Boolean models and simultaneous inequalities

S. S. Kutateladze

The Farkas Lemma, also known as the Farkas—Minkowski Lemma, plays a key role in linear programming and the relevant areas of optimization (cp. [1]). Some rather simple proof of the lemma is given in [2]. The aim of this talk is to demonstrate how Boolean models may be applied to simultaneous linear inequalities with operators. This particular theme is another illustration of the deep and powerful technique of “stratified validity” which is characteristic of Boolean valued analysis [3].

Assume that X is a real vector space, Y is a *Kantorovich space* also known as a Dedekind complete vector lattice or a complete Riesz space. Let $\mathbb{B} := \mathbb{B}(Y)$ be the *base* of Y, i. e., the complete Boolean algebras of positive projections in Y; and let $m(Y)$ be the universal completion of Y. Denote by $L(X, Y)$ the space of linear operators from X to Y. In case X is furnished with some Y-seminorm on X, by $L^{(m)}(X, Y)$ we mean the *space of dominated operators* from X to Y. As usual, $\{T \le 0\} := \{x \in X : Tx \le 0\}$ for $T \in L(X, Y)$.

Theorem 1. *Assume that $A_1, \dots, A_N$ and B belong to $L^{(m)}(X, Y)$.*

The following are equivalent:

(1) *Given $b \in \mathbb{B}$, the operator inequality $bBx \le 0$ is a consequence of the simultaneous linear operator inequalities $bA_1x \le 0, \dots, bA_Nx \le 0$, i. e.,*

$$\{bB \le 0\} \supset \{bA_1 \le 0\} \cap \dots \cap \{bA_N \le 0\}.$$

(2) *There are positive orthomorphisms $\alpha_1, \dots, \alpha_N \in \operatorname{Orth}(m(Y))$ such that*

$$B = \sum_{k=1}^{N} \alpha_k A_k;$$

i. e., B lies in the operator convex conic hull of $A_1, \dots, A_N$.

Theorem 2. *Take A and B in $L(X, Y)$. The following are equivalent:*

(1) $(\exists \alpha \in m(Y))\ B = \alpha A$;

(2) *There is a projection $\varkappa \in \mathbb{B}$ such that*

$$\{\varkappa bB \le 0\} \supset \{\varkappa bA \le 0\};$$

$$\{\neg\varkappa bB \le 0\} \supset \{\neg\varkappa bA \ge 0\}$$

for all $b \in \mathbb{B}$.

References

[1] Floudas C. A., Pardalos P. M. (eds.) Encyclopedia of Optimization. Berlin and New York: Springer, 2009.

[2] Bartl D. A short algebraic proof of the Farkas lemma. SIAM J. Optim., 19 (2008), N. 1, 234–239.

[3] Kusraev A. G., Kutateladze S. S. Introduction to Boolean Valued Analysis. Moscow: Nauka, 2005.

Sobolev Institute of Mathematics

E-mail: sskut@math.nsc.ru