The Growth Points of Boolean Valued Analysis: Cantor's Continuum Problem and Kantorovich Spaces

A. E. Gutman
A. G. Kusraev
S. S. Kutateladze

Geometry Days in Novosibirsk—2014
Sobolev Institute of Mathematics
(Novosibirsk, September 26, 2014)

Three Remarkable Events

- 1878: G. Cantor (Continuum Hypothesis, CH).
- 1935: L. V. Kantorovich (Dedekind complete vector lattices).
- 1963: P. J. Cohen (Forcing, Independence of CH).
- Contents:
  - ✓ some historical remarks;
  - ✓ the interplay;
  - ✓ the results.

Three Remarkable Events

- 1878: G. Cantor (Continuum Hypothesis, CH).
- 1935: L. V. Kantorovich (Dedekind complete vector lattices).
- 1963: P. J. Cohen (Forcing, Independence of CH).
- Contents:
  - ✓ some historical remarks;
  - ✓ the interplay;
  - ✓ the results.

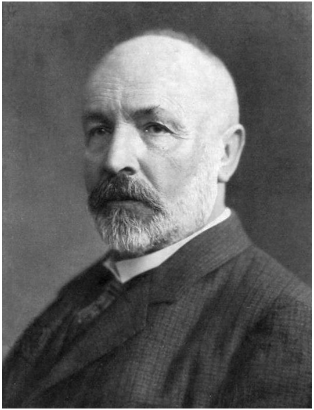
Three Remarkable Events

- 1878: G. Cantor (Continuum Hypothesis, CH).
- 1935: L. V. Kantorovich (Dedekind complete vector lattices).
- 1963: P. J. Cohen (Forcing, Independence of CH).
- Contents:
  - ✓ some historical remarks;
  - ✓ the interplay;
  - ✓ the results.

Three Remarkable Events

- 1878: G. Cantor (Continuum Hypothesis, CH).
- 1935: L. V. Kantorovich (Dedekind complete vector lattices).
- 1963: P. J. Cohen (Forcing, Independence of CH).
- Contents:
  - ✓ some historical remarks;
  - ✓ the interplay;
  - ✓ the results.

Georg Ferdinand Ludwig Philipp Cantor



German mathematician, the inventor of set theory
(March 3, 1845, Saint Petersburg, Russian Empire—
January 6, 1918, Halle, German Empire)

G. Cantor: The Problem of Uniqueness (1869–1972)

- Cantor's first ten papers were on number theory (PhD, 1867).
- Heirich Eduard Heine (1869) proposed the problem of uniqueness of representation of a function by a trigonometric series.
- Uniqueness Problem (Heine, Dirichlet, Lipschitz, and Riemann):

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) = 0 \quad (x \in \mathbb{R} \setminus F)$$

$$\implies a_n = b_n = 0 \quad (n \in \mathbb{N})?$$

- Definition. If YES then F is called a set of uniqueness.
- Cantor solved the problem if $F = \varnothing$ (1870) and F is finite (1871).

G. Cantor: The Problem of Uniqueness (1869–1972)

- Cantor's first ten papers were on number theory (PhD, 1867).
- Heirich Eduard Heine (1869) proposed the problem of uniqueness of representation of a function by a trigonometric series.
- Uniqueness Problem (Heine, Dirichlet, Lipschitz, and Riemann):

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) = 0 \quad (x \in \mathbb{R} \setminus F)$$

$$\implies a_n = b_n = 0 \quad (n \in \mathbb{N})?$$

- Definition. If YES then F is called a set of uniqueness.
- Cantor solved the problem if $F = \varnothing$ (1870) and F is finite (1871).

G. Cantor: The Problem of Uniqueness (1869–1972)

- Cantor's first ten papers were on number theory (PhD, 1867).
- Heirich Eduard Heine (1869) proposed the problem of uniqueness of representation of a function by a trigonometric series.
- Uniqueness Problem (Heine, Dirichlet, Lipschitz, and Riemann):

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) = 0 \quad (x \in \mathbb{R} \setminus F)$$

$$\implies a_n = b_n = 0 \quad (n \in \mathbb{N})?$$

- Definition. If YES then F is called a set of uniqueness.
- Cantor solved the problem if $F = \varnothing$ (1870) and F is finite (1871).

G. Cantor: The Problem of Uniqueness (1869–1972)

- Cantor's first ten papers were on number theory (PhD, 1867).
- Heirich Eduard Heine (1869) proposed the problem of uniqueness of representation of a function by a trigonometric series.
- Uniqueness Problem (Heine, Dirichlet, Lipschitz, and Riemann):

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = 0 \quad (x \in \mathbb{R} \setminus F)$$

$$\implies a_n = b_n = 0 \quad (n \in \mathbb{N})?$$

- Definition. If YES then F is called a set of uniqueness.
- Cantor solved the problem if $F = \varnothing$ (1870) and F is finite (1871).

G. Cantor: The Problem of Uniqueness (1869–1972)

- Cantor's first ten papers were on number theory (PhD, 1867).
- Heirich Eduard Heine (1869) proposed the problem of uniqueness of representation of a function by a trigonometric series.
- Uniqueness Problem (Heine, Dirichlet, Lipschitz, and Riemann):

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) = 0 \quad (x \in \mathbb{R} \setminus F)$$

$$\implies a_n = b_n = 0 \quad (n \in \mathbb{N})?$$

- Definition. If YES then F is called a set of uniqueness.
- Cantor solved the problem if $F = \varnothing$ (1870) and F is finite (1871).

Cantor–Lebesgue Theorem (1872–1903)

- **Definition.** $F^{(0)} := F,\ F^{(1)} := F',\ \ldots\ F^{(n+1)} := (F^{(n)})'.$

Cantor–Bendixson derivative $F' = \{x \in F :\ x = \lim_n x_n,\ (x_n) \subset F\}$.

- G. Cantor (1872). Let $F \subset \mathbb{R}$ be closed.

  If $F^{(n)} = \varnothing$ for some $n \in \mathbb{N} \implies F$ is a set of uniqueness.
- G. Cantor. *Über die Ausdehnung eines Satzes aus der Theorie der trigonometrischen Reihen.* Math. Ann. **5**, 123 (1872).
- It is a natural intension to continue $\omega, \omega+1, \ldots, \omega^2, \ldots, \omega^\omega \ldots$, since the Cantor–Bendixson process does not terminate in finitely many steps. This led Cantor to the development of the theory of ordinals.
- Theorem (Lebesgue, 1903)

  Each countable closed set $F \subset \mathbb{R}$ is a set of uniqueness.

Cantor–Lebesgue Theorem (1872–1903)

- Definition. $F^{(0)} := F,\ F^{(1)} := F',\ \dots\ F^{(n+1)} := (F^{(n)})'.$

Cantor–Bendixson derivative $F' = \{x \in F :\ x = \lim_n x_n,\ (x_n) \subset F\}$.

- G. Cantor (1872). Let $F \subset \mathbb{R}$ be closed.

  If $F^{(n)} = \varnothing$ for some $n \in \mathbb{N} \implies F$ is a set of uniqueness.

- G. Cantor. *Über die Ausdehnung eines Satzes aus der Theorie der trigonometrischen Reihen.* Math. Ann. **5**, 123 (1872).
- It is a natural intension to continue $\omega, \omega+1, \dots, \omega^2, \dots, \omega^\omega \dots,$ since the Cantor–Bendixson process does not terminate in finitely many steps. This led Cantor to the development of the theory of ordinals.
- Theorem (Lebesgue, 1903)

  Each countable closed set $F \subset \mathbb{R}$ is a set of uniqueness.

Cantor–Lebesgue Theorem (1872–1903)

- Definition. $F^{(0)} := F,\ F^{(1)} := F',\ \ldots\ F^{(n+1)} := (F^{(n)})'$.

Cantor–Bendixson derivative $F' = \{x \in F :\ x = \lim_n x_n,\ (x_n) \subset F\}$.

- G. Cantor (1872). Let $F \subset \mathbb{R}$ be closed.

