Nonstandard Tools for Nonsmooth Analysis

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Agenda

- Analysis is the technique of differentiation and integration. Differentiation discovers trends, and integration forecasts the future from trends. Analysis relates to the universe, reveals the glory of the Lord, and implies equality and smoothness.
- Optimization is the choice of what is most preferable. Nonsmooth analysis is the technique of optimization which speaks about the humankind, reflects the diversity of humans, and involves inequality and obstruction. The list of the main techniques of nonsmooth analysis contains subdifferential calculus (cp. [1, 2]).
- Calculus reduces forecast to numbers, which is scalarization in modern parlance. Spontaneous solutions are often labile and rarely optimal. Thus, nonsmooth analysis deals with inequality, scalarization and stability. Some aspects of the latter are revealed by the tools of nonstandard models to be discussed in this talk.

Nonstandard Tool Kits

- A model within set theory is *nonstandard* if the membership between the objects of the model differs from that of the originals. In fact the nonstandard tools of today use a couple of set-theoretic models simultaneously. The most popular are *infinitesimal analysis* (cp. [3, 4]) and *Boolean-valued analysis* (cp. [5, 6]).
- Infinitesimal analysis provides us with a novel understanding for the method of indivisibles or monadology, synthesizing the two approaches to calculus which belong to the inventors.
- Boolean valued analysis originated with the famous works by Paul Cohen on the continuum hypothesis and distinguishes itself by the technique of ascending and descending, cyclic envelopes and mixings, and B-sets.

The Best Is Divine

- Leibniz wrote to Samuel Clarke:[1]
- God can produce everything that is possible or whatever does not imply a contradiction, but he wills only to produce what is the best among things possible.

[1] See [7, p. 54]; cp. [8].

Enter the Reals

- Choosing the best, we use preferences. To optimize, we use infima and suprema for bounded sets which is practically the *least upper bound property*. So optimization needs ordered sets and primarily boundedly complete lattices.
- To operate with preferences, we use group structure. To aggregate and scale, we use linear structure.
- All these are happily provided by the *reals* $\mathbb{R}$, a one-dimensional Dedekind complete vector lattice. A Dedekind complete vector lattice is a *Kantorovich space*.
- Since each number is a measure of quantity, the idea of reducing to numbers is of a universal importance to mathematics. Model theory provides justification of the *Kantorovich heuristic principle* that the members of his spaces are numbers as well (cp. [9] and [10]).

Enter Inequality and Convexity

- Life is inconceivable without numerous conflicting ends and interests to be harmonized. Thus the instances appear of multiple criteria decision making. It is impossible as a rule to distinguish some particular scalar target and ignore the rest of them. This leads to vector optimization problems, involving order compatible with linearity.
- Linear inequality implies linearity and order. When combined, the two produce an ordered vector space. Each linear inequality in the simplest environment of the sort is some half-space. Simultaneity implies many instances and so leads to the intersections of half-spaces. These yield polyhedra as well as arbitrary convex sets, identifying the theory of linear inequalities with convexity.
- Convexity reigns in optimization, feeding generation, separation, calculus, and approximation. Generation appears as duality; separation, as optimality; calculus, as representation; and approximation, as stability [11].

Legendre in Disguise

- Assume that X is a vector space, E is an ordered vector space, $f : X \to E^{\bullet}$ is some operator, and $C := \operatorname{dom}(f) \subset X$ is a convex set. A *vector program* (C, f) is written as follows:

$$x \in C, \quad f(x) \to \inf.$$

- The standard sociological trick includes (C, f) into a parametric family yielding the *Legendre transform* or *Young–Fenchel transform* of f:

$$f^{*}(l) := \sup_{x \in X} (l(x) - f(x)),$$

  with $l \in X^{\#}$ a linear functional over X. The epigraph of f^{*} is a convex subset of $X^{\#}$ and so f^{*} is convex. Observe that $-f^{*}(0)$ is the value of (C, f).

Order Omnipresent

- A convex function is locally a positively homogeneous convex function, a *sublinear functional*. Recall that $p : X \to \mathbb{R}$ is sublinear whenever

$$\operatorname{epi} p := \{(x,\ t) \in X \times \mathbb{R} \mid p(x) \leq t\}$$

  is a cone. Recall that a numeric function is uniquely determined from its epigraph.

- Given $C \subset X$, put

$$H(C) := \{(x,\ t) \in X \times \mathbb{R}^{+} \mid x \in tC\},$$

  the *Hörmander transform* of C. Now, C is convex if and only if $H(C)$ is a cone. A space with a cone is a (*pre*)*ordered vector space*.
  The order, the symmetry, the harmony enchant us. . . .

