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PARAMETRIZATION OF ISOPERIMETRIC-TYPE PROBLEMS IN CONVEX GEOMETRY

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Аннотация. Choice of parametrization is considered for the isoperimetric-type extremal problems of optimal location of compact convex sets under many subsidiary constraints. Comparison is given between two parametrizations using support and surface area functions.

The aim of this article is to attract attention to the optimization problems that arise in analysis of the isoperimetric-type problems of the theory of convex surfaces. The extremal problems of geometry, a classical object of variational calculus, are of profound interest due to various applied problems involving optimal location of figures.

A particular extremal problem presupposes no a priori information about the algebraic structure of the set of feasible solutions. To pose such a problem we need only some preorder on the range of the target function of the problem. Clearly, no theoretic analysis is reasonable in this generality. The present-day methods for studying variational problems utilize the structure of a vector space. In this event, many simplifications are available in the case of convexity of the feasible set and the target of the problem in question.

Study of an extremal problem of convex geometry proceeds in the same steps as general analysis of an arbitrary optimization problem of a practical origin:

(a) The problem is *parametrized*; i. e., some vector space is chosen whose terms paraphrase the initial problem. Abusing the language, we call this vector space as well as our choice *parametrization*;

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(b) The polars are described of the cones of feasible directions. Simultaneously, we calculate subdifferentials or directional derivatives of the target and constraints of the problem.

(c) The Euler–Lagrange equation is formulated which we treat as the *optimality criterion* for a feasible solution.

Step (c) implies translation of the results from the language of the parametrization into the original terms of our problem. From a theoretical standpoint, to solve an extremal problem means customarily to derive Euler–Lagrange equations. Parametrization usually determines the shape of optimality criteria and so the power of the relevant analysis of the initial problem. Extremal problems of convex geometry provide a unique situation in which we possess two principally distinct parametrizations. The corresponding Minkowski and Blaschke structures are the main topic of the further exposition.

The present article was written at the request of the Siberian Conference on Industrial and Applied Mathematics in Memory of L. V. Kantorovich. In the recent years the author was involved in editing the collected works by L. V. Kantorovich in the field of applied functional analysis [1] and scientific papers by A. D. Alexandrov in the theory of mixed volumes of convex surfaces [2]. The two projects of the Gordon and Breach Publishers revived the author's old interest in the related problems of geometry and mathematical programming, so invoking the present article.

1. Vector Structures on the Set of Convex Surfaces

Dealing with the set $\mathcal{V}_N$ of compact convex sets in the N-dimensional Euclidean space $\mathbb{R}^N$, which are called *convex figures* for the sake of brevity, we usually distinguish two well-known parametrizations.

The first rests on the classical *Minkowski duality* which identifies a convex figure $\mathfrak{x}$ in $\mathbb{R}^N$ with its *support function* $\mathfrak{x}(z) := \sup\{(x,z) \mid x \in \mathfrak{x}\}$ for $z \in \mathbb{R}^N$. Considering the members of $\mathbb{R}^N$ as singletons, we assume that $\mathbb{R}^N$ lies in $\mathcal{V}_N$. The Minkowski duality induces in $\mathcal{V}_N$ the structure of a cone in the space $C(S_{N-1})$ of continuous functions on the Euclidean unit sphere S_{N-1}, the boundary of the unit ball $\mathfrak{z}_N$. This parametrization is the *Minkowski structure.* Addition of the support functions of convex figures amounts to passing to the algebraic sum of the latter, also called the *Minkowski addition.* It is worth observing that the *linear span* $[\mathcal{V}_N]$ of the cone $\mathcal{V}_N$ is dense in $C(S_{N-1})$.

The second parametrization, *Blaschke structure*, results from identifying the coset of translates $\{z + \mathfrak{x} \mid z \in \mathbb{R}^N\}$ of a *convex body* $\mathfrak{x}$, which is by definition a convex figure with nonempty interior, and the corresponding measure on the unit sphere which we call the *surface area function* of the coset of $\mathfrak{x}$ and denote by $\mu(\mathfrak{x})$. The soundness of this parametrization rests on the celebrated Alexandrov Theorem of recovering a convex surface from its surface area function. Each surface area function is an *Alexandrov measure*. So we call a positive measure on the unit sphere which is supported by no great hypersphere and which annihilates singletons. The last property of a measure is referred to as translation invariance in the theory of convex surfaces. Thus, each Alexandrov measure is a translation-invariant additive functional over the cone $\mathcal{V}_N$. The cone of positive translation-invariant measures in the dual $C'(S_{N-1})$ of $C(S_{N-1})$ is denoted by $\mathcal{A}_N$. We now agree on some preliminaries.

Given $\mathfrak{x}, \mathfrak{y} \in \mathcal{V}_N$, we let the record $\mathfrak{x} =_{\mathbb{R}^N} \mathfrak{y}$ mean that $\mathfrak{x}$ and $\mathfrak{y}$ are equal up to translation or, in other words, are translates of one another. We may say that $=_{\mathbb{R}^N}$ is the equivalence associated with the preorder $\geq_{\mathbb{R}^N}$ on $\mathcal{V}_N$ symbolizing the possibility of inserting one figure into the other by translation. Arrange the factor set $\mathcal{V}_N/\mathbb{R}^N$ which consists of the cosets of translates of the members of $\mathcal{V}_N$. Clearly, $\mathcal{V}_N/\mathbb{R}^N$ is a cone in the factor space $[\mathcal{V}_N]/\mathbb{R}^N$ of the vector space $[\mathcal{V}_N]$ by the subspace $\mathbb{R}^N$.