  If $F^{(n)} = \varnothing$ for some $n \in \mathbb{N} \implies F$ is a set of uniqueness.
- G. Cantor. *Über die Ausdehnung eines Satzes aus der Theorie der trigonometrischen Reihen.* Math. Ann. **5**, 123 (1872).
- It is a natural intension to continue $\omega, \omega + 1, \ldots, \omega^2, \ldots, \omega^\omega \ldots$, since the Cantor–Bendixson process does not terminate in finitely many steps. This led Cantor to the development of the theory of ordinals.
- Theorem (Lebesgue, 1903)

  Each countable closed set $F \subset \mathbb{R}$ is a set of uniqueness.

Cantor–Lebesgue Theorem (1872–1903)

- Definition. $F^{(0)} := F,\ F^{(1)} := F',\ \dots\ F^{(n+1)} := (F^{(n)})'.$

Cantor–Bendixson derivative $F' = \{x \in F:\ x = \lim_n x_n,\ (x_n) \subset F\}$.

- G. Cantor (1872). Let $F \subset \mathbb{R}$ be closed.

  If $F^{(n)} = \varnothing$ for some $n \in \mathbb{N} \implies F$ is a set of uniqueness.
- G. Cantor. *Über die Ausdehnung eines Satzes aus der Theorie der trigonometrischen Reihen.* Math. Ann. **5**, 123 (1872).
- It is a natural intension to continue $\omega, \omega+1, \dots, \omega^2, \dots, \omega^\omega \dots$, since the Cantor–Bendixson process does not terminate in finitely many steps. This led Cantor to the development of the theory of ordinals.
- Theorem (Lebesgue, 1903)
  Each countable closed set $F \subset \mathbb{R}$ is a set of uniqueness.

Cantor–Lebesgue Theorem (1872–1903)

- Definition. $F^{(0)} := F,\ F^{(1)} := F',\ \dots\ F^{(n+1)} := (F^{(n)})'.$

Cantor–Bendixson derivative $F' = \{x \in F :\ x = \lim_n x_n,\ (x_n) \subset F\}.$

- G. Cantor (1872). Let $F \subset \mathbb{R}$ be closed.

  If $F^{(n)} = \varnothing$ for some $n \in \mathbb{N} \implies F$ is a set of uniqueness.

- G. Cantor. *Über die Ausdehnung eines Satzes aus der Theorie der trigonometrischen Reihen.* Math. Ann. **5**, 123 (1872).

- It is a natural intension to continue $\omega, \omega + 1, \dots, \omega^2, \dots, \omega^\omega \dots$, since the Cantor–Bendixson process does not terminate in finitely many steps. This led Cantor to the development of the theory of ordinals.

- Theorem (Lebesgue, 1903)
  Each countable closed set $F \subset \mathbb{R}$ is a set of uniqueness.

Cantor's Set Theory: The Beginning (1874)

- Remark. An arbitrary countable set: S. Bernstein (1908), W. H. Young (1909); the union of countable many sets of uniqueness: N. Bari (1923).
- After 1872 Cantor never returned to the uniqueness problem.
- His search for the extensions allowing exceptional points led him to the creation of set theory including the concepts of ordinal and cardinal and the method of transfinite induction.
- Cantor (1874): *Uber eine Eigenschaft des Inbegriffes aller reellen algebraischen Zahlen*. J. Reine und Angew. Math, Bd. 77, S. 258–262.
- The beginning of set theory as a branch of mathematics.
- Explanation of the size of a set in terms of equipollency of two sets.

Cantor's Set Theory: The Beginning (1874)

- Remark. An arbitrary countable set: S. Bernstein (1908), W. H. Young (1909); the union of countable many sets of uniqueness: N. Bari (1923).
- After 1872 Cantor never returned to the uniqueness problem.
- His search for the extensions allowing exceptional points led him to the creation of set theory including the concepts of ordinal and cardinal and the method of transfinite induction.
- Cantor (1874): *Uber eine Eigenschaft des Inbegriffes aller reellen algebraischen Zahlen*. J. Reine und Angew. Math, Bd. 77, S. 258–262.
- The beginning of set theory as a branch of mathematics.
- Explanation of the size of a set in terms of equipollency of two sets.

Cantor's Set Theory: The Beginning (1874)

- Remark. An arbitrary countable set: S. Bernstein (1908), W. H. Young (1909); the union of countable many sets of uniqueness: N. Bari (1923).
- After 1872 Cantor never returned to the uniqueness problem.
- His search for the extensions allowing exceptional points led him to the creation of set theory including the concepts of ordinal and cardinal and the method of transfinite induction.
- Cantor (1874): *Uber eine Eigenschaft des Inbegriffes aller reellen algebraischen Zahlen*. J. Reine und Angew. Math, Bd. 77, S. 258–262.
- The beginning of set theory as a branch of mathematics.
- Explanation of the size of a set in terms of equipollency of two sets.

Cantor's Set Theory: The Beginning (1874)

- Remark. An arbitrary countable set: S. Bernstein (1908), W. H. Young (1909); the union of countable many sets of uniqueness: N. Bari (1923).
- After 1872 Cantor never returned to the uniqueness problem.
- His search for the extensions allowing exceptional points led him to the creation of set theory including the concepts of ordinal and cardinal and the method of transfinite induction.
- Cantor (1874): *Uber eine Eigenschaft des Inbegriffes aller reellen algebraischen Zahlen*. J. Reine und Angew. Math, Bd. 77, S. 258–262.
- The beginning of set theory as a branch of mathematics.
- Explanation of the size of a set in terms of equipollency of two sets.

Cantor's Set Theory: The Beginning (1874)

- Remark. An arbitrary countable set: S. Bernstein (1908), W. H. Young (1909); the union of countable many sets of uniqueness: N. Bari (1923).
- After 1872 Cantor never returned to the uniqueness problem.
- His search for the extensions allowing exceptional points led him to the creation of set theory including the concepts of ordinal and cardinal and the method of transfinite induction.
- Cantor (1874): *Uber eine Eigenschaft des Inbegriffes aller reellen algebraischen Zahlen*. J. Reine und Angew. Math, Bd. 77, S. 258–262.
- The beginning of set theory as a branch of mathematics.
- Explanation of the size of a set in terms of equipollency of two sets.

Cantor's Set Theory: The Beginning (1874)

- Remark. An arbitrary countable set: S. Bernstein (1908), W. H. Young (1909); the union of countable many sets of uniqueness: N. Bari (1923).
- After 1872 Cantor never returned to the uniqueness problem.
- His search for the extensions allowing exceptional points led him to the creation of set theory including the concepts of ordinal and cardinal and the method of transfinite induction.
- Cantor (1874): *Uber eine Eigenschaft des Inbegriffes aller reellen algebraischen Zahlen*. J. Reine und Angew. Math, Bd. 77, S. 258–262.
- The beginning of set theory as a branch of mathematics.
- Explanation of the size of a set in terms of equipollency of two sets.

Cantor's Set Theory: Cardinals

- $\mathrm{Card}(A)$ = the cardinal number of A.
- $\mathrm{Card}(A) = \mathrm{Card}(B) \iff A \sim B$

  A and B can be put into a one-to-one correspondence.
- $\mathrm{Card}(A) < \mathrm{Card}(B) \iff A \sim B_0 \subset B$ but not $B \sim A_0 \subset A$.
- $\mathrm{Card}(A) \leqslant \mathrm{Card}(B) \iff$

  either $\mathrm{Card}(A) < \mathrm{Card}(B)$ or $\mathrm{Card}(A) = \mathrm{Card}(B)$.

Cantor's Set Theory: Cardinals

- $\mathrm{Card}(A)$ = the cardinal number of A.
- $\mathrm{Card}(A) = \mathrm{Card}(B) \iff A \sim B$
  A and B can be put into a one-to-one correspondence.
- $\mathrm{Card}(A) < \mathrm{Card}(B) \iff A \sim B_0 \subset B$ but not $B \sim A_0 \subset A$.
- $\mathrm{Card}(A) \leqslant \mathrm{Card}(B) \iff$
  either $\mathrm{Card}(A) < \mathrm{Card}(B)$ or $\mathrm{Card}(A) = \mathrm{Card}(B)$.

