathscr{B}$ the relation $\mu(b) \neq 0$ implies the existence of $b_0 \in \mathscr{B}$ such that $b_0 \leqslant b$, $\mu(b_0) \neq 0$ and $\mu(b - b_0) \neq 0$.

3.11.2. *Assume that $\mathscr{B}$ is a complete Boolean algebra and $\mu : \mathscr{B} \to \mathbb{R}$ is a nonatomic order continuous additive function. Then for all $b \in \mathscr{B}$ and $0 \leqslant \alpha \leqslant \mu(b)$ there exists $b_\alpha \in \mathscr{B}$ with $b_\alpha \leqslant b$ and $\alpha = \mu(b_\alpha)$.*

$\triangleleft$ Let $b \in \mathscr{B}$ and $0 \leqslant \alpha \leqslant \mu(b)$ be given. Put $D := \{d \in \mathscr{B} : d \leqslant b,\ \mu(d) \leqslant \alpha\}$ and, given $c, d \in \mathscr{B}$, put $c \preccurlyeq d$ whenever $\mu(c-d) = 0$. It can easily be checked involving order continuity of μ that every chain in an ordered set $(D, \preccurlyeq)$ has an upper bound in D. Thus, by the Kuratowski–Zorn Lemma, D has a maximal element, say b_α; i.e., if $c \in D$ and $b_\alpha \leqslant c$, then $\mu(c - b_\alpha) = 0$. We claim that $\mu(b_\alpha) = \alpha$. Indeed, otherwise $\mu(b - b_\alpha) = \mu(b) - \mu(b_\alpha) \geqslant \alpha - \mu(b_\alpha) > 0$ and, since μ is nonatomic, there is $c_1 \leqslant b_0 := b - b_\alpha$ with $0 < \mu(c_1) < \mu(b_0)$. Moreover, with the choice $b_1 := c_1$ or $b_1 := b_0 - c_1$ this yields $0 < \mu(b_1) \leqslant (1/2)\mu(b_0)$. Repeating the same argument, we obtain a sequence (b_n) in $\mathscr{B}$ such that

$b_0 \geqslant b_n \geqslant b_{n-1}$ and $0 < \mu(b_n) \leqslant (1/2)\mu(b_{n-1}) \leqslant (1/2^n)\mu(b_0)$ for all $n \in \mathbb{N}$. Choose n with $(1/2^n)\mu(b_0) \leqslant \alpha - \mu(b_\alpha)$ and put $d := b_n \vee b_\alpha$. Then $d \leqslant b$, $b_\alpha \leqslant d$, and $\mu(d - b_\alpha) = \mu(b_n) > 0$. This contradicts the maximality of b_α. $\triangleright$

3.11.3. Theorem. *An order bounded functional f on a vector lattice X is diffuse if and only if for all $0 \leqslant x \in X$ and $0 < \varepsilon \in \mathbb{R}$ there is a finite disjoint collection of positive functionals $f_1, \dots, f_N \in X^\sim$ such that*

$$|f| = f_1 + \dots + f_N, \quad f_k(x) \leqslant \varepsilon \quad (k := 1, \dots, N).$$

$\triangleleft$ Assume that h is a nonzero lattice homomorphism with $h \leqslant |f|$ and choose $x \in X_+$ such that $h(x) = 1$. If $|f| = f_1 + \dots + f_N$ for a collection of pairwise disjoint positive functionals $f_1, \dots, f_N \in X^\sim$, then $h \leqslant f_k$ for some $1 \leqslant k \leqslant n$. Thus $1 = h(x) \leqslant f_k(x)$ and the above necessary condition cannot be fulfilled.

To prove the sufficiency, take a diffuse $f \in X^\sim$ and fix $x \in X_+$ and $0 < \varepsilon \in \mathbb{R}$. There is no loss of generality in assuming that f is positive and $f_0(x) > 0$ for every component $f_0 \in \mathbb{C}(f)$. Put $\mathscr{B} := \mathbb{C}(f)$ and define $\mu : \mathscr{B} \to \mathbb{R}$ by $\mu(b) := b(x)$. Clearly, μ is order continuous nonatomic additive function on $\mathscr{B}$. Pick $0 = \alpha_0 < \alpha_1 < \dots < \alpha_N = \mu(f)$ with $\alpha_\imath - \alpha_{\imath-1} < \varepsilon$ ($\imath := 1, \dots, N$). By 3.11.2 we can choose a finite sequence $b_1 \leqslant \dots \leqslant b_N$ in $\mathscr{B}$ such that $\mu(b_\imath) = \alpha_\imath$ for all $\imath := 1, \dots, N$. If $b_0 := b_N^*$ then $\mu(b_0) = \mu(f) - \mu(b_N) = 0$. It remains to put $f_\imath := b_\imath - b_{\imath-1}$ ($\imath := 1, \dots, N$) and observe that $f_1 + \dots + f_N = f$ and $f_\imath(x) = \mu(f_\imath) = \mu(b_\imath) - \mu(b_{\imath-1}) = \alpha_\imath - \alpha_{\imath-1} < \varepsilon$ for all $\imath := 1, \dots, N$. $\triangleright$

3.11.4. Theorem. *Let X and Y be vector lattices with Y Dedekind complete. For $T \in L^\sim(X, Y)$, the following are equivalent:*

(1) *T is diffuse.*

(2) *For every $x \in X$ we have*

$$\bigwedge \left\{ \bigvee_{\imath=1}^{n} T_\imath |x| : \ |T| = \sum_{\imath=1}^{n} T_\imath, \ T_\imath \perp T_j \ (\imath \neq j), \ n \in \mathbb{N} \right\} = 0.$$

(3) *Given $x \in X_+$, $0 < \varepsilon \in \mathbb{R}$, and $\pi \in \mathbb{P}(Y)$ with $\pi|T|x \neq 0$, there exist a nonzero projection $\rho \leqslant \pi$ and pairwise disjoint positive operators $T_1, \dots, T_N$ from X to ρY such that $\rho|T| = T_1 + \dots + T_N$ and $T_k x \leqslant \varepsilon |T| x$ for all $k := 1, \dots, N$.*

(4) *Given $x \in X_+$, $0 < \varepsilon \in \mathbb{R}$, and $\pi \in \mathbb{P}(Y)$ with $\pi|T|x \neq 0$, there exists a countable partition (π_n) of π such that for every $n \in \mathbb{N}$ the operator $\pi_n|T|$ decomposes into the sum of pairwise disjoint positive operators $T_{1,n}, \dots, T_{n,n}$ from X to $\pi_n Y$ satisfying $T_{k,n}x \leqslant \varepsilon|T|x$ for all $k := 1, \dots, n$.*

$\lhd$ It is an easy exercise to check the equivalences (2) $\Longleftrightarrow$ (3) $\Longleftrightarrow$ (4). The proof of (1) $\Longleftrightarrow$ (4) is obtained by interpreting Theorem 3.11.3 within $\mathbb{V}^{(\mathbb{B})}$ where $\mathbb{B} := \mathbb{P}(Y)$. By the Gordon Theorem we can take $Y = \mathscr{R}{\downarrow}$ without loss of generality. Moreover, the problem reduces easily to the case of positive T not involving Boolean valued arguments. Put $\tau := T{\uparrow}$ and note that, according to Corollary 3.3.6 (4), T is diffuse if and only if $[\![\tau \text{ is diffuse}]\!] = \mathbb{1}$. Theorem 3.11.3 is valid within $\mathbb{V}^{(\mathbb{B})}$ by transfer, and so the sentence "τ is diffuse" is equivalent to the formula

$$(\forall\, x \in X^\wedge)(\forall\, 0 < \varepsilon \in \mathbb{R}^\wedge)(\exists\, n \in \mathbb{N}^\wedge)(\exists\, \nu)\varphi(x, \varepsilon, n, \nu, \tau, X)$$

where $\varphi(x, \varepsilon, n, \nu, \tau, X)$ stands for the assertion: $\nu : \{1, \dots, n\} \to X^\sim_+$, $\tau = \nu(1) + \dots + \nu(n)$ and $\nu(\imath) \perp \nu(\jmath)$, $(\imath \neq \jmath)$, $\nu(\imath)x \leqslant \varepsilon\tau(x)$ for all $\imath, \jmath := 1, \dots, n$. By 1.5.2 quantifications over $X^\wedge$, $\mathbb{R}^\wedge$, and $\mathbb{N}^\wedge$ can be replaced by order operations in $\mathbb{B}$ over X, $\mathbb{R}$, and $\mathbb{N}$:

$$\mathbb{1} = \bigwedge_{x \in X} \bigwedge_{0 < \varepsilon \in \mathbb{R}} \bigvee_{n \in \mathbb{N}} [\![(\exists\, \nu)\varphi(x^\wedge, \varepsilon^\wedge, n^\wedge, \nu, \tau, X^\wedge)]\!].$$

This amounts to saying that for all $x \in X$ and $0 < \varepsilon \in \mathbb{R}$ we can choose a countable partition of unity (b_n) in $\mathbb{B}$ with $b_n \leqslant [\![(\exists\, \nu)\varphi(\dots)]\!]$.

In view of the maximum principle, for each $n \in \mathbb{N}$ there exists $\nu_n \in \mathbb{V}^{(\mathbb{B})}$ such that $b_n \leqslant [\![\varphi(x^\wedge, \varepsilon^\wedge, n^\wedge, \nu_n, \tau, X^\wedge)]\!]$. Passing to relative subalgebra $\mathbb{B}_n := [\mathbb{0}, b_n]$ and taking 1.3.7 and 1.4.6 into account we see that the last inequality is fulfilled if and only if $\varphi(x^\wedge, \varepsilon^\wedge, n^\wedge, \bar\nu_n, \bar\tau, X^\wedge)$ with $\bar\nu_n := b_n \wedge \nu_n \in \mathbb{V}^{(\mathbb{B}_n)}$ and $\bar\tau := b_n \wedge \tau \in \mathbb{V}^{(\mathbb{B}_n)}$ is true within $\mathbb{V}^{(\mathbb{B}_n)}$ or, in more details (with $\langle n \rangle := \{1, \dots, n\}$ and $e := \bar\tau(x^\wedge)$ for short),

$$\mathbb{V}^{(\mathbb{B}_n)} \models \bar\nu_n : \langle n \rangle^\wedge \to (X^\wedge)^\sim_+ \wedge \bar\tau = \bar\nu_n(1) + \dots + \bar\nu_n(n),$$
$$\mathbb{V}^{(\mathbb{B}_n)} \models (\forall\, \imath, \jmath \in \langle n \rangle^\wedge)(\imath \neq \jmath \to \bar\nu_n(\imath) \perp \bar\nu_n(\jmath)) \wedge (\bar\nu_n(\imath)x^\wedge \leqslant \varepsilon^\wedge e).$$

By 1.5.8 the modified descent $\mathscr{T} := \bar\nu_n{\downarrow}$ maps $\{1, \dots, n\}$ into $(X^\wedge)^\sim_+{\downarrow}$.

Put $\pi_n := \chi^{-1}(b_n)$ and $T_\imath := \mathscr{T}(\imath){\downarrow}$ and observe that, by Theorem 3.3.5 (5), $\{T_1, \dots, T_n\}$ is a disjoint collection of positive operators from X to $\pi_n Y$ with

$$\pi_n T = T_1 + \dots + T_n.$$

Moreover, for each $x \in X$ we have

$$b_n \leqslant [\![T_i x = \mathscr{T}(\imath)x^\wedge = \bar{\nu}_n(\imath^\wedge)x^\wedge]\!] \wedge [\![\bar{\nu}_n(\imath^\wedge)x^\wedge \leqslant \varepsilon^\wedge e]\!] \leqslant [\![T_i x \leqslant \varepsilon^\wedge e]\!],$$

so that $T_i x \leqslant \varepsilon T x$.

Thus we arrived at the conclusion that (4) holds if and only if $[\![\tau \text{ is diffuse }]\!] = \mathbb{1}$, which completes the proof. $\triangleright$

3.11.5. Theorem. *Let X, Y, and Z be vector lattices with Z Dedekind complete. For $B \in BL^\sim(X,Y;Z)$, the following are equivalent:*

(1) *B is diffuse.*

(2) *For all $x \in X$ and $y \in Y$ the identity holds*

$$\bigwedge \Bigg\{ \bigvee_{\imath=1}^{n} B_\imath(|x|,|y|) : |B| = \sum_{\imath=1}^{n} B_\imath,\ B_\imath \perp B_\jmath\,(\imath \neq \jmath),\ n \in \mathbb{N} \Bigg\} = 0.$$

(3) *Given $x \in X_+$, $y \in Y_+$, $0 < \varepsilon \in \mathbb{R}$, and $\pi \in \mathbb{P}(Z)$ with $\pi|B|(x,y) \neq 0$, there exist a nonzero projection $\rho \leqslant \pi$ and pairwise disjoint positive bilinear operators $B_1, \dots, B_N$ from $X \times Y$ to ρZ such that*

$$\rho|B| = B_1 + \dots + B_N;$$
$$B_k(x,y) \leqslant \varepsilon|B|(x,y)$$

for all $k := 1, \dots, N$.

(4) *Given $x \in X_+$, $y \in Y_+$, $0 < \varepsilon \in \mathbb{R}$, and $\pi \in \mathbb{P}(Z)$ with $\pi|B|(x,y) \neq 0$, there exists a countable partition (π_n) of π such that for every $n \in \mathbb{N}$ the operator $\pi_n|B|$ decomposes into the sum of pairwise disjoint positive operators $B_{1,n}, \dots, B_{n,n}$ from $X \times Y$ to $\pi_n Z$ satisfying*

$$B_{k,n}(x,y) \leqslant \varepsilon|B|(x,y)$$

for all $k := 1, \dots, n$.

$\triangleleft$ The proof can be obtained by reasoning along the lines of Theorem 3.11.4. Alternatively, it can be reduced to the case of linear operators by applying Fremlin's Theorem 3.2.8. $\triangleright$

3.12. Variations on the Theme

In this section we apply the Boolean value approach to the three types of problems: the multiplicative representation of a lattice multimorphism, the characterization of disjointness preserving sets of operators, and the Sobczyk–Hammer type decomposition for measures with values in Dedekind complete vector lattices.

3.12.A. Representation of Lattice Multimorphisms

3.12.A.1. Let X and Y be vector lattices. Recall that a bilinear operator $B : X \times X \to Y$ is said to be *orthosymmetric* if $x \perp y$ implies $B(x, y) = 0$ for all $x, y \in X$, *symmetric* if $B(x, y) = B(y, x)$ for all $x, y \in X$, and *positive semidefinite* if $B(x, x) \geqslant 0$ for every $x \in X$. A bilinear operator B is said to be a *lattice bimorphism* if the partial mappings $B(x, \cdot) : y' \mapsto b(x, y')$ $(y' \in Y)$ and $B(\cdot, y) : x' \mapsto B(x', y)$ $(x' \in X)$ are lattice homomorphisms for all $0 \leqslant x \in X$ and $0 \leqslant y \in Y$.

3.12.A.2. Given a lattice bimorphism $\beta : X \times Y \to \mathbb{R}$, there are two lattice homomorphisms $\sigma : X \to \mathbb{R}$ and $\tau : Y \to \mathbb{R}$ such that

$$\beta(x, y) = \sigma(x)\tau(y) \quad (x \in X,\ y \in Y).$$

If, in addition, $X = Y$ and β is symmetric then we can take $\sigma = \tau$.

$\triangleleft$ We assume that the lattice bimorphism $\beta : X \times Y \to \mathbb{R}$ is nonzero, since otherwise we have nothing to prove. Choose $0 \leqslant x_0 \in X$ so that $\tau := \beta(x_0, \cdot)$ be a nonzero lattice homomorphism.

Take $u \in X_+$ and put $e = x_0 + u$. It is clear that the three lattice homomorphisms β_{x_0}, β_u, and β_e are connected by the equality $\beta_e = \beta_{x_0} + \beta_u$. By the Kutateladze Theorem 3.1.11 ((1) $\Longleftrightarrow$ (6)), for appropriate $r, s \in \mathbb{R}_+$ we have $\tau = \beta_{x_0} = r\beta_e$ and $\beta_u = s\beta_e$. Since $\tau \neq 0$, we have $r > 0$ and so $\beta_u = \gamma\tau$, where $\gamma := s/r$.

So, for every $u \in X_+$ there exists a number $\gamma(u) \geqslant 0$ such that $\beta_u = \gamma(u)\tau$; i.e., $\beta(u, y) = \gamma(u)\tau(y)$ for all $u \in X_+$ and $y \in Y$. Hence, in particular, $\gamma(x_0) = 1$.

Since, for $y \in Y_+$, the functional $\beta(\cdot, y)$ is a lattice homomorphism; therefore, for all $u, u' \in X_+$ and $\lambda \in \mathbb{R}_+$ we have

$$\gamma(u + u')\tau(y) = \gamma(u)\tau(y) + \gamma(u')\tau(y),$$

$$\gamma(u \vee u')\tau(y) = \gamma(u)\tau(y) \vee \gamma(u')\tau(y),$$

$$\gamma(\lambda u)\tau(y) = \lambda\gamma(u)\tau(y) \quad (y \in Y_+).$$

Hence, γ is additive, positive homogeneous, and join preserving. Put

$$\sigma(x) := \gamma(x^+) - \gamma(x^-) \quad (x \in X).$$

Then the functional σ extends γ onto the whole lattice X and σ is a lattice homomorphism. Moreover, for $x \in X$ and $y \in Y$ we have

$$\beta(x, y) = \beta(x^+, y) - \beta(x^-, y) = \gamma(x^+)\tau(y) - \gamma(x^-)\tau(y) = \sigma(x)\tau(y).$$

If $X = Y$ and β is symmetric then $\sigma(x)\tau(y) = \tau(x)\sigma(y)$ for all $x \in X$ and $y \in Y$; hence, $\tau(y) = \tau(x_0)\sigma(y)$. By putting $\rho := \sqrt{\tau(x_0)}\sigma$ we obtain the representation $\beta(x, y) = \rho(x)\rho(y)$. $\triangleright$

3.12.A.3. Theorem. *Given an arbitrary lattice bimorphism $B : X \times Y \to Z$, there are two lattice homomorphisms $S : X \to Z^{\mathfrak{u}}$ and $T : Y \to Z^{\mathfrak{u}}$ such that*

$$B(x, y) = S(x)T(y) \quad (x \in X,\ y \in Y).$$

If, in addition, $X = Y$ and B is symmetric then we can take $S = T$.

$\triangleleft$ The reduction of the general case to the scalar case is carried out by means of Boolean valued analysis. To apply the latter, it is important to observe that 3.12.A.2 remains valid on replacing X and Y with vector lattices over an ordered field $\mathbb{F}$ satisfying the inclusions $\mathbb{Q} \subset \mathbb{F} \subset \mathbb{R}$. Furthermore, it is worth taking account of the fact that the functionals β, σ, and τ act into $\mathbb{R}$ and are $\mathbb{F}$-linear. The Kutateladze Theorem, which is a key tool of the above proof, remains valid for those functionals.

Turning to the general case and recalling the Gordon Theorem, we can assume that the universally complete vector lattice $Z^{\mathfrak{u}}$ is the descent $\mathscr{R}{\downarrow}$ of the reals $\mathscr{R}$ from the Boolean valued model $\mathbb{V}^{(\mathbb{B})}$ with $\mathbb{B} := \mathbb{P}(Z^{\mathfrak{u}})$. Take the ordered field $\mathbb{F}$ to be the standard name $\mathbb{R}^\wedge$ of $\mathbb{R}$. Then $X^\wedge$ and $Y^\wedge$ are vector lattices over $\mathbb{F}$ within $\mathbb{V}^{(\mathbb{B})}$. The technique of ascending and descending (cp. 1.6.8) yields existence of $\beta \in \mathbb{V}^{(\mathbb{B})}$ such that

$$[\![\, \beta : X^\wedge \times Y^\wedge \to \mathscr{R} \,]\!] = \mathbb{1},$$
$$[\![\, \beta \text{ is a lattice } \mathbb{F}\text{-bimorphism} \,]\!] = \mathbb{1},$$
$$[\![\, \beta(x^\wedge, y^\wedge) = B(x, y) \,]\!] = \mathbb{1} \quad (x \in X,\ y \in Y).$$

The above fact on the structure of real lattice homomorphisms is valid within $\mathbb{V}^{(\mathbb{B})}$ according to the transfer principle 1.4.1. Applying the maximum principle 1.4.2, we find elements $\sigma, \tau \in \mathbb{V}^{(\mathbb{B})}$ such that

$$
\begin{gathered}
[\![\sigma : X^{\wedge} \to \mathscr{R}]\!] = [\![\tau : Y^{\wedge} \to \mathscr{R}]\!] = \mathbb{1}, \\
[\![\sigma \text{ and } \tau \text{ are lattice } \mathbb{F}\text{-homomorphisms}]\!] = \mathbb{1}, \\
[\![(\forall\, x \in X^{\wedge})\, (\forall\, y \in Y^{\wedge})\, \beta(x,y) = \sigma(x)\tau(y)]\!] = \mathbb{1}.
\end{gathered}
$$

Let S and T denote the modified descents of σ and τ as defined in 1.5.8:

$$
\begin{gathered}
S : X \to \mathscr{R}{\downarrow}, \quad T : Y \to \mathscr{R}{\downarrow}, \\
[\![S(x) = \sigma(x^{\wedge})]\!] = [\![T(y) = \tau(y^{\wedge})]\!] = \mathbb{1} \quad (x \in X,\ y \in Y).
\end{gathered}
$$

By Corollary 3.3.6 (1), S and T are lattice homomorphisms. Moreover, the representation $B(x,y) = S(x)T(y)$ is valid, since for $x \in X$ and $y \in Y$ we have

$$
[\![B(x,y) = \beta(x^{\wedge}, y^{\wedge}) = \sigma(x^{\wedge})\tau(y^{\wedge}) = S(x)T(y)]\!] = \mathbb{1}.
$$

The fact that B is symmetric amounts to the fact that β is symmetric within $\mathbb{V}^{(\mathbb{B})}$; therefore, we can take σ and τ coincident, which is equivalent to the equality $S = T$. $\triangleright$

3.12.A.4. Say that a bilinear operator $B : X \times Y \to G$ is *disjointness preserving* if for all $x \in X$ and $y \in Y$ we have

$$
\begin{gathered}
x_1 \perp x_2 \implies B(x_1, y) \perp B(x_2, y) \quad (x_1, x_2 \in X), \\
y_1 \perp y_2 \implies B(x, y_1) \perp B(x, y_2) \quad (y_1, y_2 \in Y).
\end{gathered}
$$

It is clear that a bilinear operator B is disjointness preserving if and only if $B(x, \cdot) : Y \to G$ and $B(\cdot, y) : X \to G$ are disjointness preserving for all $x \in X$ and $y \in Y$. A positive disjointness preserving bilinear operator is a lattice bimorphism, since $B(x, \cdot)$ and $B(\cdot, y)$ are lattice homomorphisms for $x \geqslant 0$ and $y \geqslant 0$.

3.12.A.5. Corollary. *Each order bounded disjointness preserving bilinear operator $B : X \times Y \to Z$ is representable as the product*

$$
B(x,y) = S(x)T(y) \quad (x \in X,\ y \in Y),
$$

where one of the two operators $S : X \to Z^{\mathrm{u}}$ and $T : Y \to Z^{\mathrm{u}}$ can be chosen to be a lattice homomorphism, while the other be bounded and *disjointness preserving. If, in addition, $X = Y$ and B is symmetric, then we can take $T = \pi S - \pi^{\perp} S$, where π is a band projection in $Z^{\mathfrak{u}}$.*

$\lhd$ Let a bilinear operator B be order bounded and disjointness preserving. By Theorems 3.2.8 and 3.4.4 there exists a projection π in $Z^{\mathfrak{u}}$ such that $\pi B = B^+$ and $\pi^{\perp} B = -B^-$; moreover, B^+ and B^- are lattice bimorphisms. According to Theorem 3.12.A.3 we have the representations

$$B^+(x, y) = S_1(x)T_1(y), \quad B^-(x, y) = S_2(x)T_2(y) \quad (x \in X,\ y \in Y),$$

where $S_1, S_2 : X \to Z^{\mathfrak{u}}$ and $T_1, T_2 : Y \to Z^{\mathfrak{u}}$ are lattice homomorphisms. Denote the bilinear operator $(x, y) \mapsto S(x)T(y)$ by $S \odot T$. We have

$$\begin{aligned} B = B^+ - B^- = \pi B + \pi^{\perp} B &= \pi S_1 \odot T_1 - \pi^{\perp} S_2 \odot T_2 \\ &= \pi S_1 \odot \pi T_1 - \pi^{\perp} S_2 \odot \pi^{\perp} T_2. \end{aligned}$$

Put $S = \pi S_1 - \pi^{\perp} S_2$ and $T = \pi T_1 + \pi^{\perp} T_2$. Then S is order bounded and disjointness preserving, T is a lattice homomorphism, and

$$\begin{aligned} S \odot T &= (\pi S_1 - \pi^{\perp} S_2) \odot (\pi T_1 + \pi^{\perp} T_2) \\ &= \pi S_1 \odot \pi T_1 + \pi S_1 \odot \pi^{\perp} T_2 - \pi^{\perp} S_2 \odot \pi T_1 - \pi^{\perp} S_2 \odot \pi^{\perp} T_2 \\ &= \pi S_1 \odot \pi T_1 - \pi^{\perp} S_2 \odot \pi^{\perp} T_2 = B \end{aligned}$$

as required. The rest is obvious. $\rhd$

3.12.A.6. Corollary. *Let X and Y be vector lattices, T an order bounded disjointness preserving operator from X to Y, and Y_0 a vector sublattice of Y generated by $T(X)$. Then there exists a unique algebra and lattice homomorphism $\widetilde{T}$ from $\mathrm{Orth}(X)$ to $\mathrm{Orth}(Y_0)$ such that*

$$\widetilde{T}(\pi)(Tx) = T(\pi x) \quad (\pi \in \mathrm{Orth}(X),\ x \in X).$$

$\lhd$ Assume without loss of generality that $Y = Y_0$. Moreover, the proof can be reduced to the case of positive B according to Theorem 3.4.3. It suffices to note that if $T : X \to Y$ is a lattice homomorphism, then the bilinear operator $B : \mathrm{Orth}(X) \times X \to Y$ defined as $(\pi, x) \mapsto T(\pi x)$ is a lattice bimorphism and apply the above result. Indeed, by Theorem 3.12.A.3 there are lattice homomorphisms $\bar{S} : \mathrm{Orth}(X) \to Y^{\mathfrak{u}}$ and $\bar{T} : X \to Y^{\mathfrak{u}}$ such that $T(\pi x) = \bar{S}(\pi)\bar{T}(x)$ for all $x \in X$ and $\pi \in \mathrm{Orth}(X)$. It follows that the element $u := \bar{S}(I_X)$

is an order unit in $Y^{\mathfrak{u}}$ and $\bar{T}(X)^{\perp\perp} = Y^{\mathfrak{u}}$, since $Tx = u\bar{T}x$ for all $x \in X$, and hence u^{-1} exists in the f-algebra $Y^{\mathfrak{u}}$. Denote now by $\widetilde{T}(\pi)$ the multiplication operator $y \mapsto u^{-1}\bar{S}(\pi)y$ $(y \in Y)$ and observe that $\widetilde{T}(\pi)$ is an orthomorphism on Y. Indeed, if $y = Tx$ then $\widetilde{T}(\pi)y = \bar{S}(\pi)u^{-1}Tx = \bar{S}(\pi)\bar{T}x = T(\pi x) \in Y$, so that $\widetilde{T}(\pi)$ sends $T(X)$ and hence the whole Y into Y. Clearly, $\widetilde{T}(I_X) = I_Y$, since by the same reason $\widetilde{T}(I_X)Tx = \bar{S}(I_X)u^{-1}Tx = \bar{S}(I_X)\bar{T}x = Tx$. Moreover, $\widetilde{T}$ is a lattice homomorphism, since so is $\bar{S}$. Given $\pi, \rho \in \mathbb{P}(Y)$ and $x \in X$ we have

$$
\begin{aligned}
u^{-1}\bar{S}(\pi\rho)\bar{T}x = u^{-1}T(\pi\rho x) = u^{-1}\bar{S}(\pi)\bar{T}(\rho x)& \\
= u^{-1}\bar{S}(\pi)u^{-1}T(\rho x) = u^{-1}\bar{S}(\pi)u^{-1}\bar{S}(\rho)\bar{T}x&,
\end{aligned}
$$

whence $\widetilde{T}(\pi\rho) = \widetilde{T}(\pi)\widetilde{T}(\rho)$ and $\widetilde{T}$ is an f-algebra homomorphism. Observe finally that $T(\pi x) = \bar{S}(\pi)\bar{T}x = u^{-1}\bar{S}(\pi)Tx = \widetilde{T}(\pi)Tx$. $\triangleright$

3.12.A.7. Theorem. *Assume that X, Y, and Z are vector lattices with Z having the projection property. An order bounded bilinear operator $B : X \times Y \to Z$ is disjointness preserving if and only if for every $\pi \in \mathbb{P}(Z)$ the subspaces $X_\pi := \bigcap\{\ker(\pi B(\cdot, y)) : y \in Y\}$ and $Y_\pi := \bigcap\{\ker(\pi B(x, \cdot)) : x \in X\}$ are order ideals respectively in X and Y, and the kernel of every stratum πB of B with $\pi \in \mathbb{P}(Z)$ is representable as*

$$
\ker(\pi B) = \bigcup \big\{X_\sigma \times Y_\tau : \ \sigma, \tau \in \mathbb{P}(Z);\ \sigma \vee \tau = \pi\big\}.
$$

$\triangleleft$ The necessity is immediate from 3.12.A.5. The proof of the sufficiency can be deduced from a corresponding scalar result by means of interpreting it within the appropriate Boolean valued model similar to that in Theorem 3.4.2. As to the scalar case, the following is true:

An order bounded bilinear functional $\beta : X \times Y \to \mathbb{R}$ is disjointness preserving if and only if $\ker(\beta) = (X_0 \times Y) \cup (X \times Y_0)$ *for some order ideals $X_0 \subset X$ and $Y_0 \subset Y$.*

Indeed, assume that the latter is fulfilled and take $y \in Y$. If $y \in Y_0$ then $\beta(\cdot, y) \equiv 0$, otherwise $\ker(\beta(\cdot, y)) = X_0$ and $\beta(\cdot, y)$ is disjointness preserving by 3.4.1 (7). Similarly, $\beta(x, \cdot)$ is disjointness preserving for all $x \in X$. The converse follows from 3.12.A.2. $\triangleright$

3.12.B. Disjointness Preserving Sets of Operators.

3.12.B.1. A nonempty subset $\mathscr{D}$ of $L^\sim(X, Y)$ is called *n-disjoint* in $L^\sim(X, Y)$ if $|T_0x_0| \wedge \dots \wedge |T_nx_n| = 0$ for all $T_0, \dots, T_n \in \mathscr{D}$ and pairwise disjoint $x_0, \dots, x_n \in X$. An n-disjoint set $\mathscr{M}$ in $L^\sim(X,Y)$ is said to be *maximal* if every n-disjoint set in $L^\sim(X,Y)$ including $\mathscr{M}$ coincides with $\mathscr{M}$. A 1-disjoint set of operators is also called disjointness preserving. More precisely, a nonempty subset $\mathscr{D}$ of $L^\sim(X,Y)$ is *disjointness preserving* in $L^\sim(X,Y)$ if $S(u) \perp T(v)$ for all $S, T \in \mathscr{D}$ and $u, v \in X$ with $u \perp v$.

Observe some immediate consequences of the definition. An order bounded operator T from X into Y is n-disjoint if and only if the singleton $\{T\}$ is an n-disjoint set in $L^\sim(X,Y)$. Therefore, each member of an n-disjoint set in $L^\sim(X,Y)$ is an order bounded n-disjoint operator. Moreover, the nonempty subset $\mathscr{D}$ of $L^\sim(X,Y)$ is n-disjoint in $L^\sim(X,Y)$ if and only if each collection of $n+1$ elements $\{T_0, \dots, T_n\}$ of the members of $\mathscr{D}$ is n-disjoint.

3.12.B.2. Suppose that X is a vector sublattice of Y. A mapping $T : X \to Y$ is called an *orthomorphism* from X to Y if T is order bounded and $x \perp y$ implies $Tx \perp y$ for all $x \in X$ and all $y \in Y$. The set of all orthomorphisms from X to Y is denoted by $\operatorname{Orth}(X,Y)$. It is easily seen that if T is an orthomorphism from X to Y then $T(X) \subset X^{\perp\perp}$. Moreover, the representation holds:

$$\operatorname{Orth}(X,Y) = \{T|_X : T \in \operatorname{Orth}(Y^{\mathfrak{u}}),\ T(X) \subset Y\}.$$

Indeed, the universal completion $Y^{\mathfrak{u}}$ of a vector lattice Y is an f-algebra with a multiplicative unit. Each orthomorphism T from X to Y extends uniquely to an orthomorphism $\hat{T}$ on $Y^{\mathfrak{u}}$. Each orthomorphism on $Y^{\mathfrak{u}}$ is a multiplication operator. Therefore, if $T \in \operatorname{Orth}(X,Y)$, then there exists some $y \in X^{\mathfrak{u}}$ such that $T(x) = yx$ holds for all $x \in X$, so that $y \cdot X \subset Y$ (cp. Aliprantis and Burkinshaw [28, Theorem 2.63]).

3.12.B.3. Consider an example. For $\mathscr{D} \subset L^\sim(Y,Z)$ and $T : X \to Y$ put $\mathscr{D} \circ T := \{S \circ T : S \in \mathscr{D}\}$. If $T, T_1, \dots, T_n$ are lattice homomorphisms from X to Y then $\operatorname{Orth}(T(X),Y) \circ T$ is a disjointness preserving set and the set $\operatorname{Orth}(T_1(X),Y) \circ T_1 + \dots + \operatorname{Orth}(T_n(X),Y) \circ T_n$ is n-disjoint. The next aim is to demonstrate that this example is typical.

3.12.B.4. *Given n pairwise disjoint nonzero real lattice homomorphisms $h_1, \dots, h_n$ on a vector lattice X, there exist pairwise disjoint elements $x_1, \dots, x_n \in X$ such that $h_\imath(x_\jmath) = \delta_{ij}$ for all $\imath, \jmath := 1, \dots, n$ (with $\delta_{\imath,\jmath}$ the Kronecker symbol).*

$\triangleleft$ Pick $u_\imath \in X_+$ with $h_\imath(u_\imath) > 0$ and put $u := u_1 + \dots + u_n$. By the Kakutani–Kreĭns Representation Theorem the order ideal X_u in X

generated by u can be identified with a norm dense vector sublattice of $C(Q)$ containing constants and separating points, where $C(Q)$ is the Banach lattice of continuous functions on a Hausdorff compact topological space Q. Moreover, u corresponds under this identification to the identically one function $\mathbb{1} \in C(Q)$. Then the restrictions $h_1|_{X_u}, \dots, h_n|_{X_u}$ are pairwise disjoint lattice homomorphisms. Let $\hat{h}_\imath$ stand for the extension of $h_\imath|_{X_u}$ to $C(Q)$ by norm continuity. Clearly, $\hat{h}_1, \dots, \hat{h}_n$ are also pairwise disjoint nonzero lattice homomorphisms and so there exist distinct points $q_1, \dots, q_n \in Q$ such that $\hat{h}_\imath$ coincides with the Dirac measure $\delta_{q_\imath} : x \mapsto x(q_\imath)$ $(x \in C(Q))$. By the Tietze–Urysohn Theorem we can find pairwise disjoint continuous functions $y_1, \dots, y_n \in C(Q)$ such that $y_\imath(q_\imath) = 1$ and $0 \leqslant y_\imath(q) \leqslant 1$ for all $q \in Q$ and $\imath := 1, \dots, n$. Take $\bar{y}_\imath \in X_u$ so that $\|y_\imath - \bar{y}_\imath\| < \varepsilon < 1/2$ and note that $h_\imath(\bar{y}_\imath) - \varepsilon > 1 - 2\varepsilon > 0$ and $\bar{y}_\imath - \varepsilon\mathbb{1} \leqslant y_\imath$. Put $x_\imath := (h_\imath(\bar{y}_\imath) - \varepsilon)^{-1}(\bar{y}_\imath - \varepsilon\mathbb{1}) \vee 0$ and observe that $\{x_1, \dots, x_n\} \subset X$ is the required collection. $\triangleright$

3.12.B.5. *A nonempty set $\mathscr{D}$ in $X^\sim$ is n-disjoint if and only if there exist pairwise disjoint lattice homomorphisms $h_1, \dots, h_n : X \to \mathbb{R}$ such that $\mathscr{D} \subset \mathbb{R} \cdot h_1 + \ldots + \mathbb{R} \cdot h_n$. Moreover, $\mathscr{D}$ is maximal if and only if either $\mathscr{D} = \mathrm{Hom}(X, \mathbb{R}) = \{0\}$ or $\mathscr{D} = \mathbb{R} \cdot h_1 + \dots + \mathbb{R} \cdot h_m$ with nonzero $h_1, \dots, h_m$ and $m := \min\{n, \mathrm{cat}(X^\sim)\}$, where $\mathrm{cat}(X^\sim)$ stands for the cardinality of atoms in $\mathbb{P}(X^\sim)$ (see 3.10.4). In this event the collection $\{h_1, \dots, h_m\}$ is unique up to permutation.*

$\triangleleft$ The sufficiency is obvious, so only the necessity will be proved. Suppose that $\mathscr{D} \neq \{0\}$. There is no loss of generality in assuming that $f \in \mathscr{D}$ implies $|f| \in \mathscr{D}$. According to Theorem 3.7.7 each functional in $\mathscr{D}$ is decomposable into a sum of disjointness preserving components. Let $\mathscr{D}_0$ stand for the set of all such components of all functionals in $\mathscr{D}$. We claim that, assuming n-disjointness of $\mathscr{D}$, there is at most n nonzero pairwise disjoint members in $\mathscr{D}_0$. Let $\{h_1, \dots, h_m\}$ be a disjoint collection of nonzero lattice homomorphisms in $\mathscr{D}_0$. By the above we can pick m nonzero pairwise disjoint elements $x_0, \dots, x_m \in X_+$ such that $h_\imath(x_\jmath) = \delta_{\imath,\jmath}$ for all $1 \leqslant \imath, \jmath \leqslant m$. By construction, for each $\imath \leqslant m$ we can choose $0 \leqslant f_\imath \in \mathscr{D}$ with $f_\imath = h_\imath + \cdots$, so that $f_\imath(x_\imath) = h_\imath(x_\imath) + \cdots \geqslant h_\imath(x_\imath) = 1$. It follows that $|f_\imath(x_\imath)| \wedge \dots \wedge |f_m(x_m)| \geqslant 1$ and so $m \leqslant n$ by assumption. Evidently, $\mathbb{R} \cdot h_1 + \dots + \mathbb{R} \cdot h_m$ is a maximal n-disjoint set in $X^\sim$ including $\mathscr{D}$. $\triangleright$

3.12.B.6. *Given a nonempty set $\mathscr{D}$ in $L^\sim(X, \mathscr{R}{\downarrow})$, put $\mathscr{D}{\uparrow} := \{T{\uparrow} : T \in L^\sim(X, \mathscr{R}{\downarrow})\}$ and $\Delta := (\mathscr{D}{\uparrow}){\uparrow}$. If $\mathscr{D}$ is n-disjoint in $L^\sim(X, \mathscr{R}{\downarrow})$ for* *some natural* $n \in \mathbb{N}$, *then* $[\![\Delta$ *is* $n^\wedge$*-disjoint in* $(X^\wedge)^\sim]\!] = \mathbb{1}$. *Moreover,* $\mathscr{D}$ *is maximal if and only if* $[\![\Delta$ *is maximal* $]\!] = \mathbb{1}$.

$\lhd$ Assuming that $\mathscr{D}$ is n-disjoint in $L^\sim(X, \mathscr{R}{\downarrow})$, prove that Δ is $n^\wedge$-disjoint in $(X^\wedge)^\sim$ within $\mathbb{V}^{(\mathbb{B})}$. The sentence "$\Delta$ is $n^\wedge$-disjoint in $(X^\wedge)^\sim$" can be written as

$$\Phi \equiv \big(\forall\, \tau : \{0, \dots, n\}^\wedge \to \Delta\big)\big(\forall\, \varkappa : \{0, \dots, n\}^\wedge \to X^\wedge\big)$$
$$\Big((\forall\, \imath, \jmath \leqslant n^\wedge)(\imath \neq \jmath \to \varkappa(\imath) \perp \varkappa(\jmath)) \to \bigwedge_{\imath \leqslant n^\wedge} |\tau(\imath)\varkappa(\imath)| = 0\Big).$$

We have to prove that $[\![\Phi]\!] = \mathbb{1}$. Calculating the Boolean truth values for the universal quantifiers and taking 1.5.9 into account, we see that $[\![\Phi]\!] = \mathbb{1}$ if and only if $[\![|\mathscr{T}(0)k(0)| \wedge \dots \wedge |\mathscr{T}(n)k(n)| = 0]\!] = \mathbb{1}$ for all mappings $\mathscr{T} = \tau{\downarrow} : \{0, \dots, n\} \to \Delta{\downarrow}$ and $k = \varkappa{\downarrow} : \{0, \dots, n\} \to X^\wedge{\downarrow}$ with $[\![\varkappa(\imath^\wedge) \perp \varkappa(\jmath^\wedge)]\!] = \mathbb{1}$ for all $\imath \neq \jmath$. Since $\Delta{\downarrow} = \operatorname{mix}(\mathscr{D}{\uparrow})$ and $X^\wedge = \operatorname{mix}(\{x^\wedge{\downarrow} :\, x \in X\})$, there exists a partition of unity $(b_\xi)_{\xi \in \Xi}$ in $\mathbb{B}$ and for each $\imath = 0, \dots, n$ there are families $(T_{\xi,\imath})_{\xi \in \Xi}$ in $L^\sim(X, \mathscr{R}{\downarrow})$ and $(x_{\xi,\imath})_{\xi \in \Xi}$ in X such that $\mathscr{T}(\imath) = \operatorname{mix}_{\xi \in \Xi}(b_\xi T_{\xi,\imath}{\uparrow})$ and $k(\imath) = \operatorname{mix}_{\xi \in \Xi}(b_\xi x^\wedge_{\xi,\imath})_{\xi \in \Xi}$. Note that $b_\xi \leqslant [\![\varkappa(\imath^\wedge) = k(\imath) = x^\wedge_{\xi,\imath}]\!]$, so that $x_{\xi,\imath} \perp x_{\xi,\jmath}$ whenever $b_\xi \neq 0$ and $\imath \neq \jmath$. Thus, from the n-disjointness of $\mathscr{D}$ we deduce that

$$\begin{aligned} b_\xi &\leqslant [\![\, |T_{\xi,0} x_{\xi,0}| \wedge \dots \wedge |T_{\xi,n} x_{\xi,n}| = 0]\!] \\ \wedge &\bigwedge_{\imath \leqslant n} [\![\mathscr{T}(\imath) x^\wedge_{\xi,\imath} = T_{\xi,\imath}(x_{\xi,\imath})]\!] \wedge [\![k(\imath) = x^\wedge_{\xi,\imath}]\!] \\ &\leqslant [\![|\mathscr{T}(0)k(0)| \wedge \dots \wedge |\mathscr{T}(n)k(n)| = 0]\!]. \end{aligned}$$

This yields the required relation, since $\bigvee_{\xi \in \Xi} b_\xi = \mathbb{1}$. $\rhd$

3.12.B.7. *Given a nonempty* $\mathscr{D} \subset L(X, \mathscr{R}{\downarrow})$, *put* $R_{\mathscr{D}} := \bigvee\{R_T :\, T \in \mathscr{D}\}$. *Then*

$$R_{\mathscr{D}} = \chi([\![\Delta \neq \{0\}]\!]).$$

$\lhd$ This is immediate from 3.8.4 and Definition of Δ in 3.12.B.6. $\rhd$

We have gathered now all of the ingredients for proving the main result of this section.

3.12.B.8. Theorem. *Let* X *and* Y *be vector lattices with* Y *having the projection property. A nonempty set* $\mathscr{D}$ *in* $L^\sim(X, Y)$ *is* n*-disjoint if and only if there exist pairwise disjoint lattice homomorphisms* $T_1, \dots, T_n$ *from* X *to* $Y^{\mathfrak{u}}$ *such that* $\mathscr{D}$ *is contained in* $\operatorname{Orth}(T_1(X), Y) \circ$

$T_1 + \dots + \mathrm{Orth}(T_n(X), Y) \circ T_n$. *Moreover, $\mathscr{D}$ is maximal if and only if, additionally, there is a partition of unity $\pi_0, \dots, \pi_n$ in $\mathbb{P}(Y^{\mathrm{u}})$ such that*

$$\pi_0 \circ \mathscr{D} = \mathrm{Hom}(X, \pi_0 Y) = \{0\},$$
$$\mathscr{D} = \mathrm{Orth}(T_1(X), Y) \circ T_1 + \dots + \mathrm{Orth}(T_n(X), Y) \circ T_n,$$
$$\pi_m + \dots + \pi_n = R_{T_m} \quad (m := 1, \dots, n).$$

The collection $T_1, \dots, T_n$ in this representation is unique up to $\mathbb{P}(Y)$-permutation.

$\lhd$ The claim reduces to the case of Y universally complete, since by the Gordon Theorem we can assume that $Y = \mathscr{R}{\downarrow}$ without loss of generality.

Let $\mathscr{D}$ be an n-disjoint set in $L^{\sim}(X, \mathscr{R}{\downarrow})$ and Δ is defined as in 3.12.B.6. Working within $\mathbb{V}^{(\mathbb{B})}$ and using the transfer principle we conclude that Δ is $n^{\wedge}$-disjoint in $(X^{\wedge})^{\sim}$ and, by 3.12.B.5, $\Delta \subset \mathscr{R} \cdot \tau(1^{\wedge}) + \dots + \mathscr{R} \cdot \tau(n^{\wedge})$ for some $\tau : \{1, \dots, n\}^{\wedge} \to \mathrm{Hom}(X^{\wedge}, \mathscr{R})$. Just as in 3.12.B.6 put $\mathscr{T} := \tau{\downarrow}$ and note that $\mathscr{T}$ sends $\{1, \dots, n\}$ to $\mathrm{Hom}(X^{\wedge}, \mathscr{R}){\downarrow}$. If $T \in \mathscr{D}$ then $[\![T{\uparrow} \in \Delta]\!] = \mathbb{1}$, so that there exists $\alpha \in \mathbb{V}^{(\mathbb{B})}$ with

$$[\![\alpha : \{1, \dots, n\}^{\wedge} \to \mathscr{R}]\!] = \Big[\!\!\Big[T{\uparrow} = \sum\nolimits_{\imath \leqslant n^{\wedge}} \alpha(\imath)\mathscr{T}(\imath)\Big]\!\!\Big] = \mathbb{1}.$$

Put $\alpha_\imath := \alpha{\downarrow}(\imath)$ and $T_\imath := \mathscr{T}(\imath){\downarrow}$ for all $\imath := 1, \dots, n$. Then $\alpha_1, \dots, \alpha_n \in \mathscr{R}{\downarrow}$, $T_1, \dots, T_n \in \mathrm{Hom}(X, \mathscr{R}{\downarrow})$, so that the chain of internal identities

$$Tx = T{\uparrow}x^{\wedge} = \sum\nolimits_{\imath \leqslant n^{\wedge}} \alpha(\imath)\mathscr{T}(\imath)x^{\wedge} = \sum\nolimits_{\imath \leqslant n} \alpha_\imath \mathscr{T}(\imath){\downarrow}x = \sum\nolimits_{\imath \leqslant n} \alpha_i T_i x$$

with arbitrary $x \in X$ yields the required representation $T = \sum_{\imath=1}^{n} \alpha_i T_\imath$. Actually we have proved more: It is clear from the above argument that the double descent $(\Lambda{\downarrow}){\downarrow} := \{\tau{\downarrow} : \tau \in \Lambda{\downarrow}\}$ of $\Lambda := \mathscr{R} \cdot \tau(1^{\wedge}) + \dots + \mathscr{R} \cdot \tau(n^{\wedge})$ consists of all operators representable as $\sum_{\imath \leqslant n} \alpha_i T_\imath$ for some $\alpha_1, \dots, \alpha_n \in \mathscr{R}{\downarrow}$.

Assume now that $\mathscr{D}$ is maximal. Then $[\![\Delta \text{ is maximal}]\!] = \mathbb{1}$ by 3.12.B.6. The maximality condition in 3.12.B.5 can be symbolized as follows:

$$\Psi \equiv \big(\Delta = \mathrm{Hom}(X^{\wedge}, \mathscr{R}) = \{0\}\big)$$
$$\vee\big(\exists\, m \in \{1, \dots, n\}^{\wedge}\big)\Big((\forall\, \imath \leqslant m)(\tau(\imath) \neq 0)$$
$$\wedge(\Delta = \mathbb{R} \cdot \tau(1) + \dots + \mathbb{R} \cdot \tau(m))\Big).$$

Put $b_0 := [\![\Delta = \operatorname{Hom}(X^\wedge, \mathscr{R}) = \{0\}]\!]$. By transfer $[\![\Psi]\!] = \mathbb{1}$, and the calculation of Boolean valued truth values yields

$$b_0^* = [\![\Delta \neq 0]\!] = \bigvee_{m=1}^{n} [\![\Delta = \mathbb{R} \cdot \mathscr{T}(1) + \dots + \mathbb{R} \cdot \mathscr{T}(m)]\!] \wedge \bigwedge_{\imath=1}^{m} [\![\mathscr{T}(\imath) \neq 0]\!].$$

It follows that there exists a finite partition of unity $\{b_0, b_1, \dots, b_n\}$ in $\mathbb{B}$ such that $b_m \leqslant [\![\Delta = \mathbb{R} \cdot \mathscr{T}(1) + \dots + \mathbb{R} \cdot \mathscr{T}(m)]\!]$ and $b_m \leqslant [\![\mathscr{T}(\imath) \neq 0]\!]$ $(\imath \leqslant m)$ for all $m := 1, \dots, n$. Put $\pi_m := \chi(b_m)$ and observe that $\{\pi_0, \pi_1, \dots, \pi_n\}$ is a partition of unity in $\mathbb{P}(\mathscr{R}{\downarrow})$. Note also that $(\Lambda{\downarrow}){\downarrow} = \operatorname{mix}(\mathscr{D})$, so that $(\Lambda{\downarrow}){\downarrow} = \mathscr{D}$, since $\mathscr{D}$ is maximal. Combing the above and using 3.12.B.6, we see that

$$R_{\mathscr{D}} = \pi_0^{\perp} = \pi_1 + \dots + \pi_n, \quad \pi_m \leqslant R_{T_1} \circ \ldots \circ R_{T_m},$$
$$\pi_m \circ \mathscr{D} = \pi_m \circ (\mathscr{R}{\downarrow} \cdot T_1 + \dots + \mathscr{R}{\downarrow} \cdot T_m) \quad (m := 1, \dots, n).$$

The first identity gives $\pi_0 \circ \mathscr{D} = \operatorname{Hom}(X, \pi_0 Y) = \{0\}$. The second yields $\pi_m + \dots + \pi_n \leqslant R_{T_m}$ $(m := 1, \dots, n)$. Replacing T_m by $(\pi_m + \dots + \pi_n) T_m$, if need be, and summing the third identities over m brings about the required maximality conditions. $\triangleright$

3.12.C. Atomic Decomposition of Vector Measures.

3.12.C.1. Let $\mathscr{A}$ be a Boolean algebra and let Y be a vector lattice. By a vector measure we mean an arbitrary mapping $\mu : \mathscr{A} \to Y$ which is *finitely additive,* i.e., $\mu(a_1 \vee a_2) = \mu(a_1) + \mu(a_2)$ for all disjoint $a_1, a_2 \in \mathscr{A}$. A measure μ is *bounded* if $\mu(\mathscr{A})$ is an order bounded subset of Y. Denote by $\operatorname{ba}(\mathscr{A}, Y)$ the space of all bounded Y-valued measures and put $\operatorname{ba}(\mathscr{A}) := \operatorname{ba}(\mathscr{A}, \mathbb{R})$. A measure $\mu \in \operatorname{ba}(\mathscr{A}, Y)$ is *positive* if $\mu(a) \geqslant 0$ for all $a \in \mathscr{A}$. It is well known that $\operatorname{ba}(\mathscr{A}, Y)$ is a Dedekind complete vector lattice whose positive cone coincides with the set of positive measures. Moreover, $|\mu|(a) = \sup\{\mu(b) : b \in \mathscr{A},\ b \leqslant a\}$ for all $a \in \mathscr{A}$.

A measure $\mu \in \operatorname{ba}(\mathscr{A}, Y)$ is said to be *disjointness preserving* if $a_1 \wedge a_2 = \mathbb{0}$ implies $|\mu(a_1)| \wedge |\mu(a_2)| = 0$ for all $a_1, a_2 \in \mathscr{A}$. We say that μ is *diffuse* if μ is disjoint from all disjointness preserving measures and *atomic* if μ lies in the band generated by disjointness preserving measures.

3.12.C.2. Theorem. *Let $\mathscr{A}$ be a Boolean algebra and let Y be a universally complete vector lattice represented as $Y = \mathscr{R}{\downarrow}$. Given $\mu \in \operatorname{ba}(\mathscr{A}, Y)$, the modified ascent $m := \mu{\uparrow}$ is an order bounded finitely additive real measure on $\mathscr{A}^\wedge$ within $\mathbb{V}^{(\mathbb{B})}$; i.e., $[\![m \in \operatorname{ba}(\mathscr{A}^\wedge, \mathscr{R})]\!] = \mathbb{1}$.*

The mapping $\mu \mapsto \mu{\uparrow}$ is a lattice isomorphism between the Dedekind complete vector lattices $\mathrm{ba}(\mathscr{A}, Y)$ *and* $\mathrm{ba}(\mathscr{A}^{\wedge}, \mathscr{R}){\downarrow}$.

$\lhd$ The proof can be extracted from Theorem 3.3.3. $\rhd$

Henceforth m will denote the bounded measure from $\mathscr{A}^{\wedge}$ into $\mathscr{R}$ within $\mathbb{V}^{(\mathbb{B})}$ corresponding to $\mu \in \mathrm{ba}(\mathscr{A}, Y)$ by the above theorem. Observe some immediate consequences:

(1) μ is disjointness preserving if and only if m is disjointness preserving within $\mathbb{V}^{(\mathbb{B})}$;

(2) μ is atomic if and only if m is atomic within $\mathbb{V}^{(\mathbb{B})}$;

(3) μ is diffuse if and only if m is diffuse within $\mathbb{V}^{(\mathbb{B})}$.

3.12.C.3. Hammer–Sobczyk Decomposition Theorem. *Let μ be a finitely additive real measure on a Boolean algebra $\mathscr{A}$. Then there exist a sequence $(\nu_n)_{n\in\mathbb{N}}$ of pairwise disjoint $\{0,1\}$-valued measures on $\mathscr{A}$, a sequence $(r_n)_{n\in\mathbb{N}}$ of reals, and a diffuse measure μ_0 on $\mathscr{A}$, such that $\sum_{n=1}^{\infty} |r_n| < \infty$ and, $\mu = \mu_0 + \sum_{n=1}^{\infty} r_n \nu_n$. Furthermore, this decomposition is unique.*

$\lhd$ See Rao K. P. S. B. and Rao M. B. [342, Theorem 5.2.7]. $\rhd$

3.12.C.4. Take a measure $\mu \in \mathrm{ba}(\mathscr{A}, Y)$ and a nonzero element $\pi \in \mathbb{P}(Y)$. The symbol $[e]$ stands for the projection onto the band $\{e\}^{\perp\perp}$ generated by the element $e \in Y$. An element $a \in \mathscr{A}$ is called a π-*atom* of the measure μ if $\pi \leqslant [|\mu|(a)]$ and for all $a_0 \in \mathscr{A}$, $a_0 \leqslant a$, the elements $\pi\mu(a_0)$ and $\pi\mu(a \setminus a_0)$ are disjoint.

In case $Y = \mathbb{R}$ we speak of *atoms* instead of π-atoms. More precisely, an atom of a measure $\mu \in \mathrm{ba}(\mathscr{A})$ is an element $a_0 \in \mathscr{A}$ such that $\mu(a_0) \neq 0$ and for every $a \in \mathscr{A}$, $a \leqslant a_0$, either $\mu(a) = 0$, or $\mu(a_0 \setminus a) = 0$.

3.12.C.5. *Fix $b \in \mathbb{B}$ and put $\pi := \chi(b)$. An element $a \in \mathscr{A}$ is a π-atom of the measure μ if and only if $b \leqslant [\![a^{\wedge}$ is an atom of $m]\!] = \mathbb{1}$.*

$\lhd$ The sentence "$a^{\wedge}$ is an atom of m" can be formalized as

$$\Phi(m, a^{\wedge}, \mathscr{A}^{\wedge}) \equiv |m|(a^{\wedge}) \neq 0 \wedge (\forall a_0 \in \mathscr{A}^{\wedge})(a_0 \leqslant a^{\wedge} \\ \to (m(a_0) = 0 \vee m(a^{\wedge} \setminus a_0) = 0).$$

Thus, the estimate $b \leqslant [\![\Phi(m, a^{\wedge}, \mathscr{A})]\!]$ amounts to the system of inequalities $b \leqslant [\![|m|(a) \neq 0]\!]$ and $b \leqslant \mathbb{1} \Rightarrow [\![m(a_0^{\wedge}) = 0 \vee m(a^{\wedge} \setminus a_0^{\wedge}) = 0]\!]$ for all $a_0 \in \mathscr{A}$, $a_0 \leqslant a$ or, equivalently,

$$b \leqslant [\![|\mu|(a) \neq 0]\!], \quad b \leqslant [\![\mu(a_0) = 0]\!] \vee [\![\mu(a \setminus a_0) = 0]\!] \quad (a_0 \leqslant a).$$

The first inequality, means that $b \leqslant [|\mu|(a)]$, and the remaining one is satisfied if and only if $b = b_1 \vee b_2$ for some $b_1, b_2 \in \mathbb{B}$ with $\chi(b_1)\mu(a_0) = 0$, $\chi(b_2)\mu(a \setminus a_0) = 0$, and $b_1 \wedge b_2 = 0$. To ensure this, we need only to put $b_1 := [\![\mu(a_0) = 0]\!] \wedge b$, and $b_2 := [\![\mu(a \setminus a_0) = 0]\!] \wedge b$. The identities $\chi(b_1)\mu(a_0) = 0$ and $\chi(b_2)\mu(a \setminus a_0) = 0$ are equivalent to the inequalities $\chi(b_2) \geqslant [\mu(a_0)]$ and $\chi(b_1) \geqslant [\mu(a \setminus a_0)]$, which in turn mean that $\pi\mu(a_0)$ and $\pi\mu(a \setminus a_0)$ are disjoint. $\triangleright$

3.12.C.6. *Let Y be a Dedekind complete vector lattice. Boolean homomorphisms $\mu_1, \mu_2 : \mathscr{A} \to \mathbb{P}(Y)$ are disjoint in the vector lattice* $\mathrm{ba}(\mathscr{A}, \mathrm{Orth}(Y))$ *if and only if there exist a partition of unity $(\pi_\xi)_{\xi \in \Xi}$ in $\mathbb{P}(Y)$ and a family $(a_\xi)_{\xi \in \Xi}$ in $\mathscr{A}$, such that $\pi_\xi \mu_1(a_\xi) = 0$ and $\pi_\xi \mu_2(a_\xi^*) = 0$ for all $\xi \in \Xi$.*

$\triangleleft$ Assume that $Y^{\mathfrak{u}} = \mathscr{R}{\downarrow}$ and put $m_\imath := \mu_\imath{\uparrow}$ $(\imath = 1, 2)$. Since $\mathbb{B}$ is the descent of the two-element Boolean algebra $\{0,1\}^{\mathbb{B}} \in \mathbb{V}^{(\mathbb{B})}$ by 1.8.1, $\mathbb{V}^{(\mathbb{B})} \models$ "m_1 and m_2 are $\{0,1\}$-valued measures." Clearly, μ_1 and μ_2 are disjoint in $\mathrm{ba}(\mathscr{A}, \mathrm{Orth}(Y))$ if and only if $\mathbb{V}^{(\mathbb{B})} \models$"$m_1$ and m_2 are disjoint elements of the vector lattice $\mathrm{ba}(\mathscr{A}^\wedge)$" by Theorem 3.12.C.2 (cp. 3.3.5 (5)). At the same time, the disjointness of m_1 and m_2 is equivalent to

$$
\begin{aligned}
\mathbb{1} &= [\![(\exists\, a \in \mathscr{A})(m_1(a) = 0 \,\wedge\, m_2(a^*) = 0)]\!] \\
&= \bigvee_{a \in \mathscr{A}} [\![m_1(a^\wedge) = 0]\!] \wedge [\![m_2((a^*)^\wedge) = 0]\!] \\
&= \bigvee_{a \in \mathscr{A}} [\![\mu_1(a) = 0]\!] \wedge [\![\mu_2(a^*) = 0]\!].
\end{aligned}
$$

This amounts to saying that there exist a partition of unity $(b_\xi)_{\xi \in \Xi}$ in $\mathbb{B}$ and a family $(a_\xi)_{\xi \in \Xi}$ in $\mathscr{A}$, such that $b_\xi \leqslant [\![\mu_1(a_\xi) = 0]\!]$ and $b_\xi \leqslant [\![\mu_2(a_\xi^*) = 0]\!]$ for all $\xi \in \Xi$. This is equivalent to the desired condition with $\pi_\xi := \chi(b_\xi)$ by 2.2.4 ($\mathbb{G}$). $\triangleright$

3.12.C.7. *For every measure $\mu \in \mathrm{ba}(\mathscr{A}, Y)$ the following are equivalent:*

(1) *μ is disjointness preserving.*

(2) *There exists a Boolean homomorphism $h : \mathscr{A} \to \mathbb{B}$ such that $\mu(a) = h(a)\mu(\mathbb{1})$ for all $a \in \mathscr{A}$.*

(3) *If $b \leqslant [|\mu|(a)]$ for some $a \in \mathscr{A}$ and $b \in \mathbb{B}$, then a is a $\chi(b)$-atom of μ.*

$\lhd$ Recall 3.12.C.5 and use the arguments similar to 3.12.C.6. $\rhd$

We are now in a position to state the $\mathbb{B}$-atomic decomposition theorem and the fact that the $\mathbb{B}$-atomic component of a vector measure is the sum of a disjoint sequence of "spectral measures."

3.12.C.8. Theorem. *Assume that Y is a Dedekind complete vector lattice. For every measure $\mu \in \mathrm{ba}(\mathscr{A}, Y)$ there exist a diffuse measure $\mu_0 \in \mathrm{ba}(\mathscr{A}, Y)$, a sequence $(\nu_n)_{n\in\mathbb{N}}$ of pairwise disjoint Boolean homomorphisms from $\mathscr{A}$ into $\mathbb{P}(Y)$, and a sequence $(y_n)_{n\in\mathbb{N}}$ in Y, such that $|y_{n+1}| \leqslant |y_n|$ $(n \in \mathbb{N})$, the series $\sum_{k=1}^{\infty} |y_n|$ is o-convergent, and*

$$\mu(a) = \mu_0(a) + \sum_{n=1}^{\infty} \nu_n(a) y_n \quad (a \in \mathscr{A}).$$

This representation is unique in the following sense: If $\mu_0 \in \mathrm{ba}(\mathscr{A}, Y)$, $(\bar{\nu}_n)_{n\in\mathbb{N}}$ and $(\bar{y}_n)_{n\in\mathbb{N}}$ obey the above conditions, then $\bar{\mu}_0 = \mu_0$ and there exists an $\mathbb{N} \times \mathbb{N}$ matrix $(\pi_{k,n})$ whose rows and columns are partitions on unity in $\mathbb{P}(Y)$ such that for all $a \in \mathscr{A}$ and $n \in \mathbb{N}$ we have

$$\bar{\nu}_n(a) = \bigvee_{k=1}^{\infty} \pi_{k,n} \nu_k(a), \quad \bar{y}_n = \sum_{k=1}^{\infty} \pi_{k,n} y_k.$$

$\lhd$ Theorem 3.12.C.2 enables us to "scalarize" the problem. By transfer we can apply the Hammer–Sobczyk Decomposition Theorem to the measure $m := \mu\!\uparrow \in \mathrm{ba}(\mathscr{A}^{\wedge}, \mathscr{R})$ within $\mathbb{V}^{(\mathbb{B})}$. Using the maximum principle, we can pick $m_0 \in \mathbb{V}^{(\mathbb{B})}$, $(h_n) \subset \mathbb{V}^{(\mathbb{B})}$, and $(y_n) \subset \mathscr{R}$ such that the following hold within $\mathbb{V}^{(\mathbb{B})}$:

$$m_0 : \mathscr{A}^{\wedge} \to \mathscr{Y} \text{ is a diffuse measure;}$$

$$h_n : \mathscr{A}^{\wedge} \to \{0, 1\} \text{ is a sequence of pairwise disjoint measures;}$$

$$|y_{n+1}| \leqslant |y_n| \ \ (n \in \mathbb{N}) \text{ and the series } \sum_{n=1}^{\infty} |y_n| \text{ converges;}$$

$$m(a) = m_0(a) + \sum_{n=1}^{\infty} h_n(a) y_n \quad (a \in \mathscr{A}^{\wedge}).$$

Let μ_0 and ν_n be the modified descents of the measures m_0 and h_n, respectively. Clearly, $\mu_0 \in \mathrm{ba}(\mathscr{A}, Y)$ is diffuse by 3.12.C.2(3). From 3.12.C.2 and 3.12.C.7 it follows that (ν_n) is a sequence of pairwise disjoint

Boolean homomorphisms. The o-convergence of the series $\sum_{n=1}^{\infty} |y_n|$ and the required representation of μ are deduced using 2.4.7. The proof of uniqueness is based on the argument similar to that of 3.10.3. $\triangleright$

Let us conclude the section with the characterization of diffuse measures which is similar to that for diffuse operators (cp. 3.11.4).

3.12.C.9. Theorem. *Let Y be a Dedekind complete vector lattice. For every measure $\mu \in \operatorname{ba}(\mathscr{A}, Y)$ the following are equivalent:*

(1) *μ is diffuse.*

(2) *μ has no summands of the form $a \mapsto h(a)y$ $(a \in \mathscr{A})$ with $0 \neq y \in Y$ and $h : \mathscr{A} \to \mathbb{B}$ a Boolean homomorphism.*

(3) *For all $0 \leqslant e \in Y$ and $\pi \in \mathbb{P}(Y)$ with $\pi e \neq 0$ there exist a nonzero projection $\pi_0 \leqslant \pi$ and a finite disjoint family measures $\mu_1, \dots, \mu_n \in \operatorname{ba}(\mathscr{A}, Y)$ such that $\mu = \mu_1 + \dots + \mu_n$ and $\pi_0 |\mu_k|(\mathbb{1}) \leqslant e$ $(k := 1, \dots, n)$.*

3.13. Comments

3.13.1. (1) In 1936, Kantorovich [193] laid grounds for the theory of regular operators in vector lattices. Also, the Riesz–Kantorovich Theorem (3.1.2) appeared in this article for the first time. Riesz [346] formulated an analogous assertion for the space of continuous linear functionals over the lattice $C[a, b]$ in his famous talk at the International Mathematical Congress in Bologna in 1928 and thereby became enlisted in the cohort of the founders of the theory of ordered vector spaces.

(2) Abramovich in [1] developed a version of the calculus of 3.1.4 in which suprema and infima can be taken over partitions of the argument into disjoint parts. For the modulus of a regular operator, this fact was independently established by Luxemburg and Zaanen in [298].

(3) The problem of dominated extension of linear operators originates with the Hahn–Banach Theorem (see a nice survey by Buskes [77] in which the history, interconnections, and numerous generalizations of the Hahn–Banach Theorem are collected). Theorem 3.1.7 was discovered by Kantorovich [191] in 1935. In fact, the converse is also true: *A preordered vector space Y has dominated extension property* (*i.e., Theorem* 3.1.7 *holds true for all U, V, p, and S*) *if and only if Y has the least upper bound property*. This fact was first established by Bonnice and Silvermann [62] and To [392]; an elegant proof with decisive simplifications is due to Ioffe [177]; for more details also see Kusraev and Kutateladze [247].

(4) Theorem 3.1.7 can be considered as an exemplar application of the heuristic transfer principle for Dedekind complete vector lattices (see 2.13.1 (2, 3)). It claims that the *Kantorovich principle* is valid in relation to the classical Dominated Extension Theorem; i.e., we may replace the reals in the standard Hahn–Banach Theorem by an arbitrary Dedekind complete vector lattice and a linear functional by a linear operator with values in this lattice.

(5) Theorem 3.1.9 determines the least (or minimal) extension operator $\mathscr{E}$ from $L^{\sim}(X_0, Y)_+$ to $L^{\sim}(X, Y)_+$ which is additive and positively homogeneous, so that $\mathscr{E}$ can be extended by differences to a positive operator to the whole $L^{\sim}(X_0, Y)$ (for the properties of $\mathscr{E}$ see, for instance, Aliprantis and Burkinshaw [28] and Kusraev [228]). The extension theorem for positive order continuous operators (Theorem 2.1.10) is due to Veksler [396]. Theorem 3.1.13 was proved independently by Kutateladze [263, 265], Lipecki [284], and Luxemburg and Schep [296]. Various approaches to Hahn–Banach type theorems for lattice homomorphisms are discussed in Bernau [52].

(6) The theory of positive operators with a vast field of applications is thoroughly covered in many books. The following (incomplete) list of monographs that deal with vector lattices and positive (order bounded) operators provides an impression of the subject: Abramovich and Aliprantis [5]; Abramovich, Arenson, and Kitover [6]; Abramovich and Kitover [8]; Akilov and Kutateladze [22]; Aliprantis and Burkinshow [28]; Fremlin [124, 126]; Jameson [178]; Kantorovich, Vulikh, and Pinsker [196]; Krasnosel′skiĭ [206]; Krasnosel′skiĭ, Zabreĭko, Pustylnik, and Sobolevskiĭ [208]; Krasnosel′skiĭ, Lifshits, and Sobolev [207]; Kriger [209]; Kusraev [222, 228]; Kutateladze (ed.) [272]; Lacey [275]; Lindenstrauss and Tzafriri [280, 281]; Luxemburg and Zaanen [297]; Meyer-Nieberg [311]; Nagel (ed.) [316]; Nakano [319, 318]; Peressini [335]; Schaefer [355, 356]; Schwarz [361]; Vulikh [403]; and Zaanen [427, 428].

3.13.2. (1) The study of the order properties of bilinear operators in vector lattices was started more than a half-century ago. The first publication on the topic by Nakano was in 1953. But this article had not attracted specialists and the new achievements had appeared only in the 1970s in the articles by Fremlin [121, 123], Wittstock [416, 417], Cristescu [97], and Kusraev [215]. In two decades after that the bilinear operators were not topical within the theory of operators in vector lattices. But it stands to reason to mention that several particular cases (like bilinear functionals, multiplication on a lattice ordered algebra, and various tensor products) were studied from time to time by various authors; see Schaefer [357, 358]. A more detailed history and the state of the art in the area are reflected in the surveys by Boulabiar, Buskes, and Triki [68, 69]; Bu, Buskes, and Kusraev [72]; and Kusraev [234, 235].

(2) In spite of the nice universal property (Theorem 3.2.8), Fremlin's tensor product has the essential disadvantage: The isomorphism of 3.2.9 does not preserve order continuity. For an order continuous $T \in L_r(X \bar{\otimes} Y, Z)$ the bilinear operator $T\otimes \in BL_r(X, Y; Z)$ is also order continuous but the converse may fail. An example can be extracted from Fremlin [123] in which he introduced also a construction for the "projective" tensor product $X \overset{\vartriangle}{\otimes} Y$ of Banach lattices X and Y as the completion of Fremlin's tensor product $X \bar{\otimes} Y$ under "positive-projective" norm $\|\cdot\|_{|\pi|}$ (see [123, Theorem 1 E]. If $X = L^2([0,1])$, then $X \bar{\otimes} X$ is order dense in $X \overset{\vartriangle}{\otimes} X$ but the norm of $X \overset{\vartriangle}{\otimes} X$ is not order continuous (cp. [123, 4 B and 4 C]). Thus, there exists a (norm continuous) positive linear functional $l \in (X \overset{\vartriangle}{\otimes} X)'$ which is not order continuous. Clearly, the restriction l_0 of l to $X \bar{\otimes} X$ is not order continuous, too. At the same time the positive bilinear functional $b = l_0\otimes$ is separately order continuous, since X has an order continuous norm.

(3) The class of orthosymmetric bilinear operators on vector lattices was introduced in Buskes and van Rooij [81] and received much attention in the succeeding years. An inseparable companion of the orthosymmetric bilinear operators turns out to be the concept of square of vector lattice, developed by the same authors in another paper by Buskes and Rooij [82]; Definition 3.2.11 and Theorems 3.2.12 and 3.2.13 are taken from this paper.

(4) For $\alpha, s, t \in \mathbb{R}$, $\alpha > 0$, we denote $t^\alpha := |t|^\alpha \operatorname{sgn}(t)$ and $\sigma_\alpha(s,t) := (s^{1/\alpha} + t^{1/\alpha})^\alpha$. In a uniformly complete vector lattice X, we introduce new vector operations $\oplus$ and $*$, while the original order $\leqslant$ remain unchanged: $x \oplus y := \sigma(x, y)$, $\lambda * x := \lambda^\alpha x$ $(x, y \in X; \lambda \in \mathbb{R})$. Then $X^{(\alpha)} := (X, \oplus, *, \leqslant)$ is again a vector lattice. If X is uniformly complete then $X^{\odot} = X^{(1/2)}$ and $x \odot y := (xy)^{1/2}$ for all $x, y \in X$. If $(X, \|\cdot\|)$ is a Banach lattice then we can define also a homogeneous function $\|\cdot\|_\alpha : X^{(\alpha)} \to \mathbb{R}$ by $\|x\|_\alpha := \|x\|^{1/\alpha}$. The pair $(X^{(\alpha)}, \|\cdot\|_\alpha)$ is called an *α-convexification* of X if $\alpha > 1$ and *α-concavification* if $\alpha < 1$; cp. Lindenstrauss and Tzafriri [282, pp. 53, 54] and Szulga [374, Definition 4.4.1]. For the homogeneous functional calculus in uniformly complete vector lattices and Banach lattices see Lindenstrauss and Tzafriri [282, Theorem 1.d.1] and Buskes, de Pagter, and van Rooij [79].

(5) A Banach lattice E is called *p-concave* ($1 \leqslant p < \infty$) if there is a constant $C < \infty$ such that for every finite collection $\{x_1, \dots, x_n\} \subset X$ the inequality holds (Lindenstrauss and Tzafriri [282, Definition 1.d.3]):

$$\left(\sum_{k=1}^{n} \|x_k\|^p \right)^{\frac{1}{p}} \leqslant C \left\| \left(\sum_{k=1}^{n} |x_k|^p \right)^{\frac{1}{p}} \right\|.$$

If X is a p-convex Banach lattice for some $p \geqslant 1$ then $(X^{(\alpha)}, \|\cdot\|_\alpha)$ is also a Banach lattice provided that $\alpha p \geqslant 1$; see Szulga [374, Proposition 4.8]. In particular, $X^{\odot}$ equipped with the norm $\|x \odot x\|_{\odot} := \|x\|^2$ becomes a Banach lattice if $q > 2$ (also see Bu, Buskes, Popov, Tcacius, and Troitsky [73]).

(6) The theory of positive bilinear operators partially presented in this book can be developed for positive multilinear operators. In particular, Fremlin's tensor product of two vector lattices and Buskes–van Roij square of a vector lattice together with their universal properties for the classes of positive bilinear operators (see Theorems 3.2.6 and 3.2.8) and positive orthosymmetric bilinear operators (Theorems 3.2.12 and 3.2.13) were extended to multilinear case by Schep [359] and Boulabiar and Buskes [65], respectively.

3.13.3. In 1935 Kantorovich's in his first definitive article on vector lattices [191] wrote: "In this note, I define the new type of space that I call a semiordered linear space. The introduction of such a space allows us to study linear operations of one abstract class (those with values in such a space) as linear functionals." Theorem 3.3.3 with technical Corollaries 3.3.5 and 3.3.6 presents one of the mathematical realization of this heuristic principle. The two other mathematical forms of Kantorovich's heuristic principle will be presented in Chapters 4 and 5.

3.13.4. (1) The first appearance of disjointness preserving operators in the literature occurred in 1943 in the article [402] by Vulikh implicitly under the name *multiplicative linear operations*. A systematic study of disjointness preserving operators began from the articles [11, 12] by Abramovich, Veksler, and Koldunov. Various aspects of disjointness preserving operators have been studied by many authors over the years, and we indicate only a portion of these publications: on multiplicative representations on function spaces (Abramovich, Veksler, and Koldunov [11, 12], Araujo, Beckenstein, and Narici [33], Abramovich, Arenson, and Kitover [6], Henriksen and Smith [171]); on weight-shift factorization (Gutman [157]–[160], [162]); on spectral theory (Arendt [34], Arendt and Hart [36], Huijsmans and de Pagter [174], Meyer-Nieberg [311]); on the inverses of disjointness preserving operators (Abramovich and Kitover [8]); on the various properties of the band generated by disjointness preserving operators and the corresponding band projections (Huijsmans and de Pagter [174], Kolesnikov [200, 201], Tabuev [376]); on polar decomposition (Abramovich, Arenson, and Kitover [8], Boulabiar and Buskes [64], Grobler and Huijsmans [151]). Sometimes order bounded disjointness preserving operators are called *Lamperti operators* [34, 66] or *separating mappings* [33, 171]. For more historical remarks and references we refer to the survey [63] by Boulabiar.

(2) Theorem 3.4.2 is taken from Kusraev and Kutatetladze [251]. A bilinear version of this result which can proved by using similar arguments see below in 3.12.A.7. The first proof of the Theorem 3.4.3 was given in [310] by Meyer. This proof is based upon the Kuratowski–Zorn Lemma (i.e., the Axiom of Choice). Later, the two proofs of Theorem 3.4.3, free of the Kuratowski–Zorn Lemma, were obtained by Bernau in [51] and de Pagter in [329], respectively. Theorem 3.4.4 is immediate from Abramovich, Arenson, and Kitover [6, Theorem 3.3]. In Section 3.4 we show that Boolean valued approach provides a new insight into this circle of problems. Theorems 3.4.8 and 3.4.9 were obtained in Kusraev and Kutateladze [251].

(3) It was proved by Kusraev and Tabuev in [257] that a bilinear version of Meyer's Theorem is also true: *Let X, Y, and G be vector lattices and let $B : X \times Y \to G$ be an order bounded disjointness preserving bilinear operator. Then b possesses the positive part B^+, the negative part B^-, and the modulus $|B|$ which are lattice bimorphisms. Moreover, $B^+(x, y) = B(x, y)^+$ and $B^-(x, y) = B(x, y)^-$ for $0 \leqslant x \in X$, $0 \leqslant y \in Y$, and $|B|(|x|, |y|) = |B(x, y)|$ for all $x \in X$ and $y \in Y$. In particular, B is regular.*

(4) Combining Theorem 3.9.11 with Corollary 3.12.A.5 yields the following result due to Kusraev and Tabuev [258].

Theorem. *Let X, Y, and Z be vector lattices with Z Dedekind complete and $B : X \times Y \to Z$ an order bounded disjointness preserving bilinear operator. Then there exist a partition of unity $(\rho_\xi)_{\xi\in\Xi}$ in the Boolean algebra $\mathbb{P}(Z)$ and families of positive elements $(e_\xi)_{\xi\in\Xi}$ in X and $(f_\xi)_{\xi\in\Xi}$ in Y such that the representation holds*

$$B(x,y) = o\text{-}\sum_{\xi\in\Xi} \rho_\xi W \cdot \sigma(x/e_\xi) \cdot \tau(y/f_\xi) \quad (x \in X,\ y \in Y),$$

where σ and τ are shift operators into Z^{u} with $\mathscr{D}(\sigma)$ and $\mathscr{D}(\tau)$ being order dense ideals in X^{u} and Y^{u}, respectively, and $W : Z^{\mathrm{u}} \to Z^{\mathrm{u}}$ is the operator of multiplication by $o\text{-}\sum_{\xi\in\Xi} \rho_\xi B(e_\xi, f_\xi)$ (see 3.9.10).

(5) The noncommutative analogs of disjointness preserving operators have been studied as well. We present only one result of Wolff [415]. Let A and B be two C^*-algebras. A bounded linear operator $T : A \to B$ is called *disjointness preserving* if $T(x^*) = (Tx)^*$ for all $x \in A$ and $yz = 0$ implies $T(y)T(z) = 0$ for all hermitian $x, y \in A$. Let I_h and $M(I_h)$ stand for the principal ideal in the commutant $\{h\}'$ of h generated by h and the multiplier algebra of I_h, respectively. Assume now that A is unital and T is a disjointness preserving operator from A to B which sends the unity of A to $h \in B$. Then $T(A) \subset I_h$ and there exists a Jordan $*$-homomorphism S from A into $M(I_h)$ such that $S(1_A) = 1_{M(I_h)}$ and $Tx = hS(x)$ for all $x \in A$.

3.13.5. (1) Theorem 3.5.4 was established by Kutateladze in [273]. The Moreau–Rockafellar Formula is one of the key tools in subdifferential calculus; various aspects and applications can be found in Kusraev and Kutateladze [247]. Theorem 3.5.8 was stated and proved in this form in Kutateladze [261]. Obviously, the Riesz space in this theorem may be viewed over an arbitrary dense subfield of the reals $\mathbb{R}$.

(2) Descending Theorem 3.5.4 from an appropriate Boolean valued universe or, which is equivalent, using the characterization of the modules admitting convex analysis, we can arrive to an analogous description for a dominated module homomorphism with kernel a Riesz subspace in modules over an almost rational subring of the orthomorphism ring of the range (cp. Abramovich, Arenson, and Kitover [6, Theorem 3.3] and Kutateladze [266]).

(3) From Theorem 3.5.4 it is immediate that the Stone Theorem cannot be abstracted far beyond the limits of AM-spaces. Indeed, if each closed Riesz subspace of a Banach lattice is an intersection of two-point relations then there are sufficiently many Riesz homomorphisms to separate the points of the Banach lattice under consideration (cp. Schaefer [356, Chapter 3, Section 9]).

3.13.6. Theorem 3.6.2 was obtained by Kutateladze in [274]. The history of the property 3.6.1 is as follows: In 1955 Grothendieck [156] distinguished the subspaces that satisfy 3.6.1 in the space $C(Q,\mathbb{R})$ of continuous real functions on a compact space Q. He determined such a subspace as the set of functions f satisfying some family of relations of the form $f(q^1_\alpha) = \lambda_\alpha f(q^2_\alpha)$ $(q^1_\alpha, q^2_\alpha \in Q;\ \lambda_\alpha \in \mathbb{R},\ \alpha \in \mathrm{A})$. These subspaces were discovered by Grothendieck and gave the examples of the L_1-predual Banach spaces other than AM-spaces. In 1969 Lindenstrauss and Wulpert characterized these subspaces by using 2.6.1. They also introduced the concept of G-space (cp. [283]).

3.13.7. **(1)** The notion of n-disjoint operator between vector lattices (Definition 3.7.1) as well as the main results of Section 3.7 (Theorem 3.7.7 and Corollaries 3.7.9 and 3.7.10 without assuming that the summands are pairwise disjoint) are due to Bernau, Huijsmans, and de Pagter [53] (see also Bernau [52]). Radnaev [341, 340] noticed that, first, the disjointness preserving operators $T_1,\dots,T_n$ in the decomposition $T = T_1 + \dots + T_n$ can be chosen pairwise disjoint and, second, this decomposition is unique up to "mixture permutation" (cp. 3.7.8). Similar results for dominated operators on lattice normed spaces are collected in Kusraev [228, 5.2.1 and Theorem 5.2.7].

(2) Radnaev [341, 340] found a purely algebraic approach to the proof of Theorem 3.7.7 (see [228, 2.1.10, 5.2.6, 5.2.7]). In the same article he gave various characterizations of n-disjoint operators (see [228, 5.2.1 (2) and 5.2.5]) employing Kutateladze's *canonical sublinear operator method* [262, 264] (also see Rubinov [352]).

3.13.8. **(1)** In [331] de Pagter and Schep raised the problem of finding conditions for the sum of two order bounded disjointness preserving operators to be disjointness preserving. In Section 3.8 the problem is examined for arbitrary finite sums in the more general setting of n-disjoint operators. The Boolean valued approach to the problem and the main results are taken from Kusraev and Kutateladze [251]. The equivalence (1) $\Longleftrightarrow$ (4) in Theorem 3.8.6 is essentially $(a) \Longleftrightarrow (c)$ in de Pagter and Schep [331, Proposition 2.13 (5)].

(2) Let X, Y, and Z be vector lattices with Z Dedekind complete. Say that the finite collections $\{x_0, x_1, \dots, x_n\} \subset X$ and $\{y_0, y_1, \dots, y_n\} \subset Y$ are *bidisjoint,* if for every two naturals $0 \leqslant i, j \leqslant n$,

$i \neq j$, either $x_i \perp x_j$, or $y_i \perp y_j$. A bilinear operator B from $X \times Y$ to Z is called *n-disjoint*, if $|B(x_0, y_0)| \wedge |B(x_1, y_1)| \wedge \cdots \wedge |B(x_n, y_n)| = 0$, for all bidisjoint collections $\{x_0, x_1, \ldots, x_n\}$ in X and $\{y_0, y_1, \ldots, y_n\}$ in Y. For a regular bilinear operator $B : X \times Y \to Z$ there exists a linear regular operator $T : X \bar{\otimes} Y \to Z$ such that $B = T\otimes$, where $X \bar{\otimes} Y$ is the Fremlin tensor product of X and Y (see Theorem 3.2.6). It was proved in Kusraev and Tabuev [257] that B is n-disjoint if and only if T is n-disjoint. These facts enable us to transfer some results on regular n-disjoint linear operators to regular n-disjoint bilinear operators. In particular, some versions of Theorems 3.8.6 and 3.8.7 hold for bilinear operators.