There is a natural bijection between $\mathcal{V}_N/\mathbb{R}^N$ and $\mathcal{A}_N$. Namely, we identify the coset of singletons with the zero measure. To the straight line segment with endpoints x and y, we assign the measure

$$|x-y|(\varepsilon_{(x-y)/|x-y|} + \varepsilon_{(y-x)/|x-y|}),$$

where $|\cdot|$ stands for the Euclidean norm and the symbol ε_z for $z \in S_{N-1}$ stands for the *Dirac measure* supported at z. If the dimension of the affine span $\operatorname{Aff}(\mathfrak{x})$ of a representative $\mathfrak{x}$ of a coset in $\mathcal{V}_N/\mathbb{R}^N$ is greater than unity, then we assume that $\operatorname{Aff}(\mathfrak{x})$ is a subspace of $\mathbb{R}^N$ and identify this class with the surface area function of $\mathfrak{x}$ in $\operatorname{Aff}(\mathfrak{x})$ which is some measure on $S_{N-1} \cap \operatorname{Aff}(\mathfrak{x})$ in this event. Extending the measure by zero to a measure on S_{N-1}, we obtain the member of $\mathcal{A}_N$ that we assign to the coset of all translates of $\mathfrak{x}$. The fact that this correspondence is one-to-one follows easily from the Alexandrov Theorem.

The vector space structure on the set of regular Borel measures induces in $\mathcal{A}_N$ and, hence, in $\mathcal{V}_N/\mathbb{R}^N$ the structure of a cone or, strictly speaking, the structure of a commutative $\mathbb{R}_+$-operator semigroup with cancellation. This structure on $\mathcal{V}_N/\mathbb{R}^N$ is called the *Blaschke structure*. Note that the sum of the surface area functions of $\mathfrak{x}$ and $\mathfrak{y}$ generates a unique class $\mathfrak{x}\#\mathfrak{y}$ which is referred to as the *Blaschke sum* of $\mathfrak{x}$ and $\mathfrak{y}$.

Let $C(S_{N-1})/\mathbb{R}^N$ stand for the factor space of $C(S_{N-1})$ by the subspace of all restrictions of linear functionals on $\mathbb{R}^N$ to S_{N-1}. Denote by $[\mathcal{A}_N]$ the space $\mathcal{A}_N - \mathcal{A}_N$ of translation-invariant measures. It is easy to see that $[\mathcal{A}_N]$ is also the linear span of the set of Alexandrov measures. The spaces $C(S_{N-1})/\mathbb{R}^N$ and $[\mathcal{A}_N]$

are set in duality by the canonical bilinear form

$$\langle f, \mu \rangle = \frac{1}{N} \int_{S_{N-1}} f d\mu \quad (f \in C(S_{N-1})/\mathbb{R}^N,\ \mu \in [\mathcal{A}_N]).$$

For $\mathfrak{x} \in \mathcal{V}_N/\mathbb{R}^N$ and $\mathfrak{y} \in \mathcal{A}_N$, the quantity $\langle \mathfrak{x}, \mathfrak{y} \rangle$ coincides with the *mixed volume* $V_1(\mathfrak{y}, \mathfrak{x})$. The space $[\mathcal{A}_N]$ is usually furnished with the weak topology induced by the above indicated duality with $C(S_{N-1})/\mathbb{R}^N$.

By the *dual* K^* of a given cone K in a vector space X in duality with another vector space Y, we mean the set of all positive linear functionals on K; i. e., $K^* := \{y \in Y \mid (\forall x \in K) \langle x, y \rangle \geq 0\}$. Recall also that to a convex subset U of X and a point $\bar{x}$ in U there corresponds the cone

$$U_{\bar{x}} := \mathrm{Fd}(U, \bar{x}) := \{h \in X \mid (\exists \alpha \geq 0)\ \bar{x} + \alpha h \in U\}$$

which is called the *cone of feasible directions* of U at $\bar{x}$. Fortunately, description is available for all dual cones we need.

1.1. *The dual $\mathcal{A}_N^*$ of $\mathcal{A}_N$ is the positive cone of $C(S_{N-1})/\mathbb{R}^N$.*

1.2. *Let $\bar{\mathfrak{x}} \in \mathcal{A}_N$. Then the dual $\mathcal{A}_{N,\bar{\mathfrak{x}}}^*$ of the cone of feasible directions of $\mathcal{A}_N n$ at $\bar{\mathfrak{x}}$ may be represented as follows*

$$\mathcal{A}_{N,\bar{\mathfrak{x}}}^* = \{f \in \mathcal{A}_N^* \mid \langle \bar{\mathfrak{x}}, f \rangle = 0\}.$$

Assume that μ and ν are positive measures on the sphere S_{N-1}. Say that μ is *linearly stronger than* ν and write $\mu \gg_{\mathbb{R}^N} \nu$ if to each decomposition of ν into the sum of finitely many positive terms $\nu = \nu_1 + \ldots + \nu_m$ there exists a decomposition of μ into the sum of finitely many terms $\mu = \mu_1 + \ldots + \mu_m$ such that $\mu_k - \nu_k \in (\mathbb{R}^N)^*$ for all $k = 1, \ldots, m$.

1.3. *Let $\mathfrak{x}$ and $\mathfrak{y}$ be convex figures. Then*

(1) $\mu(\mathfrak{x}) - \mu(\mathfrak{y}) \in \mathcal{V}_N^* \leftrightarrow \mu(\mathfrak{x}) \gg_{\mathbb{R}^N} \mu(\mathfrak{y})$;

(2) If $\mathfrak{x} \geq_{\mathbb{R}^N} \mathfrak{y}$ then $\mu(\mathfrak{x}) \gg_{\mathbb{R}^N} \mu(\mathfrak{y})$;

(3) $\mathfrak{x} \geq_{\mathbb{R}^2} \mathfrak{y} \leftrightarrow \mu(\mathfrak{x}) \gg_{\mathbb{R}^2} \mu(\mathfrak{y})$.

1.4. *Let $\bar{\mathfrak{x}}$ and $\mathfrak{y}$ be convex figures. Then*

(1) If $\mathfrak{y} - \bar{\mathfrak{x}} \in \mathcal{A}_{N,\bar{\mathfrak{x}}}^*$ then $\mathfrak{y} =_{\mathbb{R}^N} \bar{\mathfrak{x}}$;

(2) If $\mu(\mathfrak{y}) - \mu(\bar{\mathfrak{x}}) \in \mathcal{V}_{N,\bar{\mathfrak{x}}}^*$ then $\mathfrak{y} =_{\mathbb{R}^N} \bar{\mathfrak{x}}$.

In the sequel we never distinguish between a convex figure, the respective coset of translates in $\mathcal{V}_N/\mathbb{R}^N$, and the corresponding measure in $\mathcal{A}_N$.

