Scientific and Pedagogical Contributions of S. L. Sobolev

Sergeĭ L′vovich Sobolev will always rank among the most prominent scientists of the twentieth century who tremendously influenced the outlook of the modern science and culture. Sobolev discovered a new concept of derivative that changed differential calculus, the mathematical cornerstone of the natural sciences. He enriched the researcher's intellectual thesaurus with the marvelous concepts and technologies that opened ways to many intractable problems of long standing.

Sobolev was a founding father of various mathematical schools and centers throughout the world as well as discoverer of new promising sections of applied mathematics, mechanics, and computational mathematics.

Sobolev was born in St. Petersburg on October 6, 1908 in the family of Lev Aleksandrovich Sobolev, a solicitor. Sobolev's grandfather on his father's side descended from a family of Siberian Cossacks.

Sobolev was bereaved of his father in the early childhood and was raised by his mother Natal′ya Georgievna who was a highly-educated teacher of literature and history. His mother also had the second speciality: she graduated from a medical institute and worked as a tutor at the First Leningrad Medical Institute. She cultivated in Sobolev the decency, indefatigability, and endurance that characterized him as a scholar and personality.

Sobolev fulfilled the program of secondary school at home, revealing his great attraction to mathematics. During the Civil War he and his mother lived in Kharkov. When living there, he studied at the preparatory courses of an evening technical school for one semester. At the age of 15 he completed the obligatory programs of secondary school in mathematics, physics, chemistry, and other natural sciences, read the classical pieces of the Russian and world literature as well as many books on philosophy, medicine, and biology.

After the family had transferred from Kharkov to Petrograd in 1923, Sobolev entered the graduate class of School No. 190 and finished with honors in 1924, continuing his study at the First State Art School in the piano class. At the same year he entered the Faculty of Physics and Mathematics of Leningrad State University (LSU) and attended the lectures of Professors N. M. Günter, V. I. Smirnov, G. M. Fikhtengolts, and others. He made his diploma on the analytic solutions of a system of differential equations with two independent variables under the supervision of Günter. Those years LSU was already a large mathematical research center maintaining the remarkable traditions of the Petersburg mathematical school famous for the profound discoveries by P. L. Chebyshev, A. M. Lyapunov, and A. A. Markov.

After graduation from LSU in 1929, Sobolev started his work at the Theoretical Department of the Leningrad Seismological Institute. In a close cooperation with Smirnov he then solved some fundamental problems of wave propagation. It was Smirnov whom Sobolev called his teacher alongside Günter up to his terminal days.

In 1930 Sobolev published an article on wave propagation in an inhomogeneous medium in the Proceedings of the Seismological Institute. This article and his subsequent publications on the same subject remain remarkable from a mathematical viewpoint as originating the celebrated Sobolev method for solving the Cauchy problem for second order hyperbolic equations. Many important solutions of the wave equation, e.g. solutions of the zero degree of homogeneity, are functionally invariant. Reflecting functionally invariant solutions in a plane boundary under the classical boundary conditions, we obtain functionally invariant solutions again. Using the new method, Sobolev and Smirnov explicitly solved the famous Lamb problem about the displacement of an elastic half-plane under a concentrated impulse. The three-dimensional axisymmetric case of the Lamb problem was also solved by applying the superposition principle. Indeed, if a plane step wave is incident on a corner (equalling zero before the wave front and unity behind the latter) then the solution has the zero degree of homogeneity. The technique of homogeneous functionally invariant solutions turned out rather convenient here.

Since 1932 Sobolev worked at the Steklov Mathematical Institute in Leningrad; and since 1934, in Moscow. He continued the study of hyperbolic equations and proposed a new method for solving the Cauchy problem for a hyperbolic equation with variable coefficients. This method was based on a generalization of the Kirchhoff formula. Research into hyperbolic equations led Sobolev to revising the classical concept of a solution to a differential equation. The concept of a generalized or weak solution of a differential equation was considered earlier. However, it was exactly in the works by Sobolev that this concept was elaborated and applied systematically. Sobolev posed and solved the Cauchy problem in spaces of functionals, which was based on the revolutionary extension of the Eulerian concept of function and declared 1935 as the date of the birth of the theory of distributions.

In 1933–1935 Sobolev published a series of articles on the Cauchy problem for hyperbolic equations, demonstrating the unique solvability of the Cauchy problem in spaces of generalized functions. These works played an important role in development of the modern theory of partial differential equations.

