KANTOROVICH SPACES AND OPTIMIZATION

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Abstract. This is an expository talk on interaction between mathematical programming and vector lattices at the Kantorovich Memorial (St. Petersburg, January 9–13, 2004).

1. Optimization and Order

Subdifferential calculus is a branch of functional analysis whose main source is the theory of extremal problems. Consider an abstract minimization problem:

$$x \in X, \quad f(x) \to \inf .$$

Here X is a vector space and $f : X \to \overline{\mathbb{R}}$ is a numeric function with possibly infinite values. We are usually interested in the quantity $\inf f(x)$, the *value* of the problem, and in a *solution* or *optimum plan*; i.e., a point $\bar{x}$ satisfying $f(\bar{x}) = \inf f(X)$. It is a rare occurrence to solve an arbitrary extremal problem explicitly, i.e. to exhibit the value of the problem and one of the solutions. As a rule we must simplify the initial problem by reducing it on using the details of the objective function. The hypothesis behind this reduction is as follows. Introducing an auxiliary function l, we considers the problem:

$$x \in X, \quad f(x) - l(x) \to \inf .$$

This perturbed problem is clearly as complicated as the initial problem provided that l is a linear functional over X, i.e., an element of the *algebraic dual* $X^{\#}$ of X. In other words, addressing the minimization problem for f, we assume known the mapping $f^* : X^{\#} \to \overline{\mathbb{R}}$ given by the relation

$$f^*(l) := \sup_{x \in X} (l(x) - f(x)).$$

This f^* is called the *Young–Fenchel transform* of f. Observe that $-f^*(0)$ is precisely the value of the initial extremal problem.

This reduction makes the original problem into that of change-of-variable in the Young–Fenchel transform, i.e., into calculation of the aggregate $(f \circ G)$, where $G : Y \to X$ is some operator from Y to X. We emphasize that f^* is a convex

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function on the dual of X. This circumstance alone prompts us to await the most convincing results in the key case of convexity of the initial objective function.

Defining the *subdifferential* of f at a point $\bar{x}$ by

$$\partial f(\bar{x}) := \{l \in X^{\#} : \ (\forall x \in X)\,(lx - l\bar{x} \leqslant f(x) - f(\bar{x}))\},$$

we can proceed as follows: A point $\bar{x}$ is a solution to the initial minimization problem if and only if

$$0 \in \partial f(\bar{x}).$$

The last containment is called the *Fermat criterion*. It is worth noting that the Fermat criterion is of little avail if we lack effective tools for calculating the subdifferential $\partial f(\bar{x})$. Putting it otherwise, we arrive at the question of deriving rules for calculation of the subdifferential of a composite mapping $\partial(f \circ G)(\bar{y})$. Furthermore, the adequate understanding of G as a convex mapping requires that some structure of an ordered vector space be present in X.

Thus, we are driven with necessity to studying convex operators between ordered vector spaces. The main task of subdifferential calculus consists in finding explicit rules for calculation of the Young-Fenchel transform and subdifferential of a composite mapping.

2. Dominated Extension

The question of calculating the subdifferential of a composite mapping is closely connected with the Hahn–Banach Dominated Extension Theorem which can be written within subdifferential calculus as follows:

$$\partial(p \circ \iota)(0) = \partial p(0) \circ \iota$$

Here $p : X \to \mathbb{R}$ is a sublinear functional and ι the identity embedding of some subspace of X. Thus, we arrive at the fundamental problem of dominated extension.

2.1. Let p be a sublinear operator from a vector space X into a preordered vector space E with $\operatorname{dom}(p) = X$. The collection of all linear operators from X into E dominated by p is called the *support set* or *subdifferential* of p and denoted by ∂p; symbolically,

$$\partial p := \{T \in L(X, E) : (\forall x \in X)\, Tx \leq p(x)\},$$

where $L(X, E)$ is the space of all linear operators from X into E. We say that the members of ∂p *support* p. Assume that X_0 is a subspace of X and $T_0 : X_0 \to E$ is a linear operator such that $T_0 x \leq p(x)$ for all $x \in X_0$. If for all X, X_0, T_0 and p there exists an operator $T \in \partial p$ extending T_0 from X_0 to the whole of X, then we say that E *admits dominated extension*.