3.13.9. (1) In Section 3.9 we present the Boolean valued approach to Gutman's representation theory for disjointness preserving operators [160, 162]. The main idea of [160, 162] can be worded as follows: The *shadow* of an operator $T : X \to Y$ between vector lattices is the mapping $\mathrm{sh} := \mathrm{sh}_T : \mathbb{B}(X) \to \mathbb{B}(Y_T)$ defined as $\mathrm{sh}(B) = T(B)^{\perp\perp}$ $(B \in \mathbb{B}(X))$. If X and Y have the projection property then we can also define $\mathrm{sh} : \mathbb{P}(X) \to \mathbb{P}(Y_T)$ by $\mathrm{sh}(\pi) = [T\pi(X)]$; i.e., $\mathrm{sh}(\pi)$ is the band projection onto $(T\pi(X))^{\perp\perp}$. In the latter case a linear operator $T : X \to Y$ is disjointness preserving if and only if its shadow sh_T is a Boolean homomorphism. The shadow generates the so-called shift operator which is a lattice homomorphism on a certain order dense ideal of the universal completion of the departure vector lattice. Both are closely related with the initial disjointness preserving operator and concentrate, in a sense, its multiplicative properties. Using these simplest types of operators, we can construct weighted shift operators; i.e., the composites $W \circ S \circ w$ of two orthomorphisms w and W and a shift operator S. Moreover, an arbitrary disjointness preserving operator is representable as the strongly disjoint sum of weighted shift operators (see [228, Subsections 5.3.2, 5.3.6, and 5.3.10]).

(2) The shadow of an operator (without introducing the term) was first considered by Luxemburg in [291] for lattices homomorphisms and then by Kusraev in [224] for a disjointness preserving operators in lattice normed spaces. The shadow of an operator may fail to be a Boolean homomorphism unless X has the projection property or T is order continuous. It was proved by de Pagter and Schep in [331, Proposition 2.8] that the shadow sh_T of a lattice homomorphism T is a Boolean homomorphism if and only if T has the unique positive linear extension to the Dedekind completion of X.

3.13.10. (1) Theorem 3.10.10, the main result of Section 3.10, was proved by Tabuev in [376, Theorem 2.2] with standard tools. The pseudoembedding operators are closely connected with the so-called order narrow operators. A linear operator $T : X \to Y$ is *order narrow* if for every $x \in X_+$ there exists a net (x_α) in X such that $|x_\alpha| = x$ for all α and (Tx_α) is order convergent to zero in Y; see [308, Definition 3.1]. The main result by Maslyuchenko, Mykhaylyuk, and Popov in [308, Theorem 11.7 (ii)] states that if X and Y are Dedekind complete vector lattices with X atomless and Y an order ideal of some order continuous Banach lattice then an order bounded order continuous operator is order narrow if and only if it is pseudoembedding.

(2) The term *pseudoembedding operator* stems from a result by Rosenthal [349] which asserts that a nonzero bounded linear operator in L^1 is a pseudoembedding if and only if it is a near isometric embedding when restricted to a suitable $L^1(A)$-subspace. Systematic study of narrow operators was started by Plichko and Popov in [337]. For a detailed presentation of the theory of narrow operators see the recent book by Popov and Randrianantoanina [338] and the references therein.

3.13.11. (1) Theorems 3.11.4 and 3.11.5 are due to Tabuev [375, 376]. The characterization of diffuse functionals in 3.11.3 belongs to H. Gordon [141]. For positive operators in Banach function spaces over separable metric space the diffuse decomposition was first investigated by Weis [406, 407]. The general decomposition and projection results were obtained by Huijsmans and de Pagter in [174] and by Tabuev in [375]. The projection onto the band of diffuse and pseudoembedding operators was studied by Kolesnikov [201].

(2) The main tools in Huijsmans and de Pagter [174] and Tabuev [375] are sublinear operators p_T and q_T respectively defined as

$$p_T(x) := \inf\big\{Tx_1 \vee \dots \vee Tx_n : |x| \leqslant x_1 \vee \dots \vee x_n;\\ x_1, \dots, x_n \in E_+,\ n \in \mathbb{N}\big\},$$

$$q_T(x) := \inf\Big\{T_1|x| \vee \dots \vee T_n|x| : T = \sum_{k=1}^{n} T_k,\\ T_1, \dots, T_n \in L^\sim(E, F)_+,\ T_k \perp T_l \quad (k \neq l)\Big\},$$

where $x \in X$ and T is a positive operator from X to Y. Then T is diffuse if and only if $p_T(x) = 0$ for all $x \in X$ [174] if and only if $q_T(x) = 0$ for all $x \in X$ [375]. In Maslyuchenko, Mykhaylyuk, and Popov [308] the operator p_T is also called the *Enflo–Starbird function* in the wake of Enflo and Starbird [113].

3.13.12. (1) Theorem 3.12.A.3 and Corollary 3.12.A.5 were obtained in Kusraev and Tabuev [258]. Corollary 3.12.A.6, a special case of Theorem 4.12.A.3, is due to Hart [167]. Theorem 3.12.A.7 is a recent result by Kusraev and Kutateladze [251].

(2) Theorem 3.12.B.8, the main result of 3.12.B, was proved in Kusraev and Kutateladze [251]. This result gives a complete description of n-disjoint sets of operators. In the particular setting of disjointness preserving operators this problem was motivated by the research of Benamor and Boulabiar [44, 46, 45] and stated explicitly in Boulabiar [63, Problem 5.8]: *Given a lattice homomorphism T from X into Y, under what conditions is $\mathscr{D} := \mathrm{Orth}(Y) \circ T$ maximal?*

(3) The decomposition theorem in 3.12.C.3 is due to Sobczyk and Hammer [366]. Theorem 3.12.C.8 tells us that every finitely additive measures with values in a Dedekind complete vector lattice can be written as the sum of two measures, one of which is diffuse, the other is a countable sum of finitely additive “spectral measure.” This is a special case of a more general Sobczyk–Hammer type decomposition theorem for finitely additive measures with values in a Banach–Kantorovich space which was obtained by Kusraev and Malyugin in [252, 253].

CHAPTER 4

BAND PRESERVING OPERATORS

WP: *When are we so happy in a vector lattice that all band preserving linear operators turn out to be order bounded?*

This question raised by Wickstead in [408] is often referred to as the *Wickstead problem*. The answer depends on the vector lattice in which the operator in question acts. There are several results that guarantee automatic order boundedness for a band preserving operator between concrete classes of vector lattices. The goal of this chapter is to examine the Wickstead problem in universally complete vector lattices for various classes of band preserving operators: projections, algebraic operators, derivations, algebra homomorphisms, etc. Boolean valued representation of band preserving operators reduces this task to examining the classical Cauchy functional equation.

4.1. Orthomorphisms

In this section we introduce the class of band preserving operators and briefly overview some properties of orthomorphisms.

4.1.1. *Let X and Y be vector sublattices of a vector lattice Z. For a linear operator T from X into Y the following are equivalent:*

(1) *$x \perp z$ implies $z \perp Tx$ for all $x \in X$ and $z \in Z$.*

(2) *$Tx \in \{x\}^{\perp\perp}$ for all $x \in X$ (with the disjoint complements taken in Z).*

(3) *$T(K \cap X) \subset K$ for all bands $K \in \mathbb{B}(Z)$.*

$\lhd$ The implications (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) are immediate from the definitions. To ensure that (3) $\Longrightarrow$ (1), put $K := \{z\}^{\perp}$ and note that for an arbitrary $z \in Z$ the relation $x \perp z$ and $x \in K$ are equivalent, so that $Tx \in T(K \cap X) \subset K$ by (3), whence $z \perp Tx$. $\rhd$

4.1.2. A linear operator T from X to Y is called *band preserving*[1] provided that one (and hence all) of the conditions 4.1.1 (1–3) holds. If T is band preserving and $Tx \perp X$ for some $x \in X$, then $Tx \in X^{\perp} \subset \{x\}^{\perp}$ and, in view of 4.1.1 (2), $Tx \in \{x\}^{\perp\perp} \cap \{x\}^{\perp} = \{0\}$. Thus, $T(X) \subset Y \cap X^{\perp\perp}$ and we will only deal in the sequel with the case $Z = X^{\perp\perp}$.

A band preserving operator T need not be order bounded (cp. Theorems 4.4.9, 4.6.4, and 4.7.6 below). An order bounded band preserving operator $\pi : X \to Y$ is called an *orthomorphism* from X to Y. Let $\mathrm{Orth}(X, Y)$ signify the set of all orthomorphisms from X to Y. An orthomorphism $T \in \mathrm{Orth}(X, Y)$ is called an *extended orthomorphism* of Y whenever X is an order dense ideal of Y and a *weak orthomorphism* of Y provided that X is an order dense sublattice of Y. In the case $Y = X$ we speak of the *orthomorphisms* of X and put $\mathrm{Orth}(X) := \mathrm{Orth}(X, X)$. Let $\mathscr{Z}(X)$ stand for the set of regular operators $T \in L(X)$ satisfying $-cI_X \leqslant T \leqslant cI_X$ for some $c \in \mathbb{R}_+$. By 3.1.11 (6) $\mathscr{Z}(X) \subset \mathrm{Orth}(X)$. The space $\mathscr{Z}(X)$ is often called the *ideal center* of X, since $\mathscr{Z}(X)$ coincides with the order ideal in $\mathrm{Orth}(X)$ generated by the identity operator I_X.

4.1.3. *Let X be a vector lattice. Then $\mathrm{Orth}(X)$ is a semiprime commutative f-algebra with the composite as ring multiplication and with the identity operator as weak order unit.*

$\lhd$ Since an orthomorphism is disjointness preserving, it is also regular by the Meyer Theorem 3.4.3. Moreover, the collection of orthomorphisms $\mathrm{Orth}(X)$ is a vector lattice under the induced order from the space of regular operators.

The vector lattice $\mathrm{Orth}(X)$ has some natural multiplicative structure: given π and ρ in $\mathrm{Orth}(X)$, put $\pi\rho := \pi \circ \rho$. It follows easily from the definition that $\pi\rho \in \mathrm{Orth}(X)$. Obviously, $\mathrm{Orth}(X)$ is a lattice ordered algebra (cp. 2.3.1) and the identity operator is the ring unity. It remains to check that if $\pi \perp \rho$ then $\sigma\pi \perp \rho$ and $\pi\sigma \perp \rho$ for all $\pi, \rho, \sigma \in \mathrm{Orth}(X)_+$. Indeed, if $\pi \wedge \rho = 0$ then $\pi x \wedge \rho x = (\pi \wedge \rho)x = 0$ for all $x \in X_+$. Since σ is band preserving, the relation $\pi x \perp \rho x$ implies $\sigma(\pi x) \perp \rho x$ or $(\sigma\pi)x \wedge \varrho x = 0$. Hence $(\sigma\pi) \perp \varrho$. At the same time, putting $x_0 := x \vee \sigma x$ we deduce

$$0 \leqslant (\pi\sigma)x \wedge \rho x \leqslant \pi x_0 \wedge \rho x_0 = 0,$$

which implies that $(\pi\sigma) \perp \rho$. Commutativity and semiprimeness of $\mathrm{Orth}(X)$ can be seen from 4.3.8. $\rhd$

[1] In the Russian literature the term *nonextending* is also in use.

4.1.4. A vector lattice X is said to have a *cofinal family of band projections* (or a *cofinal family of projection bands*) if for every nonzero band $B \subset X$ there exists a nonzero projection band $B_0 \subset B$. Equivalently, X has a cofinal family of band projections if for each nonzero band B in X there exists a nonzero band projection π on X such that $\pi(X) \subset B$. The space of continuous functions $C(K)$ is a vector lattice with a cofinal family of band projections whenever K is a zero-dimensional compact space.

4.1.5. *Let Z be a vector lattice with a cofinal family of band projections and let X and Y be vector sublattices of Z. For a linear operator $T : X \to Y$ the following are equivalent:*

(1) *T is band preserving.*

(2) *$\pi x = 0$ implies $\pi T x = 0$ for all $x \in X$ and $\pi \in \mathbb{P}(Z)$.*

(3) *$\pi x = \pi y$ implies $\pi T x = \pi T y$ for all $x, y \in X$ and $\pi \in \mathbb{P}(Z)$.*

$\lhd$ If T is band preserving and $\pi x = 0$ for a band projection $\pi \in \mathbb{P}(Z)$, then $Tx \in \{x\}^{\perp\perp} \subset \ker(\pi)$ by 4.1.1 (2), so that $\pi T x = 0$. Conversely, assume that T is not band preserving, while (2) holds. Then, according to 4.1.1 (2) there is $x \in X$ such that $Tx \notin \{x\}^{\perp\perp}$ and so the band $K := \{Tx\}^{\perp\perp} \cap \{x\}^{\perp}$ is nonzero. By hypothesis there is a projection band $\pi \in \mathbb{P}(Z)$ with $\pi(Z) \subset K$, so that $\pi T x \neq 0$ and $\pi x = 0$; a contradiction. Thus (1) and (2) are equivalent. The equivalence of (2) and (3) follows trivially from the linearity of T. $\rhd$

4.1.6. *Let Z be a vector lattice with a cofinal family of band projections, X an order ideal of Z, and Y a vector sublattice of Z. For a linear operator $T : X \to Y$ the following are equivalent:*

(1) *T is band preserving.*

(2) *$\pi T x = T \pi x$ for all $x \in X$ and $\pi \in \mathbb{P}(Z)$.*

(3) *$\pi T \pi^{\perp} x = 0$ for all $x \in X$ and $\pi \in \mathbb{P}(Z)$ with $\pi^{\perp} := I_X - \pi$.*

$\lhd$ (1) $\Longrightarrow$ (2): If T is band preserving then $T(K \cap X) \subset K$ and $T(K^{\perp} \cap X) \subset K^{\perp}$ for every band $K \in \mathbb{B}(Z)$. Assume that K is a projection band. Put $\pi := [K]$ and $\pi' := [K^{\perp}]$. Then $\pi|_X \in \mathbb{P}(X)$, $\pi'|_X \in \mathbb{P}(X)$, and $\pi'|_X = I_X - \pi|_X$. Consequently, $Tx = T\pi x + T\pi' x$ with $T\pi x \in K$ and $T\pi' x \in K^{\perp}$, so that $\pi T x = \pi T \pi x = T \pi x$.

(2) $\Longrightarrow$ (3): Replace x by $\pi^{\perp} x$ in (2) to get (3).

(3) $\Longrightarrow$ (1): If (3) holds then $\pi T x = \pi T \pi x$ for all $x \in X$ and $\pi \in \mathbb{P}(Z)$. Therefore, $\pi x = 0$ implies trivially that $\pi T x = 0$ and we are done by 4.1.5 (2). $\triangleright$

4.1.7. *Let X and Y be vector sublattices of a laterally complete vector lattice Z with $Z = X^{\perp\perp}$. Each band preserving linear operator T from X into Y extends uniquely to a band preserving linear operator T^λ from $\lambda(X)$ to $\lambda(Y)$. Moreover, T is order bounded if and only if so is T^λ and in this case $|T^\lambda| = |T|^\lambda$.*

$\triangleleft$ An arbitrary $x \in \lambda(X)$ may be presented as $x = o\text{-}\sum_{\xi \in \Xi} \pi_\xi x_\xi$ where $(\pi_\xi)_{\xi \in \Xi}$ is a partition of unity in $\mathbb{P}(Z)$ and $(x_\xi)_{\xi \in \Xi}$ is a family in X; see 2.5.3. Define $T^\lambda : \lambda(X) \to \lambda(Y)$ by putting

$$T^\lambda x := o\text{-}\sum_{\xi \in \Xi} \pi_\xi T x_\xi \quad \Big(x = o\text{-}\sum_{\xi \in \Xi} \pi_\xi x_\xi \Big),$$

or equivalently, $\pi_\xi T^\lambda x := \pi_\xi T x_\xi$ for all $\xi \in \Xi$. The definition is sound. Indeed, given another representation $x = o\text{-}\sum_{\eta \in \mathrm{H}} \rho_\eta y_\eta$ with a partition of unity $(\rho_\eta)_{\eta \in \mathrm{H}}$ in $\mathbb{P}(Z)$ and a family $(y_\eta)_{\eta \in \mathrm{H}}$ in X, we have $\pi_\xi \rho_\eta x_\xi = \pi_\xi \rho_\eta x = \pi_\xi \rho_\eta y_\eta$ and so $\pi_\xi \rho_\eta T x_\xi = \pi_\xi \rho_\eta T y_\eta$ by 4.1.5 (3). Consequently,

$$\begin{aligned} o\text{-}\sum_{\xi \in \Xi} \pi_\xi T x_\xi &= o\text{-}\sum_{\xi \in \Xi} o\text{-}\sum_{\eta \in \mathrm{H}} \pi_\xi \rho_\eta T x_\xi \\ &= o\text{-}\sum_{\eta \in \mathrm{H}} o\text{-}\sum_{\xi \in \Xi} \pi_\xi \rho_\eta T y_\eta = o\text{-}\sum_{\eta \in \mathrm{H}} \rho_\eta T y_\eta. \end{aligned}$$

To show the linearity of T^λ, take $x, y \in \lambda(X)$ and observe that there are families (x_ξ) and (y_ξ) in X and a common partition of unity (π_ξ) in $\mathbb{P}(Z)$ such that $\pi_\xi x = \pi_\xi x_\xi$ and $\pi_\xi y = \pi_\xi y_\xi$ for all ξ. It follows that $\pi_\xi(\alpha x + \beta y) = \pi_\xi(\alpha x_\xi + \beta y_\xi)$ with $\alpha, \beta \in \mathbb{R}$. From the definition of T^λ we deduce

$$\begin{aligned} \pi_\xi T^\lambda(\alpha x + \beta y) &= \pi_\xi T(\alpha x_\xi + \beta y_\xi) \\ &= \alpha \pi_\xi T x_\xi + \beta \pi_\xi T y_\xi = \pi_\xi(\alpha T^\lambda x + \beta T^\lambda y), \end{aligned}$$

whence $T^\lambda(\alpha x + \beta y) = \alpha T^\lambda x + \beta T^\lambda y$.

To see that T^λ is band preserving, observe that if $\pi x = 0$ then $\pi \pi_\xi x_\xi = \pi_\xi \pi x = 0$ and so $\pi_\xi(\pi T^\lambda x) = \pi \pi_\xi T x_\xi = 0$ by 4.1.5 (2). As ξ is arbitrary, $\pi T^\lambda x = 0$, as required.

Suppose that $\hat{T}$ is another band preserving linear operator from $\lambda(X)$ to $\lambda(Y)$ with $\hat{T}|_X = T$. If $x \in \lambda(X)$ is as above, then $\pi_\xi x = \pi_\xi x_\xi$ with $x_\xi \in X$ and, according to 4.1.5 (3), $\pi_\xi \hat{T}x = \pi_\xi \hat{T}x_\xi = \pi_\xi T x = \pi_\xi T^\lambda x_\xi = \pi_\xi T^\lambda x$. Since ξ is arbitrary, it follows that $\hat{T}x = T^\lambda x$.

If T is order bounded then by the Meyer Theorem $|T|$ exists and $|Tx| = |T|(|x|)$ for all $x \in X$. By what we have proved $|T|$ has a unique extension $|T|^\lambda$ to $\lambda(X)$ and $|T|^\lambda$ is positive. Given $x \in \lambda(X)$ as above, we have

$$|T^\lambda(x)| = o\text{-}\sum_{\xi\in\Xi} \pi_\xi |Tx_\xi| = o\text{-}\sum_{\xi\in\Xi} \pi_\xi |T|(|x_\xi|) = |T|^\lambda(|x|).$$

From this we see that T^λ is order bounded and $|T^\lambda| = |T|^\lambda$. Conversely, if T^λ is order bounded then $|T^\lambda|$ exists and $|Tx| = |T^\lambda x| = |T^\lambda|(|x|)$ for all $x \in D$. Thus, T is order bounded. $\triangleright$

4.1.8. *Let X, Y, Z, T, and T^λ be the same as in* 4.1.7 *and X is order dense in Z. Then $\lambda(X) = X^\lambda$ and T is injective if and only if so is T^λ.*

$\triangleleft$ We need only to check that T^λ is injective whenever T is injective. It follows from the definition of T^λ that T^λ is injective if and only if $\pi T x = 0$ implies $\pi x = 0$ for all $x \in X$ and $\pi \in \mathbb{P}(Z)$. Suppose the contrary to our claim that there are $x \in X$ and $\pi \in \mathbb{P}(Z)$ such that $\pi T x = 0$, while $\pi x \neq 0$. We can assume further that $x > 0$ because $Tx^+ \perp Tx^-$ and so $\pi T x^+ = \pi T x^- = 0$, while either $\pi x^+ \neq 0$ or $\pi x^- \neq 0$. Choose $x_0 \in X$ with $0 < x_0 \leqslant \pi x$, put $\rho := [x_0]$, and note that $\rho^\perp T x_0 = 0$ because $Tx_0 \in \{x_0\}^{\perp\perp}$. At the same time $\rho \leqslant \pi$ and so $\rho T x_0 = 0$. Thus, $Tx_0 = 0$, while $x_0 \neq 0$; a contradiction. $\triangleright$

4.2. The Cauchy Functional Equation

Here we shortly address the celebrated Cauchy equation of the classical calculus. In the next section we demonstrate that the contracting operators in universally complete vector lattices are solutions in disguise of the Cauchy equation and the Wickstead problem amounts to that of regularity of all solutions to the equation under some extra condition of regularity type.

4.2.1. By $\mathbb{F}$ we denote either $\mathbb{R}$ or $\mathbb{C}$. The *Cauchy functional equation* with $f : \mathbb{F} \to \mathbb{F}$ unknown has the form

$$f(x + y) = f(x) + f(y) \quad (x, y \in \mathbb{F}).$$

Clearly, every solution to the equation is automatically $\mathbb{Q}$-homogeneous; i.e., it satisfies another functional equation:

$$f(qx) = qf(x) \quad (q \in \mathbb{Q},\ x \in \mathbb{F}).$$

In the sequel we will be interested in a more general situation. Namely, we will consider the simultaneous functional equations

$$\begin{cases} f(x+y) = f(x) + f(y) & (x, y \in \mathbb{F}), \\ f(px) = pf(x) & (p \in \mathbb{P},\, x \in \mathbb{F}), \end{cases} \tag{L}$$

where $\mathbb{P}$ is a subfield of $\mathbb{F}$ (which includes $\mathbb{Q}$). In case $\mathbb{F} = \mathbb{C}$ we assume that $i \in \mathbb{P}$, so that $\mathbb{Q} \oplus i\mathbb{Q} \subset \mathbb{P}$. Denote by $\mathbb{F}_{\mathbb{P}}$ the field $\mathbb{F}$ which is considered as a vector space over $\mathbb{P}$. Clearly, solutions to the simultaneous equations (L) are precisely $\mathbb{P}$-linear functions from $\mathbb{F}_{\mathbb{P}}$ to $\mathbb{F}_{\mathbb{P}}$.

4.2.2. *Let $\mathscr{E}$ be a Hamel basis for a vector space $\mathbb{F}_{\mathbb{P}}$, and let $\mathscr{F}(\mathscr{E}, \mathbb{F})$ be the space of all functions from $\mathscr{E}$ to $\mathbb{F}$. The solution set of (L) is a vector space over $\mathbb{F}$ isomorphic with $\mathscr{F}(\mathscr{E}, \mathbb{F})$. Such an isomorphism can be implemented by sending a solution f to the restriction $f|_{\mathscr{E}}$ of f to $\mathscr{E}$.*

$\triangleleft$ The solution set of (L) coincides with the space $L_{\mathbb{P}}(\mathbb{F})$ of all $\mathbb{P}$-linear operators in $\mathbb{F}_{\mathbb{P}}$. Suffice it to mention that $L_{\mathbb{P}}(\mathbb{F})$ and $\mathscr{F}(\mathscr{E}, \mathbb{F})$ are isomorphic vector spaces.

Let $\mathscr{F}_0(\mathscr{E}, \mathbb{P})$ be the set of finitely supported functions φ from $\mathscr{E}$ to $\mathbb{P}$; i.e., each $\varphi : \mathscr{E} \to \mathbb{P}$ is such that the support $\{e \in \mathscr{E} :\ \varphi(e) \neq 0\}$ of φ is finite. Then $\mathscr{F}_0(\mathscr{E}, \mathbb{P})$ is a vector space over $\mathbb{P}$ isomorphic with $\mathbb{F}_{\mathbb{P}}$. Such an isomorphism can be implemented by sending $\varphi \in \mathscr{F}_0(\mathscr{E}, \mathbb{P})$ to the sum $x_\varphi := \sum_{e \in \mathscr{E}} \varphi(e)e$. The inverse isomorphism $x \mapsto \varphi$ is determined by expansion of $x \in X$ in $\mathscr{E}$.

Given $\psi \in \mathscr{F}(\mathscr{E}, \mathbb{F})$, put

$$f_\psi(x_\varphi) := \sum_{e \in \mathscr{E}} \varphi(e)\psi(e) \quad (\varphi \in \mathscr{F}_0(\mathscr{E}, \mathbb{F})).$$

This yields an isomorphism of $\mathscr{F}(\mathscr{E}, \mathbb{F})$ to $L_{\mathbb{P}}(\mathbb{F})$. The inverse isomorphism takes the form $f \mapsto f|_{\mathscr{E}}$. The definitions of the isomorphisms $\varphi \mapsto x_\varphi$ and $\psi \mapsto f_\psi$ are meaningful, since there are only finitely many nonzero terms under the sign of summation. $\triangleright$

4.2.3. Corollary. *Let $\mathbb{P}$ be a subfield of $\mathbb{R}$. The general form of a $\mathbb{P}$-linear function $f : \mathbb{R} \to \mathbb{R}$ is given as*

$$f(x) = \sum_{e \in \mathscr{E}} x_e \phi(e) \quad \text{if } \; x = \sum_{e \in \mathscr{E}} x_e e,$$

where $\phi : \mathscr{E} \to \mathbb{R}$ is an arbitrary function.

$\lhd$ This is immediate from 4.2.2. $\rhd$

4.2.4. Theorem. *Each solution of (L) is either $\mathbb{F}$-linear or everywhere dense in $\mathbb{F}^2 := \mathbb{F} \times \mathbb{F}$.*

$\lhd$ A solution f of (L) is $\mathbb{F}$-linear if and only if f has presentation $f(x) = cx$ $(x \in \mathbb{F})$, with $c := f(1)$. If f is not $\mathbb{F}$-linear, then there are $x_1, x_2 \in \mathbb{F}$ satisfying $f(x_1)/x_1 \neq f(x_2)/x_2$. This yields the linear independence of $v_1 := (x_1, f(x_1))$ and $v_2 := (x_2, f(x_2))$ in the vector space $\mathbb{F}^2$ over $\mathbb{F}$. Indeed, if $\alpha_1 v_1 + \alpha_2 v_2 = 0$ for some $\alpha_1, \alpha_2 \in \mathbb{F}$, then $\alpha_1 x_1 + \alpha_2 x_2 = 0$ and $\alpha_1 f(x_1) + \alpha_2 f(x_2) = 0$, while the two simultaneous equations has only the trivial solution $\alpha_1 = \alpha_2 = 0$, since the relevant determinant $x_1 f(x_2) - x_2 f(x_1)$ is other than zero by hypothesis. Thus, each pair $(x, y) \in \mathbb{F}^2$ admits the presentation $(x, y) = \alpha_1 v_1 + \alpha_2 v_2$ for some $\alpha_1, \alpha_2 \in \mathbb{F}$. Since $\mathbb{P}$ is dense in $\mathbb{F}$, each neighborhood of (x, y) contains a vector of the form $p_1 v_1 + p_2 v_2$, with $p_1, p_2 \in \mathbb{P}$. (Recall that $\mathbb{Q} \oplus i\mathbb{Q} \subset \mathbb{P}$ whenever $\mathbb{F} = \mathbb{C}$.) Therefore, $\{p_1 v_1 + p_2 v_2 : p_1, p_2 \in \mathbb{P}\}$ is dense in $\mathbb{F}^2$. At the same time this set lies in f, since the $\mathbb{P}$-linearity of f implies that

$$\begin{aligned} p_1 v_1 + p_2 v_2 = (p_1 x_1 + p_2 x_2, p_1 f(x_1) + p_2 f(x_2)) \\ = (p_1 x_1 + p_2 x_2, f(p_1 x_1 + p_2 x_2)) \end{aligned}$$

for all $p_1, p_2 \in \mathbb{P}$. $\rhd$

4.2.5. *Let $\mathscr{E}$ be a Hamel basis of the space $\mathbb{F}_{\mathbb{P}}$ and let $\phi : \mathscr{E} \to \mathbb{F}$ be an arbitrary function. The unique $\mathbb{P}$-linear extension $f : \mathbb{F} \to \mathbb{F}$ of ϕ is continuous if and only if $\phi(e)/e = c = \mathrm{const}$ for all $e \in \mathscr{E}$. Moreover, in this event f admits the representation $f(x) = cx$ for all $x \in \mathbb{F}$.*

$\lhd$ By the $\mathbb{P}$-linearity of f we have $f(p) = pf(1)$ $(p \in \mathbb{P})$. If f is continuous, then using the denseness of $\mathbb{P}$ in $\mathbb{F}$, we arrive at the desired presentation with $c := f(1)$. If $g(e) = ce$ for all $e \in \mathscr{E}$ then the function $x \mapsto cx$ is a $\mathbb{P}$-linear extension of g, and by uniqueness of such extension $f(x) = cx$ $(x \in \mathbb{F})$, whence the continuity of f follows. $\rhd$

Thus, for a solution f to (L) to admit the presentation $f(x) = cx$ $(x \in \mathbb{R})$ we must impose some condition of regularity and continuity exemplifies such a condition. Let us list some other regularity conditions. First, agree to call an additive function $f : \mathbb{R} \to \mathbb{R}$ *order bounded* if f is bounded when restricted to every interval $[a, b] \subset \mathbb{R}$.

4.2.6. *Each solution f of (L) in case $\mathbb{F} = \mathbb{R}$ admits the representation $f(x) = cx$ $(x \in \mathbb{R})$ with some $c \in \mathbb{R}$ if and only if one of the following is fulfilled:*

(1) *f is continuous at some point.*

(2) *f is increasing or decreasing.*

(3) *f is order bounded.*

(4) *f is bounded above or below on some interval $]a, b[\subset \mathbb{R}$ with $a < b$.*

(5) *f is bounded above or below on some measurable subset of positive Lebesgue measure.*

(6) *f is Lebesgue measurable.*

$\triangleleft$ We start with demonstrating (4) by way of example. Necessity is obvious. To prove sufficiency, assume that f is bounded above by a real M on $]a, b[$. Then the open set $\{(s, t) \in \mathbb{R}^2 : a < s < b,\ M < t\}$ is disjoint from f, and so f cannot be dense in $\mathbb{R}^2$. But if f fails to admit the desired representation then f is dense in $\mathbb{R}^2$.

Clearly (1), (2), and (3) follow from (4). The proofs of (5) and (6) are available in [14, Ch. 2, Theorem 8] and [211, Theorem 9.4.3]. $\triangleright$

4.2.7. Turn now to $\mathbb{F} = \mathbb{C}$ and let $\mathbb{P} := \mathbb{P}_0 + i\mathbb{P}_0$ for some subfield of a subfield $\mathbb{P}_0 \subset \mathbb{R}$. In this event the solution set of (L) is the complexification of the solution set of (L) in the case that $\mathbb{P} := \mathbb{P}_0$.

In more detail, if $g : \mathbb{R} \to \mathbb{R}$ is a $\mathbb{P}_0$-linear function, then g extends uniquely to the $\mathbb{P}$-linear function $\widetilde{g} : \mathbb{C} \to \mathbb{C}$ defined as $\tilde{g}(z) = g(x) + ig(y)$ $(z = x + iy \in \mathbb{C})$. Conversely, if $f : \mathbb{C} \to \mathbb{C}$ is a $\mathbb{P}$-linear function, then there exists a unique pair of $\mathbb{P}_0$-linear functions $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ satisfying $f(z) = \tilde{f}_1(z) + i\tilde{f}_2(z)$ $(z \in \mathbb{C})$.

Therefore, each solution f to (L) has the form $f = \tilde{f}_1 + i\tilde{f}_2$, where $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ are $\mathbb{P}_0$-linear and $f_i(\mathbb{R}) \subset \mathbb{R}$ $(i = 1, 2)$. Say that f is *monotone* or *bounded* provided that so are f_1 and f_2. It is now easy to see that the following are true:

(1) The function f is dense in $\mathbb{C}^2$ if and only if both f_1 and f_2 are dense in $\mathbb{R}^2$.[2]

(2) A solution f of (L) in case $\mathbb{F} = \mathbb{C}$, $\mathbb{P} = \mathbb{P}_0 + i\mathbb{P}_0$, and $\mathbb{P}_0 \subset \mathbb{R}$ admits the representation $f(x) = cx$ $(x \in \mathbb{C})$ with some $c \in \mathbb{C}$ if and only if one of the conditions 4.2.4 (1–6) is fulfilled.

4.2.8. Theorem. *Let $\mathbb{P}$ be a subfield of $\mathbb{F}$, while $\mathbb{P} := \mathbb{P}_0 + i\mathbb{P}_0$ for some subfield $\mathbb{P}_0 \subset \mathbb{R}$, in case $\mathbb{F} = \mathbb{C}$. The following are equivalent:*

(1) $\mathbb{F} = \mathbb{P}$.

(2) *Every solution to (L) is order bounded.*

$\lhd$ It is trivial that (1) $\Longrightarrow$ (2). Prove the converse by way of contradiction. The assumption $\mathbb{F} \neq \mathbb{P}$ implies that each Hamel basis $\mathscr{E}$ for the vector space $\mathbb{F}_{\mathbb{P}}$ contains at least two nonzero distinct elements $e_1, e_2 \in \mathscr{E}$. Define the function $\psi : \mathscr{E} \to \mathbb{F}$ so that $\psi(e_1)/e_1 \neq \psi(e_2)/e_2$. Then the $\mathbb{P}$-linear function $f = f_\psi : \mathbb{F} \to \mathbb{F}$, coinciding with ψ on $\mathscr{E}$, would exist by 4.2.2 and be dense in $\mathbb{F}^2$ by 4.2.4. Therefore, f_ψ could not be order bounded (cp. 4.2.6 and 4.2.7). $\rhd$

4.2.9. Consider the two more collections of simultaneous functional equations:

$$\begin{cases} f(x+y) = f(x) + f(y) \quad (x, y \in \mathbb{F}), \\ f(px) = pf(x) \quad (p \in \mathbb{P},\ x \in \mathbb{F}), \\ f(xy) = f(x)f(y) \quad (x, y \in \mathbb{F}), \end{cases} \tag{A}$$

$$\begin{cases} f(x+y) = f(x) + f(y) \quad (x, y \in \mathbb{F}), \\ f(px) = pf(x) \quad (p \in \mathbb{P},\ x \in \mathbb{F}), \\ f(xy) = f(x)y + xf(y) \quad (x, y \in \mathbb{F}). \end{cases} \tag{D}$$

The nonzero solutions to (A) are called $\mathbb{P}$-*automorphisms* of the field $\mathbb{F}$, while the solutions of D are called $\mathbb{P}$-*derivatives* of $\mathbb{F}$. The identity automorphism and the zero derivation are called *trivial*. The problem of existence of nontrivial solutions to (A) and (D) needs more delicate results from field theory which will be presented in Section 4.11.

4.2.10. Sometimes it is important to deal with f satisfying the equation $f(x+y) = f(x) + f(y)$ only for $(x, y) \in G$, where G is a subset of $\mathbb{R} \times \mathbb{R}$. In this case the term *restricted Cauchy functional equation* is in common parlance (cp. [211, § 13.6]).

[2] Recall that each function $f : X \to Y$ from X to Y is a subset of $X \times Y$.

4.3. Representation of Band Preserving Operators

Here we will demonstrate that a band preserving operator can be represented in an appropriate Boolean valued model as a solution of the restricted Cauchy functional equation. This fact enables us to study the properties of band preserving operators with the help of the theory of functional equations.

4.3.1. We consider a pair of vector lattices X and Y with Y a nonzero vector sublattice of the universal completion $X^{\mathfrak{u}}$. Let $L_{\mathrm{bp}}(X,Y)$ be the set of all band preserving linear operators from X to Y. Clearly, $L_{\mathrm{bp}}(X,Y)$ is a vector space. Moreover, $L_{\mathrm{bp}}(X,Y)$ becomes a faithful unitary module over the f-algebra $A := \mathrm{Orth}(Y)$ on letting $\pi T := \pi \circ T$, since the multiplication by an element of A is band preserving and the composite of band preserving operators is band preserving too.

The part of $L_{\mathrm{bp}}(X,Y)$ comprising all order bounded operators is denoted by $L^{\sim}_{\mathrm{bp}}(X,Y)$. Clearly, $L^{\sim}_{\mathrm{bp}}(X,Y)$ is an A-submodule of $L_{\mathrm{bp}}(X,Y)$. Moreover, according to the Meyer Theorem, $L^{\sim}_{\mathrm{bp}}(X,Y)$ is a vector sublattice of $L^{\sim}(X,Y)$. Denote $L_{\mathrm{bp}}(X) := L_{\mathrm{bp}}(X,X)$ and $L^{\sim}_{\mathrm{bp}}(X) := L^{\sim}_{\mathrm{bp}}(X,X)$.

4.3.2. Let $\mathscr{R}_{\mathbb{R}}$ stand for the reals $\mathscr{R}$ within $\mathbb{V}^{(\mathbb{B})}$ considered as a vector space over the field $\mathbb{R}^{\wedge}$. Thus, the expression "$\mathscr{X}$ is a subspace of $\mathscr{R}_{\mathbb{R}}$" means that $\mathscr{X}$ is an $\mathbb{R}^{\wedge}$-subspace of $\mathscr{R}$. Actually, in this case $\mathscr{X}$ is a totally ordered vector space over the ordered field $\mathbb{R}^{\wedge}$ or, trivially, a vector sublattice of $\mathscr{R}_{\mathbb{R}}$.

Let $\mathscr{X}$ and $\mathscr{Y}$ be nonzero vector sublattices of $\mathscr{R}_{\mathbb{R}}$. By $L_{\mathbb{R}^{\wedge}}(\mathscr{X},\mathscr{Y})$ we denote the element of $\mathbb{V}^{(\mathbb{B})}$ that represents the space of all $\mathbb{R}^{\wedge}$-linear operators from $\mathscr{X}$ to $\mathscr{Y}$. Then $L_{\mathbb{R}^{\wedge}}(\mathscr{X},\mathscr{Y})$ is a vector space over $\mathbb{R}^{\wedge}$ within $\mathbb{V}^{(\mathbb{B})}$, and $L_{\mathbb{R}^{\wedge}}(\mathscr{X},\mathscr{Y}){\downarrow}$ is a unitary semiprime module over the f-algebra $\mathbb{R}^{\wedge}{\downarrow}$. Just as in 4.3.1, denote by $L^{\sim}_{\mathbb{R}^{\wedge}}(\mathscr{X},\mathscr{Y})$ the part of $L_{\mathbb{R}^{\wedge}}(\mathscr{X},\mathscr{Y})$ consisting of order bounded functions.

Recall that $\mathscr{R}{\downarrow}$ is a universally complete vector lattice and a semiprime f-algebra with unity $\mathbb{1} := 1^{\wedge}$, while $X := \mathscr{X}{\downarrow}$ and $Y := \mathscr{Y}{\downarrow}$ are laterally complete vector sublattices in $\mathscr{R}{\downarrow}$. Moreover, $X^{\mathfrak{u}} = Y^{\mathfrak{u}} = \mathscr{R}{\downarrow}$, so that we can define $L_{\mathrm{bp}}(X,Y)$. The main purpose of this section is to demonstrate that $L_{\mathrm{bp}}(X,Y)$ is isomorphic to $L_{\mathbb{R}^{\wedge}}(\mathscr{X},\mathscr{Y}){\downarrow}$.

4.3.3. *Let X be a vector lattice, $X^{\mathfrak{u}} = \mathscr{R}{\downarrow}$, and let Y be a vector sublattice of $X^{\mathfrak{u}}$. A linear operator $T : X \to Y$ is band preserving if and* *only if T is extensional; i.e., $[\![x_1 = x_2]\!] \leqslant [\![Tx_1 = Tx_2]\!]$ for all $x_1, x_2 \in X$.*

$\lhd$ Let $\mathbb{B} \simeq \mathbb{B}(X)$ and $\chi : \mathbb{B} \to \mathbb{P}(\mathscr{R}{\downarrow})$ is the same as in the Gordon Theorem. Then, in view of the properties of χ in 2.2.4 ($\mathbb{G}$), T is extensional if and if $\chi(b)x_1 = \chi(b)x_2$ implies $\chi(b)Tx_1 = \chi(b)Tx_2$ for all $x_1, x_2 \in X$ and $b \in \mathbb{B}$. But the latter means that T is band preserving by 4.1.5. $\rhd$

4.3.4. Theorem. *Let X and Y be vector sublattices of $\mathscr{R}{\downarrow}$ and $\mathscr{R}{\downarrow} = X^{\mathrm{u}}$. The mapping $T \mapsto T{\uparrow}$ defines the isomorphisms of $L_{\mathrm{bp}}(X, Y)$ to $L_{\mathbb{R}^\wedge}(\mathscr{X}, \mathscr{Y}){\downarrow}$ and $L^{\sim}_{\mathrm{bp}}(X, Y)$ to $L^{\sim}_{\mathbb{R}^\wedge}(\mathscr{X}, \mathscr{Y}){\downarrow}$. Both isomorphisms are onto whenever X and Y are laterally complete and in this case the inverse isomorphisms are defined by the mapping $\tau \mapsto \tau{\downarrow}$.*

$\lhd$ Put $\bar{X} := \mathscr{X}{\downarrow}$ and $\bar{Y} := \mathscr{Y}{\downarrow}$ and observe that $\bar{X} = \mathrm{mix}(X) = X^{\lambda}$ and $\bar{Y} = \mathrm{mix}(Y) = \lambda(Y)$; see 2.5.3. By 4.1.7 each $T \in L_{\mathrm{bp}}(X, Y)$ admits the unique band preserving extension $\bar{T} \in L_{\mathrm{bp}}(\bar{X}, \bar{Y})$. Therefore, $L_{\mathrm{bp}}(X, Y)$ can naturally be identified with a subspace of $L_{\mathrm{bp}}(\bar{X}, \bar{Y})$. Each $T \in L_{\mathrm{bp}}(X, Y)$ is extensional by 4.3.3, and so T has the ascent $\tau := T{\uparrow}$ presenting the unique mapping from $\mathscr{X}$ into $\mathscr{Y}$ such that $[\![\tau(x) = Tx]\!] = \mathbb{1}$ for all $x \in X$ (see 1.6.5). Using this identity and the definition of the ring structure on $\mathscr{R}{\downarrow}$ (cp. 2.2.2), we see that

$$
\begin{aligned}
\tau(x \oplus y) &= T(x + y) = Tx + Ty = \tau(x) \oplus \tau(y),\\
\tau(\lambda^\wedge \odot x) &= T(\lambda \cdot x) = \lambda \cdot Tx = \lambda^\wedge \odot \tau(x)
\end{aligned}
$$

hold within $\mathbb{V}^{(\mathbb{B})}$ for all $x, y \in X$ and $\lambda \in \mathbb{R}$. Hence, $[\![\, \tau \in L_{\mathbb{R}^\wedge}(\mathscr{X}, \mathscr{Y}) \,]\!] = \mathbb{1}$; i.e., $[\![\, \tau : \mathscr{X} \to \mathscr{Y}$ is an $\mathbb{R}^\wedge$-linear function $]\!] = \mathbb{1}$, where $\oplus$ and $\odot$ stand for the operations on $\mathscr{X}$ and $\mathscr{Y}$, while $+$ and $\cdot$ are the operations on X and Y.

Conversely, if $\tau \in L_{\mathbb{R}^\wedge}(\mathscr{X}, \mathscr{Y}){\downarrow}$ then the descent $\tau{\downarrow} : \bar{X} \to \bar{Y}$ is extensional by 1.5.6. A similar argument as above shows that if τ is $\mathbb{R}^\wedge$-linear within $\mathbb{V}^{(\mathbb{B})}$ then $\tau{\downarrow}$ is a linear operator. Now it is clear from 1.6.7 that the ascent functor as well as the descent functor defines a one-to-one correspondence between $L_{\mathrm{bp}}(\bar{X}, \bar{Y})$ and $L_{\mathbb{R}^\wedge}(\mathscr{X}, \mathscr{Y}){\downarrow}$.

Show that the above one-to-one correspondences preserve addition and scalar multiplication. This can be done by simple calculations, revealing isomorphisms, which is immediate from the identities

$$
\begin{aligned}
(S + T){\uparrow}x = (S + T)x &= Sx + Tx\\
&= S{\uparrow}x \oplus T{\uparrow}x = (S{\uparrow} \oplus T{\uparrow})x \quad (x \in \mathscr{R}{\downarrow});
\end{aligned}
$$

$$(\alpha\cdot S)\!\uparrow\! x = (\alpha\cdot S)x = \alpha\cdot(Sx) = \alpha\odot(S\!\uparrow\! x) = (\alpha\odot S\!\uparrow)x \quad (\alpha, x \in \mathscr{R}\!\downarrow),$$

where $\oplus$ and $\odot$ stand for the operations on $\mathscr{Y}$ and $L_{\mathbb{R}^\wedge}(\mathscr{X},\mathscr{Y})\!\downarrow$, while $+$ and $\cdot$ are the operations on $\bar{Y}$ and $L_{\mathrm{bp}}(\bar{X},\bar{Y})$.

Thus, $L_{\mathrm{bp}}(\bar{X},\bar{Y})$ and $L_{\mathbb{R}^\wedge}(\mathscr{X},\mathscr{Y})\!\downarrow$ are isomorphic. It remains to show that the ascent and descent preserve order boundedness. Take $\tau \in L_{\mathbb{R}^\wedge}(\mathscr{X},\mathscr{Y})\!\downarrow$. The sentence "$\tau$ is order bounded within $\mathbb{V}^{(\mathbb{B})}$" can be written as (with $[c,d]^\diamond$ stand for a order interval within $\mathbb{V}^{(\mathbb{B})}$)

$$\begin{aligned}\mathbb{1} &= [\![(\forall\, a \in \mathscr{X}_+)(\exists\, b \in \mathscr{Y}_+)\tau([-a,a]^\diamond) \subset [-b,b]^\diamond]\!]\\ &= \bigwedge_{a\in\bar{X}} [\![(\exists\, b \in \mathscr{Y}_+)\tau([-a,a]^\diamond) \subset [-b,b]^\diamond]\!].\end{aligned}$$

By the maximum principle for every $a \in \bar{X}_+$ there exists $b_a \in \bar{Y}$ such that $[\![\tau([-a,a]^\diamond) \subset [-b_a,b_a]^\diamond]\!] = \mathbb{1}$. The last identity is equivalent to $\tau\!\downarrow\![-a,a] \subset [-b_a,b_a]$ because of the three relations:

$$\begin{gathered}[c,d]^\diamond\!\downarrow = [c,d],\\ [\![A \subset B]\!] = \mathbb{1} \iff A\!\downarrow \subset B\!\downarrow,\\ \tau(A)\!\downarrow = \tau\!\downarrow\!(A\!\downarrow).\end{gathered}$$

The first relation is immediate from the definition of the descent of an order in 2.2.3, the second is easily deduced with the help of 1.5.2, and the third follows from 1.5.3. $\triangleright$

4.3.5. *Put* $L_{\mathrm{bp}}(X) := L_{\mathrm{bp}}(X,X)$ *and* $\mathrm{End}(\mathscr{R}_{\mathbb{R}}) := L_{\mathbb{R}^\wedge}(\mathscr{R},\mathscr{R})$. *The modules* $L_{\mathrm{bp}}(\mathscr{R}\!\downarrow)$ *and* $\mathrm{End}(\mathscr{R}_{\mathbb{R}})\!\downarrow$ *are isomorphic, and such an isomorphism can be implemented by sending a band preserving operator to its ascent. Moreover, the isomorphism preserves order boundedness.*

$\triangleleft$ This is immediate from 4.3.4. $\triangleright$

We now formulate a few corollaries to *Boolean valued representation of an order bounded operators* obtained in Theorem 3.3.3. An operator $T \in L^\sim(X,Y)$ is said to be *disjointness preserving* if $x \perp y$ implies $Tx \perp Ty$ for all $x,y \in X$. Let $L^\sim_{dp}(X,Y)$ stand for the set of all order bounded disjointness preserving operators from X to Y.

4.3.6. Theorem. *If* $T : X \to X$ *is an injective band preserving operator on a vector lattice* X, *then its inverse* $T^{-1} : T(X) \to X$ *is also band preserving.*

$\triangleleft$ This is immediate from 4.1.7 and 4.1.8 and Theorem 4.3.4. If X^λ and T^λ are the same as in 4.1.8 then we can assume that $X^\lambda = \mathscr{X}\!\downarrow$

and $T^\lambda = \tau{\downarrow}$ for some $\mathscr{X}, \tau \in \mathbb{V}^{(\mathbb{B})}$ with $[\![\mathscr{X}$ is a subspace of $\mathscr{R}_\mathbb{R}]\!] = \mathbb{1}$ and $[\![\tau : \mathscr{X} \to \mathscr{X}$ is an $\mathbb{R}^\wedge$-linear function$]\!] = \mathbb{1}$. Observe that τ is injective within $\mathbb{V}^{(\mathbb{B})}$ if and only if for every $x \in X^\lambda$ we have $[\![\tau x = 0 \leftrightarrow x = 0]\!] = \mathbb{1}$ or, equivalently, $[\![\tau{\downarrow}(x) = 0]\!] = [\![x = 0]\!]$. By 2.2.4 ($\mathbb{G}$) and 1.5.6 this amounts to saying that $T^\lambda(x) = 0 \Longleftrightarrow x = 0$ for all $x \in X^\lambda$ or, which is the same, T^λ is injective. By 4.1.8 and our hypothesis T^λ is injective. Consequently, $[\![\tau$ is injective$]\!] = \mathbb{1}$ and by the transfer and maximum principles there exists $\tau^{-1} \in \mathbb{V}^{(\mathbb{B})}$ such that $[\![\tau(\mathscr{X})$ is a subspace of $\mathscr{R}_\mathbb{R}$ and $\tau^{-1} : \tau(\mathscr{X}) \to \mathscr{X}$ is $\mathbb{R}^\wedge$-linear function$]\!] = \mathbb{1}$. By 2.5.1 (1) $X_\tau := \tau(\mathscr{X}){\downarrow}$ is a vector sublattice of X^λ. It follows from 1.5.5 (1) that $(T^\lambda)^{-1} = (\tau{\downarrow})^{-1} = (\tau^{-1}){\downarrow}$, so that T^λ is a linear operator from X_τ to X^λ. By 4.3.4 $(T^\lambda)^{-1}$ is band preserving. We arrive at the desired conclusion by appealing again to 4.1.7, since $T(X) \subset X_\tau$ and the restriction of $(T^\lambda)^{-1}$ to $T(X)$ is band preserving. $\triangleright$

4.3.7. *Assume that $\mathscr{X}$ and $\mathscr{Y}$ are $\mathbb{P}$-linear subspaces of $\mathbb{R}$. A $\mathbb{P}$-linear function $\tau : \mathscr{X} \to \mathscr{Y}$ is order bounded if and only if there exists $p \in \mathbb{P}_+$ such that $|\tau(x)| \leqslant p|x|$ for all $x \in \mathscr{X}$. In this case τ admits the representation $\tau(x) = cx$ $(x \in \mathscr{X})$ for some $c \in \mathbb{R}$.*

$\triangleleft$ Sufficiency is obvious, so only the necessity should be proved. If τ is order bonded then there are $0 < q \in \mathbb{P}$ and $0 < e \in \mathscr{X}$ such that $\tau([-e, e]) \subset [-q, q]$. Given $x \in \mathscr{X}$, take an arbitrary $\alpha \in \mathbb{P}$ with $\alpha \geqslant |x|$ and choose $p \in \mathbb{P}_+$ such that $q/e \leqslant p$. Since $ex/\alpha \in [-e, e]$, we have $\tau(ex/\alpha) \in [-q, q]$ or $|\tau(x)| \leqslant (q/e)\alpha \leqslant p\alpha$. Thus $|\tau(x)| \leqslant p|x|$ as $\alpha \geqslant |x|$ is arbitrary. In particular, τ is uniformly continuous and admits the unique continuous extension $\bar\tau : \mathbb{R} \to \mathbb{R}$. From the continuity and $\mathbb{P}$-linearity of $\bar\tau$ we see that $\bar\tau(x) = \tau(x) = cx$ $(x \in \mathscr{X})$ where $c := \bar\tau(1)$. $\triangleright$

4.3.8. *Let X be vector lattice and let Y be a sublattice of $X^{\mathfrak{u}}$. A band preserving operator $T : X \to Y$ is order bounded if and only if T can be presented as $Tx = c \cdot x$ $(x \in X)$ for some fixed $c \in X^{\mathfrak{u}}$.*

$\triangleleft$ If T admits the above representation then $T([-a, a]) \subset [-|c|a, |ca|]$ for all $a \in X_+$, so that T is order bounded. To prove the converse assume that $X^u = \mathscr{R}{\downarrow}$ and put $\mathscr{X} := X{\uparrow}$ and $\mathscr{Y} := Y{\uparrow}$. Working within $\mathbb{V}^{(\mathbb{B})}$ and using transfer, apply 4.3.7 to the function $\tau := T{\uparrow}$ from $\mathscr{X}$ to $\mathscr{Y}$ which is $\mathbb{R}^\wedge$-linear and order bounded. Thus, $[\![(\exists\, c \in \mathscr{R})(\forall\, x \in \mathscr{X})\tau(x) = c \odot x]\!] = \mathbb{1}$. By the maximum principle there exists $c \in \mathscr{R}{\downarrow}$ such that $[\![\tau(x) = c \odot x]\!] = \mathbb{1}$ for all $x \in \mathscr{X}{\downarrow}$. It follows that for all $x \in X \subset \mathscr{X}{\downarrow}$

we have

$$\mathbb{1} = [\![Tx = \tau(x)]\!] \wedge [\![\tau(x) = c \odot x]\!] \leqslant [\![Tx = c \cdot x]\!]$$

and so $Tx = cx$. $\triangleright$

4.3.9. *Let X be a vector lattice with $X^{\mathrm{u}} = \mathscr{R}{\downarrow}$ and let Y be a nonzero vector sublattice of X^{u}. Every band preserving operator from X to Y is order bounded if and only if $\mathscr{X} := X{\uparrow}$ is a one-dimensional subspace of $\mathscr{R}$ over $\mathbb{R}^{\wedge}$ within $\mathbb{V}^{(\mathbb{B})}$, with $\mathbb{B} := \mathbb{B}(X)$. In symbols,*

$$L_{\mathrm{bp}}(X, Y) = L^{\sim}_{\mathrm{bp}}(X, Y) \Longleftrightarrow \mathbb{V}^{(\mathbb{B})} \models (\exists\, e \in \mathscr{X})\, \mathscr{X} = \mathbb{R}^{\wedge} e.$$

$\triangleleft$ $\Longleftarrow$: If $\mathscr{X} = \mathbb{R}^{\wedge} e$ for some $e \in \mathscr{X}$ then every $\mathbb{R}^{\wedge}$-linear function $\tau : \mathscr{X} \to \mathscr{Y}$ within $\mathbb{V}^{(\mathbb{B})}$ evidently admits the representation $\tau(x) = cx$ for all $x \in \mathscr{X}$ with $c = \tau(e)$ and, by 4.3.7, τ is order bounded. By 4.3.4 every band preserving operator from X to Y is order bounded.

$\Longrightarrow$: If there is no $e \in \mathscr{X}$ with $\mathbb{R}^{\wedge} e \neq \mathscr{X}$, then each Hamel basis $\mathscr{E}$ for the vector space $\mathscr{X}$ over $\mathbb{R}^{\wedge}$ has at least two distinct elements $e_1 \neq e_2$. Defining some function $\phi_0 : \mathscr{E} \to \mathscr{Y}$ so that $\phi_0(e_1)/e_1 \neq \phi_0(e_2)/e_2$, we can extend ϕ_0 to an $\mathbb{R}^{\wedge}$-linear function $\phi : \mathscr{X} \to \mathscr{Y}$ as in 4.2.2 which could not be order bounded by 4.2.4 and 4.3.7. Therefore, the descent of ϕ would be a band preserving linear operator that fails to be order bounded by 4.3.4. $\triangleright$

4.3.10. *Let X be a vector sublattice of a vector lattice Y and let $T : X \to Y$ be a band preserving linear operator. Then there is a band B of Y^{λ} such that the restriction of T to $X \cap B$ is order bounded and the restriction of T to $X \cap B^{\perp}$ has the property*

$$(\forall\, x \in X \cap B^{\perp}_{+})\, (\forall\, n \in \mathbb{N})\, (\exists\, u_n \in \lambda(X) \cap B^{\perp}_{+})$$
$$\text{such that } u_n \leqslant x \text{ and } |T^{\lambda} u_n| \geqslant nx. \quad (*)$$

In particular, the restriction $T|_J$ of T to every nonzero order ideal J of $X \cap B^{\perp}$ is not order bounded.

$\triangleleft$ There is no loss of generality in assuming that $X = Y^{\mathrm{u}} = \mathscr{R}{\downarrow}$. Let an operator T^{λ} from $\lambda(X)$ to Y^{λ} is defined as in 4.1.7. Put $\tau := T^{\lambda}{\uparrow}$, $b = [\![\tau : \mathscr{X} \to \mathscr{Y}$ is order bounded$]\!]$ and $\pi_b := \chi(b)$. Then $[\![b \wedge \tau : b \wedge \mathscr{X} \to b \wedge \mathscr{Y}$ is order bounded$]\!] = \mathbb{1}_b := b$ by 1.3.7. Moreover, by 2.3.6, $(b \wedge \mathscr{Y}){\downarrow}$ can naturally be identified with the band $B := \pi_b(\mathscr{Y}{\downarrow}) = \pi_b(Y^{\lambda})$, while $(b \wedge \tau){\downarrow}$ can be identified with the restriction of T^{λ} to $\lambda(X) \cap B$. Thus, the restriction is order bounded together with its restriction to $X \cap B$ which coincides with $T|_{X \cap B}$.

Observe further that $b^* = [\![\tau : \mathscr{X} \to \mathscr{Y}$ is not order bounded$]\!]$ and by 1.3.7 we again have $[\![b^* \wedge \tau : b^* \wedge \mathscr{X} \to b^* \wedge \mathscr{Y}$ is not order bounded$]\!] = \mathbb{1}_{b^*} := b^*$. Moreover, $\pi'_b := I_{Y^\lambda} - \pi_b = \chi(b^*)$ and, by 2.3.6 $(b^* \wedge \mathscr{Y}){\downarrow}$ can be identified with the band $B^\perp := \pi'_b(Y^\lambda)$ and $(b^* \wedge \tau){\downarrow}$ can be identified with the restriction of T^λ to $\lambda(X) \cap B^\perp$. For brevity, put $\tau_0 := b^* \wedge \tau$, $\mathscr{X}_0 := b^* \wedge \mathscr{X}$, and $\mathscr{Y}_0 := b^* \wedge \mathscr{Y}$. Since $\mathscr{X}_0$ is linearly ordered and Archimedean, the fact that τ_0 is not bounded can be formalized as follows:

$$(\forall\, 0 \leqslant x \in \mathscr{X}_0)\, (\forall\, n \in \mathbb{N}^\wedge)\, (\exists\, u_n \in \mathscr{X}_0)\, \big(0 \leqslant u_n \leqslant x \wedge |\tau(y_n)| \geqslant nx\big).$$

By transfer this sentence is true within $\mathbb{V}^{(\mathbb{B})}$. Calculation of the Boolean truth values of the two universal quantifiers and application of the maximum principle to the existential quantifier leads to the assertion: for all $0 \leqslant x \in X \cap B^\perp$ and $n \in \mathbb{N}$ there exists $u_n \in \lambda(X) \cap B^\perp$ such that $|T^\lambda(u_n)| \geqslant nx$, which is precisely $(*)$.

Assume that there are $0 < x \in X \cap B^\perp$ and $y \in Y_+$ such that $|Tu| \leqslant y$ for all $u \in [0, x]$. Then $|T^\lambda(v)| \leqslant y$ for all $v \in \lambda(X)$ with $0 \leqslant v \leqslant x$. Indeed, if $v \in [0, x]$ and $\pi_\xi v = \pi_\xi v_\xi$ $(\xi \in \Xi)$ for a family $(v_\xi)_{\xi \in \Xi}$ in X and a partition of unity $(\pi_\xi)_{\xi \in \Xi}$ in $\mathbb{P}(Y^\lambda)$, then we have also $\pi_\xi v = \pi_\xi u_\xi$ with $u_\xi = x \wedge v_\xi \in X \cap [0, x]$ $(\xi \in \Xi)$. It follows that

$$|\pi_\xi T^\lambda(v)| = |\pi_\xi T u_\xi| \leqslant \pi_\xi y$$

and so $|T^\lambda(v)| \leqslant y$. If a sequence (u_n) is chosen in accordance with $(*)$, then $nx \leqslant |T^\lambda(u_n)| \leqslant y$ $(n \in \mathbb{N})$; a contradiction. Consequently, the restriction of T to any nonzero order ideal in $X \cap B^\perp$ is not order bounded. $\triangleright$

4.4. Dedekind Cuts and Continued Fractions

The behavior of Dedekind cuts and continued fractions in a Boolean valued model clarifies that $\mathbb{R}^\wedge$ coincides with the internal reals $\mathscr{R} \in \mathbb{V}^{(\mathbb{B})}$ if and only if the complete Boolean algebra $\mathbb{B}$ is σ-distributive.

4.4.1. Consider an ordered set L. A *Dedekind cut* in L is a pair (a, b) of nonempty subsets $a \subset L$ and $b \subset L$ such that a consists of all lower bounds of b and b consists of all upper bounds of a (in symbols,

$a = \underline{b}$ and $b = \overline{a}$). Denote by $\widehat{L}$ the set of all Dedekind cuts in L and introduce the order on $\widehat{L}$ by putting $(a,b) \leqslant (a',b')$ for $(a,b),(a',b') \in \widehat{L}$ if and only if $a \subset a'$ or, equivalently, $b' \subset b$. Assign to each $u \in L$ the Dedekind cut $\hat{u} := \big((\leftarrow,u],[u,\rightarrow)\big)$, where $(\leftarrow,u] := \{v \in L :\ v \leqslant u\}$ and $[u,\rightarrow) := \{v \in L :\ v \geqslant u\}$. Then $\widehat{L}$ is an order complete lattice; i.e., each nonempty upper bounded subset has supremum, and each nonempty bounded below subset has infimum. Moreover, $u \mapsto \hat{u}$ is a one-to-one mapping of L to $\widehat{L}$ preserving suprema and infima and for every cut $(a,b) \in \hat{L}$ we have

$$\sup\{\hat{u} :\ u \in a\} = (a,b) = \inf\{\hat{v} :\ v \in b\}.$$

The order complete lattice $\widehat{L}$ is called a *Dedekind completion* of L.

4.4.2. In particular, if $L := \mathbb{Q}$ then the Dedekind completion $\widehat{\mathbb{Q}}$ is isomorphic to $\mathbb{R}$. If (α_1,β_1) and (α_2,β_2) are Dedekind cuts in $\mathbb{Q}$ then $(\alpha_1,\beta_1) + (\alpha_2,\beta_2) = (\alpha_0,\beta_0)$ with $\alpha_0 := \underline{\beta_1 + \beta_2}$ and $\beta_0 := \overline{\alpha_1 + \alpha_2}$; if, in addition, $(\alpha_i,\beta_i) \geqslant \hat{0}$ $(i := 1,2)$ then $(\alpha_1,\beta_1)\cdot(\alpha_2,\beta_2) = (\alpha,\beta)$ with $\alpha := \underline{\beta_1 \cdot \beta_2}$ and $\beta := \overline{\alpha_1 \cdot \alpha_2}$. Here and below we put $u + v := \{x + y :\ x \in u,\ y \in v\}$ and $u \cdot v := \{x \cdot y :\ x \in u,\ y \in v\}$.

Assume now that L is a vector lattice. Introduce the addition and scalar multiplication on $\widehat{L}$ $\big($with (a,b), (a_1,b_1), $(a_2,b_2) \in \widehat{L}$ and $t \in \mathbb{R}\big)$ as follows:

$$(a_1,b_1) + (a_2,b_2) := \big(\underline{b_1 + b_2}, \overline{a_1 + a_2}\big),$$

$$t(a,b) := \begin{cases} (ta,tb), & \text{if } \ t > 0,\\ t(a,b) := (tb,ta), & \text{if } \ t < 0,\\ t(a,b) := (L_-,L_+), & \text{if } \ t = 0. \end{cases}$$

With these operators, $\widehat{L}$ becomes a Dedekind complete vector lattice and the mapping $\iota : u \mapsto \hat{u}$ is a lattice isomorphism of L to $\widehat{L}$. Moreover, $(\widehat{L},\iota)$ is a Dedekind completion of the vector lattice L.

4.4.3. *If $\mathscr{Q}$ is the rationals within $\mathbb{V}^{(\mathbb{B})}$, then*

$$\mathbb{V}^{(\mathbb{B})} \models \mathscr{Q} = \mathbb{Q}^{\wedge}.$$

$\lhd$ By transfer and the maximum principle there are $\mathscr{Z}$ and $\mathscr{Q} \in \mathbb{V}^{(\mathbb{B})}$ such that $[\![\mathscr{Z}$ is the ring of integers$]\!] = [\![\mathscr{Q}$ is the ring of rationals$]\!] = \mathbb{1}$. We have to show that

$$[\![\mathscr{Z} = \mathbb{Z}^{\wedge}]\!] = [\![\mathscr{Q} = \mathbb{Q}^{\wedge}]\!] = \mathbb{1}.$$

We knew already that $[\![\aleph_0 = (\omega_0)^\wedge]\!] = \mathbb{1}$ (cp. 1.9.9 (1)). So, using the fact that the definition $\mathbb{N} := \omega \setminus \{0\}$ is a bounded ZF-formula, we can write within $\mathbb{V}^{(\mathbb{B})}$ that

$$\aleph_0 \setminus \{0\} = \omega^\wedge \setminus \{0\} = (\omega \setminus \{0\})^\wedge = \mathbb{N}^\wedge.$$

Hence, $\mathbb{N}^\wedge$ is the set of naturals within $\mathbb{V}^{(\mathbb{B})}$. Let $\bar{\omega} = \{\dots, -n, \dots, -1, 0\}$ be an isomorphic copy of ω with the reverse order: $-n \leqslant -m \Longleftrightarrow m \leqslant n$. Then the set of integers can be defined as the direct sum (= disjoint union) $\mathbb{Z} := \bar{\omega} + \mathbb{N}$. Since the direct sum as well as $\bar{\omega}$ is given by a bounded formula, we can write within $\mathbb{V}^{(\mathbb{B})}$as follows:

$$\mathscr{Z} = \overline{\aleph_0} + \mathbb{N}^\wedge = \overline{\omega^\wedge} + (\bar{\omega} + \mathbb{N})^\wedge = \mathbb{Z}^\wedge.$$

Recall that the set of rationals is defined as the factor set $\mathbb{Q} := \mathbb{Z} \times \mathbb{N}/R$, where the coset of (m, n) stands for the rational m/n, and the equivalence of the pairs $(m, n)R(m', n')$ means that $mn' = nm'$. This definition is also written as a bounded formula and so within $\mathbb{V}^{(\mathbb{B})}$ we have

$$\mathscr{Q} = \mathscr{Z} \times \mathbb{N}^\wedge / R^\wedge = \mathbb{Z}^\wedge \times \mathbb{N}^\wedge / R^\wedge = (\mathbb{Z} \times \mathbb{N}/R)^\wedge = \mathbb{Q}^\wedge.$$

By analogy we see that the equality $\mathscr{Q} = \mathbb{Q}^\wedge$ within $\mathbb{V}^{(\mathbb{B})}$ may be viewed as the coincidence of the respective algebraic systems, since the ring and field axioms are given by bounded formulas. $\triangleright$

4.4.4. *For all $a \subset \mathbb{Q}$ and $b \subset \mathbb{Q}$, the following holds:*

$$\begin{aligned}&(a, b) \text{ is a Dedekind cut in } \mathbb{Q}\\ &\qquad\Longleftrightarrow [\![\, (a^\wedge, b^\wedge) \text{ is a Dedekind cut in } \mathbb{Q}^\wedge \,]\!] = \mathbb{1}.\end{aligned}$$

$\triangleleft$ The formula $\varphi(a, b, \mathbb{Q}) := (a \subset \mathbb{Q}) \wedge (b \subset \mathbb{Q}) \wedge (a = \underline{b}) \wedge (b = \overline{a})$ stating that a and b comprise a Dedekind cut in $\mathbb{Q}$, is bounded. Indeed, the formula $a \subset \mathbb{Q}$ is bounded (see 1.1.4) and $a = \underline{b}$ can be written as

$$(\forall\, r \in a)(\forall\, s \in b)(r \leqslant s) \wedge (\forall\, r \in \mathbb{Q})((\forall\, s \in b)(r \leqslant s \to r \in a)$$

which is also a bounded formula. Similarly, $b = \overline{a}$ is a bounded formula. So we are done by restricted transfer (cp. 1.4.7). $\triangleright$

4.4.5. *If $A \sim B$ and $\mathscr{P}(B^\wedge) = \mathscr{P}(B)^\wedge$ then $\mathscr{P}(A^\wedge) = \mathscr{P}(A)^\wedge$.*

$\triangleleft$ Given a mapping $\beta : A \to B$ define $\tilde{\beta} : \mathscr{P}(A) \to \mathscr{P}(B)$ as $\tilde{\beta} : C \mapsto \beta(C)$. If β is a bijection then $\tilde{\beta}$ is also a bijection. Moreover, by restricted transfer, the mappings $\beta^\wedge : A^\wedge \to B^\wedge$ and $\tilde{\beta}^\wedge := (\tilde{\beta})^\wedge : \mathscr{P}(A)^\wedge \to \mathscr{P}(B)^\wedge$ are one-to-one within $\mathbb{V}^{(\mathbb{B})}$. By transfer the mapping $\widetilde{\beta^\wedge} : \mathscr{P}(A^\wedge) \to \mathscr{P}(B^\wedge)$ is one-to-one too. Clearly, $\mathscr{P}(A)^\wedge$ is a subset of $\mathscr{P}(A^\wedge)$. It remains to show that the restriction of $\widetilde{\beta^\wedge}$ to $\mathscr{P}(A)^\wedge$ coincides with $\tilde{\beta}^\wedge$:

$$[\![(\forall\, u \in \mathscr{P}(A)^\wedge)\widetilde{\beta^\wedge}(u) = \tilde{\beta}^\wedge(u)]\!] = \bigwedge_{u\in\mathscr{P}(A)} [\![\widetilde{\beta^\wedge}(u^\wedge) = \tilde{\beta}^\wedge(u^\wedge)]\!]$$

$$= \bigwedge_{C\subset A} [\![\widetilde{\beta^\wedge}(C^\wedge) = \tilde{\beta}^\wedge(C^\wedge)]\!] = \bigwedge_{C\subset A} [\![\beta^\wedge(C^\wedge) = \tilde{\beta}(C)^\wedge]\!] = \mathbb{1}. \ \triangleright$$

4.4.6. *If* $\mathbb{B}$ *is* σ*-distributive then* $\mathbb{V}^{(\mathbb{B})} \models \mathscr{R} \subset \mathbb{R}^\wedge$.

$\triangleleft$ Assume that $\mathbb{B}$ is σ-distributive. By 1.9.13 (3) $\mathscr{P}(\omega^\wedge) = \mathscr{P}(\omega)^\wedge$ and $\mathscr{P}(\mathbb{Q}^\wedge) = \mathscr{P}(\mathbb{Q})^\wedge$ by 4.4.5. To demonstrate the desired inclusion we are to show only that $[\![t \in \mathscr{R}]\!] = \mathbb{1}$ implies $[\![t \in \mathbb{R}^\wedge]\!] = \mathbb{1}$. Assume that $[\![t \in \mathscr{R}]\!] = \mathbb{1}$; i.e., t is a Dedekind cut within $\mathbb{V}^{(\mathbb{B})}$. We then see within $\mathbb{V}^{(\mathbb{B})}$ that

$$\big(\exists\, a \in \mathscr{P}(\mathscr{Q})\big)\big(\exists\, \tilde{a} \in \mathscr{P}(\mathscr{Q})\big)\, \varphi(a, \tilde{a}, \mathscr{Q}) \wedge t = (a, \tilde{a}),$$

where φ is the same as in 4.4.4. Considering that $\mathscr{P}(\mathscr{Q}) = \mathscr{P}(\mathbb{Q})^\wedge$ in view of 4.4.3 and calculating the truth value of the above formula, we infer

$$\mathbb{1} = \bigvee_{a\subset\mathbb{Q}} \bigvee_{\tilde{a}\subset\mathbb{Q}} [\![\varphi(a^\wedge, \tilde{a}^\wedge, \mathbb{Q}^\wedge)]\!] \wedge [\![t = (a, \tilde{a})^\wedge]\!].$$

Choose a partition of unity $(b_\xi) \subset \mathbb{B}$ and two families (a_ξ) and $(\tilde{a}_\xi)$ in $\mathscr{P}(\mathbb{Q})$ so that

$$b_\xi \leqslant [\![\varphi(a_\xi^\wedge, \tilde{a}_\xi^\wedge, \mathbb{Q}^\wedge)]\!] \wedge [\![t = (a_\xi, \tilde{a}_\xi)^\wedge]\!].$$

It follows that $t = \operatorname{mix}_\xi b_\xi(a_\xi, \tilde{a}_\xi)^\wedge$, and $b_\xi \leqslant [\![\varphi(a_\xi^\wedge, \tilde{a}_\xi^\wedge, \mathbb{Q}^\wedge)]\!]$. If $b_\xi \neq \mathbb{0}$ then $[\![\varphi(a_\xi^\wedge, \tilde{a}_\xi^\wedge, \mathbb{Q}^\wedge)]\!] = \mathbb{1}$, since $\varphi(x_1, x_2, x_3)$ is a bounded formula and the truth value $[\![\varphi(x_1^\wedge, x_2^\wedge, x_3^\wedge)]\!]$ of a bounded formula may be either $\mathbb{0}$ or $\mathbb{1}$ by the definitions and rules of calculating truth values. By 4.4.4 $\varphi(a_\xi, \tilde{a}_\xi, \mathbb{Q})$; i.e., $(a_\xi, \tilde{a}_\xi)$ is a Dedekind cut. Evidently, $b_\xi \leqslant [\![t = (a_\xi, \bar{a}_\xi)^\wedge \in \mathbb{R}^\wedge]\!]$. Hence, $[\![t \in \mathbb{R}^\wedge]\!] = \mathbb{1}$. $\triangleright$

4.4.7. To prove the converse implication in 4.4.6 we use continued fractions. Put

$$\mathbb{I} := \{t \in \mathbb{R} \ :\ 0 < t < 1,\ t \text{ is irrational}\},$$

$$\mathscr{I} := \{t \in \mathscr{R} :\ 0 < t < 1,\ t \text{ is irrational}\}\ \big(\text{within } \mathbb{V}^{(\mathbb{B})}\big).$$

It is well known that there is a bijection $\lambda : \mathbb{I} \to \mathbb{N}^{\mathbb{N}}$ sending a real t to the sequence $\lambda(t) = a : \mathbb{N} \to \mathbb{N}$ of partial continued fractions of the continued fraction expansion of t:

$$t = \cfrac{1}{a(1) + \frac{1}{a(2)+\frac{1}{a(3)+\cdots}}}.$$

Given the two sequences $a : \mathbb{N} \to \mathbb{N}$ and $s : \mathbb{N} \to \mathbb{I}$, consider the bounded formula $\varphi_0(a, s, t, \mathbb{N})$ stating that $s(1) = t^{-1}$ and

$$a(n) = \left[\frac{1}{s(n)}\right], \quad s(n+1) = \frac{1}{s(n)} - a(n),$$

for all $n \in \mathbb{N}$, where $[\alpha]$ is the integer part of $0 < \alpha \in \mathbb{R}$ which is expressed by the bounded formula $\psi(\alpha, [\alpha], \mathbb{N})$:

$$[\alpha] \in \mathbb{N} \wedge [\alpha] \leqslant \alpha \wedge (\forall\, n \in \mathbb{N})(n \leqslant \alpha \to n \leqslant [\alpha]).$$

The equality $\lambda(t) = a$ means the existence of a sequence $s : \mathbb{N} \to \mathbb{I}$ such that $\varphi_0(a, s, t, \mathbb{N})$. Call the bijection λ the *continued fraction expansion.* By transfer, the continued fraction expansion $\tilde{\lambda} : \mathscr{I} \to (\aleph_0)^{\aleph_0} = (\mathbb{N}^{\wedge})^{\mathbb{N}^{\wedge}}$ exists within $\mathbb{V}^{(\mathbb{B})}$.

4.4.8. *Within $\mathbb{V}^{(\mathbb{B})}$, the restriction of $\tilde{\lambda}$ to $\mathbb{I}^{\wedge}$ coincides with $\lambda^{\wedge}$; i.e.,*

$$\mathbb{V}^{(\mathbb{B})} \models (\forall\, t \in \mathbb{I}^{\wedge})\, \tilde{\lambda}(t) = \lambda^{\wedge}(t).$$

$\lhd$ The desired is true if $\tilde{\lambda}(t^{\wedge}) = \lambda(t)^{\wedge}$ for all $t \in \mathbb{I}$. By the definition of $\tilde{\lambda}$ we have to demonstrate the validity within $\mathbb{V}^{(\mathbb{B})}$ of the formula: $(\exists\, s \in \mathscr{I}^{\mathbb{N}^{\wedge}})\, \varphi_0(\lambda(t)^{\wedge}, s, t^{\wedge}, \mathbb{N}^{\wedge})$. By the definition of λ there is a sequence $\sigma : \mathbb{N} \to \mathbb{I}$ satisfying $\varphi_0(\lambda(t), \sigma, t, \mathbb{N})$. Since φ_0 is bounded, $\mathbb{1} = [\![\varphi_0(\lambda(t)^{\wedge}, \sigma^{\wedge}, t^{\wedge}, \mathbb{N}^{\wedge})]\!]$. Note that $\sigma^{\wedge} : \mathbb{N}^{\wedge} \to \mathbb{I}^{\wedge} \subset \mathscr{I}$; i.e., $[\![\sigma^{\wedge} \in \mathscr{I}^{\mathbb{N}^{\wedge}}]\!] = \mathbb{1}$. Summarizing the above, we can write

$$[\![\, (\exists\, s \in \mathscr{I}^{\mathbb{N}^{\wedge}})\, \varphi_0(\lambda(t)^{\wedge}, s, t^{\wedge}, \mathbb{N}^{\wedge})\,]\!] \geqslant [\![\varphi_0(\lambda(t)^{\wedge}, \sigma^{\wedge}, t^{\wedge}, \mathbb{N}^{\wedge})]\!] = \mathbb{1}. \rhd$$

4.4.9. Theorem. *Assume that X is a universally complete vector lattice, $\mathbb{B} := \mathbb{P}(X)$, and $\mathscr{R}$ stands for the reals within $\mathbb{V}^{(\mathbb{B})}$. Then the following are equivalent:*

(1) *$\mathbb{B}$ is σ-distributive.*

(2) *$\mathscr{R} = \mathbb{R}^\wedge$ within $\mathbb{V}^{(\mathbb{B})}$.*

(3) *Every band preserving linear operator in X is order bounded.*

$\triangleleft$ The implication (1) $\Longrightarrow$ (2) amounts to 4.4.6. Prove that $\mathbb{V}^{(\mathbb{B})} \models \mathscr{R} = \mathbb{R}^\wedge$ implies σ-distributivity of $\mathbb{B}$.

By hypothesis $\operatorname{im}(\tilde{\lambda}) = \mathscr{I} = \mathbb{I}^\wedge = \operatorname{im}(\lambda^\wedge)$ within $\mathbb{V}^{(\mathbb{B})}$. Hence, $\tilde{\lambda}$ and $\lambda^\wedge$ are bijections, $\tilde{\lambda}$ extends $\lambda^\wedge$ by 4.4.8, and their images coincide. Clearly, the domains coincide in this event too (and, moreover, $\tilde{\lambda} = \lambda^\wedge$). Therefore, $(\mathbb{N}^\mathbb{N})^\wedge = (\mathbb{N}^\wedge)^{\mathbb{N}^\wedge}$. By 1.9.13 (2) $\mathbb{B}$ is σ-distributive.

The equivalence (2) $\Longleftrightarrow$ (3) follows from 4.3.7. $\triangleright$

4.5. Hamel Bases in Boolean Valued Models

As can be seen from 4.3.9, the important feature of a vector lattice is the internal dimension of its Boolean valued representation considered as a vector lattice over $\mathbb{R}^\wedge$. It stands to reason to find out what construction in a vector lattice corresponds to a Hamel basis for within the Boolean valued representation.

4.5.1. Let X be a vector lattice with a cofinal family of band projections. We will say that $x, y \in X$ are *distinct* at $\pi \in \mathbb{P}(X)$ provided that $\pi|x-y|$ is a weak order unit in $\pi(X)$ or, equivalently, if $\pi(X) \subset |x-y|^{\perp\perp}$. Clearly, x and y differ at π whenever $\rho x = \rho y$ implies $\pi\rho = 0$ for all $\rho \in \mathbb{P}(X)$. A subset $\mathscr{E}$ of X is said to be *locally linearly independent* provided that, for an arbitrary nonzero band projection π in X and each collection of the elements $e_1, \dots, e_n \in \mathscr{E}$ that are pairwise distinct at π, and each collection of reals $\lambda_1, \dots, \lambda_n \in \mathbb{R}$, the condition $\pi(\lambda_1 e_1 + \dots + \lambda_n e_n) = 0$ implies that $\lambda_k = 0$ for all $k := 1, \dots, n$. In other words, $\mathscr{E}$ is locally linearly independent if for all $\pi \in \mathbb{P}(X)$ every subset of $\pi(\mathscr{E})$, consisting of nonzero members pairwise distinct at π, is linearly independent.

An inclusion maximal locally linearly independent subset of X is called a *local Hamel basis* for X.

4.5.2. *Each vector lattice X with a cofinal family of band projections has a local Hamel basis for X.*

$\triangleleft$ It suffices to apply the Kuratowski–Zorn Lemma to the inclusion ordered set of all locally linearly independent subsets of X. $\triangleright$

4.5.3. *A locally linearly independent set $\mathscr{E}$ in X is a local Hamel basis for X if and only if for every $x \in X$ there exists a partition of* *unity* $(\pi_\xi)_{\xi\in\Xi}$ *in* $\mathbb{P}(X)$ *such that for every* $\xi \in \Xi$ *the projection* $\pi_\xi x$ *is a finite linear combination of nonzero elements of* $\pi_\xi\mathscr{E}$ *pairwise distinct at* π. *This representation of* $\pi_\xi x$ *is unique in the band* $\pi_\xi(X)$.

$\lhd$ $\Longleftarrow$: Assume that $\mathscr{E} \subset X$ is locally linearly independent but fails to be a Hamel basis. Then we can find $x \in X$ such that $\mathscr{E} \cup \{x\}$ is locally linearly independent. Therefore, there is no nonzero band projection π for which πx is a linear combination of nonzero elements from $\pi\mathscr{E}$ pairwise distinct at π. This contradicts the existence of a partition of unity with the above mentioned properties.

$\Longrightarrow$: If $\mathscr{E}$ is a local Hamel basis for X then $\mathscr{E} \cup \{x\}$ is not locally linearly independent for an arbitrary $x \in X$. Thus, there exist a nonzero band projection π and $e_1, \dots, e_n \in \mathscr{E}$ such that either $\rho x = 0$ for some nonzero band projection $\rho \leqslant \pi$, or $\rho x = \rho e_k$ for some $k \in \{1, \dots, n\}$ and nonzero band projection $\rho \leqslant \pi$, or $\pi(\lambda_0 x + \lambda_1 e_1 + \dots + \lambda_n e_n) = 0$ for some $\lambda_0, \lambda_1, \dots, \lambda_n \in \mathbb{R}$, while $\pi e_1, \dots, \pi e_n, \pi x$ are nonzero and pairwise distinct at π and not all $\lambda_0, \lambda_1, \dots, \lambda_n$ are equal to zero. In the latter case the equality $\lambda_0 = 0$ contradicts the local linear independence of $\mathscr{E}$, so that $\lambda_0 \neq 0$. In all cases there is a nonzero band projection π such that πx is representable as a linear combination of $\pi e_1, \dots, \pi e_n$. The set of such band projections π is minorizing in $\mathbb{P}(X)$, since in above reasoning we can replace x by σx with an arbitrary band projection $\sigma \in \mathbb{P}(X)$. The existence of the required partition of unity follows from the fact that every minorizing subset of a complete Boolean algebra admits a disjoint refinement (the exhaustion principle). $\rhd$

4.5.4. The claim of 4.5.3 admits the reformulation: A locally linearly independent set $\mathscr{E}$ in $\mathbb{X}$ is a local Hamel basis if and only if for every $x \in \mathbb{X}$ there exist a partition of unity $(\pi_\xi)_{\xi\in\Xi}$ in $\mathbb{P}(\mathbb{X})$ and a family of reals $(\lambda_{\xi,e})_{\xi\in\Xi, e\in\mathscr{E}}$ such that

$$x = o\text{-}\sum_{\xi\in\Xi} \left(\sum_{e\in\mathscr{E}} \lambda_{\xi,e} \pi_\xi e \right)$$

and for every $\xi \in \Xi$ the set $\{e \in \mathscr{E} : \lambda_{\xi,e} \neq 0\}$ is finite and consists of nonzero elements pairwise distinct at π_ξ. Moreover, the representation is unique in the sense that if x admits another representation

$$x = o\text{-}\sum_{\omega\in\Omega} \left(\sum_{e\in\mathscr{E}} \varkappa_{\omega,e} \rho_\omega e \right),$$

and for every $\omega \in \Omega$ the set $\{e \in \mathscr{E} : \varkappa_{\omega,e} \neq 0\}$ is finite and consists of nonzero elements pairwise distinct at ρ_ω, then for all $\xi \in \Xi$, $\omega \in \Omega$, and $e \in \mathscr{E}$ the relation $\pi_\xi \rho_\omega e \neq 0$ implies $\lambda_{\xi,e} = \varkappa_{\omega,e}$.