Cantor's Set Theory: Cardinals

- $\mathrm{Card}(A)$ = the cardinal number of A.
- $\mathrm{Card}(A) = \mathrm{Card}(B) \iff A \sim B$
  A and B can be put into a one-to-one correspondence.
- $\mathrm{Card}(A) < \mathrm{Card}(B) \iff A \sim B_0 \subset B$ but not $B \sim A_0 \subset A$.
- $\mathrm{Card}(A) \leqslant \mathrm{Card}(B) \iff$
  either $\mathrm{Card}(A) < \mathrm{Card}(B)$ or $\mathrm{Card}(A) = \mathrm{Card}(B)$.

Cantor's Set Theory: Cardinals

- $\mathrm{Card}(A)$ = the cardinal number of A.
- $\mathrm{Card}(A) = \mathrm{Card}(B) \iff A \sim B$
  A and B can be put into a one-to-one correspondence.
- $\mathrm{Card}(A) < \mathrm{Card}(B) \iff A \sim B_0 \subset B$ but not $B \sim A_0 \subset A$.
- $\mathrm{Card}(A) \leqslant \mathrm{Card}(B) \iff$
  either $\mathrm{Card}(A) < \mathrm{Card}(B)$ or $\mathrm{Card}(A) = \mathrm{Card}(B)$.

What Is the Continuum?

- The physical continuum (time).
- The geometrical continuum (straight line).
- The arithmetical continuum (reals $\mathbb{R}$).
- The set-theoretical continuum (powerset $\mathcal{P}(\mathbb{N})$ of naturals $\mathbb{N}$).
- $\mathbb{N} := \{0, 1, \ldots, n, \ldots\}, \quad 2^{\mathbb{N}} := \{ f : \mathbb{N} \to \{0, 1\} \}$.
- Theorem. $2^{\mathbb{N}} \sim \mathcal{P}(\mathbb{N}) \sim \mathbb{R}$.

What Is the Continuum?

- The physical continuum (time).
- The geometrical continuum (straight line).
- The arithmetical continuum (reals $\mathbb{R}$).
- The set-theoretical continuum (powerset $\mathcal{P}(\mathbb{N})$ of naturals $\mathbb{N}$).
- $\mathbb{N} := \{0, 1, \ldots, n, \ldots\}, \quad 2^{\mathbb{N}} := \{ f : \mathbb{N} \to \{0, 1\} \}$.
- Theorem. $2^{\mathbb{N}} \sim \mathcal{P}(\mathbb{N}) \sim \mathbb{R}$.

What Is the Continuum?

- The physical continuum (time).
- The geometrical continuum (straight line).
- The arithmetical continuum (reals $\mathbb{R}$).
- The set-theoretical continuum (powerset $\mathcal{P}(\mathbb{N})$ of naturals $\mathbb{N}$).
- $\mathbb{N} := \{0, 1, \ldots, n, \ldots\}, \quad 2^{\mathbb{N}} := \big\{ f : \mathbb{N} \to \{0, 1\} \big\}$.
- Theorem. $2^{\mathbb{N}} \sim \mathcal{P}(\mathbb{N}) \sim \mathbb{R}$.

What Is the Continuum?

- The physical continuum (time).
- The geometrical continuum (straight line).
- The arithmetical continuum (reals $\mathbb{R}$).
- The set-theoretical continuum (powerset $\mathcal{P}(\mathbb{N})$ of naturals $\mathbb{N}$).
- $\mathbb{N} := \{0, 1, \ldots, n, \ldots\}$, $2^{\mathbb{N}} := \{ f : \mathbb{N} \to \{0, 1\} \}$.
- Theorem. $2^{\mathbb{N}} \sim \mathcal{P}(\mathbb{N}) \sim \mathbb{R}$.

What Is the Continuum?

- The physical continuum (time).
- The geometrical continuum (straight line).
- The arithmetical continuum (reals $\mathbb{R}$).
- The set-theoretical continuum (powerset $\mathcal{P}(\mathbb{N})$ of naturals $\mathbb{N}$).
- $\mathbb{N} := \{0, 1, \ldots, n, \ldots\}, \quad 2^{\mathbb{N}} := \big\{ f : \mathbb{N} \to \{0, 1\} \big\}$.
- Theorem. $2^{\mathbb{N}} \sim \mathcal{P}(\mathbb{N}) \sim \mathbb{R}$.

What Is the Continuum?

- The physical continuum (time).
- The geometrical continuum (straight line).
- The arithmetical continuum (reals $\mathbb{R}$).
- The set-theoretical continuum (powerset $\mathcal{P}(\mathbb{N})$ of naturals $\mathbb{N}$).
- $\mathbb{N} := \{0, 1, \ldots, n, \ldots\}, \quad 2^{\mathbb{N}} := \big\{ f : \mathbb{N} \to \{0, 1\} \big\}$.
- Theorem. $2^{\mathbb{N}} \sim \mathcal{P}(\mathbb{N}) \sim \mathbb{R}$.

Cantor's Continuum Hypothesis (1878)

- Definitions. $\mathrm{Card}(\{0, 1 \ldots, n-1\}) := n$.

  ✓ A is finite $\iff$ $\mathrm{Card}(A) = \mathrm{Card}(n)$ for some $n \in \mathbb{N}$.

  ✓ A is countable $\iff$ $\mathrm{Card}(A) = \mathrm{Card}(\mathbb{N})$.

  ✓ A is continual $\iff$ $\mathrm{Card}(A) = \mathrm{Card}(\mathbb{R})$.

- Continuum Hypothesis (CH).

  Every $A \subset [0, 1]$ is either finite, or countable, or continual.

- Cantor (1878): *Ein Beitrag zur Mannigfaltigketslehre*. J. Reine und Angew. Math, Bd. 84, S. 242–258.

Cantor's Continuum Hypothesis (1878)

- Definitions. $\mathrm{Card}(\{0, 1 \ldots, n-1\}) := n$.

  ✓ A is finite $\iff$ $\mathrm{Card}(A) = \mathrm{Card}(n)$ for some $n \in \mathbb{N}$.

  ✓ A is countable $\iff$ $\mathrm{Card}(A) = \mathrm{Card}(\mathbb{N})$.

  ✓ A is continual $\iff$ $\mathrm{Card}(A) = \mathrm{Card}(\mathbb{R})$.

- Continuum Hypothesis (CH).

  Every $A \subset [0, 1]$ is either finite, or countable, or continual.

- Cantor (1878): *Ein Beitrag zur Mannigfaltigketslehre*. J. Reine und Angew. Math, Bd. 84, S. 242–258.

Cantor's Continuum Hypothesis (1878)

- Definitions. $\mathrm{Card}(\{0, 1 \ldots, n-1\}) := n$.

  ✓ A is finite $\iff \mathrm{Card}(A) = \mathrm{Card}(n)$ for some $n \in \mathbb{N}$.

  ✓ A is countable $\iff \mathrm{Card}(A) = \mathrm{Card}(\mathbb{N})$.

  ✓ A is continual $\iff \mathrm{Card}(A) = \mathrm{Card}(\mathbb{R})$.

- Continuum Hypothesis (CH).

  Every $A \subset [0, 1]$ is either finite, or countable, or continual.

- Cantor (1878): *Ein Beitrag zur Mannigfaltigketslehre*. J. Reine und Angew. Math, Bd. 84, S. 242–258.