An ordered vector space enjoys the *least upper bound property* if each order bounded subset possesses a least upper bound. A vector lattice with the least upper bound property is *Dedekind complete*. The concept of Dedekind complete vector lattice stems from L. V. Kantorovich (1912–1986) and so the competing terms *Kantorovich space* and *K-space* are appropriate. The next theorem contains a complete solution to the problem of dominated extension for linear operators with values in a preordered vector space.

2.2. Theorem. *A preordered vector space admits dominated extension if and only if it enjoys the least upper bound property.*

Historically this theorem was established in two steps.

2.3. Hahn–Banach–Kantorovich Theorem. *Every Kantorovich space admits dominated extension.*

This theorem by L. V. Kantorovich can be considered as the first theorem of the theory of K-spaces.

2.4. Bonnice–Silvermann–To Theorem. *Each ordered vector space admitting dominated extension is a K-space.*

2.5. The problem of dominated extension originated with the Hahn–Banach Theorem (see [137] for its history). Theorem 1.4.13 (1) as stated was discovered by L. V. Kantorovich in 1935 and was perceived as a generalization serving somewhat bizarre purpose. Now it became a truism that convex analysis and the theory of ordered vector spaces are boon companions. The equivalence between the dominated extension and least upper bound properties was first established by W. Bonnice and R. Silvermann and T.-O. To; a crucial simplification is due to A. D. Ioffe.

3. Ordered Structures Admitting Convex Analysis

Studying subdifferentials, we encounter some algebraic structures with richer structure than that of vector spaces. It is worth noting that the support set of each sublinear operator is *operator-convex* rather than simply convex, i.e. the subdifferential of an operator satisfies the analog of the usual convexity property with scalars replaced by the so-called *multiplicators*, i.e., positive operators dominated by the identity operator. In other words, dealing with the conventional convex objects in vector spaces, we are automatically driven to more involved abstractions of convexity; i.e., convexity in modules over rings (in particular, over the multiplication ring of a K-space with a strong order unit).

There is a deeper reason for interest in convexity-type objects in modules. Applications often supply us with the situation in which the divisibility hypothesis is utterly unreasonable. Such are all problems of integer programming. In this connection it seems to be of considerable importance to clarify the extent to which we may preserve the machinery of subdifferential calculus in abstract algebraic systems.

3.1. We thus let A be an arbitrary lattice-ordered ring with positive unity $\mathbf{1}_A$. This means that A is a ring with some order $\leq$ on A making A into a lattice. Moreover, addition and multiplication are assumed compatible with the order in a conventional and quite natural fashion. In particular, the positive elements A^+ of the ring A constitute a semigroup with respect to addition on A. We now consider a *module* X over A or, in short, an A-*module* X. This module, as well as all to follow, is always assumed *unitary*, i.e. $\mathbf{1}_A x = x$ for all $x \in X$.

3.2. Consider an operator $p : X \to E^{\cdot}$, where $E := E^{\cdot} \cup \{+\infty\}$, as above, and E is an ordered A-module (a little thought about this notion will prompt its natural definition). An operator is called A-*sublinear* or *modular-sublinear* when the ring A is understood from the context provided that

$$p(\pi x + \rho y) \leq \pi p(x) + \rho p(y)$$

for all $x, y \in X$ and $\pi, \rho \in A^+$. It is easy to see that the equality $p(\pi x) = \pi p(x)$ may fail for some $x \in X$ and $\pi \in A^+$ with $\pi \neq 0$, which makes an essential difference with the case of $\mathbb{R}$-sublinear operators. If $p(\pi x) = \pi p(x)$ for all $x \in X$ and $\pi \in A^+$ then p is called an A^+-*homogeneous operator.*

3.3. An A-module E is said to possess the A-*extension property* if for all A-modules X and Y, A-sublinear operator $p : Y \to E$, and homomorphism $T \in \mathrm{Hom}_A(X, Y)$, the following *Hahn-Banach formula* is valid:

$$\partial^A(p \circ T) = \partial^A p \circ T.$$

Moreover, if the subdifferential $\partial p(y)$ is nonempty for each $y \in Y$ then we say that E *admits convex analysis.*

Let E be an ordered abelian group (i.e. an ordered $\mathbb{Z}$-module). Put $E_b := E^+ - E^+$ and assume that E_b is an erased K-space. Recall that by an *erased* K-*space* we mean a group that results from some K-space by neglecting multiplication by real numbers; i.e. by forgetting part of information about the space.