4.5.5. *Assume that $\mathscr{E}, \mathscr{X} \in \mathbb{V}^{(\mathbb{B})}$, $[\![\, \mathscr{X}$ is a vector subspace of the vector space $\mathscr{R}_{\mathbb{R}} \,]\!] = \mathbb{1}$, $[\![\, \mathscr{E} \subset \mathscr{X} \,]\!] = \mathbb{1}$, and $X := \mathscr{X}{\downarrow}$. Then $[\![\, \mathscr{E}$ is a linearly independent subset of the vector space $\mathscr{X}$ (over $\mathbb{R}^{\wedge}$) $]\!] = \mathbb{1}$ if and only if $\mathscr{E}{\downarrow}$ is a locally linearly independent subset of X.*

$\lhd$ $\Longleftarrow$: Put $\mathscr{E}' := \mathscr{E}{\downarrow}$ and assume that $\mathscr{E}'$ is locally linearly independent. Given a natural n, let the formula $\varphi(n, \tau, \sigma)$ expresses the following: τ and σ are mappings from $\langle n \rangle := \{0, 1, \dots, n-1\}$ into $\mathbb{R}^{\wedge}$ and $\mathscr{E}$ respectively, $\sigma(k) \neq \sigma(l)$ for different k and l in $\langle n \rangle$, and $\sum_{k \in \langle n \rangle} \tau(k)\sigma(k) = 0$. Denote the formula

$$(\forall\, \tau)(\forall\, \sigma)\big(\varphi(n, \tau, \sigma) \to (\forall\, k \in n)\, \tau(k) = 0\big)$$

by $\psi(n)$. Then the linear independence of $\mathscr{E}$ within $\mathbb{V}^{(\mathbb{B})}$ amounts to the equality

$$\mathbb{1} = [\![\, (\forall\, n \in \mathbb{N}^{\wedge})\, \psi(n) \,]\!] = \bigwedge_{n \in \mathbb{N}} [\![\, \psi(n^{\wedge}) \,]\!].$$

Hence, we are left with proving that $[\![\, \psi(n^{\wedge}) \,]\!] = \mathbb{1}$ for all $n \in \mathbb{N}$. Calculate the truth values, using the construction of the formula ψ and the rules of Boolean valued analysis (cp. 1.5.2). The result is as follows:

$$\bigwedge \Big\{ [\![\, (\forall\, k \in \langle n \rangle^{\wedge})\, \tau(k) = 0 \,]\!] : \ \tau, \sigma \in \mathbb{V}^{(\mathbb{B})};\ [\![\, \varphi(n^{\wedge}, \tau, \sigma) \,]\!] = \mathbb{1} \Big\}.$$

Take some $\tau, \sigma \in \mathbb{V}^{(\mathbb{B})}$ and $n \in \mathbb{N}$ such that $[\![\varphi(n^{\wedge}, \tau, \sigma)]\!] = \mathbb{1}$. Then $[\![\tau : \langle n \rangle^{\wedge} \to \mathbb{R}^{\wedge}]\!] = \mathbb{1}$ and $[\![\, \sigma : \langle n \rangle^{\wedge} \to \mathscr{E} \,]\!] = \mathbb{1}$. Moreover, $[\![\, \sigma(k) \neq \sigma(l)$ for distinct k and l in $\langle n \rangle^{\wedge}$, and $\sum_{k \in n^{\wedge}} \tau(k)\sigma(k) = 0 \,]\!] = \mathbb{1}$.

Let $t : \langle n \rangle \to \mathbb{R}^{\wedge}{\downarrow}$ and let $s : \langle n \rangle \to \mathscr{E}'$ stand for the modified descents of τ and σ (cp. 1.5.8). Then

$$\begin{aligned} \mathbb{1} &= [\![(\forall\, k, l \in \langle n \rangle^{\wedge})\big(k \neq l \to \sigma(k) \neq \sigma(l)\big)]\!] \\ &= \bigwedge_{\substack{k, l \in \langle n \rangle \\ k \neq l}} [\![\sigma(k^{\wedge}) \neq \sigma(l^{\wedge})]\!] = \bigwedge_{\substack{k, l \in \langle n \rangle \\ k \neq l}} [\![s(k) \neq s(l)]\!], \end{aligned}$$

and so $s(k)$ and $s(l)$ differ at the identity projection for k and l distinct. Furthermore,

$$\left[\!\!\left[\sum_{k=0}^{n-1} t(k)s(k) = 0 \right]\!\!\right] = \left[\!\!\left[\sum_{k \in \langle n \rangle^{\wedge}} \tau(k)\sigma(k) = 0 \right]\!\!\right] = \mathbb{1}.$$

Hence, $\sum_{k=0}^{n-1} t(k)s(k) = 0$. Since $t(k) \in \mathbb{R}^\wedge{\downarrow}$ for all $k \in \langle n\rangle$, there is a partition of unity $(b_\xi)_{\xi\in\Xi}$ in $\mathbb{B}$ and, to each $k \in \langle n\rangle$, there is a numerical family $(\lambda_{\xi,k})_{\xi\in\Xi}$ such that

$$t(k) = o\text{-}\sum_{\xi\in\Xi} \lambda_{\xi,k}\chi(b_\xi)\mathbb{1} \quad (k := 0, 1, \dots, n-1).$$

Inserting these expressions into the equality $\sum_{k=0}^{n-1} t(k)s(k) = 0$, we obtain

$$0 = \sum_{k=0}^{n-1}\left(o\text{-}\sum_{\xi\in\Xi} \lambda_{\xi,k}\chi(b_\xi)\mathbb{1}\right)s(k) = o\text{-}\sum_{\xi\in\Xi}\chi(b_\xi)\sum_{k=0}^{n-1}\lambda_{\xi,k}s(k).$$

Consequently, $\chi(b_\xi)\sum_{k=0}^{n-1}\lambda_{\xi,k}s(k) = 0$ and, since $s(k)$ and $s(l)$ differ at $\chi(b_\xi)$ for distinct $k, l \in \langle n\rangle$, by the definition of local linear independence we have $\lambda_{\xi,k} = 0$ $(k = 0, 1, \dots, n-1)$. Thus $t(k) = 0$ $(k = 0, 1, \dots, n-1)$, and so

$$\mathbb{1} = \bigwedge_{k\in\langle n\rangle} [\![t(k) = 0]\!] = \bigwedge_{k\in\langle n\rangle} [\![\tau(k^\wedge) = 0]\!] = [\![(\forall\, k \in \langle n\rangle^\wedge)\, \tau(k) = 0]\!],$$

which was required.

$\Longrightarrow$: Assume that $[\![\, \mathscr{E}$ is an $\mathbb{R}^\wedge$-linearly independent set in $\mathscr{X}\,]\!] = \mathbb{1}$. Consider $\pi \in \mathbb{P}(X)$, $n \in \mathbb{N}$, $t : \langle n\rangle \to \mathbb{R}$, and $s : \langle n\rangle \to \mathscr{E}'$ such that $\pi \neq 0$, $s(k)$ and $s(l)$ are distinct at π for different $k, l \in \langle n\rangle$, and $\pi\sum_{k=0}^{n-1} t(k)s(k) = 0$. Our goal is now to prove that $t(k) = 0$ $(k := 0, \dots, n-1)$.

Let $\tau, \sigma \in \mathbb{V}^{(\mathbb{B})}$ be the modified ascents of t and s (cp. 1.6.8). Then, within $\mathbb{V}^{(\mathbb{B})}$, we have $\tau : \langle n\rangle^\wedge \to \mathbb{R}^\wedge$, $\sigma : \langle n\rangle^\wedge \to \mathscr{E}$, and

$$\Big((\forall\, k, l \in \langle n\rangle^\wedge)\,\big(k \neq l \to \sigma(k) \neq \sigma(l)\big) \wedge \sum_{k\in\langle n\rangle^\wedge} \tau(k^\wedge)\sigma(k^\wedge) = 0\Big)$$
$$\to (\forall\, k \in \langle n\rangle^\wedge)\, \tau(k) = 0.$$

Calculating the truth value of the latter formula, we obtain

$$b := \bigwedge_{\substack{k,l\in\langle n\rangle\\ k\neq l}} [\![s(k) \neq s(l)]\!] \wedge \left[\!\!\left[\sum_{k=0}^{n-1} t(k)s(k) = 0\right]\!\!\right] \leqslant \bigwedge_{k=0}^{n-1} [\![t(k)^\wedge = 0]\!].$$

According to the initial properties of π, s, and t, by 2.2.4 ($\mathbb{G}$) we have $\pi \leqslant \chi(b)$ implying that $\pi t(k)^\wedge = 0$ for all $k \in \langle n \rangle$ again by 2.2.4 ($\mathbb{G}$). Since $\pi \neq 0$; therefore, $t(k) = 0$ for all $k := 0, \dots, n-1$. $\triangleright$

4.5.6. *Let $\mathscr{E}_0$ be a locally linearly independent subset of X and $\mathscr{E} := \mathscr{E}_0\uparrow$. Then $[\![\mathscr{E}$ is $\mathbb{R}^\wedge$-linearly independent in $\mathscr{X}]\!] = \mathbb{1}$. In particular,* $\operatorname{mix}(\mathscr{E}_0)$ *is locally linearly independent.*

$\triangleleft$ By 4.5.5 it suffices to show that $\mathscr{E}_0' := \operatorname{mix}(\mathscr{E}_0) = \mathscr{E}\downarrow = \mathscr{E}_0\uparrow\downarrow$ is locally linearly independent. Take some nonzero band projection π in X, elements $e_1, \dots, e_n \in \mathscr{E}_0'$ that differ at π, and reals $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ satisfying $\pi(\lambda_1 e_1 + \dots + \lambda_n e_n) = 0$. There are a partition of unity (b_ξ) in $\mathbb{B}$ and families $(g_{\xi,k}) \subset \mathscr{E}_0$ such that $e_k = o\text{-}\sum_\xi \chi(b_\xi) g_{\xi,k}$. Clearly, $\rho := \pi\chi(b_\eta) \neq 0$ for some index η. The elements $g_{\eta,1}, \dots, g_{\eta,n}$ differ pairwise at ρ and $\rho(\lambda_1 g_{\eta,1} + \dots + \lambda_n g_{\eta,n}) = 0$. Since $\mathscr{E}_0$ is locally linearly independent, $\lambda_1 = \dots = \lambda_n = 0$. $\triangleright$

4.5.7. Theorem. *Assume that $\mathscr{E}, \mathscr{X} \in \mathbb{V}^{(\mathbb{B})}$, $[\![\mathscr{E} \subset \mathscr{X}]\!] = \mathbb{1}$, $[\![\mathscr{X}$ is a vector subspace of $\mathscr{R}_\mathbb{R}]\!] = \mathbb{1}$, and $X := \mathscr{X}\downarrow$. Then $[\![\mathscr{E}$ is a Hamel basis for the vector space $\mathscr{X}$ (over $\mathbb{R}^\wedge$)$]\!] = \mathbb{1}$ if and only if $\mathscr{E}\downarrow$ is a local Hamel basis for X.*

$\triangleleft$ This is immediate from 4.5.5 and 4.5.6. $\triangleright$

4.6. Locally One-Dimensional Vector Lattices

In this section we examine locally one-dimensional vector lattices and show that a universally complete vector lattice is locally one-dimensional if and only if all band preserving operators in it are automatically order bounded.

4.6.1. A vector lattice X is said to be *locally one-dimensional* if for every two nondisjoint $x_1, x_2 \in X$ there exist nonzero components u_1 and u_2 of x_1 and x_2 respectively such that u_1 and u_2 are proportional.

Every atomic vector lattice is evidently locally one-dimensional, but the converse is not true. Below in 4.7.7–4.7.10 we will demonstrate that there exists a purely nonatomic locally one-dimensional universally complete vector lattice.

An element $x \in X$ is *locally constant* with respect to $u \in X_+$ if there exist a numerical family $(\lambda_\xi)_{\xi\in\Xi}$ and partition $(\pi_\xi)_{\xi\in\Xi}$ of $[x]$ in $\mathbb{P}(X)$ such that $\pi_\xi x = \lambda_\xi \pi_\xi u$ for all $\xi \in \Xi$. In this event $x = o\text{-}\sum_{\xi\in\Xi} \lambda_\xi \pi_\xi u$.

4.6.2. *Let X be a vector lattice with a cofinal family of band projections, let X^λ be a lateral completion of X, and let $\mathscr{X} \in \mathbb{V}^{(\mathbb{B})}$ be a Boolean valued representation of X with $\mathbb{B} := \mathbb{P}(X)$. The following are equivalent:*

(1) *X is locally one-dimensional.*

(2) *$\mathbb{V}^{(\mathbb{B})} \models$ "$\mathscr{X}$ is a one-dimensional vector lattice over $\mathbb{R}^\wedge$."*

(3) *There is a singleton local Hamel basis for X^λ.*

(4) *Every pair of locally independent members in X is disjoint.*

$\lhd$ We can assume without loss of generality that $X \subset X^\lambda = \mathscr{X}{\downarrow}$ and $X \neq \{0\}$.

(1) $\Longrightarrow$ (2): Given $x, y \in X$, put $b_0 := [\![|x| \wedge |y| \neq 0]\!]$ and $X_0 := \{|x| \wedge |y|\}^{\perp\perp}$. Since X has a cofinal family of projection bands, it follows from (1) that there exists a partition $(X_\xi)_{\xi\in\Xi}$ in $\mathbb{B}(X)$ of X_0 such that $[X_\xi]x = \alpha_\xi[X_\xi]y$ with some $0 \neq \alpha_\xi \in \mathbb{R}$ for all $\xi \in \Xi$. Put $b_\xi := \chi^{-1}([X_\xi])$ and $\alpha_0 := \mathrm{mix}_{\xi\in\Xi}\, b_\xi \alpha_\xi^\wedge$ and observe that $b_0 = \bigvee_{\xi\in\Xi} b_\xi$, $b_\xi \leqslant [\![x = \alpha_\xi^\wedge y]\!]$ $(\xi \in \Xi)$, and $[\![\alpha_0 \in \mathbb{R}^\wedge]\!] = \mathbb{1}$. From this we deduce

$$
\begin{aligned}
b_\xi \leqslant [\![x = \alpha_\xi^\wedge y]\!] \wedge [\![\alpha_0 = \alpha_\xi^\wedge]\!] \wedge [\![\alpha_0 \in \mathbb{R}^\wedge]\!] \\
\leqslant [\![x = \alpha_0 y]\!] \wedge [\![\alpha_0 \in \mathbb{R}^\wedge]\!] \leqslant [\![(\exists\, \alpha \in \mathbb{R}^\wedge) x = \alpha y]\!] \\
= [\![x \text{ and } y \text{ are proportional}]\!].
\end{aligned}
$$

Thus, we have proved that $b_0 \leqslant [\![x \text{ and } y \text{ are proportional}]\!]$ or, what is the same, $[\![|x| \wedge |y| \neq 0]\!] \Rightarrow [\![x \text{ and } y \text{ are proportional}]\!] = \mathbb{1}$ for all $x, y \in X$. A simple calculation completes the proof:

$$
\begin{aligned}
&[\![\mathscr{X} \text{ is a one-dimensional vector lattice over } \mathbb{R}^\wedge]\!] \\
&= [\![(\forall\, x \in \mathscr{X})(\forall\, y \in \mathscr{X})|x| \wedge |y| \neq 0 \to x \text{ and } y \text{ are proportional}]\!] \\
&= \bigwedge_{x,y\in X} [\![|x| \wedge |y| \neq 0]\!] \Rightarrow [\![x \text{ and } y \text{ are proportional}]\!] = \mathbb{1}.
\end{aligned}
$$

(2) $\Longrightarrow$ (3): Working within $\mathbb{V}^{(\mathbb{B})}$ choose a nonzero $e \in \mathscr{X}$ so that $\mathscr{X} \simeq \mathbb{R}^\wedge e$. Then $e \in X^\lambda$ and $\{e\}$ is a local Hamel basis for X^λ by 4.5.6, since $[\![\{e\} \text{ is a Hamel basis for } \mathscr{X}]\!] = \mathbb{1}$.

(3) $\Longrightarrow$ (4): Let $\{e\}$ be a singleton local Hamel basis for X^λ and consider a pair of locally independent members $x, y \in X$. It follows that there exist a partition of unity $(\pi_\xi)_{\xi\in\Xi}$ in $\mathbb{P}(X^\lambda)$ and numerical families $(\alpha_\xi)_{\xi\in\Xi}$ and $(\beta_\xi)_{\xi\in\Xi}$ such that $\pi_\xi x = \alpha_\xi \pi_\xi e$ and $\pi_\xi y = \beta_\xi \pi_\xi e$ for all

$\xi \in \Xi$. If x and y are not disjoint then there exists $\eta \in \Xi$ with $\alpha_\eta \beta_\eta \neq 0$. Choose nonzero band projection $\pi \in \mathbb{P}(X)$ with $\pi \leqslant \pi_\eta$. Then πx and πy are proportional; a contradiction.

(4) $\Longrightarrow$ (1): Take a pair of nondisjoint elements $x_1, x_2 \in X$. By (4) the set $\{x_1, x_2\}$ is locally linearly dependent. Thus, there exists a nonzero band projection $\pi \in \mathbb{P}(X)$ such that $\{\pi x_1, \pi x_2\}$ is a linearly dependent pair of distinct elements. It follows that $u_1 := \pi x_1$ and $u_2 := \pi x_2$ are proportional components of x_1 and x_2. $\rhd$

4.6.3. *For each laterally complete vector lattice X with a weak order unit $\mathbb{1}$ the following are equivalent:*

(1) *X is locally one-dimensional.*

(2) *All elements of X_+ are locally constant with respect to $\mathbb{1}$.*

(3) *All elements of X_+ are locally constant with respect to an arbitrary weak order unit $e \in X$.*

(4) *$\{\mathbb{1}\}$ is a local Hamel basis for X.*

(5) *Every local Hamel basis for X consists of pairwise disjoint members.*

$\lhd$ The equivalence (1) $\Longleftrightarrow$ (4) and the implication (1) $\Longrightarrow$ (5) are immediate from 4.6.2. Obviously, (3) $\Longrightarrow$ (2). To prove the converse, note that, given $x \in X$, we can choose a partition of unity $(\pi_\xi)_{\xi \in \Xi}$ in $\mathbb{P}(X)$ such that for each $\xi \in \Xi$ both $\pi_\xi x$ and $\pi_\xi e$ are multiples of $\pi_\xi \mathbb{1}$. So, $\pi_\xi x$ is a multiple of $\pi_\xi e$ and (2) $\Longrightarrow$ (3). A similar argument shows that $\{\mathbb{1}\}$ is a local Hamel basis if and only if so is $\{e\}$ for every order unit $e \in X$. Thus, if (5) holds and $\mathscr{E}$ is a local Hamel basis for X then $e := \sup \mathscr{P}$ exists and $\{e\}$ is a local Hamel basis for X. It follows that (5) $\Longrightarrow$ (4). Clearly, (4) $\Longrightarrow$ (2) by 4.5.3. To complete the proof, we have to show (2) $\Longrightarrow$ (5). If (5) fails then we can choose a nonzero band projection π and a local Hamel basis containing two members e_1 and e_2 such that both πe_1 and πe_2 are nonzero multiples of $\pi \mathbb{1}$. Consequently, $\pi(\lambda_1 e_1 + \lambda_2 e_2) = 0$ for some $\lambda_1, \lambda_2 \in \mathbb{R}$ and we arrive at the contradictory conclusion that $\{e_1, e_2\}$ is not locally linearly independent. $\rhd$

4.6.4. Theorem. *Let X be a universally complete vector lattice. Then the following are equivalent:*

(1) *X is locally one-dimensional.*

(2) *Every band preserving operator on X is order bounded.*

$\triangleleft$ By the Gordon Theorem we can assume that $X = \mathscr{R}{\downarrow}$ with $\mathscr{R} \in \mathbb{V}^{(\mathbb{B})}$ and $\mathbb{B} \simeq \mathbb{P}(X)$. Thus, the problem reduces to existence of a discontinuous solution to the Cauchy functional equation in 4.2.1. From 4.3.5 we see that 4.6.4 (i) $\Longleftrightarrow$ 4.2.8 (i) $(i = 1, 2)$ if in 4.2.8 replace $\mathbb{R}$ by $\mathscr{R}$ and $\mathbb{P}$ by $\mathbb{R}^{\wedge}$. Thus the claim follows from 4.2.8 by transfer. $\triangleright$

4.6.5. It is worth comparing the above proof of 4.6.4 with the standard proof that does not involve Boolean valued representation.

$\triangleleft$ (1) $\Longrightarrow$ (2): Recall that a linear operator $T : X \to X$ is band preserving if and only if $\pi T = T\pi$ for every band projection π in X $\big($cp. 4.1.1 (4)$\big)$. Assume that T is band preserving and put $\rho := T\mathbb{1}$. Since an arbitrary $e \in X_+$ can be expressed as $e = \sup_{\xi\in\Xi} \lambda_\xi \pi_\xi \mathbb{1}$, we deduce

$$\pi_\xi Te = T(\pi_\xi e) = T(\lambda_\xi \pi_\xi \mathbb{1}) = \lambda_\xi \pi_\xi T(\mathbb{1}) = \pi_\xi(e)T(\mathbb{1}) = \pi_\xi e\rho,$$

whence $Te = \rho e$. It follows that T is a multiplication operator in X which is obviously order bounded.

(2) $\Longrightarrow$ (1): Assume that (1) is false. According to 4.6.4 (4) there is a local Hamel basis $\mathscr{E}$ for X containing two members e_1 and e_2 that are not disjoint. Then the band projection $\pi := [e_1] \wedge [e_2]$ is nonzero. (Here and below $[e]$ is the band projection onto $\{e\}^{\perp\perp}$.) For an arbitrary $x \in X$ there exists a partition of unity $(\pi_\xi)_{\xi\in\Xi}$ such that $\pi_\xi x$ is a finite linear combination of elements of $\mathscr{E}$. Assume the elements of $\mathscr{E}$ have been labelled so that $\pi_\xi x = \lambda_1 \pi_\xi e_1 + \lambda_2 \pi_\xi e_2 + \cdots$. Define Tx to be the unique element in X with $\pi_\xi Tx := \lambda_1 \pi\pi_\xi e_2$. It is easy to check that T is a well defined linear operator from X into itself.

Take $x, y \in X$ with $x \perp y$ and let $(\pi_\xi)_{\xi\in\Xi}$ be a partition of unity such that both $\pi_\xi x$ and $\pi_\xi y$ are finite linear combinations of elements from $\mathscr{E}$. Refining the partition of unity if need be, we can also require that at least one of the elements $\pi_\xi x$ and $\pi_\xi y$ equals zero for all $\xi \in \Xi$. If $\pi_\xi y \neq 0$ then $\pi_\xi x = 0$, and so the corresponding $\lambda_1 e_1$ is equal to zero. If $\pi\pi_\xi \neq 0$ then $\lambda_1 = 0$, and in any case $\pi_\xi Tx = 0$. It follows that $Tx \perp y$ and T is band preserving. If T were order bounded then T would be presentable as $Tx = ax$ $(x \in X)$ for some $a \in X$ (cp. 4.1.6 (4)). In particular, $Te_2 = ae_2$ and, since $Te_2 = 0$ by definition, we have $0 = [e_2]|a| \geqslant \pi|a|$. Thus $\pi e_2 = T(\pi e_1) = a\pi e_1 = 0$, contradicting the definition of π. $\triangleright$

4.6.6. *Let $\mathbb{P}$ is a proper subfield of $\mathbb{R}$. There exists an $\mathbb{P}$-linear subspace $\mathscr{X}$ in $\mathbb{R}$ such that $\mathscr{X}$ and $\mathbb{R}$ are isomorphic vector spaces over $\mathbb{P}$ but they are not isomorphic as ordered vector spaces over $\mathbb{P}$.*

$\triangleleft$ Recall that the real field $\mathbb{R}$ has no proper subfield of which it is a finite extension; see, for example, Coppel [96, Lemma 17]. It follows that $\mathbb{R}$ is an infinite dimensional vector space over the field $\mathbb{P}$. Let $\mathscr{E}$ be a Hamel basis of a $\mathbb{P}$-vector space $\mathbb{R}$. Since $\mathscr{E}$ is infinite, we can choose a proper subset $\mathscr{E}_0 \subsetneq \mathscr{E}$ of the same cardinality: $|\mathscr{E}_0| = |\mathscr{E}|$. If $\mathscr{X}$ denotes the $\mathbb{P}$-subspace of $\mathbb{R}$ generated by $\mathscr{E}_0$, then $\mathscr{X}_0 \subsetneq \mathbb{R}$ and $\mathscr{X}$ and $\mathbb{R}$ are isomorphic as vector spaces over $\mathbb{P}$. If $\mathscr{X}$ and $\mathbb{R}$ were isomorphic as ordered vector spaces over $\mathbb{P}$, then $\mathscr{X}$ would be order complete and, as a consequence, we would have $\mathscr{X} = \mathbb{R}$; a contradiction. $\triangleright$

4.6.7. Theorem. *Let X be a nonlocally one-dimensional universally complete vector lattice. Then there exist a vector sublattice $X_0 \subset X$ and a band preserving linear bijection $T : X_0 \to X$ such that T^{-1} is also band preserving but X_0 and X are not lattice isomorphic.*

$\triangleleft$ We can assume without loss of generality that $X = \mathscr{R}{\downarrow}$ and $[\![\mathscr{R} \neq \mathbb{R}^\wedge]\!] = \mathbb{1}$. By 4.6.6 there exist an $\mathbb{R}^\wedge$-linear subspace $\mathscr{X}$ in $\mathscr{R}$ and $\mathbb{R}^\wedge$-linear isomorphism τ from $\mathscr{X}$ onto $\mathscr{R}$, while $\mathscr{X}$ and $\mathscr{R}$ are not isomorphic as ordered vector spaces over $\mathbb{R}^\wedge$. Put $X_0 := \mathscr{X}{\downarrow}$, $T := \tau{\downarrow}$ and $S := \tau^{-1}{\downarrow}$. The mappings S and T and are band preserving and linear by 4.3.4. Moreover, $S = (\tau{\downarrow})^{-1} = T^{-1}$ by 1.5.5 (2). It remains to observe that X_0 and X are lattice isomorphic if and only if $\mathscr{X}$ and $\mathscr{R}$ are isomorphic as ordered vector spaces. $\triangleright$

4.6.8. Let γ be a cardinal. A vector lattice X is said to be *Hamel γ-homogeneous* whenever there exists a local Hamel basis of cardinality γ in X consisting of weak order units pairwise distinct at I_X. (Two elements $x, y \in X$ are distinct at I_X if $|x - y|$ is a weak order unit in X; see 4.5.1.) For $\pi \in \mathbb{P}(X)$ denote by $\varkappa(\pi)$ the least cardinal γ for which πX is Hamel γ-homogeneous. Say that X is *strictly Hamel γ-homogeneous* whenever X is Hamel γ-homogeneous and $\varkappa(\pi) = \gamma$ for all nonzero $\pi \in \mathbb{P}(X)$.

4.6.9. Theorem. *Let X be a universally complete vector lattice. There is a band X_0 in X such that $X_0^\perp$ is locally one-dimensional and there exists a partition of unity $(\pi_\gamma)_{\gamma\in\Gamma}$ in $\mathbb{P}(X_0)$ with Γ a set of infinite cardinals such that $\pi_\gamma X_0$ is strictly Hamel γ-homogeneous for all $\gamma \in \Gamma$.*

$\triangleleft$ Assume that $X = \mathscr{Y}{\downarrow}$ with $\mathbb{B} = \mathbb{P}(X)$. Put $b_0 := [\![\mathscr{R} \neq \mathbb{R}^\wedge]\!]$ and $X_0 := (b_0 \wedge \mathscr{R}){\downarrow}$; see 2.3.6. Then $b_0^* = [\![\mathscr{R} = \mathbb{R}^\wedge]\!]$ and $X_0^\perp = (b_0^* \wedge \mathscr{R}){\downarrow}$, so that the band $X_0^\perp$ is locally one-dimensional by 4.6.2 (1, 2). Next we can assume by passing to the model $\mathbb{V}^{([\mathbb{0},b_0])}$ that $b_0 = \mathbb{1}$ and $X_0 = X$. Thus by 1.3.7 we have $[\![\mathscr{R} \neq \mathbb{R}^\wedge]\!] = \mathbb{1}$ and therefore

$[\![\mathscr{R}$ is an infinite dimensional vector space over $\mathbb{R}^\wedge]\!] = \mathbb{1}$; i.e., the algebraic dimension of $\mathscr{R}$ is an infinite cardinal, say α, within $\mathbb{V}^{(\mathbb{B})}$. By 1.9.11 there exists a set Γ of infinite cardinals and a partition of unity $(b_\gamma)_{\gamma\in\Gamma}$ such that $b_\gamma \leqslant [\![\alpha = \gamma^\wedge]\!]$ for all $\gamma \in \Gamma$. It follows that $b_\gamma \leqslant [\![\gamma^\wedge$ is the algebraic dimension $\dim_{\mathbb{R}^\wedge}(\mathscr{R})$ of $\mathscr{R}$ over $\mathbb{R}^\wedge]\!]$. Put $\pi_\gamma := \chi(b_\gamma)$. Again, passing to the model $\mathbb{V}^{([\mathbb{0},b_\gamma])}$ and making use of 1.3.7 and 2.3.6, we find that $b_\gamma X = (b_\gamma \wedge \mathscr{R}){\downarrow}$ and $[\![\dim_{\mathbb{R}^\wedge}(b_\gamma \wedge \mathscr{R}) = \gamma^\wedge]\!] = \mathbb{1}$, so that we can assume $X = \pi_\gamma X$ and $[\![\dim_{\mathbb{R}^\wedge}(\mathscr{R}) = \gamma^\wedge]\!] = \mathbb{1}$.

Let $\mathscr{E}$ be a Hamel basis for $\mathscr{R}$ over $\mathbb{R}^\wedge$ and let $\sigma : \gamma^\wedge \to \mathscr{E}$ be a bijection within $\mathbb{V}^{(\mathbb{B})}$. Then the modified descent $s := \sigma{\downarrow} : \gamma \to \mathscr{E}{\downarrow}$ is an injection. Put $\mathscr{E}_0 := \sigma{\downarrow}(\gamma)$ and by $s := \sigma{\downarrow}|_\gamma$. Clearly, $s : \gamma \to \mathscr{E}_0$ is a bijection and it remains to ensure that $\mathscr{E}_0$ is a local Hamel basis in X consisting of weak order units pairwise distinct at I_X.

Since no Hamel basis contains the zero element, we have

$$\mathbb{1} = [\![(\forall\,\beta \in \gamma^\wedge)(\sigma(\beta) \neq 0)]\!] = \bigwedge_{\beta\in\gamma} [\![\sigma(\beta^\wedge) \neq 0^\wedge]\!] = \bigwedge_{\beta\in\gamma} [\![s(\beta) \neq 0]\!].$$

It follows that $e_{s(\beta)} = [\![s(\beta) \neq 0]\!] = \mathbb{1}$ and so $s(\beta)$ is a weak order unit for all $\beta < \gamma$. Similarly, interpreting in $\mathbb{V}^{(\mathbb{B})}$ the fact that σ is one-to-one and using the equivalence

$$\beta_1 \neq \beta_2 \Longleftrightarrow [\![\beta_1^\wedge \neq \beta_2^\wedge]\!] = \mathbb{1},$$

we deduce

$$\begin{aligned}
\mathbb{1} =& [\![(\forall\,\beta_1, \beta_2 \in \gamma^\wedge)(\beta_1 \neq \beta_2 \leftrightarrow \sigma(\beta_1) \neq \sigma(\beta_2))]\!] \\
=& \bigwedge_{\beta_1,\beta_2\in\gamma} [\![\beta_1^\wedge \neq \beta_2^\wedge]\!] \Leftrightarrow [\![\sigma(\beta_1^\wedge) \neq \sigma(\beta_2^\wedge)]\!] \\
=& \bigwedge \big\{ [\![|s(\beta_1) - s(\beta_2)| \neq 0]\!] :\ \beta_1, \beta_2 \in \gamma, \beta_1 \neq \beta_2 \big\}.
\end{aligned}$$

Thus, $|s(\beta_1) - s(\beta_2)|$ is a weak order unit in X, since $e_{|s(\beta_1)-s(\beta_2)|} = [\![|s(\beta_1) - s(\beta_2)| \neq 0]\!] = \mathbb{1}$ for all $\beta_1, \beta_2 \in \gamma$, $\beta_1 \neq \beta_2$.

Thus, $\pi_\gamma X_0$ is Hamel γ-homogeneous. To complete the proof we have to ensure that $\pi_\gamma X_0$ is strictly Hamel γ-homogeneous. This is immediate from the following: $\pi_\gamma X_0$ is strictly γ-homogeneous if and only if $b_\gamma \leqslant [\![\dim(\mathscr{X}) = \gamma^\wedge]\!]$. The latter can be proved as in [228, Theorem 8.3.11]. $\triangleright$

4.7. σ-Distributive Boolean Algebras

In this section we demonstrate that a universally complete vector lattice is locally one-dimensional if and only if the Boolean algebra of its band projections is σ-distributive; moreover, such vector lattice may be chosen purely nonatomic.

4.7.1. Let $\mathbb{B}$ be an arbitrary Boolean algebra. A subset of $\mathbb{B}$ with supremum unit is called a *cover* of $\mathbb{B}$. The partitions of unity in $\mathbb{B}$ are referred to as partitions of $\mathbb{B}$ for brevity. Let C be a cover of $\mathbb{B}$. A subset C_0 of $\mathbb{B}$ is said to be *refined* from C if, for each $c_0 \in C_0$, there exists $c \in C$ such that $c_0 \leqslant c$. An element $b \in \mathbb{B}$ is *refined* from C provided that $\{b\}$ is refined from C; i.e., $b \leqslant c$ for some $c \in C$. If $(C_n)_{n\in\mathbb{N}}$ is a sequence of covers of $\mathbb{B}$ and $b \in \mathbb{B}$ is refined from each of the covers C_n $(n \in \mathbb{N})$, then we say that b is refined from $(C_n)_{n\in\mathbb{N}}$. We also refer to a cover whose all elements are refined from $(C_n)_{n\in\mathbb{N}}$ as refined from the sequence.

4.7.2. *Let $\mathbb{B}$ be a σ-complete Boolean algebra. The following are equivalent:*

(1) *$\mathbb{B}$ is σ-distributive.*

(2) *There is a (possibly, uncountable) cover refined from each sequence of countable covers of $\mathbb{B}$.*

(3) *There is a (possibly, infinite) cover refined from each sequence of finite covers of $\mathbb{B}$.*

(4) *There is a cover refined from each sequence of two-element partitions of $\mathbb{B}$.*

$\lhd$ A proof of (1) $\Longleftrightarrow$ (2) can be found in Sikorski [365, 19.3]). Item (4) is a paraphrase of 1.9.12 (3) in the definition of σ-distributivity. The implications (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) are obvious. $\rhd$

4.7.3. *Let $\mathbb{B}$ be a complete Boolean algebra. The following are equivalent:*

(1) *$\mathbb{B}$ is σ-distributive.*

(2) *There is a (possibly, uncountable) partition refined from each sequence of countable partitions of $\mathbb{B}$.*

(3) *There is a (possibly, infinite) partition refined from each sequence of finite partitions of $\mathbb{B}$.*

(4) *There is a partition refined from each sequence of two-element partitions of* $\mathbb{B}$.

$\triangleleft$ The claim follows from 4.7.2 in view of the exhaustion principle. $\triangleright$

4.7.4. Let Q stand for the Stone space of $\mathbb{B}$ and denote by $\operatorname{Clop}(Q)$ the Boolean algebra of all clopen sets in Q. We say that a function $g \in C_\infty(Q)$ is *refined from a cover* C of the Boolean algebra $\operatorname{Clop}(Q)$ if, for every two points $q', q'' \in Q$ satisfying the equality $g(q') = g(q'')$, there exists an element $U \in C$ such that $q', q'' \in U$. If $(C_n)_{n\in\mathbb{N}}$ is a sequence of covers of $\operatorname{Clop}(Q)$ and a function g is refined from each of the covers C_n ($n \in \mathbb{N}$), then we say that g is *refined from* $(C_n)_{n\in\mathbb{N}}$.

4.7.5. *There is a function of* $C(Q)$ *refined from each sequence of finite covers of* $\operatorname{Clop}(Q)$.

$\triangleleft$ Let $(C_n)_{n\in\mathbb{N}}$ be a sequence of finite covers of $\operatorname{Clop}(Q)$. By induction, it is easy to construct a sequence of partitions $P_m = \{U_1^m, U_2^m, \dots, U_{2^m}^m\}$ of $\operatorname{Clop}(Q)$ with the following properties:

(1) for every $n \in \mathbb{N}$, there is $m \in \mathbb{N}$ such that the partition P_m is refined from C_n;

(2) $U_j^m = U_{2j-1}^{m+1} \vee U_{2j}^{m+1}$ for all $m \in \mathbb{N}$ and $j \in \{1, 2, \dots, 2^m\}$.

Given $m \in \mathbb{N}$, define the $\mathbb{2}$-valued function $\chi_m \in C(Q)$ as follows:

$$\chi_m := \sum_{i=1}^{2^{m-1}} \chi(U_{2i}^m),$$

where $\chi(U)$ is the *characteristic function* of $U \subset Q$ also denoted by 1_U in the sequel. Since the series $\sum_{m=1}^\infty \frac{1}{3^m}\chi_m$ is uniformly convergent, its sum g belongs to $C(Q)$. We will show that g is refined from $(C_n)_{n\in\mathbb{N}}$. By property (1) of the sequence $(P_m)_{m\in\mathbb{N}}$, it suffices to establish that g is refined from $(P_m)_{m\in\mathbb{N}}$.

Assume the contrary and consider the least $m \in \mathbb{N}$ such that g is not refined from P_m. In this case, there are two points $q', q'' \in Q$ satisfying the equality $g(q') = g(q'')$ and belonging to distinct elements of P_m. Since g is refined from P_{m-1} (for $m > 1$), from property (2) of the sequence $(P_m)_{m\in\mathbb{N}}$ it follows that q' and q'' belong to some adjacent elements of P_m, i.e. elements of the form U_j^m and U_{j+1}^m, with $j \in \{1, \dots, 2^m - 1\}$. For definiteness, suppose that q' belongs to an element with an even index and q'', to that with an odd index; i.e., $\chi_m(q') = 1$ and $\chi_m(q'') = 0$. Since $\chi_i(q') = \chi_i(q'')$ for all

$i \in \{1, \dots, m-1\}$; therefore,

$$
\begin{aligned}
g(q') - g(q'') &= \frac{1}{3^m} + \sum_{i=m+1}^{\infty} \frac{1}{3^i}\bigl(\chi_i(q') - \chi_i(q'')\bigr) \\
&\geqslant \frac{1}{3^m} - \sum_{i=m+1}^{\infty} \frac{1}{3^i} = \frac{1}{2 \cdot 3^m} > 0,
\end{aligned}
$$

which contradicts the equality $g(q') = g(q'')$. $\triangleright$

4.7.6. Theorem. *A universally complete vector lattice X is locally one-dimensional if and only if the complete Boolean algebra $\mathbb{P}(X)$ is σ-distributive.*

$\triangleleft$ Let Q be the Stone space of the Boolean algebra $\mathbb{P}(X)$. Suppose that X is locally one-dimensional and consider an arbitrary sequence $(P_n)_{n\in\mathbb{N}}$ of finite partitions of $\mathrm{Clop}(Q)$. By 4.7.3, to prove the σ-distributivity of X, it suffices to refine a cover of $\mathrm{Clop}(Q)$ from $(P_n)_{n\in\mathbb{N}}$. By 4.7.5, we can refine $g \in C_\infty(Q)$ from the sequence $(P_n)_{n\in\mathbb{N}}$. Since X is locally one-dimensional, there exists a partition $(U_\xi)_{\xi\in\Xi}$ of $\mathrm{Clop}(Q)$ such that g is constant on each of the sets U_ξ. Show that $(U_\xi)_{\xi\in\Xi}$ is refined from $(P_n)_{n\in\mathbb{N}}$. To this end, fix arbitrary indices $\xi \in \Xi$ and $n \in \mathbb{N}$ and establish that U_ξ is refined from P_n. We may assume that $U_\xi \neq \varnothing$. Let q_0 be an element of U_ξ. Finiteness of P_n allows us to find an element U of P_n such that $q_0 \in U$. It remains to observe that $U_\xi \subset U$. Indeed, if $q \in U_\xi$ then $g(q) = g(q_0)$ and, since g is refined from P_n, the points q and q_0 belong to the same element of P_n; i.e., $q \in U$.

Assume now that the Boolean algebra $\mathbb{P}(X)$ is σ-distributive and consider an arbitrary $g \in C_\infty(Q)$. By the definition of locally one-dimensional vector lattice, it suffices to construct a partition $(U_\xi)_{\xi\in\Xi}$ of $\mathrm{Clop}(Q)$ such that g is constant on each of the sets U_ξ. Given a natural n and integer m, denote by U_m^n the interior of the closure of the set of all points $q \in Q$ for which $\frac{m}{n} \leqslant g(q) < \frac{m+1}{n}$ and put $P_n := \bigl\{U_m^n : m \in \mathbb{Z}\bigr\}$. By 4.7.3, from the sequence $(P_n)_{n\in\mathbb{N}}$ of countable partitions of $\mathrm{Clop}(Q)$, we can refine some partition $(U_\xi)_{\xi\in\Xi}$. Clearly, this is a desired partition. $\triangleright$

4.7.7. Theorem. *There exists a purely nonatomic σ-distributive complete Boolean algebra. There exists a purely nonatomic locally one-dimensional universally complete vector lattice.*

$\triangleleft$ According to 4.7.6 we have only to prove the existence of a purely nonatomic σ-distributive complete Boolean algebra. An algebra of this kind is constructed below in 4.7.9 and 4.7.10. $\triangleright$

4.7.8. A Boolean algebra $\mathbb{B}$ is *σ-inductive* provided that each decreasing sequence of nonzero elements of $\mathbb{B}$ has a nonzero lower bound. A subalgebra $\mathbb{B}_0$ of $\mathbb{B}$ is *dense* if, for every nonzero $b \in \mathbb{B}$, there exists a nonzero element $b_0 \in \mathbb{B}_0$ such that $b_0 \leqslant b$.

As is well known, to every Boolean algebra $\mathbb{B}$ there is a complete Boolean algebra $\widehat{\mathbb{B}}$ including $\mathbb{B}$ as a dense subalgebra (cp. Sikorski [365, Section 35]). This $\widehat{\mathbb{B}}$ is unique up to isomorphism and called a *completion* of $\mathbb{B}$. Obviously, a completion of a purely nonatomic Boolean algebra is purely nonatomic. Moreover, the following lemma tells us that a completion of a σ-inductive algebra is σ-distributive.

4.7.9. *If a σ-complete Boolean algebra $\mathbb{B}$ has a σ-inductive dense subalgebra then $\mathbb{B}$ is σ-distributive.*

$\triangleleft$ Let $\mathbb{B}_0$ be a σ-inductive dense subalgebra of $\mathbb{B}$. Consider an arbitrary sequence $(C_n)_{n\in\mathbb{N}}$ of countable covers of $\mathbb{B}$, denote by C the set of all elements in $\mathbb{B}$ that are refined from $(C_n)_{n\in\mathbb{N}}$, and assume by way of contradiction that C is not a cover of $\mathbb{B}$. Then there is a nonzero element $b \in \mathbb{B}$ disjoint from all elements of C.

By induction, we construct the sequences $(b_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$ as follows: Let c_1 be an element of C_1 such that $b \wedge c_1 \neq 0$. Since $\mathbb{B}_0$ is dense, there is an element $b_1 \in \mathbb{B}_0$ such that $0 < b_1 \leqslant b \wedge c_1$. Suppose that b_n and c_n are already constructed. Let c_{n+1} be an element of C_{n+1} such that $b_n \wedge c_{n+1} \neq 0$. As b_{n+1} we take an arbitrary element of $\mathbb{B}_0$ such that $0 < b_{n+1} \leqslant b_n \wedge c_{n+1}$.

Thus, we have constructed sequences $(b_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$ such that $b_n \in \mathbb{B}_0$, $b_n \leqslant c_n \in C_n$ and $0 < b_{n+1} \leqslant b_n \leqslant b$ for all $n \in \mathbb{N}$. Since $\mathbb{B}_0$ is σ-inductive, $\mathbb{B}_0$ contains a nonzero element b_0 that satisfies $b_0 \leqslant b_n$ for all $n \in \mathbb{N}$. By the inequalities $b_0 \leqslant c_n$, we see that b_0 is refined from $(C_n)_{n\in\mathbb{N}}$; i.e., b_0 belongs to C. On the other hand, $b_0 \leqslant b$, which contradicts the disjointness of b from all elements of C. $\triangleright$

4.7.10. *Let $\mathbb{B}$ be the quotient Boolean algebra $\mathscr{P}(\mathbb{N})/\mathscr{I}$ where $\mathscr{I}$ is the ideal of $\mathscr{P}(\mathbb{N})$ comprising all finite subsets of $\mathbb{N}$. Then the completion $\widehat{\mathbb{B}}$ of $\mathbb{B}$ is purely nonatomic and σ-inductive.*

$\triangleleft$ In view of 4.7.9 we have to prove that $\mathbb{B}$ is σ-distributive. The pure nonatomicity of $\mathbb{B}$ is obvious. In order to prove that $\mathbb{B}$ is σ-inductive, it suffices to consider an arbitrary decreasing sequence $(b_n)_{n\in\mathbb{N}}$ of infinite subsets of $\mathbb{N}$ and construct an infinite subset $b \subset \mathbb{N}$ such that the difference $b\backslash b_n$ is finite for each $n \in \mathbb{N}$. We can easily obtain the desired set $b := \{m_n : n \in \mathbb{N}\}$ by induction, letting $m_1 := \min b_1$ and $m_{n+1} := \min\{m \in b_{n+1} : m > m_n\}$. $\triangleright$

4.7.11. *Let (Ω, Σ, μ) be a Maharam measure space. The Boolean algebra $\mathbb{B} := \mathbb{B}(\Omega, \Sigma, \mu) := \Sigma/\mu^{-1}(0)$ of measurable sets modulo negligible sets is σ-distributive if and only if $\mathbb{B}$ is atomic $\big($and so isomorphic to the boolean $\mathscr{P}(A)$ of a nonempty set $A\big)$.*

$\triangleleft$ Indeed, suppose that $\mathbb{B}$ is not atomic. By choosing a nonzero atomless coset $b_0 \in \mathbb{B}$ of finite measure, taking an instance $B_0 \in b_0$, and replacing (Ω, Σ, μ) with $(B_0, \Sigma_0, \mu|_{\Sigma_0})$, where $\Sigma_0 = \{B \cap B_0 : B \in \Sigma\}$, we can assume that μ is finite and $\mathbb{B}$ is atomless. Define a strictly positive countably additive function $\nu : \mathbb{B} \to \mathbb{R}$ by $\nu(b) = \mu(B)$, where $b \in \mathbb{B}$ is the coset of $B \in \Sigma$. Since every finite atomless measure admits halving, by induction it is easy to construct a sequence of finite partitions $P_m := \{b_1^m, b_2^m, \dots, b_{2^m}^m\}$ of $\mathbb{1} \in \mathbb{B}$ with $\mathbb{1} = b_1^1 \vee b_2^1$, $\nu(b_1^1) = \nu(b_2^1)$, and $b_j^m = b_{2j-1}^{m+1} \vee b_{2j}^{m+1}$, $\nu(b_{2j-1}^{m+1}) = \nu(b_{2j}^{m+1})$, for all $m \in \mathbb{N}$ and $j \in \{1, 2, \dots, 2^m\}$. Since $\nu(b_j^m) \to 0$ as $m \to \infty$ for each j, there is no partition refined from $(P_m)_{m \in \mathbb{N}}$. It remains to refer to 4.7.3 (1, 3). $\triangleright$

4.8. Band Preserving Projections

In this section we describe the band preserving projection operators on a Dedekind complete vector lattice. First we expatiate on the concept of component (see 2.1.8).

4.8.1. Let X be a vector lattice and $u \in X$. An element $v \in X$ is said to be a *component* or *fragment* of u if $|v| \wedge |u - v| = 0$. The collection of all components of u is denoted by $\mathbb{C}(u)$. This notation agrees with that in 2.1.8, since $\mathbb{C}(u) \subset X_+$ whenever $u \geqslant 0$. A subset $X_0 \subset X$ is called *componentwise closed* in X if $\mathbb{C}(u)$ is contained in X_0 for each $u \in X_0$. If X has the principal projection property then $\mathbb{C}(u) = \{\pi u : \pi \in \mathbb{P}(X)\}$. Thus, in this event, X_0 is componentwise closed in X if and all if X is invariant under each band projection, i.e., if $\pi(X_0) \subset X_0$ for all $\pi \in \mathbb{P}(X)$.

Let X be a vector lattice with the principal projection property. A projection P on X is band preserving if and only if $\ker(P)$ and $\operatorname{im}(P)$ are componentwise closed sublattices of X.

◁ By 4.1.6 P is band preserving precisely when $\ker(P)$ and $\operatorname{im}(P)$ are invariant under all band projections. But the latter is equivalent to saying that $\ker(P)$ and $\operatorname{im}(P)$ are componentwise closed. Thus, the claim is true if the componentwise closed sublattices are replaced by componentwise closed subspaces. To complete the proof observe that if a vector lattice has the projection property then for all $x, y \in X$ the representations $x \vee y = \pi x + (I_X - \pi)y$ and $x \wedge y = \pi y + (I_X - \pi)x$ hold with $\pi := \sup\{\rho \in \mathbb{P}(X) : \rho x \geqslant \rho y\}$. Thus, every pair of elements $x, y \in X$ lies in a subspace together with $x \vee y$ and $x \wedge y$, as the latter are the sums of components of x and y. ▷

4.8.2. *Let P be a band preserving linear operator on a vector lattice X. Assume that $X^\lambda = \mathscr{X}{\downarrow}$ for a vector subspace $\mathscr{X}$ of $\mathscr{R}_{\mathbb{R}}$ and $p = P^\lambda{\uparrow}$. Then P is a projection if and only if so is p within $\mathbb{V}^{(\mathbb{B})}$.*

◁ Observe that $(P \circ P)^\lambda = P^\lambda \circ P^\lambda$. Indeed, given a family (x_ξ) in X and a partition of unity (π_ξ) in $\mathbb{P}(X^\lambda)$ with $\pi_\xi x = \pi_\xi x_\xi$ for all ξ, we have $\pi_\xi P^\lambda x = \pi_\xi P x_\xi$ by definition of P^λ. Considering that P^λ commutes with all band projections in X^λ, we can write

$$\begin{aligned}\pi_\xi(P^\lambda \circ P^\lambda)x = P^\lambda(\pi_\xi P^\lambda x) = P^\lambda(\pi_\xi P x_\xi) \\ = \pi_\xi P^\lambda(P x_\xi) = \pi_\xi(P \circ P)x_\xi,\end{aligned}$$

so that the required relation follows from the definition of P^λ.

It remains to note that the relations $P^\lambda \circ P^\lambda = P^\lambda$ and $[\![p \circ p = p]\!] = \mathbb{1}$ are equivalent, since $P^\lambda = p{\downarrow}$, $P^\lambda \circ P^\lambda = (p \circ p){\downarrow}$, and $[\![(P^\lambda \circ P^\lambda){\uparrow} = p \circ p]\!] = \mathbb{1}$ according to 1.6.4, 1.5.5 (1), and 1.6.6. ▷

4.8.3. *Let X be a laterally complete vector lattice, and let $\mathscr{X} \in \mathbb{V}^{(\mathbb{B})}$ be the Boolean valued representation of X with $\mathbb{B} := \mathbb{P}(X)$. Assume that $\operatorname{Vec}(\mathscr{X})$ stands for the collection of all $\mathscr{X}_0 \in \mathbb{V}^{(\mathbb{B})}$ such that $[\![\mathscr{X}_0$ is a vector subspace of $\mathscr{X}$ (over $\mathbb{R}^\wedge$)$]\!] = \mathbb{1}$ and $\operatorname{Lat}(X)$ stands for the set of vector sublattices of X which are componentwise closed and laterally complete. Then the mapping $\mathscr{X}_0 \mapsto \mathscr{X}_0{\downarrow}$ is a one-to-one correspondence from $\operatorname{Vec}(\mathscr{X}){\downarrow}$ onto $\operatorname{Lat}(X)$.*

◁ This is immediate from 2.5.3 and 1.6.6. ▷

4.8.4. *Let $\mathbb{P}$ be a subfield of $\mathbb{R}$ and let $\mathscr{X}$ be a subspace of $\mathbb{R}_{\mathbb{P}}$. The following are equivalent:*

(1) *$\mathscr{X} = \mathbb{P}e$ for some $0 \neq e \in \mathscr{X}$; i.e., $\mathscr{X}$ is one-dimensional.*

(2) *There are no $\mathbb{P}$-subspaces in $\mathscr{X}$ other than $\{0\}$ and $\mathscr{X}$.*

(3) *There are no $\mathbb{P}$-linear projection on $\mathscr{X}$ other then* 0 *and* $I_{\mathscr{X}}$.

(4) *All $\mathbb{P}$-linear projections on $\mathscr{X}$ commute.*

(5) *The composite of two $\mathbb{P}$-linear projections on $\mathscr{X}$ is a $\mathbb{P}$-linear projection as well.*

$\triangleleft$ The implications (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) $\Longrightarrow$ (5) are trivial. To ensure the remaining implication (5) $\Longrightarrow$ (1), assume that $\mathscr{X}$ is not one-dimensional; i.e., a Hamel basis $\mathscr{E}$ for $\mathscr{X}$ contains at least two members $e_1, e_2 \in \mathscr{E}$. Define the two projections p and q in $\mathscr{X}$ by putting $p(e_1) = p(e_2) = (e_1 + e_2)/2$, $q(e_1) = e_1$, $q(e_2) = 0$, and $p(e) = q(e) = 0$ for all $e \in \mathscr{E} \setminus \{e_1, e_2\}$. Then p and q do not commute, since $p(q(e_1)) = (e_1 + e_2)/2$ and $q(p(e_1)) = e_1/2$. $\triangleright$

4.8.5. Theorem. *For a laterally complete vector lattice X the following are equivalent:*

(1) *X is locally one-dimensional.*

(2) *Each laterally complete componentwise closed sublattice in X is a band.*

(3) *Each band preserving projection on X is a band projection.*

(4) *All band preserving projections on X commute.*

(5) *The composite of two band preserving projections on X is a projection.*

$\triangleleft$ There is no loss of generality in assuming that $X = \mathscr{X}{\downarrow}$ with $\mathscr{X}$ a subspace of $\mathscr{R}_{\mathbb{R}}$ within $\mathbb{V}^{(\mathbb{B})}$, $\mathbb{B} := \mathbb{P}(X)$. By transfer we can apply 4.8.4 within $\mathbb{V}^{(\mathbb{B})}$ on replacing $\mathbb{P}$ by $\mathbb{R}^{\wedge}$ and $\mathbb{R}$ by $\mathscr{R}$. The rest follows from 4.6.2 (1, 2), 4.8.2, and 4.8.3. $\triangleright$

4.8.6. Corollary. *Let X be a universally complete vector lattice which is not locally one-dimensional. Then there exists a projection operator P on X such that P commutes with all band projections but, nevertheless, P is not a band projection.*

$\triangleleft$ This follows from the equivalence (1) $\Longleftrightarrow$ (2) in 4.8.5, since P is band preserving if and only if P commutes with all band projections; see 4.1.6. $\triangleright$

4.8.7. *Let X be a vector lattice with the principal projection property and let $T : X \to X$ be a band preserving operator. For a disjoint family $(y_\xi)_{\xi\in\Xi}$ in $(\mathrm{im}(T))_+$ there exists a disjoint family $(x_\xi)_{\xi\in\Xi}$ in X_+ such that $y_\xi = Tx_\xi$ for all $\xi \in \Xi$.*

◁ Observe that if $Tu \geqslant 0$ for some $u \in X$ then $Tu^- = 0$. Indeed, a band preserving operator is disjointness preserving, so that $Tu^+ \perp Tu^-$ and so $Tu^+ - Tu^- = Tu \geqslant 0$ implies $Tu^- = 0$. Now, given a disjoint family $(y_\xi)_{\xi\in\Xi}$ in $(\mathrm{im}(T))_+$, for every $\xi \in \Xi$ choose $u_\xi \in X$ with $y_\xi = Tu_\xi$ and put $x_\xi := \pi_\xi u_\xi^+$ with $\pi := [y_\xi]$. Then $(x_\xi)_{\xi\in\Xi}$ is a disjoint family in X_+ and, by 4.1.6, $y_\xi = \pi P u_\xi = P\pi_\xi u_\xi^+ = Px_\xi$ for all $\xi \in \Xi$. ▷

4.8.8. *Let X be a laterally complete vector lattice. A subspace X_0 of X is the range of a band preserving projection operator if and only if X_0 is componentwise closed and laterally complete sublattice. Moreover, in this event there exists a componentwise closed and laterally complete sublattice $X_1 \subset X$ such that $X = X_0 \oplus X_1$.*

◁ If X_0 is a componentwise closed and laterally complete sublattice of X then, in view of 4.8.3, $X_0 = \mathscr{X}_0{\downarrow}$ for some vector subspace $\mathscr{X}_0 \subset \mathscr{X}$ within $\mathbb{V}^{(\mathbb{B})}$. Working within $\mathbb{V}^{(\mathbb{B})}$ choose some complementary subspace $\mathscr{X}_1 \subset \mathscr{X}$ and let p be a projection on $\mathscr{X}$ with $\mathrm{im}(p) = \mathscr{X}_0$ and $\ker(p) = \mathscr{X}_1$. By 4.8.2 $P := p{\downarrow}$ is a band preserving projection and $\mathrm{im}(P) = \mathrm{im}(p){\downarrow} = \mathscr{X}_0{\downarrow} = X_0$.

Conversely, assume that $X_0 = P(X)$ for some band preserving projection P on X. By 4.8.1 X_0 is componentwise closed. To show that X_0 is laterally complete take a disjoint family $(y_\xi)_{\xi\in\Xi}$ in $(X_0)_+$ and, using 4.8.7, choose a disjoin family $(x_\xi)_{\xi\in\Xi}$ in X_+ such that $y_\xi = Px_\xi$ for all $\xi \in \Xi$. As X is laterally complete, there exists $x := \sup_{\xi\in\Xi} x_\xi$. Clearly, $y = Px$ is the least upper bound of the family $(y_\xi)_{\xi\in\Xi}$, since $\pi_\xi y = P\pi_\xi x = P\pi_\xi x_\xi = y_\xi$ for all $\xi \in \Xi$.

It remains to observe that $X = X_0 \oplus X_1$, whenever $X_1 := \mathscr{X}_1{\downarrow}$ and $\mathscr{X}_1$ is an (arbitrary) complementary subspace of $\mathscr{X}$ within $\mathbb{V}^{(\mathbb{B})}$. ▷

4.8.9. *Let X be a Dedekind complete vector lattice. The following are equivalent:*

(1) *Each principal band in X is universally complete.*

(2) *For each $x \in X_+$, for each disjoint sequence (x_n) in $\mathbb{C}(x)$, and for each sequence (λ_n) in $\mathbb{R}_+$ there exists in X the element*

$$\sum_{n=1}^{\infty} \lambda_n x_n = \sup_{m\in\mathbb{N}} \sum_{n=1}^{m} \lambda_n x_n.$$

◁ Only the implication (2) $\Longleftrightarrow$ (1) is nontrivial. Assume that (2) is fulfilled and verify that for an arbitrary $e \in X_+$ the band $B := \{e\}^{\perp\perp}$ is universally complete. Take $0 \leqslant x \in B^{\mathrm{u}}$ and let $(e^x_\lambda)_{\lambda\in\mathbb{R}}$ stands for the spectral system of x with respect to e (considered as a unit element in B^{u}). Fix a partition of the real line $\beta := (t_n)_{n\in\mathbb{Z}}$; i.e., $t_n < t_{n+1}$ $(n \in \mathbb{N})$ and $\lim_{n\to\pm\infty} t_n = \pm\infty$. Observe that $\underline{x}(\beta) \leqslant x \leqslant \overline{x}(\beta)$ where

$$\underline{x}(\beta) := \sum_{n\in\mathbb{Z}} t_n(e^x_{t_{n+1}} - e^x_{t_n}), \quad \underline{x}(\beta) := \sum_{n\in\mathbb{Z}} t_{n+1}(e^x_{t_{n+1}} - e^x_{t_n}).$$

By (2) we have $\underline{x}(\beta), \overline{x}(\beta) \in X \cap B^{\mathrm{u}} = B$ and hence $B^{\mathrm{u}} \subset B$. $\triangleright$

A Dedekind complete vector lattice X satisfying any of the equivalent conditions in 4.8.9 is called *principally universally complete.*

4.8.10. *A projection P on a principally universally complete vector lattice X is band preserving if and only the following hold:*

(1) $\ker(P)$ *and* $\mathrm{im}(P)$ *are componentwise closed.*

(2) *For every principal band B in X the intersections $B\cap\ker(P)$ and $B \cap \mathrm{im}(P)$ are laterally complete.*

$\triangleleft$ According to 4.8.8 the above conditions (1) and (2) are equivalent to saying that the restriction of P to every principal band is band preserving. In particular, $Px \in \{x\}^{\perp\perp}$ for all $x \in X$. From this it is immediate that $P(B) \subset B$ for every band $B \in \mathbb{B}(X)$, because $x \in B$ implies $Px \in \{x\}^{\perp\perp} \subset B$. $\triangleright$

4.8.11. Theorem. *Let X be a Dedekind complete vector lattice and let P be a band preserving projection operator on X. Then there exists a unique pair of complimentary bands X_1 and X_2 such that the following hold:*

(1) *X_1 is the maximal band such that the restriction of P to X_1 is order bounded and, in particular, $P|_{X_1}$ is a band projection.*

(2) *X_2 principally universally complete and the restriction $P|_{X_2}$ is described as in* 4.8.10.

$\triangleleft$ Take $X = Y$ in 4.3.10 and put $X_1 := B$ and $X_2 := B^\perp$. In view of 4.8.10 we have only to prove that X_2 is principally universally complete. Take $0 \leqslant x \in X_2$, a disjoint sequence (x_n) of components of x, and a sequence (λ_n) of positive scalars. According to 4.3.10 for each $n \in \mathbb{N}$ we can find $y_n \in X$ such that $0 \leqslant y_n \leqslant x_n$ and $|Py_n| \geqslant n\lambda_n x_n$. Obviously, $\sum_{n=k}^m (1/n)y_n \leqslant (1/m)x$ for all $k \leqslant m \in \mathbb{N}$, and so the series $\sum_{n=1}^\infty (1/n)y_n$ converges uniformly to some $y \in Y$. Since the terms of the series are pairwise disjoint, we have $|Py| \geqslant |P((1/n)y_n)| \geqslant \lambda_n x_n$ for all $n \in \mathbb{N}$, whence $\sum_{n=1}^{\infty} \lambda_n x_n$ exists in X. Appealing to 4.8.9 completes the proof. $\triangleright$

4.9. Algebraic Band Preserving Operators

In this section a description of algebraic orthomorphisms on a vector lattice is given and the Wickstead problem for algebraic operators is examined.

4.9.1. Let $\mathbb{P}[x]$ be a ring of polynomials in variable x over a field $\mathbb{P}$. An operator T on a vector space X over a field $\mathbb{P}$ is said to be *algebraic* if there exists a nonzero $\varphi \in \mathbb{P}[x]$, a polynomials with coefficients in $\mathbb{P}$, for which $\varphi(T) = 0$.

For an algebraic operator T there exists a unique polynomial φ_T such that $\varphi_T(T) = 0$, the leading coefficient of φ_T equals to 1, and φ_T divides each polynomial ψ with $\psi(T) = 0$. The polynomial φ_T is called the *minimal polynomial* of T. The simple examples of algebraic operators yield a projection P (an idempotent operator, $P^2 = P$) in X with $\varphi_P(\lambda) = \lambda^2 - \lambda$ whenever $P \neq 0, I_X$, and a nilpotent operator S ($S^m = 0$ for some $m \in \mathbb{N}$) in X with $\varphi_S(\lambda) = \lambda^k$, $k \leqslant m$.

For an operator T on X, the set of all eigenvalues of T will be denoted throughout by $\sigma_p(T)$. A real λ is a root of φ_T if and only if $\lambda \in \sigma_p(T)$. In particular, $\sigma_p(T)$ is finite.

4.9.2. *Let X be a vector lattice and $b - a^2 > 0$ for some $a, b \in \mathbb{R}$. Then $T^2 + 2aT + bI$ is a weak order unit in* $\operatorname{Orth}(X)$ *for every $T \in$* $\operatorname{Orth}(X)$.

$\triangleleft$ Since $I := I_X$ is a weak order unit in $\operatorname{Orth}(X)$, so is $(b - a^2)I$. Moreover, in $\operatorname{Orth}(X)$ the inequalities hold:

$$0 < (b - a^2)I \leqslant (b - a^2)I + (T + aI)^2 = T^2 + 2aT + bI.$$

Consequently, $T^2 + 2aT + bI$ is a weak order unit in $\operatorname{Orth}(X)$ as well. $\triangleright$

4.9.3. *Let X be a vector lattice and let T in* $\operatorname{Orth}(X)$ *be algebraic. Then*

$$\varphi_T(x) = \prod_{\lambda \in \sigma_p(T)} (x - \lambda).$$

$\triangleleft$ We claim that there are no quadratic polynomials in the factorization of T into irreducible elements in $\mathbb{R}[X]$. Otherwise, there would exist $a, b \in \mathbb{R}$ with $b - a^2 > 0$ and a nonzero polynomial $\psi \in \mathbb{R}[X]$ such that $\varphi_T(x) = (x^2 + 2ax + b)\psi(x)$. This would entail that $(T^2 + 2aT + bI)\psi(T) = \varphi_T(T) = 0$. But $\psi(T) \in \operatorname{Orth}(X)$ and so $\psi(T) = 0$ by 4.9.2, which contradicts the minimality of φ_T. Accordingly,

$$\varphi_T(x) = \prod_{\lambda \in \sigma_p(T)} (x - \lambda)^{n_\lambda}$$

for some $n_\lambda \in \mathbb{N}$ $(\lambda \in \sigma_p(T))$. Choose n a common multiple of the collection $\{n_\lambda : \lambda \in \sigma_p(T)\}$. Obviously, φ_T divides the polynomial $\bigl(\prod_{\lambda \in \sigma_p(T)} (x - \lambda)\bigr)^n$ and therefore $\bigl(\prod_{\lambda \in \sigma_p(T)} (T - \lambda I)\bigr)^n = 0$ in $\operatorname{Orth}(X)$. Since the f-algebra $\operatorname{Orth}(X)$ is semiprime by 4.1.3, we find $\prod_{\lambda \in \sigma_p(T)} (T - \lambda I) = 0$, whence the desired identity follows. $\triangleright$

4.9.4. *Consider the universally complete vector lattice $X = \mathscr{R}{\downarrow}$. Let T be a band preserving linear operator on X and let τ be an $\mathbb{R}^\wedge$-linear function on $\mathscr{R}$. For $\varphi \in \mathbb{R}[x]$, $\varphi(x) = a_0 + a_1 x + \dots + a_n x^n$ define $\hat{\varphi} \in \mathbb{R}^\wedge[x]$ by $\hat{\varphi}(x) = a_0^\wedge + a_1^\wedge x + \dots + a_n^\wedge x^{n^\wedge}$. Then*

$$\hat{\varphi}(\tau){\downarrow} = \varphi(\tau{\downarrow}), \quad \varphi(T){\uparrow} = \hat{\varphi}(T{\uparrow}).$$

$\triangleleft$ It follows from 1.5.5 (1) and 1.6.4 that $(\tau^{n^\wedge}){\downarrow} = (\tau{\downarrow})^n$ and $(T^n){\uparrow} = (T{\uparrow})^{n^\wedge}$. It remains to apply 4.3.5. $\triangleright$

4.9.5. A linear operator T on a vector lattice X is said to be *diagonal* if $T = \lambda_1 P_1 + \dots + \lambda_m P_m$ for some collections of reals $\lambda_1, \dots, \lambda_m$ and projection operators $P_1, \dots, P_m$ on X with $P_\imath \circ P_\jmath = 0$ $(\imath \neq \jmath)$. In the equality above, we can and will assume that $P_1 + \dots + P_n = I_X$ and that $\lambda_1, \dots, \lambda_m$ are pairwise different. An algebraic operator T is diagonal if and only if the minimal polynomial of T have the form $\varphi_T(x) = (x - \lambda_1) \cdots (x - \lambda_m)$ with pairwise distinct $\lambda_1, \dots, \lambda_m \in \mathbb{R}$.

We call an operator T on X *strongly diagonal* if there exist pairwise disjoint band projections $P_1, \dots, P_m$ and reals $\lambda_1, \dots, \lambda_m$ such that $T = \lambda_1 P_1 + \dots + \lambda_m P_m$. In particular, each strongly diagonal operator on X is an orthomorphism. It is easily seen that the set of all strongly diagonal operators on X is an f-subalgebra of $\operatorname{Orth}(X)$.

4.9.6. *Let $T = \lambda_1 P_1 + \dots + \lambda_m P_m$ be a diagonal operator on a vector lattice X. Then T is band preserving if and only if the projection operators $P_1, \dots, P_m$ are band preserving.*

$\triangleleft$ The sufficiency is obvious. To prove the necessity, observe first that if T is band preserving then so is T^n for all $n \in \mathbb{N}$ and so $\varphi(T)$

is band preserving for every polynomial $\varphi \in \mathbb{R}[x]$. Next, make use of the representation $P_j = \varphi_j(T)$ $(j := 1, \dots, m)$, where $\varphi_j \in \mathbb{R}[x]$ is an interpolation polynomial defined by $\varphi_j(\lambda_k) = \delta_{jk}$ with δ_{jk} the Kronecker symbol. $\triangleright$

4.9.7. *Let X be a vector lattice. A linear operator T on X is strongly diagonal if and only if T is an algebraic orthomorphism on X.*

$\triangleleft$ The necessity follows from 4.9.5. Let T be an orthomorphism in X and $\varphi(T) = 0$, where φ is a minimal polynomial of T, so that $\varphi(\lambda) = (\lambda - \lambda_1) \cdots (\lambda - \lambda_m)$ with $\lambda_1, \dots, \lambda_m \in \mathbb{R}$. Since T admits the unique extension to an orthomorphism on $X^{\mathfrak{u}}$, we can assume without loss of generality that $X = X^{\mathfrak{u}} = \mathscr{R}{\downarrow}$ and $\tau = T{\uparrow}$. Then $[\![\tau(x) = \lambda_0 x \ (x \in \mathscr{R})]\!] = \mathbb{1}$ for some $\lambda_0 \in \mathscr{R}$. It is seen from 4.9.4 that $\hat{\varphi}(\lambda_0) = 0$ and so $(\lambda_0 - \lambda_1^{\wedge}) \cdots (\lambda_0 - \lambda_m^{\wedge}) = 0$ or $\lambda_0 \in \{\lambda_1^{\wedge}, \dots, \lambda_m^{\wedge}\}$ within $\mathbb{V}^{(\mathbb{B})}$. Put $P_l := \chi(b_l)$ with $b_l := [\![\lambda_0 = \lambda_l^{\wedge}]\!]$ and observe that $\{P_1, \dots, P_m\}$ is a partition of unity in $\mathbb{P}(X)$. Moreover, given $x \in X$, we can estimate $b_l \leqslant [\![Tx = \tau x = \lambda_0 x]\!] \wedge [\![\lambda_0 = \lambda_l^{\wedge}]\!] \leqslant [\![Tx = \lambda_l^{\wedge} x]\!]$, so that $P_l T x = P_l(\lambda_l x) = \lambda_l P_l(x)$. Summing up over $l = 1, \dots, m$, we get $Tx = \lambda_1 P_1 x + \dots + \lambda_m P_m$. $\triangleright$

4.9.8. Theorem. *Let X be a universally complete vector lattice. The following assertions are equivalent:*

(1) *The Boolean algebra $\mathbb{P}(X)$ is σ-distributive.*

(2) *Every algebraic operator in $L_{\mathrm{bp}}(X)$ is order bounded.*

(3) *Every algebraic operator in $L_{\mathrm{bp}}(X)$ is strongly diagonal.*

(4) *Every diagonal operator in $L_{\mathrm{bp}}(X)$ is strongly diagonal.*

(5) *Every projection operator in $L_{\mathrm{bp}}(X)$ is a band projection.*

(6) *Every nilpotent operator in $L_{\mathrm{bp}}(X)$ is order bounded.*

(7) *Every nilpotent operator in $L_{\mathrm{bp}}(X)$ is trivial.*

$\triangleleft$ (1) $\Longrightarrow$ (2): Follows from 4.6.4 and 4.7.7.

(2) $\Longrightarrow$ (3): Follows from 4.9.7.

(3) $\Longrightarrow$ (4): A diagonal operator is algebraic by definition (cp. 4.9.5).

(4) $\Longrightarrow$ (5): This is evident.

(5) $\Longleftrightarrow$ (1): Follows from 4.8.5 ((1) $\Longleftrightarrow$ (3)).

(2) $\Longrightarrow$ (6): A nilpotent operator is algebraic by definition.

(6) $\Longrightarrow$ (7): A nilpotent orthomorphism is trivial; i.e., the f-algebra $\mathrm{Orth}(X)$ is semiprime (cp. 4.1.3).

(7) $\Longrightarrow$ (1): Arguing for a contradiction, assume that $\mathbb{P}(X)$ is not σ-distributive and construct a nonzero band preserving nilpotent operator

in X. By 4.4.9 ((1) $\iff$ (2)) $\mathbb{V}^{(\mathbb{B})} \models \mathscr{R} \neq \mathbb{R}^\wedge$ and in this case $\mathscr{R}$ is an infinite-dimensional vector space over $\mathbb{R}^\wedge$ within $\mathbb{V}^{(\mathbb{B})}$; see 4.6.6. Let $\mathscr{E} \subset \mathscr{R}$ be a Hamel basis and choose an infinite sequence $(e_n)_{n\in\mathbb{N}}$ of pairwise distinct elements in $\mathscr{E}$. Fix a natural $m > 1$ and define an $\mathbb{R}^\wedge$-linear function $\tau : \mathscr{R} \to \mathscr{R}$ within $\mathbb{V}^{(\mathbb{B})}$ by letting $\tau(e_{km+i}) = e_{km+i-1}$ if $2 \leqslant i \leqslant m$, $\tau(e_{km+1}) = 0$ for all $k := 0, 1, \dots$, and $\tau(e) = 0$ if $e \neq e_n$ for all $n \in \mathbb{N}$. In other words, if $\mathscr{R}_0$ is the $\mathbb{R}^\wedge$-linear subspace of $\mathscr{R}$ generated by the sequence $(e_n)_{n\in\mathbb{N}}$, then $\mathscr{R}_0$ is an invariant subspace for τ and τ is the linear operators associated to the infinite block matrix $\operatorname{diag}(A, \dots, A, \dots)$ with equal blocks in the principal diagonal and A a square matrix of dimension m,

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

It follows that τ is discontinuous and $\tau^m = 0$ by construction. Consequently, $T := \tau{\downarrow}$ is a band preserving linear operator in X and $T^m = 0$ by 4.9.4, but T is not order bounded; a contradiction. $\triangleright$

4.10. Band Preserving Operators on Complex Vector Lattices

Consider some properties of band preserving operators in a complex vector lattice.

4.10.1. A vector lattice X is called *square-mean closed* if for all $x, y \in X$ the set $\{(\cos\theta)x + (\sin\theta)y : 0 \leqslant \theta < 2\pi\}$ has a supremum $\mathfrak{s}(x, y)$ in X. Every uniformly complete vector lattice is square-mean closed. But a square-mean closed Archimedean vector lattice need not be relatively uniformly complete.

Recall that a *complex vector lattice* is the complexification

$$X_{\mathbb{C}} := X \oplus iX := \{x + iy : \ x, y \in X\}$$

of a real square-mean closed vector lattice X; see 2.3.3. Thus, each element $z \in X_{\mathbb{C}}$ in a complex vector lattice has the absolute value $|z|$ defined as

$$|z| := \mathfrak{s}(x, y) \quad (z := x + iy \in X_{\mathbb{C}}).$$

Clearly, $|z| = \sqrt{x^2 + y^2}$ in the sense of homogeneous functional calculus and so $|x| \vee |y| \leqslant |z| \leqslant |x| + |y|$. The mapping $z \mapsto |z|$ of $X_{\mathbb{C}}$ to X satisfies the relations ($\lambda \in \mathbb{C}$; $z, z_1, z_2 \in X_{\mathbb{C}}$; $\bar{z} := x - iy$):

(1) $|z| \geqslant 0$; $|z| = 0 \Longleftrightarrow z = 0$;

(2) $|\lambda z| = |\lambda||z|$; $|z| = |\bar{z}|$;

(3) $|z_1 + z_2| \leqslant |z_1| + |z_2|$;

(4) $\big||z_1| - |z_2|\big| \leqslant |z_1 - z_2|$.

A subset $A \subset X_{\mathbb{C}}$ is *order bounded* if the set $\{|z| : z \in X_{\mathbb{C}}\}$ is order bounded in X. As in the real case, the notion of disjointness of elements $z := x + iy$ and $w := u + iv$ in $X_{\mathbb{C}}$ is defined by the formula $z \perp w \Longleftrightarrow |z| \wedge |w| = 0$ and is equivalent to the relation $\{x, y\} \perp \{u, v\}$. The disjoint complement $A^{\perp}$ of a nonempty set $A \subset X_{\mathbb{C}}$ is defined by $A^{\perp} := \{z \in X_{\mathbb{C}} : z \perp w$ for all $w \in A\}$. Say that $X_{\mathbb{C}}$ is Dedekind complete (σ-complete) if X is Dedekind complete (σ-complete).

4.10.2. A vector *sublattice* of $X_{\mathbb{C}}$ is a vector subspace $Y \subset X_{\mathbb{C}}$ such that $z \in Y$ implies $\bar{z} \in Y$ and $|z| \in Y$. An *ideal* J in $X_{\mathbb{C}}$ is defined as the linear subspace which is solid: $|w| \leqslant |z|$ with $w \in X_{\mathbb{C}}$ and $z \in J$ implies $w \in J$. As in the real case, a *band* in $X_{\mathbb{C}}$ can be defined as $\{z \in X_{\mathbb{C}} : (\forall\, w \in V)\, z \perp w\}$, where V is a nonempty subset of $X_{\mathbb{C}}$. The sublattices, ideals, and bands of $X_{\mathbb{C}}$ are precisely the complexifications of sublattices, ideals, and bands of X (cp. Schaefer [356, Chapter II, § 11] and Zaanen [427, Section 91] for more detail). A band B is a projection band if $X_{\mathbb{C}} = B \oplus B^{\perp}$. Each projection band B is the range of a projection P on $X_{\mathbb{C}}$ with kernel $B^{\perp}$ called a *band projection.* As in the real case $\mathbb{B}(X_{\mathbb{C}})$ and $\mathbb{P}(X_{\mathbb{C}})$ stand respectively for the Boolean algebras of all band and all band projections in $X_{\mathbb{C}}$.

4.10.3. Let X and Y be real vector spaces considered as real subspaces of $X_{\mathbb{C}}$ and $Y_{\mathbb{C}}$, respectively. Each $\mathbb{R}$-linear operator $T : X \to Y$ admits the unique extension to the $\mathbb{C}$-linear operator $T_{\mathbb{C}} : X_{\mathbb{C}} \to Y_{\mathbb{C}}$ defined as

$$T_{\mathbb{C}}(x + iy) := Tx + iTy \quad (x + iy \in X_{\mathbb{C}}).$$

The operator $T_{\mathbb{C}}$ is usually identified with T, so that the vector space $L(X, Y)$ of $\mathbb{R}$-linear operators from X to Y is viewed as a real vector subspace of $L(X_{\mathbb{C}}, Y_{\mathbb{C}})$ comprising the operators satisfying $T(X) \subset Y$.

With this agreement in mind it is easily seen that an operator $T \in L(X_{\mathbb{C}}, Y_{\mathbb{C}})$ is uniquely representable as $T = T_1 + iT_2$, where $T_1, T_2 \in$

$L(X,Y)$, that is,

$$Tz = T_1x - T_2y + i(T_2x + T_1y) \quad (z = x + iy \in X_{\mathbb{C}}).$$

Thus, the space $L(X_{\mathbb{C}}, Y_{\mathbb{C}})$ of $\mathbb{C}$-linear operators is isomorphic to the complexification of the real space $L(X,Y)$ of $\mathbb{R}$-linear operators; i.e., $L(X_{\mathbb{C}}, Y_{\mathbb{C}}) = L(X,Y)_{\mathbb{C}}$.

4.10.4. Assume now that X and Y are real vector lattices. An operator $T = T_1 + iT_2$ is *positive* provided that $T_1 \geqslant 0$ and $T_2 = 0$ and *order bounded* provided that for every $e \in X_+$ there is $f \in Y_+$ satisfying $|Tz| \leqslant f$ whenever $z \in X_{\mathbb{C}}$, $|z| \leqslant e$. The space $L^{\sim}(X_{\mathbb{C}}, Y_{\mathbb{C}})$ of all order bounded linear operators from $X_{\mathbb{C}}$ into $Y_{\mathbb{C}}$ is the complexification of the space of all order bounded linear operators from X into Y:

$$L^{\sim}(X_{\mathbb{C}}, Y_{\mathbb{C}}) = L^{\sim}(X,Y)_{\mathbb{C}} = L^{\sim}(X,Y) \oplus iL^{\sim}(X,Y).$$

An operator $T = T_1 + iT_2 \in L(X_{\mathbb{C}}, Y_{\mathbb{C}})$ is said to be *regular* if T_1 and T_2 are regular. If Y is Dedekind complete then $L^{\sim}(X_{\mathbb{C}}, Y_{\mathbb{C}})$ is also a Dedekind complete complex vector lattice. In particular, every operator $T = T_1 + iT_2$ has the modulus $|T|$ and the Riesz–Kantorovich formula holds true; i.e., for every $u \in X_+$ we have

$$|T|u = |T_1 + iT_2|u = \sup_{|z| \leqslant u} |Tz| = \sup_{|x+iy| \leqslant u} |(T_1 + iT_2)(x + iy)|.$$

A *lattice homomorphism* is an operator $T = T_1 + iT_2 \in L(X_{\mathbb{C}}, Y_{\mathbb{C}})$ with $T_2 = 0$ and T_1 a lattice homomorphism from X to Y. Clearly, T is a lattice homomorphism if and only if $|Tz| = |T|(|z|)$ for all $x \in X_{\mathbb{C}}$. It is also worth mentioning that if $P = P_1 + iP_2$ is a projection onto the band $B = B_1 + iB_2$ then $P_2 = 0$ and P_1 is a projection onto the band B. More details can be found in Abramovich and Aliprantis [5, Chapter 3], Schaefer [356, Chapter II] and Zaanen [427, Section 92].

Suppose that Y is a sublattice of a vector lattice X. A linear operator T from $Y_{\mathbb{C}}$ to $X_{\mathbb{C}}$ is *band preserving* provided that

$$z \perp w \Longrightarrow Tz \perp w \quad (z \in Y_{\mathbb{C}},\ w \in X_{\mathbb{C}}),$$

where the disjointness relations are understood in $X_{\mathbb{C}}$ (cp. 4.1.1).

4.10.5. *A linear operator $T := T_1 + iT_2$ from $Y_{\mathbb{C}}$ to $X_{\mathbb{C}}$ is band preserving if and only if such are the real linear operators T_1 and T_2 from Y to X.*

$\lhd$ Assume that T_1 and T_2 are band preserving. If $z := x + iy$ and $w := u+iv$ are disjoint then $\{x, y\} \perp \{u, v\}$. Therefore, $\{x, y\} \perp \{T_1u - T_2v, T_1v+T_2u\}$. Hence, $z \perp Tw$, since $Tw = (T_1u - T_2v) + i(T_1v + T_2u)$.

Conversely, if T is band preserving and $x \in X$ and $y \in Y$ are disjoint then $x \perp Ty = T_1y + iT_2y$ hence, $x \perp \{T_1y, T_2y\}$, so that T_1 and T_2 are band preserving. $\rhd$

4.10.6. In particular, if X is a vector lattice enjoying the principal projection property and Y is an order dense ideal of X then a linear operator $T = T_1 + iT_2 : Y_{\mathbb{C}} \to X_{\mathbb{C}}$ is band preserving if and only if $\pi T_k z = T_k \pi z$ ($z \in Y_{\mathbb{C}}$, $k = 1, 2$) for all $\pi \in \mathbb{P}(X_{\mathbb{C}})$. An order bounded band preserving operator on $X_{\mathbb{C}}$ is called an *orthomorphism* and the set of all orthomorphisms on $X_{\mathbb{C}}$ is denoted by $\mathrm{Orth}(X_{\mathbb{C}})$. Clearly, $\mathrm{Orth}(X_{\mathbb{C}})$ is the complexification of $\mathrm{Orth}(X)$; i.e., $\mathrm{Orth}(X_{\mathbb{C}}) = \mathrm{Orth}(X)_{\mathbb{C}}$.

4.10.7. Define *a complex f-algebra* to be the complexification $A_{\mathbb{C}}$ of a real square-mean closed f-algebra A (cp. 4.10.1). The multiplication on A extends naturally to $A_{\mathbb{C}}$ by the formula

$$(x + iy)(u + iv) = (xu - yv) + i(xv + yu),$$

and so $A_{\mathbb{C}}$ becomes a commutative complex algebra. Moreover, $|z_1z_2| = |z_1||z_2|$ $(z_1, z_2 \in A_{\mathbb{C}})$. In this situation $A_{\mathbb{C}}$ is called a *complex f-algebra* (cp. Beukers, Huijsmans, and de Pagter [53]; Zaanen [427]). A complex f-algebra $A_{\mathbb{C}}$ is semiprime whenever $z \perp w$ is equivalent to $zw = 0$ for all $z, w \in A_{\mathbb{C}}$.

If Z is a universally complete vector lattice with a fixed order unit $\mathbb{1} \in Z$ then there is a unique multiplication on Z which makes Z into an f-algebra and $\mathbb{1}$ into the multiplicative unity. Thus, $Z_{\mathbb{C}}$ is an example of a complex f-algebra. We will always keep this circumstance in mind while considering a universally complete vector lattice as an f-algebra.