  Leibniz

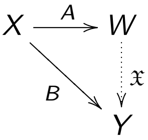
- Thus, convexity and order are intrinsic to nonsmooth analysis.

Environment for Inequality

- Assume that X is a real vector space, Y is a *Kantorovich space*. Let $\mathbb{B} := \mathbb{B}(Y)$ be the *base* of Y, i.e., the complete Boolean algebras of positive projections in Y; and let $m(Y)$ be the universal completion of Y. Denote by $L(X, Y)$ the space of linear operators from X to Y. In case X is furnished with some Y-seminorm on X, by $L^{(m)}(X, Y)$ we mean the *space of dominated operators* from X to Y. As usual, $\{T \leq 0\} := \{x \in X \mid Tx \leq 0\}$; $\ker(T) = T^{-1}(0)$ for $T : X \to Y$. Also, $P \in \mathrm{Sub}(X, Y)$ means that P is *sublinear*, while $P \in \mathrm{PSub}(X, Y)$ means that P is *polyhedral*, i.e., finitely generated. The superscript $^{(m)}$ suggests domination.

Kantorovich's Theorem

- Find $\mathfrak{X}$ satisfying



- **(1):** $(\exists \mathfrak{X})\ \mathfrak{X}A = B \leftrightarrow \ker(A) \subset \ker(B)$.
- **(2):** *If W is ordered by W_+ and $A(X) - W_+ = W_+ - A(X) = W$, then*[2]

$$(\exists \mathfrak{X} \geq 0)\ \mathfrak{X}A = B \leftrightarrow \{A \leq 0\} \subset \{B \leq 0\}.$$

[2] Cp. [2, p. 51].

The Farkas Alternative

- *Let X be a Y-seminormed real vector space, with Y a Kantorovich space. Assume that $A_1, \dots, A_N$ and B belong to $L^{(m)}(X, Y)$. Then one and only one of the following holds:*
  (1) *There are $x \in X$ and $b, b' \in \mathbb{B}$ such that $b' \le b$ and*

  $$b' Bx > 0, bA_1 x \le 0, \dots, bA_N x \le 0.$$

  (2) *There are positive orthomorphisms $\alpha_1, \dots, \alpha_N \in \mathrm{Orth}(m(Y))_+$ such that $B = \sum_{k=1}^{N} \alpha_k A_k$.*

- **Theorem 1.** *Let X be a Y-seminormed real vector space, with Y a Kantorovich space. Assume given some dominated operators $A_1, \dots, A_N, B \in L^{(m)}(X, Y)$ and elements $u_1, \dots, u_N, v \in Y$. The following are equivalent:*
  (1) *For all $b \in \mathbb{B}$ the inhomogeneous operator inequality $bBx \le bv$ is a consequence of the consistent simultaneous inhomogeneous operator inequalities $bA_1x \le bu_1, \dots, bA_Nx \le bu_N$, i.e.,*

$$\{bB \le bv\} \supset \{bA_1 \le bu_1\} \cap \cdots \cap \{bA_N \le bu_N\}.$$

  (2) *There are positive orthomorphisms $\alpha_1, \dots, \alpha_N \in \mathrm{Orth}(m(Y))$ satisfying*

$$B = \sum_{k=1}^{N} \alpha_k A_k; \quad v \ge \sum_{k=1}^{N} \alpha_k u_k.$$

Boolean Modeling

- Cohen's final solution of the problem of the cardinality of the continuum within ZFC gave rise to the Boolean valued models by Scott, Solovay, and Vopěnka.[3]
- Takeuti coined the term "Boolean valued analysis" for applications of the models to analysis.

[3] Cp. [4].

Scott's Comments

- Scott forecasted in 1969:[4]

  We must ask whether there is any interest in these nonstandard models aside from the independence proof; that is, do they have any mathematical interest? The answer must be yes, but we cannot yet give a really good argument.

- In 2009 Scott wrote:[5]

  At the time, I was disappointed that no one took up my suggestion. And then I was very surprised much later to see the work of Takeuti and his associates. I think the point is that people have to be trained in Functional Analysis in order to understand these models. I think this is also obvious from your book and its references. Alas, I had no students or collaborators with this kind of background, and so I was not able to generate any progress.

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[4] Cp. [13].

[5] Letter of April 29, 2009 to S. S. Kutateladze.