It is worth noting that the volume $V(\mathfrak{x}) := \langle \mathfrak{x}, \mathfrak{x} \rangle$ of a convex figure $\mathfrak{x}$ is a homogeneous polynomial of degree N with respect to the Minkowski structure. That is why to calculate the subdifferential of $V(\cdot)$ is an easy matter. The particular feature of the Minkowski structure is an intricate construction of the dual of the cone of compact convex sets whose description bases on the relation $\gg_{\mathbb{R}^N}$ in the space of measures $[\mathcal{A}_N]$. If we use the Blaschke addition in the space of dimension $N \geq 3$ then the dual of the cone of Alexandrov measures is rather simple whereas volume fails to be a homogeneous polynomial, which complicates analysis.

In the sequel we use the following notations:

$$p : \mathfrak{x} \mapsto V^{1/N}(\mathfrak{x}) \quad (\mathfrak{x} \in \mathcal{V}_N/\mathbb{R}^N);$$
$$\widehat{p} : \mathfrak{x} \mapsto V^{(N-1)/N}(\mathfrak{x}) \quad (\mathfrak{x} \in \mathcal{A}_N).$$

The *Minkowski inequality* is thus paraphrased as

$$\langle \mathfrak{x}, \mathfrak{y} \rangle \geq p(\mathfrak{x})\widehat{p}(\mathfrak{y}).$$

1.5. Brunn–Minkowski Theorem. *The functional p is superlinear on the cone $\mathcal{V}_N$.*

The following important proposition was most likely known to H. Minkowski.

1.6. *The functional $\widehat{p}$ is superlinear on the cone $\mathcal{A}_N$.*

1.7. Herglotz Theorem. *The function p is concave on the convex set $\mathcal{A}_N$.*

Since the *surface area* of $\mathfrak{x}$ may be written as $S(\mathfrak{x}) = N\langle \mathfrak{z}_N, \mathfrak{x} \rangle$, the isoperimetric problem becomes a convex program with respect to the Blaschke structure.

1.8. Isoperimetric Problem.

(1) $\mathfrak{x} \in \mathcal{A}_N$*;*
(2) $\langle \mathfrak{z}_N, \mathfrak{x} \rangle = b$*;*
(3) $\widehat{p}(\mathfrak{x}) \to \max$.

The simplest example of a convex program with respect to the Minkowski structure is the Urysohn problem of finding which of the convex bodies with equal integral width has greatest volume.

1.9. Urysohn Problem.

(1) $\mathfrak{x} \in \mathcal{V}_N$*;*
(2) $\langle \mathfrak{x}, \mathfrak{z}_N \rangle = b$*;*
(3) $p(\mathfrak{x}) \to \max$.

Since all dual cones are available, we are left with calculating the directional derivative of volume. Denote by $p_{\bar{\mathfrak{x}}}$ the directional derivative of $\mathfrak{y} \in \mathcal{V}_N \mapsto \widehat{p}(\bar{\mathfrak{x}})p(\mathfrak{y})$ at $\bar{\mathfrak{x}}$. Similarly, let $\widehat{p}_{\bar{\mathfrak{x}}}$ stand for the directional derivative of $\mathfrak{y} \in \mathcal{A}_N \mapsto \widehat{p}(\mathfrak{y})p(\mathfrak{x})$ at $\bar{\mathfrak{x}}$.

1.10. *The following hold:*

(1) $\widehat{p}_{\bar{\mathfrak{x}}}(\mathfrak{g}) = \langle \bar{\mathfrak{x}}, \mathfrak{g} \rangle$ *for all* $\mathfrak{g} \in \mathcal{A}_{N,\bar{\mathfrak{x}}}$*;*
(2) $p_{\bar{\mathfrak{x}}}(g) = \langle g, \bar{\mathfrak{x}} \rangle$ *for all* $g \in \mathcal{V}_{N,\bar{\mathfrak{x}}}$.

2. Minkowski Structure in Programming Isoperimetric-Type Problems

In this section we set forth the technique of manipulating the simplest operator-type constraints in isoperimetric-type problems by means of the Minkowski structure. We bear in mind the constraints of the following sort: "a solution lies in a given convex set," "a solution is centrally symmetric" etc., which supplement an extremal problems whose target function and the rest of the constraints are stated in terms of mixed volumes. The inclusion in a polyhedron may alternatively be treated as finitely many inequality-type constraints on the support function of a solution. Straightforward implementation of this possibility leads to technical inconveniences. Namely, the resulting Euler–Lagrange equations involve solutions of some linear programs. Moreover, it is impossible in general to state the external isoperimetric problem for a polyhedron using finitely many pointwise inequalities. We suggest that an inclusion-type constraint be treated as inequality in the space of compact convex sets. This entails simplification since we encounter a sole Lagrange multiplier that accounts for violation of the inequality-type constraint.

For the sake of simplicity we start with presenting the technique of deriving optimality criteria for plane problems. We proceed further with discussing a rather routine manner in which this technique changes in many dimensions.

2.1. Internal Isoperimetric Problem. *Among the convex figures lying in a fixed convex body* $\mathfrak{x}_0$ *and having perimeter equal to* $S(\bar{\mathfrak{x}})$*, find a figure of maximal area.*

Recall that a convex body is a convex figure with nonempty interior. Sometimes this object is identified with its boundary and so referred to as a (closed) *convex surface*. Existence is easy for this and analogous problems on using the Blaschke Choice Theorem which proclaims that the set of convex figures lying in a fixed convex figure is compact. Uniqueness of a solution to within translation rests on the strict convexity of volume which amounts to the conditions of equality holding in the isoperimetric inequality.