4.10.8. Given an algebra A over a field $\mathbb{P}$ and a subalgebra A_0 of A, we call a $\mathbb{P}$-linear operator $D : A_0 \to A$ a $\mathbb{P}$-*derivation* (or simply a *derivation* if $\mathbb{P}$ is meant) provided that

$$D(uv) = D(u)v + uD(v) \quad (u, v \in A_0).$$

A $\mathbb{P}$-*endomorphism of an algebra* A is a $\mathbb{P}$-linear multiplicative operator $M : A \to A$; i.e., M is $\mathbb{P}$-linear and satisfy the equation

$$M(uv) = M(u)M(v) \quad (u, v \in A).$$

A bijective $\mathbb{P}$-endomorphism is a $\mathbb{P}$-*automorphism*. We simply speak of endomorphisms and automorphisms whenever $\mathbb{P}$ is meant.

The kernel of a derivation is a subalgebra and the kernel of an automorphism is a ring ideal. A nonzero derivation is called *nontrivial*. The identical automorphism is commonly referred to as the *trivial automorphism*. If $\mathbb{P} = \mathbb{R}$ or $\mathbb{P} = \mathbb{C}$ in the above definitions of a $\mathbb{P}$-derivation then we speak of real derivation and complex derivation, respectively.

Let Z stand for a real universally complete vector lattice with a fixed f-algebra multiplication and X be an f-subalgebra of Z.

4.10.9. *Let $D \in L(X_{\mathbb{C}}, Z_{\mathbb{C}})$ and $D = D_1 + iD_2$. The operator D is a complex derivation if and only if D_1 and D_2 are real derivations from X into Z. If X is minorizing in Z and $X^{\perp\perp} = Z$ then each derivation from $X_{\mathbb{C}}$ into $Z_{\mathbb{C}}$ is a band preserving operator.*

$\lhd$ To ensure that the first assertion holds we only have to insert $D := D_1 + iD_2$ in the equality $D(uv) = D(u)v + uD(v)$, take $u := x \in X$ and $v := y \in X$, and then equate the real and imaginary parts of the resulting relation. According to this fact and 4.10.5, it remains only to establish that every real derivation is a band preserving operator. Let $D : X \to Z$ be a real derivation. Take disjoint $x, y \in X$. Since the relation $x \perp y$ in an f-algebra implies $xy = 0$, we have $0 = D(xy) = D(x)y + xD(y)$. But the elements $D(x)y$ and $xD(y)$ are disjoint as well by the definition of f-algebra; therefore, $D(x)y = 0$ and $xD(y) = 0$. Hence, since the f-algebra X is semiprime, we obtain $D(x) \perp y$ and $x \perp D(y)$. Now, consider disjoint $x \in X$ and $z \in Z$. By hypothesis, the order ideal I generated by $(X \cap \{x\}^{\perp}) \cup \{x\}$ is order dense in Z. Therefore, without loss of generality we may assume $|z| = \sup_\alpha y_\alpha$ for some family (y_α) in X_+. We have $y_\alpha \perp D(x)$ as just proved and consequently, $z \perp D(x)$. $\rhd$

4.10.10. *Put $X := \mathscr{R}{\downarrow}$ and let $L_{\mathrm{bp}}(X_{\mathbb{C}})$ be the set of all band preserving linear operators in $X_{\mathbb{C}}$. Denote by $\mathrm{End}(\mathscr{C}_{\mathbb{C}})$ the member of $\mathbb{V}^{(\mathbb{B})}$ that depicts the $\mathbb{C}^{\wedge}$-vector space of all $\mathbb{C}^{\wedge}$-linear mappings from $\mathscr{C}$ into $\mathscr{C}$. Then the faithful unitary $X_{\mathbb{C}}$-modules $L_{\mathrm{bp}}(X_{\mathbb{C}})$ and $\mathrm{End}(\mathscr{C}_{\mathbb{C}}){\downarrow}$ are put into isomorphy by sending a band preserving operator to its ascent.*

$\lhd$ Recall that $\mathscr{C} \in \mathbb{V}^{(\mathbb{B})}$ is defined as $\mathscr{C} := \mathscr{R} \oplus i\mathscr{R}$ and by the Gordon Theorem the descent $\mathscr{C}{\downarrow} = \mathscr{R}{\downarrow} \oplus i\mathscr{R}{\downarrow}$ is a universally complete complex vector lattice and a complex f-algebra simultaneously. Moreover, $[\![\, \mathbb{C}^{\wedge} = \mathbb{R}^{\wedge} \oplus i\mathbb{R}^{\wedge}$ is a dense subfield of $\mathscr{C}\,]\!] = \mathbb{1}$. (We write i instead of $i^{\wedge}$.) It is easy to observe that

$$L_{\mathrm{bp}}(X_{\mathbb{C}}) = L_{\mathrm{bp}}(X)_{\mathbb{C}}, \quad [\![\mathrm{End}(\mathscr{C}_{\mathbb{C}}) = \mathrm{End}(\mathscr{R})_{\mathbb{C}}]\!] = \mathbb{1}.$$

The claim follows from 4.3.5 and 4.10.5. $\triangleright$

4.11. Automorphisms and Derivations on the Complexes

Here we recall the information on field theory which we need for further analysis of the two collections of simultaneous functional equations (A) and (D) in Section 4.2.

4.11.1. Consider some fields K and L. If K is a subfield of L, then L is an *extension* of K. An extension L of a field K is called *algebraic* provided that each element of L is a root of some nonzero polynomial (in a sole variable) with coefficients in K. In other words, an extension L of K is algebraic in case every $x \in L$ is algebraic over K; i.e., to each $x \in L$ there are finitely many $a_0, \dots, a_n \in K$, $n \geqslant 1$, some of them nonzero, such that $a_0 + a_1 x + \dots + a_n x^n = 0$. An extension L of K is *transcendental* over K if L is not algebraic.

Recall that a field K is *algebraically closed* provided that each nonconstant polynomial with coefficients in K has at least one root in K. In other words, K is algebraically closed if and only if every algebraic extension of K is K.

The *algebraic closure* of a field K is an extension of K that is algebraic over K and algebraically closed. It is proved in field theory that each field K has some algebraic closure that is unique up to K-isomorphism (cp. Bourbaki [70] and Van der Waerden [405]).

4.11.2. Let L be an extension of a field K. The pairwise distinct $x_1, \dots, x_n \in L$ are called *algebraically independent* over K provided that for each polynomial P in n variables with coefficients in K from $P(x_1, \dots, x_n) = 0$ it follows that $P \equiv 0$; i.e., all coefficients of P are equal to zero.

The definition prompts us to say that the algebraic independence of $x_1, \dots, x_n$ amounts to the linear independence over K of the set of al monomials of the form $x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$, where $n \in \mathbb{N}$ and $i_1, \dots, i_n \in \mathbb{N}$.

A subset $\mathscr{E}$ of L is called *algebraically independent* provided that every finite subset of $\mathscr{E}$ is algebraically independent. So, the empty set is algebraically independent. An inclusion maximal subset $\mathscr{E}$ of L algebraically independent over K is called a *transcendence basis* for L. Let $K(\mathscr{E})$ stand for the inclusion least subfield of L which includes K and $\mathscr{E} \subset L$. In this event we say that $K(\mathscr{E})$ results from K by adjunction of $\mathscr{E}$. In case $L = K(\mathscr{E})$ and $\mathscr{E}$ is algebraically independent, L is called a *pure extension* of K, while $\mathscr{E}$ is a *pure transcendence basis* of L over K.

4.11.3. Steinitz Theorem. *Each extension L of a field K has a transcendence basis $\mathscr{E}$ over K. In this event L is an algebraic extension of the pure extension $K(\mathscr{E})$.*

$\lhd$ See Bourbaki [70, Chapter 5, Section 5, Theorem 1]. $\rhd$

4.11.4. Isomorphism Extension Theorem. *Assume that L is an extension of a field K and $\mathscr{E}$ is a transcendence basis for L over K. Assume further that $\imath$ is an isomorphism of K to some field K' and L' is an algebraically closed extension of K'. Then to each algebraically independent family $(l_e)_{e\in\mathscr{E}}$ of elements of L' there is an isomorphism $\imath'$ of L to L' extending $\imath$ and satisfying the condition $\imath'(e) = l_e$ for all $e \in \mathscr{E}$.*

$\lhd$ See Bourbaki [70, Chapter 5, Section 4, Proposition 1]. $\rhd$

4.11.5. A mapping $d : K \to L$ is a *derivation* of $K \subset L$ to L provided that $d(x + y) = d(x) + d(y)$ and $d(xy) = d(x)y + xd(y)$ for all $x, y \in K$. The general result on extension of derivations to be formulated in the next subsection uses the concept of *separable extension*. We will not expatiate upon the formal definition of separable extension and relevant information, but the interested reader can find all details in Zariski and Samuel [429]. For our ends, it suffices to mention that if K is algebraically closed or has characteristic zero, then every extension of K is separable.

4.11.6. Derivation Extension Theorem. *Let k be a subfield of L, while K is an extension of k lying in L. For a derivation d from k to L the following hold:*

(1) *If K is a pure transcendental extension of k with a pure transcendence basis $\mathscr{E} \subset K$ over k, then to each family $(l_e)_{e\in\mathscr{E}}$ of elements of L there corresponds the unique derivation D from K to L extending d such that $De = l_e$ for all $e \in \mathscr{E}$.*

(2) *If K is a separable algebraic extension of k, then to d there corresponds the unique derivation D from K to L extending d.*

$\lhd$ See Bourbaki [70, Chapter 5, Section 9, Propositions 4 and 5]. $\rhd$

4.11.7. *Let $\mathbb{C}$ be a transcendental extension of a field $\mathbb{P}$. Then there is a nontrivial $\mathbb{P}$-automorphism of $\mathbb{C}$.*

$\lhd$ Let $\mathscr{E}$ be a transcendence basis for the extension $\mathbb{C}$ over $\mathbb{P}$. Since $\mathbb{C}$ is an algebraically closed extension of $\mathbb{P}(\mathscr{E})$, each $\mathbb{P}$-automorphism ϕ of the field $\mathbb{P}(\mathscr{E})$ extends to a $\mathbb{P}$-automorphism Φ of the field $\mathbb{C}$ by Theorem 4.11.4 (see Bourbaki [70, Chapter 5, § 5, Theorem 1]. It is clear that if ϕ is nontrivial then so is Φ.

To construct a nontrivial $\mathbb{P}$-automorphism in $\mathbb{P}(\mathscr{E})$, we firstly consider the case when $\mathscr{E}$ contains only one element e; i.e., when $\mathbb{C}$ is an algebraic extension of a simple transcendental extension $\mathbb{P}(e)$. Take $a, b, c, d \in \mathbb{P}$ such that $ad - bc \neq 0$. Then $e' = (ae + b)/(ce + d)$ is a generator of the field $\mathbb{P}(e)$ which diffes from e. The field $\mathbb{P}(e) = \mathbb{P}(e')$ is isomorphic to the field of rational fractions in one variable t; consequently, the linear-fractional substitution $t \mapsto (at+b)/(ct+d)$ defines a $\mathbb{P}$-automorphism ϕ of the field $\mathbb{P}(e)$ which sends e to e' (cp. Van der Waerden [405, Section 39]).

Assume now that $\mathscr{E}$ contains at least two distinct elements e_1 and e_2 and take an arbitrary one-to-one mapping $\phi_0 : \mathscr{E} \to \mathscr{E}$ for which $\phi_0(e_1) = e_2$. Again, using the circumstance that $\mathbb{C}$ is an algebraically closed extension of $\mathbb{P}(\mathscr{E})$, we can construct a $\mathbb{P}$-automorphism ϕ of $\mathbb{C}$ such that $\phi_0(e) = \phi(e)$ for all $e \in \mathscr{E}$ (see Theorem 4.11.4). Clearly, ϕ is nontrivial. $\rhd$

4.11.8. *Let $\mathbb{C}$ be a transcendental extension of a field $\mathbb{P}$. Then there is a nontrivial $\mathbb{P}$-derivation on $\mathbb{C}$.*

$\lhd$ We again use a transcendence basis $\mathscr{E}$ for the extension $\mathbb{C}$ over $\mathbb{P}$. It is well known that each derivation of $\mathbb{P}$ extends onto a purely transcendental extension; moreover, this extension is defined uniquely by prescribing arbitrary values at the elements of a transcendence basis (see Theorem 4.11.6 (1)). Thus, for every mapping $d : \mathscr{E} \to \mathbb{C}$, there is a unique derivation $D : \mathbb{P}(\mathscr{E}) \to \mathbb{C}$ such that $D(e) = d(e)$ for all $e \in \mathscr{E}$ and $D(x) = 0$ for $x \in \mathbb{P}$. Now, $\mathbb{C}$ is a separable algebraic extension of $\mathbb{P}(\mathscr{E})$; consequently, D admits the unique extension to some derivation $\overline{D} : \mathbb{C} \to \mathbb{C}$ by Theorem 4.11.6 (2). It is obvious that the freedom in the choice of d guarantees that $\overline{D}$ is nontrivial. $\rhd$

4.11.9. Theorem. *Let $\mathbb{C}$ be an extension of an algebraically closed* *subfield* $\mathbb{P}$. *Then the following are equivalent:*

(1) $\mathbb{P} = \mathbb{C}$.

(2) *Every* $\mathbb{P}$*-linear function in* $\mathbb{C}$ *is order bounded.*

(3) *There is no nontrivial* $\mathbb{P}$*-derivation on* $\mathbb{C}$.

(4) *There is no nontrivial* $\mathbb{P}$*-automorphism of* $\mathbb{C}$.

(5) *Every* $\mathbb{P}$*-endomorphism of* $\mathbb{C}$ *is the zero or the identity function.*

$\lhd$ If $\mathbb{P} = \mathbb{C}$ then every $\mathbb{P}$-linear function $f : \mathbb{C} \to \mathbb{C}$ is of the form $f(z) = cz$ $(z \in \mathbb{C})$ for some $c \in \mathbb{C}$; therefore (1) $\Longrightarrow$ (2) and (1) $\Longrightarrow$ (3) trivially. If f is multiplicative then $c^2 = c$ and hence $c = 0$ or $c = 1$, whence (1) $\Longrightarrow$ (4) and (1) $\Longrightarrow$ (5). The converse implications follows from 4.11.7, 4.11.8, and Theorem 4.2.8. $\rhd$

4.12. Automorphisms and Derivations on Complex f-Algebras

In this section we characterize the universally complete complex f-algebras admitting nontrivial automorphisms and derivations. The results are obtained by means of Boolean valued interpretation of some properties of the complexes that appeared in the previous section.

4.12.1. Theorem. *The field* $\mathbb{C}^\wedge$ *is algebraically closed in* $\mathscr{C}$ *within* $\mathbb{V}^{(\mathbb{B})}$. *In particular, the following dichotomy holds within* $\mathbb{V}^{(\mathbb{B})}$: *either* $\mathbb{C}^\wedge = \mathscr{C}$ *or* $\mathscr{C}$ *is a transcendental extension of* $\mathbb{C}^\wedge$.

$\lhd$ The second part is obvious from the first. Prove that the field $\mathbb{C}^\wedge$ is algebraically closed in $\mathscr{C}$. Working within $\mathbb{V}^{(\mathbb{B})}$, assume that $z_0 \in \mathscr{C}$ is a root of a nonzero polynomial with coefficients in $\mathbb{C}^\wedge$. We can formalize this assertion as follows:

$$\varphi(z_0) \equiv (\exists\, n \in \omega)(\exists\, \varkappa : \langle n \rangle \to \mathbb{C}^\wedge) \\ \Bigg(\sum_{l \in \langle n \rangle} \varkappa(l) z_0^l = 0 \Bigg) \wedge ((\exists\, l \in \langle n \rangle) \varkappa(l) \neq 0),$$

where $\langle n \rangle := \{0, 1, \dots, n-1\}$. Thus, $[\![\varphi(z_0)]\!] = \mathbb{1}$, and eliminating the Boolean estimates for quantifiers by means of the maximum principle 1.4.2, we find a countable partition of unity $(b_n) \subset \mathbb{B}$ and a sequence

$(\varkappa_n) \subset \mathbb{V}^{(\mathbb{B})}$ for which

$$[\![\varkappa_n : \langle n\rangle^\wedge \to \mathbb{C}^\wedge]\!] \geqslant b_n, \quad [\![(\exists\, l \in \langle n\rangle^\wedge)\varkappa_n(l) \neq 0)]\!] \geqslant b_n,$$

$$[\![\varkappa_n(0^\wedge) + \varkappa_n(1^\wedge)z_0 + \dots + \varkappa_n((n-1)^\wedge)z_0^{(n-1)^\wedge} = 0]\!] \geqslant b_n \quad (n \in \omega).$$

It suffices to establish the inequality $[\![z_0 \in \mathbb{C}^\wedge]\!] \geqslant b_n$ for a fixed $n \in \omega$. In the arguments below, without loss of generality we can assume that $b_n = \mathbb{1}$, since otherwise we can replace $\mathbb{B}$ with the Boolean algebra $\mathbb{B}_n := [\mathbb{0}, b_n]$ with unity b_n and $\mathbb{V}^{(\mathbb{B})}$ with $\mathbb{V}^{(\mathbb{B}_n)}$ with application of 1.3.7 to the complete Boolean homomorphism $\pi : b \mapsto b \wedge b_n$ from $\mathbb{B}$ to $\mathbb{B}_n$.

Note that $X := \mathbb{C}^\wedge{\downarrow}$ is an f-subalgebra in $\mathscr{C}{\downarrow}$ and consists of piecewise constant elements. More exactly, an element $z \in \mathscr{C}$ belongs to X if and only if z has the representation $z = o\text{-}\sum_\xi \lambda_\xi \pi_\xi(\mathbb{1})$, where (π_ξ) is a partition of unity in $\mathbb{B} = \mathbb{P}(\mathscr{C}{\downarrow})$ and (λ_ξ) is a family of complex numbers with the same set of indices.

Let $k_n : \{0, 1, \dots, n-1\} \to X$ be the modified descent of $\varkappa_n$; see 1.5.8. Since $k_n(0), k_n(1), \dots, k_n(n-1) \in X$, we can choose a partition of unity $(\pi_\xi) \subset \mathbb{B}$, $\pi_\xi \neq 0$, such that $k_n(l) = o\text{-}\sum_\xi \lambda_{l,\xi}\pi_\xi(\mathbb{1})$, $l := 0, \dots, n-1$. If $\lambda_{0,\xi} = \lambda_{1,\xi} = \dots = \lambda_{n-1,\xi} = 0$ for some ξ then $[\![k_n(l) = 0]\!] \geqslant [\![k_n(l) = \lambda_{l,\xi}^\wedge]\!] \wedge [\![\lambda_{l,\xi}^\wedge = 0^\wedge]\!] \geqslant \pi_\xi$ for all l; consequently,

$$\bigvee_{l=0}^{n-1} [\![\varkappa_n(l^\wedge) \neq 0]\!] = \bigvee_{l=0}^{n-1} [\![k_n(l) \neq 0]\!] = \Bigg(\bigwedge_{l=0}^{n-1} [\![k_n(l) = 0]\!]\Bigg)^* \leqslant \pi_\xi^* < \mathbb{1}.$$

But this contradicts the relation

$$\mathbb{1} = [\![(\exists\, l \in \langle n\rangle^\wedge)\varkappa_n(l) \neq 0)]\!] = \bigvee_{l=0}^{n-1} [\![\varkappa_n(l^\wedge) \neq 0]\!].$$

The relation

$$[\![\varkappa_n(0^\wedge) + \varkappa_n(1^\wedge)z_0 + \dots + \varkappa_n((n-1)^\wedge)z_0^{(n-1)^\wedge} = 0]\!] = \mathbb{1}$$

implies the equality $k_n(0) + k_n(1)z_0 + \dots + k_n(n-1)z_0^{n-1} = 0$; therefore, using the above representation for k_n, we obtain a family of equations with constant complex coefficients

$$\lambda_{0,\xi} + \lambda_{1,\xi}\pi_\xi z_0 + \dots + \lambda_{n-1,\xi}\pi_\xi z_0^{n-1} = 0;$$

moreover, for each ξ, not all of $\lambda_{0,\xi}, \dots, \lambda_{n-1,\xi}$ are zero.

Let Q be a clopen set in the Stone space of the Boolean algebra $\mathbb{B}$ which corresponds to the projection π_ξ. Then the Dedekind complete vector lattice $\pi_\xi \mathscr{C}{\downarrow}$ is isomorphic to $C_\infty(Q, \mathbb{C})$; moreover, the element $\pi_\xi(\mathbb{1})$ goes into the identically one function on Q. If $f \in C_\infty(Q, \mathbb{C})$ is the image of an element $\pi_\xi z_0$ under the indicated isomorphism then we arrive at the relation

$$\lambda_{0,\xi} + \lambda_{1,\xi} f(q) + \dots + \lambda_{n-1,\xi} f(q)^{n-1} = 0 \quad (q \in Q).$$

By the Fundamental Theorem of Algebra, the continuous function f has at most n values; consequently, f is a step-function. But then the element $\pi_\xi z_0$ is piecewise constant and so it belongs to X. Clearly, $z_0 \in X$ and hence $\mathbb{1} = [\![z \in X{\uparrow}]\!] = [\![z \in \mathbb{C}^\wedge]\!]$. $\triangleright$

4.12.2. Thus, under the canonical embedding of the complexes into the Boolean valued model, either $\mathbb{C}^\wedge = \mathscr{C}$ or the field of complexes is a transcendental extension of some subfield of $\mathscr{C}$. The same is true for the reals. To analyze this situation, we need the notion of an algebraic or transcendence basis of a field over some subfield.

Let $\mathbb{P}$ be a subfield of $\mathbb{C}$ such that $\mathbb{C}$ is a transcendental extension of $\mathbb{P}$. By the Steinitz Theorem, there is a transcendence basis $\mathscr{E} \subset \mathbb{C}$. This means that $\mathscr{E}$ is algebraically independent over $\mathbb{P}$ and $\mathbb{C}$ is an algebraic extension of the field $\mathbb{P}(\mathscr{E})$ obtained by addition of the elements of $\mathscr{E}$ to $\mathbb{P}$. The field $\mathbb{P}(\mathscr{E})$ is a *pure extension* of $\mathbb{P}$.

4.12.3. Let $\mathscr{D}(\mathscr{C}{\downarrow})$ be the set of all complex derivations on the f-algebra $\mathscr{C}{\downarrow}$ and let $\mathscr{M}_N(\mathscr{C}{\downarrow})$ be the set of all complex band preserving automorphisms of $\mathscr{C}{\downarrow}$. Let $\mathscr{D}_{\mathbb{C}^\wedge}(\mathscr{C})$ and $\mathscr{M}_{\mathbb{C}^\wedge}(\mathscr{C})$ be the elements of $\mathbb{V}^{(\mathbb{B})}$ that depict the sets of all $\mathbb{C}^\wedge$-derivations and all $\mathbb{C}^\wedge$-automorphisms in $\mathscr{C}$. Clearly, $\mathscr{D}(\mathscr{C}{\downarrow})$ is a module over $\mathscr{C}{\downarrow}$ and $[\![\mathscr{D}_{\mathbb{C}^\wedge}(\mathscr{C})$ is a complex vector space$]\!] = \mathbb{1}$.

The descent and ascent produce isomorphisms between $\mathscr{D}_{\mathbb{C}^\wedge}(\mathscr{C}){\downarrow}$ and $\mathscr{D}(\mathscr{C}{\downarrow})$ as well as bijections between $\mathscr{M}_{\mathbb{C}^\wedge}(\mathscr{C}){\downarrow}$ and $\mathscr{M}_N(\mathscr{C}{\downarrow})$.

$\triangleleft$ The proof follows from 4.10.10. We only have to note that an operator $T \in \mathrm{End}_N(\mathscr{C}{\downarrow})$ is a complex derivation (automorphism) if and only if $[\![\tau := T{\uparrow}$ is a $\mathbb{C}^\wedge$-derivation ($\mathbb{C}^\wedge$-automorphism)$]\!] = \mathbb{1}$. $\triangleright$

4.12.4. *An order bounded derivation and an order bounded band preserving automorphism of a universally complete f-algebra $X_\mathbb{C}$ are trivial.*

$\triangleleft$ We may assume that $X_\mathbb{C} = \mathscr{C}{\downarrow}$. If T is a derivation (a band preserving automorphism) of the f-algebra $X_\mathbb{C}$ then $[\![\tau := T{\uparrow}$ is a $\mathbb{C}^\wedge$-

derivation ($\mathbb{C}^\wedge$-automorphism) of $\mathscr{C}\,]\!] = \mathbb{1}$. Moreover, T is order bounded if and only if $[\![\,\tau$ is order bounded in $\mathscr{C}\,]\!] = \mathbb{1}$. But every order bounded $\mathbb{C}^\wedge$-derivation on the field $\mathscr{C}$ is zero and every order bounded $\mathbb{C}^\wedge$-automorphism is the identity mapping. In the first case we have $T = 0$ and in the second, $T = I$. $\triangleright$

4.12.5. *If $\mathbb{V}^{(\mathbb{B})} \models \mathbb{C}^\wedge \neq \mathscr{C}$ then there exist a nontrivial derivation and a nontrivial band preserving automorphism on the universally complete complex f-algebra $\mathscr{C}{\downarrow}$.*

$\triangleleft$ It follows from the condition $\mathbb{C}^\wedge \neq \mathscr{C}$ that $\mathscr{C}$ is a transcendental extension of $\mathbb{C}^\wedge$ within $\mathbb{V}^{(\mathbb{B})}$ (cp. 3.12.1). By 4.11.9, there exist a nontrivial $\mathbb{C}^\wedge$-derivation $\delta : \mathscr{C} \to \mathscr{C}$ and a nontrivial $\mathbb{C}^\wedge$-automorphism $\alpha : \mathscr{C} \to \mathscr{C}$. If $D := \delta{\downarrow}$ and $A := \alpha{\downarrow}$ then, according to 4.12.3, D is a nontrivial derivation and A is a nontrivial band preserving automorphism of the f-algebra $\mathscr{C}{\downarrow}$. $\triangleright$

4.12.6. Theorem. *Let $\mathbb{B}$ be a complete Boolean algebra, $\mathscr{C}$ the complexes within $\mathbb{V}^{(\mathbb{B})}$, and $X := \mathscr{C}{\downarrow}$ a universally complete complex f-algebra, the descent of $\mathscr{C}$. Then the following are equivalent:*

(1) *$\mathbb{B}$ is σ-distributive.*

(2) $\mathbb{V}^{(\mathbb{B})} \models \mathscr{C} = \mathbb{C}^\wedge$.

(3) *Every band preserving linear operators on X is order bounded.*

(4) *There is no nontrivial derivation on X.*

(5) *There is no nontrivial band preserving automorphism on X.*

(6) *Every band preserving endomorphism of X is a band projection.*

$\triangleleft$ By Theorem 4.4.9 a Boolean algebra $\mathbb{B}$ is σ-distributive if and only if $\mathbb{V}^{(\mathbb{B})} \models \mathscr{R} = \mathbb{R}^\wedge$. At the same time, by restricted transfer 1.4.7 we have $\mathbb{V}^{(\mathbb{B})} \models \mathbb{R}^\wedge \oplus i\mathbb{R}^\wedge = \mathbb{C}^\wedge$. Thus $\mathbb{V}^{(\mathbb{B})} \models \mathscr{C} = \mathbb{C}^\wedge$ if and only if $\mathbb{V}^{(\mathbb{B})} \models \mathscr{R} = \mathbb{R}^\wedge$. It follows that (1) $\iff$ (2).

Observe that the assertion 4.12.6$\,(k+1)$ is the interpretation of 4.11.9$\,(k)$ within $\mathbb{V}^{(\mathbb{B})}$ for $k = 1, \dots, 5$. We now get the other equivalences by appealing to 4.10.10 and 4.12.5. $\triangleright$

4.12.7. Corollary. *Let X be a universally complete real vector lattice with a fixed structure of an f-algebra. Then for the complex f-algebra $X_{\mathbb{C}}$ the following are equivalent:*

(1) *$\mathbb{B} := \mathbb{P}(X)$ is a σ-distributive Boolean algebra.*

(2) *There is no nontrivial complex derivation on $X_{\mathbb{C}}$.*

(3) *There is no nontrivial band preserving complex automorphisms of $X_{\mathbb{C}}$.*

4.12.8. Using the same arguments as above, we can show that some analogs of 4.12.1 and 4.11.8 hold for the reals. More precisely, the following are valid:

(1) $[\![\mathbb{R}^\wedge$ is algebraically closed in $\mathscr{R}]\!] = \mathbb{1}$;

(2) If $\mathbb{V}^{(\mathbb{B})} \models \mathbb{R}^\wedge \neq \mathscr{R}$, then

$$\mathbb{V}^{(\mathbb{B})} \models \text{“}\mathscr{R} \text{ is a transcendental extension of } \mathbb{R}^\wedge\text{”};$$

(3) If $\mathbb{R}$ is a transcendental extension of a field $\mathbb{P}$ then there is a nontrivial $\mathbb{P}$-derivation on $\mathbb{R}$.

But 4.11.7 is not valid for the reals: there is no nontrivial automorphism on $\mathbb{R}$. This is connected with the fact that $\mathbb{R}$ is not an algebraically closed field.

4.12.9. A derivation (an automorphism) S on X is called *essentially nontrivial* provided that $\pi S = 0$ ($\pi S = \pi I_X$) imply $\pi = 0$ for every band projection $\pi \in \mathbb{P}(X)$. A complete Boolean algebra $\mathbb{B}$ is said to be *purely non-σ-distributive* if none of its relative Boolean algebras $[\mathbb{0}, b]$ with nonzero $b \in \mathbb{B}$ is σ-distributive.

Assume that Z is a universally complete real vector lattice and $\mathbb{P}(X)$ is purely non-σ-distributive. Then, by Theorem 4.12.7, for every band projection $\pi \in \mathbb{P}(X)$ there exist a nontrivial complex derivation and a nontrivial band preserving complex automorphisms on $\pi Z_{\mathbb{C}}$. Therefore, we can find also an essentially nontrivial complex derivation and a an essentially nontrivial band preserving complex automorphisms on $Z_{\mathbb{C}}$ making use of the exhausting principle (= every minorizing set in a complete Boolean algebra admit a disjoint refinement).

4.12.10. *For each complete Boolean algebra $\mathbb{B}$ there exists an element $b \in \mathbb{B}$ such that the relative Boolean algebra $\mathbb{B}_0 := [\mathbb{0}, b]$ is σ-distributive, while the relative Boolean algebra $[\mathbb{0}, b^*]$ is purely non-σ-distributive.*

$\lhd$ Put $b = [\![\mathscr{R} = \mathbb{R}^\wedge]\!]$ and note that $\mathbb{V}^{(\mathbb{B}_0)} \models \mathscr{R} = \mathbb{R}^\wedge$ (we use the same symbols $\mathscr{R}$ and $\mathbb{R}^\wedge$ within $\mathbb{V}^{(\mathbb{B})}$ and $\mathbb{V}^{(\mathbb{B}_0)}$ for reals and standard reals). By Theorem 4.4.9 $\mathbb{B}_0 := [\mathbb{0}, b]$ is σ-distributive. If $d \in \mathbb{B}$, $d \leqslant b^*$, and $[\mathbb{0}, d]$ is σ-distributive then again by Theorem 4.4.9 $d \leqslant [\![\mathscr{R} = \mathbb{R}^\wedge]\!] \wedge b^* = b \wedge b^* = \mathbb{0}$. $\rhd$

4.12.11. Corollary. *If (Ω, Σ, μ) is an atomless Maharam measure space then the following hold:*

(1) *There exists an essentially nontrivial $\mathbb{R}$-derivation on $L^0_{\mathbb{R}}(\Omega, \Sigma, \mu)$.*

(2) *There exists an essentially nontrivial $\mathbb{C}$-derivation on $L^0_{\mathbb{C}}(\Omega, \Sigma, \mu)$.*

(3) *The identity operator is the only automorphism of $L^0_{\mathbb{R}}(\Omega, \Sigma, \mu)$.*

(4) *There exists an essentially nontrivial band preserving automorphism of $L^0_{\mathbb{C}}(\Omega, \Sigma, \mu)$.*

$\lhd$ This is immediate from 4.12.10, Corollary 4.12.7, and Remarks in 4.12.8 and 4.12.9 in view of 4.7.11. $\rhd$

4.13. Involutions and Complex Structures

The main result of this section tells us that in a real non-locally-one-dimensional universally complete vector lattice there are band preserving complex structures and nontrivial band preserving involutions.

4.13.1. A linear operator T on a vector lattice X is called *involutory* or an *involution* if $T \circ T = I_X$ (or, equivalently, $T^{-1} = T$) and is called a *complex structure* if $T \circ T = -I_X$ (or, equivalently, $T^{-1} = -T$). The operator $P - P^{\perp}$, where P is a projection operator on X and $P^{\perp} = I_X - P$, is an involution. The involution $P - P^{\perp}$ with band projections P is referred to as *trivial*.

4.13.2. *Let X be a Dedekind complete vector lattice. Then there is no order bounded band preserving complex structure in X and there is no nontrivial order bounded band preserving involution in X.*

$\lhd$ An order bounded band preserving operator T on a universally complete vector lattice X with weak unit $\mathbb{1}$ is a multiplication operator: $Tx = ax$ $(x \in X)$ for some $a \in X$. It follows that T is an involution if and only if $a^2 = \mathbb{1}$ and so there is a band projection P on E with $a = P\mathbb{1} - P^{\perp}\mathbb{1}$ or $T = P - P^{\perp}$. If T is a complex structure on E then the corresponding equation $a^2 = -\mathbb{1}$ has no solution. $\rhd$

4.13.3. Theorem. *Let $\mathbb{F}$ be a proper subfield of $\mathbb{R}$ and let $B \subset \mathbb{R}$ be a nonempty finite set. Then there exists a discontinuous $\mathbb{F}$-linear function $f : \mathbb{R} \to \mathbb{R}$ such that $f \circ f = I_{\mathbb{R}}$ and $f(x) = x$ for all $x \in B$.*

$\lhd$ Let $\mathscr{E} \subset \mathbb{R}$ be a Hamel basis of $\mathscr{R}$ over $\mathbb{R}^{\wedge}$. Every $x \in B$ can be written in the form $x = \sum_{e \in \mathscr{E}} \lambda_e(x)e$, where $\lambda_e(x) \in \mathbb{F}$ for all $e \in \mathscr{E}$.

Put $\mathscr{E}(x) := \{e \in \mathscr{E} : \lambda_e(x) \neq 0\}$ and $\mathscr{E}_0 = \bigcup_{x\in B} \mathscr{E}(x)$. Since B is finite, so is also $\mathscr{E}_0$. Hence $\mathscr{E}\backslash\mathscr{E}_0$ has infinite cardinality. There exists a decomposition $\mathscr{E}_1 \cup \mathscr{E}_2 = \mathscr{E}\backslash\mathscr{E}_0$, where $\mathscr{E}_1$ and $\mathscr{E}_2$ disjoint sets both having the same cardinality. Hence there exists a one-to-one mapping g_0 from $\mathscr{E}_1$ onto $\mathscr{E}_2$ with the inverse $g_0^{-1} : \mathscr{E}_2 \to \mathscr{E}_1$.

Define the function $g : \mathscr{E} \to \mathscr{E}$ as follows:

$$g(e) = \begin{cases} g_0(e), & \text{for } e \in \mathscr{E}_1, \\ g_0^{-1}(e), & \text{for } e \in \mathscr{E}_2, \\ e, & \text{for } e \in \mathscr{E}_0. \end{cases} \tag{4.1}$$

Let $f : \mathbb{R} \to \mathbb{R}$ stand for the $\mathbb{F}$-linear extension of g. For $h \in \mathscr{E}_0$ we have $g(h)/h = 1$, and for $h_1 \in \mathscr{E}_1$ we have $g(h_1) = g_0(h_1) \in \mathscr{E}_2$, so that $g(h) \neq h$ and $g(h)/h \neq 1$. By 4.2.5 f is discontinuous.

For arbitrary $h \in \mathscr{E}_1$ we have $g(h) = g_0(h) \in \mathscr{E}_2$, whence $g(g(h)) = g_0^{-1}(g_0(h)) = h$. Similarly, for $h \in \mathscr{E}_2$ we have $g(h) = g_0^{-1}(h) \in \mathscr{E}_1$ and $g(g(h)) = g_0(g_0^{-1}(h)) = h$. Obviously we have $g(g(h)) = h$ for $h \in \mathscr{E}_0$. Thus $g(g(h)) = h$ for all $h \in \mathscr{E}$. Now take an arbitrary $x \in \mathbb{R}$ and write down the representation $x = \sum_{e\in\mathscr{E}} x_e e$ with $x_e \in \mathbb{P}$. Using $\mathbb{F}$-linearity of f and the relation $f|_{\mathscr{E}} = g$ we deduce

$$f(f(x)) = \sum_{e\in\mathscr{E}} x_e f(g(e)) = \sum_{e\in\mathscr{E}} x_e g(g(e)) = \sum_{e\in\mathscr{E}} x_e g(e) = x.$$

Observe further that if $x \in B$ then $e \in \mathscr{E}_0$ whenever $x_e \neq 0$. Therefore, we have

$$f(x) = \sum_{e\in\mathscr{E}} x_e f(e) = \sum_{e\in\mathscr{E}_0} x_e g(e) = \sum_{e\in\mathscr{E}_0} x_e e = x.$$

Thus $f(f(x)) = x$ for all $x \in \mathbb{R}$ and $f(x) = x$ for $x \in B$. $\triangleright$

4.13.4. Theorem. *Let $\mathbb{F}$ be a proper subfield of $\mathbb{R}$. Then there exists a discontinuous $\mathbb{F}$-linear function $f : \mathbb{R} \to \mathbb{R}$ such that $f \circ f = -I_{\mathbb{R}}$.*

$\triangleleft$ The proof is similar to that of Theorem 4.13.3 with minor modifications: put $\mathscr{E}_0 = \varnothing$ and define

$$g(e) = \begin{cases} -g_0(e), & \text{for } e \in \mathscr{E}_1, \\ g_0^{-1}(e), & \text{for } e \in \mathscr{E}_2. \end{cases}$$

If $f:\mathbb{R}\to\mathbb{R}$ is the $\mathbb{F}$-linear extension of a function g and $x=\sum_{e\in\mathscr{E}} x_e e$ then, taking it into account that $f(g_0(e))=e$ and $f(g_0^{-1}(c))=-c$ for $e\in\mathscr{E}_1$ and $c\in\mathscr{E}_2$, we get

$$
\begin{aligned}
f(f(x)) = \sum_{e\in\mathscr{E}_1} x_e f(-g_0(e)) + \sum_{c\in\mathscr{E}_2} x_c f(g_0^{-1}(c))& \\
= -\sum_{e\in\mathscr{E}_1} x_e e - \sum_{c\in\mathscr{E}_2} x_c e = -x.&
\end{aligned}
$$

Thus, f is the sought complex structure. ▷

Interpreting Theorems 2.4.3 and 2.4.4 in a Boolean valued model yields the following result.

4.13.5. Theorem. *Let X be a universally complete real vector lattice that is not locally one-dimensional. Then*

(1) *For every nonempty finite set $B\subset X$ there exists a band preserving involution T on X with $T(x)=x$ for all $x\in B$.*

(2) *There exists a band preserving complex structure on X.*

◁ Assume that $X=\mathscr{R}{\downarrow}$. Take a one-to-one function $\nu:\langle N\rangle\to X$ with $B=\operatorname{im}(\nu)$ and $\langle N\rangle:=\{1,\dots,N-1\}$. The function $\sigma:=\nu{\uparrow}:\langle N\rangle^{\wedge}\to X$ may fail to be one-to-one within $\mathbb{V}^{(\mathbb{B})}$ but $B{\uparrow}$ is again finite, as $B{\uparrow}=\operatorname{im}(\nu{\uparrow})$ by 1.2.7. By transfer, Theorem 4.13.3 is valid within $\mathbb{V}^{(\mathbb{B})}$, so there exists an $\mathbb{R}^{\wedge}$-linear function $\tau:\mathscr{R}\to\mathscr{R}$ such that $\tau\circ\tau=I_{\mathscr{R}}$ and $\tau(x)=x$ for all $x\in B{\uparrow}$ or, what is the same, $\tau\circ\sigma=\sigma$. From 1.2.3, 1.6.9, and 1.5.6 we now deduce

$$
\begin{aligned}
\mathbb{1} &= [\![(\forall\, x\in B{\uparrow})\tau(x)=x]\!] = [\![(\forall\, n\in\langle N\rangle^{\wedge})\tau(\sigma(n))=\sigma(n)]\!] \\
&= \bigwedge_{n\in\langle N\rangle} [\![\tau(\nu{\uparrow}(n^{\wedge}))=\nu{\uparrow}(n^{\wedge})]\!] = \bigwedge_{n\in\mathbb{N}} [\![\tau(\nu(n))=\nu(n)]\!] \\
&= \bigwedge_{n\in\langle N\rangle} [\![\tau{\downarrow}(\nu(n))=\nu(n)]\!].
\end{aligned}
$$

It follows that if $T:=\tau{\downarrow}$ then $T\circ T=I_X$ by 1.2.4 and $T(\nu(n))=\nu(n)$ for all $n\in\{1,\dots,N-1\}$ as required in 4.13.5 (1). The second claim is proved in a similar way using Theorem 4.13.4. ▷

4.13.6. Corollary. *Let X be a universally complete vector lattice. Then the following are equivalent:*

(1) *X is locally one-dimensional.*

(2) *There is no nontrivial band preserving involution on X.*

(3) *There is no band preserving complex structure on X.*

4.13.7. Corollary. *Let X be a universally complete real vector lattice. Then X admits a structure of complex vector space with a band preserving complex multiplication.*

$\lhd$ A complex structure T on X allows us to define on X a structure of a vector space over the complexes $\mathbb{C}$, by setting $(\alpha + i\beta)x = \alpha x + \beta T(x)$ for all $z = \alpha + i\beta \in \mathbb{C}$ and $x \in X$. If T is band preserving then the mapping $x \mapsto zx$ $(x \in X)$ is evidently band preserving for every fixed $z \in \mathbb{C}$. $\rhd$

4.13.8. Corollary. *If (Ω, Σ, μ) is an atomless Maharam measure space then $L^0(\Omega, \Sigma, \mu)$ admits a structure of a complex vector space with band preserving complex multiplication.*

$\lhd$ This is immediate from 4.7.11 and Corollary 4.13.7. $\rhd$

4.14. Variations on the Theme

In this section we briefly consider the band preserving phenomenon in some natural environments (the endomorphisms of lattice ordered modules, bilinear operators on vector lattices, and derivations in AW^*-algebras) and state some problems that may be viewed as versions of the Wickstead problem which are referred to as module, bilinear, and noncommutative Wickstead problem.

4.14.A. Lattice Ordered Modules

This subsection deals with the module Wickstead problem stated as follows:

4.14.A.1. WP(A): *When are all band preserving K-linear endomorphisms of a lattice ordered K-module X order bounded?*

Here K is a lattice ordered ring, and X is a lattice ordered module over K. Little is known about this problem. Boolean valued analysis provides the transfer principle which might translate WP(A) to WP. Below we describe the class of lattice ordered modules for which this transfer works perfectly.

4.14.A.2. An annihilator ideal of K is a subset of the form $S^{\perp} := \{k \in K : (\forall\, s \in S) ks = 0\}$ with a nonempty subset $S \subset K$. A subset S of K is called *dense* provided that $S^{\perp} = \{0\}$; i.e., the equality $k \cdot S :=$

$\{k \cdot s : s \in S\} = \{0\}$ implies $k = 0$ for all $k \in K$. A ring K is said to be *rationally complete* whenever, to each dense ideal $J \subset K$ and each group homomorphism $h : J \to K$ such that $h(kx) = kh(x)$ for all $k \in K$ and $x \in J$, there is an element r in K satisfying $h(x) = rx$ for all $x \in J$. A ring K is rationally complete if and only if K is selfinjective (cp. [249, Theorem 8.2.7 (3)]).

4.14.A.3. If $\mathscr{K}$ is an ordered field within $\mathbb{V}^{(\mathbb{B})}$ then $\mathscr{K}{\downarrow}$ is a rationally complete semiprime f-ring, and there is an isomorphism χ of $\mathbb{B}$ onto the Boolean algebra $\mathbb{B}(\mathscr{K}{\downarrow})$ of the annihilator ideals (coinciding in the case under consideration with the Boolean algebra of all bands) of $\mathscr{K}{\downarrow}$ such that

$$b \leqslant [\![x = 0]\!] \iff x \in \chi(b^*) \quad (x \in K,\ b \in \mathbb{B})$$

(cp. [249, Theorem 8.3.1]). Conversely, assume that K is a rationally complete semiprime f-ring and $\mathbb{B}$ stands for the Boolean algebra $\mathbb{B}(K)$ of all annihilator ideals (bands) of K. Then there is $\mathscr{K} \in \mathbb{V}^{(\mathbb{B})}$, called the *Boolean valued representation of* K, such that $[\![\, \mathscr{K}$ is an ordered field $]\!] = \mathbb{1}$ and the lattice ordered rings K and $\mathscr{K}{\downarrow}$ are isomorphic (cp. [249, Theorem 8.3.2]).

4.14.A.4. A K-module X is *separated* provided that for every dense ideal $J \subset K$ the identity $Jx = \{0\}$ implies $x = 0$. Recall that a K-module X is *injective* whenever, given a K-module Y, a K-submodule $Y_0 \subset Y$, and a K-homomorphism $h_0 : Y_0 \to X$, there exists a K-homomorphism $h : Y \to X$ extending h_0. The *Baer criterion* says that a K-module X is injective if and only if for each ideal $J \subset K$ and each K-homomorphism $h : J \to X$ there exists $x \in X$ with $h(a) = xa$ for all $a \in J$; see Lambek [276].

4.14.A.5. Let $\mathscr{X}$ be a vector lattice over an ordered field $\mathscr{K}$ within $\mathbb{V}^{(\mathbb{B})}$, and let $\chi : \mathbb{B} \to \mathbb{B}(\mathscr{K}{\downarrow})$ be a Boolean isomorphism from 4.14.A.3. Then $\mathscr{X}{\downarrow}$ is a separated unital injective lattice ordered module over $\mathscr{K}{\downarrow}$ satisfying

$$b \leqslant [\![x = 0]\!] \iff \chi(b)x = \{0\} \quad (x \in \mathscr{X}{\downarrow},\ b \in \mathbb{B}).$$

Conversely, let K be a rationally complete semiprime f-ring, $\mathbb{B} := \mathbb{B}(K)$, and let $\mathscr{K}$ be the Boolean valued representation of K. Assume that X is a unital separated injective lattice ordered K-module. Then there exists some $\mathscr{X} \in \mathbb{V}^{(\mathbb{B})}$ such that $[\![\, \mathscr{X}$ is a vector lattice over the ordered field $\mathscr{K}\,]\!] = \mathbb{1}$ and there are algebraic and order isomorphisms $\jmath : K \to \mathscr{K}{\downarrow}$

and $\imath : X \to \mathscr{X}{\downarrow}$ such that

$$\imath(ax) = \jmath(a)\imath(x) \quad (a \in K,\ x \in X)$$

(cp. [249, Theorems 8.3.12 and 8.3.13]). Thus, the Boolean transfer principle is applicable to unital separated injective lattice ordered modules over rationally complete semiprime f-rings.

4.14.A.6. Consider an example. Let $\mathbf{B}$ be a complete Boolean algebra and let $\mathbb{B}$ be a complete subalgebra of $\mathbf{B}$. We say that $\mathbf{B}$ is $\mathbb{B}$-σ-*distributive* if for every sequence $(b_n)_{n\in\mathbb{N}}$ in $\mathbf{B}$ we have

$$\bigvee_{\varepsilon\in\mathbb{B}^{\mathbb{N}}} \bigwedge_{n\in\mathbb{N}} \varepsilon(n)b_n = \mathbb{1},$$

where $\varepsilon(n)b_n := \big(\varepsilon(n) \wedge b_n\big) \vee \big(\varepsilon(n)^* \wedge b_n^*\big)$ and b^* is the complement of $b \in \mathbf{B}$. Clearly, the $\{\mathbb{0}, \mathbb{1}\}$-$\sigma$-distributivity of $\mathbf{B}$ means that $\mathbf{B}$ is σ-distributive $\big($cp. 1.9.12 (3)$\big)$.

4.14.A.7. Theorem. *Let X be a universally complete vector lattice with a fixed order unit $\mathbb{1}$ and let K be an order closed sublattice containing $\mathbb{1}_K := \mathbb{1}$. Put $\mathbf{B} := \mathbb{C}(\mathbb{1})$ and $\mathbb{B} := \mathbb{C}(\mathbb{1}_K)$. Then K is a rationally complete f-algebra, X is an injective lattice ordered K-module, and the following are equivalent:*

(1) $\mathbf{B}$ *is $\mathbb{B}$-σ-distributive.*

(2) *Every element $x \in X_+$ is locally K-constant; i.e., $x = \sup_{\xi\in\Xi} a_\xi \pi_\xi \mathbb{1}$ for some family $(a_\xi)_{\xi\in\Xi}$ of elements of K and a disjoint family $(\pi_\xi)_{\xi\in\Xi}$ of band projections in X.*

(3) *Every band preserving K-linear endomorphism of X is order bounded.*

$\triangleleft$ We only sketch the proof. Let $\mathscr{X}$ and $\mathscr{K}$ be the same as in 4.14.A.5. There exist $\mathscr{B} \in \mathbb{V}^{(\mathbb{B})}$ such that $[\![\, \mathscr{B}$ is a complete Boolean algebra isomorphic to $\mathbb{P}(\mathscr{X})\,]\!] = \mathbb{1}$ and $\mathscr{B}{\downarrow}$ is a complete Boolean algebra isomorphic to $\mathbf{B}$ (see 1.10.4). Moreover, $\mathbf{B}$ is $\mathbb{B}$-σ-distributive if and only if $\mathscr{B}$ is σ-distributive within $\mathbb{V}^{(\mathbb{B})}$. We are done with interpreting 4.4.9 and 4.6.4 within $\mathbb{V}^{(\mathbb{B})}$. $\triangleright$

4.14.B. The Bilinear Wickstead Problem

Let us characterize those universally complete vector lattice in which all band preserving bilinear operators are symmetric or order bounded. No new ideas are required here and all run along the lines of Section 4.6.

The needed information about bilinear operators on vector lattices is in Bu, Buskes, and Kusraev [72]; also see Buskes and Kusraev [78].

4.14.B.1. Let X be a vector lattice. A bilinear operator $B : X\times X \to X$ is *separately band preserving* provided that the mappings $B(\cdot, e) : x \mapsto B(x, e)$ and $B(e, \cdot) : x \mapsto B(e, x)$ $(x \in X)$ are band preserving for all $e \in X$ or, which is the same, provided that $B(L \times X) \subset L$ and $B(X \times L) \subset L$ for every band L in X.

4.14.B.2. *Assume that X is a vector lattice and $B : X \times X \to X$ is a bilinear operator. Then the following are equivalent:*

(1) *B is separately band preserving.*

(2) *$B(x, y) \in \{x\}^{\perp\perp} \cap \{y\}^{\perp\perp}$ for all $x, y \in X$.*

(3) *$B(x, y) \perp z$ for all $z \in X$ provided that $x \perp z$ or $y \perp z$.*

If X has the principal projection property, then (1)–(3) *are equivalent also to each of the two assertions:*

(4) *$\pi B(x, y) = B(\pi x, \pi y)$ for every $\pi \in \mathbb{P}(X)$ and all $x, y \in X$.*

(5) *$\pi B(x, y) = B(\pi x, y) = B(x, \pi y)$ for all $\pi \in \mathbb{P}(X)$ and $x, y \in X$.*

$\lhd$ We omit the routine arguments which are similar to 4.1.1 and 4.1.6. $\rhd$

4.14.B.3. Let X and Y be vector lattices. Recall that a bilinear operator B from $X \times X$ to Y is *orthosymmetric* provided that $|x| \wedge |y| = 0$ implies $B(x, y) = 0$ for arbitrary $x, y \in X$ (cp. Buskes and van Roij [81]). The difference of two positive orthosymmetric bilinear operators is *orthoregular* (cp. Buskes and Kusraev [78], and Kusraev [233]). Recall also that a bilinear operator b is *symmetric* or *antisymmetric* provided that $B(x, y) = B(y, x)$ or $B(x, y) = -B(y, x)$ for all $x, y \in X$.

4.14.B.4. The following important property of orthosymmetric bilinear operators was established in Buskes and van Rooj [81, Corollary 2]: *If X and Y are vector lattices then every orthosymmetric positive bilinear operator from $X \times X$ into Y is symmetric.*

It is evident from 4.14.B.2 that a separately band preserving bilinear operator is orthosymmetric. Hence, all orthoregular separately band preserving operators are symmetric by the above result. At the same time an order bounded separately band preserving bilinear operator B is regular with $B^+(x, y) = B(x, y)^+$ and $B^-(x, y) = B(x, y)^-$ for all $x, y \in X$ (see Kusraev and Tabuev [257, Theorem 3.4]). This brings up the following question:

4.14.B.5. WP(B): *Under what conditions are all separately band preserving bilinear operators in a vector lattice symmetric? Order bounded?*

In the case of a universally complete vector lattice the answer is similar to the linear case and is presented below in 4.14.B.7. The general case was not examined yet.

4.14.B.6. Let $BL_{\mathrm{bp}}(X)$ stand for the set of all separately band preserving bilinear operators from $X \times X$ to X, where $X := \mathscr{R}{\downarrow}$. Clearly, $BL_{\mathrm{bp}}(X)$ becomes a faithful unitary module over X provided that we define gB as $gB : (x, y) \mapsto g \cdot B(x, y)$ for all $x, y \in X$. Denote by $BL(\mathscr{R}_{\mathbb{R}})$ the element of $\mathbb{V}^{(\mathbb{B})}$ that depicts the space of all $\mathbb{R}^{\wedge}$-bilinear mappings from $\mathscr{R} \times \mathscr{R}$ into $\mathscr{R}$. Then $BL(\mathscr{R}_{\mathbb{R}})$ is a vector space over $\mathbb{R}^{\wedge}$ within $\mathbb{V}^{(\mathbb{B})}$, and $BL(\mathscr{R}_{\mathbb{R}}){\downarrow}$ is a faithful unitary module over X. Just as in 4.3.5 it can be proved that the modules $BL_{\mathrm{bp}}(X)$ and $BL(\mathscr{R}_{\mathbb{R}}){\downarrow}$ are isomorphic by sending each band preserving bilinear operator to its ascent.

4.14.B.7. Theorem. *For a universally complete vector lattice X the following are equivalent:*

(1) $\mathbb{B}(X)$ *is σ-distributive.*

(2) *There is no antisymmetric operator in $BL_{\mathrm{bp}}(X)$.*

(3) *All operators in $BL_{\mathrm{bp}}(X)$ are symmetric.*

(4) *All operators in $BL_{\mathrm{bp}}(X)$ are order bounded.*

$\lhd$ The implication (1) $\Longrightarrow$ (4) can be proved as in 4.6.5, (4) $\Longrightarrow$ (3) is immediate from 4.14.B.4, while (3) $\Longrightarrow$ (2) is trivial.

To prove the remaining implication (2) $\Longrightarrow$ (1), we can assume that $X = \mathscr{R}{\downarrow}$. Suppose that $\mathbb{B}$ is not σ-distributive. Then $\mathbb{R}^{\wedge} \neq \mathscr{R}$ by 4.4.9 and a separately band preserving antisymmetric bilinear operator can be constructed on using the bilinear version of 4.2.8. Indeed, within $\mathbb{V}^{(\mathbb{B})}$, a Hamel basis $\mathscr{E}$ for $\mathscr{R}$ over $\mathbb{R}^{\wedge}$ contains at least two distinct elements $e_1 \neq e_2$. Define the function $\beta_0 : \mathscr{E} \times \mathscr{E} \to \mathscr{R}$ so that $1 = \beta_0(e_1, e_2) = -\beta_0(e_2, e_1)$, and $\beta(e_1', e_2') = 0$ for all other pairs $(e_1', e_2') \in \mathscr{E} \times \mathscr{E}$ $\big($in particular, $0 = \beta_0(e_1, e_1) = \beta_0(e_2, e_2)\big)$. Then β_0 can be extended to an $\mathbb{R}^{\wedge}$-bilinear function $\beta : \mathscr{R} \times \mathscr{R} \to \mathscr{R}$. The descent B of β is a separately band preserving bilinear operator in X by 4.14.B.6, the bilinear version of 4.3.5. Moreover, B is nonzero and antisymmetric, since β is nonzero and antisymmetric by construction. This contradiction proves that $\mathbb{R}^{\wedge} = \mathscr{R}$ and $\mathbb{B}$ is σ-distributive. $\rhd$

4.14.B.8. (1) *There exists a nonatomic universally complete vector lattice in which all separately band preserving bilinear operators are symmetric and order bounded.*

(2) *If* (Ω, Σ, μ) *is an atomless Maharam measure space then there exists an essentially nontrivial antisymmetric separately band preserving bilinear operator in* $L^0_{\mathbb{R}}(\Omega, \Sigma, \mu)$.

$\vartriangleleft$ It follows from Theorems 4.14.B.7, 4.7.7 and 4.7.11. $\vartriangleright$

4.14.C. The Noncommutative Wickstead Problem

The relevant information on the theory of Baer $*$-algebras and AW^*-algebras can be found in Berberian [50], Chilin [89], and Kusraev [228].

4.14.C.1. A *Baer $*$-algebra* is a complex involutive algebra A such that, for each nonempty $M \subset A$, there is a projection, i.e., a hermitian idempotent, p satisfying $M^\perp = pA$ where $M^\perp := \{y \in A : (\forall x \in M)\, xy = 0\}$ is the right annihilator of M. Clearly, this amounts to saying that each left annihilator has the form ${}^\perp M = Aq$ for an appropriate projection q. To each left annihilator L in a Baer $*$-algebra there is a unique projection $q_L \in A$ such that $x = xq_L$ for all $x \in L$ and $q_L y = 0$ whenever $y \in L^\perp$. The mapping $L \mapsto q_L$ is an isomorphism between the poset of left annihilators and the poset of all projections. Thus, the poset $\mathbb{P}(A)$ of all projections in a Baer $*$-algebra is an order complete lattice. $\big($Clearly, the formula $q \leqslant p \Longleftrightarrow q = qp = pq$, sometimes pronounced as "p contains q," specifies some order on the set of projections $\mathbb{P}(A)$.$\big)$

An element z in A is *central* provided that z commutes with every member of A; i.e., $(\forall x \in A)\, xz = zx$. The *center* of a Baer $*$-algebra A is the set $\mathscr{Z}(A)$ comprising central elements. Clearly, $\mathscr{Z}(A)$ is a commutative Baer $*$-subalgebra of A, with $\lambda\mathbb{1} \in \mathscr{Z}(A)$ for all $\lambda \in \mathbb{C}$. A *central projection* of A is a projection belonging to $\mathscr{Z}(A)$. Put $\mathbb{P}_c(A) := \mathbb{P}(A) \cap \mathscr{Z}(A)$.

4.14.C.2. A *derivation* on a Baer $*$-algebra A is a linear operator $d : A \to A$ satisfying $d(xy) = d(x)y + xd(y)$ for all $x, y \in A$. A derivation d is *inner* provided that $d(x) = ax - xa$ $(x \in A)$ for some $a \in A$. Clearly, an inner derivation vanishes on $\mathscr{Z}(A)$ and is $\mathscr{Z}(A)$-linear; i.e., $d(ex) = ed(x)$ for all $x \in A$ and $e \in \mathscr{Z}(A)$.

Consider a derivation $d : A \to A$ on a Baer $*$-algebra A. If $p \in A$ is a central projection then $d(p) = d(p^2) = 2pd(p)$. Multiplying this identity by p we have $pd(p) = 2pd(p)$ so that $d(p) = pd(p) = 0$. Consequently, every derivation vanishes on the linear span of $\mathbb{P}_c(A)$, the set of all central projections. In particular, $d(ex) = ed(x)$ whenever $x \in A$

and e is a linear combination of central projections. Even if the linear span of central projections is dense in a sense in $\mathscr{Z}(A)$, the derivation d may fail to be $\mathscr{Z}(A)$-linear.

This brings up the natural question: *Under what conditions is every derivation Z-linear on a Baer $*$-algebra A provided that Z is a Baer $*$-subalgebra of $\mathscr{Z}(A)$?*

4.14.C.3. An *AW^*-algebra* is a C^*-algebra with unity $\mathbb{1}$ which is also a Baer $*$-algebra. More explicitly, an AW^*-algebra is a C^*-algebra whose every right annihilator has the form pA, with p a projection. Clearly, $\mathscr{Z}(A)$ is a commutative AW^*-subalgebra of A. If $\mathscr{Z}(A) = \{\lambda\mathbb{1} : \lambda \in \mathbb{C}\}$ then the AW^*-algebra A is an *AW^*-factor*.

4.14.C.4. *A C^*-algebra A is an AW^*-algebra if and only if the following hold:*

(1) *Each orthogonal family in $\mathbb{P}(A)$ has a supremum;*

(2) *Each maximal commutative $*$-subalgebra of $A_0 \subset A$ is a Dedekind complete f-algebra* (*or, equivalently, coincides with the least norm closed $*$-subalgebra containing all projections of A_0*).

4.14.C.5. Given an AW^*-algebra A, define the two sets $C(A)$ and $S(A)$ of measurable and locally measurable operators, respectively. Both are Baer $*$-algebras; cp. Chilin [89]. Suppose that Λ is an AW^*-subalgebra in $\mathscr{Z}(A)$, and Φ is a Λ-valued trace on A_+. Then we can define another Baer $*$-algebra, $L(A, \Phi)$, of Φ-measurable operators. The center $\mathscr{Z}(A)$ is a vector lattice with a strong unit, while the centers of $C(A)$, $S(A)$, and $L(A, \Phi)$ coincide with the universal completion of $\mathscr{Z}(A)$. If d is a derivation on $C(A)$, $S(A)$, or $L(A, \Phi)$ then $d(px) = pd(x)$ $\big(p \in \mathbb{P}_c(A)\big)$ so that d can be considered as band preserving in a sense (cp. 4.1.1 and 4.10.4). The natural question arises concerning these algebras:

4.14.C.6. WP(C): *When are all derivations on $C(A)$, $S(A)$, or $L(A, \Phi)$ inner?* This question may be regarded as the *noncommutative Wickstead problem.*

4.14.C.7. The classification of AW^*-algebras into types is determined from the structure of their lattices of projections $\mathbb{P}(A)$; see Kusraev [228] and Sakai [353]. We only recall the definition of type I AW^*-algebra. A projection $\pi \in A$ is *abelian* if $\pi A \pi$ is a commutative algebra. An algebra A has *type* I provided that each nonzero projection in A contains a nonzero abelian projection.

A C^*-algebra A is $\mathbb{B}$-*embeddable* provided that there is a type I AW^*-algebra N and a $*$-monomorphism $\imath : A \to N$ such that $\mathbb{B} = \mathbb{P}_c(N)$ and

$\imath(A) = \imath(A)''$, where $\imath(A)''$ is the bicommutant of $\imath(A)$ in N. Note that in this event A is an AW^*-algebra and $\mathbb{B}$ is a complete subalgebra of $\mathbb{P}_c(A)$.

4.14.C.8. Theorem. *Let A be a type I AW^*-algebra, let Λ be an AW^*-subalgebra of $\mathscr{Z}(A)$, and let Φ be a Λ-valued faithful normal semifinite trace on A. If the complete Boolean algebra $\mathbb{B} := \mathbb{P}(\Lambda)$ is σ-distributive and A is $\mathbb{B}$-embeddable, then every derivation on $L(A, \Phi)$ is inner.*

$\triangleleft$ We briefly sketch the proof. Let $\mathscr{A} \in \mathbb{V}^{(\mathbb{B})}$ be the Boolean valued representation of A. Then $\mathscr{A}$ is a von Neumann algebra within $\mathbb{V}^{(\mathbb{B})}$. Since the Boolean valued interpretation preserves classification into types, $\mathscr{A}$ is of type I. Let φ stand for the Boolean valued representation of Φ. Then φ is a $\mathscr{C}$-valued faithful normal semifinite trace on $\mathscr{A}$ and the descent of $L(\mathscr{A}, \varphi)$ is $*$-Λ-isomorphic to $L(A, \Phi)$; cp. Korol′ and Chilin [205]. Suppose that d is a derivation on $L(A, \Phi)$ and δ is the Boolean valued representation of d. Then δ is a $\mathscr{C}$-valued $\mathbb{C}^{\wedge}$-linear derivation on $L(\mathscr{A}, \varphi)$. Since $\mathbb{B}$ is σ-distributive, $\mathscr{C} = \mathbb{C}^{\wedge}$ within $\mathbb{V}^{(\mathbb{B})}$ and δ is $\mathscr{C}$-linear. But it is well known that every derivation on a type I von Neumann algebra is inner; cp. Albeverio, Ajupov, and Kudaybergenov [23]. Therefore, d is also inner. $\triangleright$

4.15. Comments

4.15.1. The theory of orthomorphisms stems from Nakano [320]. Orthomorphisms have been studied by many authors under various names (cp. Aliprantis and Burkinshaw [28]): *dilatators* (Nakano [320]), *essentially positive operators* (Birkhoff [58]), *polar preserving endomorphisms* (Conrad and Diem [93]), *multiplication operators* (Buck [74] and Wickstead [408]), and *stabilisateurs* (Meyer [310]). The main stages of this development as well as the various aspects of the theory of orthomorphisms are reflected in the books: Abramovich and Kitover [8]. Bigard, Keimel, and Wolfenstein [57], Aliprantis and Burkinshaw [28], Zaanen [427, Chapter 20], de Pagter [327], etc.; also see the survey papers by Bukhvalov [75, Section 2.2] and Gutman [160, Chapter 6].

4.15.2. (1) Functional equations occur practically in all branches of mathematics and have a wide variety of applications not only in mathematics but also in other disciplines. The first functional equations for determining linear and quadratic functions appeared in the medieval centuries for using in applications. The first systematic treatment of the theory of functional equations appeared in Cauchy [88]. For more historical details we refer to Aczél and Dhombres [14]. The state-of-the art of the theory can be grasped from the books: Aczél and Dhombres [14], Castillo and Ruiz-Cobo [87], Czerwik [100], Kuczma [211], Hyers, Isac, and Rassias [176], Kannappan [190], and Székelyhidi [373].

(2) Hamel [164] first succeeded in proving the existence of discontinuous additive functions on $\mathbb{R}$. Using the Zermelo Well-Ordering Theorem, Hamel showed that $\mathbb{R}$, viewed as a vector space over the rationals $\mathbb{Q}$, possesses a basis, a *Hamel basis*. Actually Hamel proved Theorem 4.2.2 for $\mathbb{P} = \mathbb{Q}$, whence the existence of a discontinuous additive function follows easily. Recall also that the Zermelo Well-Ordering Theorem, the Kuratowski–Zorn Lemma, and the axiom of choice are equivalent; see [180]. Blass [61] showed that the axiom of choice follows if we assume that each linear space over an arbitrary field has a basis.

(3) Theorem 4.2.4 is in Aczél and Dhombres [14, Theorem 2.3]. It is also true that the image of every open interval by a noncontinuous solution of (L) is dense in $\mathbb{R}$. These results show that solutions to (L) are either very regular or extremely pathological.

4.15.3. (1) The main result of Section 4.3 (Theorem 4.3.4) was established by Kusraev [229]. The problem whether or not the inverse of an injective band preserving operator on a vector lattice is also band preserving was posed by Abramovich in 1992. Huijsmans and Wickstead [175, Theorems 2 and 3] handled the problem under the additional assumption that the domain vector lattice either is uniformly complete or have the principal projection property. Later in Abramovich and Kitover [8, Theorem 7.4] the result was generalized to vector lattices with a cofinal family of band projections. Its final form, stated in Theorem 4.3.6, was obtained by the same authors [9, Theorem 3.3]. Theorem 4.3.10 and Corollary 4.10.11 amount essentially to Theorem 14.8 in Abramovich and Kitover [8].

(2) It follows from 4.3.8 that every orthomorphism is order continuous. Order continuity of an extended orthomorphism was established independently by Bigard and Keimel in [56] and by Conrad and Diem in [93] using functional representation. A direct proof was found by Luxemburg and Schep [295]. Commutativity of every Archimedean f-algebra was proved by Birkhoff and Pierce [60]; this paper also introduced the concept of f-algebra. The lattice ordered algebras were surveyed by

Boulabiar, Buskes, and Triki [68, 69]. The fact that $\mathrm{Orth}(D, X)$ is a vector lattice under the pointwise algebraic and lattice operations was also obtained in Bigard and Keimel [56] and Conrad and Diem [93]. Extensive is the bibliography on the theory of orthomorphisms; and so we indicate only a portion of it: Abramovich, Veksler, and Koldunov [11], Abramovich and Wickstead [13], Bernau [51], Bigard and Keimel [56], Duhoux and Meyer [111], Gutman [161, 162], Huijsmans and de Pagter [173], Huijsmansand Wickstead [175], Luxemburg [291], Luxemburg and Schep [295], Mittelmeyer and Wolff [312], de Pagter [329, 330], Wickstead [408, 410], and Zaanen [426].

4.15.4. (1) In Section 4.4 we follow Kusraev [229]. The property of λ in 4.4.8 is usually referred to as *absolute definability.* Gordon [138] called a continuous function *absolutely definable* if it possesses an analogous property. For instance, the functions e^x, $\log x$, $\sin x$, and $\cos x$ are absolutely definable. In particular, these functions reside in every Boolean valued universe, presenting the mappings from $\mathscr{R}$ to $\mathscr{R}$ that are continuations of the corresponding functions $\exp^\wedge(\cdot)$, $\log^\wedge(\cdot)$, $\sin^\wedge(\cdot)$, and $\cos^\wedge(\cdot)$ from $\mathbb{R}^\wedge$ into $\mathbb{R}^\wedge$. Practically all functions admitting a constructive definition are absolutely definable.

(2) Instead of using continued fraction expansions in Section 4.4 we can involve binary expansions. In this event we have to construct a bijection of $\mathscr{P}(\omega)$ onto some set of reals and apply 1.9.13 (3) in place of 1.9.13 (2).

4.15.5. (1) The terms "local linear independence" and "local Hamel basis" were coined in McPolin and Wickstead [309]. They appeared in Abramovich, Veksler and Koldunov [11] under the names *d-independence* and *d-basis.* Originally the concept was introduced by Cooper [94]. For this concept we choose the terms d-independence and d-basis, since it is somewhat weaker than that introduced in Kusraev [229] and presented in Section 4.5: A local Hamel basis in the sense of Definition 4.5.1 is what one gets interpreting a classical Hamel basis in a Boolean valued model, while a d-basis appears by interpreting a Hamel basis together with the zero element.

(2) More precisely, consider a universally complete vector lattice X represented as the reals $\mathscr{R}$ in the Boolean valued universe $\mathbb{V}^{(\mathbb{B})}$ with $\mathbb{B} = \mathbb{P}(X)$; if $\mathscr{E}$ is an internal Hamel basis for $\mathscr{R}$ over $\mathbb{R}^\wedge$, then $\mathscr{E}{\downarrow}$ is a local Hamel basis in the sense of 4.5.1 (Theorem 4.5.7), while $(\mathscr{E} \cup \{0\}){\downarrow}$ is a d-basis of X. Theorem 4.5.7, the main result of Section 4.5 was obtained in Kusraev [229, Proposition 4.6 (1)]. The representation in 4.5.3 and 4.5.4 is referred to as a *d-expansion* with respect to the local Hamel basis (d-basis). More details about d-bases and d-expansions are given in Abramovich and Kitover [8].

(3) The notions of d-independence and d-basis can be introduced in an arbitrary vector lattice (see Abramovich and Kitover [10]). A collection $(x_\gamma)_{\gamma\in\Gamma}$ of elements in a vector lattice X is *d-independent* provided that for each band B in X, each finite subset $\{\gamma_1,\dots,\gamma_n\}$ of Γ, and each family of nonzero scalars $c_1,\dots,c_n$ the condition $\sum_{\imath=1}^n c_\imath x_{\gamma_\imath}\perp B$ implies that $x_{\gamma_\imath}\perp B$ for $\imath=1,\dots,n$. A d-independent system $(x_\gamma)_{\gamma\in\Gamma}$ is a *d-basis* provided that for each $x\in X$ there is a full system $(B_\alpha)_{\alpha\in\mathrm{A}}$ of pairwise disjoint bands in X and a system of elements $(y_\alpha)_{\alpha\in\mathrm{A}}$ in X such that each y_α is a linear combination of elements in $(x_\gamma)_{\gamma\in\Gamma}$ and $(x-y_\alpha)\perp B_\alpha$ for all $\alpha\in\mathrm{A}$.

(4) The dimension $\delta(\mathscr{R})$ of the vector space $\mathscr{R}$ over $\mathbb{R}^\wedge$ is a cardinal within $\mathbb{V}^{(\mathbb{B})}$. The object $\delta(\mathscr{R})$ carries important information on the interconnection of the Boolean algebra $\mathbb{B}$ and the reals $\mathbb{R}$. Using the properties of Boolean valued ordinals, we obtain the representation $\delta(\mathscr{R})=\mathrm{mix}_\xi\, b_\xi\alpha_\xi^\wedge$, where (b_ξ) is a partition of unity in $\mathbb{B}$ and (α_ξ) is a family of standard cardinals. This representation is an instance of a "decomposition series" of $\mathbb{B}$ such that the principal ideals $[\mathbb{0},b_\xi]$ are "α_ξ-homogeneous" in a sense.

4.15.6. (1) For locally one-dimensional vector lattices the term *essentially one-dimensional* is also in use; see Abramovich and Kitover [8]. Proposition 4.6.2 establishes the Boolean valued status of locally one dimensionality: A laterally complete vector lattice is locally one-dimensional if and only if its Boolean valued representation is a one-dimensional vector space over the field $\mathbb{R}^\wedge$. Theorem 4.6.7 gives a negative answer to the following problem (Problem B in [9]): Is there a bijective disjointness preserving linear operator between vector lattices with a disjointness preserving inverse which is not order isomorphism? The existence of such an operator was demonstrated in Abramovich and Kitover [8, Theorem 13.4] with the help of d-basis. Theorems 4.3.4 and 4.5.7 enables us to reduce the problem to the easy exercise with a classical Hamel basis (see 4.6.6). Theorem 4.6.9 is due to Kusraev.

(2) An orthomorphism is a band preserving operator that is order-bounded. In [408] Wickstead raised the question whether every band preserving operator must be order bounded automatically. Existence of an unbounded band preserving operator was announced for the first time in [11, Theorem 1]. Later, it was clarified that the situation described in the paper is typical in a sense. Namely, it was established by Abramovich, Veksler, and Koldunov in [12, Theorem 2.1] and by McPolin and Wickstead in [309, Theorem 3.2] that all band preserving operators in a universally complete vector lattice are automatically bounded if and only if this vector lattice is locally one-dimensional (Theorem 5.1.2). (The definitions of locally one-dimensional K-space and local Hamel basis, as well as the equivalence conditions (1)–(4) from 5.1.1, are presented in McPolin and Wickstead [309].)

(3) It is seen from Theorem 4.3.4 and Corollary 4.3.5 that, at least in the case of a universally complete vector lattice, the claim of the Wickstead problem reduces to simple properties of reals and cardinals within $\mathbb{V}^{(\mathbb{B})}$. But even the reader who mastered the technique (of ascending and descending) of Boolean valued analysis might find the above demonstration bulky as compared with the standard proof in Abramovich, Veksler, and Koldunov [12]; McPolin and Wickstead [309], and Gutman [161]. But the aim of the exposition is to demonstrate that the Boolean approach to the problem reveals new insights and new interconnections.

(4) Wickstead's problem admits different answers depending on the spaces in which the operators in question are considered. There are many results that guarantee automatic boundedness for a band preserving operator in the particular classes of vector lattices. According to Abramovich, Veksler, and Koldunov [12, Theorem 2.1] (see also [12], [11]) every band preserving operator from a Banach lattice to a normed vector lattice is bounded. This claim remains valid if the Banach lattice of departure is replaced by a relatively uniformly complete vector lattice [12]. In McPolin and Wickstead [309] a similar result is obtained for the band preserving operators in a relatively uniformly complete vector lattice endowed with a locally convex locally solid topology.

(5) Consider a band preserving operator $S : \mathscr{R}{\downarrow} \to \mathscr{R}{\downarrow}$ satisfying the Cauchy exponential equation: $S(x + y) = S(x)S(y)$ for all $x, y \in \mathscr{R}{\downarrow}$. If, moreover, S enjoys the condition $S(\lambda x) = S(x)^\lambda$ for all $0 < \lambda \in \mathbb{R}$ and $x \in \mathscr{R}{\downarrow}$; then we call S an *exponential operator*. Say that S is order bounded if S sends order bounded sets to order bounded sets. If σ is the ascent of S then σ is exponential within $\mathbb{V}^{(\mathbb{B})}$. Therefore, in the class of functions bounded above on some nondegenerate interval we see that $\sigma = 0$ or $\sigma(x) = e^{cx}$ for all $x \in \mathscr{R}$ and some $c \in \mathscr{R}$. This implies the following (see Gutman, Kusraev, and Kutateladze [163]):

Each band preserving exponential operator S on $\mathscr{R}{\downarrow}$ is order bounded *(and so S may be presented as $S(x) = e^{cx}$ for all $x \in \mathscr{R}{\downarrow}$ and some $c \in \mathscr{R}{\downarrow}$ or S is identically zero).*

(6) An analogous situation takes place if S satisfies the *Cauchy logarithmic equation* $S(xy) = S(x) + S(y)$ for all $0 \ll x, y \in \mathscr{R}{\downarrow}$ and enjoys the condition $S(x^\lambda) = \lambda S(x)$ for all $\lambda \in \mathbb{R}$ and $x \gg 0$. (The record $0 \ll x$ means that $0 \leqslant x$ and $x^{\perp\perp} = \mathscr{R}{\downarrow}$.) We call an S of this sort a *logarithmic operator*. We may now formulate another equivalent claim as follows:

Every band preserving logarithmic operator S on $\{x \in \mathscr{R}{\downarrow} : x \gg 0\}$ is order bounded (and, consequently, S may presented as $S(x) = c \log x$ for all $0 \ll x \in \mathscr{R}{\downarrow}$ with some $c \in \mathscr{R}{\downarrow}$).

4.15.7. (1) In Section 4.7 we follow Gutman [161]. The claim of 4.6.4 can be considered as a solution to the Wickstead problem about the order boundedness of all band preserving operators. But the new notion of locally one-dimensional vector lattice crept into the answer. The novelty of this notion led to the conjecture that it coincides with that of a discrete (= atomic) vector lattice. In 1981 Abramovich, Veksler and Koldunov [12, Theorem 2.1] gave a proof for existence of an order unbounded band preserving operator in every nondiscrete universally complete vector lattice, thus seemingly corroborating the conjecture that a locally one-dimensional vector lattice is discrete (also cp. [4, Section 5]). But the proof was erroneous. Later in 1985, McPolin and Wickstead [309, Section 3] gave an example of a nondiscrete locally one-dimensional vector lattice, confuting the conjecture. But there was an error in the example. Finally, Wickstead [13] stated the conjecture as an open problem in 1993.

(2) This problem was solved by Gutman [161]: He constructed an atomless Dedekind complete vector lattice with a singleton d-basis. Moreover, Gutman gave a purely algebraic description of locally one dimensional universally complete vector lattices (see Theorem 4.7.6). Proposition 4.7.11 is also due to Gutman (see Gutman, Kusraev, and Kutateladze [163]).

4.15.8. (1) It follows from Theorems 4.8.5 and 4.1.7 that a vector lattice with the projection property is locally one-dimensional if and only if each band preserving projection operator on it is a band projection. Thus, a vector lattice with the projection property is locally one-dimensional if and only if all band preserving projection operators are is order bounded.

(2) Theorem 4.8.9, the main result of Section 4.8, is due to Abramovich and Kitover [7, Theorem 3.4]. But the description of the unbounded part of a band preserving projection operator P in [7, Theorem 3.4] relies upon Theorem 3.2 in [7] which is incorrect. Indeed, as can be seen from the proof of 4.8.7, a componentwise closed and laterally complete sublattice $X_0 \subset X$ admits infinitely many band preserving projections P with $X_0 = \mathrm{im}(P)$ or $X_0 = \ker(P)$ each of which is defined by the particular choice of $\mathscr{X}_1$.

(3) Since the space of $\mathbb{R}^\wedge$-linear functions in $\mathscr{R}$ admits a complete description that uses a Hamel basis $\big($cp. 2.1.7 (2)$\big)$; therefore, $\mathrm{End}_N(\mathscr{R}{\downarrow})$ may be described completely by means of a (strict) local Hamel basis. But this approach will evoke some problems of unicity.

4.15.9. (1) Theorem 4.9.8, the main result of Section 4.9, was proved by Kusraeva [259] using the d-basis machinery from Abramovich and Kitover [10, Theorem 3.4]. Our proof utilizes a Hamel basis within a Boolean valued model. Propositions 4.9.2, 4.9.3, and 4.9.7 are taken from Boulabiar, Buskes, and Sirotkin (Lemma 3.1, Proposition 3.2, and Theorem 3.3 of [67], respectively).

(2) Boulabiar, Buskes, and Sirotkin in [67, Theorem 4.6] proved among other things that an Archimedean vector lattice X is Kaplansky complete if and only if each locally algebraic orthomorphism on X is a strongly diagonal operator. Recall that a vector lattice X is *Kaplansky complete* if for every countable infinite disjoint set E in X_+ there exist $u \in X_+$ and an infinite set $F \subset E$ such that $u \wedge f = 0$ for all $f \in F$, and a linear operator T on X is *locally algebraic* if for every $u \in X$, there exists a nonzero polynomial $\varphi \in \mathbb{R}(x)$ (depending on u) such that $\varphi(T)(u) = 0$. Thus Kaplansky completeness amounts to saying that every locally algebraic orthomorphism is algebraic.

4.15.10. The definition of complex vector lattice in Section 4.10 is due to Lotz [287]. Complex vector and Banach lattices are treated in the books by Abramovich and Aliprantis [5, Section 3.2], Meyer-Nieberg [311, Section 2.2], Schaefer [356, Chap. II, Section 11], and Zaanen [427, Sections 91 and 92] and [428, Chapter 6]. An axiomatic approach to complex vector lattice was used in Mittelmeyer and Wolff [312].

4.15.11. (1) Detailed presentation of a portion of field theory in 4.11.1–4.11.9 is in Bourbaki [70, Chapter V], Van der Warden [405], and Zariski and Samuel [429, Chapter II]. Theorem 4.11.9 in the present form is from Gutman, Kusraev, and Kutateladze [163, Theorem 3.2.7].

(2) Two arbitrary transcendence bases for a field over a subfield have the same cardinality called the *transcendence degree* (cp. [429, Chapter II, Theorem 25]). Let $\tau(\mathscr{C})$ be the transcendence degree of $\mathscr{C}$ over $\mathbb{C}^{\wedge}$ within $\mathbb{V}^{(\mathbb{B})}$. The Boolean valued cardinal $\tau(\mathscr{C})$ carries some information on the connection between the Boolean algebra $\mathbb{B}$ and the complexes $\mathscr{C}$. Each Boolean valued cardinal is a mixture of relatively standard cardinals; i.e., the representation $\tau(\mathscr{C}) = \mathrm{mix}_{\xi}\, b_{\xi}\alpha_{\xi}^{\wedge}$ holds, where (b_{ξ}) is a partition of unity of $\mathbb{B}$ and (α_{ξ}) is some family of cardinals (cp. 1.9.7 and 1.9.11). Moreover, for $\mathbb{B}_{\xi} := [\mathbb{0}, b_{\xi}]$ we have $\mathbb{V}^{(\mathbb{B}_{\xi})} \models \tau(\mathscr{C}) = \alpha_{\xi}^{\wedge}$. In this connection, it would be interesting to characterize the complete Boolean algebras $\mathbb{B}$ such that $\tau(\mathscr{C}) = \alpha^{\wedge}$ within $\mathbb{V}^{(\mathbb{B})}$ for some cardinal α.

(3) Given $\mathscr{E} \subset X$, denote by $\langle \mathscr{X} \rangle$ the set of elements of the form $e_1^{n_1} \cdots e_k^{n_k}$, where $e_1, \dots, e_k \in \mathscr{E}$ and $k, n_1, \dots, n_k \in \mathbb{N}$. A set $\mathscr{E} \subset X$ is *locally algebraically independent* provided that $\langle \mathscr{E} \rangle$ is locally linearly independent in the sense of 4.5.1. This notion, presenting the external interpretation of the internal notion of algebraic independence (or transcendence), seems to turn out useful in studying the descents of fields (cp. Kusraev and Kutateladze [249, Section 8.3]) or the general regular rings (cp. Goodearl [131]).

4.15.12. (1) Theorems 4.12.1 and 4.12.6 as well as Corollaries 4.12.7 and 4.12.11 were obtained by Kusraev [231] (see also [163, 232]). In particular, if μ is an atomless Maharam measure then the algebra $L^0_{\mathbb{C}}(\Omega, \Sigma, \mu)$ admits nontrivial derivations which are evidently not inner and also not continuous with respect to the topology of convergence in measure. Ber, Chilin, and Sukochev [48] proved independently that the algebra $L^0_{\mathbb{C}}([0,\,1])$ of all (classes of equivalence of) measurable complex functions on the interval $[0,\,1]$ admit nontrivial derivations. Some extensions of this result and interesting related questions are discussed in Ber, de Pagter, and Sukochev [49].

(2) Using the same arguments as in Section 4.12, we can infer from 4.12.8 that if $\mathbb{R}^{\wedge} \neq \mathscr{R}$ then there are nontrivial derivations on the real f-algebra $\mathscr{R}{\downarrow}$. Thus, in the class of universally complete real vector lattices with a fixed structure of an f-algebra the absence of nontrivial derivations is equivalent to the σ-distributivity of the base of the algebra under consideration. At the same time there are no nontrivial band preserving automorphisms of the f-algebra $\mathscr{R}{\downarrow}$, regardless of the properties of its base (see 4.12.8).