The Least Uncountable Cardinal

- Notation. $\omega_0 := \mathrm{Card}(\mathbb{N}), \quad 2^{\omega_0} = \mathrm{Card}(\mathcal{P}(\mathbb{N})) = \mathrm{Card}(2^{\mathbb{N}})$.
- Theorem (Cantor, 1874). The continuum is uncountable:

$$\boxed{\omega_0 < 2^{\omega_0}} \quad (\mathrm{Card}(\mathbb{N}) < \mathrm{Card}(\mathbb{R})).$$

- Proof: By the Diagonal Argument.
- Cardinals are well ordered $\implies$

  $\exists$ the least uncountable cardinal ω_1:

$$\mathrm{Card}(\mathbb{N}) = \boxed{\omega_0 < \omega_1 \leq 2^{\omega_0}} = \mathrm{Card}(\mathbb{R}).$$

Cantor's Continuum Problem (1978)

- The Continuum Problem: Is there any cardinal number between $\mathrm{Card}(\mathbb{N})$ and $\mathrm{Card}(\mathcal{P}(\mathbb{N}))$?

$$\mathrm{Card}(\mathbb{N}) = \omega_0 < \boxed{\omega_1 < 2^{\omega_0}} = \mathrm{Card}(\mathbb{R})\,?$$

- The Continuum Hypothesis, CH $\equiv \omega_1 = 2^{\omega_0}$, says that there is no such cardinal number:

$$\mathrm{Card}(\mathbb{N}) = \omega_0 < \boxed{\omega_1 = 2^{\omega_0}} = \mathrm{Card}(\mathbb{R}).$$

Cantor's Continuum Problem (1978)

- The Continuum Problem: Is there any cardinal number between $\mathrm{Card}(\mathbb{N})$ and $\mathrm{Card}(\mathcal{P}(\mathbb{N}))$?

$$\mathrm{Card}(\mathbb{N}) = \omega_0 < \boxed{\omega_1 < 2^{\omega_0}} = \mathrm{Card}(\mathbb{R})\,?$$

- The Continuum Hypothesis, CH $\equiv \omega_1 = 2^{\omega_0}$, says that there is no such cardinal number:

$$\mathrm{Card}(\mathbb{N}) = \omega_0 < \boxed{\omega_1 = 2^{\omega_0}} = \mathrm{Card}(\mathbb{R}).$$

Cantor: Naive Set Theory

- Cantor: "I see it, but I don't believe it!" $\boxed{[0,1] \sim [0,1]^n}$.
- Hadamard, Hurwitz: Zürich, ICM-1897, Applications.
- Hilbert: "No one shall expel us from the Paradise that Cantor has created."
- Hilbert: 23 Mathematical Problems, Paris, ICM-1900.
- Problem 1: Cantor's Continuum Hypothesis (CH).

  "The investigations of Cantor...suggest a very plausible theorem [namely, CH], which in spite of the most strenuous of efforts, no one has succeeded in proving."

Cantor: Naive Set Theory

- Cantor: "I see it, but I don't believe it!" $\boxed{[0,1] \sim [0,1]^n}$.
- Hadamard, Hurwitz: Zürich, ICM-1897, Applications.
- Hilbert: "No one shall expel us from the Paradise that Cantor has created."
- Hilbert: 23 Mathematical Problems, Paris, ICM-1900.
- Problem 1: Cantor's Continuum Hypothesis (CH).

  "The investigations of Cantor...suggest a very plausible theorem [namely, CH], which in spite of the most strenuous of efforts, no one has succeeded in proving."

Cantor: Naive Set Theory

- Cantor: "I see it, but I don't believe it!" $\boxed{[0,1] \sim [0,1]^n}$.
- Hadamard, Hurwitz: Zürich, ICM-1897, Applications.
- Hilbert: "No one shall expel us from the Paradise that Cantor has created."
- Hilbert: 23 Mathematical Problems, Paris, ICM-1900.
- Problem 1: Cantor's Continuum Hypothesis (CH).

  "The investigations of Cantor...suggest a very plausible theorem [namely, CH], which in spite of the most strenuous of efforts, no one has succeeded in proving."

Cantor: Naive Set Theory

- Cantor: "I see it, but I don't believe it!" $\boxed{[0,1] \sim [0,1]^n}$.
- Hadamard, Hurwitz: Zürich, ICM-1897, Applications.
- Hilbert: "No one shall expel us from the Paradise that Cantor has created."
- Hilbert: 23 Mathematical Problems, Paris, ICM-1900.
- Problem 1: Cantor's Continuum Hypothesis (CH).

  "The investigations of Cantor...suggest a very plausible theorem [namely, CH], which in spite of the most strenuous of efforts, no one has succeeded in proving."

Cantor: Naive Set Theory

- Cantor: "I see it, but I don't believe it!" $\boxed{[0,1] \sim [0,1]^n}$.
- Hadamard, Hurwitz: Zürich, ICM-1897, Applications.
- Hilbert: "No one shall expel us from the Paradise that Cantor has created."
- Hilbert: 23 Mathematical Problems, Paris, ICM-1900.
- Problem 1: Cantor's Continuum Hypothesis (CH).

  "The investigations of Cantor...suggest a very plausible theorem [namely, CH], which in spite of the most strenuous of efforts, no one has succeeded in proving."

Zermelo–Fraenkel Axiomatic Set Theory

- ZF = Zermelo–Fraenkel axiomatic set theory (Zermelo, 1907; Fraenkel and Skolem, 1922: von Neumann, 1925).

  AC = Axiom of Choice (Zermelo, 1904).

  $\mathrm{ZFC} = \mathrm{ZF} + \mathrm{AC}$ is the common foundation of mathematics.
- $\mathbb{V}$ is the universe of sets, the von Neumann universe.
- $\mathbb{L}$ is the universe of constructible sets.
- $\mathbb{V}^{(\mathbb{B})}$ is the universe of Boolean valued sets.
- $\mathbb{L} \subset \mathbb{V} \subset \mathbb{V}^{(\mathbb{B})} \qquad \left(\mathbb{V} \rightleftarrows \mathbb{V}^{(\{0,1\})} \subset \mathbb{V}^{(\mathbb{B})}\right)$.
- Theorem. $\mathbb{V} \models \mathrm{ZFC}$.

Zermelo–Fraenkel Axiomatic Set Theory

- ZF = Zermelo–Fraenkel axiomatic set theory (Zermelo, 1907; Fraenkel and Skolem, 1922: von Neumann, 1925).

  AC = Axiom of Choice (Zermelo, 1904).

  $\mathrm{ZFC} = \mathrm{ZF} + \mathrm{AC}$ is the common foundation of mathematics.
- $\mathbb{V}$ is the universe of sets, the von Neumann universe.
- $\mathbb{L}$ is the universe of constructible sets.
- $\mathbb{V}^{(\mathbb{B})}$ is the universe of Boolean valued sets.
- $\mathbb{L} \subset \mathbb{V} \subset \mathbb{V}^{(\mathbb{B})}$ $\quad \big(\mathbb{V} \rightleftarrows \mathbb{V}^{(\{0,1\})} \subset \mathbb{V}^{(\mathbb{B})}\big)$.
- Theorem. $\mathbb{V} \models \mathrm{ZFC}$.

Zermelo–Fraenkel Axiomatic Set Theory

- ZF = Zermelo–Fraenkel axiomatic set theory (Zermelo, 1907; Fraenkel and Skolem, 1922: von Neumann, 1925).

  AC = Axiom of Choice (Zermelo, 1904).

  $\mathrm{ZFC} = \mathrm{ZF} + \mathrm{AC}$ is the common foundation of mathematics.
- $\mathbb{V}$ is the universe of sets, the von Neumann universe.
- $\mathbb{L}$ is the universe of constructible sets.
- $\mathbb{V}^{(\mathbb{B})}$ is the universe of Boolean valued sets.
- $\mathbb{L} \subset \mathbb{V} \subset \mathbb{V}^{(\mathbb{B})}$ $\quad \left(\mathbb{V} \rightleftarrows \mathbb{V}^{(\{0,1\})} \subset \mathbb{V}^{(\mathbb{B})}\right)$.
- Theorem. $\mathbb{V} \models \mathrm{ZFC}$.

Zermelo–Fraenkel Axiomatic Set Theory

- ZF = Zermelo–Fraenkel axiomatic set theory (Zermelo, 1907; Fraenkel and Skolem, 1922: von Neumann, 1925).

  AC = Axiom of Choice (Zermelo, 1904).