2.2. Optimality Criterion. *A feasible convex body* $\bar{\mathfrak{x}}$ *is a solution to the internal isoperimetric problem if and only if there are a convex figure* $\mathfrak{x} \in \mathcal{V}_2$ *and a real* $\bar{\alpha} \in \mathbb{R}_+$ *satisfying*

(1) $\bar{\mathfrak{x}} =_{\mathbb{R}^2} \mathfrak{x} + \bar{\alpha}\mathfrak{z}_2$*;*

(2) $\bar{\mathfrak{x}}(z) = \mathfrak{x}_0(z)$ *for all* z *in* $\operatorname{supp}(\mathfrak{x})$*.*

Here $\operatorname{supp}(\mathfrak{x})$ stands for the *support* of a convex figure $\mathfrak{x}$; i. e., the support of the measure $\mu(\mathfrak{x})$, the surface area function of $\mathfrak{x}$. The claimed optimality criterion coincides with the subdifferential conditions for a maximum point of the Lagrangian of the internal isoperimetric problem. The convex figure $\mathfrak{x}$, determining a solution, is the *Lagrange multiplier* corresponding to the inclusion-type constraint of the problem. This Lagrange multiplier is called a *critical figure* in view of its role in construction of an optimal solution. The support $\operatorname{supp}(\mathfrak{x})$ of a critical figure $\mathfrak{x}$ lies in the support $\operatorname{supp}(\mathfrak{x}_0)$ of $\mathfrak{x}_0$.

Abstracting Problem 2.1, we may replace the condition on the perimeter calculated with respect to the classical Euclidean metric with a constraint on the perimeter in an arbitrary Minkowski geometry defined by a possibly asymmetric conical segment. Such a perimeter is simply the mixed area with an appropriate convex figure. A solution to the so-modified problem remains to be some "parallel" set to a critical figure. In the general case a solution is the sum of a critical figure and a scaled polar of the unit disk of the original Minkowski geometry.

Indeed, since

$$\mathfrak{x}_0 = \bigcap_{z\in\operatorname{supp}(\mathfrak{x}_0)} \{\mathfrak{y} \in \mathbb{R}^2 \mid (\mathfrak{y}, z) \le \mathfrak{x}_0(z)\},$$

the condition $\mathfrak{x} \le \mathfrak{x}_0$ follows from the inequalities $\mathfrak{x}(z) \le \mathfrak{x}_0(z)$ for all $z \in \operatorname{supp}(\mathfrak{x}_0)$. To demonstrate, note that by the Minkowski duality

$$\mathfrak{x} = \bigcap_{z\in S_1} \{\mathfrak{y} \in \mathbb{R}^2 \mid (\mathfrak{y}, z) \le \mathfrak{x}(z)\} \le \bigcap_{z\in\operatorname{supp}(\mathfrak{x}_0)} \{\mathfrak{y} \in \mathbb{R}^2 \mid (\mathfrak{y}, z) \le \mathfrak{x}(z)\} \le$$
$$\le \bigcap_{z\in\operatorname{supp}(\mathfrak{x}_0)} \{\mathfrak{y} \in \mathbb{R}^2 \mid (\mathfrak{y}, z) \le \mathfrak{x}_0(z)\} = \mathfrak{x}_0.$$

Moreover, the above implies that the measure $\bar{\mu}$, equal to $\mu(\mathfrak{x})$, is supported by $\operatorname{supp}(\mathfrak{x}_0)$.

If $\mathfrak{x}_0$ is a polygon then the last remark shows that each critical figure belongs to the *Lindelöf family generated by* $\mathfrak{x}_0$, i. e., it presents a polygon whose nonzero sides are parallel to the edges of $\mathfrak{x}_0$.

It is immaterial that we consider a sole constraint on a general mixed volume. The case of an arbitrary number of constraints leads to essentially the same formula in complete accord with the general theory of extremal problems. However, geometrical intuition is of no avail in this case since the Lagrange multipliers whose number equals to the number of constrained mixed areas or volumes are in fact a solution to a linear system of equations of the same order. By way of illustration, we state a corresponding analog of Problem 2.1.

2.3. Generalized Internal Isoperimetric Problem. *Let $\mathfrak{x}_0$ be a fixed convex body. Assume also that $\mathfrak{y}_1, \dots, \mathfrak{y}_m$ are convex figures. Among the convex figures lying in $v\,\mathfrak{x}_0$ and such that $\langle \mathfrak{y}_k, \mathfrak{x}\rangle \le \langle \mathfrak{y}_k, \bar{\mathfrak{x}}\rangle$ for all $k = 1, \dots, m$, find a convex figure having greatest area.*

Since the gradient of the functional $V_1(\mathfrak{y}_k, \cdot\,)$ is proportional to $\mu(\mathfrak{y}_k)$, we come to the following

2.4. Optimality Criterion. *A feasible convex body $\bar{\mathfrak{x}}$ is a solution to the generalized internal isoperimetric problem if and only if there are some $\mathfrak{x} \in \mathcal{V}_2$ and $\bar{\alpha}_1, \dots, \bar{\alpha}_m \in \mathbb{R}_+$ satisfying*

(1) $\bar{\mathfrak{x}} =_{\mathbb{R}^2} \mathfrak{x} + \sum_{k=1}^{m} \bar{\alpha}_k \mathfrak{y}_k$;

(2) $\bar{\mathfrak{x}}(z) = \mathfrak{x}_0(z)$ *for all* $z \in \operatorname{supp}(\mathfrak{x})$.

2.5. External Isoperimetric Problem. *Among the convex figures that include a fixed convex figure $\mathfrak{x}_0$ and has perimeter equal to $S(\bar{\mathfrak{x}})$, find a convex figure of greatest area.*

An equivalent convex program is stated as follows:

(1) $\mathfrak{x} \in \mathcal{V}$;

(2) $-\mathfrak{x} \leq -\mathfrak{x}_0$;

(3) $S(\mathfrak{x}) \leq S(\bar{\mathfrak{x}})$;

(4) $p(\mathfrak{x}) \to \max$.

The presence of the minus sign in (2) complicates the optimality criterion rather than its derivation since the dual $\mathcal{V}^*_{2,\bar{\mathfrak{x}}}$, of the cone of feasible directions at an irregular convex figure $\bar{\mathfrak{x}}$ differs from zero in general. To obviate this obstacle by appealing to 1.4(2) is impossible in contrast to the case of Problem 2.1, since easy examples show that not all elements of $\mathcal{V}^*_{2,\bar{\mathfrak{x}}}$ are of the shape $\mathfrak{y} - \bar{\mathfrak{x}}$.