(3) It is well known that if Q is a compact space then there are no nontrivial derivations on the algebra $C(Q, \mathbb{C})$ of continuous complex functions on Q; for example see Aczél and Dhombres [14, Chapter 19, Theorem 21]. At the same time, we see from 4.12.6 (1, 4) that if Q is an extremally disconnected compact space and the Boolean algebra of the clopen sets of Q is not σ-distributive then there is a nontrivial derivation on $C_\infty(Q, \mathbb{C})$.

(4) Let $L^0_{\mathbb{C}}(\Omega, \Sigma, \mu)$ be the space of all (cosets of) measurable complex functions, and let $L^\infty_{\mathbb{C}}(\Omega, \Sigma, \mu)$ be the space of essentially bounded measurable complex functions. Then $L^\infty_{\mathbb{C}}(\Omega, \Sigma, \mu)$ is isomorphic to some $C(Q, \mathbb{C})$; consequently, there are no nontrivial derivations on it. If the Boolean algebra $\mathbb{B}(\Omega, \Sigma, \mu)$ of measurable sets modulo negligible sets is not atomic (and therefore it is not σ-distributive; cp. 4.7.11); then, by 4.12.6 (4), there exist nontrivial derivations on $L^0_{\mathbb{C}}(\Omega, \Sigma, \mu)$ (cp. [48, 230, 219]). The same is true about the spaces $L^\infty(\Omega, \Sigma, \mu)$ and $L^0(\Omega, \Sigma, \mu)$ of measurable real functions.

(5) Consider a band preserving operator $S : X \to X$ with $X := \mathscr{C}{\downarrow}$ satisfying the Cauchy functional equation $S(u + v) = S(u)S(v)$ for all $u, v \in X$. If, in addition, S satisfies the condition $S(\lambda u) = S(u)^\lambda$ for arbitrary $\lambda \in \mathbb{C}$ and $u \in X$ then we say that S is *exponential*. Say that S is order bounded if S sends order bounded sets to order bounded sets. If σ is the ascent of S then σ is exponential within $\mathbb{V}^{(\mathbb{B})}$; therefore, in the class of functions bounded from above on a nonzero interval, we have either $\sigma = 0$ or $\sigma(x) = e^{cx}$ $(x \in \mathscr{C})$ for some $c \in X$; see Aczél and Dhombres [14, Chapter 5, Theorem 5]. Hence, we conclude that X *is locally one-dimensional if and only if every band preserving exponential operator in* $X := \mathscr{C}{\downarrow}$ *is order bounded* (*and consequently has the form* $S = 0$ *or* $S(x) = e^{cx}$, $x \in C$, *for some* $c \in X$).

(6) Kurepa [214] proved that an additive function $f : \mathbb{R} \to \mathbb{R}$ is a derivation if and only if $f(x) = -x^2 f(1/x)$ for all $0 \neq x \in \mathbb{R}$. Interpreting this fact in a Boolean valued model and using the method of Section 4.12 we arrive at the following result: *A band preserving linear operator* T *in a universally complete real vector lattice* X *is a derivation if and only if* $Tx = -x^2 T(x^{-1})$ *for all invertible* $x \in X$. In a similar fashion, we can prove some vector lattice counterparts of various characterizations and properties of derivations in $\mathbb{R}$ and $\mathbb{C}$.

4.15.13. (1) See Theorem 4.13.3 in Kuczma [211, Theorem 12.5.2]. Theorem 4.13.5, the main result of Section 4.13, was proved by Kusraeva [260] using the technique of d-bases. In view of Corollary 2.4.7 a few points to note about spaces without complex structure should be made.

(2) Recall that a (real) Banach space X is said to admit *complex* *structure* if there exists a bounded linear operator T on X with $T^2 = -I_X$. This enables us to define on X a structure of vector space over $\mathbb{C}$ as in 4.13.7. Moreover, we can define a complex norm on X which is equivalent to the original. A finite-dimensional vector space admits complex structure if and only if the dimension of the space is even. In the infinite-dimensional setting, there are real Banach spaces admitting no complex structure. This is the case of the James space, as it was shown by Dieudonné [104]. More examples of this kind have been constructed over the years, including uniformly convex examples (Szarek [371]), the hereditary indecomposable space of Gowers and Maurey [143], etc. (see also [142, 144]).

(3) It is worth observing at this point that some complex Banach spaces cannot be obtained as the complexification of any real Banach space. Bourgain [71] proved the existence of such space using probabilistic arguments; the first explicit example was given by Kalton [189]. A finite-dimensional version of this result was independently developed by Szarek [372].

4.15.14. (1) Concerning the Boolean valued representation of lattice ordered rings and modules see Kusraev and Kutateladze [249], Ozawa [325], and Takeuti [384]. Definition 4.14.A.6 and Theorem 4.14.A.7 were introduced in Gutman, Kusraev, and Kutateladze [163, Theorem 4.3.6].

(2) Recently, Chilin and Karimov [90] obtained a classification result for regular laterally complete modules over a universally complete f-algebra $\Lambda := C_\infty(Q)$. They introduced the *passport* $\Gamma(X) = (b_\gamma)_{\gamma\in\Gamma}$ for a such module X which is the uniquely defined partition of unity in $\mathbb{B} := \mathbb{P}(\Lambda)$ indexed by a set of pairwise different cardinals with $b_\gamma X$ being strictly γ-homogeneous for all $\gamma \in \Gamma$ (cp. 4.6.8). Then they proved that Λ-modules are isomorphic if and only if their passports coincide [90, Theorem 4.3]. It can easily be seen that a regular laterally complete (in the sense of [90]) Λ-module X is represented as $X = \mathscr{X}{\downarrow}$ where $\mathscr{X}$ is a real vector space in $\mathbb{V}^{(\mathbb{B})}$. Moreover, $\dim(\mathscr{X}) \in \mathbb{V}^{(\mathbb{B})}$, the algebraic dimension of $\mathscr{X}$, is an internal cardinal and, according to 1.9.11, we have $\dim(\mathscr{X}) = \operatorname{mix}_{\gamma\in\Gamma} b_\gamma \gamma^\wedge$ where Γ is a set of cardinals and $(b_\gamma)_{\gamma\in\Gamma}$ is the passport of X. Thus, the passport $\Gamma(X)$ is the interpretation of the algebraic dimension $\dim(\mathscr{X})$, whence the result ensues.

(3) The characterization in Theorem 4.14.B.7 of a universally complete vector lattice in which all band preserving bilinear operators are symmetric is due to Kusraev [236]. The following corollary is immediate from this fact and 4.7.7: *Let* (Ω, Σ, μ) *be a nonatomic Maharam* *measure space and let* $L^0(\Omega, \Sigma, \mu)$ *be the vector space of all cosets of* (*almost everywhere equal*) *real measurable functions. Then there exists an essentially nontrivial separately band preserving antisymmetric bilinear operator in* $L^0(\Omega, \Sigma, \mu)$.

(4) Theorem 4.14.C.8 is taken from Gutman, Kusraev, and Kutateladze [163, Theorem 4.3.6]. This fact lies in an interesting area of the theory of noncommutative integration stemming from Segal [362]. Considerable attention is given to derivations in various algebras of measurable operators associated with an AW^*-algebra and a central valued trace. We mention only the article [23] by Albeverio, Ajupov, and Kudaybergenov and the article [49] by Ber, de Pagter, and Sukochev.

CHAPTER 5

Order Continuous Operators

The approach of Chapter 3 is not applicable directly to order continuous operators since we lose order continuity in ascending an operator (see 3.3.2). The technique of ascending in Chapter 4 preserves order continuity, but this approach treats a narrow class of band preserving operators. In this chapter we pursue another approach that rests on Maharam's ideas. This chapter focuses on Maharam operators in Dedekind complete vector lattices. The Maharam operators possess the most important properties of conditional expectation and enjoy the Radon–Nikodým type theorem and the Hahn type decomposition theorem. Surprisingly, each injective Banach lattice admits some Maharam operator that completely determines the structural particularities of the lattice. We will also consider some classes of operators whose definitions depend implicitly or explicitly on Maharam operators.

5.1. Order Bounded Module Homomorphisms

Here we will address the Boolean valued representation of f-module homomorphisms and suggest some construction of f-modules that corresponds to the natural embedding into the order bidual in the case of order bounded functionals.

5.1.1. Assume that A is an f-algebra, while X and Y are f-modules over A. A linear mapping $T : X \to Y$ is called an A-*module homomorphism* or A-*linear* if $T(ax) = aTx$ for all $x \in X$ and $a \in A$. Denote by $L_A(X, Y)$ the A-module of all A-linear operators from X to Y and define $L_A^\sim(X, Y)$ as the submodule of $L_A(X, Y)$ consisting of order bounded operators; i.e., $L_A^\sim(X, Y) := L_A(X, Y) \cap L^\sim(X, Y)$. Let $L_{n,A}^\sim(X, Y)$ be the part of $L_A^\sim(X, Y)$ comprising order continuous operators. If Y is Dedekind complete then $L_A^\sim(X, Y)$ and $L_{n,A}^\sim(X, Y)$ are f-modules over A as indicated in 2.11.1.

5.1.2. *If X and Y are f-modules over A, with Y Dedekind complete; then $L^{\sim}_A(X,Y)$ and $L^{\sim}_{n,A}(X,Y)$ are bands in $L^{\sim}(X,Y)$.*

$\lhd$ Given $a \in A$, define $\hat{a} \in \mathrm{Orth}(X)$ and $\bar{a} \in \mathrm{Orth}(Y)$ by putting $\hat{a}x := ax$ $(x \in X)$ and $\bar{a}y := ay$ $(y \in Y)$. Now define the endomorphisms R_a and L_a of $L^{\sim}(X,Y)$ as

$$R_a(T) := T \circ \hat{a}, \quad L_a(T) := \bar{a} \circ T \quad (T \in L^{\sim}(X,Y)).$$

Observe that R_a and L_a are orthomorphisms in $L^{\sim}(X,Y)$ (see 5.3.2 (1)) and an order bounded operator $T \in L^{\sim}(X,Y)$ belongs to $L^{\sim}_A(X,Y)$ if and only if $R_a(T) = L_a(T)$ for all $a \in A$, so that

$$L^{\sim}_A(X,Y) = \bigcap_{a\in A} \ker(R_a - L_a).$$

Thus, $L^{\sim}_A(X,Y)$ is a band, since the kernel $\ker(R_a - L_a)$ of the orthomorphism $R_a - L_a$ is a band. It remains to note that $L^{\sim}_{n,A}(X,Y)$ is the intersection of the two bands in $L^{\sim}(X,Y)$, namely, $L^{\sim}_A(X,Y)$ and $L^{\sim}_n(X,Y)$. $\rhd$

5.1.3. Assume that X and Y are unital f-modules over a Dedekind complete f-algebra A with unit which we identify with an f-subalgebra and an order ideal in $A^{\mathfrak{u}} := \mathscr{R}{\downarrow}$. Put $\mathbb{B} := \mathbb{P}(A)$ and let $\mathscr{X}, \mathscr{Y} \in \mathbb{V}^{(\mathbb{B})}$ stand for the Boolean valued representations of X and Y, respectively. Recall that $[\![\mathscr{X}$ and $\mathscr{Y}$ are real vector lattices$]\!] = \mathbb{1}$. So, $\mathscr{X}{\downarrow}$ and $\mathscr{Y}{\downarrow}$ are f-modules over $A^{\mathfrak{u}}$, and there are f-module isomorphisms $\imath$ from X to $X' := \mathscr{X}{\downarrow}$ and $\jmath$ from Y to $Y' := \mathscr{Y}{\downarrow}$ such that $\mathscr{X}{\downarrow} = \mathrm{mix}(\imath(X))$ and $\mathscr{Y}{\downarrow} = \mathrm{mix}(\imath(Y))$ (cp. 2.11.4 and 2.11.9). Let $L(\mathscr{X},\mathscr{Y})$ stand for the element in $\mathbb{V}^{(\mathbb{B})}$ uniquely defined by the relation: $\mathscr{T} \in L(\mathscr{X},\mathscr{Y}){\downarrow}$ if and only if $[\![\mathscr{T}$ is a linear operator from $\mathscr{X}$ to $\mathscr{Y}]\!] = \mathbb{1}$. Then $L(\mathscr{X},\mathscr{Y}){\downarrow}$ is an $A^{\mathfrak{u}}$-module and, given $\mathscr{T} \in L(\mathscr{X},\mathscr{Y}){\downarrow}$, the descent $\mathscr{T}{\downarrow}$ is an $A^{\mathfrak{u}}$-linear operator from X' to Y'. The spaces $L(\mathscr{X},\mathscr{Y})$, $L(X,Y)$, and $L(X',Y')$ are considered as ordered vector spaces with the cones of positive operators.

5.1.4. Theorem. *For each A-linear operator $T : X \to Y$ there exists a unique $\mathscr{T} \in \mathbb{V}^{(\mathbb{B})}$ such that $[\![\mathscr{T}$ is a linear operator from $\mathscr{X}$ to $\mathscr{Y}]\!] = \mathbb{1}$, $T' := \mathscr{T}{\downarrow}$ is an $A^{\mathfrak{u}}$-linear operator from X' to Y', and $\jmath \circ T = T' \circ \imath$. The mapping $T \mapsto \mathscr{T}$ is an order preserving injection from $L_A(X,Y)$ into $L(\mathscr{X},\mathscr{Y}){\downarrow}$ and the mapping $\mathscr{T} \mapsto \mathscr{T}{\downarrow}$ is an order preserving bijection from $L(\mathscr{X},\mathscr{Y}){\downarrow}$ onto $L_A(X',Y')$.*

$\triangleleft$ By Theorem 2.11.9 we can assume without loss of generality that $X \subset X'$ and $Y \subset Y'$ with $X' = \operatorname{mix}(X)$ and $Y' = \operatorname{mix}(Y)$, while $\imath$ and $\jmath$ are the embeddings. If $b = [\![x_1 = x_2]\!]$ and $\pi := \jmath(b)$ with $\jmath$ defined as in 2.11.4, then $\pi x_1 = \pi x_2$ and so $\pi T x_1 = T(\pi x_1) = T(\pi_2 x_2) = \pi_2 T x_2$. It follows that $b \leqslant [\![Tx_1 = Tx_2]\!]$. Thus T is extensional and we can define $\mathscr{T} := T{\uparrow}$. In view of 1.6.5 $\mathscr{T}$ is a mapping from $\mathscr{X}$ to $\mathscr{Y}$ within $\mathbb{V}^{(\mathbb{B})}$, since $X{\uparrow} = \mathscr{X}$ and $Y{\uparrow} = \mathscr{Y}$. By 1.6.6 we have $T'|_X = T$. If $\bar{\mathscr{T}}$ is one more linear operator from $\mathscr{X}$ to $\mathscr{Y}$ within $\mathbb{V}^{(\mathbb{B})}$ and $(\bar{\mathscr{T}}{\downarrow}) \circ \imath = \jmath \circ T$ then T' and $\bar{\mathscr{T}}{\downarrow} : X' \to Y'$ coincide on $\imath(X)$ and so $T' = \bar{\mathscr{T}}{\downarrow}$ by 1.6.5, since both T' and $\bar{\mathscr{T}}{\downarrow}$ are extensional. It follows that $\mathscr{T} = \bar{\mathscr{T}}$.

Let $\odot$ and $\oplus$ stand respectively for scalar multiplication and addition on $\mathscr{X}$, $\mathscr{Y}$, and $\mathscr{R}$. From the properties of ascents and descents it follows that the identities $\mathscr{T}x = Tx$, $a \odot x = ax$, $a \odot y = ay$, $x_1 \oplus x_1 = x_1 + x_2$, and $y_1 \oplus y_2 = y_1 + y_2$ hold within $\mathbb{V}^{(\mathbb{B})}$ for all $x, x_1, x_2 \in X$, $y_1, y_2 \in Y$, and $a \in A$. Combining these and the formula $\mathscr{R}{\downarrow} = \operatorname{mix}(A)$, we deduce from 1.5.2 and 1.6.2 that $\mathscr{T}$ is linear within $\mathbb{V}^{(\mathbb{B})}$ if and only if T is A-linear:

$$
\begin{aligned}
&[\![(\forall\, a_1, a_2 \in \mathscr{R})(\forall\, x_1, x_2 \in \mathscr{X})\\
&\qquad\qquad \mathscr{T}(a_1 \odot x_1 \oplus a_2 \odot x_2) = a_1 \odot \mathscr{T}(x_1) \oplus a_2 \odot \mathscr{T}(x_2)]\!]\\
&\quad = \bigwedge_{a_1, a_2 \in A}\ \bigwedge_{x_1, x_2 \in X} [\![T(a_1 x_1 + a_2 x_2) = a_1 T(x_1) + a_2 T(x_2)]\!] = \mathbb{1}.
\end{aligned}
$$

The same argument implies that $\mathscr{T}$ is linear within $\mathbb{V}^{(\mathbb{B})}$ if and only if T' is $A^{\mathfrak{u}}$-linear. It follows from 1.6.7 that $\mathscr{T} \mapsto \mathscr{T}{\downarrow}$ is a bijection between $L(\mathscr{X}, \mathscr{Y}){\downarrow}$ and $L_A(X', Y')$, while $T \mapsto \mathscr{T}{\uparrow}$ is an injection from $L_A(X, Y)$ into $L(\mathscr{X}, \mathscr{Y}){\downarrow}$. The linearity of these mappings may be proved by the argument similar to that in the proof of Theorem 3.3.3. Finally, by 1.5.1 we have

$$
\begin{aligned}
&[\![\mathscr{T}(\mathscr{X}_+) \subset \mathscr{Y}_+]\!] = \mathbb{1} \Longleftrightarrow (\mathscr{T}(\mathscr{X}_+)){\downarrow} \subset (\mathscr{Y}_+){\downarrow}\\
&\qquad\qquad \Longleftrightarrow \mathscr{T}{\downarrow}((\mathscr{X}{\downarrow})_+) \subset (\mathscr{Y}{\downarrow})_+ \Longleftrightarrow T'(X'_+) \subset Y'_+,
\end{aligned}
$$

whence $\mathscr{T}$ is positive within $\mathbb{V}^{(\mathbb{B})}$ if and only if T' is positive. $\triangleright$

5.1.5. We make some additional remarks using the same notation as in Theorem 5.1.4. Note first that, since $L_A(X', Y')$ and $L(\mathscr{X}, \mathscr{Y}){\downarrow}$ are linearly and order isomorphic, T or T' is regular if and only if $\mathscr{T}$ is regular within $\mathbb{V}^{(\mathbb{B})}$.

(1) The order boundedness of $\mathscr{T}$ may be written as $(\forall\, a \in \mathscr{X}_+)(\exists\, b \in \mathscr{Y}_+)\mathscr{T}([-a,a]) \subset [-b,b]$. Interpreting this formula within $\mathbb{V}^{(\mathbb{B})}$ and considering that $\mathscr{T}([-a,a]){\downarrow} = T'([-a,a])$, it is easy to show that T' is order bounded if and only if so is $\mathscr{T}$ within $\mathbb{V}^{(\mathbb{B})}$. In particular, the order boundedness of T implies that $\mathscr{T}$ is order bounded within $\mathbb{V}^{(\mathbb{B})}$. But the converse is not true: If $\mathscr{T}$ is order bounded then T have the following "slicewise" order boundedness property: for each $a \in X_+$ there exist a partition of unity (π_ξ) in $\mathbb{P}(A)$ and a disjoint family (b_ξ) in Y_+ such that $\pi_\xi T([-a,a]) \subset [-b_\xi, b_\xi]$ for all ξ.

(2) *A positive operator $T \in L^r_A(X,Y)$ is order continuous if and only if so is $\mathscr{T} \in L^r(\mathscr{X},\mathscr{Y})$ within $\mathbb{V}^{(\mathbb{B})}$.*

$\lhd$ According to 3.1.2 (1) there is no loss of generality in assuming that T and T' are positive. Observe that T and T' are or are not order continuous simultaneously, since X and Y are order dense sublattices in X' and Y', respectively. If $[\![\mathscr{A}$ is a downward directed set in $\mathscr{X}$ and $\inf(\mathscr{A}) = 0]\!] = \mathbb{1}$, then $A := \mathscr{A}{\downarrow}$ is also downward directed and $\inf(A) = 0$ in X'. The order continuity of T' implies that $\inf(\mathscr{T}(\mathscr{A})) = \inf(T'(A)) = 0$ within $\mathbb{V}^{(\mathbb{B})}$. Conversely, if $\mathscr{T}$ is order continuous within $\mathbb{V}^{(\mathbb{B})}$ and $A \subset X'$ is downward directed with $\inf(A) = 0$, then $[\![\mathscr{A} := A{\uparrow}$ is upward directed and $\inf(\mathscr{A}) = 0]\!] = \mathbb{1}$; consequently, for $y := \inf(T'(A))$ we have $[\![a = \inf(\mathscr{T}(\mathscr{A})) = 0]\!] = \mathbb{1}$, whence $a = 0$. $\rhd$

Given a real vector lattice $\mathscr{X}$ within $\mathbb{V}^{(\mathbb{B})}$, denote by $\mathscr{X}^\sim$ and $\mathscr{X}^\sim_n$ the internal vector lattices of order bounded and order continuous functionals on $\mathscr{X}$, respectively. More precisely, $[\![\sigma \in \mathscr{X}^\sim]\!] = \mathbb{1}$ and $[\![\sigma \in \mathscr{X}^\sim_n]\!] = \mathbb{1}$ mean that $[\![\sigma : \mathscr{X} \to \mathscr{R}$ is an order bounded functional$]\!] = \mathbb{1}$ and $[\![\sigma : \mathscr{X} \to \mathscr{R}$ is an order continuous functional$]\!] = \mathbb{1}$, respectively.

5.1.6. Corollary. *Let Y be a Dedekind complete vector lattice and let X be a unital f-module over $A := \mathscr{Z}(Y)$. For every $T \in L^\sim_A(X,Y)$ there exists a unique $\tau \in \mathbb{V}^{(\mathbb{B})}$ such that $[\![\tau : \mathscr{X} \to \mathscr{R}$ is $\mathscr{R}$-linear and order bounded$]\!] = \mathbb{1}$, $T' := \tau{\downarrow}$ is an order bounded $A^{\mathfrak{u}}$-linear operator from X' to $\mathscr{R}{\downarrow}$, and $\jmath \circ T = T' \circ \imath$. The mapping $T \mapsto \tau$ is an injection from $L^\sim_A(X,Y)$ into $(\mathscr{X}^\sim){\downarrow}$ and from $L^\sim_{n,A}(X,Y)$ into $(\mathscr{X}^\sim_n){\downarrow}$. The mapping $\tau \mapsto \tau{\downarrow}$ is a lattice isomorphism of $(\mathscr{X}^\sim){\downarrow}$ onto $L^\sim_A(X',Y')$ and $(\mathscr{X}^\sim_n){\downarrow}$ onto $L^\sim_{n,A}(X',Y')$.*

5.1.7. Theorem 5.1.4 and Corollary 5.1.6 enable us to treat some classes of module homomorphisms on f-modules over A as $\mathbb{R}$-linear operators and, whenever A is the center of the range vector lattice, as

$\mathbb{R}$-linear functionals. As an illustration consider the interpretation of Nakano's results concerning the embedding a vector lattice into the order continuous bidual. To do this we need to recall some more definitions and facts.

Consider an order ideal $\mathscr{J}$ in $L^{\sim}_{n,A}(X,Y)$. Note that then $\mathscr{J}$ is a f-submodule of $L^{\sim}_{n,A}(X,Y)$. For an arbitrary $x \in X$ define the operator $\hat{x} : \mathscr{J} \to Y$ by $\hat{x}(T) := Tx$ for all $T \in \mathscr{J}$. Clearly, $\hat{x} \in L^{\sim}_{A}(\mathscr{J}, Y)$, since $\hat{x} = (x^{+})\hat{} - (x^{-})\hat{}$, $\hat{x}$ is a positive operator whenever $x \in X_{+}$, and $\hat{x}(aT) = T(ax) = a\hat{x}(T)$ for all $a \in A$ and $T \in \mathscr{J}$. Moreover, $\hat{x} \in L^{\sim}_{n,A}(\mathscr{J}, Y)$. Indeed, if a net T_α in $\mathscr{J}$ is decreasing and $\inf_\alpha T_\alpha = 0$, then $o\text{-}\lim_\alpha \hat{x}(T_\alpha) = 0$, since $|\hat{x}(T_\alpha)| = |T_\alpha(x)| \leqslant T_\alpha(|x|)$ and $o\text{-}\lim T_\alpha(|x|) = 0$ in Y for all $x \in X$.

The mapping $x \mapsto \hat{x}$ is called the *natural embedding* of X to $L^{\sim}_{A}(\mathscr{J}, Y)$.

An f-module X over $A := \mathscr{Z}(Y)$ is said to be *Y-perfect with respect to $\mathscr{J}$* (or *Y-perfect* whenever $\mathscr{J} = L^{\sim}_{n,A}(X,Y)$) if the natural embedding is a bijection of X onto $L^{\sim}_{n,A}(\mathscr{J}, Y)$. In case $Y = \mathbb{R}$ we say that X is *perfect with respect to* $\mathscr{J}$.

5.1.8. Theorem. *Let X be a vector lattice, and let $\mathscr{J}$ be an ideal of $X^{\sim}_{n}$. Then the embedding $x \mapsto \hat{x}$ is an order continuous lattice homomorphism from X to $\mathscr{J}^{\sim}_{n}$ whose range is an order dense vector sublattice of $\mathscr{J}^{\sim}_{n}$. The natural embedding is an injection if and only if $\mathscr{J}$ separates the points of X.*

$\lhd$ See Aliprantis and Burkinshaw [28, Theorem 1.70] and Zaanen [427, Theorem 109.3]. $\rhd$

5.1.9. Theorem. *A vector lattice X is perfect with respect to an order ideal $\mathscr{J}$ in $X^{\sim}_{n}$ if and only if $\mathscr{J}^{\sim}_{n}$ separates the points of X and, given an increasing net (x_α) in X_{+} such that $\sup_\alpha f(x_\alpha) < \infty$ for each $0 \leqslant f \in \mathscr{J}$, there exists some $x \in X$ satisfying $x = \sup_\alpha x_\alpha$ in X.*

$\lhd$ See Aliprantis and Burkinshaw [28, Theorem 1.71] and Zaanen [427, Theorem 110.1]. $\rhd$

5.1.10. Theorem. *Let X be an f-module over $A := \mathscr{Z}(Y)$ with Y a Dedekind complete vector lattice and $\mathscr{J}$ an order ideal in $L^{\sim}_{n,A}(X,Y)$ separating the points of X. Then*

(1) *The natural embedding is an order continuous A-linear lattice isomorphism from X to $L^{\sim}_{n,A}(\mathscr{J}, Y)$ whose range is an order dense sublattice of $L^{\sim}_{n,A}(\mathscr{J}, Y)$.*

(2) *X is Y-perfect with respect to $\mathscr{J}$ if and only if, given an increasing net (x_α) in X_+ with $\sup_\alpha Tx_\alpha$ existing in Y for all $T \in \mathscr{J}$, there exists $x \in X$ such that $x = \sup_\alpha x_\alpha$.*

$\lhd$ We use the same notation as in Theorem 5.1.4 and Corollary 5.1.6 and identify $\mathbb{B}$ with $\mathbb{P}(A)$. Assume that $X = X'$ and $Y = Y'$. Put $\mathfrak{J} := \{T{\uparrow} : T \in \mathscr{J}\}{\uparrow}$ and observe that $\mathfrak{J}$ is an order ideal in $\mathscr{X}_n^\sim$. Indeed, according to Corollary 5.1.6, $L_{n,A}^\sim(X', Y')$ and $(\mathscr{X}_n^\sim){\downarrow}$ are f-module isomorphic under the ascent and the isomorphism sends $\mathscr{J}$ onto an order ideal in $(\mathscr{X}_n^\sim){\downarrow}$, say $\bar{\mathscr{J}}$; therefore, $[\![\mathfrak{J} = \bar{\mathscr{J}}{\uparrow}$ is an order ideal in $(\mathscr{X}_n^\sim)]\!] = \mathbb{1}$. By Corollary 5.1.6 the mapping $\tau \mapsto \tau{\downarrow}$ is a lattice isomorphism of $(\mathfrak{J}_n^\sim){\downarrow}$ onto $L_{n,A}^\sim(\mathscr{J}', Y')$ where $\mathscr{J}' := \mathfrak{J}{\downarrow}$.

Show that $[\![\,\mathfrak{J}$ is point separating$]\!] = \mathbb{1}$. Take $x \in X$ and put $b_x := b = [\![x \neq 0]\!]$. From the Kutatowski–Zorn Lemma it is easy to derive that there exist a partition $(b_\xi)_{\xi \in \Xi \cup \{\xi_0\}}$ of unity in $\mathbb{P}(A)$ and a family $(T_\xi)_{\xi \in \Xi \cup \{\xi_0\}}$ in $\mathscr{J}$ such that $T_{\xi_0} = 0$, $b_{\xi_0} = b^\perp$, and $b_\xi \leqslant [\![T_\xi x \neq 0]\!]$ for all $\xi \in \Xi$. Define $\mathscr{T}_x := \mathscr{T} \in \mathscr{X}_n^\sim$ as $\mathscr{T} := \operatorname{mix}_{\xi \in \Xi \cup \{\xi_0\}}(b_\xi(T_\xi{\uparrow}))$. Clearly, $b_\xi \leqslant [\![\mathscr{T} = T_\xi{\uparrow}]\!] \wedge [\![T_\xi{\uparrow} \in \mathfrak{J}]\!] \leqslant [\![\mathscr{T} \in \mathfrak{J}]\!]$ for all $\xi \in \Xi \cup \{\xi_0\}$ and $b_\xi \leqslant [\![\mathscr{T}x = T_\xi{\uparrow}x]\!] \wedge [\![T_\xi{\uparrow}x \neq 0]\!] \leqslant [\![\mathscr{T}x \neq 0]\!]$ for all $\xi\Xi$, so that $[\![\mathscr{T}_x \in \mathfrak{J}]\!] = \mathbb{1}$ and $b_x \leqslant [\![\mathscr{T}x \neq 0]\!]$. The result follows from simple calculation through 1.2.3, 1.5.2, and 1.6.2:

$$[\![\,\mathfrak{J} \text{ is point separating}\,]\!] = [\![(\forall\, x \in \mathscr{X})\big(x \neq 0 \to (\exists\, \tau \in \mathfrak{J})\tau(x) \neq 0\big)]\!]$$

$$= \bigwedge_{x \in X'} [\![x \neq 0]\!] \Rightarrow \bigvee_{\tau \in \mathfrak{J}\downarrow} [\![\tau(x) \neq 0]\!] \geqslant \bigwedge_{x \in X'} b_x \Rightarrow [\![\mathscr{T}_x(x) \neq 0]\!] = \mathbb{1}.$$

Let Φ and ϕ stand respectively for the natural embeddings of X to $L_A^\sim(\mathscr{J}, Y)$ and of $\mathscr{X}$ to $\mathfrak{J}_n^\sim$. Then $[\![\phi(x)\tau = \tau x = Tx = \Phi(x)T]\!] = \mathbb{1}$ for all $x \in X$, $T \in \mathscr{J}$, and $\tau := T{\uparrow}$. It follows that $\phi := \hat{\Phi}{\uparrow}$ with $\hat{\Phi} : X \to \mathfrak{J}_n^\sim{\downarrow}$ defined as $[\![\hat{\Phi}(x)\tau := \Phi(x)(T) = Tx]\!] = \mathbb{1}$ for $\tau = T{\uparrow}$ and $T \in \mathscr{J}$. From this we can easily derive that $\phi(\mathscr{X}) = \{\Phi(x){\uparrow} : x \in X\}{\uparrow}$ or, equivalently, $\phi(\mathscr{X}){\downarrow} = \operatorname{mix}\{\Phi(x){\uparrow} : x \in X\}$.

By transfer, Theorems 5.1.8 and 5.1.9 are true within $\mathbb{V}^{(\mathbb{B})}$. Thus, by Theorem 5.1.8 $[\![\,\phi : \mathscr{X} \to \mathfrak{J}_n^\sim$ is an order continuous lattice isomorphism and $\phi(\mathscr{X})$ is an order dense sublattice of $\mathfrak{J}_n^\sim\,]\!] = \mathbb{1}$. Clearly $\hat{\Phi} = \phi{\downarrow}$ is a lattice isomorphism and in view of 5.1.5 (2) it is also order continuous. Moreover, $\hat{\Phi}(X)$ is an order dense sublattice in $\mathfrak{J}_n^\sim{\downarrow}$, since $\hat{\Phi}(X) = \phi(\mathscr{X}){\downarrow}$ by 1.5.3. It follows that $\Phi(X)$ is order dense sublattice in $L_{n,A}^\sim(\mathscr{J}', Y')$.

The necessity in (2) is straightforward. To prove the sufficiency, assume that for each increasing net (x_α) in X_+ with $\sup_\alpha Tx_\alpha$ existing in Y for all $T \in \mathscr{J}$ there exists $x \in X$ such that $x = \sup_\alpha x_\alpha$. Then it can easily be seen that, given an increasing net (x_α) in $\mathscr{X}_+$ such that $\sup_\alpha \tau(x_\alpha) < \infty$ for each $0 \leqslant \tau \in \mathfrak{J}$, there exists some $x \in \mathscr{X}$ satisfying $x = \sup_\alpha x_\alpha$ in $\mathscr{X}$. By Theorem 5.1.9 we have $[\![\, \mathscr{X}$ is perfect$]\!] = [\![\phi(\mathscr{X}) = \mathfrak{J}_n^\sim]\!] = \mathbb{1}$. Thus, $\hat{\Phi}(X) = \mathfrak{J}_n^\sim{\downarrow}$ and $\Phi(X) = L_{n,A}^\sim(\mathscr{J}', Y')$ by passing to descents. $\triangleright$

5.2. Maharam Operators

Under discussion is some class of the order continuous positive operators that behave like functionals in many aspects. We establish a Radon–Nikodým-type Theorem for these operators.

5.2.1. Throughout this section X and Y are vector lattices with Y Dedekind complete. A positive operator $T : X \to Y$ is said to have the *Maharam property* (or T is said to be *order interval preserving*) whenever $T[0, x] = [0, Tx]$ for every $0 \leqslant x \in X$; i.e., if for every $0 \leqslant x \in X$ and $0 \leqslant y \leqslant Tx$ there is some $0 \leqslant u \in X$ such that $Tu = y$ and $0 \leqslant u \leqslant x$. A *Maharam operator* is an order continuous positive operator whose modulus enjoys the Maharam property.

Say that a linear operator $S : X \to Y$ is *absolutely continuous* with respect to T and write $S \preccurlyeq T$ if $|S|x \in \{|T|x\}^{\perp\perp}$ for all $x \in X_+$. It can easily be seen that if $S \in \{T\}^{\perp\perp}$ then $S \preccurlyeq T$, but the converse may be false.

5.2.2. The null ideal $\mathscr{N}_T$ of an order bounded operator $T : X \to Y$ is defined by $\mathscr{N}_T := \{x \in X : |T|(|x|) = 0$. Observe that $\mathscr{N}_T$ is indeed an ideal in X. The disjoint complement of $\mathscr{N}_T$ is referred to as the *carrier* of T and is denoted by $\mathscr{C}_T$, so that $\mathscr{C}_T := \mathscr{N}_T^\perp$. An operator T is called *strictly positive* whenever $X = \mathscr{C}_T$; i.e., $0 < x \in X$ implies $0 < |T|(x)$. Clearly, $|T|$ is strictly positive on $\mathscr{C}_T$. Sometimes we find it convenient to put $X_T := \mathscr{C}_T$ and $Y_T := (\operatorname{im} T)^{\perp\perp}$.

A positive operator $T : X \to Y$ is said to have the *Levi property* if $\sup x_\alpha$ exists in X for every increasing net $(x_\alpha) \subset X_+$, provided that the net (Tx_α) is order bounded in Y. For an order bounded order continuous operator T from X to Y denote by $\mathscr{D}_m(T)$ the largest ideal of the universal completion $X^{\mathfrak{u}}$ onto which we can extend the operator

T by order continuity. For a positive order continuous operator T we have $X = \mathscr{D}_m(T)$ if and only if T has the Levi property.

The following theorem describes an important property of Maharam operators, enabling us to embed them into an appropriate Boolean valued universe as order continuous functionals.

5.2.3. Theorem. *Let X and Y be some vector lattices with Y having the projection property and let T be a Maharam operator from X to Y. Then there exist an order closed subalgebra $\mathscr{B}$ of $\mathbb{B}(X_T)$ consisting of projection bands and a Boolean isomorphism h from $\mathbb{B}(Y_T)$ onto $\mathscr{B}$ such that $T(h(L)) \subset L$ for all $L \in \mathbb{B}(Y_T)$.*

$\lhd$ Without loss of generality, we can assume that T is strictly positive and $Y = Y_T$. For each band L in Y, we put $h(L) := \{x \in X : T(|x|) \in L\}$. Clearly, $h(L)$ is a vector subspace of X with $h(\{0\}) = 0$ and $h(Y) = X$. Moreover, $T(h(L)) \subset L$ for all $L \in \mathbb{B}(Y_T)$ by the very definition of h.

Prove that $h(L)$ is a band in X for every $L \in \mathbb{B}(X)$. Indeed, if $x \in h(L)$, $u \in X$, and $|u| \leqslant x$ then $T(|u|) \leqslant T(x) \in L$; i.e., $u \in L$, which proves that $h(L)$ is an order ideal. Suppose that a set $A \subset h(L) \cap X_+$ is directed upwards and bounded from above by $x_0 \in X_+$. Then the set $T(A) \subset L_+$ is bounded above by $T(x_0)$. Consequently, taking o-continuity of T into account, we obtain

$$T(\sup(A)) = \sup\{T(x) : \ x \in A\} \in L.$$

Thus, $\sup(A) \in L$. Hence, $h(L)$ is a band in X.

It is easily seen that the mapping $h : \mathbb{B}(Y) \to \mathbb{B}(X)$ is increasing: $L_1 \subset L_2$ implies $h(L_1) \subset h(L_2)$. We now demonstrate that h is injective. To this end, we suppose that $h(L_1) = h(L_2)$ for some $L_1, L_2 \in \mathbb{B}(Y)$ and, nevertheless, $L_1 \neq L_2$, say $L_1 \cap L_2^{\perp} \neq \varnothing$. Take an element $0 < y \in L_1$ such that $y \perp L_2$. Since $y \in L_1 \subset Y = T(X)^{\perp\perp}$, there exist $0 < y_1 \in Y$ and $0 < x \in X$ such that $y_1 \leqslant y \wedge T(x)$. If $y_2 := T(x) - y_1$ then, by the Maharam property, $x = x_1 + x_2$ and $T(x_l) = y_l$ $(l := 1, 2)$ for some $0 < x_l \in X$ $(l := 1, 2)$. But then $x_1 \in h(L_1)$ and $x_1 \notin h(L_2)$, contradicting the assumption $h(L_1) = h(L_2)$. This proves the injectivity of h.

Consider the inclusion ordered set $\mathscr{B} := \operatorname{im}(h)$; i.e., $\mathscr{B} := \{K \in \mathbb{B}(X) : K = h(L),\ L \in \mathbb{B}(Y)\}$. The above established fact means that h is an isomorphism of the ordered sets $\mathbb{B}(Y)$ and $\mathscr{B}$. Clarify what operations in $\mathscr{B}$ correspond to the Boolean operations in $\mathbb{B}(Y)$ under the order isomorphism h. First of all, observe that

$$h(\inf(\mathscr{U})) = h\Big(\bigcap \mathscr{U}\Big) = \bigcap\{h(L) :\ L \in \mathscr{U}\} \quad (\mathscr{U} \subset \mathbb{B}(Y)).$$

Further, let $L_1 \oplus L_2$ be a disjoint decomposition of the vector lattice Y. Then $h(L_1) \cap h(L_2) = \{0\}$. Given $x \in X$, we have the representation $Tx = y_1 + y_2$ with $y_l := [L_l](y)$ $(l := 1,2)$. Hence, by the Maharam property for T, there exist u_1 and $u_2 \in X_+$ such that $|x| = u_1 + u_2$ and $T(u_l) = y_l$ $(l := 1,2)$. Furthermore, for some $x_1, x_2 \in X$, we have $x = x_1 + x_2$ and $|x_l| = u_l$ $(l := 1,2)$. This yields $x_1 \in h(L_1)$ and $x_2 \in h(L_2)$. Consequently, X is the algebraic direct sum of $h(L_1)$ and $h(L_2)$. Moreover, if $x_l \in h(L_l)$ $(l := 1,2)$ then $T(|x_1| \wedge |x_2|) \leqslant T(|x_1|) \wedge T(|x_2|) \in L_1 \cap L_2 = \{0\}$. Hence $T(|x_1| \wedge |x_2|) = 0$ and, since T is strictly positive, we obtain $x_1 \perp x_2$. So, the bands $h(L_1)$ and $h(L_2)$ form a disjoint decomposition of the vector lattice X. Thus, $h(L^\perp) = h(L)^\perp$ for all $L \in \mathbb{B}(Y)$. Since the mapping $h : \mathbb{B}(Y) \to \mathscr{B}$ preserves infima and complements, it is an order continuous monomorphism of $\mathbb{B}(Y)$ onto an o-closed subalgebra $\mathscr{B}$ of the Boolean algebra $\mathbb{B}(X)$. The proof is complete. $\triangleright$

5.2.4. It is worth pointing out some corollaries to Theorem 5.2.3. Assume that X, Y, and T are as in 5.2.3.

(1) *If $S : X \to Y$ is a positive operator absolutely continuous with respect to T then $S(h(L)) \subset L$ for all $L \in \mathbb{B}(Y_T)$.*

$\triangleleft$ Given $L \in \mathbb{B}(Y_T)$ and $x \in h(L)$, we evidently have

$$|S(x)| \leqslant S(|x|) \in \{T(|x|)\}^{\perp\perp} \subset L$$

and thus $S(h(L)) \subset L$. $\triangleright$

(2) *There exists a Boolean isomorphism h from $\mathbb{P}(Y)$ onto an order closed subalgebra of $\mathbb{P}(X)$ such that $\pi S = Sh(\pi)$ for all $\pi \in \mathbb{P}(Y)$ whenever $S : X \to Y$ is a positive operator absolutely continuous with respect to T.*

$\triangleleft$ Let $\mathbb{B}$ be a Boolean algebra of projections onto the bands in $\mathscr{B}$. Denote by the same symbol h the respective isomorphism from $\mathbb{P}(Y)$ onto $\mathbb{B} \subset \mathbb{P}(X)$. It follows from (1) that $\pi^\perp \circ S \circ h(\pi) = 0$ or $S \circ h(\pi) = \pi \circ S \circ h(\pi)$. Replacing π by $\pi^\perp$ we obtain $\pi \circ S = \pi \circ S \circ h(\pi)$. We thus arrive at the sought relation $\pi \circ S = S \circ h(\pi)$. $\triangleright$

5.2.5. *Let X and Y be Dedekind complete vector lattices and T a Maharam operator from X to Y. Then there exists an f-module* *structure over $\mathscr{Z}(Y)$ on X such that an order bounded operator S from X to Y is absolutely continuous with respect to T if and only if S is $\mathscr{Z}(M)$-linear.*

$\lhd$ We can assume that T is strictly positive, because otherwise the f-module multiplication on X_T can be extended to the whole X by putting $ax := a\pi x$ for all $x \in X$ and $a \in \mathscr{Z}(Y)$ where π is the band projection onto X_T. The Boolean isomorphism h from 5.2.4 (2) can uniquely be extended to an f-algebra isomorphism from $\mathscr{Z}(Y)$ onto an f-subalgebra in $\mathscr{Z}(X)$. Denote this isomorphism by the same symbol h. An f-module structure over $\mathscr{Z}(Y)$ on X is induced by putting $\alpha x := h(\alpha)\,x$ for all $\alpha \in \mathscr{Z}(Y)$ and $x \in X$. Take an operator S absolutely continuous with respect to T. If $\alpha := \sum_{l=1}^{n} \lambda_l \pi_l$, where $\lambda_1, \dots, \lambda_n \in \mathbb{R}_+$ and $\{\pi_1, \dots, \pi_n\}$ is a partition of unity in $\mathbb{P}(Y)$, then, in view of 5.2.4 (2), $\pi_l \circ \alpha \circ S = \pi_l \circ S\,(\lambda_l h(\pi_l)\,) = \pi_l \circ S \circ h(\alpha)$ for all l. Summing over l yields $\alpha \circ S = S \circ h(\alpha)$. By the Freudenthal Spectral Theorem the set of orthomorphisms α of the above form is uniformly dense in $\mathscr{Z}(Y)$. Since S is uniformly continuous, we conclude that $\alpha \circ S = S \circ h(\alpha)$ for all $\alpha \in \mathscr{Z}(Y)$. It follows that S is $\mathscr{Z}(Y)$-linear. Conversely, assume that S is $\mathscr{Z}(Y)$-linear, $x \in X_+$, and π is the band projection onto $\{Tx\}^{\perp}$. Then $0 = \pi T x = T h(\pi) x$ by 5.2.4 (2), so that $h(\pi)x = 0$ due to the strict positivity of T. Thus, $\pi S x = S \pi x = S h(\pi) x = 0$ and $Sx \in \{Tx\}^{\perp\perp}$. $\rhd$

5.2.6. *Let X, Y, and T be as in 5.2.5. For a band $K \in \mathbb{B}(X_T)$ the following are equivalent:*

(1) *$Tu = Tv$ and $u \in K$ imply that $v \in K$ for all $u, v \in X_+$.*

(2) *$T(K'_+) \subset T(K_+)^{\perp\perp}$ yields $K' \subset K$ for all $K' \in \mathbb{B}(X_T)$.*

(3) *$K = h(L)$ for some $L \in \mathbb{B}(Y)$.*

$\lhd$ (1) $\Longrightarrow$ (2) Arguing for a contradiction, assume that $T(K'_+) \subset T(K_+)^{\perp\perp}$ for some $K' \in \mathbb{B}(X_T)$ not contained in K. Then there exists $0 < v \in K'$ with $v \perp K$. It follows that $Tv \in T(K'_+) T(K_+)^{\perp\perp} = T(K)^{\perp\perp}$ and, since $T(K)$ is an order ideal in Y, we can choose $0 < u \in K$ such that $0 < Tu \leqslant Tv$. By the Maharam property there is $0 < u_0 \leqslant v$ with $Tu_0 = Tu$. By (1) we have $u_0 \in K$ and this is a contradiction, since $u_0 \in K$ and $u_0 \leqslant v \in K'$ imply $u_0 = 0$.

(2) $\Longrightarrow$ (3) Put $L := T(K_+)^{\perp\perp}$ and observe that $K \subset h(L)$ by definition of h (cp. 5.2.3). If $K' := K^{\perp} \cap h(L)$ then $K' \subset h(K)$ and $T(K'_+) \subset L = T(K_+)^{\perp\perp}$, so that $K' \subset K$ in view of (2). It follows that $K' = \{0\}$ and $K = h(L)$.

(3) $\Longrightarrow$ (1) If $u \in K_+$ then $Tu \in L$ by the definition of h. Given $v \in X_+$ with $Tu = Tv$, we have $Tv \in L$ and $v \in K$. $\triangleright$

5.2.7. A band $K \in \mathbb{B}(X_T)$ (as well as the corresponding band projection $[K] \in \mathbb{P}(X_T)$) is said to be T-*saturated* if (hence each) of the conditions 5.2.6 (1–3) is fulfilled. The set of all T-saturated bands (band projections) is denoted by $\mathbb{B}_T(X)$ (respectively $\mathbb{P}_T(X)$).

A band projection $\pi' \in \mathbb{P}(X_T)$ is T-saturated if and only if $T\pi' \leqslant T\pi$ implies $\pi' \leqslant \pi$ for all $\pi' \in \mathbb{P}(X_T)$. It follows that the isomorphism $h : \mathbb{P}(Y) \to \mathbb{P}_T(X)$ can be defined as

$$h(\rho) = \bigvee \{\pi \in \mathbb{P}(X) :\ T\pi \leqslant \rho T\} \quad (\rho \in \mathbb{P}(Y)).$$

We now present the main result of this section stating that Maharam operators can be embedded into $\mathbb{V}^{(\mathbb{B})}$, turning thereby into order continuous functionals. This *Boolean valued representation of Maharam operators* enables one to obtain various facts about the Maharam operators from the corresponding ZFC theorems on functionals.

5.2.8. Theorem. *Let X be a Dedekind complete vector lattice, $Y := \mathscr{R}{\downarrow}$, and let $T : X \to Y$ be a positive Maharam operator with $Y = Y_T$. Then there are $\mathscr{X}, \tau \in \mathbb{V}^{(\mathbb{B})}$ satisfying the following:*

(1) *$[\![\, \mathscr{X}$ is a Dedekind complete vector lattice and $\tau : \mathscr{X} \to \mathscr{R}$ is an order continuous strictly positive functional with the Levi property $]\!] = \mathbb{1}$.*

(2) *$\mathscr{X}{\downarrow}$ is a Dedekind complete vector lattice and a unital f-module over the f-algebra $\mathscr{R}{\downarrow}$.*

(3) *$\tau{\downarrow} : \mathscr{X}{\downarrow} \to \mathscr{R}{\downarrow}$ is a strictly positive Maharam operator with the Levi property and an $\mathscr{R}{\downarrow}$-module homomorphism.*

(4) *There exists an order continuous lattice homomorphism $\varphi : X \to \mathscr{X}{\downarrow}$ such that $\varphi(X)$ is an order dense ideal of $\mathscr{X}{\downarrow}$ and $T = \tau{\downarrow} \circ \varphi$.*

$\triangleleft$ Assume without loss of generality that T is strictly positive. By Corollary 5.1.6 and 5.2.5 there exist a Dedekind complete vector lattice $\bar{\mathscr{X}}$ and an order continuous $\mathscr{R}$-linear functional $\bar{\tau}$ on $\bar{\mathscr{X}}$ within $\mathbb{V}^{(\mathbb{B})}$ and there is a lattice isomorphism φ from X into $X' := \bar{\mathscr{X}}{\downarrow}$ such that $T = \bar{\tau}{\downarrow} \circ \varphi$. By transfer there exist an order ideal $\mathscr{X}$ in $\bar{\mathscr{X}}^{\mathfrak{u}}$, including $\bar{\mathscr{X}}$, and a strictly positive order continuous functional $\tau : \mathscr{X} \to \mathscr{R}$ with the Levi property such that $\tau|_{\bar{\mathscr{X}}} = \bar{\tau}$. Clearly,

$$T = \bar{\tau}{\downarrow} \circ \varphi = (\tau|_{\bar{\mathscr{X}}}){\downarrow} \circ \varphi = \tau{\downarrow}|_{X'} \circ \varphi = \tau{\downarrow} \circ \varphi.$$

Moreover, $\varphi(X)$ is an order dense ideal in X' and so in $\mathscr{X}{\downarrow}$. Using Corollary 5.1.6 and 5.2.5 again, we conclude that $\tau{\downarrow}$ is a Maharam operator. The Levi property for strict positiveness of $\tau{\downarrow}$ are easily deduced from that of τ within $\mathbb{V}^{(\mathbb{B})}$. $\triangleright$

5.3. Representation of Order Continuous Operators

Theorem 5.2.8 together with the Boolean valued transfer principle enables us to assert that each fact about order continuous positive linear functionals on a Dedekind complete vector lattice has its counterpart for Maharam operators that can be demonstrated on using the descending–ascending machinery. The aim of this section is to prove an operator version of the next result.

5.3.1. Theorem. *Let X be a vector lattice and let $X_n^{\sim}$ separate the points of X. Then there exist order dense ideals L and X' in $X^{\mathfrak{u}}$ and a linear functional $\tau : L \to \mathbb{R}$ such that the following hold:*

(1) $X' = \{x' \in X' :\ xx' \in L$ *for all* $x \in X\}$.

(2) *τ is strictly positive, order continuous, and has the Levi property.*

(3) *For every $\sigma \in X_n^{\sim}$ there exists a unique $x' \in X'$ such that*

$$\sigma(x) = \tau(x \cdot x') \quad (x \in X).$$

(4) *$\sigma \mapsto x'$ is a lattice isomorphism of $X_n^{\sim}$ onto X'.*

$\triangleleft$ The proof may be found in Vulikh and Lozanovskiĭ [404, Theorem 2.1]. It can also be extracted from Vulikh [403, Theorem IX.3.1] or Kusraev [228, Theorem 3.4.8]. $\triangleright$

5.3.2. To translate Theorem 5.3.1 into a result for operators we need some preparations. Let X and Y be f-modules over an f-algebra A. Recall that for $a \in A$ the orthomorphisms $\hat{a} \in \operatorname{Orth}(X)$ and $\bar{a} \in \operatorname{Orth}(Y)$ are defined as $\hat{a} : x \mapsto ax$ $(x \in X)$ and $\bar{a} : y \mapsto ay$ $(y \in Y)$, while the mappings R_a and L_a on $L^{\sim}(X,Y)$ are defined by $R_a(T) := T\hat{a}$ and $L_a(T) := \bar{a}T$; see 3.1.2.

(1) *The maps $a \mapsto R_a$ and $a \mapsto L_a$ are f-algebra homomorphisms from A to* $\operatorname{Orth}\big(L^{\sim}(X,Y)\big)$.

$\triangleleft$ It is easy to note that $R_a \in \operatorname{Orth}\big(L^{\sim}(X,Y)\big)$ whenever $\hat{a} \in \mathscr{Z}(X)_+$. For an arbitrary $a \in A_+$ the sequence (π_n) in $\mathscr{Z}(X)$ with

$\pi_n := \hat{a} \wedge (nI_X)$ converges $\hat{a}^2$-uniformly to $\hat{a}$ and, for all $S, T \in L^{\sim}(X, Y)$ with $S \wedge (T\pi_n) = 0$, we have $S \wedge (T\pi_n) = 0$. Therefore, $S \wedge T = 0$ implies $S \wedge R_a(T) = S \wedge (T\hat{a}) = 0$, since $(T\pi_n)$ converges $T\hat{a}^2$-uniformly to $T\hat{a}$. It follows that $R_a \in \mathrm{Orth}\big(L^{\sim}(X, Y)\big)$. Moreover, the mapping $a \mapsto R_a$ is evidently a positive algebra homomorphism. It remains to observe that a positive algebra homomorphism is a lattice homomorphism. The case of the mapping $a \mapsto L_a$ is treated similarly. $\triangleright$

(2) *Let Y be a Dedekind complete vector lattice, $A := \mathscr{Z}(Y)$, $\bar{A} := \mathrm{Orth}(Y)$, and $A_0 := \mathrm{St}_0(\mathbb{P}(Y))$. If X is an f-module over $\bar{A}$ then*

$$L^{\sim}_{A_0}(X, Y) = L^{\sim}_{A}(X, Y) = L^{\sim}_{\bar{A}}(X, Y).$$

$\triangleleft$ It suffices to ensure that $L^{\sim}_{A_0}(X, Y) \subset L^{\sim}_{\bar{A}}(X, Y)$, because the converse inclusion is evident. Observe that $L^{\sim}_{A_0}(X, Y) \subset L^{\sim}_{A}(X, Y)$, since A_0 is uniformly dense in A by the Freudenthal Spectral Theorem and every order bounded linear operator is uniformly continuous. An arbitrary $a \in \bar{A}_+$ is the a^2-uniform limit of the sequence (a_n) in $\mathscr{Z}(Y)$ with $a_n := a \wedge (nI_Y)$. If $T \in L^{\sim}_{A}(X, Y)$ then $T(ax)$ is the $|T|(a^2x)$-uniform limit of $(T(a_n x))$ for all $x \in X_+$, so that $aTx = T(ax)$. It follows that $T \in L^{\sim}_{\bar{A}}(X, Y)$. $\triangleright$

5.3.3. Say that a set $\mathscr{T} \subset L^{\sim}(X, Y)$ *separates the points* of X or is *point separating* on X whenever, given nonzero $x \in X$, there exists $T \in \mathscr{T}$ such that $Tx \neq 0$. In the case of Y Dedekind complete and $\mathscr{T} \subset L^{\sim}(X, Y)$ a sublattice, this is equivalent to saying that for every nonzero $x \in X_+$ there is a positive operator $T \in \mathscr{T}$ with $Tx \neq 0$.

Assume that $A := \mathscr{Z}(Y)$ and X is an f-module over A. If $L^{\sim}_{n,A}(X, Y)$ separates the points of X, then X is unital and $\{\hat{\pi} : \pi \in \mathbb{P}(Y)\}$ is an order closed subalgebra in $\mathbb{P}(X)$.

$\triangleleft$ We have only to ensure that the Boolean homomorphism $\pi \mapsto \hat{\pi}$ from $\mathbb{P}(Y)$ to $\mathbb{P}(X)$ is order continuous. Take a decreasing family (π_α) in $\mathbb{P}(Y)$ with $\inf_\alpha \pi_\alpha = 0$ and suppose that $0 \leqslant u \leqslant \hat{\pi}_\alpha x$ for all α with some fixed $x, u \in X_+$. Then $0 = \inf_\alpha \pi_\alpha T x = \inf_\alpha T(\hat{\pi}_\alpha x) \geqslant Tu \geqslant 0$ for all $0 \leqslant T \in L^{\sim}_{n,A}(X, Y)$. Hence $u = 0$ and $\inf_\alpha \hat{\pi}_\alpha x = 0$ for all $x \in X_+$. $\triangleright$

5.3.4. *Denote $X := \mathscr{X}{\downarrow}$ and $A := \mathscr{R}{\downarrow}$. The mapping assigning to each member $\sigma \in \mathscr{X}^{\sim}{\downarrow}$ its descent $S := \sigma{\downarrow}$ is a lattice isomorphism of $\mathscr{X}^{\sim}{\downarrow}$ and $\mathscr{X}^{\sim}_n{\downarrow}$ onto $L^{\sim}_A(X, \mathscr{R}{\downarrow})$ and $L^{\sim}_{n,A}(X, \mathscr{R}{\downarrow})$, respectively. Moreover, $[\![\mathscr{X}^{\sim}$ (resp. $\mathscr{X}^{\sim}_n$) separates the points of $\mathscr{X}]\!] = \mathbb{1}$ if and only if $L^{\sim}_A(X, \mathscr{R}{\downarrow})$ (resp. $L^{\sim}_{n,A}(X, \mathscr{R}{\downarrow})$) separates the points of X.*

$\lhd$ The first statement follows from Corollary 5.1.6, so that we need only verify the second one. Observe first that $L_A^\sim(X, \mathscr{R}{\downarrow})$ coincides with the space $L_{\mathrm{Ext}}(X, \mathscr{R}{\downarrow})$ of all extensional order bounded linear operators from X to $\mathscr{R}{\downarrow}$. Take $x, y \in X$ and put $b = [\![x = y]\!]$. Then $\chi(b)x = \chi(b)y$ and, given $S \in L_A^\sim(X, \mathscr{R}{\downarrow})$, we have $\chi(b)Sx = S(\chi(b)x) = S(\chi(b)y) = \chi(b)Ty$, so that $b \leqslant [\![Sx = Sy]\!]$ and $S \in L_{\mathrm{Ext}}(X, \mathscr{R}{\downarrow})$. Thus, $L_A^\sim(X, \mathscr{R}{\downarrow}) \subset L_{\mathrm{Ext}}(X, \mathscr{R}{\downarrow})$ and the converse inclusion follows from 5.3.2 (2).

Now, formalize the claim that $\mathscr{X}^\sim$ is point separating: $\varphi(\mathscr{X}) \equiv (\forall\, x \in \mathscr{X})\big(((\forall\, \sigma \in \mathscr{X}^\sim)\sigma(x) = 0) \to x = 0\big)$. In view of 1.5.2 and 1.5.6

$$[\![\varphi(\mathscr{X})]\!] = \bigwedge_{x \in X} b_x \Rightarrow [\![x = 0]\!]$$

where $b_x := \bigwedge\{[\![Sx = 0]\!] : S \in L_{\mathrm{Ext}}(X, \mathscr{R}{\downarrow})\}$. Thus, $\mathscr{X}^\sim$ is point separating within $\mathbb{V}^{(\mathbb{B})}$ if and only if $b_x \leqslant [\![x = 0]\!]$ for all $x \in X$. The latter is equivalent to saying that, given $x \in X$, we have $\chi(b)x = 0$ whenever $\chi(b)Sx = 0$ for all $S \in L_A^\sim(X, \mathscr{R}{\downarrow})$. This implies that $L_A^\sim(X, \mathscr{R}{\downarrow})$ separates the points of X. $\rhd$

5.3.5. Theorem. *Let X be an f-module over $A := \mathscr{Z}(Y)$ with Y a Dedekind complete vector lattice and let $L_{n,A}^\sim(X, Y)$ separate the points of X. Then there exist an order dense ideal L in $X^{\mathfrak{u}}$ and a strictly positive Maharam operator $T : L \to Y$ such that the order ideal $X' = \{x' \in X' : (\forall\, x \in X)\, xx' \in L\} \subset X^{\mathfrak{u}}$ is lattice isomorphic to $L_{n,A}^\sim(X, Y)$. The isomorphism is implemented by assigning the operator $S_{x'} \in L_{n,A}^\sim(X, Y)$ to an element $x' \in X'$ by the formula*

$$S_{x'}(x) = T(xx') \quad (x \in X).$$

If there exists a strictly positive $T_0 \in L_{n,A}^\sim(X, Y)$ then we can choose L and T such that $X \subset L$ and $T|_X = T_0$.

$\lhd$ According to Theorem 3.1.10 $L_{n,A}^\sim(X, Y)$ is isomorphic to $L_{n,A}^\sim(X^\delta, Y)$, so that there is no loss of generality in assuming that X is Dedekind complete. By Gordon's Theorem 2.4.2 we can assume also that $Y^{\mathfrak{u}} = \mathscr{R}{\downarrow}$. Of course, in this event we can identify $A^{\mathfrak{u}}$ with $Y^{\mathfrak{u}}$.

In view of Theorem 2.11.9 there exists a Dedekind complete real vector lattice $\mathscr{X}$ within $\mathbb{V}^{(\mathbb{B})}$ with $\mathbb{B} = \mathbb{P}(Y)$ such that $\mathscr{X}{\downarrow}$ is an f-module over $A^{\mathfrak{u}}$, and there is an f-module isomorphism h from X to $\mathscr{X}{\downarrow}$ satisfying $\mathscr{X}{\downarrow} = \mathrm{mix}(h(X))$. By 5.3.4 $\mathscr{X}_n^\sim$ separates the points of $\mathscr{X}$.

The transfer principle tells us that Theorem 5.3.1 is true within $\mathbb{V}^{(\mathbb{B})}$, so that there exist an order dense ideal $\mathscr{L}$ in $\mathscr{X}^{\mathfrak{u}}$ and a strictly positive linear functional $\tau : \mathscr{L} \to \mathscr{R}$ with the Levi property such that the order ideal $\mathscr{X}' = \{x' \in \mathscr{X}^{\mathfrak{u}} : \ x'\mathscr{X} \subset \mathscr{L}\}$ is lattice isomorphic to $\mathscr{X}_n^{\sim}$; moreover, the isomorphism is implemented by assigning the functional $\sigma_{x'} \in \mathscr{X}_n^{\sim}$ to $x' \in \mathscr{X}'$ using the rule $\sigma_{x'}(x) := \tau(xx')$ $(x \in \mathscr{X})$.

Put $\hat{X} := \mathscr{X}{\downarrow}$, $\hat{L} := \mathscr{L}{\downarrow}$, $\hat{T} := \tau{\downarrow}$, and $\hat{X}' := \mathscr{X}'{\downarrow}$. By Theorem 2.11.9 we can identify the universally complete vector lattices $X^{\mathfrak{u}}$, $\hat{X}^{\mathfrak{u}}$, and $\mathscr{X}^{\mathfrak{u}}{\downarrow}$ as well as X with a laterally dense sublattice in $\hat{X}$. Then $\hat{L}$ is an order dense ideal in $\hat{X}^{\mathfrak{u}}$ and an f-module over $A^{\mathfrak{u}}$, while $\hat{T} : \hat{L} \to Y^{\mathfrak{u}}$ is a strictly positive Maharam operator with the Levi property. Since the multiplication on $X^{\mathfrak{u}}$ is the descent of the internal multiplication in $\mathscr{X}^{\mathfrak{u}}$, we have the representation $\hat{X}' = \{x' \in X^{\mathfrak{u}} : x'\hat{X} \subset \hat{L}\}$. Moreover, $\hat{X}'$ is f-module isomorphic to $L^{\sim}_{n,A}(\hat{X}, Y^{\mathfrak{u}})$ by assigning to $x' \in \hat{X}$ the operator $\hat{S}_{x'} \in L^{\sim}_{n,A}(\hat{X}, Y^{\mathfrak{u}})$ defined as $\hat{S}_{x'}(x) = \hat{T}(xx')$ $(x \in \hat{X})$. Putting

$$
\begin{aligned}
L &:= \{x \in \hat{L} : \hat{T}x \in Y\}, \quad T := \hat{T}|_L,\\
X' &:= \{x' \in \hat{X}' : x'X \subset L\},
\end{aligned}
$$

we see that if $x' \in X'$ then $S_{x'} := \hat{S}_{x'}|_X$ belongs to $L^{\sim}_{n,A}(X, Y)$. Conversely, an arbitrary $S \in L^{\sim}_{n,A}(X, Y)$ has the representation $Sx = \hat{T}(xx')$ $(x \in X)$ with some $x' \in \hat{X}'$, so that $\hat{T}(xx') \in Y$ for all $x \in X$ and so $x' \in X'$, $xx' \in L$ for all $x \in X$, and $Sx = T(xx')$ $(x \in X)$ by the above definitions. $\triangleright$

5.3.6. Corollary. *Given* $\pi \in \mathbb{P}(X)$*, define* $\hat{\pi} \in \mathbb{P}(L^{\sim}_{n,A}(X, Y))$ *as* $\hat{\pi} : S \mapsto S \circ \pi$*. Under the hypotheses of* 5.3.3 *the mapping* $\pi \mapsto \hat{\pi}$ *is a Boolean isomorphism of* $\mathbb{P}(X)$ *onto* $\mathbb{P}(L^{\sim}_{n,A}(X, Y))$*.*

$\triangleleft$ Let γ stand for the lattice isomorphism from $L^{\sim}_{n,A}(X, Y))$ onto X' in Theorem 5.3.5. Denote by $\tilde{\pi}$ the unique band projection on X' which agrees with π on $X \cap X'$. Then by Theorem 5.3.5 we have

$$
T(\gamma(\hat{\pi}S)x) = \hat{\pi}(S)x = S(\pi x) = T(\gamma(S)\pi x) = T(x\tilde{\pi}\gamma(S))
$$

for all $x \in X$, so that $\hat{\pi}(S) = \gamma^{-1}(\tilde{\pi}\gamma(S))$ for all $S \in L^{\sim}_{n,A}(X, Y)$. It remains to observe that $j(\pi) : S \mapsto \gamma^{-1}(\tilde{\pi}\gamma(S))$ is a band projection in $L^{\sim}_{n,A}(X, Y)$ and the mapping $\pi \mapsto j(\pi)$ is a Boolean isomorphism of $\mathbb{P}(X)$ onto $\mathbb{P}(L^{\sim}_{n,A}(X, Y))$. $\triangleright$

The following two results are immediate from Corollary 5.3.6.

5.3.7. Hahn Decomposition Theorem. *Let $T : X \to Y$ be a Maharam operator. Then there is a band projection $\pi \in \mathbb{P}(X)$ such that $T^+ = T \circ \pi = |T| \circ \pi$ and $T^- = -T \circ \pi^\perp = |T| \circ \pi^\perp$. In particular, $|T| = T \circ (\pi - \pi^\perp)$ and $T = |T| \circ (\pi - \pi^\perp)$.*

$\lhd$ There is no loss of generality in assuming that $|T|$ is strictly positive. Let $[T^+]$ stands for the band projection onto the band in $L^\sim_{n,A}(X,Y)$ generated by T^+. In view of Corollary 5.3.6 there is $\pi \in \mathbb{P}(X)$ such that $\hat\pi = [T^+]$. Now, by definitions we have $T^+ = [T^+](T) = \hat\pi(T) = T \circ \pi$ and $T^- = -[T^+]^\perp(T) = -\hat\pi^\perp(T) = -T \circ \pi^\perp$. $\rhd$

5.3.8. Nakano Theorem. *Let $T_1, T_2 : X \to Y$ be order bounded operators such that $T := |T_1| + |T_2|$ is a Maharam operator. Then T_1 and T_2 are disjoint if and only if so are their carriers; symbolically,*

$$T_1 \perp T_2 \Longleftrightarrow \mathscr{C}_{T_1} \perp \mathscr{C}_{T_2}.$$

$\lhd$ Again, assume without loss of generality that T is strictly positive. By Corollary 5.3.6 there is $\pi_i \in \mathbb{P}(X)$ such that $\hat\pi_i = [T_i]$. Clearly, $T_1 \perp T_2$ if and only if $\hat\pi_1 \perp \hat\pi_2$ or, equivalently, $\pi_1(X) \perp \pi_2(X)$. It remains to observe that $T_i = \hat\pi_i(T)$ and the carrier of $\hat\pi_i(T)$ coincides with the band $\pi_i(X)$ $(i := 1, 2)$. $\rhd$

5.3.9. Radon–Nikodým Theorem. *Let X and Y be Dedekind complete vector lattices and let T be a positive Maharam operator. For an order bounded order continuous operator S from X to Y the following are equivalent:*

(1) $S \in \{T\}^{\perp\perp}$.

(2) $S \preccurlyeq T$.

(3) *There exists an orthomorphism $\rho \in \mathrm{Orth}^\infty(X)$ such that $Sx = T(\rho x)$ for all $x \in \mathscr{D}(\rho)$.*

$\lhd$ The implication (1) $\Longrightarrow$ (2) is trivial. For the proof that (2) $\Longrightarrow$ (3), we can assume without loss of generality that T is strictly positive. In view of 5.2.5 there exists an f-module structure over $A := \mathscr{Z}(Y)$ on X such that an order bounded order continuous operator $S : X \to Y$ is absolutely continuous with respect to T if and only if S is in $L_{n,A}(X,Y)$. Moreover, $L_{n,A}(X,Y)$ separates the points of X. Thus, by Theorem 5.3.5, there exist an order dense ideal $\hat L$ in $X^{\mathfrak{u}}$ including X and a strictly positive Maharam operator $\hat T : L \to Y$ such that $\hat T|_X = T$ and $S(x) = \hat T(xx')$ $(x \in X)$ for some $x \in X^{\mathfrak{u}}$. It remains to put $\mathscr{D}(\rho) := \{u \in X : ux' \in X\}$ and $\rho x = xx'$ $(x \in \mathscr{D}(\rho))$.

To see (3) $\Longrightarrow$ (1), suppose that $S_0 \in \{T\}^{\perp}$ and $0 \leqslant S_0 \leqslant |S|$ for some $S_0 \in L^{\sim}(X,T)_+$. Then S_0 is absolutely continuous with respect to T and, by what has just been proved, there exists $\rho_0 \in \mathrm{Orth}^{\infty}(X)$ such that $S_0x = \hat{T}(\rho_0 x)$ for all $x \in \mathscr{D}(\rho_0) \cap X$. From 3.1.5 we deduce

$$0 = (T \wedge S_0)x = \inf_{0 \leqslant u \leqslant x} T((I-\rho_0)u + \rho_0 x) \leqslant T\rho_0 x = S_0 x$$

for all $0 \leqslant x \in X \cap \mathscr{D}(\rho_0)$, so that $S_0 = 0$ by order continuity of S. $\triangleright$

5.4. Conditional Expectation Type Operators

Conditional expectation operators have many remarkable properties related to the order structure of the underlying function space. Boolean valued analysis enables us to demonstrate that these property are shared by a much more general class of operators.

5.4.1. Let Z be a universally complete vector lattice with (weak order) unit $\mathbb{1}$. Recall that a universally complete vector lattice Z is a semiprime f-algebra with a multiplicative unit $\mathbb{1}$. Assume that $\Phi : L^1(\Phi) \to Y$ is a strictly positive Maharam operator with the Levi property. We will write $L^0(\Phi) := Z$ whenever $L^1(\Phi)$ is an order dense ideal in Z. Also, denote by $L^{\infty}(\Phi)$ the order ideal in Z generated by $\mathbb{1}$. Consider an order ideal $X \subset Z$ and we will always assume that $L^{\infty}(\Phi) \subset X \subset L^1(\Phi)$. The *associate* space X' is defined as the set of all $x' \in L^0(\Phi)$ for which $xx' \in L^1(\Phi)$ for all $x \in X$. Clearly, X' is also an order ideal in Z.

Throughout this section (Ω, Σ, μ) is a finite measure space and $L^0(\Omega, \Sigma, \mu)$ is the Dedekind complete vector lattice of Σ-measurable real functions on Ω with the usual identification of μ-equivalent functions. The corresponding L^p-spaces $L^p(\Omega, \Sigma, \mu)$ with $1 \leqslant p \leqslant \infty$ are order dense ideals of $L^0(\Omega, \Sigma, \mu)$. An *ideal space* (of measurable functions) is an order ideal $\mathscr{X}$ of the vector lattice $L^0(\Omega, \Sigma, \mu)$, so that $\mathscr{X}$ is a Dedekind complete vector lattice. We will assume that $L^{\infty}(\Omega, \Sigma, \mu) \subset \mathscr{X} \subset L^1(\Omega, \Sigma, \mu)$. If $\varphi : L^1(\Omega, \Sigma, \mu) \to \mathbb{R}$ is defined as $\varphi(x) := \int_{\Omega} x \, d\mu$ then φ is an order continuous functional with the Levi property and, according to the above notation, $L^p(\varphi) = L^p(\Omega, \Sigma, \mu)$ for $1 \leqslant p \leqslant \infty$.

5.4.2. *Let $\Phi : L^1(\Phi) \to Y$ be a Maharam operator, and let X_0 be an order closed sublattice of $L^1(\Phi)$ with $X^{\perp\perp} = X_0^{\perp\perp}$. Put $A := \mathrm{Orth}(Y_{\Phi})$ and $\Phi_0 := \Phi|_{X_0}$. Then the following are equivalent:*

(1) *Φ_0 has the Maharam property.*

(2) *A band projection in $L^1(\Phi)$ is Φ-saturated if and only if its restriction onto X_0 is a Φ_0-saturated band projection in X_0.*

(3) *X_0 is invariant under each Φ-saturated projection in $L^1(\Phi)$.*

(4) *X_0 is A-submodule and Φ_0 is A-linear with respect to the f-module structure over A on X induced by Φ.*

$\lhd$ There is no loss of generality in assuming $Y = Y_\Phi$ and $X = \mathscr{C}_\Phi$. Note that the relation $X^{\perp\perp} = X_0^{\perp\perp}$ implies $\Phi(X)^{\perp\perp} = \Phi(X_0)^{\perp\perp}$. Indeed, if $L := \Phi(X_0)^\perp \neq \{0\}$ then, by Theorem 5.2.3, $h(L) \perp X_0$ and $h(L) \neq \{0\}$ contradicting $X_0^\perp = \{0\}$. Denote by h and h_0 respectively the Boolean isomorphisms from $\mathbb{P}(Y)$ onto the Boolean algebras of Φ-saturated projections in $L^1(\Phi)$ and Φ_0-saturated projections in X_0 existing by Theorem 5.2.3 and 5.2.7. It is easily seen that $h_0(\pi) \leqslant h(\pi)|_{X_0}$ for all $\pi \in \mathbb{P}(Y)$. At the same time $\Phi_0 h_0(\pi) = \pi\Phi_0 = (\pi\Phi)|_{X_0} = (\Phi h(\pi))|_{X_0}$ and so $\Phi(h(\pi)x - h_0(\pi)x) = 0$ for every $0 \leqslant x \in X_0$. Since Φ is strictly positive, we conclude that $h(\pi)x = h_0(\pi)x$. It follows that $h_0(\pi) = h(\pi)|_{X_0}$, so that the restriction of each Φ-saturated projection onto X_0 is a Φ_0-saturated projection in X_0. The converse follows from the fact that a band projection in X_0 has the unique extension to a band projection in $L^1(\Phi)$.

Note that (2) $\Longrightarrow$ (3) is trivial, while (3) $\Longrightarrow$ (4) and (4) $\Longrightarrow$ (1) can easily be deduced by the argument similar to that in 5.2.5. $\rhd$

5.4.3. Theorem. *Let (Ω, Σ, μ) be a probability space and let $\mathscr{X}_0$ be a norm closed vector sublattice in $L^1(\Omega, \Sigma, \mu)$ containing 1_Ω. Then there exists a unique σ-subalgebra Σ_0 of Σ such that $\mathscr{X}_0 = L^1(\Omega, \Sigma_0, \mu_0)$ with $\mu_0 = \mu|_{\mathscr{X}_0}$.*

$\lhd$ See, for example, Douglas [110, Lemma 1]. $\rhd$

5.4.4. Theorem. *Let $\Phi : L^1(\Phi) \to Y$ be a strictly positive Maharam operator with $Y = Y_\Phi$ and let Z_0 be an order closed sublattice in $L^0(\Phi)$. If $\mathbb{1} \in X_0 := L^1(\Phi) \cap Z_0$ and the restriction $\Phi_0 := \Phi|_{X_0}$ has the Maharam property then $X_0 = L^1(\Phi_0)$ and there exists an operator $\mathrm{E}(\cdot|Z_0)$ from $L^1(\Phi)$ onto $L^1(\Phi_0)$ such that the following hold:*

(1) *$\mathrm{E}(\cdot|Z_0)$ is an order continuous positive linear projection.*

(2) *$\mathrm{E}(h(\pi)x|Z_0) = h(\pi)\mathrm{E}(x|Z_0)$ for all $\pi \in \mathbb{P}_\Phi(X)$ and $x \in L^1(\Phi)$; i.e., $\mathrm{E}(\cdot|Z_0)$ commutes with all Φ-saturated projections.*

(3) *$\Phi(xy) = \Phi(y\mathrm{E}(x|Z_0))$ for all $x \in L^1(\Phi)$ and $y \in L^\infty(\Phi_0)$.*

(4) $\Phi_0(|\mathrm{E}(x|Z_0)|) \leqslant \Phi(|x|)$ *for all* $x \in L^1(\Phi)$.

(5) $\mathrm{E}(v\mathrm{E}(x|Z_0)|Z_0) = \mathrm{E}(v|Z_0)\mathrm{E}(x|Z_0)$ *for all* $x \in L^1(\Phi)$ *and* $v \in L^\infty(\Phi)$*; i.e.,* $\mathrm{E}(\cdot|Z_0)$ *satisfies the averaging identity.*

$\lhd$ Put $\mathbb{B} := \mathbb{P}(Y)$. In view of Theorem 5.2.8 we can assume that $Y = \mathscr{R}{\downarrow}$, $\Phi = \varphi{\downarrow}$, $L^0(\Phi) = L^0(\varphi){\downarrow}$, and $L^1(\Phi) = L^1(\varphi){\downarrow}$ for some strictly positive order continuous functional $\varphi : L^1(\phi) \to \mathscr{R}$ in $\mathbb{V}^{(\mathbb{B})}$. Moreover, there exists a Boolean isomorphism χ from $\mathbb{B}$ onto $\mathbb{P}_\Phi(L^1(\Phi))$ such that the relations $b \leqslant [\![x \leqslant y]\!]$ and $\chi(b)x \leqslant \chi(b)y$ are equivalent for all $b \in \mathbb{B}$ and $x, y \in L^1(\Phi)$. Say that a band projection $\pi \in \mathbb{P}(L^0(\Phi))$ is Φ-saturated whenever so is its restriction to $L^1(\Phi)$. By hypothesis, Z_0 is a universally complete vector lattice. Moreover, Z_0 is invariant under each Φ-saturated projection in $L^0(\Phi)$ because so is X_0. This two properties of Z_0 amount to saying that $\chi(b)(Z_0) \subset Z_0$ for all $b \in \mathbb{B}$ and $o\text{-}\sum_\xi \chi(b_\xi)z_\xi \in Z_0$ for every family (z_ξ) in Z_0 and every partition of unity (b_ξ) in $\mathbb{B}$. It follows that $\mathscr{Z}_0 := Z_0{\uparrow}$ is an internal order closed sublattice of $L^0(\varphi)$ with $\mathbb{1} \in \mathscr{X}_0 := L^1(\varphi) \cap \mathscr{Z}_0$ and $Z_0 = \mathscr{Z}_0{\downarrow}$. Of course, $\mathscr{X}_0$ is an order closed sublattice of $L^1(\varphi)$ and $\mathscr{X}_0{\downarrow} = (L^1(\varphi) \cap \mathscr{Z}_0){\downarrow} = L^1(\varphi){\downarrow} \cap \mathscr{Z}_0{\downarrow} = L^1(\Phi) \cap Z_0 = X_0$.

In view of the Kakutani Representation Theorem we can assume further that $L^1(\varphi) = L^1(\Omega, \Sigma, \mu)$ for some probability space (Ω, Σ, μ). By Theorem 5.4.3 $\mathscr{X}_0 = L^1(\Omega, \Sigma_0, \mu_0) = L^1(\varphi_0)$ for some σ-subalgebra Σ_0 of Σ, where $\mu_0 := \mu|_{\Sigma_0}$ and $\varphi_0(x) = \int_\Omega x(\omega)\,d\mu_0(\omega)$ for all $x \in L^1(\Omega, \Sigma, \mu)$. In particular, $\mathscr{Z}_0 = L^0(\Omega, \Sigma, \mu)$, $\varphi_0 = \varphi|_{\mathscr{X}_0}$, and $\Phi_0 = \varphi_0{\downarrow}$. According to the classical Radon–Nikodým Theorem there exists the conditional expectation operator $\mathscr{E}(\cdot|\Sigma_0)$ with respect to Σ_0 acting from $L^1(\varphi)$ onto $L^1(\varphi_0)$.

Denote by $\mathrm{E}(\cdot|Z_0)$ the descent of the internal conditional expectation operator $\mathscr{E}(\cdot|\Sigma_0)$. The required conditions 5.4.4 (1–5) can be obtained by interpretation of the elementary properties of the conditional expectation operator within $\mathbb{V}^{(\mathbb{B})}$. $\rhd$

5.4.5. We will call the operator $\mathrm{E}(\cdot|Z_0)$ which is defined by Theorem 5.4.4 the *conditional expectation operator with respect to* Z_0. Say that the sublattice Z_0 in $L^0(\Phi)$ is Φ-*ample* whenever the conditions in Theorem 5.4.4 which ensures the existence of the conditional expectation operator $\mathrm{E}(\cdot|Z_0)$ with respect to Z_0 are fulfilled. It follows from 5.4.2 that Z_0 is Φ-ample if and only if Z_0 is order closed in $L^0(\Phi)$, $\mathbb{1} \in L^1(\Phi) \cap Z_0$, and $Z_0 \cap L^1(\Phi)$ is a submodule of $L^1(\Phi)$ with respect to the f-module structure over $\mathscr{Z}(Y)$ on $L^1(\Phi)$ induced by Φ.

Take $w \in X'$ and observe that $\mathrm{E}(wx|Z_0) \in L^1(\Phi_0)$ is well defined for all $x \in X$. If additionally $\mathrm{E}(wx|Z_0) \in X$ for every $x \in X$ then we can define the linear operator $T : X \to X$ by putting $Tx = \mathrm{E}(wx|Z_0)$ $(x \in X)$ called a *weighted conditional expectation operator.* Clearly, T is order bounded and order continuous. Moreover, for all $x \in X_+$ we have

$$T^+x = \mathscr{E}(w^+x|Z_0), \quad T^-x = \mathscr{E}(w^-x|Z_0), \quad |T|x = \mathscr{E}(|w|x|Z_0).$$

In particular, T is positive if and only if so is w. Putting $x := wx$ and $y := \mathbb{1}$ in 5.4.4 (3), we get $\Phi(wx) = \Phi(wx\mathbb{1}) = \Phi(\mathscr{E}(wx|Z_0)) = \Phi(Tx)$ for all $x \in X$. Now, x can be chosen to be a component of $\mathbb{1}$ with $wx = w^+$ or $wx = w^-$, so that $T = 0$ implies $\Phi(w^+) = 0$ and $\Phi(w^-) = 0$, since Φ is strictly positive. Thus $w \in X'$ is uniquely determined by T.

Say that T satisfies the *averaging identity*, if $T(y \cdot Tx) = Ty \cdot Tx$ for all $x \in X$ and $y \in L^\infty(\Phi)$. Let us give a characterization of weighted conditional expectation operators on x. We start with the case of an ideal function space.

5.4.6. Theorem. *Let (Ω, Σ, μ) be a finite measure space and let $\mathscr{X}$ be an order ideal in $L^1(\Omega, \Sigma, \mu)$ including $L^\infty(\Omega, \Sigma, \mu)$. For a linear operator $\mathscr{T}$ on $\mathscr{X}$ the following are equivalent:*

(1) *$\mathscr{T}$ is order continuous, satisfies the averaging identity, and keeps invariant $L^\infty(\Omega, \Sigma, \mu)$.*

(2) *There exist $w \in \mathscr{X}'$ and a σ-subalgebra Σ_0 of Σ such that $\mathscr{T}x = \mathscr{E}(wx|\Sigma_0)$ for all $x \in \mathscr{X}$.*

$\lhd$ See Dodds, Huijsmans, and de Pagter [105, Proposition 3.1]. $\rhd$

5.4.7. Theorem. *Let $\Phi : L^1(\Phi) \to Y$ be a strictly positive Maharam operator and let X be an order dense ideal in $L^1(\Phi)$ including $L^\infty(\Phi)$. For a linear operator T on X the following are equivalent:*

(1) *T is order continuous, satisfies the averaging identity, leaves invariant $L^\infty(\Phi)$, and commutes with all Φ-saturated projections.*

(2) *There exist $w \in X'$ and a Φ-ample sublattice Z_0 in $L^0(\Phi)$ such that $Tx = \mathrm{E}(wx|Z_0)$ for all $x \in X$.*

$\lhd$ (1) $\Longrightarrow$ (2): Just as in the proof of Theorem 5.4.4, we can assume that $X = \mathscr{X}{\downarrow}$, where $\mathscr{X}$ is an order ideal in $L^1(\Omega, \Sigma, \mu)$ including $L^\infty(\Omega, \Sigma, \mu)$ for some probability space (Ω, Σ, μ). By hypotheses T commutes with all projections $\chi(b)$ $(b \in \mathbb{B})$ and so it is extensional. Therefore, $\mathscr{T} := T{\uparrow}$ is an internal mapping in $\mathscr{X}$ and $T = \mathscr{T}{\downarrow}$. Moreover, $\mathscr{T}$ is linear, order continuous, satisfies the averaging identity, and keeps invariant $L^\infty(\Omega,\Sigma,\mu)$. By Theorem 5.4.6 there exist $w \in \mathscr{X}'$ and a σ-subalgebra Σ_0 of Σ such that $\mathscr{T}x = \mathscr{E}(wx|\Sigma_0)$ for all $x \in \mathscr{X}$. It remains to note that $\mathrm{E}(\cdot|Z_0)$ is the descent of $\mathscr{E}(\cdot|\Sigma_0)$.