Boolean Valued Universe

- Let $\mathbb{B}$ be a complete Boolean algebra. Given an ordinal α, put

$$V_\alpha^{(\mathbb{B})} := \{x \mid (\exists \beta \in \alpha)\ x : \mathrm{dom}(x) \to \mathbb{B}\ \&\ \mathrm{dom}(x) \subset V_\beta^{(\mathbb{B})}\}.$$

- The *Boolean valued universe* $\mathbb{V}^{(\mathbb{B})}$ is

$$\mathbb{V}^{(\mathbb{B})} := \bigcup_{\alpha \in \mathrm{On}} V_\alpha^{(\mathbb{B})},$$

  with On the class of all ordinals.

- The truth value $[\![\varphi]\!] \in \mathbb{B}$ is assigned to each formula φ of ZFC relativized to $\mathbb{V}^{(\mathbb{B})}$.

Descending and Ascending

- Given φ, a formula of ZFC, and y, a member of $\mathbb{V}^{\mathbb{B}}$; put $A_\varphi := A_{\varphi(\cdot,\ y)} := \{x \mid \varphi(x,\ y)\}$.
- The *descent* $A_\varphi{\downarrow}$ of a class A_φ is

$$A_\varphi{\downarrow} := \{t \mid t \in \mathbb{V}^{(\mathbb{B})} \ \&\ [\![\varphi(t,\ y)]\!] = \mathbb{1}\}.$$

- If $t \in A_\varphi{\downarrow}$, then it is said that t *satisfies* $\varphi(\cdot,\ y)$ *inside* $\mathbb{V}^{(\mathbb{B})}$.
- The *descent* $x{\downarrow}$ of $x \in \mathbb{V}^{(\mathbb{B})}$ is defined as

$$x{\downarrow} := \{t \mid t \in \mathbb{V}^{(\mathbb{B})} \ \&\ [\![t \in x]\!] = \mathbb{1}\},$$

  i.e. $x{\downarrow} = A_{\cdot \in x}{\downarrow}$. The class $x{\downarrow}$ is a set.
- If x is a nonempty set inside $\mathbb{V}^{(\mathbb{B})}$ then

$$(\exists z \in x{\downarrow})[\![(\exists t \in x)\ \varphi(t)]\!] = [\![\varphi(z)]\!].$$

- The *ascent* functor acts in the opposite direction.

The Reals Within or Kantorovich's Scalars

- There is an object $\mathcal{R}$ inside $\mathbb{V}^{(\mathbb{B})}$ modeling $\mathbb{R}$, i.e.,

$$[\![\mathcal{R} \text{ is the reals}]\!] = \mathbb{1}.$$

- Let $\mathcal{R}{\downarrow}$ be the descent of the carrier $|\mathcal{R}|$ of the algebraic system $\mathcal{R} := (|\mathcal{R}|, +, \cdot, 0, 1, \leq)$ inside $\mathbb{V}^{(\mathbb{B})}$.
- Implement the descent of the structures on $|\mathcal{R}|$ to $\mathcal{R}{\downarrow}$ as follows:

$$x + y = z \leftrightarrow [\![x + y = z]\!] = \mathbb{1};$$
$$xy = z \leftrightarrow [\![xy = z]\!] = \mathbb{1};$$
$$x \leq y \leftrightarrow [\![x \leq y]\!] = \mathbb{1};$$
$$\lambda x = y \leftrightarrow [\![\lambda^{\wedge} x = y]\!] = \mathbb{1} \ (x, y, z \in \mathcal{R}{\downarrow},\ \lambda \in \mathbb{R}).$$

- **Gordon Theorem.**[6] *$\mathcal{R}{\downarrow}$ with the descended structures is a universally complete vector lattice with base $\mathbb{B}(\mathcal{R}{\downarrow})$ isomorphic to $\mathbb{B}$.*

[6] Cp. [4, p. 349].

Art of Invention

- Leibniz wrote about his version of calculus that "the difference from Archimedes style is only in expressions which in our method are more straightforward and more applicable to the art of invention."
- Nonstandard analysis has the two main advantages: it "kills quantifiers" and it produces the new notions that are impossible within a single model of set theory.
- By way of example let us turn to the nonstandard presentations of Kuratowski–Painlevé limits and the concept of infinitesimal optimality.
- Recall that the central concept of Leibniz was that of a *monad* (cp. [15]). In nonstandard analysis the monad $\mu(\mathscr{F})$ of a standard filter $\mathscr{F}$ is the intersection of all standard elements of $\mathscr{F}$.

