Calculus of Tangents and Beyond

A. G. Kusraev and S. S. Kutateladze

Vladikavkaz and Novosibirsk

August 15, 2017

Agenda

- Optimization is the choice of what is most preferable. Geometry and local analysis of nonsmooth objects are needed for variational analysis which embraces optimization. These involved admissible directions and tangents as the limiting positions of the former. The calculus of tangents is one of the main techniques of optimization (cp. [1, 2]).
- Calculus reduces forecast to numbers, which is scalarization in modern parlance. Spontaneous solutions are often labile and rarely optimal. Thus, optimization as well as calculus of tangents deals with inequality, scalarization, and stability. Some aspects of the latter are revealed by the tools of nonstandard models to be touched slightly in this talk (cp. [3]–[6]).

The Best Is Divine

- Leibniz wrote to Samuel Clarke:[1]
- God can produce everything that is possible or whatever does not imply a contradiction, but he wills only to produce what is the best among things possible.

[1] See [7, p. 54]; cp. [8].

Enter the Reals

- Choosing the best, we use preferences. To optimize, we use infima and suprema for bounded sets which is practically the *least upper bound property*. So optimization needs ordered sets and primarily boundedly complete lattices.
- To operate with preferences, we use group structure. To aggregate and scale, we use linear structure.
- All these are happily provided by the *reals* $\mathbb{R}$, a one-dimensional Dedekind complete vector lattice. A Dedekind complete vector lattice is a *Kantorovich space*.
- Since each number is a measure of quantity, the idea of reduction to numbers is of a universal importance to mathematics. Model theory provides justification of the *Kantorovich heuristic principle* that the members of his spaces are numbers as well (cp. [9] and [10]).

Enter Inequality and Convexity

- Life is inconceivable without numerous conflicting ends and interests to be harmonized. Thus the instances appear of multiple criteria decision making. It is impossible as a rule to distinguish some particular scalar target and ignore the rest of them. This leads to vector optimization problems, involving order compatible with linearity.
- Linear inequality implies linearity and order. When combined, the two produce an ordered vector space. Each linear inequality in the simplest environment of the sort is some half-space. Simultaneity implies many instances and so leads to the intersections of half-spaces. These yield polyhedra as well as arbitrary convex sets, identifying the theory of linear inequalities with convexity.
- Convexity, stemmimg from harpedonapters, reigns in optimization, feeding generation, separation, calculus, and approximation. Generation appears as duality; separation, as optimality; calculus, as representation; and approximation, as stability (cp. [11]–[13]).

Legendre in Disguise

- Assume that X is a vector space, E is an ordered vector space, $E^\bullet$ is E with an adjoined top, $f : X \to E^\bullet$ is some operator, and $C := \mathrm{dom}(f) \subset X$ is a convex set. A *vector program* (C, f) is written as follows:

$$x \in C, \quad f(x) \to \inf.$$

- The standard sociological trick includes (C, f) into a parametric family yielding the *Legendre transform* or *Young–Fenchel transform* of f:

$$f^*(l) := \sup_{x \in X} (l(x) - f(x)),$$

with $l \in X^\#$ a linear functional over X. The epigraph of f^* is a convex subset of $X^\#$ and so f^* is convex. Observe that $-f^*(0)$ is the value of (C, f).

Order Omnipresent

- A convex function is locally a positively homogeneous convex function, a *sublinear functional*. Recall that $p : X \to \mathbb{R}$ is sublinear whenever

$$\operatorname{epi} p := \{(x,\ t) \in X \times \mathbb{R} \mid p(x) \leq t\}$$

  is a cone. Recall that a numeric function is uniquely determined from its epigraph.

- Given $C \subset X$, put

$$H(C) := \{(x,\ t) \in X \times \mathbb{R}^+ \mid x \in tC\},$$

  the *Hörmander transform* of C. Now, C is convex if and only if $H(C)$ is a cone. A space with a cone is a (*pre*)*ordered vector space*.
  The order, the symmetry, the harmony enchant us. . . .

  Leibniz

- Thus, convexity and order are tightly intertwined.