(2) $\Longrightarrow$ (1): If the claims of (2) are true, then the operator T is well defined on X by $Tx = \mathrm{E}(wx|Z_0)$ $(x \in X)$. The required properties of T follow easily from Theorem 5.4.4. $\triangleright$

5.4.8. *A linear operator $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ is a strictly positive order continuous projection if and only if $\mathscr{T}$ can be written uniquely in the form $\mathscr{T} = \mathscr{T}_1 + \mathscr{T}_2$ with $\mathscr{T}_1$ and $\mathscr{T}_2$ satisfying the conditions:*

(1) *There exist a σ-subalgebra Σ_0 of Σ and a unique pair of functions $0 \leqslant w \in \mathscr{X}'$ and $0 \leqslant k \in L^1(\Omega,\Sigma,\mu)$ such that*

$$\mathscr{E}(wk|\Sigma_0) = \mathscr{E}(k|\Sigma_0) = [k](\mathbb{1}), \quad [w] = [k],$$
$$\mathscr{T}_1 x = k\mathscr{E}(wx|\Sigma_0) \quad (x \in \mathscr{X}).$$

(2) *$\mathscr{T}_1$ is a positive order continuous operator with $\mathscr{T}_1\mathscr{T}_2 = \mathscr{T}_2$, $\mathscr{T}_2\mathscr{T}_1 = 0$, and $\mathscr{C}_{\mathscr{T}_2} = (I_{\mathscr{X}} - [k])(\mathscr{X})$.*

$\triangleleft$ See Dodds, Huijsmans, and de Pagter [105, Proposition 3.8 and Corollary 3.9]. $\triangleright$

5.4.9. *A a linear operator $T : X \to X$ is a strictly positive order continuous projection commuting with all Φ-saturated band projections if and only if T can be written uniquely in the form $T = T_1 + T_2$ with T_1 and T_2 satisfying the conditions:*

(1) *There exist an order closed sublattice Z_0 of Z and a unique pair of elements $0 \leqslant w \in X'$ and $0 \leqslant k \in L^1(\Phi)$ such that*

$$\mathrm{E}(wk|Z_0) = \mathrm{E}(k|Z_0) = [k](\mathbb{1}), \quad [w] = [k],$$
$$Tx = k\mathrm{E}(wx|Z_0) \quad (x \in X).$$

(2) *T_2 is a positive order continuous operator on X commuting with all Φ-saturated band projections such that $T_1T_2 = T_2$, $T_2T_1 = 0$, and $\mathscr{C}_{T_2} = (I_X - [k])(X)$.*

In particular, $T_1 = TR_T$, $T_2 = T(I_X - R_T)$, and $R_{T_2} \subset R_{T_1}$.

$\triangleleft$ The proof runs along the lines of the proof of Theorems 5.4.4 and 5.4.7 with obvious modifications. Apply Theorems 5.4.8 to $\mathscr{T} := T{\uparrow}$ within $\mathbb{V}^{(\mathbb{B})}$ and find $\mathscr{T}_1, \mathscr{T}_2 \in \mathbb{V}^{(\mathbb{B})}$ such that $[\![\mathscr{T}_1, \mathscr{T}_2 : \mathscr{X} \to \mathscr{X}]\!] = [\![\mathscr{T}_1 + \mathscr{T}_2 = \mathscr{T}]\!] = \mathbb{1}$, $[\![5.4.8\,(1)]\!] = \mathbb{1}$, and $[\![5.4.8\,(2)]\!] = \mathbb{1}$. Now observe that the two last identities are equivalent to 5.4.9 (1) and 5.4.9 (2), respectively. $\triangleright$

5.4.10. Theorem. *Let $T : X \to X$ be an order continuous positive projection commuting with Φ-saturated band projection. Put $\pi := [\mathscr{C}_T]$, $\pi_1 := \pi R_T$, $\pi_2 := \pi(I_X - R_T)$, $\pi_3 := I_X - \pi$, and $X_\jmath := \pi_\jmath(X)$, and let $T_{\imath\jmath}$ stands for the restriction of $\pi_\imath T$ to $X_\jmath$ $(\imath, \jmath := 1, 2, 3)$. Then the following hold:*

(1) *π_1, π_2, and π_3 are pairwise disjoint Φ-saturated band projections on X with $\pi_1 + \pi_2 + \pi_3 = I_X$.*

(2) *$T_{\imath\jmath}$ is a positive order continuous operator from $X_\jmath$ to $X_\imath$, T_{11} and T_{12} are strictly positive, $T_{2\jmath} = T_{\imath 3} = 0$ $(\imath, \jmath := 1, 2, 3)$, and*

$$T_{11}^2 = T_{11}, \quad T_{11}T_{12} = T_{12}, \quad T_{31}T_{11} = T_{31}, \quad T_{31}T_{12} = T_{32}.$$

(3) *There exist an order closed sublattice Z_0 of Z and a unique pair of elements $0 \leqslant w \in X'$ and $0 \leqslant k \in L^1(\Phi)$ such that*

$$\mathrm{E}(wk|Z_0) = \mathrm{E}(k|Z_0) = \pi_1 \mathbb{1}, \quad \pi_1 = [w] = [k],$$
$$Tx = k\mathrm{E}(wx|Z_0) \quad (x \in X).$$

Conversely, given operators $\pi_\imath$ and $T_{\imath\jmath}$ $(\imath, \jmath := 1, 2, 3)$ satisfying (1)–(3), the operator $T : X \to X$ defined as $\pi_\imath T|_{X_\jmath} = T_{\imath\jmath}$ $(\imath, \jmath := 1, 2, 3)$ is a positive order continuous projection on X commuting with all Φ-saturated band projections.

$\lhd$ Clearly, X_1, X_2, and X_3 are invariant with respect to Φ-saturated band projections because π_1, π_2, and π_3 are Φ-saturated. By definition $\pi_2 T = 0$ and $T\pi_3 = 0$, so $T_{2\jmath} = T_{\imath 3} = 0$ for all $\imath, \jmath := 1, 2, 3$. Note that $T\pi = T$ and $\pi_1 T = \pi T$, and so $T_{31}T_{11} = \pi_3 T \pi_1 T|_{X_1} = \pi_3 T \pi T|_{X_1} = T_{31}$. Similarly, $T_{31}T_{12} = T_{32}$.

The operator πT is a positive order continuous projection on X, as $(\pi T)^2 = \pi(T\pi)T = \pi T$. Denote by $\tilde{T}$ the restriction of πT onto $X_0 := X_3^\perp = \pi(X)$. If $\tilde{T}x = 0$ for some $0 \leqslant x \in X_0$ then $0 = T(\tilde{T}x) = (T\pi)(Tx) = T^2 x = Tx$ and so $x = 0$, since T is the strictly positive on X_0. It follows that $\tilde{T}$ is strictly positive order continuous projection on $X_0 = X_1 \oplus X_2$ and by 5.4.9 $\tilde{T}$ is uniquely representable in the form $\tilde{T} = \tilde{T}_1 + \tilde{T}_2$ with $\tilde{T}_1$ and $\tilde{T}_2$ satisfying 5.4.9 (1, 2). Observe now that $\pi_1 + \pi_2 = \pi$, $R_{\tilde{T}} = \pi R_T|_{X_0} = \pi_1|_{X_0}$, and $(I_{X_0} - R_{\tilde{T}}) = \pi(I_X - R_T)|_{X_0} = \pi_2|_{X_0}$. It follows from this that $T_{11} = \pi_1 \tilde{T}_1|_{X_1}$ and $T_{12} = \pi_1 \tilde{T}_2|_{X_2}$ and so $\pi_\imath$ and $T_{\imath\jmath}$ obey (1)–(3). The converse is straightforward. $\rhd$

5.5. Maharam Extension

The main problem discussed in this section is the extension of an arbitrary positive operator to an order interval preserving order continuous operator; i.e., the Maharam extension.

5.5.1. *Suppose that X is a vector lattice over a dense subfield $\mathbb{F} \subset \mathbb{R}$ and $\varphi : X \to \mathbb{R}$ is a strictly positive $\mathbb{F}$-linear functional. There exist a Dedekind complete vector lattice X^{φ} including X and a strictly positive order continuous linear functional $\bar{\varphi} : X^{\varphi} \to \mathbb{R}$ having the Levi property and extending φ such that for every $x \in X^{\varphi}$ there is a sequence (x_n) in X with $\lim_{n\to\infty} \bar{\varphi}(|x - x_n|) = 0$.*

$\lhd$ Put $d(x, y) := \varphi(|x - y|)$ and note that (X, d) is a metric space. Let X^{φ} the completion of the metric space (X, d) and let $\bar{\varphi}$ be the extension of φ to X^{φ} by continuity. It is not difficult to ensure that X^{φ} is a Banach lattice having the additive norm $\|\cdot\|^{\varphi} := \bar{\varphi}(|\cdot|)$ and including X as a norm dense $\mathbb{F}$-linear sublattice. Thus, $\bar{\varphi}$ is a strictly positive order continuous linear functional on X^{φ} with the Levi property. $\rhd$

5.5.2. Put $L^1(\varphi) := X^{\varphi}$ and let $\bar{X}$ stand for the order ideal in $L^1(\varphi)$ generated by X. Then $(L^1(\varphi), \|\cdot\|^{\varphi})$ is an AL-space and $\bar{X}$ is a Dedekind complete vector lattice. Moreover, X is norm dense in $L^1(\varphi)$ and so in $\bar{X}$.

For a nonempty subset U of a lattice L, we denote by $U^{\uparrow}$ (respectively $U^{\downarrow}$) the set of elements $x \in L$ representable in the form $x = \sup(A)$ $(x = \inf(A))$, where A is a nonempty upward (respectively downward) directed subset of U. Moreover, we put $U^{\uparrow\downarrow} := (U^{\uparrow})^{\downarrow}$ etc. If in the above definition A is countable, then we write $U^{\upharpoonleft}$, $U^{\downharpoonleft}$, and $U^{\upharpoonleft\downharpoonleft}$ instead of $U^{\uparrow}$, $U^{\downarrow}$, and $U^{\uparrow\downarrow}$. Recall that for the Dedekind completion X^{δ} we have $X^{\delta} = X^{\uparrow} = X^{\downarrow}$.

5.5.3. *An element $\bar{x} \in \bar{X}$ belongs to $X^{\downharpoonleft\upharpoonleft}$ if and only if for all $|\bar{x}| \leqslant y \in X$ and $n \in \mathbb{N}$ there exists $u_n \in X^{\downharpoonleft}$ such that $u_n \leqslant \bar{x}$ and $\bar{\varphi}(\bar{x} - u_n) \leqslant (1/n)\varphi(y)$.*

$\lhd$ Take $\bar{x} \in X^{\downharpoonleft\upharpoonleft}$ and $y \in X$ with $|\bar{x}| \leqslant y$ and observe that the set $A(\bar{x}) := \{u \in X^{\downharpoonleft} : u \leqslant \bar{x}, |u| \leqslant y\}$ is upward directed, since $X^{\downharpoonleft}$ is a sublattice of $\bar{X}$. Considering the identity $\bar{x} = \sup(A(\bar{x}))$ and order continuity of $\bar{\varphi}$ we have $\bar{\varphi}(\bar{x}) = \sup \varphi(A(\bar{x}))$, so that for every $n \in \mathbb{N}$ there is $u_n \in A(\bar{x})$ such that $u_n \leqslant \bar{x}$ and $\bar{\varphi}(\bar{x} - u_n) \leqslant (1/n)\varphi(y)$.

Conversely, assume that for some $\bar{x} \in \bar{X}$ we can choose a sequence (u_n) in $X^{\downharpoonleft}$ meeting the above conditions. Since $X^{\downharpoonleft}$ is a sublattice of $\bar{X}$, the sequence (u_n) may be chosen increasing by replacing if need be u_n

by $u_1 \vee \dots \vee u_n$. Put $u := \sup_{n \in \mathbb{N}} u_n$ and note that $u \leqslant \bar{x}$, $\bar{\varphi}(\bar{x} - u) = 0$, and $\bar{\varphi}(\bar{x}) = \sup \bar{\varphi}(u_n) = \bar{\varphi}(u)$. Since $\bar{\varphi}$ is strictly positive, $\bar{\varphi}(\bar{x} - u) = 0$ implies $\bar{x} = u \in X^{\downarrow\uparrow}$. $\triangleright$

5.5.4. *$X^{\downarrow\uparrow}$ is an order closed vector sublattice in $\bar{X}$ and $X^{\downarrow\uparrow} = X^{\uparrow\downarrow}$.*

$\triangleleft$ We show first that $X^{\downarrow\uparrow}$ is closed under countable suprema and infima. To this end note that $(X^{\downarrow\uparrow})^{\uparrow} = X^{\downarrow\uparrow}$ holds trivially and we need only prove that $(X^{\downarrow\uparrow})^{\downarrow} = X^{\downarrow\uparrow}$. Take $z \in (X^{\downarrow\uparrow})^{\downarrow}$ and pick $y \in X$ with $|z| \leqslant y$. For all $0 < \varepsilon \in \mathbb{R}$ and $n \in \mathbb{N}$ we can choose $v_n \in X^{\downarrow\uparrow}$ with $z \leqslant v_n$ and $\bar{\varphi}(v_n - z) \leqslant (\varepsilon/2^n)\bar{\varphi}(y)$. By Proposition 5.5.3 for every $n \in \mathbb{N}$ there exists $u_n \in X^{\downarrow}$ such that $u_n \leqslant v_n$, $|u_n| \leqslant y$, and $\bar{\varphi}(v_n - u_n) \leqslant (\varepsilon/2^n)\bar{\varphi}(y)$. Put $u := \inf_{n \in \mathbb{N}} u_n$ and $u'_n := \inf_{k \leqslant n} u_k$ and observe that $u \in X^{\downarrow}$, $|u| \leqslant y$, $u \leqslant z$, $u \leqslant u'_n \leqslant u_n$. Using the inequality $|z - u'_n| \leqslant \sum_{k=1}^{n} |z - u_n|$, we deduce

$$\bar{\varphi}(|z - u'_n|) \leqslant \sum_{k=1}^{n} \bar{\varphi}(|z - u_k|)$$
$$\leqslant \sum_{k=1}^{n} \Big(\bar{\varphi}(|z - v_k|) + \bar{\varphi}(|v_k - u_k|) \Big) \leqslant 2\varepsilon\bar{\varphi}(y).$$

Considering that $u = o\text{-}\lim_n u'_n$ we get

$$\bar{\varphi}(z - u) = \lim_n \bar{\varphi}(|z - u'_n|) \leqslant 2\varepsilon\bar{\varphi}(y).$$

It follows from 5.5.3 that $z \in X^{\downarrow\uparrow}$ and so $(X^{\downarrow\uparrow})^{\downarrow} \subset X^{\downarrow\uparrow}$. Observe next that by the easy identities $(A+B)^{\downarrow} = A^{\downarrow} + B^{\downarrow}$ and $(A+B)^{\uparrow} = A^{\uparrow} + B^{\uparrow}$ we have

$$X^{\downarrow\uparrow} + X^{\downarrow\uparrow} = X^{\downarrow\uparrow}, \quad X^{\downarrow\uparrow} - X^{\downarrow\uparrow} = X^{\downarrow\uparrow} + X^{\uparrow\downarrow} = X^{\downarrow\uparrow\downarrow} = X^{\downarrow\uparrow}.$$

This shows that $X^{\downarrow\uparrow}$ is a vector sublattice in $\bar{X}$. $\triangleright$

5.5.5. *The identities $\bar{X} = X^{\downarrow\uparrow} = X^{\uparrow\downarrow}$ and $L^1(\varphi) = X^{\downarrow\uparrow} = X^{\uparrow\downarrow}$ hold with both $(\cdot)^{\downarrow\uparrow}$ and $(\cdot)^{\uparrow\downarrow}$ taken in $\bar{X}$ and $L^1(\varphi)$, respectively.*

$\triangleleft$ Note that $\bar{X} = X^{\perp\perp} = \bigvee \big\{ \{x\}^{\perp\perp} : x \in X_+ \big\}$. Therefore, given $0 \leqslant \bar{x} \in \bar{X}$, we can pick a disjoint family $(u_\xi)_{\xi \in \Xi}$ in $\bar{X}$ and a family $(x_\xi)_{\xi \in \Xi}$ in X_+ such that $\bar{x} = \sup_{\xi \in \Xi} u_\xi$ and $u_\xi \in \{x_\xi\}^{\perp\perp}$ for all $\xi \in \Xi$. Moreover $u_\xi \neq 0$ holds for at most countably many ξ, since $\bar{\varphi}(\bar{x}) = \sum_{\xi \in \Xi} \bar{\varphi}(u_\xi)$. If $\{x_\xi\}^{\perp\perp} \subset X^{\downarrow\uparrow}$ (with disjoint complements taken in $\bar{X}$) then $x_\xi \in X^{\downarrow\uparrow}$ and so $\bar{x} \in X^{\downarrow\uparrow\uparrow} = X^{\downarrow\uparrow}$. Consequently, by the Freudenthal

Spectral Theorem and 5.5.4 it suffices to show that $\mathbb{C}(\bar{X}, u_0) \subset X^{\downarrow\uparrow}$ for all $0 \leqslant u_0 \in X$, where $\mathbb{C}(\bar{X}, u_0)$ stands for the Boolean algebra of components of u_0 in $\bar{X}$.

Assume now that $u_0 = u_1 + u_2$ for some disjoint $u_1, u_2 \in \mathbb{C}(\bar{X}, u)$ and put $\bar{\varphi}_\imath := \bar{\varphi} \circ [u_\imath]$ $(\imath = 0, 1, 2)$. Then $\bar{\varphi}_1$ and $\bar{\varphi}_2$ are disjoint components of $\bar{\varphi}_0$. If $\varphi_\imath$ stands for the restriction of $\bar{\varphi}_\imath$ onto $X^{\downarrow\uparrow}$, then φ_0 is an order continuous functional by 5.5.4. Moreover, φ_1 and φ_2 are disjoint components of φ_0, so that by the Nakano Theorem $u_0 = v_1 + v_2$ for some disjoint $v_1, v_2 \in \mathbb{C}(X^{\downarrow\uparrow}, u)$ with $\varphi_1(v_2) = \varphi_2(v_1) = 0$ and $\varphi_\imath$ strictly positive on $\{v_\imath\}^{\perp\perp}$. From this we deduce that $\bar{\varphi}([u_1]v_2) = \bar{\varphi}_1(u_2 \vee v_2) = \bar{\varphi}_1(u_2) + \varphi_1(v_2) - \bar{\varphi}_1(u_2 \wedge v_2) = 0$ and so $[u_1]v_2 = 0$ or $u_1 \perp v_2$. Similarly, $u_2 \perp v_2$ and we obtain $u_1 = v_1$ and $u_2 = v_2$. It follows that $\mathbb{C}(\bar{X}, u_0) = \mathbb{C}(X^{\downarrow\uparrow}, u_0) \subset X^{\downarrow\uparrow}$ and the proof is complete. $\triangleright$

5.5.6. *Let $\mathscr{X}$ be a vector lattice within $\mathbb{V}^{(\mathbb{B})}$ and $\varnothing \neq U \subset \mathscr{X}$. Then $(U^{\downarrow})\downarrow = (U\downarrow)^{\downarrow}$ and $(U^{\uparrow})\downarrow = (U\downarrow)^{\uparrow}$.*

$\triangleleft$ The two required relations are handled similarly, so that we restrict demonstration to the second. For an arbitrary $x \in \mathscr{X}\downarrow$ we have the equivalence within $\mathbb{V}^{(\mathbb{B})}$:

$$x \in U^{\uparrow} \leftrightarrow (\exists\, \sigma : \mathbb{N}^{\wedge} \to U)(\sigma \text{ is increasing and } x = \sup(\operatorname{im}(\sigma)))$$

According to the maximum principle, $[\![x \in U^{\uparrow}]\!] = \mathbb{1}$ if and only if there exists $\sigma \in \mathbb{V}^{(\mathbb{B})}$ with the properties $[\![\sigma : \mathbb{N}^{\wedge} \to U]\!] = \mathbb{1}$, $[\![\sigma \text{ is increasing}]\!] = \mathbb{1}$, and $[\![x = \sup(\operatorname{im}(\sigma))]\!] = \mathbb{1}$. Putting $s := \sigma\downarrow$ and using 1.5.9 and 1.6.8 we arrive at the assertion: $x \in U^{\uparrow}\downarrow$ if and only if there exists an increasing function $s : \mathbb{N} \to U\downarrow$ such that $x = \sup(\operatorname{im}(s))$. This gives the required result. $\triangleright$

5.5.7. Theorem. *Let X and Y be vector lattices with Y Dedekind complete and T a positive linear operator from X to Y. There exist a Dedekind complete vector lattice $\bar{X}$ and a strictly positive Maharam operator $\bar{T} : \bar{X} \to Y$ satisfying the conditions:*

(1) *There exist a lattice homomorphism $\imath : X \to \bar{X}$ and an f-algebra homomorphism $\theta : \mathscr{Z}(Y) \to \mathscr{Z}(\bar{X})$ such that*

$$\alpha T x = \bar{T}(\theta(\alpha)\imath(x)) \quad (x \in X,\ \alpha \in \mathscr{Z}(Y)).$$

(2) *$\imath(X)$ is a majorizing sublattice in $\bar{X}$ and $\theta(\mathscr{Z}(Y))$ is an o-closed sublattice and subring of $\mathscr{Z}(\bar{X})$.*

(3) *The representation* $\bar{X} = (X \odot \mathscr{Z}(Y))^{\downarrow\uparrow}$ *holds, where* $X \odot \mathscr{Z}(Y)$ *is a subspace of* $\bar{X}$ *consisting of all finite sums* $\sum_{k=1}^{n} \theta(\alpha_k)\iota(x_k)$ *with* $x_1, \dots, x_n \in X$ *and* $\alpha_1, \dots, \alpha_n \in \mathscr{Z}(Y)$.

$\lhd$ Assume without loss of generality that T is strictly positive, since otherwise we can replace T by its restriction to the carrier $\mathscr{C}_T$. By Theorem 3.3.3, there exists a positive $\mathbb{R}^\wedge$-linear functional $\tau : X^\wedge \to \mathscr{R}$ such that $[\![T(x) = \tau(x^\wedge)]\!] = \mathbb{1}$ for all $x \in X$. It is easy to see that τ is strictly positive within $\mathbb{V}^{(\mathbb{B})}$:

$$[\![(\forall x \in X_+^\wedge)\tau(x) = 0 \to x = 0]\!]$$
$$= \bigwedge_{x \in X_+} [\![\tau(x^\wedge) = 0 \to x = 0]\!] = \bigwedge_{x \in X_+} [\![T(x) = 0 \to x = 0]\!] = \mathbb{1}.$$

By 5.5.1 there exists a Dedekind complete vector lattice $\mathscr{X} := (X^\wedge)^\tau$ including $X^\wedge$ and a strictly positive order continuous linear functional $\hat{\tau} : \mathscr{X} \to \mathscr{R}$ with the Levi property extending τ. Moreover, $X^\wedge$ is dense in $\mathscr{X}$ with respect to norm $\|\cdot\|^\tau := \hat{\tau}(|\cdot|))$. By Theorem 5.2.8 $\hat{\tau}{\downarrow} : \mathscr{X}{\downarrow} \to \mathscr{R}{\downarrow}$ is a strictly positive Maharam operator with the Levi property and an $\mathscr{R}{\downarrow}$-module homomorphism. Moreover there exists a lattice isomorphism ι from X into $\mathscr{X}{\downarrow}$ such that $T = \hat{\tau}{\downarrow} \circ \iota = \tau{\downarrow}$. Denote by $\bar{X}$ and $\bar{T}$ the order ideal in $\mathscr{X}{\downarrow}$ generated by $\iota(X)$ and the restriction of $\hat{\tau}{\downarrow}$ onto $\bar{X}$, respectively. Then $\operatorname{im}(\bar{T}) \subset Y$ and $\bar{T} : \bar{X} \to Y$ is a strictly positive Maharam operator.

The $\mathscr{R}{\downarrow}$-module structure on $\mathscr{X}{\downarrow}$ induces an f-algebra homomorphism θ from $\mathscr{R}{\downarrow}$ into $\operatorname{Orth}(\mathscr{X}{\downarrow})$ such that $\theta(a)(x) = ax$ for all $a \in \mathscr{R}{\downarrow}$ and $x \in \mathscr{X}{\downarrow}$. Identify $\mathscr{Z}(Y)$ with the sublattice of corresponding multipliers in $\mathscr{R}{\downarrow}$ and denote the restriction of θ to $\mathscr{Z}(Y)$ by the same symbol we get (1), since $\alpha Tx = \alpha(\bar{T} \circ \iota)x = \bar{T}(\theta(\alpha)\iota(x))$ for all $\alpha \in \mathscr{Z}(Y)$ and $x \in X$. The assertion (2) follows from 2.11.9 and it remains to prove (3).

Let $\bar{\mathscr{X}}$ stand for the order ideal in $\mathscr{X}$ generated by $X^\wedge$. In view of 5.5.5 $\bar{\mathscr{X}}$ is a Dedekind complete vector lattice, $\bar{\mathscr{X}} = (X^\wedge)^{\downarrow\uparrow} = (X^\wedge)^{\uparrow\downarrow}$, and from 5.5.6 we get

$$\bar{\mathscr{X}}{\downarrow} = (\operatorname{mix}(X))^{\downarrow\uparrow} = (\operatorname{mix}(X))^{\uparrow\downarrow} = (\operatorname{mix}(X))^{\downarrow\uparrow} = (\operatorname{mix}(X))^{\uparrow\downarrow}.$$

Denote by M the subset in $\bar{X}$ consisting of the elements of the form of $\sum_{\xi \in \Xi} \theta(\pi_\xi)x_\xi$, where $(\pi_\xi)_{\xi \in \Xi}$ is a partition of unity in $\mathbb{P}(Y)$ and $(x_\xi)_{\xi \in \Xi}$ is an order bounded family in X. It is a routine exercise to check that

$$\bar{X} = (M)^{\downarrow\uparrow} = (M)^{\uparrow\downarrow} = (M)^{\downarrow\uparrow} = (M)^{\uparrow\downarrow}.$$

Let M_0 stand for the part of M consisting of the finite sums $\sum_{k=1}^{n}\theta(\pi_k)x_k$ with pairwise disjoint $\pi_1,\dots,\pi_n\in\mathbb{P}(Y)$. Then $M\subset M_0^{\downarrow}$ (and, of course, $M\subset M_0^{\uparrow}$). Indeed, if $\bar{x}=\sum_{\xi\in\Xi}\theta(\pi_\xi)x_\xi$ and $|x_\xi|\leqslant y$ $(\xi\in\Xi)$ for some $y\in X_+$, then $\bar{x}=\inf_{\xi\in\Xi}\{\pi_\xi x_\xi+\pi^{\perp}y\}$. It follows therefore that $\bar{x}=\inf_{\alpha\in\mathrm{A}}u_\alpha$, where A is the collection of all finite subsets of Ξ and $u_\alpha:=\sum_{\xi\in\alpha}\theta(\pi_\xi)x_\xi\in M_0$. Hence $\bar{X}\supset M_0^{\downarrow\uparrow}=(M_0^{\downarrow})^{\downarrow\uparrow}\supset M^{\downarrow\uparrow}=\bar{X}$. Consequently, $\bar{X}=(X\odot\mathscr{Z}(Y))^{\downarrow\uparrow}$, since $M_0\subset X\odot\mathscr{Z}(Y)$. $\triangleright$

5.5.8. Theorem. *If a strictly Maharam operator $\bar{S}$ from a Dedekind complete vector lattice Z to Y, while lattice homomorphisms $\varkappa: X\to Z$ and $\eta:\mathscr{Z}(Y)\to\mathscr{Z}(Z)$ satisfy the conditions 5.5.7 (1–3) in place of $\bar{T}$, ι, and θ respectively, then there exists a lattice isomorphism h from $\bar{X}$ onto an order closed sublattice in the order ideal of Z generated by $\varkappa(X)$ such that $\varkappa=h\circ\iota$ and $\bar{T}=\bar{S}\circ h$.*

$\triangleleft$ Consider the bilinear operators B and D from $X\times\mathscr{Z}(Y)$ to $\bar{X}$ and Z respectively defined as $B(x,\alpha):=\theta(\alpha)\iota(x)$ and $D(x,\alpha):=\eta(\alpha)\varkappa(x)$. Let $\bar{B}$ and $\bar{D}$ stand for the lattice homomorphisms from $Z\mathbin{\bar{\otimes}}\mathscr{Z}(Y)$ to $\bar{Z}$ and Z, respectively, uniquely determined by $\bar{B}\otimes=B$ and $\bar{D}\otimes=D$ (cp. 3.2.6 (1)). Observe that $B(x,\alpha)>0$ and $D(x,\alpha)>0$ whenever $0<x\in X$ and $0<\alpha\in\mathscr{Z}(Y)$, so that $\bar{B}$ and $\bar{D}$ are lattice isomorphisms of $Z\mathbin{\bar{\otimes}}\mathscr{Z}(Y)$ onto the vector sublattices in $\bar{X}$ and Z generated by $B(X\times\mathscr{Z}(Y))$ and $D(X\times\mathscr{Z}(Y))$, respectively (cp. 3.2.7 (1)). Under the hypotheses in (4) we have $\bar{T}(\theta(\alpha)\iota(x))=\bar{S}(\eta(\alpha)\varkappa(x))$ for all $x\in X$ and $\alpha\in\mathscr{Z}(Y)$ and so $\bar{T}\circ\bar{B}\otimes=\bar{S}\circ\bar{D}\otimes$. It follows from 3.2.6 (3) that $\bar{T}\circ\bar{B}=\bar{S}\circ\bar{D}$. Define $h:\operatorname{im}(\bar{B})\to\operatorname{im}(\bar{D})$ as $h:=\bar{D}\circ\bar{B}^{-1}$ and note that h a lattice isomorphism with $\bar{T}(u)=\bar{S}(h(u))$ for all $u\in\operatorname{im}(\bar{B})$. Moreover, h is order continuous. Indeed, given a downward directed set $A\subset\operatorname{im}(\bar{B})$ with $\inf(A)=0$ in $\bar{X}$, we have $\bar{S}(\inf(h(A)))=\inf(\bar{S}(h(A)))=\inf\bar{T}(A)=0$ and so $\inf h(A)=0$, since $\bar{S}$ is strictly positive. Note also that $h\circ\iota=\varkappa$, since denoting by I the unit element of the f-algebra $\mathscr{Z}(Y)$ we have

$$h(\iota(x))=h(B(x,I))=(\bar{D}\circ\bar{B}^{-1}\circ\bar{B})(x\otimes I)=D(x,I)=\varkappa(x).$$

Extend h from $\operatorname{im}(\bar{B})$ to $\bar{X}=B(X\otimes\mathscr{Z}(Y))^{\downarrow\uparrow}$. If (u_α) is a downward directed set in $B(X\otimes\mathscr{Z}(Y))$, $u=\inf_\alpha u_\alpha$, and there is $x\in X$ with $|u_\alpha|\leqslant\iota x$ for all α, then $|h(u_\alpha)|\leqslant\varkappa(x)$ for all α and there exists infimum of a downward directed net $(h(u_\alpha))$ in Z. Put $h(u):=\inf_\alpha h(u_\alpha)$. Similarly, if (v_α) upward directed net in $B(X\otimes\mathscr{Z}(Y))^{\downarrow}$ with $v=\sup_\alpha v_\alpha$, then we can define $h(v):=\sup_\alpha h(u_\alpha)$. The definition is sound and the relation $\bar{S} \circ h = \bar{T}$ holds because of the order continuity and strict positivity of $\bar{T}$ and $\bar{S}$. For the same reason, h is a lattice isomorphism of $\bar{X}$ onto a sublattice in Z.

Denote by $\bar{\bar{X}}$ the order ideal of Z generated by $\varkappa(X)$ and ensure that $\mathrm{im}(h)$ is an order closed sublattice in $\bar{\bar{X}}$. Since $\varkappa = h \circ \iota$ implies $\mathrm{im}(h) \subset \bar{\bar{X}}$, all we need to do is to check that $\mathrm{im}(h)$ contains suprema of all upward directed sets $V \subset \mathrm{im}(h)$ with $v_0 = \sup(V) \in \bar{\bar{X}}$. For such V we can choose $x \in X_+$ with $|v| \leqslant \varkappa(x)$ for all $v \in V \cup \{v_0\}$. Note that for arbitrary $u \in U := h^{-1}(V)$ we have $h(|u|) \leqslant \varkappa(x) = h(\iota(x))$ and so $|u| \leqslant \iota(x)$. It follows that $u_0 := \sup(U) \in \bar{X}$ and $h(u_0) = \sup(h(U)) = \sup(V) = v_0 \in \bar{\bar{X}}$. $\triangleright$

5.5.9. The pair $(\bar{X}, \bar{T})$ (or $\bar{T}$ for short) is called a *Maharam extension* of T if it satisfies 5.5.7 (1–3). The pair $(\bar{X}, \iota)$ is also called a *Maharam extension space* for T. Two Maharam extensions T_1 and T_2 of T with the respective Maharam extension spaces (X_1, ι_1) and (X_2, ι_2) are said to be *isomorphic* if there exists a lattice isomorphism h of X_1 onto X_2 such that $T_1 = T_2 \circ h$ and $\iota_2 = h \circ \iota_1$. Theorem 5.5.8 tells us that a Maharam extension is unique up to isomorphism.

5.5.10. Two simple additional remarks follow.

(1) As was shown in the proof of Theorem 5.5.7, $\bar{X} = (M)^{\downarrow\uparrow}$. It is evident from this that $\bar{X} = (X \odot \mathscr{Z}(Y))^{\downarrow\uparrow}$ whenever Y is order separable. (Recall that a vector lattice is said to be *order separable* whenever every set in it having a supremum contains a finite or countable subset with the same supremum.)

(2) Put

$$W := \{w_1 - w_2 : w_1, w_2 \in (X \odot \mathscr{Z}(Y))^{\downarrow}\}.$$

Clearly, W is a sublattice and a vector subspace of $\bar{X}$. Moreover, W is a majorizing vector sublattice, since $\iota(X) \subset W$ and $(w_1 - w_2)^+ = w_1 \vee w_2 - w_2 \in W$ for all $w_1, w_2 \in (X \odot \mathscr{Z}(Y))^{\downarrow}$. Observe also that W is an order dense in $\bar{X}$. Indeed, if $0 < \bar{x} \in \bar{X}$ then there exists an upward directed set $A \subset (X \odot \mathscr{Z}(Y))^{\downarrow}$ such that $\bar{x} = \sup(A)$. Because of $\bar{x} = \sup\{a^+ : a \in A\}$, we can pick $a \in A$ with $0 < a^+ \leqslant \bar{x}$. Thus, $\bar{X}$ is the Dedekind completion of W; i.e.,

$$\bar{X} = \big((X \odot \mathscr{Z}(Y))^{\downarrow} - (X \odot \mathscr{Z}(Y))^{\downarrow}\big)^{\delta}.$$

5.6. Properties of Maharam Extension

Now we discuss some additional structural properties of Maharam extension. In particular, description of the Boolean algebra of band projections in the Maharam extension space is presented. As an application, approximation of the Boolean algebra of components of a positive operator by elementary fragments is also given.

5.6.1. Let X and Y be vector lattices with Y Dedekind complete, $T : X \to Y$ a positive operator and $(\bar{X}, \bar{T})$ a Maharam extension of T. Consider a universal completion $\bar{X}^{\mathfrak{u}}$ of $\bar{X}$ with a fixed f-algebra structure. Let $L_1(T)$ be the greatest order dense ideal in $\bar{X}^{\mathfrak{u}}$ onto which $\bar{T}$ can be extended by order continuity. In more detail,

$$L^1(T) := \{x \in \bar{X}^{\mathfrak{u}} : \ \bar{T}([0, |x|] \cap \bar{X}) \text{ is order bounded in } Y\},$$

$$\hat{T}x := \sup\{\bar{T}u : \ u \in \bar{X},\, 0 \leqslant u \leqslant x\} \quad (x \in L^1(T)_+),$$

$$\hat{T}x = \hat{T}x^+ - \hat{T}x^- \quad (x \in L^1(T)).$$

Define the Y-valued norm $\lvert\!\lvert\cdot\rvert\!\rvert$ on $L^1(T)$ by $\lvert\!\lvert u\rvert\!\rvert := \hat{T}(|u|)$. In terms of lattice normed spaces $(L^1(T), \lvert\!\lvert\cdot\rvert\!\rvert)$ is a *Banach–Kantorovich lattice*; see 5.8.4 below (cp. Kusraev [228, Chapter 2]). In particular, $\lvert\!\lvert au\rvert\!\rvert = |a|\lvert\!\lvert u\rvert\!\rvert$ $(a \in \mathscr{Z}(Y),\ u \in L^1(T)$.

5.6.2. *Let X and Y be vector lattices with Y Dedekind complete and T a positive linear operator from X to Y. Then there exist an AL-space $\mathscr{L}$ within $\mathbb{V}^{(\mathbb{B})}$ and a lattice isomorphism h from $L^1(T)$ onto an order dense ideal in $\mathscr{L}{\downarrow}$ such that the following hold:*

(1) $[\![$*The functional* $\hat{\tau} : \mathscr{L} \to \mathscr{R}$ *defined as* $\hat{\tau}(x) := \|x^+\| - \|x^-\|$ $(x \in \mathscr{L})$ *is order continuous and has the Levi property*$]\!] = \mathbb{1}$.

(2) $[\![(h(\iota(X))){\uparrow}$ *is a norm dense* $\mathbb{R}^{\wedge}$*-linear sublattice in* $\mathscr{L}]\!] = \mathbb{1}$.

(3) $\hat{T} = \hat{\tau}{\downarrow} \circ h$, $T = \hat{T} \circ h \circ \iota$, *and* $\lvert\!\lvert\cdot\rvert\!\rvert = \|\cdot\|{\downarrow} \circ h$.

$\lhd$ This fact can be extracted from the proof of Theorem 5.5.7. $\rhd$

5.6.3. Theorem. *For a positive $T : X \to Y$ the following hold:*

(1) *$L^1(T)$ is an f-module over $\mathscr{Z}(Y)$ and $\bar{X}$ is its f-submodule.*

(2) *$\hat{T} : L^1(T) \to Y$ is a Maharam operator extending $\bar{T}$.*

(3) *The sublattice $\iota(X)$ is dense in $L^1(T)$ in the sense that for each $u \in L^1(T)$ and $0 < \varepsilon \in \mathbb{R}$ for each there exist a partition $(\pi_\xi)_{\xi\in\Xi}$ of $[\lvert\!\lvert u\rvert\!\rvert]$*

in $\mathbb{P}(Y)$ *and a family* $(x_\xi)_{\xi\in\Xi}$ *in* X *such that*

$$\left|u - \sum_{\xi\in\Xi} \pi_\xi \iota(x_\xi)\right| \leqslant \varepsilon|u|.$$

$\lhd$ Clearly, (1) and (2) follow from Theorem 5.2.8 and so all we have to show is (3). According to 5.6.2 we can assume that $L^1(T) \subset \mathscr{L}{\downarrow}$ and h is the embedding. For an arbitrary $u \in L^1(T)$ we have $[\![u \in \mathscr{L}]\!] = \mathbb{1}$. If $u = 0$ there is nothing more to prove, if not $b = [\![|u| > 0]\!] \neq \mathbb{0}$. Moreover, passing from $\mathbb{V}^{(\mathbb{B})}$ to a relative Boolean valued model $\mathbb{V}^{([\mathbb{0},b])}$ if necessary, we can assume $b = \mathbb{1}$. Interpreting the fact that $(\iota(X)){\uparrow}$ is norm dense in $\mathscr{L}$ within $\mathbb{V}^{(\mathbb{B})}$, we deduce

$$\begin{aligned}\mathbb{1} &= [\![(\forall\, 0 < \varepsilon \in \mathbb{R}^\wedge)(\exists\, x \in \iota(X){\uparrow}X^\wedge)(\|u - x\| \leqslant \varepsilon|u|)]\!]\\ &= \bigwedge_{0<\varepsilon\in\mathbb{R}} \bigvee_{x\in X} [\![\|u - \iota(x)\| \leqslant \varepsilon^\wedge|u|]\!].\end{aligned}$$

It follows that for every $0 < \varepsilon \in \mathbb{R}$ there exists a partition on unity $(b_\xi)_{\xi\in\Xi}$ in $\mathbb{B}$ and a family $(x_\xi)_{\xi\in\Xi}$ in X such that $b_\xi \leqslant [\![\|u - \iota(x_\xi)\| \leqslant \varepsilon^\wedge|u|]\!]$ for all $\xi \in \Xi$. If $\pi_\xi := \chi(b_\xi)$ then $\pi_\xi|u - \iota(x_\xi)| \leqslant \varepsilon|\pi_\xi u| = \pi_\xi|u|$ by 2.2.4 ($\mathbb{G}$). Summing up over $\xi \in \Xi$ yields the desired result. $\rhd$

5.6.4. Theorem. *For every operator* $S \in \{T\}^{\perp\perp}$ *there exists a unique operator* $\bar{S} \in \{\bar{T}\}^{\perp\perp}$ *such that* $S = \bar{S} \circ \iota$*. The mapping* $S \mapsto \bar{S}$ *implements an isomorphism of the vector lattices* $\{T\}^{\perp\perp}$ *and* $\{\bar{T}\}^{\perp\perp}$*.*

$\lhd$ Observe that the mapping $R : \bar{S} \mapsto \bar{S} \circ \iota$ from $\{\bar{T}\}^{\perp\perp}$ to $L^\sim(X, Y)$ is linear and positive and sends the order ideal generated by $\bar{T}$ into the order ideal generated by T. Moreover, $\operatorname{im}(R) \subset \{T\}^{\perp\perp}$, since $\bar{S}_\alpha \uparrow \bar{S}$ implies $\bar{S}_\alpha \circ \iota \uparrow \bar{S} \circ \iota$ for every increasing family $(\bar{S}_\alpha)$ of positive operators in $\{\bar{T}\}^{\perp\perp}$. So, all we have to show is that every $S \in \{T\}^{\perp\perp}$ admits the unique extension to $\bar{S} \in \{\bar{T}\}^{\perp\perp}$ such that $S = \bar{S} \circ \iota$.

There is no loss of generality in assuming that S is positive. Let S lie in the order ideal generated by T; i.e., $0 \leqslant S \leqslant \lambda T$ for some $\lambda \in \mathbb{R}$. Then $0 \leqslant S \circ \iota^{-1} \leqslant \lambda \bar{T}|_{\iota(X)}$, so that by Theorem 3.1.8 there exists a positive extension $\bar{S}$ of $S \circ \iota^{-1}$ to $\bar{X}$ such that $0 \leqslant \bar{S} \leqslant \lambda\bar{T}$. Clearly, $\bar{S} \circ \iota = S$ and $\bar{S} \in \{\bar{T}\}^{\perp\perp}$.

Take an increasing net (S_α) of positive operators in the order ideal generated by T such that $S := \sup S_\alpha \in \{T\}^{\perp\perp}$. On account of what was just proved there is a family $(\bar{S}_\alpha)$ in $\{\bar{T}\}^{\perp\perp}$ such that $\bar{S}_\alpha \circ \iota = S_\alpha$

for all α. If $z \in \bar{X}$ then $|z| \leqslant \iota(x)$ for some $x \in X$ and we may, therefore, estimate $|\bar{S}_\alpha z| \leqslant \bar{S}_\alpha(|z|) \leqslant Sx$. Thus it is possible to define some positive operator by putting

$$\bar{S}z := \sup \bar{S}_\alpha z \quad (z \in \bar{X}_+).$$

Obviously, $\bar{S} = \sup \bar{S}_\alpha \in \{\bar{T}\}^{\perp\perp}$ and $\bar{S} \circ \iota = S$. It remains to show that for an operator $S \in \{T\}^{\perp\perp}$ there exists at most one $\bar{S} \in \{\bar{T}\}^{\perp\perp}$ with $\bar{S} \circ \iota = S$. Assume that $\bar{S}_1 \circ \iota = S = \bar{S}_2 \circ \iota$ for some $\bar{S}_1, \bar{S}_2 \in \{\bar{T}\}^{\perp\perp}$. Then $\bar{S}_1$ and $\bar{S}_2$ coincide on $\iota(X)$. By Theorem 5.2.5 $\bar{S}_1$ and $\bar{S}_2$ are Maharam operators and so they coincides on $X \odot \mathscr{Z}(Y)$ due to $\mathscr{Z}(Y)$-linearity and coincide on $\bar{X} = (X \odot \mathscr{Z}(Y))^{\downarrow\uparrow}$ due to order continuity. $\triangleright$

The following result is a variant of the Radon–Nikodým Theorem for positive operators.

5.6.5. Theorem. *Let X and Y be vector lattices with Y Dedekind complete and let T be a positive linear operator from X to Y. For every operator $S \in \{T\}^{\perp\perp}$ there is a unique element $z = z_T \in \bar{X}^{\mathfrak{u}}$ satisfying*

$$Sx = \hat{T}(z \cdot \imath(x)) \quad (x \in X).$$

The mapping $T \mapsto z_T$ establishes a lattice isomorphism between the band $\{T\}^{\perp\perp}$ and the order dense ideal in $\bar{X}^{\mathfrak{u}}$ defined by

$$\{z \in \bar{X}^{\mathfrak{u}} : \, z \cdot \imath(X) \subset L_1(T)\}.$$

$\triangleleft$ The proof is immediate from 5.3.5 and 5.6.4. $\triangleright$

5.6.6. According to 5.6.4 and 5.3.6 the vector lattices $\bar{X}$, $L^1(T)$, $\{T\}^{\perp\perp}$, and $\{\bar{T}\}^{\perp\perp}$ have isomorphic Boolean algebras of projections. Below we will give a detailed description for bases for $\bar{X}$ and $\{T\}^{\perp\perp}$. As usual, we denote by $[\iota x]$ the band projection in $\bar{X}$ onto $\{\iota(x)\}^{\perp\perp}$.

Given an order ideal G in X and a positive operator $T \in L^\sim(X,Y)$, denote by $\pi_G(T)$ the least extension of $T|_G$ (cp. 3.1.9). Clearly, $\pi_G(T)x = \sup\{T(x \wedge g) : \, g \in G\}$ for all $x \in X_+$. Put $\pi_e := \pi_G$ whenever G is an order ideal generated by $e \in X_+$. The following representation for π_e is straightforward:

$$\pi_e Tx = \sup\nolimits_n T(ne \wedge x) \quad (x \in E^+, \ T \in L^+(E,F)),$$
$$\pi_e Tx = \pi_e Tx^+ - \pi_e Tx^- \quad (x \in E, \ T \in L^+(E,F)),$$
$$\pi_e T = \pi_e T^+ - \pi_e T^- \quad (T \in L^\sim(E,F)).$$

Denote by $\mathscr{S}(\bar{X})$ and $\mathscr{S}(T)$ the sets of all projections in $\bar{X}$ and the set of all components of T, respectively, representable as

$$\bigvee_{k=1}^{n} \rho_k[\iota x_k] \quad \text{and} \quad \bigvee_{k=1}^{n} \rho_k \pi_{x_k}(T),$$

where $x_1, \dots, x_n \in X_+$, $\rho_1, \dots, \rho_n \in \mathbb{P}(Y)$, $n \in \mathbb{N}$. Given a band K in X, denote by $\langle K \rangle$ the band projection in $\bar{X}$ onto $(\iota K)^{\perp\perp}$; i.e., $\langle K \rangle := [\iota K]$. Put

$$\langle x \rangle := [\iota(\{x\}^{\perp\perp})] \ \text{ and } \ \pi_{\langle x \rangle} := \pi_{\{x\}^{\perp\perp}} \quad (x \in X).$$

Let $\mathscr{C}(\bar{X})$ and $\mathscr{C}(T)$ denote the sets of band projections in $\bar{X}$ and components of T representable respectively as

$$\bigvee_{k=1}^{n} \rho_k \cdot \langle x_k \rangle \quad \text{and} \quad \bigvee_{k=1}^{n} \rho_k \cdot \pi_{\langle x_k \rangle},$$

where $n \in \mathbb{N}$, $\rho_1, \dots, \rho_n \in \mathbb{P}(Y)$, and $x_1, \dots, x_n \in X$. In the case of a vector lattice X with the principal projection property we may consider one more set $\mathscr{A}(T)$ consisting of the components of T representable as

$$\bigvee_{k=1}^{n} \rho_k \circ T \circ [x_k] \quad (\rho_1, \dots, \rho_n \in \mathbb{P}(Y),\ x_1, \dots, x_n \in X),$$

where $n \in \mathbb{N}$ and $[x_k]$ is the band projection in X onto $\{x_k\}^{\perp\perp}$.

5.6.7. *For all $x \in X_+$ and $K \in \mathbb{B}(X)$ the representations hold:*

(1) $\pi_x(T) = \bar{T} \circ [\iota x] \circ \iota$.

(2) $\pi_K(T) = \bar{T} \circ \langle K \rangle \circ \iota$.

(3) $\pi_{\langle x \rangle}(T) = \bar{T} \circ \langle x \rangle \circ \iota$.

◁ Indeed, using the order continuity of $\bar{\Phi}$, we deduce

$$\begin{aligned} \pi_x(T)y &= \sup\{T(y \wedge nx) : n \in \mathbb{N}\} \\ &= \sup\{\bar{T}(\iota(y) \wedge n\iota(x)) : n \in \mathbb{N}\} \\ &= \bar{T}(\sup\{\iota(y) \wedge n\iota(x)) \\ &= \bar{T} \circ [\iota x](\iota(y))). \end{aligned}$$

The proof of (2) is similar and (3) is a particular case of (2). ▷

5.6.8. *Let W be a vector lattice with a weak order unit u and the principal projection property. If $w \in W_+$ and $w = \inf(V)$ for some $V \subset W$ then*

$$[w] = \bigvee_{n\in\mathbb{N}} \bigwedge_{v\in V} \left[\left(v - \frac{1}{n}\,u\right)^+\right].$$

$\lhd$ We may assume that $W = \mathscr{R}{\downarrow}$ with $\mathscr{R} \in \mathbb{V}^{(\mathbb{B})}$ and $\mathbb{B} := \mathbb{P}(W)$. Then $V{\uparrow}$ is a numerical set and $w = \inf(V{\uparrow})$ within $\mathbb{V}^{(\mathbb{B})}$; therefore,

$$w \neq 0 \leftrightarrow 0 < w \leftrightarrow (\exists\, n \in \mathbb{N}^\wedge)(\forall\, v \in V{\uparrow})(v - (1/n)\,u)^+ \neq 0.$$

Calculating the Boolean truth values and considering 2.4.9 we deduce for traces (see 2.4.8)

$$e_w = [\![w \neq 0]\!] = \bigvee_{n\in\mathbb{N}} \bigwedge_{v\in V} e_{(v-(1/n)u)^+}.$$

The claim follows from this formula, since the band projection $[w]$ is represented in $\mathscr{R}{\downarrow}$ as multiplication by the trace e_w, while multiplication is an order continuous lattice homomorphism. $\rhd$

5.6.9. Theorem. *The following are valid:*

(1) $\mathbb{P}(\bar{X}) = \mathscr{S}(\bar{X})^{\downarrow\uparrow}$;

(2) $\mathbb{P}(\bar{X}) = \mathscr{C}(\bar{X})^{\uparrow\downarrow\uparrow}$.

$\lhd$ (1): Recall that M_0 stands for the set of finite sums $\sum_{k=1}^n \theta(\pi_k)x_k$ with pairwise disjoint $\pi_1, \dots, \pi_n \in \mathbb{P}(Y)$. By definition $[\iota y] \in \mathscr{S}(\bar{X})$ for each $y \in M_0$. If $0 \leqslant y \in M_0^{\downarrow}$, then we can choose $x \in X_+$ and $V \subset M_0$ so that $\iota x \geqslant v \geqslant y$ for all $v \in V$ and $y = \inf(V)$. Applying 5.6.8 with $w := y$ and $u := \iota x$, we have

$$[y] = \bigvee_{n\in\mathbb{N}} \bigwedge_{v\in V} \left[\left(v - \frac{1}{n}\iota x\right)^+\right].$$

Since $y_{n,v} = \big(v - (1/n)\iota x\big)^+$ belongs to M_0, it follows that $[y_{n,v}] \in \mathscr{S}$ and $[y] \in \mathscr{S}^{\downarrow\uparrow}$. An arbitrary projection $\pi \in \mathbb{P}(\bar{X})$ has the representation $\pi = \sup\{[y] \colon y \in \bar{X}_+,\ \pi y = y\}$. Thus, taking 5.5.7 (3) into consideration we arrive at the desired containment $\pi \in \big((\mathscr{S}^{\downarrow\uparrow})^{\uparrow}\big)^{\uparrow} = \mathscr{S}^{\downarrow\uparrow}$.

(2): It suffices to show that $[\iota x] \in \mathscr{C}^{\uparrow\downarrow}$ for every $x \in X_+$. Then $\mathscr{S}(\bar{X}) \subset \mathscr{C}(\bar{X})^{\uparrow\downarrow}$, so that

$$\mathbb{P}(\bar{X}) = \mathscr{S}(\bar{X})^{\downarrow\uparrow} \subset \big(\mathscr{C}(\bar{X})^{\uparrow\downarrow}\big)^{\downarrow\uparrow} = \mathscr{C}(\bar{X})^{\uparrow\downarrow\uparrow} \subset \mathbb{P}(\bar{X}).$$

Thus, what we need is only to justify the representation:

$$[\iota x] = \bigwedge_{t \in X_+} \bigvee_{n \in \mathbb{N}} \langle (nx - t)^+ \rangle.$$

Put $\sigma_t := \bigwedge_n \langle (nx - t)^+ \rangle$ and $\sigma = \bigwedge_t \sigma_t$. It is not difficult to observe that $\sigma_t \geqslant [\iota x]$ for all $t \in X_+$ For an arbitrary projection $\rho \in \mathbb{P}(\bar{X})$ with $\rho \wedge [\iota x] = 0$ put $\rho_t := \rho \wedge [\iota t]$ $(t \in X_+)$. Then $\rho_t \leqslant [\iota(t - nx)^+] \leqslant \langle (t - nx)^+ \rangle$ for every $n \in \mathbb{N}$. Since $\langle (t - nx)^+ \rangle \wedge \langle (nx - t)^+ \rangle = 0$ it follows $\rho_t \wedge \langle (nx - t)^+ \rangle = 0$ and $\rho_t \wedge \sigma_t = \bigwedge_n (\rho_t \wedge \langle (nx - t)^+ \rangle = 0$. From this we obtain

$$\rho_t \wedge \sigma = 0, \quad \rho \wedge \sigma = \sup \rho_t \wedge \sigma = 0.$$

Putting $\rho = [\iota x]^\perp$, we arrive at the desired inequalities $[\iota x] \leqslant \sigma \leqslant [\iota x]$. $\triangleright$

5.6.10. *The following are valid:*

(1) $\mathbb{C}(T) = \mathscr{S}(T)^{\downarrow\uparrow}$;

(2) $\mathbb{C}(T) = \mathscr{C}(T)^{\uparrow\downarrow\uparrow}$.

If X has the principal projection property then

(3) $\mathbb{C}(T) = \mathscr{A}(T)^{\uparrow\downarrow\uparrow}$.

$\triangleleft$ This is immediate from 5.6.7 and 5.6.9. $\triangleright$

5.7. Banach Lattices and Banach f-Modules

In this section we consider some interplay between the lattice norm and the f-module structure on a vector lattice.

5.7.1. A norm $\|\cdot\|$ on a vector lattice X is called *monotone* or a *lattice norm* if $|x| \leqslant |y|$ implies $\|x\| \leqslant \|y\|$ for all $x, y \in X$. A *normed lattice* is a vector lattice equipped with a monotone norm. A normed lattice complete with respect to the norm is called a *Banach lattice*. In a normed lattice X the lattice operations are continuous and the positive cone X_+ is closed. Every two norms making a vector lattice a Banach lattice are equivalent.

The norm dual X' of a normed lattice X is a Dedekind complete Banach lattice. Moreover, X' is an order ideal of $X^\sim$ and $X' = X^\sim$ whenever X is a Banach lattice. For arbitrary $x_0 \in X_+$ and $x'_0 \in X'_+$ we have

$$\|x'_0\| = \sup\{\langle x, x'_0 \rangle : \ x \in X_+, \ \|x\| \leqslant 1\},$$
$$\|x_0\| = \sup\{\langle x, x'_0 \rangle : \ x' \in X'_+, \ \|x'\| \leqslant 1\}.$$

5.7.2. One of the important features of Banach lattice theory is the interplay between the norm and order. A Banach lattice X is said to have

(1) an *order continuous norm* if $\lim_\alpha \|x_\alpha\| = 0$ for every decreasing net (x_α) with $\inf_\alpha x_\alpha = 0$;

(2) the *Levi property* or a *Levi norm* if $\sup_\alpha x_\alpha$ exists in X for every increasing net (x_α) in X_+ with $\|x_\alpha\| \leqslant 1$ for all α;

(3) the *Fatou property* or a *Fatou norm* if $\lim_\alpha \|x_\alpha\| = \|x\|$ for for every increasing net (x_α) in X_+ with $\sup_\alpha x_\alpha = x$;

(4) *property* (P) if there exists a contractive positive projection in X'' onto X.

We will use also the expressions "X is an order continuous (Levi, Fatou) Banach lattice." A Banach lattice with order continuous, Levi, or Fatou norm is also called *order continuous, order semicontinuous,* or *monotonically complete*, respectively. A Dedekind complete Banach lattice X with a separating order continuous dual has property (P) if and only if X has the Levi and Fatou properties.

A Banach lattice X is said to be a *Kantorovich–Banach space* (or briefly a *KB-space*) whenever every increasing norm bounded sequence of X_+ is norm convergent. This is equivalent to saying that X has an order continuous Levi norm.

Let us list some properties of order continuous norms and *KB*-spaces.

5.7.3. Theorem. *For an arbitrary Banach lattice X the following are equivalent:*

(1) *The norm on X is order continuous.*

(2) *X is Dedekind σ-complete and sequentially order continuous.*

(3) *Every monotone order bounded sequence in X is convergent.*

(4) *Every disjoint order bounded sequence in X_+ is norm convergent to zero.*

(5) *Each closed order ideal of X is a projection band.*

(6) *The null ideal $\mathscr{N}(x')$ is a band for every $x' \in X'$.*

(7) *The natural embedding $X \to X''$ sends X onto an ideal of X''.*

(8) *All norm continuous linear functionals on X are order continuous.*

$\triangleleft$ The proof can be found in Aliprantis and Burkinshaw [28, Theorems 4.9, 4.14] and Meyer-Niberg [311, Theorem 2.4.2, Corollary 2.4.4]. $\triangleright$

5.7.4. Theorem. *For a Banach lattice X the following hold:*

(1) *X is a KB-space if and only if the natural embedding $X \to X''$ sends X onto a band of X''.*

(2) *X is reflexive if and only if X and X' are both KB-spaces.*

$\lhd$ See Aliprantis and Burkinshaw [28, Theorems 4.60 and 4.70] and Meyer-Niberg [311, Theorems 2.4.12 and 2.4.15]. $\rhd$

5.7.5. Two classes of Banach lattices play a significant role in Banach lattice theory. A Banach lattice X is said to be

(1) an *AL-space* if $\|x + y\| = \|x\| + \|y\|$ for all $x, y \in X_+$ with $x \wedge y = 0$;

(2) an *AM-space* if $\|x \vee y\| = \max\{\|x\|, \|y\|\})$ for all $x, y \in X_+$ with $x \wedge y = 0$.

An AM-space has a (strong order) *unit* $u \geqslant 0$ if the order interval $[-u, u]$ is the unit ball of X.

Each AL-space is a KB-space and an AM-space has an order semicontinuous norm. A Banach lattice X is an AL-space (respectively AM-space) if and only if X' is an AM-space (AL-space).

A *lattice isometry* is a lattice isomorphism that is also an isometry. Banach lattices are *lattice isometric* if there exists a one-to-one lattice isometry between them.

5.7.6. Kakutani–Kreĭns Representation Theorem. *An AM-space is lattice isometric to a sublattice of $C(Q)$ for some Hausdorff compact topological space Q. Moreover, if an AM-space X has a strong order unit then X is lattice isometric to the whole of $C(Q)$.*

$\lhd$ See Aliprantis and Burkinshaw [28, Thorem 4.29], Meyer-Niberg [311, Theorem 2.1.3], and Semadeni [363, Theorem 13.2.3]. $\rhd$

5.7.7. Nakano–Stone Completeness Theorem. *Let K be a Hausdorff compact topological space. The vector lattice $C(Q)$ is Dedekind complete if and only if Q is extremally disconnected ($\equiv$ the closure of every open set in K is open).*[3]

$\lhd$ See Meyer-Niberg [311, Propositions 2.1.4 and 2.1.5] and Semadeni [363, Theorem 24.7.1]. $\rhd$

5.7.8. Assume that a measure space (Ω, Σ, μ) is semi-finite, that is, if $A \in \Sigma$ and $\mu(A) = \infty$ then there exists $B \in \Sigma$ with $B \subset A$

[3] An extremally disconnected Hausdorff compact space is often referred to as *Stonean*; cp. 2.8.6.

and $0 < \mu(A) < \infty$. The vector lattice $L^0(\Omega, \Sigma, \mu)$ (of μ-cosets) of μ-measurable functions on Ω is Dedekind complete if and only if (Ω, Σ, μ) is localizable. In this event $L^p(\Omega, \Sigma, \mu)$ is also Dedekind complete. (A measure space (Ω, Σ, μ) is *localizable* or *Maharam* if it is semi-finite and, for every $\mathscr{A} \subset \Sigma$, there is a $B \in \Sigma$ such that (i) $A \setminus B$ is negligible for every $A \in \mathscr{A}$; (ii) if $C \in \Sigma$ and $A \backslash C$ is negligible for every $A \in \mathscr{A}$, then $B \backslash C$ is negligible (cp. Fremlin [126]).) Observe that $\mathbb{P}(L^0(\Omega, \Sigma, \mu)) \simeq \Sigma/\mu^{-1}(0)$.

5.7.9. Kakutani Representation Theorem. *A Banach lattice is an AL-space if and only if it is lattice isometric to* $L^1(\Omega, \Sigma, \mu)$ *for some localizable measure space* (Ω, Σ, μ).

$\lhd$ See Aliprantis and Burkinshaw [28, Theorem 4.27], Meyer-Niberg [311, Theorem 2.7.1], and Semadeni [363, §2.3]. $\rhd$

5.7.10. Theorem. *If X is a Banach lattice, then* $\operatorname{Orth}(X)$ *under the order unit norm is an AM-space with unit I_X, the identity operator on X. In particular,* $\operatorname{Orth}(X) = \mathscr{Z}(X)$ *and*

$$\|T\| = \|T\|_\infty := \inf\{0 < \lambda \in \mathbb{R} : \ |T| \leqslant \lambda I_X\} \quad (T \in \mathscr{Z}(X)).$$

$\lhd$ See Aliprantis and Burkinshaw [28, Thorem 4.77] and Meyer-Niberg [311, Theorem 3.1.12]. $\rhd$

5.7.11. A *Banach f-module* over an f-algebra A is a Banach lattice that is simultaneously an f-module over A. By Definition 2.11.1 and Theorem 5.7.10, X is a Banach f-module over an f-algebra A if and only if there exists an f-algebra homomorphism $h : A \to \mathscr{Z}(X)$ such that $ax = h(a)x$ for all $a \in A$ and $x \in X$. Thus, A is considered as an f-subalgebra of $\mathscr{Z}(X)$ with the induced order unit norm $\|a\| := \|h(a)\|_\infty$ $(a \in A)$. In particular, $\|ax\| \leqslant \|a\|\|x\|$ for all $a \in A$ and $x \in X$.

Given Banach f-modules X and Y over A, denote by $\mathscr{L}(X, Y)$ and $\mathscr{L}_A(X, Y)$ respectively the spaces of all continuous linear and A-linear operators from X to Y and put $\mathscr{L}^\sim_{n,A}(X, Y) := \mathscr{L}(X, Y) \cap L^\sim_{n,A}(X, Y)$. If Y is Dedekind complete then $\mathscr{L}_A(X, Y)$ and $\mathscr{L}_{n,A}(X, Y)$ are bands in $L^\sim(X, Y)$ and Banach f-modules over A.

5.7.12. We can produce Banach f-modules by distinguishing a complete Boolean algebra of M-projections in a Banach lattice.

A band projection π in a Banach lattice X is called an *M-projection* if $\|x\| = \max\{\|\pi x\|, \|\pi^\perp x\|\}$ for all $x \in X$, where $\pi^\perp := I_X - \pi$. The collection of all M-projections forms the subalgebra $\mathbb{M}(X)$ of the Boolean algebra of all band projections $\mathbb{P}(X)$ in X. It is easily seen by induction that

$$\left\| \sum_{k=1}^{n} \pi_k x \right\| = \max_{k=1,\dots,n} \|\pi_k x\|$$

for $x \in X$ and every finite partition of unity $\pi_1, \dots, \pi_n$ in $\mathbb{M}(X)$.

An *M-module* over A is a Banach f-module over A satisfying

$$\|ax \vee by\| = \max\{\|ax\|, \|by\|\}$$

for all $x, y \in X$ and $a, b \in A$ with $a \perp b$. If A has the projection property then the f-algebra homomorphism in 5.7.11 maps $\mathbb{P}(A)$ into $\mathbb{M}(X)$; i.e., the multiplication by each $\pi \in \mathbb{P}(A)$ is an M-projection in X.

Assume that X is a Banach lattice and $\mathscr{B}$ is a complete subalgebra of the complete Boolean algebra $\mathbb{B}(X)$ consisting of projection bands and denote by $\mathbb{B}$ the corresponding Boolean algebra of band projections. Let $\Lambda := \Lambda(\mathbb{B})$ stand for a Dedekind complete AM-space with unit such that $\mathbb{P}(\Lambda)$ is isomorphic to $\mathbb{B}$. A Boolean isomorphism h from $\mathbb{P}(\Lambda)$ onto $\mathbb{B}$ can be extended to a unital f-algebra isomorphism from Λ into $\mathscr{Z}(X)$. Thus h induces an f-module structure over Λ on X.

5.7.13. We will identify $\mathbb{P}(\Lambda)$ and $\mathbb{B}$ and write $\mathbb{B} \subset L(X)$. If $(b_\xi)_{\xi \in \Xi}$ is a partition of unity in $\mathbb{B}$ and $(x_\xi)_{\xi \in \Xi}$ is a family in X, then there is at most one element $x \in X$ with $b_\xi x_\xi = b_\xi x$ for all $\xi \in \Xi$. This element is called the *mixture* of (x_ξ) by (b_ξ) and is denoted by $x = \operatorname{mix}_{\xi \in \Xi}(b_\xi x_\xi)$. Clearly, $x = o\text{-}\sum_{\xi \in \Xi} b_\xi x_\xi$. A Banach lattice X is said to be $\mathbb{B}$-*cyclic* or $\mathbb{B}$-*complete* if the mixture of every family in the unit ball $U(X)$ of X by each partition of unity in $\mathbb{B}$ (with the same index set) exists in $U(X)$.

5.8. Lattice Normed Spaces

In this section we consider the structural properties of a vector space equipped with some norm taking values in a vector lattice. The most important peculiarities of such space are connected with the norm decomposability property.

5.8.1. Consider a vector space X and a real vector lattice Λ. A mapping $\boldsymbol{|}\cdot\boldsymbol{|} : X \to \Lambda_+$ is a *vector* (Λ-*valued*) *norm* if the following hold:

(1) $\boldsymbol{|}x\boldsymbol{|} = 0 \Longleftrightarrow x = 0 \quad (x \in X)$;

(2) $\boldsymbol{|}\lambda x\boldsymbol{|} = |\lambda|\boldsymbol{|}x\boldsymbol{|} \quad (\lambda \in \mathbb{R},\ x \in X)$;

(3) $|\!|x + y|\!| \leqslant |\!|x|\!| + |\!|y|\!| \quad (x, y \in X)$.

A vector norm is called a *decomposable norm* or a *Kantorovich norm* if

(4) given $\lambda_1, \lambda_2 \in \Lambda_+$ and $x \in X$ with $|\!|x|\!| = \lambda_1 + \lambda_2$, there exist $x_1, x_2 \in X$ such that $x = x_1 + x_2$ and $|\!|x_k|\!| = \lambda_k$ $(k := 1, 2)$.

If (4) is valid only for disjoint $\lambda_1, \lambda_2 \in \Lambda_+$, then the norm is said to be *disjointly decomposable* or, in short, *d-decomposable.* In the case that X is a vector lattice, the vector norm is said to be *monotone* or a *lattice norm* whenever

(5) $|x| \leqslant |y| \Longrightarrow |\!|x|\!| \leqslant |\!|y|\!| \quad (x, y \in X)$.

A pair $\big(X, |\!|\cdot|\!|\big)$ $\big($or in brief $X\big)$ is called a *lattice normed space over* Λ if $|\!|\cdot|\!|$ is a Λ-valued norm on a vector space X. If the norm $|\!|\cdot|\!|$ is decomposable then the space X is called decomposable as well. Put $|\!|X|\!| := \{|\!|x|\!| : x \in X\}$.

5.8.2. Say that the elements $x, y \in X$ are *norm disjoint* and write $x \perp\!\!\!\perp y$ whenever $|\!|x|\!| \wedge |\!|y|\!| = 0$. A *metric band* in X is a subset of the form $M^{\perp\!\!\!\perp} := \{x \in X : (\forall\, y \in M)\ x \perp\!\!\!\perp y\}$ with $\varnothing \neq M \subset X$.

(1) *If $x, y \in X$ are norm disjoint, then $|\!|x + y|\!| = |\!|x|\!| + |\!|y|\!|$.*

$\lhd$ Indeed, the relations $|\!|x|\!| \wedge |\!|y|\!| = 0$ and $|\!|x|\!| \leqslant |\!|x + y|\!| + |\!|y|\!|$ imply

$$|\!|x|\!| \leqslant \big(|\!|x + y|\!| + |\!|y|\!|\big) \wedge |\!|x|\!| \leqslant |\!|x + y|\!| \wedge |\!|x|\!| \leqslant |\!|x + y|\!|.$$

Similarly, $|\!|y|\!| \leqslant |\!|x + y|\!|$; therefore, $|\!|x|\!| + |\!|y|\!| = |\!|x|\!| \vee |\!|y|\!| \leqslant |\!|x + y|\!|$. $\rhd$

(2) A *Boolean algebra of projections* in a vector space X is a set $\mathscr{P}$ of commuting idempotent linear operators on X in which the Boolean operations have the following form:

$$\pi \wedge \rho := \pi \circ \rho = \rho \circ \pi, \quad \pi \vee \rho = \pi + \rho - \pi \circ \rho,$$
$$\pi^* = I_X - \pi \quad (\pi, \rho \in \mathscr{P}),$$

and the zero and identity operators in X serve as the top and the bottom elements of the Boolean algebra $\mathscr{P}$.

If X is a normed space then we will assume additionally that $\mathscr{P}$ consists of contractive projections and speak of a Boolean algebra of projections in a normed space X. Suppose that $\mathscr{P}$ is isomorphic to a Boolean algebra $\mathbb{B}$. In this event we will identify the Boolean algebras $\mathscr{P}$ and $\mathbb{B}$, writing $\mathbb{B} \subset L(X)$.

(3) Let $\mathscr{B}$ stand for the set of all metric bands ordered by inclusion. It is not difficult to check that if every band of the vector lattice Λ contains the norm of some nonzero element, then $\mathscr{B}$ is a complete Boolean algebra with the mapping $K \mapsto K^{\perp\!\!\!\perp}$ $(K \in \mathscr{P})$ as Boolean complementation; see [228, 2.1.2]. Decomposability of X implies that $X = K \oplus K^{\perp\!\!\!\perp}$ for all $K \in \mathscr{P}$, so that $\mathscr{B}$ defines an isomorphic Boolean algebra of projections on X.

5.8.3. *Suppose that X is a d-decomposable lattice normed space, Λ is a vector lattice with the projection property, and $\Lambda := \lvert X\rvert^{\perp\perp}$. Then there exists a complete Boolean algebra $\mathscr{P}$ of projections in X and an isomorphism h from $\mathbb{P}(\Lambda)$ onto $\mathscr{P}$ such that*

$$b\lvert x\rvert = \lvert h(b)x\rvert \quad \big(b \in \mathbb{P}(\Lambda),\ x \in X\big).$$

Moreover, if X is a vector lattice and $\lvert\cdot\rvert$ is monotone and decomposable, then $\mathscr{P}$ is a complete subalgebra of the Boolean algebra $\mathbb{P}(X)$.

$\vartriangleleft$ Given $L \in \mathbb{B}(\Lambda)$, we let by definition $h(L) := \big\{x \in X : \lvert x\rvert \in L\big\}$. Clearly, the mapping $h : L \mapsto h(L)$ from $\mathbb{B}(\Lambda)$ to $\mathscr{B}$ preserves the intersection of every nonempty family of bands. Therefore, h preserves infima, since in the algebras under consideration they coincide with intersections. Moreover, $h\big(\{0\}\big) = \{0\}$ and $h\big(\Lambda\big) = X$. Observe that $h(L^{\perp}) = h(L)^{\perp\!\!\!\perp}$ for all $L \in \mathbb{B}(\Lambda)$. The inclusion $h(L^{\perp}) \subset h(L)^{\perp\!\!\!\perp}$ is obvious. If $0 \neq x \in h(L)^{\perp\!\!\!\perp}$ then $\lvert x\rvert$ is disjoint from all the elements of the form $\lvert y\rvert$ in L. At the same time, $x \notin h(L^{\perp})$ implies that $0 < e \leqslant \lvert x\rvert$ for some $e \in L_{+}$. Therefore, in the band $\{e\}^{\perp\perp}$ there are no elements of the form $\lvert y\rvert$, which contradicts our assumption $\Lambda := \lvert X\rvert^{\perp\perp}$. It follows from the d-decomposability assumption that X is the direct sum of K and $K^{\perp\!\!\!\perp}$ for every metric band $K \in \mathscr{B}$. Thus, to each $K \in \mathscr{B}$ there corresponds the projection π_K in X along $K^{\perp\!\!\!\perp}$. Assign $\mathscr{P} := \{\pi_K : K \in \mathscr{B}\}$. It is clear that $\mathscr{P}$ is a complete Boolean algebra of projections isomorphic to $\mathscr{B}$. Denote by the same letter h the mapping sending a band projection $\rho \in \mathbb{P}(\Lambda)$ to $\pi_K \in \mathscr{P}$ with $K := h(\rho\Lambda)$. Then h is an isomorphism of the Boolean algebras $\mathbb{P}(\Lambda)$ and $\mathscr{P}$. By the definition of h, we have $h(\pi)x \in K := h(\pi\Lambda)$; i.e., $\lvert h(\pi)x\rvert \in \pi\Lambda$. Thus, $\pi^{\perp}\lvert h(\pi)x\rvert = 0$, or $\pi\lvert h(\pi)x\rvert = \lvert h(\pi)x\rvert$. Since $h(\pi)x$ and $h(\pi^{\perp})x$ are norm disjoint, by 5.8.2 (1) we have

$$\pi\lvert x\rvert \;=\; \pi\big(\lvert h(\pi)x\rvert + \lvert h(\pi^{\perp})x\rvert\big) \;=\; \pi\lvert h(\pi)x\rvert.$$

Consequently, $\pi\lvert x\rvert = \pi\lvert h(\pi)x\rvert = \lvert h(\pi)x\rvert$.

Assume now that X is a vector lattice. From the monotonicity of the vector norm it is easily seen that $x \perp\!\!\!\perp y$ implies $x \perp y$ for all $x, y \in Y$, so that $h(L) \perp h(L^\perp)$ for every $L \in \mathbb{B}(\Lambda)$. Thus we have $h(L^\perp) \subset h(L)^\perp$. To prove the converse inclusion, assume that $x \perp h(L)$ and $x \notin h(L^\perp)$. Then $|x| \notin L^\perp$ and we can choose $0 < e \in L$ with $e \leqslant |x|$. According to the decomposability of X there exist u, $v \in X$ such that $x = u+v$, $|u| = e$, and $|v| = |x|-e$. Since $u \in h(L)$ by definition of h, we have $x \perp u$ and so $|x| \leqslant |v|$. It follows that $|x| \leqslant |v| = |x|-e$ and we get a contradiction $0 < e \leqslant 0$. Thus, we have proved that $h(L^\perp) = h(L)^\perp$. Replacing in this identity L by $L^\perp$ yields $h(L) = h(L^\perp)^\perp$. Therefore, $h(L) \in \mathbb{B}(X)$ and $\mathscr{B} \subset \mathbb{B}(X)$. By the above we get $h(L)^{\perp\!\!\perp} = h(L^\perp) = h(L)^\perp$, so that Boolean complement in $\mathscr{B}$ is induced from $\mathbb{B}(X)$. Since in both algebras $\mathbb{B}(X)$ and $\mathscr{B}$ infima coincide with set-theoretic intersections, we conclude that $\mathscr{B}$ is a complete subalgebra of $\mathscr{B}$. $\triangleright$

5.8.4. Take some $\lambda \in \Lambda_+$. A sequence (x_n) in X is said to be λ-*uniformly convergent* to $x \in X$ (respectively, λ-*uniformly Cauchy*) if for each $0 < \varepsilon \in \mathbb{R}$ there exists $n(\varepsilon) \in \mathbb{N}$ such that $|x - x_n| \leqslant \varepsilon\lambda$ for all $n(\varepsilon) \leqslant n \in \mathbb{N}$ (respectively, $|x_n - x_m| \leqslant \varepsilon\lambda$ for all $n(\varepsilon) \leqslant n, m \in \mathbb{N}$). A sequence (x_n) in X is said to be Λ-*uniformly convergent* to $x \in X$ (respectively, Λ-uniformly Cauchy) if there exists $\lambda \in \Lambda_+$ such that (x_n) converges λ-uniformly to $x \in X$ (respectively, is λ-uniformly Cauchy). Say that X is Λ-*uniformly complete* whenever every Λ-uniformly Cauchy sequence is uniformly convergent.

A subset $A \subset X$ is called *norm order bounded* if the set $\{|x| : x \in A\}$ is order bounded in Λ. A lattice normed space X over Λ is called *laterally complete* whenever, given a partition of unity (b_ξ) in $\mathbb{P}(\Lambda)$ and a norm order bounded family (x_ξ) in X there exists $x \in X$ such that $b_\xi x = b_\xi x_\xi$ for all $\xi \in \Xi$. A lattice normed space X over a Dedekind complete vector lattice Λ is said to be a *Banach–Kantorovich space* if X is decomposable, Λ-uniformly complete, and laterally complete.

5.8.5. *Let X be a decomposable uniformly Λ-complete lattice normed space over a vector lattice Λ with $\Lambda = |X|^{\perp\perp}$ and $\mathscr{P}$ is as in* 5.8.3. *Then X admits the structure of a faithful unital module over $\mathscr{Z}(\Lambda)$ such that the following hold:*

(1) *The natural representation of $\mathscr{Z}(\Lambda)$ in X defines an isomorphism of $\mathbb{P}(\Lambda)$ and $\mathscr{P}$ from* 5.8.3.

(2) *$|ax| = |a||x|$ for all $a \in \mathscr{Z}(\Lambda)$ and $x \in X$.*

If, in addition, X is a vector lattice with monotone norm, then

(3) *$\mathscr{P}$ is a complete subalgebra of the Boolean algebra $\mathbb{P}(X)$.*

(4) *X is an f-module over $\mathscr{Z}(\Lambda)$.*

$\triangleleft$ Let $a \in A := \mathscr{Z}(\Lambda)$ be a *simple element*; i.e., $a = \sum_{k=1}^{n} \lambda_k \pi_k$ where $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ and $\pi_1, \dots, \pi_n$ is a finite partition of unity in $\mathbb{P}(\Lambda)$. Then we put $ax := \sum_{k=1}^{n} \lambda_k h(\pi_k)x$. Considering 5.8.2 (1) and 5.8.3, we have

$$|ax| = \left|\sum_{k=1}^{n} \lambda_k h(\pi_k)x\right| = \sum_{k=1}^{n} |\lambda_k|\pi_k|x| = a|x|.$$

By the Freudenthal Spectral Theorem an arbitrary $a \in A$ is the uniform limit of an increasing sequence of simple elements $(a_n) \subset A$. The sequence $(a_n x) \subset X$ is uniformly Λ-fundamental, since $|a_n x - a_m x| = |a_n - a_m||x|$. Therefore, we can put $ax := \Lambda\text{-}\lim a_n x$. Moreover,

$$|ax| = |\Lambda\text{-}\lim a_n x| = r\text{-lim}\, |a_n||x| = a|x|.$$

The remaining part of the proof is straightforward. $\triangleright$

5.8.6. Let Λ be a vector lattice and let X and Y be lattice normed spaces over Λ. A linear operator T is said to be *order norm bounded* if there exists an orthomorphism $S \in \mathrm{Orth}(\Lambda)$ such that $|T(x)| \leqslant S(|x|)$ for all $x \in X$. Put $A := \mathscr{Z}(\Lambda)$.

If X and Y are decomposable and uniformly Λ-complete, then an order norm bounded linear operator $T : X \to Y$ is A-linear with respect to the module structures on X and Y defined as in 5.8.5. *In particular, $T\pi = \pi T$ for all $\pi \in \mathbb{P}(\Lambda)$.*

$\triangleleft$ If T is order norm bounded then, in view of 5.8.3, for all $x \in X$ and $\pi \in \mathbb{P}(\Lambda)$ we have

$$|\pi T(\pi^{\perp} x)| = \pi|T(\pi^{\perp} x)| \leqslant \pi S|(\pi^{\perp} x)| = \pi\pi^{\perp} S(|x|) = 0.$$

This implies $\pi T \pi^{\perp} = 0$ or $\pi T = \pi T \pi^{\perp}$. Replacing $\pi^{\perp}$ by π in the latter identity yields $T\pi = \pi T \pi^{\perp}$, so that $T\pi = \pi T$. Further, we argue as in 5.8.5 using the Freudenthal Spectral Theorem. $\triangleright$

5.8.7. Assume now that Λ is a Banach lattice and X is a lattice normed space over Λ. Then, the Λ-valued norm $|\cdot|$ enables us to define some *mixed norm* on X by putting

$$|||x||| := \| |x| \| \quad (x \in X).$$

In this situation, the normed space $(X, |||\cdot|||)$ is called a *space with mixed norm*. In view of the inequality $||x| - |y|| \leqslant |x - y|$ and monotonicity of the norm on Λ we have

$$\| |x| - |y| \| \leqslant |||x - y||| \quad (x, y \in X),$$

so that $\lvert\cdot\rvert$ is a norm continuous mapping from $(X, |||\cdot|||)$ into Λ.

A *Banach space with mixed norm over* Λ is a pair $(X, \lvert\cdot\rvert)$ such that $\lvert\cdot\rvert$ is a vector norm on X with values in a Banach lattice Λ and X is a Λ-uniformly complete lattice normed. The following proposition justifies this definition (see Kusraev [228, Proposition 7.1.2]).

(1) *Let Λ be a Banach lattice and let $\lvert\cdot\rvert$ be a Λ-valued norm on X. Then $(X, |||\cdot|||)$ is a Banach space if and only if the lattice normed space $(X, \lvert\cdot\rvert)$ is Λ-uniformly complete.*

Combining (1) and 5.8.5 we obtain the following.

(2) *Let Λ be a Dedekind complete vector lattice and let X be a decomposable Banach space with mixed norm over Λ. Then X admits a structure of a faithful unital module over $A := \mathscr{Z}(\Lambda)$ such that $|||ax||| \leqslant \|a\|_\infty |||x|||$ for all $a \in A$ and $x \in X$. In particular, X is a Banach space with the Boolean algebra of projections $\mathbb{P}(\Lambda)$.*

5.8.8. Let X be a Banach space and let $\mathbb{B}$ be a complete Boolean algebra of projections on X. Given a partition of unity $(b_\xi)_{\xi\in\Xi}$ in $\mathbb{B}$ and a family $(x_\xi)_{\xi\in\Xi}$, we refer to $x \in X$ satisfying the condition $b_\xi x = b_\xi x_\xi$ for all $\xi \in \Xi$ as a *mixture* of (x_ξ) by (b_ξ) and use the notation $x := \operatorname{mix}_{\xi\in\Xi}(b_\xi x_\xi)$. The mixture is unique if $(\forall\, \xi \in \Xi) b_\xi x = 0$ implies $x = 0$.

A Banach space X is said to be $\mathbb{B}$-*cyclic* or mix-*complete* whenever, given a partition of unity (b_ξ) in $\mathbb{B}$ and a norm bounded family (x_ξ) in X, we can find the unique element $x \in X$ such that $x = \operatorname{mix}_{\xi\in\Xi} b_\xi x_\xi$ and $\|x\| = \sup\{\|b_\xi x_\xi\| : \xi \in \Xi\}$. Clearly, this definition agrees with that in 5.7.13.

5.8.9. Let X and Y be Banach spaces with $\mathbb{B} \subset \mathscr{L}(X)$ and $\mathbb{B} \subset \mathscr{L}(Y)$. An operator $T : X \to Y$ is called $\mathbb{B}$-*linear*, if it is linear and commutes with all projections from $\mathbb{B}$. Denote the set of all bounded $\mathbb{B}$-linear operators from X into Y by $\mathscr{L}_{\mathbb{B}}(X, Y)$. Clearly $\mathscr{L}_{\mathbb{B}}(X, Y)$ is a $\mathbb{B}$-cyclic Banach space whenever so is Y. A one-to-one $\mathbb{B}$-linear operator is called a $\mathbb{B}$-*isomorphism* and an isometric $\mathbb{B}$-*isomorphism* is called a $\mathbb{B}$-*isometry.* In the case of Banach lattices, a $\mathbb{B}$-isometric lattice homomorphism is referred to as *lattice* $\mathbb{B}$-*isometry.* The space $X^\# := \mathscr{L}_{\mathbb{B}}(X, \Lambda)$ is called the $\mathbb{B}$-*dual* of X whenever $\Lambda = \Lambda(\mathbb{B})$; see 5.7.12.