  $\mathrm{ZFC} = \mathrm{ZF} + \mathrm{AC}$ is the common foundation of mathematics.
- $\mathbb{V}$ is the universe of sets, the von Neumann universe.
- $\mathbb{L}$ is the universe of constructible sets.
- $\mathbb{V}^{(\mathbb{B})}$ is the universe of Boolean valued sets.
- $\mathbb{L} \subset \mathbb{V} \subset \mathbb{V}^{(\mathbb{B})}$ $\quad \left(\mathbb{V} \rightleftarrows \mathbb{V}^{(\{0,1\})} \subset \mathbb{V}^{(\mathbb{B})}\right)$.
- Theorem. $\mathbb{V} \models \mathrm{ZFC}$.

Zermelo–Fraenkel Axiomatic Set Theory

- ZF = Zermelo–Fraenkel axiomatic set theory (Zermelo, 1907; Fraenkel and Skolem, 1922: von Neumann, 1925).

  AC = Axiom of Choice (Zermelo, 1904).

  $\mathrm{ZFC} = \mathrm{ZF} + \mathrm{AC}$ is the common foundation of mathematics.
- $\mathbb{V}$ is the universe of sets, the von Neumann universe.
- $\mathbb{L}$ is the universe of constructible sets.
- $\mathbb{V}^{(\mathbb{B})}$ is the universe of Boolean valued sets.
- $\mathbb{L} \subset \mathbb{V} \subset \mathbb{V}^{(\mathbb{B})} \qquad \big(\mathbb{V} \rightleftarrows \mathbb{V}^{(\{0,1\})} \subset \mathbb{V}^{(\mathbb{B})}\big)$.
- Theorem. $\mathbb{V} \models \mathrm{ZFC}$.

Zermelo–Fraenkel Axiomatic Set Theory

- ZF = Zermelo–Fraenkel axiomatic set theory (Zermelo, 1907; Fraenkel and Skolem, 1922: von Neumann, 1925).

  AC = Axiom of Choice (Zermelo, 1904).

  $\mathrm{ZFC} = \mathrm{ZF} + \mathrm{AC}$ is the common foundation of mathematics.
- $\mathbb{V}$ is the universe of sets, the von Neumann universe.
- $\mathbb{L}$ is the universe of constructible sets.
- $\mathbb{V}^{(\mathbb{B})}$ is the universe of Boolean valued sets.
- $\mathbb{L} \subset \mathbb{V} \subset \mathbb{V}^{(\mathbb{B})} \qquad \big(\mathbb{V} \rightleftarrows \mathbb{V}^{(\{0,1\})} \subset \mathbb{V}^{(\mathbb{B})}\big)$.
- Theorem. $\mathbb{V} \models \mathrm{ZFC}$.

Kurt Friedrich Gödel

American mathematician,
Gödel incompleteness theorems, consistency of CH with ZFC
(April 28, 1906, Brünn, Austria-Hungary —
January 14, 1978, Princeton, New Jersey, United States)

K. Gödel: The Relative Consistency of CH (1939)

- Theorem (Gödel, 1938–1940):
  ZF is consistent $\Longrightarrow$ $\mathrm{ZFC} + \mathrm{CH}$ is consistent.
- Proof:

  (1) $\mathrm{ZF} \vdash (\mathbb{L} \models \mathrm{ZFC})$.

  (2) $\mathrm{ZF} \vdash (\mathbb{L} \models \mathrm{CH})$.

  The universe of constructible sets $\mathbb{L}$ forms an "inner" model of $\mathrm{ZFC} + \mathrm{CH}$ within ZF.
- Corollary. CH cannot be disproved from the standard ZFC axioms of set theory.

K. Gödel: The Relative Consistency of CH (1939)

- Theorem (Gödel, 1938–1940):
  ZF is consistent $\Longrightarrow$ $\mathrm{ZFC} + \mathrm{CH}$ is consistent.
- Proof:

  (1) $\mathrm{ZF} \vdash (\mathbb{L} \models \mathrm{ZFC})$.

  (2) $\mathrm{ZF} \vdash (\mathbb{L} \models \mathrm{CH})$.

  The universe of constructible sets $\mathbb{L}$ forms an "inner" model of $\mathrm{ZFC} + \mathrm{CH}$ within ZF.
- Corollary. CH cannot be disproved from the standard ZFC axioms of set theory.

K. Gödel: The Relative Consistency of CH (1939)

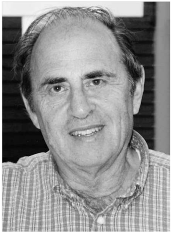
- Theorem (Gödel, 1938–1940):
  ZF is consistent $\Longrightarrow$ $\mathrm{ZFC} + \mathrm{CH}$ is consistent.
- Proof:

  (1) $\mathrm{ZF} \vdash (\mathbb{L} \models \mathrm{ZFC})$.

  (2) $\mathrm{ZF} \vdash (\mathbb{L} \models \mathrm{CH})$.

  The universe of constructible sets $\mathbb{L}$ forms an "inner" model of $\mathrm{ZFC} + \mathrm{CH}$ within ZF.
- Corollary. CH cannot be disproved from the standard ZFC axioms of set theory.

Paul Joseph Cohen



American mathematician,
Cohen forcing and independence of the Continuum hypothesis
(April 2, 1934,Long Branch, New Jersey —
March 23, 2007, Stanford, California)

P.J.Cohen: The Relative Consistency of $\neg\mathrm{CH}$ (1963)

- Theorem (Cohen, 1963).
  ZF is consistent $\Longrightarrow \mathrm{ZFC} + \neg\mathrm{CH}$ is consistent.
- Proof:
  (1) The universe of Boolean valued sets $\mathbb{V}^{(\mathbb{B})}$ forms an "inner" model of ZFC within ZFC, i.e.,
  $$\mathrm{ZFC} \vdash \left(\mathbb{V}^{(\mathbb{B})} \models \mathrm{ZFC}\right).$$
  (2) There exists a complete Boolean algebra $\mathbb{B}$ with
  $$\mathrm{ZFC} \vdash \left(\mathbb{V}^{(\mathbb{B})} \models 2^{\omega_0} \neq \omega_1\right).$$
- Corollary. CH cannot be proved within ZFC.
- Theorem. CH is logically independent of ZFC.

P.J.Cohen: The Relative Consistency of $\neg\mathrm{CH}$ (1963)

- Theorem (Cohen, 1963).
  ZF is consistent $\Longrightarrow \mathrm{ZFC} + \neg\mathrm{CH}$ is consistent.
- Proof:
  (1) The universe of Boolean valued sets $\mathbb{V}^{(\mathbb{B})}$ forms an "inner" model of ZFC within ZFC, i.e.,
  $$\mathrm{ZFC} \vdash \left(\mathbb{V}^{(\mathbb{B})} \models \mathrm{ZFC}\right).$$
  (2) There exists a complete Boolean algebra $\mathbb{B}$ with
  $$\mathrm{ZFC} \vdash \left(\mathbb{V}^{(\mathbb{B})} \models 2^{\omega_0} \neq \omega_1\right).$$
- Corollary. CH cannot be proved within ZFC.
- Theorem. CH is logically independent of ZFC.

P.J.Cohen: The Relative Consistency of $\neg\mathrm{CH}$ (1963)

- Theorem (Cohen, 1963).
  ZF is consistent $\Longrightarrow \mathrm{ZFC} + \neg\mathrm{CH}$ is consistent.
- Proof:
  (1) The universe of Boolean valued sets $\mathbb{V}^{(\mathbb{B})}$ forms an "inner" model of ZFC within ZFC, i.e.,
  $$\mathrm{ZFC} \vdash \left(\mathbb{V}^{(\mathbb{B})} \models \mathrm{ZFC}\right).$$
  (2) There exists a complete Boolean algebra $\mathbb{B}$ with
  $$\mathrm{ZFC} \vdash \left(\mathbb{V}^{(\mathbb{B})} \models 2^{\omega_0} \neq \omega_1\right).$$
- Corollary. CH cannot be proved within ZFC.
- Theorem. CH is logically independent of ZFC.