Nonoblate Cones

- Consider cones K_1 and K_2 in a topological vector space X and put $\varkappa := (K_1, K_2)$. Given a pair $\varkappa$ define the correspondence $\Phi_\varkappa$ from X^2 into X by the formula

$$\Phi_\varkappa := \{(k_1, k_2, x) \in X^3 : x = k_1 - k_2,\ k_\imath \in K_\imath\}.$$

  Clearly, $\Phi_\varkappa$ is a cone or, in other words, a conic correspondence.
- The pair $\varkappa$ is *nonoblate* whenever $\Phi_\varkappa$ is open at the zero. Since $\Phi_\varkappa(V) = V \cap K_1 - V \cap K_2$ for every $V \subset X$, the nonoblateness of $\varkappa$ means that

$$\varkappa V := (V \cap K_1 - V \cap K_2) \cap (V \cap K_2 - V \cap K_1)$$

  is a zero neighborhood for every zero neighborhood $V \subset X$.

Open Correspondences

- Since $\varkappa V \subset V - V$, the nonoblateness of $\varkappa$ is equivalent to the fact that the system of sets $\{\varkappa V\}$ serves as a filterbase of zero neighborhoods while V ranges over some base of the same filter. Let $\Delta_n : x \mapsto (x, \dots, x)$ be the embedding of X into the diagonal $\Delta_n(X)$ of X^n. A pair of cones $\varkappa := (K_1, K_2)$ is nonoblate if and only if $\lambda := (K_1 \times K_2, \Delta_2(X))$ is nonoblate in X^2.
- Cones K_1 and K_2 constitute a nonoblate pair if and only if the conic correspondence $\Phi \subset X \times X^2$ defined as

$$\Phi := \{(h, x_1, x_2) \in X \times X^2 : x_\imath + h \in K_\imath \ (\imath := 1, 2)\}$$

is open at the zero.

General Position of Cones

- Cones K_1 and K_2 in a topological vector space X are *in general position* iff
- **(1)** the algebraic span of K_1 and K_2 is some subspace $X_0 \subset X$; i.e., $X_0 = K_1 - K_2 = K_2 - K_1$;
- **(2)** the subspace X_0 is complemented; i.e., there exists a continuous projection $P : X \to X$ such that $P(X) = X_0$;
- **(3)** K_1 and K_2 constitute a nonoblate pair in X_0.

General Position of Operators

- Let σ_n stand for the rearrangement of coordinates

$$\sigma_n : ((x_1, y_1), \dots, (x_n, y_n)) \mapsto ((x_1, \dots, x_n), (y_1, \dots, y_n))$$

  which establishes an isomorphism between $(X \times Y)^n$ and $X^n \times Y^n$.
- Sublinear operators $P_1, \dots, P_n : X \to E \cup \{+\infty\}$ are *in general position* if so are the cones $\Delta_n(X) \times E^n$ and $\sigma_n(\operatorname{epi}(P_1) \times \dots \times \operatorname{epi}(P_n))$.
- Given a cone $K \subset X$, put

$$\pi_E(K) := \{T \in \mathscr{L}(X, E) : Tk \leq 0 \ (k \in K)\}.$$

  Clearly, $\pi_E(K)$ is a cone in $\mathscr{L}(X, E)$.
- **Theorem.** *Let $K_1, \dots, K_n$ be cones in a topological vector space X and let E be a topological Kantorovich space. If $K_1, \dots, K_n$ are in general position then*

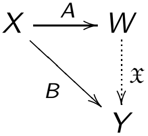
$$\pi_E(K_1 \cap \dots \cap K_n) = \pi_E(K_1) + \dots + \pi_E(K_n).$$

Environment for Inequality

- Assume that X is a real vector space, Y is a *Kantorovich space*. Let $\mathbb{B} := \mathbb{B}(Y)$ be the *base* of Y, i.e., the complete Boolean algebras of positive projections in Y; and let $m(Y)$ be the universal completion of Y. Denote by $L(X, Y)$ the space of linear operators from X to Y. In case X is furnished with some Y-seminorm on X, by $L^{(m)}(X, Y)$ we mean the *space of dominated operators* from X to Y. As usual, $\{T \le 0\} := \{x \in X \mid Tx \le 0\}$; $\ker(T) = T^{-1}(0)$ for $T : X \to Y$. Also, $P \in \mathrm{Sub}(X, Y)$ means that P is *sublinear*, while $P \in \mathrm{PSub}(X, Y)$ means that P is *polyhedral*, i.e., finitely generated. The superscript $^{(m)}$ suggests domination.