2.6. Optimality Criterion. *A feasible convex body $\bar{\mathfrak{x}}$ is a solution to the external isoperimetric problem if and only if there are a critical figure $\mathfrak{x}$ and a positive real $\bar{\alpha}$ satisfying*

(1) $\bar{\alpha}\mathfrak{z}_2 \geq_{\mathbb{R}^2} \mathfrak{x} + \bar{\mathfrak{x}}$;

(2) $\bar{\mathfrak{x}}(z) + \mathfrak{x}(z) = \bar{\alpha}\mathfrak{z}_2(z)$ *for all* $z \in \operatorname{supp}(\bar{\mathfrak{x}})$;

(3) $\bar{\mathfrak{x}}(z) = \mathfrak{x}_0(z)$ *for all* $z \in \operatorname{supp}(\mathfrak{x})$.

2.7. The Kovner–Besicovitch Body Problem. *Find the greatest centrally symmetric convex figure included in a given convex body $\mathfrak{x}_0$, the Kovner–Besicovitch body of $\mathfrak{x}_0$.*

The solution to Problem 2.7 is clearly a centrally symmetric convex figure lying in $\mathfrak{x}_0$ and having greatest area among these figures. Hence, the only delicate point in settling the problem is to describe the dual of the cone of centrally symmetric convex figures. The answer uses the Minkowski symmetrization. Recall that the *Minkowski symmetrization* of a convex figure $\mathfrak{x}$ is the convex figure $\mathfrak{x}^s$ whose support function is $z \mapsto (\mathfrak{x}(z) + \mathfrak{x}(-z))/2$.

2.8. *The inequality*

$$\int_{S_1} \mathfrak{z}\, d\mu(\mathfrak{x}) \geq \int_{S_1} \mathfrak{z}\, d\mu(\mathfrak{y})$$

holds for every centrally symmetric convex figure $\mathfrak{z}$ *if and only if some translate of the Minkowski symmetrization of* $\mathfrak{y}$ *lies in the Minkowski symmetrization of* $\mathfrak{x}$.

2.9. *The solution* $\bar{\mathfrak{x}}$ *of Problem 2.7 is the Minkowski symmetrization of a convex figure* $\mathfrak{x}$ *such that* $\bar{\mathfrak{x}}(z) = \mathfrak{x}_0(z)$ *for all* $z \in \operatorname{supp}(\mathfrak{x})$.

This proposition shows how to handle the condition that a solution is centrally symmetric in the problems like 2.1, 2.3, and 2.5.

2.10. Internal Isoperimetric Problem in the Class of Centrally Symmetric Convex Figures. *Among centrally symmetric convex figures lying in* $\mathfrak{x}_0$ *and having perimeter equal to* $S(\bar{\mathfrak{x}})$, *find a figure of greatest area.*

An equivalent program in the space of convex figures differs from the problem equivalent to Problem 2.1 only in the fact that the condition $\mathfrak{x} \in \mathcal{V}_2$ is replaced with the requirement that $\mathfrak{x}$ is centrally symmetric. The optimality criterion for the new program ia a combination of the optimality criteria for Problems 2.1 and 2.7.

2.11. Optimality Criterion. *A feasible convex body* $\mathfrak{x}$ *is a solution to Problem 2.10 if and only if there are a critical figure* $\mathfrak{x}$ *and a real* $\bar{\alpha} \in \mathbb{R}_+$ *satisfying*

(1) $\bar{\mathfrak{x}} =_{\mathbb{R}^2} \mathfrak{x}^s + \bar{\alpha}\mathfrak{z}_2$;

(2) $\bar{\mathfrak{x}}(z) = \mathfrak{x}_0(z)$ *for all* $z \in \operatorname{supp}(\mathfrak{x})$.

We may modify the problems like 2.5 in a similar fashion.

2.12. Isoperimetric Problem with a Zone. *Among the convex figure of equal perimeter and such that* $\mathfrak{x}(z) \le \mathfrak{x}_0(z)$ *for all* $z \in Z_0$, *with* Z_0 *is some, say, symmetric compact subset of* S_1, *find a convex figure having greatest area.*

In this event we treat the "zone-type" constraint as generated by the restriction operator from $C(S_1)$ to $C(Z_0)$. The method of analysis remains the same in other aspects.

2.13. Optimality Criterion. *A feasible convex body* $\bar{\mathfrak{x}}$ *is a solution to the isoperimetric problem with zone* Z_0 *if and only if there are a convex figure* $\mathfrak{x}$ *and a real* $\bar{\alpha} \in \mathbb{R}_+$ *satisfying*

(1) $\bar{\mathfrak{x}} =_{\mathbb{R}^2} \mathfrak{x} + \bar{\alpha}\mathfrak{z}_2$;

(2) $\operatorname{supp}(\mathfrak{x}) \subset Z_0$;

(3) $\bar{\mathfrak{x}}(z) = \mathfrak{x}_0(z)$ *for all* $z \in \operatorname{supp}(\mathfrak{x})$.

The above technique applies to the problem of optimal location of several figures in the cells generated by a family of planes with prescribed normals. These problems are called *problems with current polyhedra.* They belong to the class of extremal *problems with free boundary.*

2.14. Internal Isoperimetric Problem with a Current Straight Line. *Find two convex figures $\bar{\mathfrak{x}}$ and $\bar{\mathfrak{y}}$, lying in a given convex figure $\mathfrak{x}_o$, belonging to the opposite half-planes with the boundary straight line having a given outer unit normal z_0, and such that the sum of their areas is maximal while the sum of their perimeters is fixed in advance.*

2.15. Optimality Criterion. *A feasible pair of convex bodies $\bar{\mathfrak{x}}$ and $\bar{\mathfrak{y}}$ is a solution to Problem 2.14 if and only if there are convex figures $\mathfrak{x}$ and $\mathfrak{y}$ and positive reals $\bar{\alpha}$ and $\bar{\beta}$ satisfying*

(1) $\bar{\mathfrak{x}} = \mathfrak{x} + \bar{\alpha}\mathfrak{z}_2$;

(2) $\bar{\mathfrak{y}} = \mathfrak{y} + \bar{\alpha}\mathfrak{z}_2$;

(3) $\mu(\mathfrak{x}) \geq \bar{\beta}\varepsilon_{z_0}$ *and* $\mu(\mathfrak{y}) \geq \bar{\beta}\varepsilon_{z_0}$;

(4) $\bar{\mathfrak{x}}(z) = \mathfrak{x}_0(z)$ *for all* $z \in \operatorname{supp}(\mathfrak{x}) \setminus z_0$;

(5) $\bar{\mathfrak{y}}(z) = \mathfrak{x}_0(z)$ *for all* $z \in \operatorname{supp}(\mathfrak{x}) \setminus -z_0$.