Monadic Limits

- Let $F \subset X \times Y$ be an internal correspondence from a standard set X to a standard set Y. Assume given a standard filter $\mathcal{N}$ on X and a topology τ on Y. Put

$$\forall\forall(F) := {}^*\{y' \mid (\forall\, x \in \mu(\mathcal{N}) \cap \operatorname{dom}(F))(\forall\, y \approx y')(x,y) \in F\},$$
$$\exists\forall(F) := {}^*\{y' \mid (\exists\, x \in \mu(\mathcal{N}) \cap \operatorname{dom}(F))(\forall\, y \approx y')(x,y) \in F\},$$
$$\forall\exists(F) := {}^*\{y' \mid (\forall\, x \in \mu(\mathcal{N}) \cap \operatorname{dom}(F))(\exists\, y \approx y')(x,y) \in F\},$$
$$\exists\exists(F) := {}^*\{y' \mid (\exists\, x \in \mu(\mathcal{N}) \cap \operatorname{dom}(F))(\exists\, y \approx y')(x,y) \in F\},$$

  with * symbolizing standardization and $y \approx y'$ standing for the *infinite proxitity* between y and y' in τ, i.e. $y' \in \mu(\tau(y))$.
- Call $\mathrm{Q_1Q_2(F)}$ the $\mathrm{Q_1Q_2}$-*limit* of F (here $\mathrm{Q_k}$ ($k := 1, 2$) is one of the quantifiers $\forall$ or $\exists$).

Kuratowski–Painlevé Limits

- Assume for instance that F is a standard correspondence on some element of $\mathcal{N}$ and look at the $\exists\exists$-limit and the $\forall\exists$-limit. The former is the *limit superior* or *upper limit*; the latter is the *limit inferior* or *lower limit* of F along $\mathcal{N}$.
- **Theorem 2.** *If F is a standard correspondence then*

$$\exists\exists(F) = \bigcap_{U\in\mathcal{N}} \mathrm{cl}\Big(\bigcup_{x\in U} F(x)\Big);$$

$$\forall\exists(F) = \bigcap_{U\in\ddot{\mathcal{N}}} \mathrm{cl}\Big(\bigcup_{x\in U} F(x)\Big),$$

where $\ddot{\mathcal{N}}$ is the grill of a filter $\mathcal{N}$ on X, i.e., the family comprising all subsets of X meeting $\mu(\mathcal{N})$.

Convexity Is Stable

- Convexity of harpedonaptae was stable in the sense that no variation of stakes within the surrounding rope can ever spoil the convexity of the tract to be surveyed.
- Stability is often tested by perturbation or introducing various epsilons in appropriate places. One of the earliest excursions in this direction is connected with the classical Hyers–Ulam stability theorem for ε-convex functions. Exact calculations with epsilons and sharp estimates are often bulky and slightly mysterious. Some alternatives are suggested by actual infinities, which is illustrated with the conception of *infinitesimal optimality*.

Enter Epsilon

- Assume given a convex operator $f : X \to E^{\bullet}$ and a point $\overline{x}$ in the effective domain $\operatorname{dom}(f) := \{x \in X \mid f(x) < +\infty\}$ of f.
- Given $\varepsilon \geq 0$ in the positive cone E_+ of E, by the ε-*subdifferential* of f at $\overline{x}$ we mean the set

$$\partial_\varepsilon f(\overline{x}) := \Big\{T \in L(X, E) \mid$$
$$(\forall x \in X)(Tx - f(x) \leq T\overline{x} - f(\overline{x}) + \varepsilon)\Big\}.$$

Topological Setting

- The usual subdifferential $\partial f(\overline{x})$ is the intersection:

$$\partial f(\overline{x}) := \bigcap_{\varepsilon \geq 0} \partial_\varepsilon f(\overline{x}).$$

In topological setting we use continuous operators, replacing $L(X, E)$ with $\mathscr{L}(X, E)$.

General Position

- Some cones K_1 and K_2 in a topological vector space X are *in general position* provided that
- **(1)** the algebraic span of K_1 and K_2 is some subspace $X_0 \subset X$; i.e., $X_0 = K_1 - K_2 = K_2 - K_1$;
- **(2)** the subspace X_0 is complemented; i.e., there exists a continuous projection $P : X \to X$ such that $P(X) = X_0$;
- **(3)** K_1 and K_2 constitute a nonoblate pair in X_0.
- Finally, observe that the two nonempty convex sets C_1 and C_2 are *in general position* if so are their Hörmander transforms $H(C_1)$ and $H(C_2)$.