P.J.Cohen: The Relative Consistency of $\neg\mathrm{CH}$ (1963)

- Theorem (Cohen, 1963).
  ZF is consistent $\Longrightarrow \mathrm{ZFC} + \neg\mathrm{CH}$ is consistent.
- Proof:
  (1) The universe of Boolean valued sets $\mathbb{V}^{(\mathbb{B})}$ forms an "inner" model of ZFC within ZFC, i.e.,
  $$\mathrm{ZFC} \vdash \left(\mathbb{V}^{(\mathbb{B})} \models \mathrm{ZFC}\right).$$
  (2) There exists a complete Boolean algebra $\mathbb{B}$ with
  $$\mathrm{ZFC} \vdash \left(\mathbb{V}^{(\mathbb{B})} \models 2^{\omega_0} \neq \omega_1\right).$$
- Corollary. CH cannot be proved within ZFC.
- Theorem. CH is logically independent of ZFC.

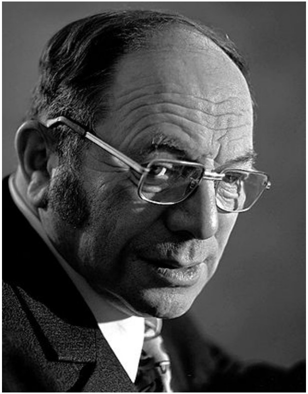
Boolean Valued Models (1967)

- D. Scott, R. Solovay, and P. Vopěnka (1967).
  - ✓ A comprehensive presentation of the Cohen forcing method.
  - ✓ This gave rise to the Boolean valued models of set theory.
- D. Scott (1977): "It was in 1963 that we were hit by a real bomb, however, when Paul J. Cohen discovered his method of 'forcing', which started a long chain reaction of independence results ... Set theory could never be the same after Cohen."

Boolean Valued Models (1967)

- D. Scott, R. Solovay, and P. Vopěnka (1967).
  - ✓ A comprehensive presentation of the Cohen forcing method.
  - ✓ This gave rise to the Boolean valued models of set theory.
- D. Scott (1977): "It was in 1963 that we were hit by a real bomb, however, when Paul J. Cohen discovered his method of 'forcing', which started a long chain reaction of independence results ... Set theory could never be the same after Cohen."

Leonid Vitaliyevich Kantorovich



Russian mathematician, Nobel Laureate in Economics,
linear programming, Kantorovich spaces
(January 19, 1912, Saint Petersburg, Russian Empire —
April 7, 1986, Moscow, USSR)

Kantorovich Spaces (1935)

- Definition. A *vector lattice* is a real vector space X with some partial order $\leq$ such that there exist

  ✓ $x \vee y := \sup\{x, y\}$, the *join* of x and y anf

  ✓ $x \wedge y := \inf\{x, y\}$, the *meet* of x and y for all $x, y \in X$,

  while the *positive cone*

  ✓ $X_+ := \{x \in X : \ x \geq 0\}$ of X has the properties

  ✓ $X_+ + X_+ \subset X_+, \quad \mathbb{R}_+ \cdot X_+ \subset X_+$.

- Definition. A vector lattice X is a *Kantorovich space* if each nonempty order bounded set in X has the supremum and infinmum:

$$U \subset [a, b] := \{x \in X : \ a \leq x \leq b\} \implies \exists \sup(U), \inf(U) \in X.$$

Kantorovich Spaces (1935)

- Definition. A *vector lattice* is a real vector space X with some partial order $\leq$ such that there exist

  ✓ $x \vee y := \sup\{x, y\}$, the *join* of x and y anf

  ✓ $x \wedge y := \inf\{x, y\}$, the *meet* of x and y for all $x, y \in X$,

  while the *positive cone*

  ✓ $X_+ := \{x \in X : \ x \geq 0\}$ of X has the properties

  ✓ $X_+ + X_+ \subset X_+, \quad \mathbb{R}_+ \cdot X_+ \subset X_+.$

- Definition. A vector lattice X is a *Kantorovich space* if each nonempty order bounded set in X has the supremum and infinmum:

  $$U \subset [a, b] := \{x \in X : \ a \leq x \leq b\} \implies \exists \sup(U), \inf(U) \in X.$$

Kantorovich Spaces: Examples

- Examples. $L^p(\Omega, \Sigma, \mu)$, l^p $(1 \le p \le \infty)$, c, and c_0.
- Theorem (Stone, 1937, 1948; Ogasawara, 1944). $C(Q)$ is a Kantorovich space $\iff$ Q is extremally disconnected.
- Definition. Let H be a Hilbert space and $B_{sa}(H)$ the space of all selfadjoint bounded linear operators in H. The order on $B_{sa}(H)$:

$$S \le T \iff (\forall h \in H)\,(Sh, h) \le (Th, h) \quad \big(S, T \in B_{sa}(H)\big).$$

- Example. $(B_{sa}(H), \le)$ is an ordered vector space.
  Each strongly closed subalgebra $A \subset B_{sa}(H)$ is a Kantorovich space.
- Example. Let $\overline{A}$ is a set of all densely defined selfadjoint linear operators in H whose spectral functions are in A.
  Then $\overline{A}$ is a Kantorovich space.

Kantorovich Spaces: Examples

- Examples. $L^p(\Omega, \Sigma, \mu)$, l^p $(1 \leq p \leq \infty)$, c, and c_0.
- Theorem (Stone, 1937, 1948; Ogasawara, 1944). $C(Q)$ is a Kantorovich space $\iff$ Q is extremally disconnected.
- Definition. Let H be a Hilbert space and $B_{sa}(H)$ the space of all selfadjoint bounded linear operators in H. The order on $B_{sa}(H)$:

$$S \leq T \iff (\forall h \in H)\,(Sh, h) \leq (Th, h) \quad \big(S, T \in B_{sa}(H)\big).$$

- Example. $(B_{sa}(H), \leq)$ is an ordered vector space. Each strongly closed subalgebra $A \subset B_{sa}(H)$ is a Kantorovich space.
- Example. Let $\overline{A}$ is a set of all densely defined selfadjoint linear operators in H whose spectral functions are in A. Then $\overline{A}$ is a Kantorovich space.

Kantorovich Spaces: Examples

- Examples. $L^p(\Omega, \Sigma, \mu)$, l^p $(1 \leq p \leq \infty)$, c, and c_0.
- Theorem (Stone, 1937, 1948; Ogasawara, 1944). $C(Q)$ is a Kantorovich space $\iff$ Q is extremally disconnected.
- Definition. Let H be a Hilbert space and $B_{sa}(H)$ the space of all selfadjoint bounded linear operators in H. The order on $B_{sa}(H)$:

$$S \leq T \iff (\forall h \in H)\ (Sh, h) \leq (Th, h)\ \ \big(S, T \in B_{sa}(H)\big).$$

- Example. $(B_{sa}(H), \leq)$ is an ordered vector space. Each strongly closed subalgebra $A \subset B_{sa}(H)$ is a Kantorovich space.
- Example. Let $\overline{A}$ is a set of all densely defined selfadjoint linear operators in H whose spectral functions are in A. Then $\overline{A}$ is a Kantorovich space.

Kantorovich Spaces: Examples

- Examples. $L^p(\Omega, \Sigma, \mu)$, l^p $(1 \le p \le \infty)$, c, and c_0.
- Theorem (Stone, 1937, 1948; Ogasawara, 1944). $C(Q)$ is a Kantorovich space $\iff$ Q is extremally disconnected.
- Definition. Let H be a Hilbert space and $B_{sa}(H)$ the space of all selfadjoint bounded linear operators in H. The order on $B_{sa}(H)$:

$$S \le T \iff (\forall h \in H)\ (Sh, h) \le (Th, h) \ \ \big(S, T \in B_{sa}(H)\big).$$

- Example. $(B_{sa}(H), \le)$ is an ordered vector space. Each strongly closed subalgebra $A \subset B_{sa}(H)$ is a Kantorovich space.
- Example. Let $\overline{A}$ is a set of all densely defined selfadjoint linear operators in H whose spectral functions are in A. Then $\overline{A}$ is a Kantorovich space.