We proceed with discussing the particular features of analysis in many dimensions.

Considering the general isoperimetric problem in the space $[\mathcal{V}_N]$, we are in a position to derive only a necessary condition for an extremum in general. The point is that the spatial isoperimetric problem is already " convex on the wrong side," i. e., it reduces to a problem of maximizing a convex function over a convex set. Indeed, the functional $\mathfrak{x} \mapsto S^{1/(N-1)}(\mathfrak{x})$ is concave by the Brunn–Minkowski Theorem, where $S(\mathfrak{x})$ stands as usual for the surface area of $\mathfrak{x}$. To make the spatial isoperimetric problem into a convex program we need another vector structure, namely, the Blaschke structure. For this reason, the above methods, settling the planar case, may be fully abstracted only to convex isoperimetric-type problems. For instance, as an analog of the internal isoperimetric problem we pose the following

2.16. Internal Urysohn Problem.

(1) $\mathfrak{x} \leq \mathfrak{x}_0$;

(2) $\langle \mathfrak{x}, \mathfrak{z}_N \rangle \leq b$;

(3) $p(\mathfrak{x}) \to \max$.

Another distinction of the plane from the spaces of higher dimension is the fact that the Minkowski and Blaschke sums are translates of one another only in the plane. Recall that in the above Euler–Lagrange equations we encounter functional parameters, and so in our particular case, the Blaschke addition. This means that, paraphrasing the plane optimality criteria in space, we must substitute Blaschke sums for Minkowski sums. For example, Problem 2.10 will involve the *Blaschke symmetrization* in space.

The next particularity of many dimensions is the presence of some translation-invariant positive but degenerate measures which may be treated as surface area functions only in lesser dimensions. For this reason, the Euler–Lagrange equations will now involve critical measures rather than critical figures as before.

The final particularity is the fact that, in the case $N \geq 3$, the condition $\mu(\mathfrak{x}) \gg_{\mathbb{R}^N} \mu(\mathfrak{y})$ does not imply in general that $\mathfrak{x} \geq_{\mathbb{R}^N} \mathfrak{y}$. The nature of this effect will become clearer in the sequel.

By way of summarizing, we may say that the duality analysis of isoperimetric-type problems in many dimensions has the sole specific feature: the technique of surface area functions replaces the technique of support functions.

Let us illustrate the above by example.

2.17. External Urysohn Problem. *Among the convex figures, including $\mathfrak{x}_0$ and having integral width fixed, find a convex body of greatest volume.*

2.18. Optimality Criterion. *A feasible convex body $\bar{\mathfrak{x}}$ is a solution to Problem 2.17 if and only if there are a positive critical measure μ and a positive real $\bar{\alpha} \in \mathbb{R}_+$ satisfying*

(1) $\bar{\alpha}\mu(\mathfrak{z}_N) \gg_{\mathbb{R}^N} \mu(\bar{\mathfrak{x}}) + \mu$;
(2) $V(\bar{\mathfrak{x}}) + \frac{1}{N}\int_{S_{N-1}} \bar{\mathfrak{x}} d\mu = \bar{\alpha} V_1(\mathfrak{z}_N, \bar{\mathfrak{x}})$;
(3) $\bar{\mathfrak{x}}(z) = \mathfrak{x}_0(z)$ for all z in the support of μ.

If, in particular, $\mathfrak{x}_0 = \mathfrak{z}_{N-1}$ then the sought body is a *spherical lens*, that is, the intersection of two balls of the same radius; while the critical measure is the restriction of the surface area function of the ball of radius $\bar{\alpha}^{1/(N-1)}$ to the complement of the support of the lens to S_{N-1}. If $\mathfrak{x}_0 = \mathfrak{z}_1$ and $N = 3$ then our result implies that we should seek a solution in the class of the so-called spindle-shaped constant-width surfaces of revolution (cf. [3, p. 157]).

Note also that, combining the tricks of the current section, we may write down the Euler–Lagrange equations for a wide class of isoperimetric-type extremal problems. In particular events, these are reasonable to apply together with another technique of geometry and mathematical programming. To illustrate this, we exhibit a rather typical example:

2.19. *Among convex figures of fixed thickness and integral width, find a convex body of greatest volume.*

Recall that the *thickness* $\Delta(\mathfrak{x})$ of a convex figure $\mathfrak{x}$ is defined as follows:

$$\Delta(\mathfrak{x}) := \inf_{z \in S_{N-1}} (\mathfrak{x}(z) + \mathfrak{x}(-z)).$$

Observe first that Problem 2.19 is stated as "convex on the wrong side." However, applying the Minkowski symmetrization once, we see that a solution belongs to the class of centrally symmetric convex figures for which the restricted thickness may be rewritten as inclusion-type constraint.

2.20. Optimality Criterion. *Let a positive measure μ and reals $\bar\alpha$, $\bar\beta \in \mathbb{R}_+$ satisfy the following conditions:*

(1) $\bar\alpha\mu(\mathfrak{z}_N) + \bar\beta(\varepsilon_{z_0} + \varepsilon_{-z_0}) \gg_{\mathbb{R}^N} \mu(\bar{\mathfrak{x}}) + \mu$;

(2) $V(\bar{\mathfrak{x}}) + \frac{1}{N}\int_{S_{N-1}} \bar{\mathfrak{x}}\,d\mu = \bar\alpha V_1(\mathfrak{z}_N, \bar{\mathfrak{x}}) + \frac{1}{N}\bar\beta(\bar{\mathfrak{x}}(z_0) + \bar{\mathfrak{x}}(-z_0))$;

(3) $\bar{\mathfrak{x}}(z) = \frac{1}{2}\Delta$ *for all z in the support of μ.*

Then a feasible convex body $\bar{\mathfrak{x}}$ is a solution to Problem 2.19.