3.4. To describe modules that admit convex analysis we need one more notion. The *orthomorphism ring* $\mathrm{Orth}\,(E)$ of a K-space is the band ${\mathrm{Id}_E}^{dd}$ generated by the identity operator Id_E in the ring $L^r(E)$ of regular operators in E. A subring A of $\mathrm{Orth}\,(E)$ is called *almost rational* if for every $n \in \mathbb{N}$ there exists a decreasing net of multiplicators $(\pi_\xi)_{\xi \in \Xi}$ in A such that for every $y \in E^+$ we have

$$\frac{1}{n} y = o\text{-}\lim_{\xi \in \Xi} \pi_\xi y = \inf_{\xi \in \Xi} \pi_\xi y.$$

It can be easily proven that a ring A is almost rational if and only if every A-sublinear operator is A^+-homogeneous.

3.5. Theorem. *An ordered A-module E admits convex analysis if and only if E_b is an erased K-space and the natural representation of A in E_b is a ring and lattice homomorphism onto an almost rational subring of* $\mathrm{Orth}\,(E_b)$.

4. Kantorovich Space and Boolean Valued Reals

4.1. In the history of functional analysis, the rise of the theory of ordered vector spaces is commonly attributed to the contribution of G. Birkhoff, L. V. Kantorovich, M. G. Kreĭn, H. Nakano, F. Riesz, H. Freudenthal, et al. At present, the theory of ordered vector spaces and its applications constitute a noble branch of mathematics representing, in fact, one of the main sections of contemporary functional analysis.

The credit for finding the most important instance of ordered vector spaces, an order complete vector lattice or a K-space, is due to L. V. Kantorovich. This notion appeared in Kantorovich's first fundamental article on this topic where he wrote, "In this note, I define a new type of space that I call a semiordered linear space. The introduction of such a space allows us to study linear operations of one abstract class (those with values in such a space) as linear functionals."

Here L. V. Kantorovich stated an important methodological principle, the *heuristic transfer principle* for K-spaces.

The heuristic transfer principle by L. V. Kantorovich was corroborated many times in the works of L. V. Kantorovich and his followers. Attempts at formalizing the heuristic ideas by L. V. Kantorovich have started at the initial stages of K-space theory, which result is the so called theorems of identity preservation (sometimes a less exact term "conservation" is also employed). They assert that if a proposition with finitely many function variables is proven for the reals then a similar fact is preserved true for the members of an arbitrary K-space.

Unfortunately, no satisfactory explanation was suggested for the internal mechanism controlling the phenomenon, of identity preservation. In sufficiently clear remained the limits to applying the heuristic transfer principle. The same applies to the general reasons for similarity and parallelism between the reals and their analogs in K-space. The omnipotence and omnipresence of Kantorovich's transfer principle found its full explanation within Boolean valued analysis

4.2. Boolean valued analysis is a branch of functional analysis which uses a special model-theoretic technique, the Boolean valued models of set theory. Boolean valued analysis elaborates the technique of studying properties of an arbitrary mathematical object by means of comparison between its representations in two different set-theoretic models whose construction utilizes principally distinct Boolean algebras. As such, we usually take the classical Cantorian paradise in the shape of the von Neumann universe and a specially-trimmed Boolean valued universe in which the conventional set-theoretic concepts and propositions acquire bizarre interpretations. Usage of two models for studying a single object is a family feature of the so-called *nonstandard methods of analysis.* For this reason, Boolean valued analysis means an instance of nonstandard analysis in common parlance.

Proliferation of Boolean valued analysis stems from the celebrated achievement of P. J. Cohen who proved in the beginning of the 1960s that the negation of the continuum hypothesis, CH, is consistent with the axioms of Zermelo–Fraenkel set theory, ZFC. This result by P. J. Cohen, together with consistency of CH with ZFC established earlier by K. Gödel, proves that CH is independent of the conventional axioms of ZFC.

