# Contents

# Preface

The subject of the present book is subdifferential calculus. The main source of this branch of functional analysis is the theory of extremal problems. For a start, we explicate the origin and statement of the principal problems of subdifferential calculus. To this end, consider an abstract minimization problem formulated as follows:

$$x \in X, \quad f(x) \to \inf .$$

Here $X$ is a vector space and $f : X \to \overline{\mathbb{R}}$ is a numeric function taking possibly infinite values. In these circumstances, we are usually interested in the quantity $\inf f(x)$, the *value of the problem*, and in a *solution* or an *optimum plan* of the problem (i.e., such an $\bar{x}$ that $f(\bar{x}) = \inf f(X)$), if the latter exists. It is a rare occurrence to solve an arbitrary problem explicitly, i.e. to exhibit the value of the problem and one of its solutions. In this respect it becomes necessary to simplify the initial problem by reducing it to somewhat more manageable modifications formulated with the details of the structure of the objective function taken in due account. The conventional hypothesis presumed in attempts at theoretically approaching the sought reduction is as follows. Introducing an auxiliary function $l$, one considers the next problem:

$$x \in X, \quad f(x) - l(x) \to \inf .$$

Furthermore, the new problem is assumed to be as complicated as the initial problem provided that $l$ is a linear functional over $X$, i.e., an element of the *algebraic dual* $X^{\#}$. In other words, in analysis of the minimization problem for $f$, we consider as known the mapping $f^* : X^{\#} \to \overline{\mathbb{R}}$ that is given by the relation

$$f^*(l) := \sup_{x \in X} (l(x) - f(x)).$$

The $f^*$ thus introduced is called the *Young-Fenchel transform* of the function $f$. Observe that the quantity $-f^*(0)$ presents the value of the initial extremal problem.

The above-described procedure reduces the problem that we are interested in to that of change-of-variable in the Young-Fenchel transform, i.e., to calculation of the aggregate $(f \circ G)$, where $G : Y \to X$ is some operator acting from $Y$ to $X$. We emphasize that $f^*$ is a convex function of the variable $l$. The very circumstance by itself prompts us to await the most complete results in the key case of convexity of the initial function. Indeed, defining in this event the *subdifferential of $f$ at a point* $\bar{x}$, we can conclude as follows. A point $\bar{x}$ is a solution to the initial minimization problem if and only if the next Fermat optimality criterion holds:

$$0 \in \partial f(\bar{x}).$$

It is worth noting that the stated Fermat criterion is of little avail if we lack effective tools for calculating the subdifferential $\partial f(\bar{x})$. Putting it otherwise, we arrive at the question of deriving rules for calculation of the subdifferential of a composite mapping $\partial(f \circ G)(\bar{y})$. Furthermore, the adequate understanding of $G$ as a convex mapping requires that some structure of an ordered vector space be present in $X$. (For instance, the presentation of the sum of convex functions as composition of a linear operator and a convex operator presumes the introduction into $\mathbb{R}^2$ the coordinatewise comparison of vectors.)

Thus, we are driven with necessity to studying operators that act in ordered vector spaces. Among the problems encountered on the way indicated, the central places are occupied by those of finding out explicit rules for calculation of the Young-Fenchel transform or the subdifferential of a composite mapping. Solving the problems constitutes the main topic of subdifferential calculus.

Now the case of convex operators, which is of profound import, appears so thoroughly elaborated that one might speak of the completion of a definite stage of the theory of subdifferentials.

Research of the present days is conducted mainly in the directions related to finding appropriate local approximations to arbitrary not necessarily convex operators. Most principal here is the technique based on the F. Clarke tangent cone which was extended by R. T. Rockafellar to general mappings. However, the stage of perfection is far from being obtained yet. It is worth nonetheless to mention that key technical tricks in this direction lean heavily on subdifferentials of convex mappings.

In this respect we confine the bulk of exposition to the convex case, leaving the vast territory of nonsmooth analysis practically uncharted. The resulting gaps transpire. A slight reassuring apology for us is a pile of excellent recent books and surveys treating raw spots of nonsmooth analysis. The tool-kit of subdifferential theory is quite full. It contains the principles of classical functional analysis, methods of convex analysis, methods of the theory of ordered vector spaces, measure theory, etc.

Many problems of subdifferential theory and nonsmooth analysis were recently solved on using nonstandard methods of mathematical analysis (in infinitesimal and Boolean-valued versions). In writing the book, we bear in mind the intention of (and the demand for) making new ideas and tools of the theory more available for a wider readership. The limits of every book (this one inclusively) are too narrow for leaving an ample room for self-contained and independent exposition of all needed facts from the above-listed disciplines.

We therefore choose a compromising way of partial explanations. In their selection we make use of our decade experience from lecture courses delivered in Novosibirsk and Vladikavkaz (North Ossetian) State Universities.

One more point deserves straightforward clarification, namely, the word "applications" in the title of the book. Formally speaking, it encompasses many applications of subdifferential theory. To list a few, we mention the calculation of the Young-Fenchel transform, justification of the Lagrange principle and derivation of optimality criteria for vector optimization problems. However, much more is left intact and the title to a greater extent reflects our initial intentions and fantasies as well as a challenge to further research.

The first Russian edition of this book appeared in 1987 under the title "Subdifferential Calculus" soon after L. V. Kantorovich and G. P. Akilov passed away. To the memory of the outstanding scholars who taught us functional analysis we dedicate this book with eternal gratitude.

A. G. Kusraev
S. S. Kutateladze