ε-Optimality

- **Theorem 2.** *Let* $f_1 : X \times Y \to E^\bullet$ *and* $f_2 : Y \times Z \to E^\bullet$ *be convex operators and* $\delta, \varepsilon \in E^+$. *Suppose that the convolution* $f_2 \vartriangle f_1$ *is* δ*-exact at some point* (x, y, z)*; i.e.,* $\delta + (f_2 \vartriangle f_1)(x, y) = f_1(x, y) + f_2(y, z)$. *If, moreover, the convex sets* $\operatorname{epi}(f_1, Z)$ *and* $\operatorname{epi}(X, f_2)$ *are in general position, then*

$$\partial_\varepsilon (f_2 \vartriangle f_1)(x, y) = \bigcup_{\substack{\varepsilon_1 \geq 0, \varepsilon_2 \geq 0, \\ \varepsilon_1 + \varepsilon_2 = \varepsilon + \delta}} \partial_{\varepsilon_2} f_2(y, z) \circ \partial_{\varepsilon_1} f_1(x, y).$$

Enter Monad

- Distinguish some downward-filtered subset $\mathcal{E}$ of E that is composed of positive elements. Assuming E and $\mathcal{E}$ standard, define the *monad* $\mu(\mathcal{E})$ of $\mathcal{E}$ as $\mu(\mathcal{E}) := \bigcap\{[0,\varepsilon] \mid \varepsilon \in {}^{\circ}\mathcal{E}\}$. The members of $\mu(\mathcal{E})$ are *positive infinitesimals* with respect to $\mathcal{E}$. As usual, ${}^{\circ}\mathcal{E}$ denotes the external set of all standard members of E, the *standard part* of $\mathcal{E}$.
- Assume that the monad $\mu(\mathcal{E})$ is an external cone over ${}^{\circ}\mathbb{R}$ and, moreover, $\mu(\mathcal{E}) \cap {}^{\circ}E = 0$. In application, $\mathcal{E}$ is usually the filter of order-units of E. The relation of *infinite proximity* or *infinite closeness* between the members of E is introduced as follows:

$$e_1 \approx e_2 \leftrightarrow e_1 - e_2 \in \mu(\mathcal{E}) \ \&\ e_2 - e_1 \in \mu(\mathcal{E}).$$

- Now

$$Df(\overline{x}) := \bigcap_{\varepsilon \in {}^{\circ}\mathcal{E}} \partial_{\varepsilon} f(\overline{x}) = \bigcup_{\varepsilon \in \mu(\mathcal{E})} \partial_{\varepsilon} f(\overline{x}),$$

which is the *infinitesimal subdifferential* of f at $\overline{x}$. The elements of $Df(\overline{x})$ are *infinitesimal subgradients* of f at $\overline{x}$.

Infinitesimal Solution

- Assume that there exists a limited value $e := \inf_{x \in C} f(x)$ of some program (C, f). A feasible point x_0 is called an *infinitesimal solution* if $f(x_0) \approx e$, i.e., if $f(x_0) \leq f(x) + \varepsilon$ for every $x \in C$ and every standard $\varepsilon \in \mathcal{E}$.
- *A point $x_0 \in X$ is an infinitesimal solution of the unconstrained problem $f(x) \to \inf$ if and only if $0 \in Df(x_0)$.*

Exeunt Epsilon

- **Theorem 3.** *Let $f_1 : X \times Y \to E^\bullet$ and $f_2 : Y \times Z \to E^\bullet$ be convex operators. Suppose that the convolution $f_2 \vartriangle f_1$ is infinitesimally exact at some point (x, y, z); i.e., $(f_2 \vartriangle f_1)(x, y) \approx f_1(x, y) + f_2(y, z)$. If, moreover, the convex sets* $\operatorname{epi}(f_1, Z)$ *and* $\operatorname{epi}(X, f_2)$ *are in general position then*

$$D(f_2 \vartriangle f_1)(x, y) = Df_2(y, z) \circ Df_1(x, y).$$

- Consider some *Slater regular program*

$$\Lambda x = \Lambda \bar{x}, \quad g(x) \le 0, \quad f(x) \to \inf .$$

- **Theorem 4.**[7] *A feasible point x_0 is an infinitesimal solution of a Slater regular program if and only if the following system of conditions is compatible:*

$$\beta \in L^+(F, E), \quad \gamma \in L(\mathfrak{X}, E), \quad \gamma g(x_0) \approx 0,$$
$$0 \in Df(x_0) + D(\beta \circ g)(x_0) + \gamma \circ \Lambda.$$

[7] Cp. [6, Sect. 5.7].