Intention to obviate obstacles to mastering the technique and results by P. J. Cohen led D. Scott and R. Solovay to constructing the so-called Boolean valued models of ZFC which are not only visually attractive from the standpoint of classical mathematicians but also are fully capable of establishing consistency and independence theorems. P. Vopěnka constructed analogous models in the same period of the early sixties.

4.3. Qualitatively speaking, the *notion of Boolean valued model involves a new conception of modeling* which might be referred to as *modeling by correspondence* or *long-distance modeling.* We explain the particularities of this conception as compared with the routine approach. Encountering two classical models of a single theory, we usually seek for a bijection between the universes of the models. If this bijection exist then we translate predicates and operations from one model to the other and speak about isomorphism between the models. Consequently, this conception of isomorphism implies a direct contact of the models which consists in witnessing to bijection of the universes of discourse.

Imagine that we are physically unable to compare the models pointwise simultaneously. Happily, we take an opportunity to exchange information with the owner of the other model using some means of communication, e.g., by having long-distance calls. While communicating, we easily learn that our interlocutor uses his model to operate on some objects that are the namesakes of ours, i.e., sets, membership, etc.

Since we are interested in ZFC, we ask the interlocutor whether or not the axioms of ZFC are valid in his model. Manipulating the model, he returns a positive answer. After checking that he uses the same inference rules as we do, we cannot help but acknowledge his model to be a model of the theory we are all investigating. It is worth noting that this conclusion still leaves unknown for us the objects that make up his universe and the procedures he uses to distinguish between true and false propositions about these objects.

All in all, the *new conception of modeling implies not only refusal from identification of the universes of discourse but also admission of various procedures for verification of propositions.*

To construct a Boolean valued model, we start with a complete Boolean algebra B, a cornerstone of a special Boolean valued universe $\mathbf{V}^{(B)}$ consisting of "B-valued sets" that are defined recursively as B-valued functions over available B-valued sets. This $\mathbf{V}^{(B)}$ will serve as a universe of discourse for ZFC. Also, we appoint B as the target of the truth value sending each formula of ZFC to a member of B. More explicitly, to each formula φ of ZFC whose every variable ranges now over $\mathbf{V}^{(B)}$, we put in correspondence some element $[\![\varphi]\!]$ of the parental Boolean algebra B. The quantity $[\![\varphi]\!]$ is the *truth value* of φ. We use truth values for validating formulas of ZFC. In particular, every theorem φ of ZFC acquires the greatest truth value $\mathbf{1}_B$, and we declare φ holding true inside the model $\mathbf{V}^{(B)}$.

Application of Boolean valued models to problems of analysis rests ultimately on the procedures of *ascending and descending*, the two natural functors acting between $\mathbf{V}^{(B)}$ and the von Neumann universe $\mathbf{V}$.

4.4. Gordon Theorem. *Let $\mathcal{R}$ be the reals in the model $\mathbf{V}^{(B)}$. Assume further that $\mathcal{R}{\downarrow}$ stands for the descent $|\mathcal{R}|{\downarrow}$ of the carrier of $\mathcal{R}$ furnished with the descended operations and order. Then the algebraic system $\mathcal{R}$ is a universally complete K-space.*

Moreover, there is a (canonical) isomorphism χ from the Boolean algebra B onto the Boolean algebra of band projections $\mathfrak{Pr}(\mathcal{R}{\downarrow})$ (or onto the Boolean algebra of the unit elements $\mathfrak{C}(\mathcal{R}{\downarrow})$ such that the following are valid:

$$\chi(b)x = \chi(b)y \leftrightarrow b \leq [\![x = y]\!],$$
$$\chi(b)x \leq \chi(b)y \leftrightarrow b \leq [\![x \leq y]\!]$$

for all $x, y \in \mathcal{R}{\downarrow}$ and $b \in B$.