5.8.10. Let $\Lambda = \mathscr{R}{\Downarrow}$ be the bounded part of the universally complete vector lattice $\mathscr{R}{\downarrow}$; i.e., Λ is the order-dense ideal in $\mathscr{R}{\downarrow}$ generated by the unity $\mathbb{1} := 1^\wedge \in \mathscr{R}{\downarrow}$. Take a Banach space $\mathscr{X}$ within $\mathbb{V}^{(\mathbb{B})}$ and put $\mathscr{X}{\Downarrow} := \{x \in \mathscr{X}{\downarrow} : \lvert x\rvert \in \Lambda\}$. Then $\mathscr{X}{\Downarrow}$ is a Banach–Kantorovich space called the *bounded descent* of $\mathscr{X}$. Since Λ is an order complete AM-

space with unity, $\mathscr{X}{\Downarrow}$ is a Banach space with mixed norm over Λ, hence, a $\mathbb{B}$-cyclic Banach space (cp. Kusraev [228, 7.3.3]).

5.8.11. Theorem. *For a Banach space X and a complete Boolean algebra $\mathbb{B}$ the following are equivalent:*

(1) *X is $\mathbb{B}$-cyclic with respect to a complete Boolean algebra $\mathbb{B}$ of projections on X.*

(2) *X is linearly isometric to a Banach space with mixed norm defined by a Banach–Kantorovich space over the unital Dedekind complete AM-space $\Lambda = \Lambda(\mathbb{B})$.*

(3) *X is linearly isometric to the restricted descent $\mathscr{X}{\Downarrow}$ of a Banach space $\mathscr{X}$ within $\mathbb{V}^{(\mathbb{B})}$.*

$\lhd$ The proof can be extracted from Kusraev [228, Theorems 7.3.2, 7.3.3 (1), 8.3.1, and 8.3.2]. $\rhd$

5.8.12. Theorem. *The bounded descent of the Banach space $\mathscr{L}(\mathscr{X},\mathscr{Y})$ and the $\mathbb{B}$-cyclic Banach space $\mathscr{L}_{\mathbb{B}}(X,Y)$ are isometrically $\mathbb{B}$-isomorphic. Some isomorphism is given by sending a bounded $\mathbb{B}$-linear operator $T : X \to Y$ to the $\mathscr{T} := T{\uparrow}$ defined by the relations $[\![\mathscr{T} : \mathscr{X} \to \mathscr{Y}]\!] = \mathbb{1}$ and $[\![\mathscr{T}(x) = T(x)]\!] = \mathbb{1}$ $(x \in X)$. In particular, $\mathscr{X}^*{\Downarrow}$ and $X^\#$ are isometrically $\mathbb{B}$-isomorphic.*

$\lhd$ See Kusraev [228, Theorem 8.3.6]. $\rhd$

5.9. Boolean Valued Banach Lattices

In this section we discuss briefly the question: What is the Boolean valued interpretation of Banach lattice theory? We restrict discussion only to some basic facts needed in the sequel. Some of the proofs can be extracted from Gordon [133, 134] but we will give independent proofs for the sake of completeness.

5.9.1. Theorem. *The bounded descent of a Banach lattice within $\mathbb{V}^{(\mathbb{B})}$ is a $\mathbb{B}$-cyclic Banach lattice. Conversely, if X is a $\mathbb{B}$-cyclic Banach lattice, then in $\mathbb{V}^{(\mathbb{B})}$ there exists a Banach lattice $\mathscr{X}$ that is unique up to the isometric isomorphism and whose bounded descent is lattice $\mathbb{B}$-isometric to X. Moreover, the mapping $\pi \mapsto \pi{\Downarrow}$ is an isomorphism of the Boolean algebras $\mathbb{M}(\mathscr{X}){\downarrow}$ and $\mathbb{M}(X)$; in symbols, $\mathbb{M}(\mathscr{X}){\downarrow} \simeq \mathbb{M}(\mathscr{X}{\Downarrow})$.*

$\lhd$ The Banach part of the claim follows from Theorem 5.8.11. Assume that X is a $\mathbb{B}$-cyclic Banach lattice and put $\mathscr{X}_+ := X{\uparrow}$. Given an extensional mapping f, we have $f(A)\uparrow = f\uparrow(A\uparrow)$ where $A \subset \mathrm{dom}(f)$ (cp. 1.6.3 and 1.6.5). Applying this successively to the addition $f : (x, y) \mapsto x + y$ $(x, y \in X)$ with $A := X_+ \times X_+$ and to the Λ-multiplication $f : (\lambda, x) \mapsto \lambda x$ $(\lambda \in \Lambda, x \in X)$ with $A := \Lambda_+ \times X_+$ we find $[\![\mathscr{X}_+ + \mathscr{X}_+ = \mathscr{X}]\!] = \mathbb{1}$ and $[\![\mathscr{R}_+ \cdot \mathscr{X}_+ = \mathscr{X}_+]\!] = \mathbb{1}$; i.e., $[\![\mathscr{X}_+$ is a convex cone$]\!] = \mathbb{1}$. Moreover, $[\![\mathscr{X}_+$ is pointed$]\!] = \mathbb{1}$, since $[\![\pm x \in \mathscr{X}_+$ and $\|x\| \leqslant 1]\!] = \mathbb{1}$ imply $\pm x \in \mathscr{X}_+{\downarrow} \cap X \subset X_+$. Define the order on $\mathscr{X}$ as $[\![(\forall\, x, y \in \mathscr{X})(x \leqslant y \leftrightarrow y - x \in \mathscr{X}_+)]\!] = \mathbb{1}$. By transfer $(\mathscr{X}, \mathscr{X}_+)$ is an ordered Banach space within $\mathbb{V}^{(\mathbb{B})}$. Moreover, for all $x, y \in X$ the relations $x \leqslant y$ and $[\![x \leqslant y]\!] = \mathbb{1}$ are equivalent.

Consider the sentence $\sigma \equiv (\forall\, a \in \{0, 1\})(\forall\, x, y \in \mathscr{X})\,(ax \leqslant ay \leftrightarrow (a \neq 1 \vee x \leqslant y))$ which is a very simple ZF-theorem. By transfer $[\![\sigma]\!] = \mathbb{1}$. Calculating the Boolean truth values for quantifiers we find that this is equivalent to saying that $[\![ax \leqslant ay]\!] = [\![a = 1]\!]^* \vee [\![x \leqslant y]\!]$ for all $a \in \{0, 1\}{\downarrow}$ and $x, y \in \mathscr{X}{\downarrow}$. Using the Boolean isomorphism $\chi : \mathbb{B} \to \{0, 1\}{\downarrow}$, we can replace $a \in \{0, 1\}{\downarrow}$ by $\chi(b)$ for $b \in \mathbb{B}$ and write $b^* \vee [\![x \leqslant y]\!] = [\![\chi(b)x \leqslant \chi(b)y]\!]$. Now it is easy to see that

$$b \leqslant [\![x \leqslant y]\!] \iff \chi(b)x \leqslant \chi(b)y \quad (b \in \mathbb{B},\ x, y \in \mathscr{X}{\downarrow}).$$

The last relation allows us to treat the interplay between X and $\mathscr{X}$. As an example we prove that $\mathscr{X}$ is a vector lattice; i.e., the sentence $(\forall\, x \in \mathscr{X})(\exists\, y \in \mathscr{X})y = \sup\{x, -x\}$ is true within $\mathbb{V}^{(\mathbb{B})}$. Using the rules for calculating Boolean truth values (see 1.6.2) and the maximum principle we have to prove that for every $x \in X$ there exists $y \in X$ for which $[\![y = \sup\{x, -x\}]\!] = \mathbb{1}$. Put $y = |x|$ and note that $[\![\pm x \leqslant y]\!] = \mathbb{1}$. Thus, it remains to check that $[\![(\forall\, u \in \mathscr{X})(\pm x \leqslant u \to y \leqslant u)]\!] = \mathbb{1}$. Again by and 1.2.3 and 1.6.2 it is equivalent to the relation $[\![\pm x \leqslant u]\!] \leqslant [\![y \leqslant u]\!]$ $(u \in X)$. If $b = [\![\pm x \leqslant u]\!]$ then $\pm\chi(b)x \leqslant \chi(b)u$ and $\chi(b)y \leqslant \chi(b)u$. It follows that $b \leqslant [\![y \leqslant u]\!]$.

The Λ-valued norm $\lvert\!\cdot\!\rvert$ of X is the descent of the norm $\|\cdot\|_{\mathscr{X}}$ of $\mathscr{X}$ and $\|x\|_X = \big\|\lvert x\rvert\big\|_\infty$ $(x \in X)$. Therefore, $\|\cdot\|_{\mathscr{X}}$ is a lattice norm if and only if $|x| \leqslant |y|$ implies $\lvert x\rvert \leqslant \lvert y\rvert$ for all $x, y \in X$. Let $\|\cdot\|_X$ be a lattice norm. If $[\![|x| \leqslant |y|]\!] = \mathbb{1}$ for some $x, y \in X$ then $|x| \leqslant |y|$. Now, if $\lvert x\rvert \leqslant \lvert y\rvert$ were false, there would be $\pi \in \mathbb{B}$ and $0 < \varepsilon \in \mathbb{R}$ with $\pi\lvert x\rvert > \pi(\lvert y\rvert + \varepsilon\mathbb{1})$. Therefore, $\|\pi x\|_X = \big\|\pi\lvert x\rvert\big\|_\infty \geqslant \big\|\pi\lvert y\rvert\big\|_\infty + \varepsilon > \|\pi y\|_X$, which contradicts the hypothesis. Thus, $(\mathscr{X}, \mathscr{X}_+)$ is a Banach lattice.

Assume that π is an M-projection in $\mathscr{X}$ and Π is the restriction of $\pi{\downarrow}$ to X. Then $[\![\pi \circ \pi = \pi]\!] = \mathbb{1}$, $[\![0 \leqslant \pi x \leqslant x\,(x \in \mathscr{X}_+)]\!] = \mathbb{1}$, and $\|x\| = \max\{\|\pi x\|, \|\pi^\perp x\|\}\,(x \in \mathscr{X})$. By 1.5.5 (1) and 1.5.6 $\pi{\downarrow} = (\pi \circ \pi){\downarrow} = \pi{\downarrow} \circ \pi{\downarrow}$ and hence $\Pi = \Pi \circ \Pi$. Since $[\![\pi x = \Pi x]\!] = \mathbb{1}$ $(x \in X)$, we have $0 \leqslant \Pi x \leqslant x$ for all $x \in X$. Finally, the relations $[\![\|x\| = \max\{\|\pi x\|, \|\pi^{\perp} x\|\}\ (x \in \mathscr{X})]\!] = \mathbb{1}$ and $|x| = \max\{|\Pi x|, |\Pi^{\perp} x|\}\ (x \in X)$ are equivalent, whence we deduce $\|x\| = \big\||\Pi x| \vee |\Pi^{\perp} x|\big\|_{\infty} = \max\{\|\Pi x\|, \|\Pi^{\perp} x\|\}$. Thus, Π is an M-projection in X; i.e., $\Pi \in \mathbb{M}(X)$. Conclusions in the reverse direction are similar. The remaining details are obvious. $\rhd$

5.9.2. The element $\mathscr{X} \in \mathbb{V}^{(\mathbb{B})}$ from Theorem 5.9.1 is said to be the *Boolean valued representation of* X. Let $\mathscr{X}$ and $\mathscr{Y}$ be the Boolean valued representations of $\mathbb{B}$-cyclic Banach lattices X and Y, respectively. Let $\mathscr{L}(\mathscr{X}, \mathscr{Y})$ and $\mathscr{L}^{r}(\mathscr{X}, \mathscr{Y})$ denote the elements in $\mathbb{V}^{(\mathbb{B})}$ which represent the spaces of all bounded linear operators and regular operators from $\mathscr{X}$ into $\mathscr{Y}$.

5.9.3. Corollary. *Let $\mathscr{X}$ be the Boolean valued representation of a $\mathbb{B}$-cyclic Banach lattice X. Then $\mathbb{B} = \mathbb{M}(X)$ if and only if $[\![\mathbb{M}(\mathscr{X}) = \{0, I_{\mathscr{X}}\}]\!] = \mathbb{1}$.*

$\lhd$ This is immediate from Theorem 5.9.1, since $\mathbb{B}$ is the descent of the two-element Boolean algebra $\{0, I_{\mathscr{X}}\}$ (see 1.8.1). $\rhd$

5.9.4. Corollary. *For a Banach lattice X and a complete Boolean algebra $\mathbb{B}$ the following are equivalent:*

(1) *X is lattice isometric to the bounded descent of some Banach lattice $\mathscr{X}$ within $\mathbb{V}^{(\mathbb{B})}$.*

(2) *X is lattice isometric to a Banach lattice with mixed norm defined by a Banach–Kantorovich lattice over a unital Dedekind complete AM-space $\Lambda = \Lambda(\mathbb{B})$.*

(3) *X is $\mathbb{B}$-cyclic relative to the complete Boolean algebra of M-projections $\mathbb{B}$.*

$\lhd$ See Theorems 5.8.11 and 5.9.1. $\rhd$

5.9.5. Let X be a Banach space and $\mathbb{B} \subset \mathscr{L}(X)$. A net $(x_{\alpha})_{\alpha \in \mathrm{A}}$ in X is said to be $\mathbb{B}$-*convergent* to $x \in X$ if for every $0 < \varepsilon \in \mathbb{R}$ there exists a partition of unity $(\pi_{\alpha})_{\alpha \in \mathrm{A}}$ in $\mathbb{B}$ with $\|\pi_{\alpha}(x - x_{\beta})\| \leqslant \varepsilon$ for all $\alpha, \beta \in \mathrm{A}$, $\beta \geqslant \alpha$. In this event x is called the $\mathbb{B}$-*limit* of (x_{α}). Let $\mathbb{B}\langle X_0\rangle$ stand for all $x \in X$ representable as $x := \mathrm{mix}_{\xi \in \Xi}(b_{\xi} x_{\xi})$ with an arbitrary family (x_{ξ}) in X_0 and a partition of unity (b_{ξ}) in $\mathbb{B}$. A subset $X_0 \subset X$ is $\mathbb{B}$-*dense* in X if every $x \in X$ is the $\mathbb{B}$-limit of some family in X_0. Equivalently, X_0 is $\mathbb{B}$-dense in X if $\mathbb{B}\langle X_0\rangle$ is norm dense in X.

Now take a $\mathbb{B}$-cyclic Banach lattice X. A decreasing net $(x_{\alpha})_{\alpha \in \mathrm{A}}$ in X is $\mathbb{B}$-*convergent to zero* if for every $0 < \varepsilon \in \mathbb{R}$ there exists a partition of unity $(\pi_\alpha)_{\alpha\in\mathrm{A}}$ in $\mathbb{B}$ such that $\|\pi_\alpha x_\alpha\| \leqslant \varepsilon$ for all $\alpha \in \mathrm{A}$. The norm on X is said to be $\mathbb{B}$-*continuous* if every decreasing net $(x_\alpha)_{\alpha\in\mathrm{A}}$ in X with $\inf_\alpha x_\alpha = 0$ is $\mathbb{B}$-convergent to zero. If $\mathrm{A} = \mathbb{N}$ in this definition, we say that the norm on X is σ-$\mathbb{B}$-continuous. Write $X_n^\#$ for the space of all norm bounded order continuous $\mathbb{B}$-linear operators from X to Λ.

5.9.6. Theorem. *Suppose that X is a $\mathbb{B}$-cyclic Banach lattice and $\mathscr{X} \in \mathbb{V}^{(\mathbb{B})}$ is its Boolean valued representation. Then the following hold:*

(1) *X is Dedekind complete $\Longleftrightarrow$ $[\![\mathscr{X}$ is Dedekind complete$]\!] = \mathbb{1}$.*

(2) *X is Fatou (Levi) $\Longleftrightarrow$ $[\![\mathscr{X}$ is Fatou (Levi) $]\!] = \mathbb{1}$.*

(3) *X is order $\mathbb{B}$-continuous $\Longleftrightarrow$ $[\![\mathscr{X}$ is order continuous$]\!] = \mathbb{1}$.*

(4) *X is order $\mathbb{B}$-continuous and Levi $\Longleftrightarrow$ $[\![\,\mathscr{X}$ is a KB-space$]\!] = \mathbb{1}$.*

(5) *$S \in X_n^\#$ $\Longleftrightarrow$ $[\![\,\sigma := S{\uparrow} \in \mathscr{X}_n'\,]\!] = \mathbb{1}$.*

$\lhd$ (1): Just as in the proof of Theorem 2.2.4 we can show that for $A \subset X_+$ there exists $a = \sup(A)$ if and only if $[\![$there exists $\sup(A{\uparrow})]\!] = \mathbb{1}$ and in this case $[\![a = \sup(A{\uparrow})]\!] = \mathbb{1}$. Thus, the Dedekind completeness of $\mathscr{X}$ within $\mathbb{V}^{(\mathbb{B})}$ implies that X is Dedekind complete. Conversely, suppose that X is Dedekind complete and take a set $\mathscr{A} \subset \mathscr{X}_+$ bounded above by $u \in \mathscr{X}$. There is no loss of generality in assuming that $[\![\|u\| \leqslant 1]\!] = \mathbb{1}$. Then $A := \mathscr{A}{\downarrow}$ lies in X and, taking the cancelation rule $\mathscr{A}{\downarrow}{\uparrow} = \mathscr{A}$ (see 1.6.6) into account, we get the following: there exists $a = \sup(\mathscr{A}{\downarrow})$ if and only if $[\![$there exists $\sup(\mathscr{A})]\!] = \mathbb{1}$ and in this case $[\![a = \sup(\mathscr{A})]\!] = \mathbb{1}$.

(2): We may assume without loss of generality that the upward directed sets in the definitions of Fatou norm and Levi norm are taken from the unit balls $B(X)$ and $B(\mathscr{X})$. Moreover, if $A \subset X$ is upward directed then $[\![A{\uparrow}$ is upward directed$]\!] = \mathbb{1}$ and $[\![\mathscr{A} \subset \mathscr{X}$ is upward directed$]\!] = \mathbb{1}$ implies that $\mathscr{A}{\downarrow}$ is upward directed. Finally, observe that $B(\mathscr{X}){\downarrow} = \big\{x \in \mathscr{X}{\downarrow} : |x| \leqslant \mathbb{1}\big\} = B(X)$. Let $\mathscr{X}$ have a Levi norm and take an upward directed set $A \subset B(X)$. It follows that $\big\{|a| : a \in A\big\} \subset [-\mathbb{1}, \mathbb{1}]$ and thus $[\![\{\|a\| : a \in A{\uparrow}\} \subset [-1, 1]\,]\!] = \mathbb{1}$; i.e., $[\![A{\uparrow} \subset B(\mathscr{X})]\!] = \mathbb{1}$. By hypothesis $a = \sup(A{\uparrow})$ exists in $\mathscr{X}$, whence $a = \sup(A)$. The argument for the converse is similar. To ensure the claim concerning the Fatou norm it suffices to observe that $|b| = \sup\big\{|a| : a \in A\big\}$ in Λ if and only if $\|b\| = \big\||b|\big\|_\infty = \sup\big\{\big\||a|\big\|_\infty : a \in A\big\}$, since the AM-space Λ has a Levi norm.

(3): Using the above remarks in (2) it is easy to see that $[\![\,\mathscr{X}$ has an order continuous norm$\,]\!] = \mathbb{1}$ if and only if for every downward directed set

$A \subset X_+$ with $\inf(A) = 0$ we have $\inf\{|a| : a \in A\} = 0$ in Λ. By Theorem 2.6.1 the latter property amounts to the following: for every $\varepsilon > 0$ there exists a partition of unity $(\pi_a)_{a \in A}$ in $\mathbb{B}$ such that $|\pi_a a| = \pi_a |a| < \varepsilon \mathbb{1}$ for all $a \in A$. Thus, we arrive at the desired result, since the relations $|\pi_a a| < \varepsilon \mathbb{1}$ and $\|\pi_a a\| < \varepsilon$ are equivalent.

(4): This is immediate from (2) and (3).

(5): By Theorem 5.8.12 $S \in X^{\#}$ if and only if $[\![\sigma := S{\uparrow} \in \mathscr{X}^*]\!] = \mathbb{1}$. Moreover, S and σ are positive or not simultaneously. Thus, we can confine demonstration to the case of S positive. Observe also that if $[\![\mathscr{A} \subset \mathscr{X}_+]\!] = \mathbb{1}$ and $A = \mathscr{A}{\downarrow}$ then $S(A) = \sigma(\mathscr{A}){\downarrow}$ by 1.5.5 (1) and 1.5.6 and if $A \subset X_+$ and $\mathscr{A} = A{\uparrow}$ then $[\![\sigma(A) = S(A){\uparrow}]\!] = \mathbb{1}$ by 1.6.3 and 1.6.5. Use the same argument as in (1), but with infimum instead of supremum. We see that if $\inf(A) = 0$ and S is order continuous then $[\![\inf \sigma(\mathscr{A}) = 0]\!] = \mathbb{1}$ and if $[\![\inf(\mathscr{A}) = 0$ and σ is order continuous$]\!] = \mathbb{1}$ then $\inf S(A) = 0$. $\triangleright$

5.9.7. Corollary. *For every $\mathbb{B}$-cyclic Banach lattice X the following are equivalent:*

(1) *The norm on X is $\mathbb{B}$-continuous.*

(2) *X is order σ-complete and the norm on X is σ-$\mathbb{B}$-continuous.*

(3) *Every monotone order bounded sequence in X is $\mathbb{B}$-convergent.*

(4) *Every disjoint order bounded sequence in X_+ is $\mathbb{B}$-convergent to zero.*

(5) *Every norm closed $\mathbb{B}$-complete order ideal of X is a band.*

(6) *The null ideal $\mathscr{N}(x^{\#})$ is a band for every $x^{\#} \in X^{\#}$.*

(7) *Every norm continuous $\mathbb{B}$-linear operator from X to Λ is order continuous; i.e., $X^{\#} = X^{\#}_n$.*

(8) *The natural embedding of X into $X^{\#\#}$ sends X onto an order ideal of $X^{\#\#}$.*

$\triangleleft$ This is proved by interpreting Theorem 5.7.3 within $\mathbb{V}^{(\mathbb{B})}$ and making use of Theorem 5.9.6. For example, the equivalence (1) $\Longleftrightarrow$ (8) of Theorem 5.7.3 together with Theorem 5.9.6 (3) implies that X is $\mathbb{B}$-continuous if and only if $[\![\mathscr{X}' = \mathscr{X}'_n]\!] = \mathbb{1}$. To ensure that the latter is equivalent to 5.9.7 (8), it is sufficient to observe that the $\mathbb{B}$-cyclic Banach lattices $X^{\#}_n$ and $\mathscr{X}'_n{\Downarrow}$ are lattice $\mathbb{B}$-isometric.

The natural embedding $x \mapsto \hat{x}$ of X into $X^{\#\#} := (X^{\#})^{\#}$ is defined by putting $\hat{x}(T) = Tx$ for all $T \in X^{\#}$. The equivalence (1) $\Longleftrightarrow$ (7) of Theorem 5.7.3 together with Theorem 5.9.6 (3) shows that X is $\mathbb{B}$-continuous if and only if $[\![$the natural embedding sends $\mathscr{X}$ onto an order ideal of $\mathscr{X}''\,]\!] = \mathbb{1}$. This is equivalent to saying that the natural embedding sends X onto an order ideal of $X^{\#\#}$, since $X^{\#\#}$ and $\mathscr{X}''\!\Downarrow$ are lattice $\mathbb{B}$-isometric. $\triangleright$

5.9.8. Theorem. *For a $\mathbb{B}$-cyclic Banach lattice X the following hold:*

(1) *The natural embedding of X into $X^{\#\#}$ sends X onto a band of $X^{\#\#}$ if and only if X has a $\mathbb{B}$-continuous Levi norm.*

(2) The natural embedding of X into $X^{\#\#}$ sends X onto $X^{\#\#}$ if and only if X and $X^{\#}$ have both $\mathbb{B}$-continuous Levi norms.

$\triangleleft$ Interpret Theorem 5.7.4 in $\mathbb{V}^{(\mathbb{B})}$ making use of Theorem 5.9.6. $\triangleright$

A $\mathbb{B}$-cyclic Banach lattice X is said to be $\mathbb{B}$-*reflexive* if $X = X^{\#\#}$ (or, more precisely, the natural embedding sends X onto $X^{\#\#}$).

5.9.9. Corollary. *A $\mathbb{B}$-cyclic Banach lattice X is $\mathbb{B}$-reflexive if and only if X and $X^{\#}$ have order $\mathbb{B}$-continuous Levi norms.*

5.10. Injective Banach Lattices

In this section we present several analytical and geometric characterizations of injective Banach lattices.

5.10.1. A real Banach lattice X is said to be *injective* if, for every Banach lattice Z, every closed vector sublattice $Y \subset Z$, and every positive linear operator $T : Y \to X$ there exists a positive linear extension $\hat{T} : Z \to X$ with $\|T\| = \|\hat{T}\|$. This definition is illustrated by the commutative $(T = \hat{T} \circ \iota)$ diagram:

$$\begin{tikzcd}
 & X & \\
Y \arrow[ur, "T"] \arrow[rr, "\iota"'] & & Z \arrow[ul, dashed, "\hat{T}"']
\end{tikzcd}$$

5.10.2. We now state two elementary properties of injective Banach lattices which are immediate from the definition.

(1) *If X is an injective Banach lattice and a closed vector sublattice $X_0 \subset X$ is the range of a contractive positive projection P then X_0 is an injective Banach lattice.*

$\triangleleft$ We need only take $P\hat{T}$ in Definition 5.10.1 in case $\operatorname{im}(T) \subset X_0$. $\triangleright$

(2) *If (X_α) is a family of injective Banach lattices then their l^∞-product $(X, \|\cdot\|_\infty)$ is also an injective Banach lattice (where X consists of all families $x = (x_\alpha)$ with $x_\alpha \in X_\alpha$ and $\|x\|_\infty := \sup_\alpha \|x_\alpha\| < \infty$).*

$\lhd$ Let $P_\alpha : X \to X_\alpha$ stand for the natural projection $x = (x_\alpha) \mapsto x_\alpha$. Then P_α is a contractive positive projection as X is equipped with the product order and $P_\alpha T : Y \to X_\alpha$ admits a positive extension $\hat{T}_\alpha : Z \to X_\alpha$ with $\|P_\alpha T\| = \|\hat{T}_\alpha\|$. Define $\hat{T} : Z \to X$ as $\hat{T}x := (\hat{T}_\alpha z)$ and note that $\hat{T}$ is a positive extension of T and

$$\|\hat{T}\| = \sup_\alpha \sup_{\|z\|\leqslant 1} \|\hat{T}_\alpha z\| = \sup_{\|y\|\leqslant 1} \sup_\alpha \|P_\alpha T y\| = \|T\|. \rhd$$

Next, we consider two important examples.

5.10.3. Theorem. *A Dedekind complete AM-space with unit is an injective Banach lattice.*

$\lhd$ Let X be a Dedekind complete AM-space with unity $\mathbb{1}$, let $Y_0 \subset Y$ be a closed vector sublattice of a Banach lattice Y, and let $T_0 : Y_0 \to X$ be a positive linear operator. Define $p : Y \to X$ by putting $p(y) := \|T_0\|\|y^+\|\mathbb{1}$ $(y \in Y)$. Observe that p is a sublinear operator and

$$T_0(y) = T_0(y^+) - T_0(y^-) \leqslant T_0(y^+) \leqslant \|T_0\|\|y^+\|\mathbb{1} = p(y) \quad (y \in Y_0).$$

By the Hahn–Banach–Kantorovich Theorem there exists a linear extension $T : Y \to X$ of T_0 such that $Ty \leqslant p(y)$ for all $y \in Y$. Evidently T is positive, since $-T(y) = T(-y) \leqslant p(-y) = \|T_0\|\|(-y)^+\| = 0$ whenever $y \geqslant 0$ and $\|T\| \leqslant \|T_0\|$ because of $|Ty| \leqslant T(|y|) \leqslant p(|y|) = \|T_0\|\|y\|\mathbb{1}$ for all $y \in Y$. $\rhd$

5.10.4. Theorem. *Each AL-space is an injective Banach lattice.*

$\lhd$ Slightly different proofs can be found in Lotz [288, Proposition 3.2], Haydon [169, Proposition 2A], Meyer-Niberg [311, Theorem 3.2.5], and Schaefer [357, Theorem 4.2]. $\rhd$

5.10.5. *Each Banach lattice L is lattice isometric to a closed vector sublattice of an injective Banach lattice.*

$\lhd$ Given $\alpha \in L'$, put $I_\alpha := \{x \in L : \langle |x|, \alpha\rangle = 0\}$, and equip the quotient vector lattice L/I_α with the norm $\|\tilde{x}_\alpha\|_\alpha := \langle |x|, \alpha\rangle$ where $\tilde{x}_\alpha := x + I_\alpha$ is a coset of $x \in L$. This norm is additive on the positive cone, the completion X_α of $(L/I_\alpha, \|\cdot\|_\alpha)$ is an AL-space. The l^∞-product X of the family $\{X_\alpha : 0 \leqslant \alpha \in L', \|\alpha\| \leqslant 1\}$ is an injective Banach lattice by 5.10.2 (2). It remains to observe that the mapping $x \mapsto (\tilde{x}_\alpha)$ is a lattice isometry from L into X. $\triangleright$

5.10.6. Theorem. *For a Banach lattice X the following are equivalent:*

(1) *X is injective.*

(2) *If X is lattice isometrically embedded into a Banach lattice Y and T_0 is a positive linear operator from X to a Banach lattice Z then there exists a positive linear extension $T : Y \to Z$ with $\|T_0\| = \|T\|$.*

(3) *If X is lattice isometrically embedded into a Banach lattice Y then there exists a contractive positive projection from Y onto X.*

$\triangleleft$ (1) $\Longrightarrow$ (3) and (2) $\Longrightarrow$ (3): To ensure that (3) is a special case of both (1) and (2), we need only to take $Y := X$ and $Z := Y$ in (1), $Z := X$ in (2), and T_0 the identity operator in both cases.

(3) $\Longrightarrow$ (1): By 5.10.5 we can assume that X is a closed vector sublattice of an injective Banach lattice, say L. It follows that a positive linear operator T_0 from a closed vector sublattice Y of a Banach lattice Z to $X \subset L$ admits a positive linear extension $\bar{T} : Z \to L$ with $\|\bar{T}\| = \|T_0\|$. By (3) there exists a contractive positive projection P from L onto X. The operator $T := P \circ \bar{T} : Z \to X$ has the desired properties.

(3) $\Longrightarrow$ (2): If Y, Z, and T_0 are given as in (2) then by (3) there exists a contractive positive projection P from Y onto X and the operator $T := T_0 \circ P$ is the desired extension. $\triangleright$

5.10.7. Corollary. *An injective Banach lattice is Dedekind complete and has the Fatou and Levi properties.*

$\triangleleft$ For every Banach lattice X the natural embedding $\varkappa : X \to X''$ is a lattice isometry and $\varkappa(X)$ is a closed sublattice in X''. If X is injective, then there exists a positive contractive projection from X'' onto $\varkappa(X)$; see Theorem 5.10.6 (3). Given an order or norm bounded set U in X, there exists $y := \sup \varkappa(U)$ in X'', since X'' is Dedekind complete and has the Levi property. Moreover, the identities $x := \varkappa^{-1}(Py) = \sup(U)$ and $\|x\| = \sup_{u \in U} \|u\|$ evidently are true in X because X'' has the Fatou property too. $\triangleright$

5.10.8. Corollary. *The Banach lattice of continuous function $C(K)$ on a Hausdorff compact topological space K is injective if and only if K is extremally disconnected.*

$\triangleleft$ This is immediate from 5.10.3 and 5.10.7 on using the Kakutani–

Kreĭns Representation Theorem and the Nakano–Stone Completeness Theorem. $\triangleright$

5.10.9. (1) A Banach lattice X has the *Cartwright property* if, given $x_1, x_2, y \in X_+$ with $\|x_1\| \leqslant 1$, $\|x_2\| \leqslant 1$, and $\|x_1 + x_2 + y\| \leqslant 2$, there exist $y_1, y_2 \in X_+$ such that $y_1 + y_2 = y$, $\|x_1 + y_1\| \leqslant 1$, and $\|x_2 + y_2\| \leqslant 1$.

(2) A Banach lattice X has the *splitting property* if, given $x_1, x_2, y \in X_+$ and $0 < r_1, r_2 \in \mathbb{R}$ with $\|x_1\| \leqslant r_1$, $\|x_2\| \leqslant r_2$, and $\|x_1 + x_2 + y\| \leqslant r_1 + r_2$, there exist $y_1, y_2 \in X_+$ such that $y_1 + y_2 = y$, $\|x_1 + y_1\| \leqslant r_1$, and $\|x_2 + y_2\| \leqslant r_2$.

(3) A Banach lattice X has the *finite order intersection property* if, given $z \in X_+$ and finite collections $x_1, \dots, x_n \in X_+$, $y_1, \dots, y_m \in X_+$ and strictly positive reals $r_1, \dots, r_n \in \mathbb{R}_+$, $s_1, \dots, s_m \in \mathbb{R}_+$ such that $\|x_\imath\| \leqslant r_\imath$, $\|y_\jmath\| \leqslant s_\jmath$, and $\|x_\imath + y_\jmath + z\| \leqslant r_\imath + s_\jmath$ for all $\imath := 1, \dots, n$ and $\jmath := 1, \dots, m$, there exist $u, v \in X_+$ with $z = u + v$, $\|x_\imath + u\| \leqslant r_\imath$, and $\|y_\jmath + v\| \leqslant s_\jmath$ for all $\imath := 1, \dots, n$ and $\jmath := 1, \dots, m$.

5.10.10. Theorem. *A Banach lattice has the Cartwright property if and only if it has the splitting property if and only if it has the finite order intersection property.*

$\triangleleft$ See Cartwright [85, Theorem 2.9]. $\triangleright$

5.10.11. Theorem. *A Banach lattice has the Cartwright property if and only if its bidual is injective. A Banach lattice is injective if and only if it has the Cartwright property and property* (P).

$\triangleleft$ See Cartwright [85, Theorem 3.6 and Corollary 3.8]. $\triangleright$

5.10.12. Theorem. *A Banach lattice is injective if and only if it has the Cartwright, Fatou and Levi properties.*

$\triangleleft$ See Haydon [169, Theorem 3.6 and Corollary 5.D]. $\triangleright$

5.11. Injectives: M-Structure

In this section we will demonstrate that injective Banach lattices carry M-structure in addition to their structure as Banach lattices, which determines important peculiar properties. We start with some elementary facts concerning M-projections.

5.11.1. *For a projection π in a Banach space X the following are equivalent* (*with* $\pi^\perp := I_X - \pi$) :

(1) $\|x\| = \max\{\|\pi x\|, \|\pi^\perp x\|\}$ $(x \in X)$.

(2) $\|\pi u + \pi^\perp v\| = \max\{\|\pi u\|, \|\pi^\perp v\|\}$ $(u, v \in X)$.

(3) $\|\pi u + \pi^\perp v\| \leqslant \max\{\|u\|, \|v\|\}$ $(u, v \in X)$.

$\lhd$ The equivalence (1) $\iff$ (2) is immediate: Putting $x := \pi u + \pi^\perp v$ in (1) yields (2) and, conversely, (1) is the particular case of (2) with $x = u = v$. It is easily seen from (1) that π and $\pi^\perp$ are contractive, which shows that (2) $\Longrightarrow$ (3). For the implication (3) $\Longrightarrow$ (1) observe that taking $u := \pi x$ and $v := \pi^\perp y$ in (3) yields $\|x\| \leqslant \max\{\|\pi x\|, \|\pi^\perp x\|\}$ and the reverse inequality is also true, since π and $\pi^\perp$ are evidently contractive under the assumption (3). $\rhd$

5.11.2. Theorem. *Assume that a Banach lattice X has the Fatou and Levi properties. Then $\mathbb{M}(X)$ is an order closed subalgebra of the complete Boolean algebra $\mathbb{P}(X)$. In particular, a Banach lattice having the Fatou and Levi properties is $\mathbb{B}$-cyclic with $\mathbb{B} := \mathbb{M}(X)$.*

$\lhd$ It is immediate from 5.7.12 that π and $\pi^\perp$ are M-projections or not simultaneously. If π and ρ are M-projections then, from 5.11.1 (1, 2) we deduce

$$\begin{aligned}\|x\| &= \max\{\|\pi x\|, \|\pi^\perp x\|\}\\ &= \max\{\max\{\|\rho\pi x\|, \|\rho^\perp \pi x\|\}, \|\pi^\perp x\|\}\\ &= \max\{\|\rho\pi x\|, \max\{\|\pi(x - \rho x)\|, \|\pi^\perp(x)\|\}\}\\ &= \max\{\|\rho\pi x\|, \|(I - \rho\pi)x\|\},\end{aligned}$$

so that $\pi\rho$ is an M-projection. It follows easily by induction that

$$\|x\| = \sup\{\|\pi_\alpha x\| : \alpha \in \mathrm{A}\}$$

for every finite partition of unity $(\pi_\alpha)_{\alpha\in\mathrm{A}}$ in $\mathbb{M}(X)$ and for all $x \in X$. Observe that the last identity is true for an arbitrary partition of unity $(\pi_\alpha)_{\alpha\in\mathrm{A}}$ provided X has the Fatou property. Indeed, if Θ stands for the collection of all finite subsets of A and $\rho_\theta := \sup_{\alpha\in\theta} \pi_\alpha$ then the family $(\rho_\theta|x|)_{\theta\in\Theta}$ is upward directed with $|x| = \sup_{\theta\in\Theta} \rho_\theta|x|$ and taking the Fatou property into account we deduce

$$\|x\| = \sup_{\theta\in\Theta} \|\rho_\theta|x|\,\| = \sup_{\theta\in\Theta}\sup_{\alpha\in\theta} \|\pi_\alpha|x|\,\| = \sup_{\alpha\in\mathrm{A}} \|\pi_\alpha x\|.$$

Assume now that $\pi \in \mathbb{P}(X)$ lies in the order closure of $\mathbb{M}(X)$ in $\mathbb{P}(X)$. Then there exist a partition of unity $(\pi_\alpha)_{\alpha\in\mathrm{A}}$ in $\mathbb{M}(X)$ and a subset $A_0 \subset A$ such that $\pi = \sup_{\alpha\in\mathrm{A}_0} \pi_\alpha$ and $\pi^\perp = \sup_{\alpha\in\mathrm{A}_0'} \pi_\alpha$ with $A_0' = A \setminus A_0$.

From the above we get

$$\|x\| = \sup_{\alpha\in\mathrm{A}} \|\pi_\alpha x\| = \max\left\{ \sup_{\alpha\in\mathrm{A}_0} \|\pi_\alpha x\|, \sup_{\alpha\in\mathrm{A}'_0} \|\pi_\alpha x\| \right\}$$
$$= \max\{\|\pi x\|, \|\pi^\perp x\|\}.$$

Thus $\pi \in \mathbb{M}(X)$ and $\mathbb{M}(X)$ is an order closed subalgebra of $\mathbb{P}(X)$. $\rhd$

5.11.3. Let B be a band in a Banach lattice X. An element $x \in X$ is called *maximal* in B if x is a maximal element of the set $\{y \in X : \|y\| = \|x\|\} \cap B$. We say that x is *maximal* if $B = X$ and *relatively maximal* if $B = \{x\}^{\perp\perp}$. Given $u \in X_+$, put

$$M_u := \{x \in X : \ \|u + y\| > \|u\| \text{ for all } 0 < y \leqslant |x|\}.$$

It is immediate from the definition that M_u is *solid*; i.e., $x \in M_u$ and $|y| \leqslant |x|$ imply $y \in M_u$ for all $x, y \in X$. In particular, $0 \in M_u$ and $M_0 = X$. Also, it can easily be seen that u is maximal in B if and only if $B = M_u$.

5.11.4. *Let a Banach lattice X have the Levi and Fatou properties. Given a band X_0, $0 < \varepsilon \in \mathbb{R} \cup \{\infty\}$, and $x \in X_0$, there exists a maximal element of the set*

$$V_\varepsilon(x) := \{y \in X_0 : \ y \geqslant x,\ \|y\| = \|x\|,\ \|x - y\| \leqslant \varepsilon\}.$$

$\lhd$ It is an easy consequence of the Kuratowski–Zorn Lemma. We need only observe that a linearly ordered subset $A \subset V_\varepsilon(x)$ is norm bounded and has supremum $\bar a = \sup(V_\varepsilon(x)) \leqslant y$ in X by the Levi property, while the Fatou property implies $\|\bar a\| = \sup_{a\in A} \|a\| = \|x\|$ and $\|\bar a - y\| = \sup_{a\in A} \|a - y\| \leqslant \varepsilon$, so that $\bar a \in V_\varepsilon(x)$. $\rhd$

5.11.5. *If a Banach lattice X has the Cartwright property then $\|u + v_1 + v_2\| = \|u\| + \|v_1 + v_2\|$ for all $0 \leqslant v_1, v_2 \in M_u$.*

$\lhd$ Put $s := \|v_1 + v_2\|$, $t := \|u\|$, and $r := \|v_1 + v_2 + u\| - t$. Note that $r \leqslant s$ and apply the Cartwright property with $x_1 := v_1 + v_2$, $x_2 := u$, and $y := (1 - (r/s))(v_1 + v_2)$. By 5.10.9 (2) there exist $y_1, y_2 \in X_+$ such that $y_1 + y_2 = y$, $\|u + y_2\| = t$, and $\|v_1 + v_2 + y_2\| = s$. If $y_2 \neq 0$ then either $x := y_2 \wedge v_1$ or $x := y_2 \wedge v_2$ is nonzero and $t = \|u\| \leqslant \|u + x\| \leqslant \|u + y_2\| = t$. At the same time $0 < x \leqslant v_1$ or $0 < x \leqslant v_2$ and so $x \in M_u$, implying that $\|u + x\| > \|u\|$ by definition. This contradiction shows that $y_2 = 0$ and $y_1 = y$. Hence, we arrive at the equation

$$\|v_1 + v_2\| = s = \|v_1 + v_2 + y_1\| = (1 + (1 - r/s))\|v_1 + v_2\|$$

which implies that $r = s$. $\triangleright$

5.11.6. *If a Banach lattice X has the Cartwright property then M_u is a band and $\|u + |x|\,\| = \|u\| + \|x\|$ for all $x \in X$.*

$\triangleleft$ It is an immediate consequence of 5.11.5 and the definition of M_u that M_u is an order ideal in X. Assume that $x_0 = \sup(A) \in X$ for some upward directed set $A \subset (M_u)_+$. For an arbitrary $0 < y_0 \leqslant x_0$ choose $x \in A$ such that $0 < x \wedge y_0 =: y$. Then $0 < y \leqslant x \in M_u$ and by definition $\|y_0 + u\| \geqslant \|y + u\| > \|u\|$, so that $x_0 \in M_u$ and M_u is a band. $\triangleright$

5.11.7. *Let X be a Banach lattice with the Cartwright, Levi, and Fatou properties and $0 < u \in X$. Then the representation holds* (*with $|x|/\|x\| = 0$ for $x := 0$*)*:*

$$M_u^{\perp} := \left\{ x \in X : \left\| u + |x| \frac{\|u\|}{\|x\|} \right\| = \|u\| \right\}.$$

$\triangleleft$ Without loss of generality we can assume that $\|u\| = 1$, since $M_{\lambda u} = M_u$ for all $0 < \lambda \in \mathbb{R}$. Now it suffices to prove that the element $x \in X_+$ with $\|x\| = 1$ lies in $M_u^{\perp}$ if and only if $\|u + x\| = 1$. If $\|u + x\| = 1$ and $0 \leqslant x_0 \leqslant x$ for some $x_0 \in M_u$ then by 5.11.6 we have $1 \geqslant \|u + x_0\| = \|u\| + \|x_0\| = 1 + \|x_0\|$; it follows that $x_0 = 0$ and $x \perp M_u$.

Conversely, assume that $0 \leqslant x \in M_u^{\perp}$ and $\|x\| = 1$. Then the set $U(x) := \{y \in X : 0 < y \leqslant x,\ \|u + y\| = 1\}$ is nonempty, since otherwise $x \in M_u$, contradicting the choice of x. Note that for an upward directed set $A \subset U(x)$ we have $y_0 := \sup(A) \in U(x)$, since by the Fatou property $\|u + y_0\| = \sup_{y \in A} \|u + y\| = 1$. By the Kuratowski–Zorn Lemma there exists a maximal element $\bar{y} \in U(x)$. Put $z := x - \bar{y}$ and observe that $\|\bar{y} + z\| = \|x\| = 1$ and $\|u + \bar{y} + \bar{y} + z\| = \|u + \bar{y} + x\| \leqslant 2$. Applying the Cartwright property 5.10.9 (1) with $x_1 := u + \bar{y}$, $x_2 := \bar{y}$, $y := z$, we can split z as $z = z_1 + z_2$, where $\|u + \bar{y} + z_1\| \leqslant 1$ and $\|\bar{y} + z_2\| \leqslant 1$. The maximality of $\bar{y}$ in $U(x)$ implies $z_2 = 0$ and $\|u + x\| = \|u + \bar{y} + z_1\| = 1$. $\triangleright$

5.11.8. *Let X be a Banach lattice with the Cartwright, Levi, and Fatou properties. Then M_u is an M-band for every $u \in X_+$.*

$\triangleleft$ Given $u \in X_+$ with $\|u\| = 1$, take $y \in M_u$ and $z \in M_u^{\perp}$ with $\max\{\|y\|, \|z\|\} \leqslant 1$. By 5.11.7 $\|u + z\| = 1$, and so $\|u + z + y\| \leqslant 2$. By the Cartwright property there exist $y_1, y_2 \in X_+$ with $y_1 + y_2 = y$,

$\|u + y_1\| \leqslant 1$, and $\|z + y_2\| \leqslant 1$. Since $y \in M_u$, we have $y_1 = 0$, and so $y = y_2$ and $\|y + z\| \leqslant 1$. $\triangleright$

5.11.9. Theorem. *Let X be a Banach lattice with the Cartwright, Levi, and Fatou properties. If $\mathbb{M}(X) = \{0, I_X\}$ then X is an AL-space.*

$\triangleleft$ Assume that X is not an AL-space. Then there exist $x, y \in X_+$ with $\|x + y\| < \|x\| + \|y\|$. Note that $x \neq 0$ and $y \neq 0$. Take $0 < \varepsilon < \|x\| + \|y\| - \|x + y\|$, form $V_\varepsilon(x)$ as in 5.11.4 with $X_0 = X$, and denote by u a maximal member of $V_\varepsilon(x)$. Prove that M_u is a nontrivial proper M-band. If $y \in M_u$ then $\|u + y\| = \|u\| + \|y\|$ by 5.11.6 and this yields a contradiction:

$$\|u+y\| \leqslant \|x+y\| + \|u-x\| \leqslant \|x+y\| + \varepsilon < \|x\| + \|y\| = \|u\| + \|y\|.$$

Thus $y \notin M_u$ and $M_u \neq X$. Observe now that x is not maximal, since otherwise $x = u \neq 0$ and we again arrive at a contradiction $M_u = X$. If $u - x$ were maximal, then we would have $M_{u-x} = X$ and 5.11.6 would imply $\|u\| = \|(u - x) + x\| = \|u - x\| + \|x\| > \|x\| = \|u\|$. Thus, $M_{u-x}^{\perp} \neq \{0\}$ and by 5.11.6 we can choose $0 < v \in M_{u-x}^{\perp}$ with $\|v\| = \|u - x\|$ and $\|u - x + v\| = \|u - x\| \leqslant \varepsilon$. If $v \notin M_u$ then there would exist $0 < z \leqslant y$ such that $\|u + z\| = \|u\| = \|x\|$. This contradicts maximality of u, because $\|u + z - x\| \leqslant \|u + v - x\| \leqslant \varepsilon$. It follows that $v \in M_u \neq \{0\}$. It remains to apply 5.11.8 to conclude that $[M_u]$ is a nontrivial proper M-projection; i.e., $\mathbb{M}(X) \neq \{0, I_X\}$. $\triangleright$

5.11.10. Corollary. *For an injective Banach lattice X the following are equivalent:*

(1) *X is an AL-space.*

(2) $\mathbb{M}(X) = \{0, I_X\}$.

(3) *$\mathscr{Z}_m(X)$ is one-dimensional.*

$\triangleleft$ Evidently, (2) and (3) are equivalent for every Banach lattice, while (2) $\Longrightarrow$ (1) is just Theorem 5.11.9. The remaining implication (1) $\Longrightarrow$ (2) is easy and can be extracted from Harmand, Werner, and Wener [166, Example 1.6 (a) or Theorem 1.8]. $\triangleright$

5.12. Representation of Injective Banach Lattices

The results above allow us to get the representation results for injective Banach lattices.

5.12.1. Theorem. *Let X be a $\mathbb{B}$-cyclic Banach lattice and let $\mathscr{X}$ be its Boolean valued representation in $\mathbb{V}^{(\mathbb{B})}$. Then the following hold:*

(1) $\mathbb{V}^{(\mathbb{B})} \models$ "$\mathscr{X}$ *is injective" if and only if* X *is injective.*

(2) $\mathbb{V}^{(\mathbb{B})} \models$ "$\mathscr{X}$ *is an AM-space" if and only if* X *is an AM-space.*

(3) $\mathbb{V}^{(\mathbb{B})} \models$ "$\mathscr{X}$ *is an AL-space" if and only if* X *is injective and* $\mathbb{B} \simeq \mathbb{M}(X)$.

$\lhd$ (1): Theorem 5.10.2 is valid within $\mathbb{V}^{(\mathbb{B})}$ by transfer. In view of Theorem 5.9.6 (2) we only have to show that $[\![\mathscr{X}$ has the splitting property$]\!] = \mathbb{1}$ if and only if X has the splitting property. It is easy to see that $[\![\mathscr{X}$ has the splitting property$]\!] = \mathbb{1}$ is equivalent to the following property: For all $x, y, z \in X_+$ with $|x| \leqslant \mathbb{1}$, $|y| \leqslant \mathbb{1}$, and $|x+y+z| \leqslant 2\mathbb{1}$, there exist $u, v \in X_+$ such that $z = u + v$, $|x + u| \leqslant \mathbb{1}$ and $|y + v| \leqslant \mathbb{1}$. But the latter amounts to the splitting property in X, since the relations $|x| \leqslant C\mathbb{1}$ and $\|x\|_X = \||x|\|_\infty \leqslant C$ are equivalent.

(2): Since the Λ-valued norm $|\cdot|$ in X is the restricted descent of the norm $\|\cdot\|_{\mathscr{X}}$ and the join $(x, y) \mapsto x \vee y$ in X is the descent of the similar operation on $\mathscr{X}$, it follows that $[\![\, \|\cdot\|_{\mathscr{X}}$ is an M-norm$]\!] = \mathbb{1}$ if and only if $|x \vee y| = |x| \vee |y|$ for all $x, y \in X^+$. Since $(\Lambda, \|\cdot\|_\infty)$ is an AM-space, we deduce $\|x \vee y\|_X = \||x \vee y|\|_\infty = \||x|\|_\infty \vee \||y|\|_\infty = \|x\|_X \vee \|y\|_X$.

(3): By transfer and Theorem 5.11.9 we can claim that $[\![\mathscr{X}$ is an AL-space if and only if $\mathscr{X}$ is injective and $\mathbb{M}(\mathscr{X}) = \{0, I_{\mathscr{X}}\}]\!] = \mathbb{1}$. Therefore, the result is immediate from (1), Theorem 5.9.1, and 1.8.1. $\rhd$

5.12.2. Corollary. *Let X be a Banach lattice with the Fatou and Levi properties and $\mathbb{B}$ an isomorphic copy of the complete Boolean algebra $\mathbb{M}(X)$. Then X is injective if and only if X is lattice $\mathbb{B}$-isometric to the bounded descent of some AL-space $\mathscr{X}$ from $\mathbb{V}^{(\mathbb{B})}$.*

$\lhd$ It is immediate from Theorems 5.11.2 and 5.12.1 (3). $\rhd$

5.12.3. A positive operator $T : X \to Y$ is said to have the *Levi property* if $\sup x_\alpha$ exists in X for every increasing net $(x_\alpha) \subset X_+$, provided that the net (Tx_α) is order bounded in Y. A *Maharam operator* T is an order continuous order interval preserving ($\equiv T([0, x] = [0, Tx])$ for all $x \in X_+$) operator (cp. 5.2.1).

5.12.4. Consider vector lattices X and Y, with Y order complete, and an operator $\Phi \in L_+(X, Y)$. Suppose that Φ is strictly positive ($\equiv x > 0$ implies $\Phi(x) > 0$) and put $|x| := \Phi(|x|)$ $(x \in X)$. Then $(X, |\cdot|)$ is a lattice normed space. The *bo*-completion of X denoted by $L^1(\Phi)$ is a Banach–Kantorovich lattice (cp. [228, Theorems 2.2.8 and 2.2.11]). It is easy to see that $L^1(\Phi) = X$ if and only if Φ is a strictly positive Maharam operator with the Levi property.

5.12.5. Theorem. *Let X be a Banach lattice with the complete Boolean algebra $\mathbb{B} = \mathbb{M}(X)$ of M-projections, and let Λ be a Dedekind complete unital AM-space such that $\mathbb{P}(\Lambda)$ is isomorphic to $\mathbb{B}$. Then the following are equivalent:*

(1) *X is injective.*

(2) *X is lattice $\mathbb{B}$-isometric to the bounded descent of some AL-space from $\mathbb{V}^{(\mathbb{B})}$.*

(3) *There exists a strictly positive Maharam operator $\Phi : X \to \Lambda$ with the Levi property such that $X = L^1(\Phi)$ and $\|x\| = \|\Phi(|x|)\|_\infty$ for all $x \in X$.*

(4) *There is a Λ-valued additive norm on X such that $(X, \boldsymbol{|}\cdot\boldsymbol{|})$ is a Banach–Kantorovich lattice and $\|x\| = \big\|\boldsymbol{|}x\boldsymbol{|}\big\|_\infty$ for all $x \in X$.*

$\lhd$ (1) $\Longleftrightarrow$ (2): This follows from Corollary 5.12.2 and Theorem 5.12.1 (3).

(2) $\Longrightarrow$ (3): Assume that the Boolean valued representation $\mathscr{X}$ of X is an AL-space within $\mathbb{V}^{(\mathbb{B})}$. Working within $\mathbb{V}^{(\mathbb{B})}$ and using the transfer principle, we can find a strictly positive order continuous functional $\phi : \mathscr{X} \to \mathscr{R}$ with the Levi property such that $\|x\|_{\mathscr{X}} = \phi(|x|)$ for all $x \in \mathscr{X}$. The descent $\Phi' := \phi{\downarrow}$ as well as its restriction $\Phi := \Phi'|_X : X \to \Lambda$ is a strictly positive Maharam operator with the Levi property (cp. 5.2.8). Since $\boldsymbol{|}\cdot\boldsymbol{|} = (\|\cdot\|_{\mathscr{X}}){\downarrow}$ we have $\boldsymbol{|}x\boldsymbol{|} = \Phi(|x|)$ for all $x \in X$. By the definition of restricted descent $\|x\|_X = \big\|\boldsymbol{|}x\boldsymbol{|}\big\|_\infty = \|\Phi(|x|)\|_\infty$.

(3) $\Longrightarrow$ (4): If (3) is true then some Λ-valued additive norm on X is defined by $\boldsymbol{|}x\boldsymbol{|} := \Phi(|x|)$ $(x \in X)$. The fact that $(X, \boldsymbol{|}\cdot\boldsymbol{|})$ is a Banach–Kantorovich space follows from Theorem 5.5.7.

(4) $\Longrightarrow$ (2): This is immediate from Theorems 5.8.11, 5.9.1, and 5.12.1 (3). $\rhd$

5.12.6. Corollary. *If Φ is a strictly positive Maharam operator with the Levi property taking values in a Dedekind complete AM-space Λ with unit and $|||x||| = \|\Phi(|x|)\|_\infty$ $(x \in L^1(\Phi))$, then $(L^1(\Phi), |||\cdot|||)$ is an injective Banach lattice and there is a Boolean isomorphism φ from $\mathbb{P}(\Lambda)$ onto $\mathbb{M}(L^1(\Phi))$ with $\pi \circ \Phi = \Phi \circ \varphi(\pi)$ for all $\pi \in \mathbb{B}$.*

Conversely, each injective Banach lattice X is lattice $\mathbb{B}$-isometric to $(L^1(\Phi), |||\cdot|||)$ for some strictly positive Maharam operator Φ with the Levi property taking values in a Dedekind complete AM-space Λ with unity, where $\mathbb{B} = \mathbb{P}(\Lambda) \simeq \mathbb{M}(L^1(\Phi))$.

5.12.7. Corollary. *An injective Banach lattice has an order $\mathbb{B}$-continuous norm with $\mathbb{B}$ the complete Boolean algebra of its M-projections.*

$\lhd$ It is immediate from 5.9.6 (3) and 5.12.5 (2). $\rhd$

5.12.8. Corollary. *An injective Banach lattice X has an order continuous norm if and only if X is a finite l_∞-product of AL-spaces.*

$\lhd$ It is clear from the representation $\|x\|_X = \|\Phi(|x|)\|_\infty$ $(x \in X)$ that X has an order continuous norm if and only if Λ has an order continuous norm. But the latter occurs only if Λ is finite dimensional. $\rhd$

A Maharam operator Φ in Theorem 5.12.5 (3) and Corollary 5.11.6 is not unique. If σ is an automorphism of $\mathbb{B}$ then there exists a unique lattice isomorphism $\widehat{\sigma}$ of Λ onto itself such that $\widehat{\sigma}(\pi\mathbb{1}) = \sigma(\pi)\mathbb{1}$. The operator $\widehat{\sigma}$ is called the *shift* by σ. If $\widehat{\sigma}$ is the shift in Λ by an automorphism of $\mathbb{B}$ and β is an invertible positive orthomorphism in $L_1(\Phi)$ then $\Psi = \widehat{\sigma} \circ \Phi \circ \beta$ is a strictly positive Maharam operator with the Levi property and the Banach lattices $L_1(\Phi)$ and $L_1(\Psi)$ coincide. The following result tells us that this example is exhaustive.

5.12.9. Theorem. *Let X be an injective Banach lattice and let $\Phi, \Psi : X \to \Lambda$ be strictly positive Maharam operators with the Levi property such that $\|\Phi(|x|)\|_\infty = \|x\|_X = \|\Psi(|x|)\|_\infty$ for all $x \in X$. Then there exist an automorphism σ of $\mathbb{B}$ and an invertible positive orthomorphism β in X such that $\Psi = \widehat{\sigma} \circ \Phi \circ \beta$.*

$\lhd$ If the conditions above are satisfied then $\Phi(X) = \Lambda = \Psi(X)$ and by Corollary 5.2.4 (2) there are order continuous Boolean homomorphisms $\varrho, \tau : \mathbb{B} \to \mathbb{M}(X)$ such that $\pi \circ \Phi = \Phi \circ \varrho(\pi)$ and $\pi \circ \Psi = \Psi \circ \tau(\pi)$ for all $\pi \in \mathbb{B}$. Observe that $\sigma := \tau^{-1} \circ \varrho$ is an automorphism of a Boolean algebra $\mathbb{B}$. Let $\widehat{\sigma}$ stand for the corresponding shift operator on Λ. It can easily be seen from the definitions (see Kusraev [228, 5.3.2, 5.3.3]) that $\widehat{\sigma}(\pi\lambda) = \sigma(\pi)\widehat{\sigma}(\lambda)$ for all $\pi \in \mathbb{B}$ and $\lambda \in \Lambda$. Put $\Phi_1 = \widehat{\sigma} \circ \Phi$ and note that Φ_1 is also a strictly positive Maharam operator with the Levi property. Moreover, Ψ is absolutely continuous with respect to Φ_1. Indeed, for $\pi \in \mathbb{B}$ and $x \in X$ we have

$$\begin{aligned}\sigma(\pi)\Phi_1(x) = \sigma(\pi)\widehat{\sigma}(\Phi(x)) = \widehat{\sigma}(\pi\Phi(x))& \\ = \widehat{\sigma}(\Phi(\varrho(\pi)x)) = \Phi_1(\varrho(\pi)x) &= \Phi_1(\tau\sigma(\pi)x),\end{aligned}$$

and so $\pi \circ \Phi_1 = \Phi_1 \circ \tau(\pi)$ $(\pi \in \mathbb{B})$. If $\pi\Phi_1(x) = 0$ for some $\pi \in \mathbb{B}$ and $x \in X_+$ then $\Phi_1(\tau(\pi)x) = 0$ and $\tau(\pi)x = 0$, as Φ_1 is strictly positive.

It follows that $\pi\Psi(x) = \Psi(\tau(\pi)x) = 0$ and so $\Psi(x) \in \Phi_1(x)^{\perp\perp}$ for all $x \in X_+$. By the Radon–Nikodým Theorem 5.3.9 there exists a sequence (β_n) of positive orthomorphisms in X such that $\Psi(x) = \sup_n \Phi_1(\beta_n x)$ for all $x \in X_+$. The positive orthomorphism $\beta = \sup_n \beta_n$ is well defined in X, since Φ_1 has the Levi property. Moreover, by order continuity we have $\Psi(x) = \Phi_1(\beta x)$ $(x \in X)$ or $\Psi = \widehat{\sigma} \circ \Phi \circ \beta$. $\rhd$

5.12.10. The construction of the Maharam extension of positive operators (see Section 5.5) together with Corollary 5.12.6 supplies plenty of injective Banach lattices. Recall some notation. Given a subset M of a vector lattice X, denote by $M^{\downarrow}$ the collection of all elements $x \in X$ that can be written as $x = \inf(A)$, where A is a downward directed subset of M. The set $M^{\uparrow}$ is defined similarly on using upward directed sets. We also put $M^{\downarrow\uparrow} := (M^{\downarrow})^{\uparrow}$. Write $\mathscr{Z}_m(X)$ for the order closed f-subalgebra of the center $\mathscr{Z}(X)$ generated by $\mathbb{M}(X)$.

We recall also that a subspace X_0 of a $\mathbb{B}$-cyclic Banach space X is said to be $\mathbb{B}$-*dense* if for all $x \in X$ and $0 < \varepsilon \in \mathbb{R}$ there are $x_\varepsilon \in X$, a partition of unity $(\pi_\xi)_{\xi\in\Xi}$ in $\mathbb{M}(X)$, and a family $(x_\xi)_{\xi\in\Xi}$ in X_0 such that $\|x - x_\varepsilon\| \leqslant \varepsilon$ and $\pi_\xi x_\varepsilon = \pi_\xi x_\xi$ $(\xi \in \Xi)$.

5.12.11. Theorem. *Let L be a vector lattice, let Λ be a Dedekind complete AM-space with unit, and let $\Phi : L \to \Lambda$ be a positive operator. Then there exists a unique (up to lattice isometry) injective Banach lattice X such that the following hold:*

(1) $\mathbb{M}(X) \simeq \mathbb{P}(\Lambda)$.

(2) *There are a lattice homomorphism ι from L to X and an f-algebra isomorphism h from $\mathscr{Z}(\Lambda)$ onto $\mathscr{Z}_m(X)$ such that $\|\sigma\Phi(x)\|_\infty = \|h(\sigma)\iota(x)\|$ $(x \in L_+,\ \sigma \in \mathscr{Z}(\Lambda)_+)$.*

(3) *$\iota(L)$ is $\mathbb{B}$-dense in X.*

(4) *$X = X_0^{\downarrow\uparrow}$, where X_0 comprises all finite sums $\sum_{k=1}^n \pi_k \iota(x_k)$ with $\pi_k \in \mathbb{M}(X)$ and $x_k \in L$ $(k = 1, \dots, n \in \mathbb{N})$.*

$\lhd$ The Maharam extension $\bar{\Phi}$ of Φ is a strictly positive Maharam operator by Theorem 5.5.7. If $X = L^1(\bar{\Phi})$ is the domain of $\hat{\Phi}$ (see 5.6.1) and $\|x\| = \|\tilde{\Phi}(|x|)\|_\infty$ $(x \in X)$ then X is an injective Banach lattice by Theorem 5.12.5. The properties (1)–(4) are immediate from Theorems 5.5.7 and 5.6.3. $\rhd$

5.12.12. Theorem. *Let X be an order $\mathbb{B}$-continuous $\mathbb{B}$-cyclic Banach lattice and let $X^{\mathfrak{u}}$ be its universal completion. There exists an order dense ideal L in $X^{\mathfrak{u}}$ which is an injective Banach lattice with*

$\mathbb{M}(L) \simeq \mathbb{B}$. *Moreover, if* $L = L^1(\Phi)$ *for a strictly positive Maharam operator* $\Phi : L \to \Lambda = \Lambda(\mathbb{B})$ *with the Levi property, then the mapping assigning the operator* $S_{x'} : X \to \Lambda$ *to an element* $x' \in X'$ *by* $S_{x'}(x) = \Phi(x \cdot x')$ $(x \in X)$ *is a lattice* $\mathbb{B}$*-isometry from* X' *onto* $X^{\#}$.

$\lhd$ The Boolean valued representation $\mathscr{X}$ of X is an order continuous Banach lattice within $\mathbb{V}^{(\mathbb{B})}$ (cp. Theorem 5.9.6). Working within $\mathbb{V}^{(\mathbb{B})}$ we can find an order continuous strictly positive functional $\phi : L_1(\phi) \to \mathscr{R}$ having the Levi property, with $L^1(\phi)$ an order dense ideal in the universal completion $\mathscr{X}^{\mathfrak{u}}$ of $\mathscr{X}$. Put $\mathscr{X}' := \{x' \in \mathscr{X}^{\mathfrak{u}} : x' \cdot \mathscr{X} \subset L^1(\phi)\}$. Then assigning to every element $x' \in \mathscr{X}'$ the functional $\sigma_{x'} : x \mapsto \phi(xx')$ $(x \in \mathscr{X})$ yields a lattice isometry from $\mathscr{X}'$ onto the dual $\mathscr{X}^*$. It is easy to see that $\mathscr{X}^{\mathfrak{u}}{\downarrow} = X^{\mathfrak{u}}$. Define Φ as the restriction of $\phi{\downarrow}$ to $L := L^1(\Phi) := \{x \in X^{\mathfrak{u}} : \phi{\downarrow}(x) \in \Lambda\}$. Clearly, Φ is a strictly positive Maharam operator with the Levi property and so $L^1(\phi){\Downarrow} = L^1(\Phi)$. It remains to observe that identifying X with an order dense ideal in $\mathscr{X}{\downarrow}$ we have $\mathscr{X}^*{\Downarrow} = X^{\#}$, $\mathscr{X}'{\Downarrow} = X'$, and $S_{x'} = \sigma_x{\downarrow}$. $\rhd$

5.12.13. Corollary. *An injective Banach lattice* X *is lattice* $\mathbb{B}$*-isometric to* $\mathscr{L}_{n,\mathbb{B}}(\mathscr{Z}(X), \Lambda) := \mathscr{L}_{\mathbb{B}}(\mathscr{Z}(X), \Lambda) \cap \mathscr{L}_n(\mathscr{Z}(X), \Lambda)$. *If* X *is represented as* $L_1(\Phi)$ *for a strictly positive Maharam operator* Φ *with the Levi property, then the lattice* $\mathbb{B}$*-isometry is carried out by assigning to each* $x \in X$ *the operator* $S_x : \pi \mapsto \Phi(\pi x)$ $(\pi \in \mathscr{Z}(X))$.

$\lhd$ Put $X := L_1(\Phi)$ in Theorem 5.12.12. Then $\mathscr{X} = L_1(\phi)$ and so $\mathscr{X}' = L_\infty(\phi)$. Consequently we can deduce $X' = \mathscr{X}'{\Downarrow} = L_\infty(\phi){\Downarrow} = L_\infty(\Phi)$. It remains to note that $L_\infty(\Phi)$ and $\mathscr{Z}(X)$ are lattice $\mathbb{B}$-isometric and $X^{\#} = \mathscr{L}_{n,\mathbb{B}}(X, \Lambda)$ by Corollary 5.9.7 (7). $\rhd$

5.12.14. Corollary. *Let* X *be an injective Banach lattice, while* Y *and* Λ *be Dedekind complete AM-spaces with unit such that* $\mathbb{P}(Y) \simeq \mathbb{P}(X)$ *and* $\mathbb{B} := \mathbb{P}(\Lambda) \simeq \mathbb{M}(X)$. *Then* X *is lattice* $\mathbb{B}$*-isometric to* $\mathscr{L}_{n,\mathbb{B}}(Y, \Lambda)$.

5.13. Operators Factorable Through Injective Banach Lattices

This section treats the operators that admit factorization through injective Banach lattices. In fact we implement the Boolean valued interpretation of a portion of the theory of cone absolutely summing operators.

5.13.1. Let X be a Banach lattice and let Y be a $\mathbb{B}$-cyclic Banach space. Denote by $\mathrm{Prt}_\sigma := \mathrm{Prt}_\sigma(\mathbb{B})$ and $\mathscr{P}_{\mathrm{fin}}(X)$ the set of all countable partitions of unity in $\mathbb{B}$ and the collection of all finite subsets of X, respectively. Given $T \in \mathscr{L}(X,Y)$, put

$$\sigma(T) := \sigma_{\mathbb{B}}(T) := \sup\Bigg\{ \inf_{(\pi_k)\in\mathrm{Prt}_\sigma} \sup_{k\in\mathbb{N}} \sum_{\imath=1}^{n} \|\pi_k T x_\imath\| : \\ (x_\imath) \in \mathscr{P}_{\mathrm{fin}}(X),\ \Bigg\| \sum_{\imath=1}^{n} |x_\imath| \Bigg\| \leqslant 1 \Bigg\}.$$

An operator $T \in \mathscr{L}(X,Y)$ is said to be $\mathbb{B}$-*summing* if $\sigma(T) < \infty$. Thus, T is $\mathbb{B}$-*summing* if and only if there exists a positive constant C such that for every finite collection $x_1,\dots,x_n \in X$ there is a countable partition of unity $(\pi_k)_{k\in\mathbb{N}}$ in $\mathbb{B}$ with

$$\sup_{k\in\mathbb{N}} \sum_{\imath=1}^{n} \|\pi_k T x_\imath\| \leqslant C \Bigg\| \sum_{\imath=1}^{n} |x_\imath| \Bigg\|.$$

Denote by $\mathscr{S}_{\mathbb{B}}(X,Y)$ the set of all $\mathbb{B}$-summing operators.

5.13.2. Observe that if $\mathbb{B} = \{\mathbb{0},\mathbb{1}\} \simeq \{0, I_Y\}$ then $\mathscr{S}(X,Y) := \mathscr{S}_{\{\mathbb{0},\mathbb{1}\}}(X,Y)$ is the space of *cone absolutely summing operators* (cp. [356, Ch. 4, §3, Proposition 3.3 (d)]) or (which is the same) $(1,1)$-concave operators. In this case $\sigma(T)$ takes the form

$$\varsigma(T) := \sigma_{\{\mathbb{0},\mathbb{1}\}}(T) := \\ \sup\Bigg\{ \sum_{\imath=1}^{n} \|T x_\imath\| : (x_\imath) \in \mathscr{P}_{\mathrm{fin}}(X),\ \Bigg\| \sum_{\imath=1}^{n} |x_\imath| \Bigg\| \leqslant 1 \Bigg\}.$$

A linear operator T is cone absolutely summing if and only if for every norm summable sequence (x_n) in X_+, the sequence (Tx_n) is absolutely summable in Y (cp. [356, Ch. 4, §3, Proposition 3.3]).

Let $\mathscr{X}$ and $\mathscr{Y}$ stand for the Boolean valued representation of X and Y, respectively. Write $\mathscr{S}(\mathscr{X},\mathscr{Y})$ for the space of all cone absolutely summing operators from $\mathscr{X}$ to $\mathscr{Y}$ within $\mathbb{V}^{(\mathbb{B})}$. If $\mathscr{T} \in \mathscr{S}(\mathscr{X},\mathscr{Y})\Downarrow$ and $T = \mathscr{T}\Downarrow$ then $[\![\sigma(T) = \varsigma(\mathscr{T})]\!] = \mathbb{1}$.

5.13.3. Suppose that Q is a Stonean space and X is a Banach space. Let $C_\infty(Q,X)$ be the set of cosets of continuous vector-functions u that act from comeager subsets $\operatorname{dom}(u) \subset Q$ into X. (Recall that a set is called comeager if its complement is of first category.) Vector-functions u and v are equivalent if $u(q) = v(q)$ whenever $q \in \operatorname{dom}(u) \cap \operatorname{dom}(v)$. The set $C_\infty(Q, X)$ is endowed, in a natural way, with the structure of a module over the f-algebra $C_\infty(Q)$. Moreover, the continuous extension of the pointwise norm $q \mapsto \|u(q)\|$ defines a decomposable norm $u \mapsto |u| \in C_\infty(Q)$ on $C_\infty(Q, X)$. Moreover, $C_\infty(Q, X)$ is a Banach–Kantorovich space (cp. 5.8.4).

Denote by $C_\#(Q, X)$ the part of $C_\infty(Q, X)$ that consists of vector-functions u satisfying $|u| \in C(Q)$ endowed with the norm $\|u\| := \||u|\|_\infty$ (cp. Kusraev [228, 2.3.3]). Suppose that Q is a Stonean space and X is a Banach lattice. Then the space $C_\#(Q, X)$ is a $\mathbb{B}$-cyclic Banach lattice with $\mathbb{B}$ isomorphic to the Boolean algebra $\operatorname{Clop}(Q)$ of clopen subsets of Q. For $U \in \operatorname{Clop}(Q)$ the corresponding M-projection in $C_\#(Q, X)$ is given by $u \mapsto 1_U u$.

5.13.4. Theorem. *Suppose that X is a Banach lattice and $\mathscr{X}$ is the completion of the metric space $X^\wedge$ within $\mathbb{V}^{(\mathbb{B})}$. Then $[\![\mathscr{X}$ is a Banach lattice$]\!] = \mathbb{1}$ and $\mathscr{X}{\Downarrow}$ is $\mathbb{B}$-isomorphic to $C_\#(Q, X)$. Moreover, if Y is a $\mathbb{B}$-cyclic Banach lattice, then $T \mapsto T \circ h$ is a lattice $\mathbb{B}$-isometry from $\mathscr{L}_{\mathbb{B}}(C_\#(Q, X), Y)$ onto $\mathscr{L}(X, Y)$, where h is the lattice isometry from X into $C_\#(Q, X)$ defined as $h(x) := 1_Q x$.*

$\triangleleft$ The proof is a due modification of Kusraev [228, 8.3.4]. $\triangleright$

5.13.5. Corollary. *A Banach lattice X is an AL-space if and only if $C_\#(Q, X)$ is an injective Banach lattice with $\mathbb{M}(C_\#(Q, X))$ isomorphic to the Boolean algebra $\operatorname{Clop}(Q)$. The Boolean isomorphism sends a clopen set $G \subset Q$ to the M-projection $u \mapsto u1_G$ $(u \in C_\#(Q, X))$.*

$\triangleleft$ Assume that X is an AL-space. By restricted transfer $(X^\wedge, \|\cdot\|^\wedge)$ is a normed vector lattice over $\mathbb{R}^\wedge$ within $\mathbb{V}^{(\mathbb{B})}$. Moreover, the norm $\|\cdot\|^\wedge$ is additive on the positive cone $(X^\wedge)_+$ and so $\mathscr{X}$ is an AL-space within $\mathbb{V}^{(\mathbb{B})}$. Thus, $C_\#(Q, X)$ is an injective Banach lattice with $\mathbb{M}(C_\#(Q, X))$ isomorphic to $\operatorname{Clop}(Q)$ by Theorems 5.12.1 (3) and 5.13.4. Conversely, the mapping $x \mapsto 1_Q x$ is a lattice isometry from X onto a closed vector sublattice in $C_\#(Q, X)$. Therefore, X is injective whenever $C_\#(Q, X)$ is injective. If $\pi \in \mathbb{M}(X)$ then $u \mapsto \pi \circ u1_Q$ $(u \in C_\#(Q, X))$ is an M-projection in $C_\#(Q, X)$ and $\mathbb{M}(C_\#(Q, X)) \simeq \operatorname{Clop}(Q)$ implies $\pi = 0$ or $\pi = I_X$. Thus, X is an AL-space by Theorem 5.11.10. $\triangleright$

5.13.6. Theorem. *Let X, $\mathscr{X}$, and Y be as in Theorem 5.13.4, and let $\mathscr{Y}$ be the Boolean valued representation of Y. For every*

$T \in \mathscr{S}_{\mathbb{B}}(X,Y)$ there exists a unique $\mathscr{T} := T\!\uparrow \in \mathbb{V}^{(\mathbb{B})}$ determined from the formulas

$$[\![\, \mathscr{T} \in \mathscr{S}(\mathscr{X}, \mathscr{Y}) \,]\!] = \mathbb{1}, \quad [\![\, \mathscr{T} x^{\wedge} = Tx \,]\!] = \mathbb{1} \quad (x \in X).$$

The mapping $T \mapsto \mathscr{T}$ is a $\mathbb{B}$-isometry from $\mathscr{S}_{\mathbb{B}}(X,Y)$ onto the restricted descent $\mathscr{S}(\mathscr{X}, \mathscr{Y})\!\Downarrow$.

$\lhd$ Suppose that $T \in \mathscr{L}_{\mathbb{B}}(X,Y)$ and its Boolean valued representation $\mathscr{T}$ (see Theorem 5.8.12) is cone absolutely summing; i.e., $\mathscr{T} \in \mathscr{S}(\mathscr{X}, \mathscr{Y})\!\Downarrow$. Then $\varsigma(\mathscr{T}) \in \Lambda$ and we can assume $\varsigma(\mathscr{T}) \leqslant C\mathbb{1}$ for some $0 < C \in \mathbb{R}$. Moreover, the relation

$$\big(\forall\, \theta \in \mathscr{P}_{\mathrm{fin}}(\mathscr{X})\big) \sum_{x\in\theta} \|\mathscr{T} x\| \leqslant C^{\wedge} \Bigg\| \sum_{x\in\theta} |x| \Bigg\| \tag{$*$}$$

holds in $\mathbb{V}^{(\mathbb{B})}$ and so its Boolean truth value is $\mathbb{1}$. Recall that $X^{\wedge}$ is a dense sublattice in $\mathscr{X}$. Replacing $\mathscr{X}$ and $\theta \in \mathscr{P}_{\mathrm{fin}}(\mathscr{X})$ by $X^{\wedge}$ and $\theta \in \mathscr{P}_{\mathrm{fin}}(X^{\wedge})$ respectively and using the formula $\mathscr{P}_{\mathrm{fin}}(X^{\wedge}) = \mathscr{P}_{\mathrm{fin}}(X)^{\wedge}$, we can replace the universal quantifier in $(*)$ over finite subsets of $X^{\wedge}$ within $\mathbb{V}^{(\mathbb{B})}$ by the external infimum over $\theta \in \mathscr{P}_{\mathrm{fin}}(X)$ and get

$$\mathbb{V}^{(\mathbb{B})} \models \sum_{x\in\theta^{\wedge}} \|\mathscr{T} x\| \leqslant C^{\wedge} \Bigg\| \sum_{x\in\theta^{\wedge}} |x| \Bigg\|. \tag{$**$}$$

Recall that if $\mathbb{Q}$ is the rationals then $\mathbb{Q}^{\wedge}$ may be considered as the internal rationals. Denote by $B(\mathscr{Y})$ the unit ball of $\mathscr{Y}$. Given $0 < \varepsilon \in \mathbb{R}$ and $\theta \in \mathscr{P}_{\mathrm{fin}}(X)$ we have the sentence that is a formal presentation of the fact that $\|\mathscr{T} x\| \leqslant r_x \leqslant (1+\varepsilon)\|\mathscr{T} x\|$ for a suitable rational r_x:

$$(\forall\, x \in \theta^{\wedge})(\exists\, r_x \in \mathbb{Q}^{\wedge})(r_x \leqslant (1+\varepsilon^{\wedge})\|\mathscr{T} x\|) \wedge (\mathscr{T} x \in r_x B(\mathscr{Y})).$$

Replacing quantifiers by infimum over θ and supremum over $\mathbb{Q}$ we deduce that for every $x \in \theta$ there are a countable partition of unity $(\pi_{x,k})$ and a sequence of rationals $(r_{x,k})$ such that

$$\pi_{x,k} \leqslant [\![(r_{x,k}^{\wedge} \leqslant (1+\varepsilon^{\wedge})\|\mathscr{T} x^{\wedge}\|) \wedge (\mathscr{T} x^{\wedge} \in r_{x,k}^{\wedge} B(\mathscr{Y}))]\!] \quad (k \in \mathbb{N}).$$

Let (π_k) be a common refinement of the finite collection of partitions of unity $(r_{x,k})$ $(x \in \theta)$. Then for every $x \in \theta$ there is $k(x) \in \mathbb{N}$ such that

$$\begin{gathered} \pi_k \leqslant [\![r_{x,k(x)}^{\wedge} \leqslant (1+\varepsilon^{\wedge})\|\mathscr{T} x^{\wedge}\|]\!], \\ \pi_k \leqslant [\![\mathscr{T} x^{\wedge} \in r_{x,k(x)}^{\wedge} B(\mathscr{Y})]\!] \quad (k \in \mathbb{N}). \end{gathered} \tag{$*{*}*$}$$

Since $[\![\mathscr{T}x^\wedge = Tx]\!] = \mathbb{1}$ and $r_{x,k(x)}B(Y) = \big(r^\wedge_{x,k(x)}B(\mathscr{Y})\big){\Downarrow}$, the second relation in $(***)$ implies $\pi_k Tx \in r_{x,k(x)}B(Y)$ or $\|\pi_k Tx\| \leqslant r_{x,k(x)}$. The last inequality together with $(**)$ and the first relation in $(***)$ yields

$$\Big(\sum_{x\in\theta}\|\pi_k Tx\|\Big)^\wedge \leqslant \Big(\sum_{x\in\theta} r_{x,k(x)}\Big)^\wedge$$

$$= \sum_{x\in\theta} r^\wedge_{x,k(x)} \leqslant (1+\varepsilon)^\wedge \sum_{x\in\theta}\|\mathscr{T}x^\wedge\| \leqslant ((1+\varepsilon)C)^\wedge \Big\|\sum_{x\in\theta}|x^\wedge|\Big\|$$

$$= \Big((1+\varepsilon)C\Big\|\sum_{x\in\theta}|x|\Big\|\Big)^\wedge.$$

It follows that for every finite subset $\theta \subset X$ we have

$$\inf_{(\pi_k)\in\mathrm{Prt}_\sigma}\ \sup_{k\in\mathbb{N}} \sum_{x\in\theta}\|\pi_k Tx\| \leqslant (1+\varepsilon)C\Big\|\sum_{x\in\theta}|x|\Big\|.$$

Thus, T is $\mathbb{B}$-summing and $\sigma(T) \leqslant C$, since $\varepsilon > 0$ is arbitrary.

Conversely, assume that $T \in \mathscr{S}_{\mathbb{B}}(X,Y)$ and C is a positive constant in Definition 5.13.1. Them for a finite subset $\theta \subset X$ there is a countable partition of unity (π_k) in $\mathbb{B}$ such that

$$\bigvee_{k\in\mathbb{N}}\sum_{x\in\theta}\|\!|\pi_k Tx|\!\| \leqslant \bigvee_{k\in\mathbb{N}}\sum_{x\in\theta}\|\pi_k Tx\|\pi_k\mathbb{1} \leqslant C\Big\|\sum_{x\in\theta}|x|\Big\|\mathbb{1}.$$

Using the definition of $\mathscr{T}$, we deduce from the last inequality

$$\pi_k \leqslant \Big[\!\Big[\sum_{x\in\theta}\|\mathscr{T}x^\wedge\| = \sum_{x\in\theta}\|\!|Tx|\!\|\Big]\!\Big] \wedge [\![(\forall\, x\in\theta)Tx = \pi_k Tx]\!]$$

$$\leqslant \Big[\!\Big[\sum_{x\in\theta}\|\mathscr{T}x^\wedge\| = \sum_{x\in\theta}\|\!|\pi_k Tx|\!\| \leqslant C^\wedge\Big\|\sum_{x\in\theta}|x^\wedge|\Big\|\Big]\!\Big].$$

Finally, for every $\theta \in \mathscr{P}_{\mathrm{fin}}(X)$ we have

$$\mathbb{1} = \bigvee_{k\in\mathbb{N}}\pi_k \leqslant \Big[\!\Big[\sum_{x\in\theta}\|\mathscr{T}x^\wedge\| \leqslant C^\wedge\Big\|\sum_{x\in\theta}|x^\wedge|\Big\|\Big]\!\Big]$$

and so we arrive at $(*)$, which implies that $\mathscr{T} \in \mathscr{S}(\mathscr{X},\mathscr{Y})$ and $[\![\varsigma(\mathscr{T}) \leqslant C^\wedge]\!] = \mathbb{1}$. $\triangleright$

5.13.7. Corollary. *Let X be a Banach lattice and let Y be a $\mathbb{B}$-cyclic Banach lattice. An operator $T \in \mathscr{L}_{\mathbb{B}}(X,Y)$ is $\mathbb{B}$-summing with*

$\sigma(T) \leqslant C$ if and only if there exists $\lambda \in \Lambda$ such that $\|\lambda\|_\infty \leqslant C$ and for every finite collection $x_1, \dots, x_n \in X$ we have

$$\sum_{\imath=1}^{n} |Tx_\imath| \leqslant \lambda \left\| \sum_{\imath=1}^{n} |x_\imath| \right\|.$$

5.13.8. Theorem. *Let X be a Banach lattice and let Y be a $\mathbb{B}$-cyclic Banach lattice. For $T \in \mathscr{L}_{\mathbb{B}}(X, Y)$ the following are equivalent:*

(1) *T is $\mathbb{B}$-summing and $\sigma(T) \leqslant C$.*

(2) *There exists a linear operator $S \in \mathscr{L}(X, \Lambda)$ such that $\|S\| \leqslant C$ and $\|\pi T x\| \leqslant \|\pi S(|x|)\|_\infty$ for all $x \in X$ and $\pi \in \mathbb{P}(\Lambda)$.*

(3) *There exist an injective Banach lattice L, a lattice homomorphism $T_1 \in \mathscr{L}(X, L)$ with $\mathbb{B}$-dense range, and $T_2 \in \mathscr{L}(L, Y)$ such that $\|T_1\| \leqslant C$, $\|T_2\| \leqslant 1$, and $T = T_2 \circ T_1$.*

$\lhd$ (1) $\Longrightarrow$ (2): Assume that $T \in \mathscr{L}_{\mathbb{B}}(X, Y)$ with $\sigma(T) \leqslant C$ and $\mathscr{T}$ is defined as in Theorem 5.13.6. Then $\mathscr{T} \in \mathscr{S}_{\mathbb{B}}(\mathscr{X}, \mathscr{Y})$ and by Schaefer [356, Ch. 4, § 3, Proposition 3.3 (b)] there is $\sigma \in \mathbb{V}^{(\mathbb{B})}$ such that $[\![\sigma \in \mathscr{X}', \|\sigma\| \leqslant C^\wedge$ and $\|\mathscr{T}x\| \leqslant \langle |x|, \sigma\rangle$ for all $x \in \mathscr{X}]\!] = \mathbb{1}$. If S is the bounded descent of σ then $\|S\| \leqslant C$ and $|Tx| \leqslant S(|x|)$ for all $x \in X$. The last inequality is equivalent to $(\forall\, \pi \in \mathbb{P}(\Lambda))\, \|\pi T x\| \leqslant \|\pi S(|x|)\|_\infty$.