Discrete Dynamic Systems

- Let $X_0, \dots, X_N$ be some topological vector spaces, and let $G_k : X_{k-1} \rightrightarrows X_k$ be a nonempty convex correspondence for all $k := 1, \dots, N$. The collection $G_1, \dots, G_N$ determines the *dynamic family* of processes $(G_{k,l})_{k<l\le N}$, where the correspondence $G_{k,l} : X_k \rightrightarrows X_l$ is defined as

$$G_{k,l} := G_{k+1} \circ \dots \circ G_l \quad \text{if} \quad k+1 < l;$$

$$G_{k,k+1} := G_{k+1} \quad (k := 0, 1, \dots, N-1).$$

Clearly, $G_{k,l} \circ G_{l,m} = G_{k,m}$ for all $k < l < m \le N$.
A *path* or *trajectory* of the above family of processes is defined to be an ordered collection of elements $\mathfrak{x} := (x_0, \dots, x_N)$ such that $x_l \in G_{k,l}(x_k)$ for all $k < l \le N$. Moreover, we say that x_0 is the *beginning* of $\mathfrak{x}$ and x_N is the *ending* of $\mathfrak{x}$.

Optimal Paths

- Let Z be a topological ordered vector space. Consider some convex operators $f_k : X_k \to Z$ $(k := 0, \dots, N)$ and convex sets $S_0 \subset X_0$ and $S_N \subset X_N$. Assume given a topological Kantorovich space E and a monotone sublinear operator $P : Z^N \to E^\bullet$. Given a path $\mathfrak{x} := (x_0, \dots, x_N)$, put $\mathfrak{f}(\mathfrak{x}) := (f_0(x_0), f_1(x_1) \dots, f_k(x_N))$. Let $\mathrm{Pr}_k : Z^N \to Z$ denote the projection of Z^N to the kth coordinate. Then $\mathrm{Pr}_k(\mathfrak{f}(\mathfrak{x})) = f_k(x_k)$ for all $k := 0, \dots, N$.
  Assume given a monotone sublinear operator P from X to $E^\bullet$.
  A path $\mathfrak{x}$ is *feasible* ifft the beginning of $\mathfrak{x}$ belongs to S_0 and the ending of $\mathfrak{x}$, to S_N. A path $\mathfrak{x}^0 := \left(x_0^0, \dots, x_N^0\right)$ is *infinitesimally optimal* provided that $x_0^0 \in S_0$, $x_N^0 \in S_N$, and $P \circ \mathfrak{f}$ attains an infinitesimal minimum over the set of all feasible paths.

Characteristic of an Optimal Path

- Introduce the sets

$$C_0 := S_0 \times \prod_{k=1}^{N} X_k; \quad C_1 := G_1 \times \prod_{k=2}^{N} X_k;$$

$$C_2 := X_0 \times G_2 \times \prod_{k=3}^{N} X_k; \ldots; \quad C_N := \prod_{k=0}^{N-2} X_k \times G_N;$$

$$C_{N+1} := \prod_{k=1}^{N-1} X_k \times S_N; \quad X := \prod_{k=0}^{N} X_k.$$

Characteristic of an Optimal Path

- **Theorem 6.** *Suppose that the convex sets*

$$C_0 \times E^+, \dots, C_{N+1} \times E^+$$

are in general position as well as the sets $X \times \operatorname{epi}(P)$ *and* $\operatorname{epi}(\mathfrak{f}) \times E$. *A feasible path* $\left(x_0^0, \dots, x_N^0\right)$ *is infinitesimally optimal if and only if the following system of conditions is compatible:*

$$\alpha_k \in \mathcal{L}(X_k, E), \quad \beta_k \in \mathcal{L}^+(Z, E) \quad (k := 0, \dots, N);$$
$$\beta \in \partial(P);\ \beta_k := \beta \circ \mathrm{Pr}_k;$$
$$(\alpha_{k-1}, \alpha_k) \in DG_k\left(x_{k-1}^0, x_k^0\right) - \{0\} \times D(\beta_k \circ f_k)\left(x_k^0\right) \quad (k := 1, \dots, N);$$
$$-\alpha_0 \in DS_0(x_0) + D(\beta_0 \circ f_0)(x_0); \quad \alpha_N \in DS_N(x_N).$$

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