NONSTANDARD TRENDS IN FUNCTIONAL ANALYSIS

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- The aim of this talk is to overview **Boolean valued analysis**.
- Boolean valued analysis is a branch of functional analysis which uses a special model-theoretic technique and consists in studying the properties of a mathematical object by means of comparison between its representations in two different set-theoretic models whose construction utilizes distinct Boolean algebras.
- The **von Neumann universe** (Cantorian paradise) $\mathbb{V}$ and a specially-trimmed **Boolean valued universe** $\mathbb{V}^{(\mathbb{B})}$ are taken as these models.
- The comparative analysis requires some **ascending–descending machinery** to carry out the interplay between $\mathbb{V}$ and $\mathbb{V}^{(\mathbb{B})}$.

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Nonstandard Methods of Analysis

- Using two models for studying a single object is a family feature of the so-called **nonstandard methods of analysis**. For this reason, Boolean valued analysis means an instance of nonstandard analysis in common parlance.
- The term "Boolean valued analysis" was by **G. Takeuti**.
- Proliferation of Boolean valued models is due to **P. Cohen's** final breakthrough in Hilbert's Problem Number One. His **method of forcing** was rather intricate and the inevitable attempts at simplification gave rise to the Boolean valued models by **D. Scott, R. Solovay,** and **P. Vopěnka**.

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Von Neumann Universe

The universe of sets or von Neumann universe $\mathbb{V}$ is defined by recursion on $\alpha \in \mathrm{On}$:

$$\mathbb{V}_0 := \varnothing, \quad \mathbb{V}_1 := \mathscr{P}(\varnothing) = \{\varnothing\}, \quad \mathbb{V}_2 := \{\varnothing, \{\varnothing\}\}, \dots,$$

$$\mathbb{V}_{\alpha+1} := \mathscr{P}(\mathbb{V}_\alpha);$$

$$\mathbb{V}_\beta := \bigcup_{\alpha<\beta} \mathbb{V}_\alpha \quad (\beta \text{ is a limit ordinal});$$

$$\mathbb{V} := \bigcup_{\alpha \in \mathrm{On}} \mathbb{V}_\alpha.$$

Theorem. $\mathbb{V}$ is a standard model of ZFC.

The Boolean Valued Universe $\mathbb{V}^{(\mathbb{B})}$
is defined by recursion on $\alpha \in \mathrm{On}$:

$\mathbb{B}$ is a complete Boolean algebra,

$$\mathbb{V}_0^{(\mathbb{B})} := \varnothing, \quad \mathbb{V}_1^{(\mathbb{B})} := 0 := \{\varnothing\}, \quad \mathbb{V}_2^{(\mathbb{B})} := \{\varnothing\} \cup \{(0, b) : \ b \in \mathbb{B}\}, \dots,$$

$$\mathbb{V}_{\alpha+1}^{(\mathbb{B})} := \{f : \mathrm{dom}(f) \to \mathbb{B} : \ \mathrm{dom}(f) \subset \mathbb{V}_\alpha^{(\mathbb{B})}\},$$

$$\mathbb{V}_\beta^{(\mathbb{B})} := \bigcup_{\alpha<\beta} \mathbb{V}_\alpha^{(\mathbb{B})} \quad (\beta \text{ is a limit ordinal}),$$

$$\mathbb{V}^{(\mathbb{B})} := \bigcup_{\alpha\in\mathrm{On}} \mathbb{V}_\alpha^{(\mathbb{B})}.$$

Theorem. $\mathbb{V}^{(\mathbb{B})}$ is a Boolean valued model of ZFC.

Boolean Valued Model

- **How to make statements about $x_1, \dots, x_n \in \mathbb{V}^{(\mathbb{B})}$?**

  Take a ZF-formula $\varphi = \varphi(u_1, \dots, u_n)$ and replace the variables $u_1, \dots, u_n$ by elements $x_1, \dots, x_n \in \mathbb{V}^{(\mathbb{B})}$. Then $\varphi(x_1, \dots, x_n)$ is a statement about $x_1, \dots, x_n$.

- **How to verify whether or not $\varphi(x_1, \dots, x_n)$ is true in $\mathbb{V}^{(\mathbb{B})}$?**

  There is a natural way of assigning to each such statement an element of $\mathbb{B}$, the **Boolean truth-value** $[\![\varphi(x_1, \dots, x_n)]\!] \in \mathbb{B}$

- **Definition.** $\mathbb{V}^{(\mathbb{B})} \models \varphi(x_1, \dots, x_n) \iff [\![\varphi(x_1, \dots, x_n)]\!] = \mathbb{1}$.

