One Puzzling Definition and Model Theory

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Agenda

The mainstream of mathematics deflects from a few creative ideas of Sobolev. A central place among the latter is occupied by the marvelous universal understanding of computational mathematics as the science of finite approximations to general (not necessarily metrizable) compacta. This revolutionary and challenging definition was given in the joint talk submitted by Sobolev, Lyusternik, and Kantorovich at the Third All-Union Mathematical Congress in 1956.

This talk will focus on the trends of interaction between model theory and the methods of domination, discretization, and scalarization which are in a sense some instances of approximation in disguise.

Discretization

Approximation consists in replacing functions spaces and operators by their analogs in finite dimensions. Infinitesimal methods suggest a background, providing new schemes for discretization of general compacta. As an approximation to a compact topological space we may take an arbitrary internal subset containing all standard elements of the space under approximation. It seems to be more than a coincidence that on October 6 we celebrate not only the birth of Serge Sobolev (1908–1989) but also the birth of Abraham Robinson (1918–1974).

Scalarization

Scalarization in the most general sense means reduction to numbers. Since each number is a measure of quantity, the idea of scalarization is clearly of a universal importance to mathematics. The deep roots of scalarization are revealed within Boolean valued analysis.

Boolean Methods

Cohen's final solution of the problem of the cardinality of the continuum within ZFC gave rise to the Boolean-valued models by Vopěnka, Scott, and Solovay. Takeuti coined the term "Boolean-valued analysis" for applications of the new models to functional analysis.

Let B be a complete Boolean algebra. Given an ordinal α, put

$$V_\alpha^{(B)} := \{x :$$
$$(\exists \beta \in \alpha)\ x : \mathrm{dom}(x) \to B \ \&\ \mathrm{dom}(x) \subset V_\beta^{(B)}\}.$$

The *Boolean-valued universe* $\mathbb{V}^{(B)}$ is

$$\mathbb{V}^{(B)} := \bigcup_{\alpha \in \mathrm{On}} V_\alpha^{(B)},$$

with On the class of all ordinals. The truth value $[\![\varphi]\!] \in B$ is assigned to each formula φ of ZFC relativized to $\mathbb{V}^{(B)}$.

Descent

Given φ, a formula of ZFC, and y, a subset $\mathbb{V}^{B}$; put $A_\varphi := A_{\varphi(\cdot,\ y)} := \{x : \varphi(x,\ y)\}$. The *descent* $A_\varphi{\downarrow}$ of a class A_φ is

$$A_\varphi{\downarrow} := \{t : t \in \mathbb{V}^{(B)} \ \&\ [\![\varphi(t,\ y)]\!] = \mathbb{1}\}.$$

If $t \in A_\varphi{\downarrow}$, then it is said that t *satisfies* $\varphi(\cdot,\ y)$ *inside* $\mathbb{V}^{(B)}$.

The *descent* $x{\downarrow}$ of an element $x \in \mathbb{V}^{(B)}$ is defined by the rule

$$x{\downarrow} := \{t : t \in \mathbb{V}^{(B)} \ \&\ [\![t \in x]\!] = \mathbb{1}\},$$

i.e. $x{\downarrow} = A_{\cdot \in x}{\downarrow}$. The class $x{\downarrow}$ is a set. Moreover, $x{\downarrow} \subset \mathrm{mix}(\mathrm{dom}(x))$, where mix is the symbol of the taking of the *strong cyclic hull*. If x is a nonempty set inside $\mathbb{V}^{(B)}$ then

$$(\exists z \in x{\downarrow})[\![(\exists z \in x)\ \varphi(z)]\!] = [\![\varphi(z)]\!].$$

The Reals

There is an object $\mathscr{R}$ inside $\mathbb{V}^{(B)}$ modeling $\mathbb{R}$, i. e.,

$$[\![\mathscr{R} \text{ is the reals}]\!] = \mathbb{1}.$$

Let $\mathscr{R}{\downarrow}$ be the descend of the carrier $|\mathscr{R}|$ of the algebraic system $\mathscr{R} := (|\mathscr{R}|, +, \cdot, 0, 1, \leq)$ inside $\mathbb{V}^{(B)}$. Implement the descent of the structures on $|\mathscr{R}|$ to $\mathscr{R}{\downarrow}$ as follows:

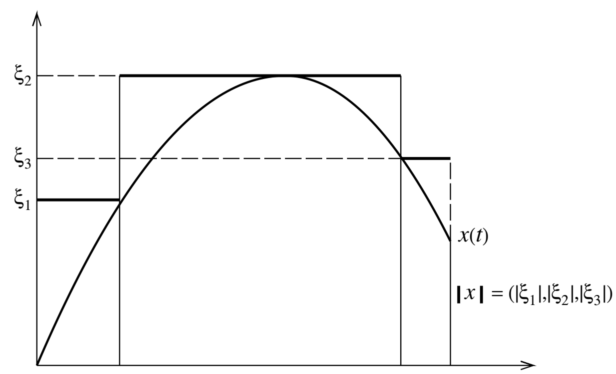
$$x + y = z \leftrightarrow [\![x + y = z]\!] = \mathbb{1};$$
$$xy = z \leftrightarrow [\![xy = z]\!] = \mathbb{1};$$
$$x \leq y \leftrightarrow [\![x \leq y]\!] = \mathbb{1};$$
$$\lambda x = y \leftrightarrow [\![\lambda^{\wedge} x = y]\!] = \mathbb{1}$$
$$(x, y, z \in \mathscr{R}{\downarrow},\ \lambda \in \mathbb{R}).$$

Gordon Theorem. *$\mathscr{R}{\downarrow}$ with the descended structures is a universally complete Kantorovich space with base $\mathscr{B}(\mathscr{R}{\downarrow})$ isomorphic to B.*

Thus, L_p-spaces ascend to the reals.

Norming Sequences

$$\mathbf{|}(\xi_1, \xi_2, \ldots)\mathbf{|} = (|\xi_1|, |\xi_2|, \ldots, |\xi_{N-1}|, \sup_{k \geq N} |\xi_k|) \in \mathbb{R}^N.$$



Domination

Let X and Y be real vector spaces lattice-normed with Dedekind complete vector lattices E and F. In other words, given are some lattice-norms $|\cdot|_X$ and $|\cdot|_Y$. Assume further that T is a linear operator from X to Y and S is a positive operator from X into Y satisfying

$$\begin{array}{ccc} X & \xrightarrow{\quad T \quad} & Y \\ {\scriptstyle |\cdot|_X}\downarrow & & \downarrow{\scriptstyle |\cdot|_Y} \\ E & \xrightarrow[\quad S \quad]{} & F \end{array}$$

Moreover, in case

$$|Tx|_Y \leq S|x|_X \quad (x \in X),$$

we call S the *dominant* or *majorant* of T.

Abstract Norm

If the set of all dominants of T has the least element, then the latter is called the *abstract norm* or *least dominant* of T and denoted by $⦀T⦀$. Hence, the least dominant $⦀T⦀$ is the least positive operator from E to F such that

$$⦀Tx⦀ \leq ⦀T⦀(⦀x⦀) \quad (x \in X).$$

Domination consists in deciphering the properties of an operator from reflections in its abstract norm.

Domination and Model Theory

These days the development of domination proceeds within the frameworks of Boolean valued analysis. All principal properties of lattice normed spaces represents the Boolean valued interpretations of the relevant properties of classical normed spaces. The most important interrelations here are as follows: Each Banach space inside a Boolean valued model becomes a universally complete Banach–Kantorovich space in result of the external deciphering of constituents. Moreover, each lattice normed space may be realized as a dense subspace of some Banach space in an appropriate Boolean valued model. Finally, a Banach space X results from some Banach space inside a Boolean valued model by a special machinery of bounded descent if and only if X admits a complete Boolean algebra of norm-one projections which enjoys the cyclicity property. The latter amounts to the fact that X is a Banach–Kantorovich space and X is furnished with a mixed norm.

Approximation

Study of stability is accomplished sometimes by introducing various epsilons in appropriate places. Exact calculations with epsilons and sharp estimates are sometimes bulky and slightly mysterious. Some alternatives are suggested by actual infinities which is lavishly provided by *infinitesimal analysis*.

As an approximation to a compact space we may take an arbitrary internal finite subset containing all standard elements of the space under approximation.

Hypodiscretization

The analysis of the equation $Tx = y$, with $T : X \to Y$ a bounded linear operator between some Banach spaces X and Y, consists in choosing finite-dimensional vector spaces X_N and Y_N and the corresponding embeddings $\imath_N$ and $\jmath_N$:

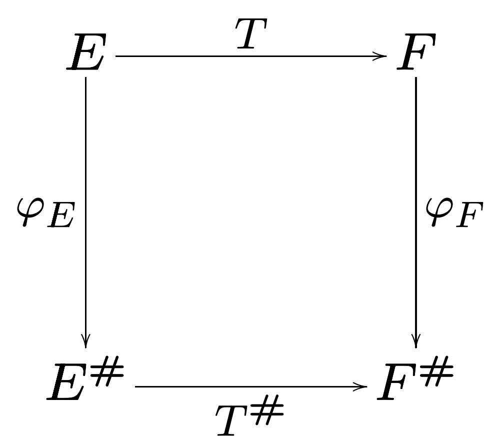
$$\begin{array}{ccc} X & \xrightarrow{\;\;T\;\;} & Y \\ {\scriptstyle \imath_N}\Big\uparrow & & \Big\uparrow{\scriptstyle \jmath_N} \\ X_N & \xrightarrow[\;\;T_N\;\;]{} & Y_N \end{array}$$

In this event, the equation $T_N x_N = y_N$ is viewed as a finite-dimensional approximation to the original problem.

Hypoapproximation or approximation by inner space is the most widespread instance of discretization.

Hyperdiscretization

Nonstandard models open up the opportunity of another type of discretization, *hyperapproximation or approximation by outer space*:



Here E and F are normed spaces over the same scalars, while T is a bounded linear operator from E to F, and $^{\#}$ symbolizes a nonstandard hull.

The Hull of a Space

Let * is the symbol of the Robinsonian standardization. Let $(E, \|\cdot\|)$ be an internal normed space over $^*\mathbb{F}$, with $\mathbb{F} := \mathbb{R}; \mathbb{C}$. As usual, $x \in E$ is a *limited* element provided that $\|x\|$ is a limited real (whose modulus has a standard upper bound by definition). If $\|x\|$ is an infinitesimal then x is also referred to as an *infinitesimal*. Denote by $\operatorname{ltd}(E)$ and $\mu(E)$ the external sets of limited elements and infinitesimals of E. The set $\mu(E)$ is the *monad* of the origin in E. Clearly, $\operatorname{ltd}(E)$ is an external vector space over $\mathbb{F}$, and $\mu(E)$ is a subspace of $\operatorname{ltd}(E)$. Put $E^\# = \operatorname{ltd}(E)/\mu(E)$ and endow $E^\#$ with the natural norm $\|\varphi x\| := \|x^\#\| := \operatorname{st}(\|x\|) \in \mathbb{F}$ for all $x \in \operatorname{ltd}(E)$ Here $\varphi := \varphi_E := (\cdot)^\# : \operatorname{ltd}(E) \to E^\#$ is the canonical homomorphism, and st takes the standard part of a limited real. This $(E^\#, \|\cdot\|)$ is an external normed space called the *nonstandard hull* of E.

The Hull of an Operator

Suppose now that E and F are internal normed spaces and $T : E \to F$ is an internal bounded linear operator. The set of reals $c(T) := \{C \in {}^*\mathbb{R} : (\forall x \in E)\|Tx\| \leq C\|x\|\}$ is internal and bounded. Recall that $\|T\| := \inf c(T)$. If the norm $\|T\|$ of T is limited then the classical normative inequality $\|Tx\| \leq \|T\|\,\|x\|$ valid for all $x \in E$, implies that $T(\mathrm{ltd}(E)) \subset \mathrm{ltd}(F)$ and $T(\mu(E)) \subset \mu(F)$. Hence, we may soundly define the descent of T to the factor space $E^{\#}$ as the external operator $T^{\#} : E^{\#} \to F^{\#}$, acting by the rule

$$T^{\#}\varphi_E x := \varphi_F Tx \quad (x \in E).$$

The operator $T^{\#}$ is linear (with respect to the members of $\mathbb{F}$) and bounded; moreover, $\|T^{\#}\| = \mathrm{st}(\|T\|)$. The operator $T^{\#}$ is called the *nonstandard hull* of T.

State of the Art

Adaptation of the ideas of model theory projects among the most important directions of developing the synthetic methods of mathematics. This approach yields new models of numbers, spaces, and types of equations. The content expands of all available theorems and algorithms. The whole methodology of mathematical research is enriched and renewed, opening up absolutely fantastic opportunities. We can now use actual infinities and infinitesimals, transform matrices into numbers, spaces into straight lines, and noncompact spaces into compact spaces, yet having still uncharted vast territories of new knowledge.

Vistas of the Future

Quite a long time had passed until the classical functional analysis occupied its present position of the language of continuous mathematics. Now the time has come of the new powerful technologies of model theory in mathematical analysis. Not all mathematicians have already gained the importance of modern tools and learned how to use them. However, there is no backward traffic in science, and the new methods are doomed to reside in the realm of mathematics for ever and in a short time they will become as elementary and omnipresent in calculuses and calculations as Banach spaces and linear operators.