4.5. This fact can be reformulated as follows: *a universally complete Kantorovich space is the interpretation of the reals in an appropriate Boolean-valued model.* Moreover, it turns out that every theorem on reals has an analog for the corresponding Kantorovich space. Theorems are transferred by means of some precisely-defined Escher-type procedures: ascent, descent and canonical embedding. Thereby, Kantorovich's assertion that "the elements of a Kantorovich space are generalized numbers" acquires a rigorous mathematical status.

5. Lower Semicontinuous Convex Operators

Let X be a Banach space. Take a convex operator $f : X \to E^{\cdot}$ and a convex subset $C \subset X$. A feasible element $x_0 \in C$ is called an *ideal optimum* or *ideal solution* of the optimization problem $x \in C,\ f(x) \to \inf$ if $f(x_0) = \inf_{x\in C} f(x)$.

Inspecting simple examples, we easily seek that such an ideal optimum may be achieved on an extremely rare occasion. This circumstance impels us to introduce various abstractions of ideal optimality bearing in mind some particular classes of applications. One of these possibilities relates to continuous vector functions with values in a Banach space X.

5.1. In this subsection Q is assumed to be an extremally disconnected compact set and E, the universally complete (extended) K-space $C_\infty(Q)$, i.e., the space of continuous functions from Q into $\overline{\mathbb{R}}$ taking the values $\pm\infty$ only on meager sets. Throughout this section X is a Banach space. Denote by the symbol $C_\infty(Q, X)$ the set of all continuous mappings $z : \operatorname{dom}(z) \to X$, where $\operatorname{dom}(z)$ is a certain dense open subset in Q (particularly, for every z). Introduce the addition operation in $C_\infty(Q, X)$ according to the following rule: if $\operatorname{dom}(z_1) \cap \operatorname{dom}(z_2) = Q_0$ then $\operatorname{dom}(z_1 + z_2) = Q_0$ and $(z_1 + z_2)(t) = z_1(t) + z_2(t)$ for all $t \in Q_0$. Moreover, put $(\lambda z)(t) = \lambda \cdot z(t)$ $(t \in \operatorname{dom}(z))$, where λ is a number. The mappings $z_1, z_2 \in C_\infty(Q, X)$ are said to be *equivalent* if they coincide on $\operatorname{dom}(z_1) \cap \operatorname{dom}(z_2)$. At last, denote by $E(X)$ the quotient set of $C_\infty(Q, X)$ by the above equivalence relation. The usual translation of operations from $C_\infty(Q, X)$ makes $E(X)$ into a vector space.

5.2. For every $u \in C_\infty(Q, X)$ the function $t \mapsto \|u(t)\|$ with $t \in \operatorname{dom}(u)$ is continuous and determines a unique element of the space $E := C_\infty(Q)$ which is one and the same for equivalent u and $v \in C_\infty(Q, X)$. Now given $z \in E(X)$, define the element $\lvert z \rvert \in E$ according to the rule

$$\lvert z \rvert (t) = \|u(t)\| \quad (u \in z,\ t \in \operatorname{dom}(u)).$$

It is easy to check the following properties of the mapping $\lvert \cdot \rvert : E(X) \to E$:

(1) $\lvert z \rvert \geq 0$; $\lvert z \rvert = 0 \leftrightarrow z = 0$;

(2) $\lvert z_1 + z_2 \rvert \leq \lvert z_1 \rvert + \lvert z_2 \rvert$;

(3) $\lvert \lambda z \rvert = |\lambda| \cdot \lvert z \rvert$.

Henceforth, we will take the liberty of identifying each equivalence class $z \in E(X)$ with one of its representatives $u \in z$. If $z \in E(X)$ and $\pi \in \mathfrak{Pr}(E)$, then we denote by the symbol πz the vector-function from $\operatorname{dom}(z)$ into X such that $(\pi z)(t) = z(t)$ for $t \in Q_\pi \cap \operatorname{dom}(z)$ and $\pi z(t) = 0$ for $t \in \operatorname{dom}(z) \setminus Q_\pi$, where Q_π is a clopen subset of Q corresponding to the projection π. Note that in this case $\lvert \pi z \rvert = \pi \lvert z \rvert$. We identify each element $x \in X$ with the constant mapping $t \mapsto x$ $(t \in Q)$ and suppose that $X \subset E(X)$. Now if (x_ξ) is a family in X and (π_ξ) is a partition of unity in $\mathfrak{Pr}(E)$, then $\sum \pi_\xi x_\xi$ is a mapping from $E(X)$ taking the value x_ξ on the set Q_{π_ξ}.