Kantorovich Spaces: Examples

- Examples. $L^p(\Omega, \Sigma, \mu)$, l^p $(1 \le p \le \infty)$, c, and c_0.
- Theorem (Stone, 1937, 1948; Ogasawara, 1944). $C(Q)$ is a Kantorovich space $\iff$ Q is extremally disconnected.
- Definition. Let H be a Hilbert space and $B_{sa}(H)$ the space of all selfadjoint bounded linear operators in H. The order on $B_{sa}(H)$:

$$S \le T \iff (\forall h \in H)\ (Sh, h) \le (Th, h) \ \big(S, T \in B_{sa}(H)\big).$$

- Example. $(B_{sa}(H), \le)$ is an ordered vector space.
  Each strongly closed subalgebra $A \subset B_{sa}(H)$ is a Kantorovich space.
- Example. Let $\overline{A}$ is a set of all densely defined selfadjoint linear operators in H whose spectral functions are in A.
  Then $\overline{A}$ is a Kantorovich space.

L. V. Kantorovich: The Heuristics (1935)

- A Kantorovich space is also called a Dedekind complete vector lattice. The concept (under the name of a "complete semiordered vector space") appeared in Kantorovich's first fundamental article on this topic:
- L. V. Kantorovich. *Dokl. Akad. Nauk SSSR.* 4(1–2) (1935), 11–14, where he wrote:
- "In this note, I define a new type of space that I call a semiordered linear space. The introduction of such a space allows us to study linear operations of one abstract class (those with values in such a space) as linear functionals."
- Here Kantorovich stated the definitive heuristic transfer principle for Dedekind complete vector lattices.

L. V. Kantorovich: The Heuristics (1935)

- A Kantorovich space is also called a Dedekind complete vector lattice. The concept (under the name of a "complete semiordered vector space") appeared in Kantorovich's first fundamental article on this topic:
- L. V. Kantorovich. *Dokl. Akad. Nauk SSSR.* 4(1–2) (1935), 11–14, where he wrote:
- "In this note, I define a new type of space that I call a semiordered linear space. The introduction of such a space allows us to study linear operations of one abstract class (those with values in such a space) as linear functionals."
- Here Kantorovich stated the definitive heuristic transfer principle for Dedekind complete vector lattices.

L. V. Kantorovich: The Heuristics (1935)

- A Kantorovich space is also called a Dedekind complete vector lattice. The concept (under the name of a "complete semiordered vector space") appeared in Kantorovich's first fundamental article on this topic:
- L. V. Kantorovich. *Dokl. Akad. Nauk SSSR.* 4(1–2) (1935), 11–14, where he wrote:
- "In this note, I define a new type of space that I call a semiordered linear space. The introduction of such a space allows us to study linear operations of one abstract class (those with values in such a space) as linear functionals."
- Here Kantorovich stated the definitive heuristic transfer principle for Dedekind complete vector lattices.

L. V. Kantorovich: The Heuristics (1935)

- A Kantorovich space is also called a Dedekind complete vector lattice. The concept (under the name of a "complete semiordered vector space") appeared in Kantorovich's first fundamental article on this topic:
- L. V. Kantorovich. *Dokl. Akad. Nauk SSSR.* 4(1–2) (1935), 11–14, where he wrote:
- "In this note, I define a new type of space that I call a semiordered linear space. The introduction of such a space allows us to study linear operations of one abstract class (those with values in such a space) as linear functionals."
- Here Kantorovich stated the definitive heuristic transfer principle for Dedekind complete vector lattices.

Boolean Valued Analysis: The Beginning (1977)

- D. Scott (1969): We must ask whether there is any interest in these nonstandard models aside from the independence proof; that is do they have any mathematical interest? The answer must be yes, but we cannot yet give a really good arguments.
- E. I. Gordon, *Dokl. Akad. Nauk SSSR*, **237**(4) (1977), 773-775.
- Theorem. The interpretation of the reals in an appropriate Boolean valued model of set theory is a Kantorovich space.

Boolean Valued Analysis: The Beginning (1977)

- D. Scott (1969): We must ask whether there is any interest in these nonstandard models aside from the independence proof; that is do they have any mathematical interest? The answer must be yes, but we cannot yet give a really good arguments.
- E. I. Gordon, *Dokl. Akad. Nauk SSSR*, **237**(4) (1977), 773-775.
- Theorem. The interpretation of the reals in an appropriate Boolean valued model of set theory is a Kantorovich space.

Boolean Valued Analysis: The Beginning (1977)

- D. Scott (1969): We must ask whether there is any interest in these nonstandard models aside from the independence proof; that is do they have any mathematical interest? The answer must be yes, but we cannot yet give a really good arguments.
- E. I. Gordon, *Dokl. Akad. Nauk SSSR*, **237**(4) (1977), 773-775.
- Theorem. The interpretation of the reals in an appropriate Boolean valued model of set theory is a Kantorovich space.

Boolean Valued Analysis: The Beginning (1978)

- G. Takeuti, *Two Applications of Logic to Mathematics*, Princeton Univ. Press, Princeton, (1978).

  ✓ The vector lattice of (cosets of) measurable functions can be considered as Boolean valued reals.

  ✓ The commutative algebra of unbounded selfadjoint operators is another sample of Boolean valued reals.

  ✓ Coined the term "Boolean valued analysis" (1979).

Gordon's Theorem (1977)

- The depth and universality of Kantorovich's principle were demonstrated within Boolean valued analysis.
- Gordon's Theorem (1977).
  Let $\mathcal{R}$ be the field of reals in $\mathbb{V}^{(\mathbb{B})}$. The algebraic structure $\mathrm{R} := \mathcal{R}{\downarrow} \in \mathbb{V}$ (with the descended operations and order) is a (universally complete) Kantorovich space with $\mathbb{B} \simeq \mathbb{P}(\mathrm{R})$.
- The converse is also true: Each Kantorovich space X is isomorphic to an order ideal in $\mathcal{R}{\downarrow}$ with $\mathcal{R} \in \mathbb{V}^{(\mathbb{B})}$ and $\mathbb{B} \simeq \mathbb{P}(X)$.

Gordon's Theorem (1977)

- The depth and universality of Kantorovich's principle were demonstrated within Boolean valued analysis.
- Gordon's Theorem (1977).
  Let $\mathcal{R}$ be the field of reals in $\mathbb{V}^{(\mathbb{B})}$. The algebraic structure $\mathrm{R} := \mathcal{R}{\downarrow} \in \mathbb{V}$ (with the descended operations and order) is a (universally complete) Kantorovich space with $\mathbb{B} \simeq \mathbb{P}(\mathrm{R})$.
- The converse is also true: Each Kantorovich space X is isomorphic to an order ideal in $\mathcal{R}{\downarrow}$ with $\mathcal{R} \in \mathbb{V}^{(\mathbb{B})}$ and $\mathbb{B} \simeq \mathbb{P}(X)$.

Gordon's Theorem (1977)

- The depth and universality of Kantorovich's principle were demonstrated within Boolean valued analysis.
- Gordon's Theorem (1977).
  Let $\mathcal{R}$ be the field of reals in $\mathbb{V}^{(\mathbb{B})}$. The algebraic structure $\mathrm{R} := \mathcal{R}{\downarrow} \in \mathbb{V}$ (with the descended operations and order) is a (universally complete) Kantorovich space with $\mathbb{B} \simeq \mathbb{P}(\mathrm{R})$.
- The converse is also true: Each Kantorovich space X is isomorphic to an order ideal in $\mathcal{R}{\downarrow}$ with $\mathcal{R} \in \mathbb{V}^{(\mathbb{B})}$ and $\mathbb{B} \simeq \mathbb{P}(X)$.

Interaction of Universes

Von Neumann Universe

Boolean Valued Universe

What Is Boolean Valued Analysis?

- Boolean valued analysis is a branch of functional analysis which uses a special model-theoretic technique and consists in studying the properties of a mathematical object by means of comparison between its representations in two different set-theoretic models whose construction utilizes distinct Boolean algebras.
- The von Neumann universe (Cantorian paradise) $\mathbb{V}$ and a specially-trimmed Boolean valued universe $\mathbb{V}^{(\mathbb{B})}$ are taken as these models.
- The comparative analysis requires some ascending–descending machinery to carry out the interplay between $\mathbb{V}$ and $\mathbb{V}^{(\mathbb{B})}$.

What Is Boolean Valued Analysis?