Sobolev suggested to solve the Cauchy problem in the space of functionals. This rejected the standard understanding that any solution is a function. Sobolev considered a differential equation as solved even in the cases when available are only the arbitrary integral indices of the behavior if the process under study. That is how science was enriched with a new understanding of the key principles of forecast and prognosis.

In 1755 Euler gave his universal definition of function which was perceived as the most general and perfect during almost two hundred years. The generalized derivatives in the sense of Sobolev are not covered by the Eulerian concept of function. Differentiation by Sobolev rests on the new understanding of interrelations between mathematical magnitudes. A distribution is defined implicitly through the integrals calculated for all members of a class of test functions to be taken in advance.

The apparatus of generalized functions gave rise to new methods in the theory of partial differential equations. These new methods open a way to solving many problems whose solution was long sought for, to putting many previously obtained results into a final shape, and to formulating and solving new problems. The new apparatus and related concepts and methods, which were developed rapidly in the 1950s by L. Schwartz, I. M. Gelfand, and other researchers, momentarily changed the outlook of many sections of the theory of differential equations. With his definition of generalized derivative, Sobolev enriched mathematics with the spaces of functions whose weak derivatives are integrable to some power. These are now called *Sobolev spaces.*

Let f and g be locally integrable functions on an open subset G of $\mathbb{R}^n$, and let α be a multi-index. A function g, denoted by $D^\alpha f$, is the *generalized derivative in the Sobolev sense* or *weak derivative* of f of order α provided that

$$\int_G f(x) D^\alpha \varphi(x)\,dx = (-1)^{|\alpha|} \int_G g(x)\varphi(x)\,dx$$

for every *test function* φ, i.e. such that the support of φ is a compact subset of G and φ is $|\alpha| = \alpha_1 + \dots + \alpha_n$ times continuously differentiable in G, where $D^\alpha\varphi$ is the classical derivative of φ of order α. The vector space W_p^l, with $p \geq 1$, of the (cosets of) locally integrable functions f on G whose all weak derivatives $D^\alpha f$ with $|\alpha| \leq l$ are p-integrable in G becomes a Banach space under the norm:

$$\|f\|_{W_p^l} = \left(\int\limits_G |f|^p \, dx \right)^{1/p} + \sum_{|\alpha|=l} \left(\int\limits_G |D^\alpha f|^p \, dx \right)^{1/p}.$$

Sobolev found the general criteria for equivalence of various norms on W_p^l and showed that these spaces are the natural environment for posing the boundary value problems for elliptic equations. This conclusion was based on his thorough study of the properties of Sobolev spaces. The most important facts are *embedding theorems*. Each embedding theorem estimates the operator norm of an embedding, yileding special inequalities between the norms of one and the same function inside various spaces. Basing on embedding theorems, Sobolev found a correct statement of boundary value problems for elliptic equations in multidimensional domains when boundary conditions are given on the manifolds of various dimensions and proved the unique existence of solutions of these problems.

The spaces of functions with weak derivatives and embedding theorems became the classical tools of the modern mathematics and brought Sobolev well-deserved world recognition.

Sobolev was an outstanding teacher. His brilliant lectures were delivered to the students of the Leningrad Electrotechnical Institute as well as the state universities of Leningrad, Moscow, and Novosibirsk. These lectures laid the grounds for his popular textbooks and monographs. The influence of the ideas and methods of Sobolev was so great that many scientists feel themselves the disciples of Sobolev despite the fact that never were his students.

The contributions of Sobolev brought him recognition in the USSR. In 1933 Sobolev was elected a corresponding member of the Academy of Sciences at the age of 24 years. In 1939 he became a full member of the Academy and remained the youngest academician for many years.

The series of Sobolev's papers on almost periodic solutions of the wave equations initiated a new area of the theory of differential equations with deals with the behavior at large time of the solutions of boundary value problems for nonstationary equations.

Inspired by military applications in the 1940s, Sobolev studying the system of differential equations describing small oscillations of a rotating fluid. He obtained the conditions for stability of a rotating body with a filled-in cavity which depend on the shape and parameters of the cavity. Moreover, he elaborated the cases in which the cavity is a cylinder or ellipsoid of rotation. This research by Sobolev signposted another area of the general theory which concerns the Cauchy and boundary value problems for the equations and systems that are not solved with respect to higher time derivatives.

In the grievous years of WW II from 1941 to 1944 Sobolev occupied the position of the director of the Steklov Mathematical Institute.