Therefore, a convex figure $\bar\alpha\mathfrak{z}_N \# \bar\beta\mathfrak{z}_{N-1}$ of given integral width and thickness is optimal for Problem 2.19. In the case $N = 3$, a solution belongs to the class of the so-called cheese-shaped constant-width surfaces of revolution (cf. [3, p. 171]).

Concluding the current section, we explain the connection of the above technique with the conventional approach resting on isoperimetric inequalities.

First of all, note that a solution to each of the above-stated convex programs generates an isoperimetric inequality of the following form

$$\varphi(\bar{\mathfrak{x}}, \bar\alpha) \geq \varphi(\tilde{\mathfrak{x}}, \bar\alpha) \quad (\tilde{\mathfrak{x}} \in \mathcal{V}_N),$$

where φ stands for the corresponding Lagrangian.

It is intuitively clear that in the case of constrained mixed volumes these inequalities must reduce to the classical inequalities of Brunn–Minkowski type. This claim admits a rigorous treatment.

By way of example, consider Problem 2.1. Rewrite the inequalities $\varphi(\bar{\mathfrak{x}}, \bar\alpha) \geq \varphi(\tilde{\mathfrak{x}}, \bar\alpha)$ as

$$V(\bar{\mathfrak{x}}) \geq (V(\bar{\mathfrak{x}})V(\bar{\mathfrak{x}}))^{1/2} + \bar\alpha(V_1(\bar{\mathfrak{x}}, \mathfrak{z}_2) - V_1(\tilde{\mathfrak{x}}, \mathfrak{z}_2)) + V_1(\mathfrak{x}, \mathfrak{x}_0) - V_1(\tilde{\mathfrak{x}}, \mathfrak{x}).$$

We have $V_1(\mathfrak{x}, \mathfrak{x}_0) = V_1(\mathfrak{x}, \bar{\mathfrak{x}})$, since $\bar{\mathfrak{x}}(z) = \mathfrak{x}_0(z)$ for all z in the support $\operatorname{supp}(\mathfrak{x})$ of $\mathfrak{x}$. Therefore,

$$V(\bar{\mathfrak{x}}) \geq (V(\bar{\mathfrak{x}})V(\tilde{\mathfrak{x}}))^{1/2} + V_1(\bar{\mathfrak{x}}, \mathfrak{x} + \bar\alpha\mathfrak{z}_2) - V_1(\mathfrak{x} + \bar\alpha\mathfrak{z}_2, \tilde{\mathfrak{x}}).$$

Since $\bar{\mathfrak{x}} =_{\mathbb{R}^2} \mathfrak{x} + \bar\alpha\mathfrak{z}_2$, the last inequality implies

$$V(\tilde{\mathfrak{x}})V(\bar{\mathfrak{x}}) \leq V_1^2(\tilde{\mathfrak{x}}, \bar{\mathfrak{x}})$$

which is the classical Minkowski inequality.

In turn, assuming that a convex figure $\bar{\mathfrak{x}}$ satisfies the conditions of the optimality criterion for Problem 2.1 and reversing the above arguments, we see that $\bar{\mathfrak{x}}$ is a solution to the internal isoperimetric problem. So, with the structure of a solution available, we may easily demonstrate the corresponding criterion by standard methods. However, it is highly unlikely that we may guess a solution to a general problem since, strictly speaking, each linear program reduces to an instance of Problem 2.3.

3. Blaschke Structure in Programming Isoperimetric-Type Problems

As was mentioned, many classical extremal problems of geometry are not convex problems in the Minkowski structure, which leads to serious inconveniences in analyzing the simplest problems. Even the classical isoperimetric problem becomes "convex on the wrong side" if posed in the Minkowski structure in three dimensions. An obvious necessary optimality condition is written down in gradients and takes the form $\bar{\alpha}\mu_1(\bar{\mathfrak{x}}, \mathfrak{z}_N) = \mu(\bar{\mathfrak{x}})$, where μ_1 is the corresponding mixed surface area function.

By the well-known Alexandrov–Volkov Theorem we may claim that the isoperimetric problem has no regular solutions other than a ball. The final conclusion about solution is impossible on using just the above argument but requires extra information. These complications become plenty when the number of subsidiary constraints grows. The idea of surpassing these obstacles belongs to W. Blaschke [3, p. 135]. His observation amounts to the following: On summing the surface area functions of convex bodies rather than the bodies themselves we transform the isoperimetric problem into a convex program. Clearly, this observation cannot solve the problem in its own right. To implement the Blaschke idea, we are to formalize the appropriate parametrization, describe the dual cone, and calculate the subdifferential of volume. These details are available from Section 1. So, we are in a position to address particularities of programming extremal problems in the Blaschke structure.

3.1. Generalized Isoperimetric Problem. *Suppose that $\mathfrak{y}_1, \dots, \mathfrak{y}_m$ are given convex bodies in $\mathbb{R}^N$ and $b_1, \dots, b_m \in \mathbb{R}_+$. Among the convex figures satisfying the inequalities $\langle \mathfrak{y}_k, \mathfrak{x} \rangle \le b_k$ $(k = 1, \dots, m)$, find a convex body of greatest volume.*

3.2. Optimality Criterion. *A feasible convex body $\bar{\mathfrak{x}}$ is a solution to the generalized isoperimetric problem if and only if there some reals $\bar{\alpha}_1, \dots, \bar{\alpha}_m \in \mathbb{R}_+$ satisfying the complementary slackness conditions and such that $\bar{\mathfrak{x}} =_{\mathbb{R}^N} \bar{\alpha}_1\mathfrak{y}_1 + \dots + \bar{\alpha}_m\mathfrak{y}_m$.*

Problem 3.1 perfectly illustrates the distinction between the ways of programming in the Blaschke and Minkowski structures. In the latter structure Problem 3.1 fails to be convex for $N \ge 3$ and a necessary optimality condition for it looks like

$$\mu(\bar{\mathfrak{x}}) = \mu_1 \left(\bar{\mathfrak{x}}, \sum_{j=1}^{m} \bar{\alpha}_j \mathfrak{y}_j \right).$$

To extract the appropriate representation of a solution from the last condition becomes possible only on assuming a priori that $\bar{\mathfrak{x}}$, $\mathfrak{y}_1, \dots, \mathfrak{y}_m$ are regular and appealing further to the Alexandrov–Volkov Theorem.