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- **Transfer Principle.** $\mathbb{V}^{(\mathbb{B})} \models \mathrm{ZFC}$. In more detail:
  $$\mathrm{ZFC} \vdash \varphi(v_1, \dots, v_n) \implies \big(\forall x_1, \dots, x_n \in \mathbb{V}^{(\mathbb{B})}\big)\ \mathbb{V}^{(\mathbb{B})} \models \varphi(x_1, \dots, x_n).$$
- **Restricted Transfer Principle.** If all quantifiers in φ are of the form $(\forall x \in y)\psi$ or $(\exists x \in y)\psi$, then for all $x_1, \dots, x_n \in \mathbb{V}$
  $$\varphi(x_1, \dots, x_n) \iff \mathbb{V}^{\mathbb{B}} \models \varphi(x_1^{\wedge}, \dots, x_n^{\wedge}).$$
- **Maximum Principle.** The supremum is attained at the formulae:
  $$[\![(\exists x)\varphi(x)]\!] := \bigvee\{[\![\varphi(u)]\!] : \ u \in \mathbb{V}^{(\mathbb{B})}\}$$
- **Corollary.** If $(\exists x)\varphi(x)$ is true within $\mathbb{V}^{(\mathbb{B})}$, then there exists $x_0 \in \mathbb{V}^{(\mathbb{B})}$ such that $\varphi(x_0)$ is true within $\mathbb{V}^{(\mathbb{B})}$.

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Principles of Boolean Valued Analysis

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Standard Name, Ascents, Descents

- **How is the interplay between $\mathbb{V}$ and $\mathbb{V}^{(\mathbb{B})}$ carried out?**

  The relevant **ascending-and-descending** technique rests on the functors of standard name, descent, and ascent.

- **Standard name functor:** $\mathbb{V} \ni X \mapsto X^\wedge \in \mathbb{V}^{(\mathbb{N})}$,

  $$\mathbb{V} \longleftrightarrow \mathbb{V}^{(\{0,1\})} \subset \mathbb{V}^{(\mathbb{B})}.$$

- **Ascent functor:** $\mathbb{V} \cap \mathscr{P}(\mathbb{V}^{(\mathbb{B})}) \ni X \mapsto \mathscr{X} := X{\uparrow} \in \mathbb{V}^{(\mathbb{B})}$.
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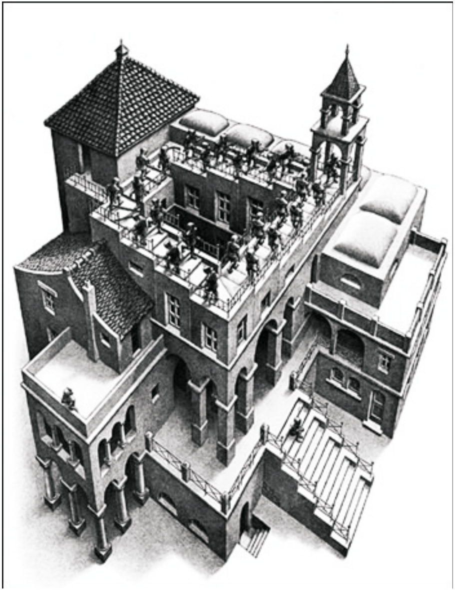
Ascending and Descending (Maurits Cornelis Escher, 1960)



- **CH:** $2^{\omega_0} = \omega_1$. **GCH:** $2^{\omega_\alpha} = \omega_{\alpha+1}$.
- **Theorem.** *There exists a CBA* $\mathbb{B}$ *with* $\mathbb{V}^{(\mathbb{B})} \models 2^{\omega_0} = \omega_2$.
- **Corollary.** Consis $(ZF) \implies$ Consis $(ZFC + (\neg CH))$.
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- **D. Scott (1969):** We must ask whether there is any interest in these nonstandard models aside from the independence proof; that is do they have any mathematical interest? The answer must be yes, but we cannot yet give a really good arguments.
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- $\mathbb{R} := (R, +, \times, \leq, 0, 1)$, $\varphi(\mathbb{R}) = \varphi(R, +, \times, \leq, 0, 1)$ is true, where $\varphi(\mathbb{R})$ is the conjunction of the axioms of the reals.
- Theorem. There is a field of reals unique up to isomorphism.
- Transfer Principle $\Longrightarrow$ $[\![(\exists \mathcal{R})\varphi(\mathcal{R})]\!] = \mathbb{1}$, i. e.