- Boolean valued analysis is a branch of functional analysis which uses a special model-theoretic technique and consists in studying the properties of a mathematical object by means of comparison between its representations in two different set-theoretic models whose construction utilizes distinct Boolean algebras.
- The von Neumann universe (Cantorian paradise) $\mathbb{V}$ and a specially-trimmed Boolean valued universe $\mathbb{V}^{(\mathbb{B})}$ are taken as these models.
- The comparative analysis requires some ascending–descending machinery to carry out the interplay between $\mathbb{V}$ and $\mathbb{V}^{(\mathbb{B})}$.

What Is Boolean Valued Analysis?

- Boolean valued analysis is a branch of functional analysis which uses a special model-theoretic technique and consists in studying the properties of a mathematical object by means of comparison between its representations in two different set-theoretic models whose construction utilizes distinct Boolean algebras.
- The von Neumann universe (Cantorian paradise) $\mathbb{V}$ and a specially-trimmed Boolean valued universe $\mathbb{V}^{(\mathbb{B})}$ are taken as these models.
- The comparative analysis requires some ascending–descending machinery to carry out the interplay between $\mathbb{V}$ and $\mathbb{V}^{(\mathbb{B})}$.

A Boolean Valued Telescope

What Is the Boolean Valued Transfer Principle?

- Let $\mathrm{X} \subset \mathbb{V}$ and $\mathbb{X} \subset \mathbb{V}^{(\mathbb{B})}$ be two classes of mathematical objects.
  Suppose we are able to prove the result:
- Boolean Valued Representation. *Every $X \in \mathrm{X}$ embeds into an Boolean valued model, becoming an object $\mathcal{X} \in \mathbb{X}$ within $\mathbb{V}^{(\mathbb{B})}$.*
- Boolean Valued Transfer Principle. Every theorem about $\mathcal{X}$ within *ZFC* has its counterpart for the original object X interpreted as a Boolean valued object $\mathcal{X}$.
- Boolean Valued Machinery. Translation of theorems from $\mathcal{X} \in \mathbb{V}^{(\mathbb{B})}$ to $X \in \mathbb{V}$ is carried out by the appropriate general operations (ascending–descending) and the principles of Boolean valued analysis.
- A. G. Kusraev and S. S. Kutateladze, *Introduction to Boolean Valued Analysis*, Moscow, Nauka (2005).

  Boolean Valued Analysis, Dordrecht, Kluwer (1999).

What Is the Boolean Valued Transfer Principle?

- Let $\mathrm{X} \subset \mathbb{V}$ and $\mathbb{X} \subset \mathbb{V}^{(\mathbb{B})}$ be two classes of mathematical objects.
  Suppose we are able to prove the result:
- Boolean Valued Representation. *Every $X \in \mathrm{X}$ embeds into an Boolean valued model, becoming an object $\mathcal{X} \in \mathbb{X}$ within $\mathbb{V}^{(\mathbb{B})}$.*
- Boolean Valued Transfer Principle. Every theorem about $\mathcal{X}$ within *ZFC* has its counterpart for the original object X interpreted as a Boolean valued object $\mathcal{X}$.
- Boolean Valued Machinery. Translation of theorems from $\mathcal{X} \in \mathbb{V}^{(\mathbb{B})}$ to $X \in \mathbb{V}$ is carried out by the appropriate general operations (ascending–descending) and the principles of Boolean valued analysis.
- A. G. Kusraev and S. S. Kutateladze, *Introduction to Boolean Valued Analysis*, Moscow, Nauka (2005).

  Boolean Valued Analysis, Dordrecht, Kluwer (1999).

What Is the Boolean Valued Transfer Principle?

- Let $\mathrm{X} \subset \mathbb{V}$ and $\mathbb{X} \subset \mathbb{V}^{(\mathbb{B})}$ be two classes of mathematical objects.
  Suppose we are able to prove the result:
- Boolean Valued Representation. *Every $X \in \mathrm{X}$ embeds into an Boolean valued model, becoming an object $\mathcal{X} \in \mathbb{X}$ within $\mathbb{V}^{(\mathbb{B})}$.*
- Boolean Valued Transfer Principle. Every theorem about $\mathcal{X}$ within *ZFC* has its counterpart for the original object X interpreted as a Boolean valued object $\mathcal{X}$.
- Boolean Valued Machinery. Translation of theorems from $\mathcal{X} \in \mathbb{V}^{(\mathbb{B})}$ to $X \in \mathbb{V}$ is carried out by the appropriate general operations (ascending–descending) and the principles of Boolean valued analysis.
- A. G. Kusraev and S. S. Kutateladze, *Introduction to Boolean Valued Analysis*, Moscow, Nauka (2005).
  Boolean Valued Analysis, Dordrecht, Kluwer (1999).

What Is the Boolean Valued Transfer Principle?

- Let $\mathrm{X} \subset \mathbb{V}$ and $\mathbb{X} \subset \mathbb{V}^{(\mathbb{B})}$ be two classes of mathematical objects.
  Suppose we are able to prove the result:
- Boolean Valued Representation. *Every $X \in \mathrm{X}$ embeds into an Boolean valued model, becoming an object $\mathcal{X} \in \mathbb{X}$ within $\mathbb{V}^{(\mathbb{B})}$.*
- Boolean Valued Transfer Principle. Every theorem about $\mathcal{X}$ within *ZFC* has its counterpart for the original object X interpreted as a Boolean valued object $\mathcal{X}$.
- Boolean Valued Machinery. Translation of theorems from $\mathcal{X} \in \mathbb{V}^{(\mathbb{B})}$ to $X \in \mathbb{V}$ is carried out by the appropriate general operations (ascending–descending) and the principles of Boolean valued analysis.
- A. G. Kusraev and S. S. Kutateladze, *Introduction to Boolean Valued Analysis*, Moscow, Nauka (2005).
  Boolean Valued Analysis, Dordrecht, Kluwer (1999).

What Is the Boolean Valued Transfer Principle?

- Let $\mathrm{X} \subset \mathbb{V}$ and $\mathbb{X} \subset \mathbb{V}^{(\mathbb{B})}$ be two classes of mathematical objects.
  Suppose we are able to prove the result:
- Boolean Valued Representation. *Every $X \in \mathrm{X}$ embeds into an Boolean valued model, becoming an object $\mathcal{X} \in \mathbb{X}$ within $\mathbb{V}^{(\mathbb{B})}$.*
- Boolean Valued Transfer Principle. Every theorem about $\mathcal{X}$ within *ZFC* has its counterpart for the original object X interpreted as a Boolean valued object $\mathcal{X}$.
- Boolean Valued Machinery. Translation of theorems from $\mathcal{X} \in \mathbb{V}^{(\mathbb{B})}$ to $X \in \mathbb{V}$ is carried out by the appropriate general operations (ascending–descending) and the principles of Boolean valued analysis.
- A. G. Kusraev and S. S. Kutateladze, *Introduction to Boolean Valued Analysis*, Moscow, Nauka (2005).

  Boolean Valued Analysis, Dordrecht, Kluwer (1999).

Some Long Standing Problems

The problem	Raised by	Reduced to (by means of BA):	Solved by
Intrinsic characterization of subdifferentials	Kutateladze 1976	Weakly compact convex sets of functionals	Kusraev Kutateladze 1982
General desintegration in Kantorovich spaces	Ioffe, Levin Neumann 1972/1977	Hahn–Banach and Radon–Nikodým theorems	Kusraev 1984
Kaplansky Problem: Homogeneity of a type I AW^*-algebra	Kaplansky 1953	Homogeneity of $B(H)$ with H Hilbert space	Ozawa 1984

Some Long Standing Problems

The problem	Raised by	Reduced to (by means of BA):	Solved by
Order boundedness of BP operators	Wickstead 1983	Cauchy type functional equations	Gutman Kusraev 1995, 2006
Maharam extension of a positive operator	Luxemburg Schep 1978	Daniel extension of an elementary integral	Akilov Kolesnikov Kusraev 1988
Classification of injective Banach lattices	Lotz Cartright 1975	Classification of AL-space (L_1 spaces)	Kusraev 2012

THANK YOU FOR ATTENTION.