5.3. A linear operator $T : X \to E$ is o-bounded or *dominated* if the image of the unit ball $\{Tx : \|x\| \leq 1\}$ is an order bounded subset of E. The set of all dominated operators from X into E is denoted by the symbol $\mathcal{L}_0(X, E)$. For every $T \in \mathcal{L}_0(X, E)$ there exists the least upper bound

$$\lvert T \rvert := \sup\{|Tx| : x \in X,\ \|x\| \leq 1\}$$

sometimes called the *abstract norm* of T. It is clear that the mapping $\lvert \cdot \rvert : \mathcal{L}_0(X, E) \to E$ satisfies the above-listed properties 5.2 (1)–(3) too.

A mapping $f : X \to \overline{E}$ is *lower semicontinuous* at a point $x_0 \in X$ if

$$f(x_0) = \sup_{n \in \mathbb{N}} \inf \{f(x) : x \in X,\ \|x - x_0\| \leq 1/n\}.$$

5.4. Theorem. *For every lower semicontinuous mapping* $f: X \to \overline{E}$ *there is a unique mapping* $\tilde{f} : E(X) \to \overline{E}$ *satisfying the conditions*

(1) *for arbitrary* $\pi \in \mathfrak{Pr}(E)$ *and* $u, v \in E(X)$ *the equality* $\pi u = \pi v$ *implies* $\pi \tilde{f}(u) = \pi \tilde{f}(v)$*;*

(2) $\tilde{f}$ *is lower semicontinuous in the following sense:*

$$(\forall u \in E(X))\, \tilde{f}(u) = \sup_{\varepsilon \downarrow 0} \inf \{\tilde{f}(v) : v \in E(X),\ |u - v| \leq \varepsilon \mathbf{1}\};$$

(3) $f(x) = \tilde{f}(x)$ *for all* $x \in X$*.*

Moreover, f *is convex, sublinear, or linear if and only if* $\tilde{f}$ *is a convex, sublinear, or linear mapping.*

5.5. It can be shown that, in an appropriate Boolean-valued model $\mathbf{V}^{(B)}$, the space $E(X)$ is represented as conventional Banach space $\mathcal{X} \in \mathbf{V}^{(B)}$ while the operator $\widetilde{f}$ is represented as an element $\varphi \in \mathbf{V}^{(B)}$ serving as a lower semicontinuous convex function $\varphi : \mathcal{X} \to \mathcal{R}^{\cdot}$ inside $\mathbf{V}^{(B)}$. This fact together with Theorem 5.4 allows us to study lower semicontinuous convex operators by means of Boolean-valued analysis.

6. A Vector Valued Variational Principle

6.1. It can be easily seen that a dominated operator $T \in \mathcal{L}_0(X, E)$ belongs to the subdifferential $\partial_\varepsilon f(x)$ if and only if $\widetilde{T} \in \partial_\varepsilon \tilde{f}(x)$. This assertion suggests the following definition. An operator $T \in \mathcal{L}_0(X, E)$ is called an *ε-subgradient of a convex operator* $f : X \to E^{\cdot}$ at a point $z \in E(X)$ if $\widetilde{T} \in \partial_\varepsilon \tilde{f}(z)$. Denote by $\partial_\varepsilon f(z)$ the set of all ε-subgradients of f at a point z:

$$\begin{aligned}\partial_\varepsilon f(z) :&= \{T \in \mathcal{L}_0(X, E) : \widetilde{T} \in \partial_\varepsilon \tilde{f}(z)\}\\ &= \{T \in \mathcal{L}_0(X, E) : (\forall x \in X)\ Tz - Tx \leq f(z) - f(x) + \varepsilon\}.\end{aligned}$$

Let $f : X \to E^{\cdot}$ be a lower semicontinuous mapping. An element $z \in E(X)$ is called a *generalized ε-solution* of the optimization problem $f(x) \to \inf$ provided that $\tilde{f}(z) \leq \inf_{x \in X} f(x) + \varepsilon$. As usual, we have the Fermat Criterion: z is a generalized ε-solution of the problem $f(x) \to \inf$ if and only if $0 \in \partial_\varepsilon f(z)$.