(2) $\Longrightarrow$ (3): Using Theorem 5.12.11 with $\Phi := S$, we only have to put $L := X$, $T_1 := \iota : X \to L$, and define $T_2 : L \to Y$ by $T_2 x := \lim_{\varepsilon \to 0} T_2 x_\varepsilon$ and $\pi_\xi T_2 x_\varepsilon = \pi_\xi T \iota(x_\xi)$ $(\xi \in \Xi)$. Evidently, by 5.12.11 (2, 3) we have $\|T_1\| \leqslant C$, $T_2 \in \mathscr{L}_{\mathbb{B}}(L, Y)$ and $\|T_2\| \leqslant 1$. Moreover, $T = T_2 \circ \iota = T_2 \circ T_1$ by definition and $T_1(X)$ is $\mathbb{B}$-dense in L by 5.12.11 (3).

(3) $\Longrightarrow$ (1): Let $T = T_2 \circ T_1$ be a factorization claimed in (3). Observe that the relation $|Tx| \leqslant S(|x|)$ $(x \in X)$ implies $|T_2 u| \leqslant |u|$ $(u \in L)$. For every finite collection $x_1, \dots, x_n \in X_+$ we have

$$\sum_{\imath=1}^{n} |T_2 \circ T_1 x_\imath| \leqslant \sum_{\imath=1}^{n} |T_1 x_\imath| = \sum_{\imath=1}^{n} \Phi \circ T_1 x_\imath$$

$$= \Phi \circ T_1 \left(\sum_{\imath=1}^{n} x_\imath \right) \leqslant C \left\| \sum_{\imath=1}^{n} x_\imath \right\| \mathbb{1}$$

and (1) follows from Corollary 5.13.7. $\rhd$

5.13.9. Corollary. *Let X_0 be a Banach sublattice of a Banach lattice X and let Y be a $\mathbb{B}$-cyclic Banach space. If $T_0 \in \mathscr{S}_{\mathbb{B}}(X_0, Y)$ then T_0 admits an extension $T \in \mathscr{S}_{\mathbb{B}}(X, Y)$ with $\sigma(T_0) = \sigma(T)$.*

5.13.10. Theorem. *Let X be a Banach lattice and let Y be a $\mathbb{B}$-cyclic Banach lattice. The following are equivalent:*

(1) *$\mathscr{S}_{\mathbb{B}}(X,Y)$ is an injective Banach lattice with a Boolean algebra of M-projections isomorphic to $\mathbb{B}$.*

(2) *X is an AM-space and Y is an injective Banach lattice with $\mathbb{B} = \mathbb{M}(Y)$.*

$\lhd$ In order to ensure the claim, we interpret in $\mathbb{V}^{(\mathbb{B})}$ the corresponding result for cone absolutely summing operators (due to Schlotterbeck; see Schaefer [356, Ch. 4, Proposition 4.5]) saying that $\mathscr{S}(\mathscr{X},\mathscr{Y})$ is an AL-space if and only if $\mathscr{X}$ is an AM-space and $\mathscr{Y}$ is an AL-space. By Theorems 5.12.1 (3) and 5.13.6 $\mathscr{S}_{\mathbb{B}}(X,Y)$ is an injective Banach lattice with $\mathbb{M}(\mathscr{S}_{\mathbb{B}}(X,Y))$ isomorphic to $\mathbb{B}$ if and only if $[\![\mathscr{S}(\mathscr{X},\mathscr{Y})$ is an AL-space$]\!] = \mathbb{1}$. Thus, the latter is equivalent to the conjunction of the two assertions: $[\![\mathscr{X}$ is an AM-space$]\!] = \mathbb{1}$ and $[\![\mathscr{Y}$ is an AL-space$]\!] = \mathbb{1}$. The claim is immediate from Theorem 5.12.1 (1, 2). $\rhd$

5.14. Variations on the Theme

In this section we sketch some further applications of the Boolean valued approach to nonassociative Radon–Nikodým type theorems, integration with respect to a measure taking values in a Dedekind complete vector lattice, and transfer in harmonic analysis.

5.14.A. The Radon–Nikodým Theorem for JB-Algebras

5.14.A.1. Let A be a vector space over some field $\mathbb{F}$. Say that A is a *Jordan algebra*, if there is given a (generally) nonassociative binary operation $A \times A \ni (x,y) \mapsto xy \in A$ on A, called *multiplication* and satisfying the following for all $x, y, z \in A$ and $\alpha \in \mathbb{F}$:

(1) $xy = yx$;

(2) $(x+y)z = xz + yz$;

(3) $\alpha(xy) = (\alpha x)y$;

(4) $(x^2 y)x = x^2(yx)$.

An element e of a Jordan algebra A is a *unit element* or a *unit* of A, if $e \neq 0$ and $ea = a$ for all $a \in A$.

5.14.A.2. Recall that a *JB-algebra* A is simultaneously a real Banach space and a Jordan algebra with unit $\mathbb{1}$ such that

(1) $\|xy\| \leqslant \|x\| \cdot \|y\| \quad (x, y \in A)$,

(2) $\|x^2\| = \|x\|^2 \quad (x \in A)$,

(3) $\|x^2\| \leqslant \|x^2 + y^2\| \quad (x, y \in A)$.

The set $A_+ := \{x^2 : x \in A\}$, presenting a convex cone, determines the structure of an ordered vector space on A such that the unit $\mathbb{1}$ of the algebra A serves as a strong order unit, and the order interval $[-\mathbb{1}, \mathbb{1}] := \{x \in A : -\mathbb{1} \leqslant x \leqslant \mathbb{1}\}$ serves as the unit ball. Moreover, the inequalities $-\mathbb{1} \leqslant x \leqslant \mathbb{1}$ and $0 \leqslant x^2 \leqslant \mathbb{1}$ are equivalent.

The intersection of all maximal associative subalgebras of A is called the *center* of A and denoted by $\mathscr{Z}(A)$. The element a belongs to $\mathscr{Z}(A)$ if and only if $(ax)y = a(xy)$ for arbitrary $x, y \in A$. If $\mathscr{Z}(A) = \mathbb{R} \cdot \mathbb{1}$, then A is said to be a *JB-factor*. The center $\mathscr{Z}(A)$ is an associative *JB*-algebra, and such an algebra is isometrically isomorphic to the real Banach algebra $C(Q)$ of continuous functions on some compact space Q.

5.14.A.3. The idempotents of a *JB*-algebra are also called *projections*. The set of all projections $\mathbb{P}(A)$ forms a complete lattice with the order defined as $\pi \leqslant \rho \iff \pi \circ \rho = \pi$. The sublattice of *central projections* $\mathbb{P}_c(A) := \mathbb{P}(A) \cap \mathscr{Z}(A)$ is a Boolean algebra. Given a complete Boolean algebra $\mathbb{B}$ denote by $\Lambda(\mathbb{B})$ a unital Dedekind complete AM-space with $\mathbb{B} \simeq \mathbb{P}(\Lambda(\mathbb{B}))$ (which is unique up to lattice isometry). Assume that $\mathbb{B}$ is a subalgebra of the Boolean algebra $\mathbb{P}_c(A)$ or, equivalently, $\Lambda(\mathbb{B})$ is a subalgebra of the center $\mathscr{Z}(A)$. Then we say that A is a $\mathbb{B}$-*JB-algebra* if, for every partition of unity $(e_\xi)_{\xi \in \Xi}$ in $\mathbb{B}$ and every family $(x_\xi)_{\xi \in \Xi}$ in A, there exists a unique $\mathbb{B}$-mixture $x := \operatorname{mix}_{\xi \in \Xi}(e_\xi x_\xi)$; i.e., the only element $x \in A$ such that $e_\xi x_\xi = e_\xi x$ for all $\xi \in \Xi$. If $\Lambda(\mathbb{B}) = \mathscr{Z}(A)$, then a $\mathbb{B}$-*JB*-algebra is also referred to as *centrally extended JB-algebra.*

The unit ball of a $\mathbb{B}$-*JB*-algebra is closed under $\mathbb{B}$-mixing. Consequently, each $\mathbb{B}$-*JB*-algebra is a $\mathbb{B}$-cyclic Banach space.

5.14.A.4. Theorem. *The restricted descent of a JB-algebra within* $\mathbb{V}^{(\mathbb{B})}$ *is a* $\mathbb{B}$-*JB-algebra. Conversely, for every* $\mathbb{B}$-*JB-algebra* A *there exists a unique (up to isomorphism) JB-algebra* $\mathscr{A}$ *within* $\mathbb{V}^{(\mathbb{B})}$ *whose restricted descent is isometrically* $\mathbb{B}$*-isomorphic to* A*. Moreover,* $[\![\mathscr{A}$ *is a JB-factor*$]\!] = \mathbb{1}$ *if and only if* $\Lambda(\mathbb{B}) = \mathscr{Z}(A)$.

$\lhd$ See Kusraev and Kutateladze [249, Theorem 12.7.6] and Kusraev [226, Theorem 3.1]. $\rhd$

5.14.A.5. Now we give two applications of the above Boolean valued representation result to $\mathbb{B}$-*JB*-algebras. Theorems 5.14.A.7 and 5.14.A.11

below appear by transferring the corresponding facts of the theory of *JB*-algebras.

Let A be a $\mathbb{B}$-*JB*-algebra and put $\Lambda := \Lambda(\mathbb{B})$. An operator $\Phi \in A^{\#}$ is called a Λ-*valued state* if $\Phi \geqslant 0$ and $\Phi(\mathbb{1}) = \mathbb{1}$. A state Φ is said to be *normal* if, for every increasing net (x_α) in A satisfying $x := \sup x_\alpha$, we have $\Phi(x) = o\text{-}\lim \Phi(x_\alpha)$. If $\mathscr{A}$ is the Boolean valued representation of A, then the ascent $\varphi := \Phi\!\uparrow$ is a bounded linear functional on $\mathscr{A}$ by Theorem 5.8.12. Moreover, φ is positive and order continuous; i.e., φ is a normal state on $\mathscr{A}$. The converse is also true: if $[\![\varphi$ is a normal state on $\mathscr{A}]\!] = \mathbb{1}$, then the restriction of the operator $\varphi\!\downarrow$ to A is a Λ-valued normal state. Now we will characterize $\mathbb{B}$-*JB*-algebras that are $\mathbb{B}$-dual spaces. Toward this end, it suffices to give Boolean valued interpretation for the following result.

5.14.A.6. Theorem. *A JB-algebra is a dual Banach space if and only if it is monotone complete and has a separating set of normal states.*

$\lhd$ See Shultz [364, Theorem 2.3]. $\rhd$

5.14.A.7. Theorem. *Let $\mathbb{B}$ be a complete Boolean algebra and let Λ be a Dedekind complete unital AM-space with $\mathbb{B} \simeq \mathbb{P}(\Lambda)$. A $\mathbb{B}$-JB-algebra A is a $\mathbb{B}$-dual space if and only if A is monotone complete and admits a separating set of Λ-valued normal states. If one of these equivalent conditions holds, then the part of $A^{\#}$ consisting of order continuous operators serves as a $\mathbb{B}$-predual space of A.*

$\lhd$ See Kusraev and Kutatelaze [249, Theorem 12.8.5] and Kusraev [226, Theorem 4.2]. $\rhd$

5.14.A.8. An algebra A satisfying one of the equivalent conditions 5.14.A.7 is called a $\mathbb{B}$-JBW-*algebra*. If, moreover, $\mathbb{B}$ coincides with the set of all central projections, then A is said to be a $\mathbb{B}$-JBW-*factor*. It follows from Theorems 5.14.A.4 and 5.14.A.7 that A is a $\mathbb{B}$-JBW-algebra ($\mathbb{B}$-JBW-factor) if and only if its Boolean valued representation $\mathscr{A} \in \mathbb{V}^{(\mathbb{B})}$ is a JBW-algebra (JBW-factor).

A mapping $\Phi : A_+ \to \Lambda \cup \{+\infty\}$ is a (Λ-valued) *weight* if the following are satisfied (under the assumptions that $\lambda + (+\infty) := +\infty + \lambda := +\infty$, $\lambda \cdot (+\infty) =: \lambda$ for all $\lambda \in \Lambda$, while $0 \cdot (+\infty) := 0$ and $+\infty + (+\infty) := +\infty$):

(1) $\Phi(x + y) = \Phi(x) + \Phi(y)$ for all $x, y \in A_+$;

(2) $\Phi(\lambda x) = \lambda \Phi(x)$ for all $x \in A_+$ and $\lambda \in \Lambda_+$.

A weight Φ is said to be a *trace* provided that

(3) $\Phi(x) = \Phi(U_s x)$ for all $x \in A_+$ and $s \in A$ with $s^2 = \mathbb{1}$.

Here, U_a is the operator from A to A defined for a given $a \in A$ as $U_a : x \mapsto 2a(ax) - a^2\ (x \in A)$. This operator is positive; i.e., $U_a(A_+) \subset A_+$. If $a \in \mathscr{Z}(A)$, then $U_a x = a^2 x\ (x \in A)$.

A weight (trace) Φ is called *normal* if $\Phi(x) = \sup_\alpha \Phi(x_\alpha)$ for every increasing net (x_α) in A_+ with $x = \sup_\alpha x_\alpha$; *semifinite* if there exists an increasing net (a_α) in A_+ with $\sup_\alpha a_\alpha = \mathbb{1}$ and $\Phi(a_\alpha) \in \Lambda$ for all α; and *bounded* if $\Phi(\mathbb{1}) \in \Lambda$. Given two Λ-valued weights Φ and Ψ on A, say that Φ is dominated by Ψ if there exists $\lambda \in \Lambda_+$ such that $\Phi(x) \leqslant \lambda\Psi(x)$ for all $x \in A_+$.

5.14.A.9. We need a few additional remarks on descents and ascents. Fix $+\infty \in \mathbb{V}^{(\mathbb{B})}$. If $\Lambda = \mathscr{R}{\Downarrow}$ and $\Lambda^{\mathfrak{u}} = \mathscr{R}{\downarrow}$ then

$$(\Lambda^{\mathfrak{u}} \cup \{+\infty\}){\uparrow} = (\Lambda \cup \{+\infty\}){\uparrow} = \Lambda{\uparrow} \cup \{+\infty\}{\uparrow} = \mathscr{R} \cup \{+\infty\}.$$

At the same time, $\Lambda^\star := (\mathscr{R} \cup \{+\infty\}){\downarrow} = \operatorname{mix}(\mathscr{R}{\downarrow} \cup \{+\infty\})$ consists of all elements of the form $\lambda_\pi := \operatorname{mix}(\pi\lambda, \pi^\perp(+\infty))$ with $\lambda \in \Lambda^{\mathfrak{u}}$ and $\pi \in \mathbb{P}(\Lambda)$. Thus, $\Lambda^{\mathfrak{u}} \cup \{+\infty\}$ is a proper subset of $\Lambda^\star$, since $x_\pi \in \Lambda \cup \{+\infty\}$ if and only if $\pi = 0$ or $\pi = I_\Lambda$.

Assume now that $A = \mathscr{A}{\downarrow}$ with $\mathscr{A}$ a *JB*-algebra within $\mathbb{V}^{(\mathbb{B})}$ and $\mathbb{B}$ isomorphic to $\mathbb{P}(A)$. Every bounded weight $\Phi : A \to \Lambda$ is evidently extensional: $b := [\![x = y]\!]$ implies $bx = by$, which in turn yields $b\Phi(x) = \Phi(bx) = \Phi(by) = b\Phi(y)$ or, equivalently, $b \leqslant [\![\Phi(x) = \Phi(y)]\!]$. But an unbounded weight may fail to be extensional. Indeed, if $\Phi(x_0) = +\infty$ and $\Phi(x) \in \Lambda$ for some $x_0, x \in A$ and $b \in \mathbb{P}(A)$ then

$$\Phi(\operatorname{mix}(bx, b^\perp x_0)) = \operatorname{mix}(b\Phi(x), b^\perp(+\infty)) \notin \Lambda \cup \{+\infty\}.$$

Given a semifinite weight Φ on A, we define its extensional modification $\widehat{\Phi} : A \to \Lambda^\star$ as follows. If $\Phi(x) \in \Lambda$ we put $\widehat{\Phi}(x) := \Phi(x)$. If $\Phi(x) = +\infty$ then $x = \sup(D)$ with $D := \{a \in A : 0 \leqslant a \leqslant x,\ \Phi(a) \in \Lambda\}$ by semifiniteness. Let b stand for the greatest element of $\mathbb{P}(\Lambda)$ such that $\Phi(bD)$ is order bounded in $\Lambda^{\mathfrak{u}}$ and put $\lambda := \sup(\Phi(bD))$. We define $\widehat{\Phi}(x)$ as $\lambda_b = \operatorname{mix}(b\lambda, b^\perp(+\infty))$; i.e., $b\widehat{\Phi}(x) = \lambda$ and $b^\perp\widehat{\Phi}(x) = b^\perp(+\infty)$. It is not difficult to check that $\widehat{\Phi}$ is extensional. Thus, for $\varphi := \widehat{\Phi}{\uparrow}$ we have $[\![\varphi : \mathscr{A} \to \mathscr{R} \cup \{+\infty\}]\!] = \mathbb{1}$ and, according to 1.6.6, $\widehat{\Phi} = \varphi{\downarrow} \neq \Phi$. But if we define $\varphi{\Downarrow}$ as $\varphi{\Downarrow}(x) = \varphi{\downarrow}(x)$ whenever $\varphi{\downarrow}(x) \in \Lambda$ and $\varphi{\Downarrow}(x) = +\infty$ otherwise, then $\Phi = (\widehat{\Phi}{\uparrow}){\Downarrow}$.

5.14.A.10. Theorem. *Let $\mathscr{A}$ be a JBW-algebra and let τ be a normal semifinite real-valued trace on $\mathscr{A}$. For each real-valued weight φ*

on $\mathscr{A}$ dominated by τ there exists a unique positive element $h \in \mathscr{A}$ such that $\varphi(a) = \tau(U_{h^{1/2}}a)$ for all $a \in \mathscr{A}_+$. Moreover, φ is bounded if and only if $\tau(h)$ is finite and φ is a trace if and only if h is a central element of $\mathscr{A}$.

$\triangleleft$ This fact was proved in King [199]. $\triangleright$

5.14.A.11. Theorem. *Let A be a $\mathbb{B}$-JBW-algebra and let T be a normal semifinite Λ-valued trace on A. For each weight Φ on A dominated by T there exists a unique positive $h \in A$ such that $\Phi(x) = \mathrm{T}(U_{h^{1/2}}x)$ for all $x \in A_+$. Moreover, Φ is bounded if and only if $\mathrm{T}(h) \in \Lambda$ and Φ is a trace if and only if h is a central element of A.*

$\triangleleft$ We present a sketch of the proof. Taking into consideration the remarks in 5.14.A.9, we put $\varphi = \widehat{\Phi}{\uparrow}$ and $\tau = \mathrm{T}{\uparrow}$. Then within $\mathbb{V}^{(\mathbb{B})}$ the following hold: τ is a semifinite normal real-valued trace on $\mathscr{A}$ and φ is real-valued weight on $\mathscr{A}$ dominated by τ. By transfer we can apply Theorem 5.14.A.10 and find $h \in \mathscr{A}$ such that $\varphi(x) = \tau(U_{h^{1/2}}x)$ for all $x \in \mathscr{A}_+$. Actually, $h \in A$ and $\varphi{\Downarrow}(x) = \tau{\Downarrow}(U_{h^{1/2}}x)$ for all $x \in A_+$. It remains to note that $\Phi = \varphi{\Downarrow}$ and $\mathrm{T} = \tau{\Downarrow}$. The details of the proof are left to the reader. $\triangleright$

5.14.B. Vector Measures and Integrals

5.14.B.1. Let Z be a universally σ-complete vector lattice with unit $\mathbb{1}$ and let Y be an arbitrary vector lattice. Fix a subalgebra $\mathscr{A}$ of the σ-complete Boolean algebra $\mathbb{C}(\mathbb{1})$ of all components of $\mathbb{1}$ in Z. A *Y-valued measure* on $\mathscr{A}$ is a mapping $\mu : \mathscr{A} \to Y \cup \{+\infty\}$ such that $\mu(\mathscr{A}) \subset Y_+ \cup \{+\infty\}$, $\mu(\mathbb{0}) = 0$ and

$$\mu\Bigg(\bigvee_{n=1}^{+\infty} a_n\Bigg) = o\text{-}\sum_{n=1}^{+\infty} \mu(a_n) := \bigvee_{n=1}^{+\infty} \sum_{k=1}^{n} \mu(a_k)$$

for an arbitrary disjoint sequence (a_n) in $\mathscr{A}$. Here, $\bigvee M$ stands for the supremum in Y whenever it exists and $+\infty$ otherwise. A measure μ is called *semifinite* if $\mu(a) = \sup\{\mu(b) :\ b \in \mathscr{A},\, b \leqslant a,\, \mu(b) \in Y\}$ for all $a \in \mathscr{A}$.

Denote by $S(\mathscr{A})$ the vector sublattice of Z comprising all $\mathscr{A}$-simple elements; i.e., $x \in S(\mathscr{A})$ means that some representation $x = \sum_{k=1}^{n} \alpha_k a_k$ holds with $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ and pairwise disjoint $a_1, \dots, a_n \in \mathscr{A}$. Put

$$I_\mu(x) := \int x\, d\mu := \sum_{k=1}^{n} \alpha_k \mu(a_k) \quad \big(x \in S(\mathscr{A})\big).$$

It is clear that this formula correctly defines the positive linear operator $I_\mu : S(\mathscr{A}) \to Y$ and $|\int x\,d\mu| \leqslant \int |x|\,d\mu$ for all $x \in S(\mathscr{A})$.

5.14.B.2. Let us define the integral at the elements that can be approximated by $\mathscr{A}$-simple elements. We say that a positive element $x \in Z$ is *integrable with respect to* μ or μ-*integrable* if there is an increasing sequence $(x_n)_{n\in\mathbb{N}}$ of positive elements in $S(\mathscr{A})$ o-converging in Z to x and the supremum $\sup_{n\in\mathbb{N}} \int x_n\,d\mu$ exists in Y. For such a sequence (x_n) the sequence of the integrals $(I_\mu(x_n))_{n\in\mathbb{N}}$ is o-fundamental, since

$$\left|\int x_n\,d\mu - \int x_m\,d\mu\right| \leqslant \int |x_n - x_m|\,d\mu \\ \leqslant \bigvee_{k=1}^{\infty}\left\{\int x_k\,d\mu\right\} - \int x_p\,d\mu \underset{p\to\infty}{\longrightarrow} 0,$$

where $p = \min\{m, n\}$. Now we can define the integral of x by putting

$$I_\mu(x) := \int x\,d\mu := \underset{n\to\infty}{o\text{-lim}} \int x_n\,d\mu.$$

The definition is sound. An element $x \in E$ is *integrable* ($=$ μ-integrable) if its positive part x^+ and negative part x^- are both integrable and in this event we put

$$I_\mu(x) := \int x\,d\mu := \int x^+\,d\mu - \int x^-\,d\mu.$$

5.14.B.3. Denote by $\mathscr{L}^1(\mu)$ and $\mathscr{L}^\infty(\mu)$ the set of all integrable elements in Z and the order ideal in $\mathfrak{S}(\mathscr{A})$ generated by the order unit, respectively. It can easily be checked that $\mathscr{L}^1(\mu)$ is an order dense ideal in $\mathfrak{S}(\mathscr{A})$ and $I_\mu : \mathscr{L}^1(\mu) \to Y$ is a positive linear operator. Define in $\mathscr{L}^1(\mu)$ the Y-valued seminorm

$$|x|_1 := \int |x|\,d|\mu| \quad \big(x \in \mathscr{L}^1(\mu)\big).$$

We say that two elements $x, y \in G$ are μ-*equivalent* if there is a unit element $e \in \mathbb{C}(\mathbb{1})$ with $\mu(\mathbb{1} - e) = 0$ and $[e]x = [e]y$. The set $\mathscr{N}(\mu)$ of all elements that are μ-equivalent to zero is a sequentially o-closed order ideal in $\mathscr{L}^1(\mu)$. It follows from the definition of integral that $\mathscr{N}(\mu) = \{x \in \mathscr{L}^1(\mu) : |x|_1 = 0\}$. Define the Dedekind σ-complete vector lattice $L^1(\mu)$ as the quotient space of $\mathscr{L}^1(\mu)$ by the σ-ideal $\mathscr{N}(\mu)$. The coset of $x \in \mathscr{L}^1(\mu)$ will be denoted by $\tilde{x}$. The Y-valued norm on $\mathscr{L}^1(\mu)$ is introduced by setting $|\tilde{x}|_1 := |x|$ $\big(x \in \mathscr{L}^1(\mu)\big)$. Thus, $\big(L^1(\mu), |\cdot|\big)$ is a lattice normed space over Y.

5.14.B.4. Put $\mathscr{A}_\circ := \{a \in \mathscr{A} : \mu(A) \neq +\infty\}$ and $N(\mu) := \{a \in \mathscr{A} : \mu(a) = 0\}$. Let $\tilde{\mathscr{A}}$ and ϕ denote the quotient algebra $\mathscr{A}/N(\mu)$ and the natural quotient mapping $\mathscr{A}/N(\mu) \to \tilde{\mathscr{A}}$, respectively. There is a unique measure $\tilde{\mu} : \tilde{\mathscr{A}} \to Y$ such that $\tilde{\mu} \circ \phi = \mu$. Given a Boolean homomorphism $h : \mathbb{B} := \mathbb{P}(Y) \to \tilde{\mathscr{A}}$, we say that μ is *modular with respect to* h, or h*-modular* if $b\tilde{\mu}(\phi a) = \tilde{\mu}(h(b) \wedge \phi(a))$ for all $a \in \mathscr{A}_\circ$ and $b \in \mathbb{B}$. Clearly, the modularity of μ means that $b\mu(a) = \mu(b' \wedge a')$ for all $a \in \mathscr{A}_\circ$, $a' \in \phi(a)$ and $b' \in h(b)$. Moreover, the modularity of μ amounts to the modularity of $\tilde{\mu}$; i.e., $b\tilde{\mu}(\tilde{a}) = \tilde{\mu}(h(b) \wedge \tilde{a})$ for all $b \in \mathbb{B}$ and $\tilde{a} \in \tilde{\mathscr{A}_\circ}$.

Let $e := \bigvee\{b \in \mathbb{B} : (\forall\, a \in \mathscr{A})\ b\mu(a) = 0\}$. Then $e\mu(\mathscr{A}) = \{0\}$ and $\mu(\mathscr{A}) \subset (\mathbb{1} - e)Y$. Moreover, $b\mu(\mathscr{A}) = \{0\}$ if and only if $h(b) \in N(\mu)$. Thus, h is injective on $[\mathbb{0}, \mathbb{1}-e]$. In the sequel we agree that $\mu(\mathscr{A}_\circ)^{\perp\perp} = Y$ and in this event h is an isomorphic embedding of $\mathbb{B}$ into $\tilde{\mathscr{A}}$.

An h-modular measure μ is said to be *ample* (with respect to h) if for every partition of unity $(b_\xi)_{\xi\in\Xi}$ in $\mathbb{B}$ and an arbitrary family $(a_\xi)_{\xi\in\Xi}$ in $\mathscr{A}$ there exists a unique (up to equivalence) element $a \in \mathscr{A}$ such that $b_\xi\mu(a \triangle a_\xi) = 0$ for all $\xi \in \Xi$. Because of the h-modularity of μ this amounts to saying that $h(b_\xi) \wedge \phi(a) = h(b_\xi) \wedge \phi(a_\xi)$ for all $\xi \in \Xi$. In particular, if μ is ample with respect to h, then h is a complete isomorphism of $\mathbb{B}$ into $\tilde{\mathscr{A}}$.

Say that μ is *Maharam* if μ is semifinite and $\tilde{\mathscr{A}}$ is Dedekind complete. It can easily be checked that a bounded modular measure is Maharam if and only if it is ample. For an unbounded measure only the necessity is true: a modular Maharam measure is ample.

A *vector measure algebra* is a triple $(\mathscr{A}, \mu, Y)$, where $\mathscr{A}$ is a Dedekind σ-complete Boolean algebra and $\mu : \mathscr{A} \to Y \cup \{+\infty\}$ is a strictly positive countably additive measure. If $Y = \mathscr{R}$ we speak of a (scalar) measure algebra $(\mathscr{A}, \mu)$. A measure algebra $(\mathscr{A}, \mu, Y)$ is also called *Maharam* or h*-modular* whenever so is the measure μ.

5.14.B.5. *For the lattice normed space* $\big(L^1(\mu), |\cdot|\big)$ *the following hold:*

(1) $L^1(\mu)$ *is uniformly* Y*-complete.*

(2) $L^1(\mu)$ *is disjointly decomposable if and only if* μ *is modular.*

(3) *$L^1(\mu)$ is a Banach–Kantorovich lattice if and only if μ is Maharam.*

$\lhd$ See Kusraev[228, 6.1.8 and 6.1.9 (3, 4)]. $\rhd$

5.14.B.6. Theorem. *Let $(\mathscr{A}, \mu, Y)$ be an h-modular Maharam measure algebra with $Y := \mathscr{R}{\downarrow}$ and h a Boolean isomorphism from $\mathbb{B} := \mathbb{P}(Y)$ into $\mathscr{A}$. Then there exist $\mathfrak{A}, m \in \mathbb{V}^{(\mathbb{B})}$ such that the following hold:*

(1) *$[\![(\mathfrak{A}, m)$ is a Maharam scalar measure algebra$]\!] = \mathbb{1}$.*

(2) *If $\mu' := m{\Downarrow}$ and $\mathscr{A}' := \mathfrak{A}{\downarrow}$ then $(\mathscr{A}', \mu', Y)$ is a h'-modular Maharam vector measure algebra with h' a Boolean isomorphism from $\mathbb{B}$ into $\mathscr{A}'$.*

(3) *There exists a Boolean isomorphism $\jmath$ from $\mathscr{A}$ onto $\mathscr{A}'$ such that $\mu = \mu' \circ \jmath$ and $h' = \jmath \circ h$.*

$\lhd$ Apply Theorem 1.10.4 with $D := \mathscr{A}$ and put $\mathfrak{A} := \mathscr{D}$, $\mathscr{A}' := D'$, $\jmath := H$, $h' := \imath'$, and $\mu' := \mu \circ \jmath^{-1}$. The h-modularity assumption implies that μ' is h'-modular, which in turn implies the estimate $[\![a_1 = a_2]\!] \leqslant [\![\mu'(a_1) = \mu'(a_2)]\!]$ provided that $\mu'(a_1), \mu'(a_2) \in Y$. Since μ' is semifinite, just as in 5.14.A.9 we define the extensional modification $\widehat{\mu}' : \mathscr{A}' \to \Lambda^\star$ of μ'. Thus, we can define $m := \mu'{\downarrow}$ and ensure that $\mu' = m{\Downarrow}$. Since $(\mathscr{A}', \mu', Y)$ is evidently an h'-modular Maharam vector measure algebra, it can be deduced using the ascending–descending machinery that $(\mathfrak{A}, m)$ is a Maharam scalar measure algebra within $\mathbb{V}^{(\mathbb{B})}$. $\rhd$

To state the next theorem we use the notations from 5.14.B.6.

5.14.B.7. Theorem. *Suppose $L^1(m), I_m \in \mathbb{V}^{(\mathbb{B})}$ have the properties that $[\![L^1(m)$ is a Banach lattice of m-integrable spectral systems from $\mathfrak{S}(\mathscr{A})]\!] = \mathbb{1}$ and $[\![I_m$ is an order continuous linear functional on $L^1(m)$ defined as $I_m : x \mapsto \int x\, dm$ $(x \in L^1(m))]\!] = \mathbb{1}$. Then the following hold:*

(1) *$L^1(m){\downarrow}$ is a Dedekind complete Banach–Kantorovich lattice over Y and $I_m{\downarrow}$ is a strictly positive Maharam operator from $L^1(m){\downarrow}$ to Y.*

(2) *There exists an isometric lattice isomorphism g from $L^1(m){\downarrow}$ onto $L^1(\mu)$ such that $(I_m{\downarrow}) \circ g = I_\mu$.*

$\lhd$ Since $L^1(m)$ and $L^1(\mu)$ are Dedekind complete, we can reduce demonstration to the case of finite measures by decomposition into a direct sum of bands with order unit. So we will assume below that m and μ take values in $\mathscr{R}$ and $Y = \mathscr{R}{\downarrow}$ respectively. Since $L^1(m)$ is a Banach lattice within $\mathbb{V}^{(\mathbb{B})}$, we see that $L^1(m){\downarrow}$ is a Banach–Kantorovich lattice (cp. Kusraev [228, Chapter 2]). Let $x\colon \mathbb{R} \to \mathscr{A}$ be a summable spectral system in $\mathscr{A}$ and $y := x{\uparrow}$. Then $[\![y\colon \mathbb{R}^{\wedge} \to \mathfrak{A}$ is an increasing mapping satisfying $\bigvee y(\mathbb{R}^{\wedge}) = 1$ and $\bigwedge y(\mathbb{R}^{\wedge}) = 0]\!] = 1$. Define $\bar{x} \in \mathbb{V}^{(\mathbb{B})}$ by the formula

$$\bar{x}(t) = \bigvee \big\{y(s)\colon s \in \mathbb{R}^{\wedge},\, s < t\big\} \quad (t \in \mathscr{R}).$$

Clearly, $\bar{x}$ is a spectral system in $\mathfrak{A}$ and $[\![\bar{x}(t^{\wedge}) = x(t)]\!] = \mathbb{1}$ $(t \in \mathbb{R}^{\wedge})$. Show that $[\![I_\mu(x) = I_m(\bar{x})]\!] = \mathbb{1}$ for all $x \in L^1(\mu)$. To this end, take $\varepsilon > 0$ and let ν_n and t_n $(\pm n \in \omega)$ enjoy the conditions

$$-\infty \longleftarrow \ \dots \nu_{-k} < \dots < \nu_0 < \dots < \nu_k \dots \ \to +\infty,$$
$$t_n \in [\nu_n, \nu_{n+1})\ (\pm n \in \omega), \qquad \sup_{\pm n \in \omega} (\nu_{n+1} - \nu_n) < \varepsilon.$$

Define $\sigma, \sigma_n \in Y = \mathscr{R}{\downarrow}$ by

$$\sigma := \sum_{-\infty}^{\infty} t_n \mu(x(\nu_{n+1}) - x(\nu_n)),$$

$$\sigma_n := \sum_{k=-n}^{n} t_k \mu(x(\nu_{k+1}) - x(\nu_k)).$$

Without loss of generality, we can assume that $e := \mu(\mathbb{1})$ is an order unit of Y. It is easy that σ_n is an integral sum for $I_m(\bar{x})$ within $\mathbb{V}^{(\mathbb{B})}$; i.e.,

$$\left[\!\!\left[\sigma_n = \sum_{k=-n^{\wedge}}^{n^{\wedge}} t_k m(\bar{x}(\nu_{k+1}^{\wedge}) - \bar{x}(\nu_k^{\wedge}))\right]\!\!\right] = \mathbb{1}.$$

Since $\sigma = \operatorname*{o\text{-}lim}_{n\to\infty} \sigma_n$; we have by 2.4.5 that $[\![\sigma = \lim_{n\to\infty} \sigma_n]\!] = \mathbb{1}$, implying that $[\![\,|\sigma - I_m(\bar{x})| < \varepsilon^{\wedge}]\!] = \mathbb{1}$. Moreover, $|I_\mu(x) - \sigma| \leqslant \varepsilon e$, and so

$$[\![\,|I_m(\bar{x}) - I_\mu(x)| \leqslant 2\,\varepsilon^{\wedge} e\,]\!] = \mathbb{1}.$$

Since $\varepsilon > 0$ is arbitrary, we see that $[\![I_m(\bar{x}) = I_\mu(x)]\!] = [\![\bar{x} \in L^1(m)]\!] = \mathbb{1}$. If $g\colon L^1(\mu) \to L^1(m){\downarrow}$ is defined as $g(x) = \bar{x}$, then $(I_m{\downarrow}) \circ g = I_\mu$. Given $x \in L^1(\mu)$, we see within $\mathbb{V}^{(\mathbb{B})}$ that

$$\|\bar{x}\| = I_m(|\bar{x}|) = I_m(\,\overline{|x|}\,) = I_\mu(|x|) = \boldsymbol{|} x \boldsymbol{|}.$$

Hence, $[\![\,\|g(x)\| = \boldsymbol{|} x \boldsymbol{|}\,]\!] = \mathbb{1}$ and we conclude that g is an isometry. The linearity of g and the preservation of the meets and joins of nonempty finite sets under g are proved by similar arguments. Thus, we will show only that g is additive. To this end, take another spectral system $y \in L^1(\mu)$. Recall that

$$(x+y)(r) = \bigvee \{x(s) \wedge y(t) \colon s, t \in \mathbb{R},\ s+t=r\}.$$

Observe also that $\overline{x+y}(r^\wedge) = (x+y)(r)$, $\bar{x}(r^\wedge) = x(r)$ and $\bar{y}(r^\wedge) = y(r)$ $(r \in \mathbb{R})$. So we have the following within $\mathbb{V}^{(\mathbb{B})}$:

$$\begin{aligned} \overline{x+y}(r^\wedge) &= (x+y)(r) \\ &= \bigvee \{x(s) \wedge y(t) \colon s, t \in \mathbb{R},\ s+t=r\} \\ &= \bigvee \{\bar{x}(s^\wedge) \wedge \bar{y}(t^\wedge) \colon s, t \in \mathbb{R},\ s+t=r\} \\ &= \bigvee \{\bar{x}(s) \wedge \bar{y}(t) \colon s, t \in \mathbb{R}^\wedge,\ s+t=r^\wedge\} = (\bar{x}+\bar{y})(r^\wedge). \end{aligned}$$

Consequently, $\overline{x+y}$ and $\bar{x}+\bar{y}$ coincide on the dense subset $\mathbb{R}^\wedge \subset \mathscr{R}$. Since each spectral system is left continuous, $\overline{x+y} = \bar{x}+\bar{y}$.

Take an arbitrary $z \in L_1(m){\downarrow}$ and put $y := z|_{\mathbb{R}^\wedge}$; i.e., $y \in \mathbb{V}^{(\mathbb{B})}$ is the restriction of the spectral system $z \colon \mathscr{R} \to \mathfrak{A}$ to $\mathbb{R}^\wedge$. If $x := y{\downarrow}$, then $x \colon \mathbb{R} \to \mathscr{A}$ is a spectral system in $\mathscr{A}$. It is easy that $\bar{x} = z$. Moreover, the m-summability of z implies the μ-summability of x. Hence, $[\![I_m(z) = I_\mu(x)]\!] = \mathbb{1}$. $\triangleright$

5.14.B.8. Theorem. *A Banach lattice X is injective if and only if there exists a modular Maharam vector measure algebra $(\mathscr{A}, \mu, \Lambda)$, where Λ is a Dedekind complete unital AM-space with $\mathbb{P}(\Lambda)$ isomorphic to $\mathbb{M}(X)$, such that X is lattice $\mathbb{P}(\Lambda)$-isometric to $L^1(\mu)$. Moreover, X admits a Banach f-module structure over $\mathscr{Z}(\Lambda)$ and the lattice $\mathbb{P}(\Lambda)$-isometry between X and $L^1(m)$ is an f-module isomorphism too.*

$\triangleleft$ This is immediate from Theorems 5.7.9, 5.12.1 and 3.14.B.7. $\triangleright$

5.14.C. Transfer in Harmonic Analysis

In what follows, G is a locally compact abelian group, τ its topology, $\tau(0)$ a basic neighborhood system of 0 in G, and G' stands for the dual group. Then G is also the dual group of G' and we write $\langle g, \gamma \rangle := \gamma(g)$ $(g \in G,\ \gamma \in G')$. We consider G as an additive group.

5.14.C.1. By restricted transfer $G^\wedge$ is a group within $\mathbb{V}^{(\mathbb{B})}$. At the same time $\tau(0)^\wedge$ may fail to be a topology of $G^\wedge$. But $G^\wedge$ becomes a topological group by defining the basic neighborhood system of $0 := 0^\wedge$ to be $\tau(0)^\wedge$. This topological group is again denoted by $G^\wedge$. Clearly,

$G^\wedge$ may fail to be locally compact. Let U be a neighborhood of 0 such that U is compact. Then U is totally bounded. It follows by restricted transfer that $U^\wedge$ is also totally bounded, since total boundedness can be expressed by a restricted formula. Therefore the completion of $G^\wedge$ is again locally compact. The completion of $G^\wedge$ is denoted by $\mathscr{G}$, and by the above observation $\mathscr{G}$ is a locally compact abelian group within $\mathbb{V}^{(\mathbb{B})}$.

5.14.C.2. Let Y be a Dedekind complete vector lattice and let $Y_{\mathbb{C}}$ be its complexification. A vector-valued function $\varphi : G \to Y$ is said to be *uniformly order continuous* on a set K if

$$\inf_{U\in\tau(0)} \sup\{|\varphi(g_1) - \varphi(g_2)| : \ g_1, g_2 \in K,\ g_1 - g_2 \in U\} = 0.$$

This amounts to saying that φ is order bounded on K and, if $e \in Y$ is an arbitrary upper bound of $\varphi(K)$, then for arbitrary $0 < \varepsilon \in \mathbb{R}$ there exists a partition of unity $(\pi_\alpha)_{\alpha\in\tau(0)}$ in $\mathbb{P}(Y)$ such that $\pi_\alpha|\varphi(g_1) - \varphi(g_2)| \leqslant \varepsilon e$ for all $\alpha \in \tau(0)$ and $g_1, g_2 \in K$, $g_1 - g_2 \in \alpha$. If, in this definition we put $g_2 = 0$, then we arrive at the definition of a mapping φ *order continuous at zero.*

Let us introduce the class of dominated mappings with values in a vector lattice Y. A mapping $\psi : G \to Y_{\mathbb{C}}$ is called *positive definite* if

$$\sum_{\jmath,k=1}^{n} \psi(g_\jmath - g_k)\ c_\jmath \bar{c}_k \geqslant 0$$

for all finite collections $g_1, \dots, g_n \in G$ and $c_1, \dots, c_n \in \mathbb{C}$ $(n \in \mathbb{N})$.

For $n = 1$, the definition readily implies that $\psi(0) \in Y_+$. For $n = 2$, we have the inequality $|\psi(g)| \leqslant \psi(0)$ $(g \in G)$. If we introduce the structure of an f-algebra with unit $\psi(0)$ in the order ideal of Y generated by $\psi(0)$ then, for $n = 3$, from the above definition we can deduce one more inequality

$$|\psi(g_1) - \psi(g_2)|^2 \leqslant 2\psi(0)(\psi(0) - \operatorname{Re}\psi(g_1 - g_2)) \quad (g_1, g_2 \in G).$$

It follows that every positive definite mapping $\psi : G \to Y_{\mathbb{C}}$ o-continuous at zero is order bounded (by the element $\psi(0)$) and uniformly o-continuous. A mapping $\varphi : G \to Y$ is called *dominated* if there exists a positive definite mapping $\psi : G \to Y_{\mathbb{C}}$ such that

$$\left| \sum_{\jmath,k=1}^{n} \varphi(g_\jmath - g_k) c_\jmath \bar{c}_k \right| \leqslant \sum_{\jmath,k=1}^{n} \psi(g_\jmath - g_k) c_\jmath \bar{c}_k$$

for all $g_1, \dots, g_n \in G$, $c_1, \dots, c_n \in \mathbb{C}$, and $n \in \mathbb{N}$. In this case we also say that ψ is a *dominant* of φ. It is easy to show that if $\varphi : G \to Y_{\mathbb{C}}$ has a dominant order continuous at zero then φ is order bounded and uniformly order continuous.

We denote by $\mathfrak{D}(G, Y_{\mathbb{C}})$ the vector space of all dominated mappings from G into $Y_{\mathbb{C}}$ whose dominants are order continuous at zero. We also consider the set $\mathfrak{D}(G, Y_{\mathbb{C}})_+$ of all positive definite mappings from G into $Y_{\mathbb{C}}$. This set is a cone in $\mathfrak{D}(G, Y_{\mathbb{C}})$ and defines the order compatible with the structure of a vector space on $\mathfrak{D}(G, Y_{\mathbb{C}})$. Actually, $\mathfrak{D}(G, Y_{\mathbb{C}})$ is a Dedekind complete complex vector lattice (see 5.14.C.13 below). We also define $\mathfrak{D}(\mathscr{G}, \mathscr{C}) \in \mathbb{V}^{(\mathbb{B})}$ to be the set of functions $\varphi : \mathscr{G} \to \mathscr{C}$ with the property that $[\![\varphi$ has a dominant continuous at zero$]\!] = \mathbb{1}$.

5.14.C.3. *Let $Y = \mathscr{R}{\downarrow}$. For every $\varphi \in \mathfrak{D}(G, Y_{\mathbb{C}})$ there exists a unique $\tilde{\varphi} \in \mathbb{V}^{(\mathbb{B})}$ such that $[\![\tilde{\varphi} \in \mathfrak{D}(\mathscr{G}, \mathscr{C})]\!] = \mathbb{1}$ and $[\![\tilde{\varphi}(x^{\wedge}) = \varphi(x)]\!] = \mathbb{1}$ for all $x \in G$. The mapping $\varphi \mapsto \tilde{\varphi}$ is a linear and order isomorphism from $\mathfrak{D}(G, Y)$ onto $\mathfrak{D}(\mathscr{G}, \mathscr{C}){\downarrow}$.*

5.14.C.4. Define $C_0(G)$ as the space of all continuous complex functions f on G vanishing at infinity. The latter means that for every $0 < \varepsilon \in \mathbb{R}$ there exists a compact set $K \subset G$ such that $|f(x)| < \varepsilon$ for all $x \in G \setminus K$. Denote by $C_c(G)$ the space of all continuous complex functions on G having compact support. Evidently, $C_c(G)$ is dense in $C_0(G)$ with respect to the norm $\| \cdot \|_\infty$.

5.14.C.5. Let us introduce one simple class of majorized or dominated operators. Let X be a normed complex vector space and let Y be a complex Banach lattice. A linear operator $T : X \to Y$ is called *majorized* or *dominated* if T sends the unit ball of X into an order bounded subset of Y. This amounts to saying that there exists $c \in Y_+$ such that $|Tx| \leqslant c\|x\|_\infty$ for all $x \in X$. The set of all dominated operators from X to Y is denoted by $L_m(X, Y)$. If Y is Dedekind complete then the element

$$\boldsymbol{|}T\boldsymbol{|} := \{|Tx| : \ x \in X, \ \|x\| \leqslant 1\}$$

exists and is called the *abstract norm* of T. Moreover, if X is a vector lattice and Y a Dedekind complete vector lattice then $L_m(X, Y)$ is a vector sublattice of $L^{\sim}(X, Y)$.

Given a positive operator $T \in L_m(C_0(G'), Y)$, we can define the mapping $\varphi : G \to Y$ by putting $\varphi(g) = T(\langle g, \cdot \rangle)$ for all $g \in G$, since the function $\gamma \mapsto \langle g, \gamma \rangle$ lies in $C_0(G')$ for every $g \in G$. It is not difficult to ensure that the so-defined mapping φ is order continuous at zero and positive definite. The converse is also true; see 5.14.C.8.

5.14.C.6. Consider a metric space (M, r). The definition of metric space can be written as a bounded formula, say $\varphi(M, r, \mathbb{R})$, so that $[\![\varphi(M^{\wedge}, r^{\wedge}, \mathbb{R}^{\wedge})]\!] = \mathbb{1}$ by restricted transfer. In other words, $(M^{\wedge}, r^{\wedge})$ is a metric space within $\mathbb{V}^{(\mathbb{B})}$. Moreover we consider the internal function $r^{\wedge} : M^{\wedge} \times M^{\wedge} \to \mathbb{R}^{\wedge} \subset \mathscr{R}$ as an $\mathscr{R}$-valued metric on $M^{\wedge}$. Denote by $(\mathscr{M}, \rho)$ the completion of the $(M^{\wedge}, r^{\wedge})$; i.e., $[\![(\mathscr{M}, \rho)$ is a complete metric space and $M^{\wedge}$ is a dense subset of $\mathscr{M}]\!] = \mathbb{1}$ and $[\![r(x)^{\wedge} = \rho(x^{\wedge})]\!] = \mathbb{1}$ for all $x \in M$.

If $(X, \|\cdot\|)$ is a real (or complex) normed vector space then $[\![X^{\wedge}$ is a vector space over the field $\mathbb{R}^{\wedge}$ (or $\mathbb{C}^{\wedge}$) and $\|\cdot\|^{\wedge}$ is a norm on $X^{\wedge}$ with values in $\mathbb{R}^{\wedge} \subset \mathscr{R}]\!] = \mathbb{1}$. So, we will consider $X^{\wedge}$ as an $\mathbb{R}^{\wedge}$-vector space with an $\mathscr{R}$-valued norm within $\mathbb{V}^{(\mathbb{B})}$. Let $\mathscr{X} \in \mathbb{V}^{(\mathbb{B})}$ stand for the (metric) completion of $X^{\wedge}$ within $\mathbb{V}^{(\mathbb{B})}$. It is not difficult to see that $[\![\mathscr{X}$ is a real (complex) Banach space including $X^{\wedge}$ as an $\mathbb{R}^{\wedge}(\mathbb{C}^{\wedge})$-linear subspace$]\!] = \mathbb{1}$, since the metric $(x, y) \mapsto \|x - y\|$ on $X^{\wedge}$ is translation invariant. Clearly, if X is a real (complex) Banach lattice then $[\![\mathscr{X}$ is a real (complex) Banach lattice including $X^{\wedge}$ as an $\mathbb{R}^{\wedge}(\mathbb{C}^{\wedge})$-linear sublattice$]\!] = \mathbb{1}$ (see 5.13.3–5.13.5).

5.14.C.7. Theorem. *Let $Y = \mathscr{C}{\downarrow}$ and let $\mathscr{X}'$ be the topological dual of $\mathscr{X}$ within $\mathbb{V}^{(\mathbb{B})}$. For every $T \in L_m(X, Y)$ there exists a unique $\tau \in \mathscr{X}'{\downarrow}$ such that $[\![\tau(x^{\wedge}) = T(x)]\!] = \mathbb{1}$ for all $x \in X$. The correspondence $T \to \phi(T) := \tau$ defines an isomorphism between the $\mathscr{C}{\downarrow}$-modules $L_m(X, Y)$ and $\mathscr{X}'{\downarrow}$. Moreover, $|T| = |\phi(T)|$ for all $T \in L_m(X, Y)$. If X is a normed lattice then $[\![\mathscr{X}'$ is a Banach lattice $]\!] = \mathbb{1}$, $\mathscr{X}'{\downarrow}$ is a vector lattice and ϕ is a lattice isomorphism.*

$\lhd$ It suffices to settle the case of the real scalars. Apply Kusraev [228, Theorem 8.3.2] to the lattice normed space $X := (X, |\cdot|)$, where $|x| = \|x\|\mathbb{1}$. By [228, Theorem 8.3.4 (1) and Proposition 8.3.4 (2)] the spaces $\mathscr{X}'{\downarrow} := \mathscr{L}^{(\mathbb{B})}(\mathscr{X}, \mathscr{R}){\downarrow}$ and $L_m(X, Y)$ are linearly isometric. To complete the proof, refer to [228, Proposition 5.5.1 (1)]. $\rhd$

5.14.C.8. Theorem. *A mapping $\varphi : G \to Y_{\mathbb{C}}$ is order continuous at zero and positive definite if and only if there exists a unique positive operator $T \in L_m(C_0(G'), Y_{\mathbb{C}})$ such that $\varphi(g) = T(\langle g, \cdot \rangle)$ for all $g \in G$.*

$\lhd$ By transfer, 5.14.C.3, and Theorem 5.14.C.7, we can replace φ and T by their Boolean valued representations $\tilde{\varphi}$ and τ. The norm completion of $C_0(G')^{\wedge}$ within $\mathbb{V}^{(\mathbb{B})}$ coincides with $C_0(\mathscr{G}')$. (This can be proved by the reasoning similar to that in Takeuti [380, Proposition 3.2].) Application of the classical Bochner Theorem (see Loomis [286, Section 36A]) to $\tilde{\varphi}$ and τ yields the desired result. $\triangleright$

5.14.C.9. We now specify the vector integral of 5.14.B for elements of some abstract Dedekind σ-complete vector lattice. Take as a universally σ-complete vector lattice Z the vector lattice $\mathbb{R}^Q$ of all real functions on a nonempty set Q. Let $\mathscr{A}$ be a σ-algebra of subsets of Q; i.e., $\mathscr{A} \subset \mathscr{P}(Q)$. We identify this algebra with the isomorphic algebra of the characteristic functions $\{1_A := \chi_A : A \in \mathscr{A}\}$ so that $S(\mathscr{A})$ is the space of all $\mathscr{A}$-simple functions on Q; i.e., $f \in S(\mathscr{A})$ means that $f = \sum_{k=1}^n \alpha_k \chi_{A_k}$ for some $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ and disjoint $A_1, \dots, A_n \in \mathscr{A}$. Let a measure μ be defined on $\mathscr{A}$ and take values in a Dedekind complete vector lattice Y. We suppose that μ is order bounded. If $f \in S(\mathscr{A})$ then we put by definition

$$I_\mu := \int f \, d\mu = \sum_{k=1}^n \alpha_k \mu(A_k).$$

As was described in 5.14.B, the integral I_μ can be extended to the spaces of μ-summable functions $\mathscr{L}^1(\mu)$ for which the more informative notations $\mathscr{L}^1(Q,\mu)$ and $\mathscr{L}^1(Q,\mathscr{A},\mu)$ are also used. On identifying equivalent functions, we obtain the Dedekind σ-complete vector lattice $L^1(\mu) := L^1(Q,\mu) := L^1(Q,\mathscr{A},\mu)$.

5.14.C.10. Assume now that Q is a topological space. Denote by $\mathscr{F}(Q)$, $\mathscr{K}(Q)$, and $\mathscr{B}(Q)$ the collections of all closed, compact, and Borel subsets of Q. A measure $\mu : \mathscr{B}(Q) \to Y$ is said to be *quasi-Radon* (*quasiregular*) if μ is order bounded and

$$|\mu|(U) = \sup\{|\mu|(K) : \ K \in \mathscr{K}(Q),\ K \subset U\}$$
$$(|\mu|(U) = \sup\{|\mu|(K) : \ K \in \mathscr{F}(Q),\ K \subset U\})$$

for every open set $U \subset Q$. If these equalities are fulfilled for all Borel $U \subset Q$ then we speak about *Radon* and *regular* measures. Say that $\mu = \mu_1 + i\mu_2 : \mathscr{B}(Q) \to Y_{\mathbb{C}}$ has one of the above properties whenever the property is enjoyed by μ_1 and μ_2. We denote by $\mathrm{qca}(Q,Y)$ the vector lattice of all σ-additive quasi-Radon measures on $\mathscr{B}(Q)$ with values in $Y_{\mathbb{C}}$. If Q is locally compact or (even completely regular) then $\mathrm{qca}(Q,Y)$ is a vector lattice; see [228, Theorem 6.2.2]. The variation $|\mu|$ of a $Y_{\mathbb{C}}$-valued (in particular, $\mathbb{C}$-valued) Borel measure μ is defined as the least positive measure $\nu : \mathscr{B}(Q) \to Y$ with $|\mu(A)| \leqslant \nu(a)$ for all $A \in \mathscr{B}(Q)$.

5.14.C.11. Theorem. *Let Y be a real Dedekind complete vector lattice and let Q be a locally compact topological space. Then for each T in $L_m(C_0(Q), Y_{\mathbb{C}})$ there exists a unique measure $\mu := \mu_T \in \operatorname{qca}(Q, Y_{\mathbb{C}})$ such that*

$$T(f) = \int\limits_Q f\,d\mu \quad (f \in C_0(Q)).$$

Moreover, $T \mapsto \mu_T$ is a lattice isomorphism from $L_m(C_0(Q), Y_{\mathbb{C}})$ onto $\operatorname{qca}(Q, Y_{\mathbb{C}})$.

$\triangleleft$ See Kusraev and Malyugin [255, Theorem 2.5]. $\triangleright$

5.14.C.12. Theorem. *Assume that G is a locally compact abelian group, G' is the dual group of G, and Y is a Dedekind complete real vector lattice. For $\varphi : G \to Y_{\mathbb{C}}$ the following are equivalent:*

(1) *φ has a dominant order continuous at zero.*

(2) *There exists a unique measure $\mu \in \operatorname{qca}(G', Y_{\mathbb{C}})$ such that*

$$\varphi(g) = \int_{G'} \chi(g)\,d\mu(\chi) \quad (g \in G).$$

$\triangleleft$ This is immediate from 5.14.C.8 and 5.14.C.11. $\triangleright$

5.14.C.13. Corollary. *The Fourier transform establishes a lattice isomorphism between the space of measures $\operatorname{qca}(G', Y)$ and the space of dominated mappings $\mathfrak{D}(G, Y_{\mathbb{C}})$. In particular, $\mathfrak{D}(G, Y_{\mathbb{C}})$ is a Dedekind complete complex vector lattice.*

5.15. Comments

5.15.1. (1) The fact is well known in the context of lattice normed spaces and operator algebras that the module homomorphisms become linear operators when ascended to a suitable Boolean valued universe; cp. [248, 249]. Here we firstly publish analogous results about Boolean valued representation of bounded homomorphisms of f-modules in Theorem 5.1.4 and Corollary 5.1.6.

(2) Theorem 5.1.10 is the Boolean valued interpretation of a portion of Nakano duality theory (Theorems 5.1.8 and 5.1.9). This result was obtained for the first time by Luxemburg and de Pagter using standard tools [293, Theorem 4.9]. Proposition 5.1.2 is taken also from Luxemburg and de Pagter [293, Lemma 4.4].

5.15.2. (1) In [299]–[302] Maharam created a powerful approach to studying positive operators (also see the survey by Maharam [303]). The concept of Maharam operator and the main ideas of Sections 5.1–5.8 stem from these papers. The concept of interval preserving operator was introduced by Maharam under the name full-valued F-integral (= full-valued function-valued integral). The Maharam idea was that we need full-valuedness for transferring the results of the classical integration theory to operators in function lattices.

(2) Luxemburg in the joint articles with Schep [295] and de Pagter [293] extended some portion of Maharam's theory to the case of positive operators in Dedekind complete vector lattices. The terms *Maharam property* and *Maharam operator* were introduced by Luxemburg and Schep in [295] and by Kusraev in [217] (see more details in Kusraev [228]). The Maharam ideas were transplanted to the environment of convex operators by Kusraev [217, 219]. This theory is presented in Kusraev and Kutateladze [247].

(3) Theorem 5.2.8 states that each Maharam operator is an interpretation of some order continuous linear functional in an appropriate Boolean valued model. This Boolean valued status of the concept of Maharam operator was announced in [217] and proved in [220] by Kusraev (see also [228]). It is worth emphasizing that Maharam's approach is notable within Boolean valued analysis for the clarity and simplicity of the idea, because a considerable part of the theory reduces to manipulating numerical measures and integrals in a suitable Boolean valued model.

(4) Therefore, the Maharam operators must play the same role in the theory of Banach f-modules as the Lebesgue integral in the theory of Banach spaces. For instance, we can introduce an analog of the Lebesgue scale of function spaces. To this end, consider a Dedekind complete vector lattice Y and a universally complete vector lattice Z with a unit (= a universally complete unital f-algebra). Let $\Phi : L^1(\Phi) \to Y$ be a strictly positive Maharam operator with $L^1(\Phi)$ an order dense ideal of Z. Take $I_Y \leqslant p \in \Lambda := \mathscr{Z}(Y)$ and define the vector lattice $L^p(\Phi) \subset Z$ and the Y-valued norm $\boldsymbol{|} \cdot \boldsymbol{|}_p$ on $L^p(\Phi)$ as

$$
\begin{aligned}
L^p(\Phi) &:= \left\{ z \in Z : \ |z|^p \in L^1(\Phi) \right\}, \\
\boldsymbol{|} z \boldsymbol{|} &:= \big(\Phi(|x|^p)\big)^{\frac{1}{p}} \quad \big(z \in L^p(\Phi)\big).
\end{aligned}
$$

The expression $|z|^p$ makes sense on evoking the generalized functional calculus as defined in Haydon, Levy, and Raynaud [170] and Tasoev [389]. It can be showed that $L^p(\Phi)$ is a Banach–Kantorovich lattice. The scale can be studied on using the Boolean valued representation or the straightforward sectionwise techniques of continuous Banach bundles (cp. Kusraev [228, Sections 2.4 and 2.5]).

5.15.3. Operator variants of the Hahn Decomposition Theorem (Theorem 5.3.7), the Nakano Theorem (Theorem 5.3.8), and the Radon–Nikodým Theorem (Theorem 5.3.9) were obtained by Luxemburg and Schep in [295]. Maharam established Theorem 5.3.9 for a full-valued integral acting between spaces of measurable functions [302]. Theorem 5.3.5 due to Kusraev [217, 219]. The proof given in 5.3.5 is just a Boolean valued interpretation of the corresponding scalar result, i.e. Theorem 5.3.1. This latter result for functionals in the order ideal generated by Φ was proved by Vulikh (see Kantorovich, Vulikh, and Pinsker [196] and Vulikh [403]); the general case was announced by Lozanovskiĭ in [289] and proved in Vulikh and Lozanovskiĭ [404]. Another proof in the scalar case was given by Rice [345].

5.15.4. (1) A detailed discussion of the properties of conditional expectation can be found in Neveu [322] and Rao [343]. Conditional expectation operators on an L_p space were characterized as averaging operators by Moy [314] and Rota [351] and as contractive projections by Douglas [110] and Ando [30]. Positive projections on a rearrangement invariant KB-space were characterized in terms of conditional expectation by Kulakova [212]. Dodds, Huijsmans, and de Pagter [105] extended the Kulakova characterization to arbitrary ideals of measurable functions. This result [105, Theorem 3.10] gives a complete description of order continuous positive projections in terms of weighted conditional extension operators.

(2) Theorem 3.10 in Dodds, Huijsmans, and de Pagter [105] is a particular case $Y = \mathbb{R}$ of Theorem 5.4.10, and so $L^\infty(\Omega, \Sigma, \mu) \subset X \subset L^1(\Omega, \Sigma, \mu)$. At the same time Theorem 5.4.10, the main result of Section 5.4 is itself is nothing else but the Boolean valued interpretation of [105, Theorem 3.10].

(3) Grobler and de Pagter [153] introduced the class of multiplication conditional expectation representable (MCE-representable) operators on ideals of measurable functions. Grobler and Rambane [155] characterized the class of order continuous order bounded operators on ideals of measurable functions, showing that multiplication operators, Riesz homomorphisms, and conditional expectations constitute the building blocks of every order continuous operator. Of course, similar results with a conditional expectation type operator of Section 5.4 can be obtained by transfer.

5.15.5. (1) The construction of Section 5.5 stems from the Maharam theory of positive operators [301, 303]. In this section we follow the articles by Akilov, Kolesnikov, and Kusraev [20, 21] of 1988. Therein the three different approaches to describing the Maharam extension were suggested: the first uses the technique of completion of a lattice normed space (see Kusraev [228, Section 2.2]); the second treats the Maharam extension as a space of filters; and the third bases on the embedding $x \mapsto \hat{x}$ of a vector lattice X to $L^{\sim}\big((L^{\sim}(X,Y),Y\big)$ by the formula $\hat{x}(T) := Tx$ $(T \in L^{\sim}(X,Y))$. The last approach is accomplished also by Luxemburg and de Pagter in the voluminous paper [293], where the problem of extending a positive operator to a Maharam operator was thoroughly studied. Regarding the functional representation of the Maharam extension space see Kolesnikov and Kusraev [202] and Kusraev [228, Section 6.3].

(2) The main result of Section 5.5 is the construction of a Maharam extension of a given positive operator. The article by Luxemburg and de Pagter [293] treats a more general situation, where $\mathscr{J}$ is a given ideal of operators in $L^{\sim}(X,Y)$, and a Dedekind complete vector lattice $\bar{X}$ is constructed such that each operator $T \in \mathscr{J}$ has the Maharam extension $\bar{T} : \bar{X} \to Y$, and $T \mapsto \bar{T}$ is a lattice homomorphism. It should be also mentioned that the main result on the Maharam extension in [293, Theorem 5.4] was presented by Luxemburg at the conference in honor of Dorothy Maharam and was announced in [292] without proof.

5.15.6. (1) The properties of the Maharam extension in Section 5.6 have their natural framework in Dedekind complete vector lattices. It would be worthy to look for topological and metric aspects of the Maharam extension. Theorems 5.6.3–5.6.5 are taken from [228, Section 3.5]. More details can be found in Akilov, Kolesnikov, and Kusraev [21] and Luxemburg and de Pagter [293].

(2) In 5.6.9 (1, 2) and 5.6.10 (1–3) every component of a positive operator is obtained from its simpler fragments by up and down procedures. Similar assertions are often referred to as *up-down theorems*. The first up-down theorem (5.6.10 (3)) was established by de Pagter [328]; also see Aliprantis and Burkinshaw [27, 28]. But it involved the two essential constraints: Y was assumed to admit a separating set of o-continuous functionals, and X was order complete (or at least with the principal projection property). The first constraint was eliminated by Kusraev and Strizhevskiĭ in [256] and the second, by Akilov, Kolesnikov, and Kusraev in [21]. Of course, in the latter case the set of simple fragments is essentially different (see 5.6.7).

(3) A set of projections $\mathscr{P} \subset \mathbb{P}(L^{\sim}(X,Y))$ is said to be *generating* if for all $T \in L^{+}(X,Y)$ and $x \in X$ we have $Tx^{+} = \sup\{pTx : p \in \mathscr{P}\}$. A general up-down theorem was obtained by Kutateladze [269]. Namely, if $\mathscr{P}$ is a generating set of projections in $L^{\sim}(X,Y)$ (where X and Y are vector lattices with Y order complete) then $\mathfrak{E}(\Phi) = \mathscr{P}^{\vee}(\Phi)^{\uparrow\downarrow\uparrow}$, where $\mathscr{P}^{\vee}(\Phi)$ comprises the components representable as $\sum_{k=1}^{\infty} \pi_k \circ (\rho_k \Phi)$ with pairwise disjoint $\pi_k \in \mathbb{P}(Y)$ and arbitrary $\rho_k \in \mathscr{P}$. All formulas from 5.6.10 can be deduced from Kutateladze's Up-Down Theorem by specifying generating sets.

5.15.7. (1) The material in 5.7.1–5.7.10 is traditional for the theory of normed lattices and can be found in Aliprantis and Burkinshaw [28], Kantorovich and Akilov [195], Meyer-Nieberg [311], Schaefer [356], and Schwarz [361]. As examples of Banach spaces with some Boolean algebra of M-projections we mention the Banach spaces with mixed norm: $L^{p,\infty}(\mu \otimes \nu)$ and $L^{\infty}(\mu, X)$, where $1 \leqslant p \leqslant \infty$, X is a Banach lattice, and μ and ν are finite or σ-finite measures.

(2) The following sufficient condition on a measure space (Ω, Σ, μ) under which $L^0(\Omega, \Sigma, \mu)$ is Dedekind complete (and hence universally complete) is used rather often. A measure space (Ω, Σ, μ) is said to have the *direct sum property* if Σ includes a family $(A_\xi)_{\xi \in \Xi}$ of pairwise disjoint measurable sets of finite measure such that for every measurable set $A \in \Sigma$ of finite measure there exist a countable set of indices $\Theta \subset \Xi$ and a measurable set $A_0 \in \Sigma$ with $\mu(A_0) = 0$ satisfying $A = A_0 \cup \big(\bigcup_{\xi \in \Theta} (A \cap A_\xi)\big)$. If a measure space (Ω, Σ, μ) has the direct sum property then the associate vector lattice $L^0(\Omega, \Sigma, \mu)$ (as well as the Boolean algebra $\Sigma/\mu^{-1}(0)$) is Dedekind complete; see Kantorovich and Akilov [195] and Kusraev [228].

5.15.8. (1) The concept of lattice normed space was introduced for the first time by Kantorovich in 1936 [192]. These are vector spaces normed by elements of a vector lattice. Somewhat earlier, Kurepa [212] considered *espaces pseudodistanciés*, i.e. spaces with a metric that takes values in an ordered vector space. The first applications of vector norms and metrics were related to successive approximations in numerical analysis; see Kantorovich [192, 194, 196], Kollatz [203], and Schröder [360]. The theory of lattice normed spaces and dominated operators on them is presented in Kusraev [228].

(2) It is worth stressing that Kantorovich [193] is the very paper in which the unusual decomposability axiom (see 5.8.1 (4)) for an abstract norm appeared for the first time. Paradoxically, this axiom was often omitted as inessential in the further research by various authors. The profound importance of 5.8.1 (4) was rediscovered in connection with Boolean valued analysis (see Kusraev [221] and [222]). Namely, the decomposability axiom implies the existence of a Boolean algebra of linear projections in a lattice normed space and so it leads to a Boolean valued representation as a normed lattice. The spaces with a fixed Boolean algebra of linear projections and a coordinated order (the so-called coordinated spaces) were studied by Cooper [94, 95].

5.15.9. (1) The tools of Section 5.9 are some combinations of those stemming from Gordon [133]–[137] and Takeuti [379, 381, 383, 384]. In particular, Theorem 5.9.1 is a combination of Theorems 2.2.4 and 5.8.11((1) $\Longleftrightarrow$ (3)). The $\mathbb{B}$-convergence in 5.9.5 is essentially the piecewise convergence or m-convergence by Takeuti (cp. [379, 383]).

(2) Order continuity does not pique much interest in the context of $\mathbb{B}$-cyclic Banach lattices. If a $\mathbb{B}$-cyclic Banach lattice $(X, \|\cdot\|)$ is order continuous, then $\mathbb{B}$ is a finite Boolean algebra and so there are finitely many order continuous Banach lattices $(X_k, \|\cdot\|_k)$ $(k := 1, \dots, n)$ such that $X = X_1 \oplus \dots \oplus X_n$ and $\|x\| = \max\{\|x_k\|_k : k = 1, \dots, n\}$ $(x = x_1 + \dots + x_n,\ x_k \in X_k)$. Indeed, assuming that $\mathbb{B}$ is infinite and denoting the Stone space of $\mathbb{B}$ by Q, we can choose an decreasing net (e_α) in $C(Q)$ such that $\inf_\alpha e_\alpha = 0$ and $\lambda := \inf_\alpha \|e_\alpha\|_\infty > 0$. By Corollary 5.9.4 X is a Banach–Kantorovich lattice over $C(Q)$ and so X is a $C(Q)$-module. The $\mathbb{B}$-completeness of X implies the existence of $x_0 \in X_+$ with $|x_0| = \mathbb{1} := 1_Q$. The net $(e_\alpha x_0)$ is decreasing, $\inf_\alpha e_\alpha x_0 = 0$, and $\|e_\alpha x_0\| = \|e_\alpha |x_0|\|_\infty = \|e_\alpha\|_\infty \geqslant \lambda > 0$, so X is not order continuous.

5.15.10. (1) The concept of injective Banach lattice was introduced by Lotz in [288]. In this article he also proved Theorems 5.10.3 and 5.10.4. Theorem 5.10.3 was earlier obtained by Abramovich [1]. A Banach lattice X is called *λ-injective* if $\|T\| \leqslant \lambda \|\hat{T}\|$ is required in 5.10.1. In this book *injective* means 1-injective; the λ-injective Banach lattices $(\lambda > 1)$ are not considered. Concerning λ-injective Banach lattices

($\lambda > 1$) we refer to Lindenstrauss and Tzafriri [282], Lindenstrauss and Wulbert [283], and Mangheni [307].

(2) By 5.10.1 the injective Banach lattices are the injective objects of the category of Banach lattices with positive contractions as morphisms. Arendt [35, Theorem 2.2] proved that the injective objects are the same if the regular operators with contractive modulus are taken as morphisms.

(3) The Banach space $C(Q)$ with Q extremally disconnected is the only (up to isometric isomorphism) injective object in the category of Banach spaces and linear contractions (see Goodner [132], Kelley [198], and Nachbin [315]). Hasumi [168] treated the complex case. Thus, Theorem 5.10.4 shows that there is an essential difference between injective Banach lattices and injective Banach spaces.

(4) A separable Banach lattice X is said to be *separably injective* if for every pair of separable Banach lattices $Y \subset Z$ and every positive (continuous) linear operator from Y to X, there exists a norm preserving positive linear extension from Z to Y. In [76, Theorem 3] Buskes made the observation that every separably injective Banach lattice is injective.

(5) The injective objects in the category of Banach spaces can be also characterized geometrically in terms of the *binary intersection property:* a Banach space is injective if and only if each collection of pairwise intersecting closed balls $\{x \in X : \|x - x_i\| \leqslant r_i\}$ has nonempty intersection; see Nachbin [315]. An important contribution to the study of injective Banach lattices was made by Cartwright [85] who founded the *order intersection property* (Definition 5.10.9 (3)) and proved Theorems 5.10.10 and 5.10.11.

5.15.11. (1) Another significant advance is due to Haydon [169]. He discovered that an injective Banach space has a mixed AM-AL-structure and, in particular, proved Theorem 5.11.9. It was also proved (Theorem 5.10.12) in this article that a Banach lattice with the Cartwright, Levi, and Fatou properties is necessarily injective; see Haydon [169, Corollary 5D]. Thus, the conjunction of the Levi and Fatou properties is equivalent to property (P) for a Banach lattice with the Cartwright property. It follows that Theorem 5.11.9 gives a completely intrinsic characterization of injective Banach lattices, while Theorem 5.10.11 contains an extrinsic property (P). In Section 5.11 we follow Haydon [169].

(2) The notion of M-projection goes back to Alfsen and Effros [24] and Ando [31] and plays a crucial role in the theory of injective Banach lattices. The dual concept of *L-projection* is defined by the norm condition $\|x\| = \|\pi x\| + \|(I-\pi)x\|$ $(x \in X)$. In a wider context of the general Banach space theory the concepts are presented in Behrends [41] and Harmand, Werner, and Wener [166]. A closed subspace J of a Banach space X is called an *M-ideal* if $J^{\perp} := \{x' \in X' : J \subset \ker(x')\}$ is the range of an L-projection on X'. The main idea is to study the structure of a Banach space by means of the collections of its M-ideals.

(3) A natural generalization of the concept of M-projection is the concept of L^p-projection, $p \geqslant 1$, introduced by Beherends [40]. A linear projection π on a Banach space X is called an *L^p-projection* if $\|x\|^p = \|\pi x\|^p + \|(I-\pi)x\|^p$ for all $x \in X$. An L^1-projection is referred to as *L-projection*. Every two L^p-projections commute and the collection of all L^p-projections forms a complete Boolean algebra. Moreover, there is a complete duality between L^p-projections in X and L^q-projections in X' with $q = p/(p-1)$. Detailed presentation of this concept is in Beherends, Danckwerts, Evans, Göbel, Greim, Meyfarth, and Müller [42].

(4) A version of Theorem 5.11.2 for general Banach spaces is also true (see Cunningham [98]): *The set of all M-projections forms a* (*generally not complete*) *Boolean algebra. The set of all L-projections forms a complete Boolean algebra.* The closed linear span of the set of L-projections on X is called the *Cunningham algebra* of X and denoted by $\mathrm{Cun}(X)$. The *centralizer* $\mathscr{Z}(X)$ of X is a commutative unital C^*-algebra which is dual to the Cunningham algebra: $T \in \mathrm{Cun}(X) \Longleftrightarrow T' \in \mathscr{Z}(X')$ and $T' \in \mathrm{Cun}(X') \Longleftrightarrow T \in \mathscr{Z}(X)$. The L-structure of X provides the *integral module representation* of X such that the operators in the Cunningham algebra correspond to the multiplication operators; see Beherends, Danckwerts, Evans, Göbel, Greim, Meyfarth, and Müller [42]. Similar consideration on using the M-structure leads to the *maximal function module representation* so that the operators from $\mathscr{Z}(X)$ correspond to multiplication operators; see Beherends [41] and Cunningham [99].

5.15.12 **(1)** In Section 5.12 we follow Kusraev [240, 242]. Theorem 5.12.1 states that each injective Banach lattice embeds into an appropriate Boolean valued model, becoming an AL-space. According to this fact and the principles of Boolean valued analysis, each theorem about the AL-space within Zermelo–Fraenkel set theory has an analog in the original injective Banach lattice interpreted as a Boolean valued AL-space. This transfer principle is a new powerful tool in studying injective Banach lattices; see Kusraev [240]–[243].

(2) Corollary 5.12.14 is essentially the Main Representation Theorem by Haydon in [169, Theorem 5C]. Another representation result by

Haydon [169, Theorem 6H] (see Theorem 5.14.B.8) tells us that an injective Banach lattice can be represented as $L^1(\mathbf{m})$ for some Λ-valued modular Maharam measure $\mathbf{m}$. This is immediate from 5.12.5, since the mapping $\Phi : L^1(\mathbf{m}) \to \Lambda$ defined by $\Phi : f \mapsto \int f\, d\mathbf{m}$ is a Maharam operator with Levi property and $L^1(\mathbf{m}) = L^1(\Phi)$; see [228, Theorem 6.1.10] and subsection 5.15.B. The Haydon Third Representation Theorem [169, Theorem 7B] can also be deduced from Theorem 5.12.5 on using the bundle representation of Banach–Kantorovich spaces; see Kusraev [228, Section 2.4] and [239].

5.15.13. In Section 5.13 we follow Kusraev [240] and [242]. The cone absolutely summing operators were introduced by Levin [278] and independently but later by Schlotterbeck; see Schaefer [356, Ch. 4]. The meticulous exposition of the general theory of p-summing operators and their relatives with various interconnections and applications can be found in Diestel, Jarchow, and Tonge [103]. Observe that if $\mathbb{B} = \{0, I_Y\}$ then $\mathscr{S}_{\mathbb{B}}(X, Y)$ is the space of cone absolutely summing operators; see [356, Ch. 4, §3, Proposition 3.3 (d)] or (which is the same) 1-concave operators; see [103, p. 330].

5.15.14.A. (1) *JB*-algebras are nonassociative real analogs of C^*-algebras and von Neumann operator algebras. The theory of these algebras stems from Jordan, von Neumann, and Wigner [186] and exists as a branch of functional analysis since the mid 1960s. The stages of its development are reflected in Alfsen, Shultz, and Størmer [25]. The theory of *JB*-algebras undergoes intensive study, and the scope of its applications widens. Among the main areas of research are the structure and classification of *JB*-algebras, nonassociative integration and quantum probability theory, the geometry of states of *JB*-algebras, etc. (see Hanshe-Olsen and Störmer [165], Ajupov [15, 16], Ajupov, Usmanov, and Rakhimov [19] as well as the references therein).

(2) The Boolean valued approach to JB-algebras was charted by Kusraev in the article [226] which contains Theorems 5.14.A.4 and 5.14.A.7 (also see [227]). These theorems are instances of the Boolean valued interpretation of the results by Shulz [364] and by Ajupov and Abdullaev [17]. In [226] Kusraev introduced the class of $\mathbb{B}$-JBW-algebras which is broader than the class of JBW-algebras. The principal distinction is that a $\mathbb{B}$-JBW-algebra has a faithful representation as an algebra of selfadjoint operators on some AW^*-module rather than on a Hilbert space as in the case of JBW-algebras (cp. Kusraev and Kutateladze [249]). The class of AJW-algebras was firstly mentioned by Topping in [393]. Theorem 5.14.A.11 was never published before.

5.15.14.B. (1) In 5.14.B we briefly present a Boolean valued approach to Wright's theory of Stone-algebra-valued measures and integrals [419, 418, 420]. The material of this subsection (excluding Theorem 5.14.B.8) is taken from Kusraev and Malyugin [252]. We can easily reveal that Wright's theory is a measure theoretic incarnation of Maharam's ideas for positive operators. Theorem 5.14.B.8 is essentially Haydon's Representation Theorem [169, Theorem 6H].

(2) According to Wright [418] a measure $\mu : \mathscr{A} \to C(Q)$ (with Q extremally disconnected and compact) is modular with respect to an algebra homomorphism $\pi : C(Q) \to L^\infty(\mu)$ if $I_\mu(\pi(a)f) = aI_\mu(f)$ for all $a \in C(Q)$ and $f \in L^1(\mu)$). Equivalence of this definition to that in 5.14.B.4 follows from 5.14.B.5 (2) and 5.8.3. It follows from 5.14.B.5 (3), definition of Banach–Kantorovich space in 5.8.4, and [228, Theorem 7.4.4] that μ is ample if and only if $L^1(\mu)$ is a Banach–Kantorovich space as well as if and only if $L^2(\mu)$ is a Kaplansky–Hilbert module. Thus, Wright's ample measure as defined in [418] is the same as that in 5.14.B.4.

(3) Wright [418] showed in [418, Theorem 4.1] that the Radon–Nikodým Theorem is true for ample measures. This was done by applying the Kaplansky Theorem [197, Theorem 5] (with $X := L^2(\mu)$) which is read as follows: *If X is a Kaplansky–Hilbert module over Λ and $f : X \to \Lambda$ a continuous Λ-linear operator, then there exists a unique element $y \in X$ such that $f(x) = \langle x \mid y \rangle$ for all $x \in X$.* An improved version of the fact was obtained by Haydon [169, 6G]. This result is immediate from Theorems 4.14.B.7 and 5.3.10.

(4) In [411] Wickstead constructed an integral with respect to a vector measure with range a universally complete vector lattice admitting the Radon–Nikodým Theorem. The resultant space of integrable functions is rather similar to the construction of a Maharam extension space of Section 5.5. The article [411] lucidly shows the obstacles to constructing an integral with values in a vector lattice. Namely, the definition of integral with range a Dedekind complete vector lattice needs as an adequate construction the completion that has some mixed structure of order and topology. This means that the completion appears in the two stages: firstly we supplement the space with all mixtures—the order stage, and secondly we adjoin the limits with respect to relative uniform convergence—the topological stage; see [228, Theorems 2.2.2 and 3.2.8]. The problem arises then to find an appropriate functional realization of such a completion; cp. [228, Section 6.3].

5.15.14.C. (1) In [380] Takeuti introduced the Fourier transform for the mappings defined on a locally compact abelian group and having as values pairwise commuting normal operators in a Hilbert space. By applying the transfer principle, he developed a general technique for translating classical results to operator-valued functions. In this way he in particular established a version of the Bochner Theorem describing the set of all inverse Fourier transforms of positive operator-valued Radon measures. Given a complete Boolean algebra $\mathbb{B}$ of projections in a Hilbert space H, denote by $(\mathbb{B})$ the space of all selfadjoint operators on H whose spectral resolutions are in $\mathbb{B}$; i.e.,

$$A \in (\mathbb{B}) \Longleftrightarrow \left(A = \int \lambda \, dE_\lambda \text{ with } (\forall \lambda \in \mathbb{R})\, E_\lambda \in \mathbb{B}\right).$$

If $Y := (\mathbb{B})$ then Theorem 5.14.C.8 is essentially Takeuti's result [380, Theorem 1.3].

(2) Kusraev and Malyugin in [255] developed Takeuti's results in the following directions: First, they considered more general arrival spaces, namely, norm complete lattice normed spaces. So the important particular cases of Banach spaces and Dedekind complete vector lattices were covered. Second, the class of dominated mappings was identified with the set of all inverse Fourier transforms of order bounded quasi-Radon vector measures. Third, the construction of a Boolean valued universe was eliminated from the definitions and statements of results.

In particular, Theorem 5.14.C.12 and Corollary 5.14.C.13 correspond to [255, Theorem 4.3] and [255, Theorem 4.4]; while their lattice normed valued versions, to [255, Theorem 4.1] and [255, Theorem 4.5].

(3) Theorem 5.15.C.7 is due to Gordon [134, Theorem 2]. Proposition 3.3 in Takeuti [380] is essentially the same result stated for the particular departure and arrival spaces; i.e., $X = L^1(G)$ and $Y = (\mathbb{B})$.

(4) Theorem 5.14.C.11 is taken from Kusraev and Malyugin [255]. In the case of Q compact, it was proved by Wright in [424, Theorem 4.1]. In this result μ cannot be chosen regular rather than quasiregular. Wright in [421, Theorem T] obtained the characterization of order complete vector lattices for which this choice is always possible: *A Dedekind complete vector lattice Y is weakly (σ,∞)-distributive if and only if each Y-valued*

Baire measure on an arbitrary compact space can be extended to a regular Y-valued Borel measure if and only if every Y-valued quasiregular Borel measure on an arbitrary compact space is regular.

(5) Quasiregular measures were introduced by Wright in [419]. He also introduced quasi-Radon measures in [425]. Wright discovered the principal distinction between Radon and quasi-Radon measures and the role of the distinction in the problem of extending the measures and operators that arrive in Dedekind complete vector lattices; see [420]–[425]. The theory of Radon vector measures was then further developed by Kusraev and Malyugin; see [254, 255] and Malyugin [305, 306].

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