As another typical example, we address the following problem with an operator-type constraint imposed on curvature. To save room, we consider a single general restriction.

3.3. Lindelöf Problem.

Assume given some convex bodies $\mathfrak{x}_0$ *and* $\mathfrak{y}$. *Among the convex figures satisfying the conditions*

(1) $\mu(\mathfrak{x}) \leq \mu(\mathfrak{x}_0)$;

(2) $\langle \mathfrak{y}, \mathfrak{x} \rangle \leq \langle \mathfrak{y}, \bar{\mathfrak{x}} \rangle$,

find a convex body of greatest volume.

It is worth observing that a program with constraints like 3.3(1) is rather complicated in the space of convex sets $[\mathcal{V}_N]$ since its feasible set lacks convexity for $N \geq 3$.

3.4. Optimality Criterion. *If there is some* $\bar{\alpha} \in \mathbb{R}_+$ *such that*

(1) $\bar{\mathfrak{x}} \geq \bar{\alpha}\mathfrak{y}$;

(2) $\bar{\mathfrak{x}}(z) = \bar{\alpha}\mathfrak{y}(z)$ *for all* $z \in \operatorname{supp}(\mu(\mathfrak{x}_0) - \mu(\mathfrak{x}))$,

then a feasible convex body is a solution to the Lindelöf problem.

4. Comparison Between the Blaschke and Minkowski Structures

Isoperimetric-type problems with subsidiary constraints on location of convex figures comprise in a sense a unique class of meaningful problems of mathematical programming which admits two essentially different parametrization. The principal features of the latter are seen from the table.

OBJECT OF PARAMETRIZATION	MINKOWSKI'S STRUCTURE	BLASCHKE'S STRUCTURE
cone of sets	$\mathcal{V}_N/\mathbb{R}^N$	$\mathcal{A}_N$
dual cone	$\mathcal{V}_N^*$	$\mathcal{A}_N^*$
positive cone	$\mathcal{A}_N^*$	$\mathcal{A}_N$
typical linear functional	$V_1(\mathfrak{z}_N, \cdot)$ (width)	$V_1(\cdot, \mathfrak{z}_N)$ (area)
concave functional (power of volume)	$V^{1/N}(\cdot)$	$V^{(N-1)/N}(\cdot)$
simplest convex program	isoperimetric problem	Urysohn's problem
operator-type constraint	inclusion of figures	inequalities on "curvatures"
Lagrange's multiplier	surface	function
differential of volume at a point $\bar{\mathfrak{x}}$ is proportional to	$V_1(\bar{\mathfrak{x}}, \cdot)$	$V_1(\cdot, \bar{\mathfrak{x}})$

This table shows that the classical isoperimetric problem is not a convex program in the Minkowski structure for $N \geq 3$. In this event a necessary optimality condition leads to a solution only under extra regularity conditions. Whereas in the Blaschke structure this problem is a convex program whose optimality criterion reads: "Each solution is a ball."

The task of choosing an appropriate parametrization for a wide class of problems is practically unstudied in general. In particular, those problems of geometry remain unsolved which combine constraints each of which is linear in one of the two vector structures on the set of convex figures. The simplest example of an unsolved "combined" problem is the internal isoperimetric problem in the space $\mathbb{R}^N$ for $N \geq 3$.

The above geometric facts make it reasonable to address the general problem of parametrizing the important classes of extremal problems of practical provenance.

COMMENTS

An extensive literature deals with the extremal problems of geometry. We mention only the classical surveys by Bonnesen and Fenchel [4] Busemann [5], and Hadwiger [6]. We distinguish the series of articles by A. D. Alexandrov on the theory of mixed volumes which is reprinted in [2]. It is in these articles that the technique of functional analysis was firstly applied to the problems of convex geometry. A. D. Alexandrov gave the most fundamental and profound applications of the

Minkowski structure to the extremal problems of convex geometry. We also mention a rather recent monographs [7–9] which survey all necessary facts from convex geometry and contain a detailed bibliography on isoperimetric-type inequalities and problems. We especially point to a brilliant monograph by Hörmander [10] which includes in particular a detailed exposition of the Brunn–Minkowski theory. By the way, Hörmander is the author of one of the first articles about parametrization of the classes of convex subsets of general topological vector spaces. The above-indicated sources contain all necessary facts from the theory of mixed volumes and surface area functions. As regards the Urysohn problem, see [11] and the survey by L. A. Lyusternik [12].

The general scheme for applying the Minkowski duality to the extremal problems of convex geometry was set forth in the survey [13] and in its expanded version, the book [14]. This scheme rests on combining the ideas of mathematical programming belonging to L. V. Kantorovich and the functional-analytical methods in the theory of convex bodies which were propounded by A. D. Alexandrov. The articles [13, 14] give a more detailed exposition of the technique of deriving optimality criteria for the problems of Section 2. Unfortunately, the results of the articles [13, 14] are stated using unnecessarily bulky descriptions for the duals to the cones of feasible directions. The simplification we use in the present article was firstly formulated in [15]. Simplest linear programs for the finitely-parametrized families of convex figures in the presence of current polyhedra as free boundaries are considered in [14]. Problem 2.14 with a current straight line is stated in the present article for the first time. A detailed analysis of external and internal isoperimetric problems with current polyhedra will appear elsewhere. We only point out that, in analyzing these problems, it is especially fruitful to combine the above tricks of programming with the classical symmetrization technique stemming from Steiner, Schwarz, et al.

As regards the Blaschke addition, cf. [3, 7, 15–18]. The construction of Section 1 is in fact implemented in [17]. Proposition 1.6 in space is indicated in [16], see also [15]. The formalism for programming in the Blaschke structure was suggested in [15]. The same article gives the closure of the directional derivative of volume. Elaborating the argument of [15], D. M. Goĭkhman proved the closure of the derivative itself in [19].

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