Sobolev was one of the first scientists who foresaw the future of computational mathematics and cybernetics. From 1952 to 1960 he held the chair of the first national department of computational mathematics at Moscow State University. This department has played a key role in the development of this important area of the today's mathematics. As early as in the pre-WW-II years Sobolev published a few papers on estimation of the sums of values of functions on a grid. These papers gave the first instances of difference analogs of embedding theorems. This direction of research, initiated by Sobolev, gained substantial development and is now an indispensable tool for estimating the errors of grid solutions. The qualitative study of solutions to difference equations and their stability for many classes of grid problems is reduced to the analysis of behavior of the Green's functions of grid problems. Sobolev discovered some exact estimates for the asymptotic behavior of the difference Green's function for the Laplace equation.

While studying the convergence and stability of algorithms for solution of the problems of mathematical physics, Sobolev introduced some fruitful concepts of approximate analysis: in particular, the concepts of regular and irregular closures of a computational algorithm. If the closure of an algorithm is regular, then we may expect that the algorithm be stable under various perturbations. These contributions by Sobolev became a source of the general theory of computational algorithms which is devoted to the abstract study of the techniques and methods for solving large systems of equations.

Addressing the problems of computational mathematics, Sobolev lavishly applied the apparatus of the modern sections of the theoretical core of mathematics. It is typical for him to pose the problems of computational mathematics within functional analysis. Winged are his words that "to conceive the theory of computations without Banach spaces is impossible just as trying to conceive it without computers."

It is worthwhile to emphasize the great role in the uprise of cybernetics, genetics, and other new areas of research in this country which was played by the publications and speeches of Sobolev who valiantly defended the new trends in science from the ideologized obscurantism.

It is difficult to overrate the contribution of Sobolev to the design of the nuclear shield of this country. From the first stages of the atomic project of the USSR he was listed among the top officials of Laboratory No. 2 which was renamed for secrecy reasons into the Laboratory of Measuring Instruments (abbreviated as LIPAN in Russian). Now

LIPAN lives as the Kurchatov Center. The main task of the joint work with I. K. Kikoin was the implementation of gaseous diffusive uranium enrichment for creation of a nuclear explosive device.

Sobolev administered and supervised various computational teams, studied the control of the industrial processes of isotope separation, struggled for the low costs of production and made decisions on many managerial and technological matters. For his contribution to the A-bomb project Sobolev twice gained a Stalin Prize of the First Degree. In January of 1952 Sobolev was awarded with the highest decoration of the USSR: he was declared the Hero of the Socialist Labor for exceptional service to the state.

Sobolev's research was inseparable from his management in science. At the end of the 1950s M. A. Lavrent′ev, S. L. Sobolev, and S. A. Khristianovich came out with the initiative to organize a new big scientific center, the Siberian Division of the Academy of Sciences. For many scientists of the first enrolment to the Siberian Division it was the example of Sobolev, his authority in science, and the attraction of his personality that yielded the final argument in deciding to move to Novosibirsk. The Siberian period of Sobolev's life in science was marked by the great achievements in the theory of cubature formulas. Approximate integration is one of the main problems in the theory of computations—the cost of computation of multidimensional integrals is extremely high. The problem of optimizing the integration formulas becomes in the up-to-date mentality the problem of minimizing of the norm of the error on some function space. Sobolev suggested new approaches to the problem and discovered marvelous classes of optimal cubature formulas.

The role of Sobolev cannot be overestimated in the rise of the Siberian mathematical school. The founder of the Institute of Mathematics of the Siberian Division and its director in the course of a quarter of century, Sobolev made a decisive contribution to the scientific destiny of the Institute which now bears his name.

Sobolev's achievements were highly appraised in this country and abroad. He was decorated with many orders, medals, and other signs of distinction. He was an honorary doctor of Humbold University in Berlin, an honorary doctor of Charles University in Prague, and an honorary doctor of the Higher School of Architecture and Construction in Weimar. Sobolev was a foreign member of the Academy of Sciences of the Institute of France, a foreign member of the Accademia Nazionale dei Lincei in Rome, a foreign member of the GDR Academy of Sciences in Berlin, an honorary member of the Edinburgh Royal Society, as well as an honorary member of the Moscow Mathematical Society and American Mathematical Society.

Sobolev merits were decorated by many medals and prizes. In 1988 he was awarded the highest prize of the Russian Academy of Sciences, the Lomonosov Gold Medal.

Sobolev died on January 3, 1989 and was buried at the Novodevichiĭ Cemetery in Moscow. His path in life is an exemplar of service to science and the homeland.

Many remarkable articles are written by the pen of Sobolev, implementing his contribution to science. To chart the creative legacy of Sobolev is the aim of this booklet.

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