The following result claims that near to each ε-solution there lies a generalized ε-solution which yields an ideal optimum to a perturbed objective function. In the case $E = \mathbb{R}$ this fact is well-known in the literature as the *Ekeland variational principle.*

6.2. Theorem. *Let* f *be a lower semicontinuous mapping from* X *into* $E^{\cdot}$*. Assume that* $f(x_0) \leq \inf\{f(x) : x \in X\} + \varepsilon$ *for some* $0 < \varepsilon \in E$ *and* $x_0 \in X$*. Then for every invertible* $0 \leq \lambda \in E$ *there exists* $z_\lambda \in E(X)$ *such that*

$$\tilde{f}(z_\lambda) \leq f(x_0), \quad |z_\lambda - x_0| \leq \lambda,$$

$$\tilde{f}(z_\lambda) = \inf \{f(x) + \lambda^{-1}\varepsilon\, |z_\lambda - x| : x \in X\}.$$

6.3. Generalized Brønsted–Rockafellar Theorem. *Let* $f : X \to E^{\cdot}$ *be a lower semicontinuous convex operator. Suppose that* $T \in \partial_\varepsilon f(x_0)$ *for some* $x_0 \in X$, $0 \le \varepsilon \in E$, *and* $T \in \mathcal{L}_0(X, E)$. *Then for every invertible* $0 \le \lambda \in E$ *there exists* $z_\lambda \in E(X)$ *and* $S_\lambda \in \mathcal{L}_0(X, E)$ *such that*

$$|z_\lambda - x_0| \le \lambda, \quad |S_\lambda - T| \le \lambda; \quad S_\lambda \in \partial f(z_\lambda).$$

6.4. A continuous vector-function $u \in \mathcal{C}(Q, X)$ with $u(q) \in C$ for all $q \in \mathrm{dom}\,(u)$ is a *generalized exposed point* of C provided that there is a dominated operator $S : X \to E$ satisfying $\sup Sx : \ x \in C = \widetilde{S}(\widetilde{u})$ (i.e. S attains its supremum on C at u). In this case we call S a *generalized supporting operator* of C. If $E = \mathbb{R}$ we speak simply about exposed points and supporting functionals.

From the Hahn–Banach Theorem we easily see that each boundary point of a closed convex set C with nonempty interior is an exposed point of C. It is not evident however that a nonempty closed convex set with empty interior has any support point. The celebrated Bishop–Phelps Theorem asserts that the set of exposed points and the set of supporting functionals of C are both dense. Happily, some vector-valued version of this fact is also true.

6.5. Generalized Bishop–Phelps Theorem. *Suppose that* C *is a nonempty closed convex subset of a Banach space* X *and a dominated linear operator* $S_0 : X \to E$ *is order bounded on* C. *For every invertible* $\varepsilon \in E^+$ *there are a dominated linear operator* $S : X \to E$ *and a continuous vector-function* $u \in \mathcal{C}(Q, X)$ *such that*

$$\|S - S_0\| \le \varepsilon \|S_0\|, \quad \sup\{Sx : \ x \in C\} = \widetilde{S}(\widetilde{u}).$$

6.6. The most frequent version of the Bishop–Phelps Theorem in Banach space theory reads: *the set of bounded linear functionals attaining their norm on the ball of a Banach space is norm dense in the dual space*. The vector-valued abstraction of this fact is as follows:

6.7. Theorem. *For every dominated linear operator* $S_0 : X \to E$ *and every real* $\varepsilon > 0$ *there exists a dominated operator* $S : X \to E$ *such that* $\|S - S_0\| \le \varepsilon \|S_0\|$ *and* $\widetilde{S}(\widetilde{u}) = \|S\|$ *for some* $u \in \mathcal{C}(Q, X)$, *with* $\|\widetilde{u}\| = \mathbf{1}$.

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