  there exists the field of reals within $\mathbb{V}^{(\mathbb{B})}$.
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- **Definition.** A *vector lattice* is a real vector space X that is equipped with a partial order $\leq$ for which there exist
  ✓ $x \vee y := \sup\{x, y\}$, the supremum,
  ✓ $x \wedge y := \inf\{x, y\}$, the infimum,
  for all vectors $x, y \in X$ and such that the *positive cone*
  ✓ $X_+ := \{x \in X : \ x \geq 0\}$ of X have the properties
  ✓ $X_+ + X_+ \subset X_+, \quad \mathbb{R}_+ \cdot X_+ \subset X_+.$
- **Definition.** If X is simultaneously a Banach space and the order is connected to the norm by the condition that
  ✓ $|x| \leq |y| \implies \|x\| \leq \|y\|$ (*monotonicity*),
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$$U \subset [a, b] := \{x \in X : \ a \le x \le b\} \implies \exists \sup(U), \inf(U) \in X.$$

- **Example 1.** $C(K)$, $L^p(\Omega, \Sigma, \mu)$, l^p $(1 \le p \le \infty)$, c_0, c.
- **Theorem (Stone, 1937, 1948; Ogasawara, 1944).** $C(K)$ is a Kantorovich space $\iff$ K is extremally disconnected.
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Gordon's Theorem (1977). Let $\mathcal{R}$ be the field of reals in $\mathbb{V}^{(\mathbb{B})}$.

- (1) The algebraic structure $\mathbf{R} := \mathcal{R}{\downarrow}$ (with the descended operations and order) is an universally complete vector lattice.
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L. V. Kantorovich: The Heuristic Transfer Principle

- Kantorovich indicated an important instance of ordered vector spaces, a Dedekind complete vector lattice, nowadays often called a **Kantorovich space**. This notion appeared in Kantorovich's first fundamental article on this topic:
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Boolean Valued Representations

Algebraic structure (space, algebra, etc.)	Boolean valued representation	Author year
Universally complete Kantorovich space	Field of reals	Gordon, 1977
Rationally complete semiprime abelian ring	Field	Gordon, 1983
Banach–Kantorovich space	Banach space	Kusraev, 1985
$\mathbb{B}$-cyclic Banach space	Banach space	Kusraev Ozawa, 1990
Unital separated injective module	Vector space	Gordon, 1991

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Von Neumann algebra	Von Neumann factor	Takeuti, 1981
Kaplansky–Hilbert module	Hilbert space	Takeuti Ozawa, 1983
$\mathbb{B}$-complete C^*-algebra	C^*-algebra	Takeuti, 1983
AW^*-algebra	AW^*-factor	Ozawa, 1984
Embeddable AW^*-algebra	Von Neumann algebra	Ozawa, 1986
$\mathbb{B}$-complete JB-algebra	JB-factor	Kusraev, 1996
Injective Banach lattice	AL-space (L_1 space)	Kusraev, 2011

Some Long Standing Problems

The problem	Rased	Reduced to (by means of BA):	Solved
Intrinsic characterization of subdifferentials	Kutateladze 1976	Weakly compact convex set of functionals	Kusraev Kutateladze 1982
General desintegration in Kantorovich spaces	Ioffe, Levin Neumann 1972/1977	Hahn–Banach and Radon–Nikidým theorems	Kusraev 1984
Kaplansky Problem: Homogeneity of a type I AW^*-algebra	Kaplansky 1953	Homogeneity of $B(H)$ with H Hilbert space	Ozawa 1984

Some Long Standing Problems

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Wickstead problem: Order boundedness of BP operators	Wickstead 1983	Cauchy type functional equations	Gutman Kusraev 1995, 2006
Maharam extension of a positive operator	Luxemburg Schep 1978	Daniel extension of an elementary integral	Akilov Kolesnikov Kusraev 1988
Classification of injective Banach lattices	Lotz Cartright 1975	Classification of AL-space (L_1 spaces)	Kusraev 2012

THANK YOU FOR ATTENTION