Kantorovich's Theorem

- Find $\mathfrak{X}$ satisfying



- **(1):** $(\exists \mathfrak{X})\ \mathfrak{X}A = B \leftrightarrow \ker(A) \subset \ker(B)$.
- **(2):** *If W is ordered by W_+ and $A(X) - W_+ = W_+ - A(X) = W$, then*[2]

$$(\exists \mathfrak{X} \geq 0)\ \mathfrak{X}A = B \leftrightarrow \{A \leq 0\} \subset \{B \leq 0\}.$$

[2] Cp. [2, p. 51].

The Farkas Alternative

- *Let X be a Y-seminormed real vector space, with Y a Kantorovich space. Assume that $A_1, \dots, A_N$ and B belong to $L^{(m)}(X, Y)$. Then one and only one of the following holds:*
  (1) *There are $x \in X$ and $b, b' \in \mathbb{B}$ such that $b' \leq b$ and*

  $$b'Bx > 0, bA_1x \leq 0, \dots, bA_Nx \leq 0.$$

  (2) *There are positive orthomorphisms $\alpha_1, \dots, \alpha_N \in \mathrm{Orth}(m(Y))_+$ such that $B = \sum_{k=1}^{N} \alpha_k A_k$.*

Inhomogeneous Inequalities

- **Theorem.** *Let X be a Y-seminormed real vector space, with Y a Kantorovich space. Assume given some dominated operators $A_1, \dots, A_N, B \in L^{(m)}(X, Y)$ and elements $u_1, \dots, u_N, v \in Y$. The following are equivalent:*
  (1) *For all $b \in \mathbb{B}$ the inhomogeneous operator inequality $bBx \leq bv$ is a consequence of the consistent simultaneous inhomogeneous operator inequalities $bA_1x \leq bu_1, \dots, bA_Nx \leq bu_N$, i.e.,*

$$\{bB \leq bv\} \supset \{bA_1 \leq bu_1\} \cap \dots \cap \{bA_N \leq bu_N\}.$$

  (2) *There are positive orthomorphisms $\alpha_1, \dots, \alpha_N \in \mathrm{Orth}(m(Y))$ satisfying*

$$B = \sum_{k=1}^{N} \alpha_k A_k; \quad v \geq \sum_{k=1}^{N} \alpha_k u_k.$$

Boolean Modeling

- The above infinite-dimensional results appear as interpretations of one-dimensional predecessors on using model theory.
- Cohen's final solution of the problem of the cardinality of the continuum within ZFC gave rise to the Boolean valued models by Scott, Solovay, and Vopěnka.[3]
- Takeuti coined the term "Boolean valued analysis" for applications of the models to analysis.

[3] Cp. [4].

Scott's Comments

- Scott forecasted in 1969:[4]

  We must ask whether there is any interest in these nonstandard models aside from the independence proof; that is, do they have any mathematical interest? The answer must be yes, but we cannot yet give a really good argument.

- In 2009 Scott wrote:[5]

  At the time, I was disappointed that no one took up my suggestion. And then I was very surprised much later to see the work of Takeuti and his associates. I think the point is that people have to be trained in Functional Analysis in order to understand these models. I think this is also obvious from your book and its references. Alas, I had no students or collaborators with this kind of background, and so I was not able to generate any progress.

[4] Cp. [14].

[5] Letter of April 29, 2009 to S. S. Kutateladze.

Art of Invention

- Leibniz wrote about his version of calculus that "the difference from Archimedes style is only in expressions which in our method are more straightforward and more applicable to the art of invention."
- Nonstandard analysis has the two main advantages: it "kills quantifiers" and it produces the new notions that are impossible within a single model of set theory.
- Let us turn to the nonstandard presentations of Kuratowski–Painlevé limits of use in tangent calculus, and explore the variations of tangents.
- Recall that the central concept of Leibniz was that of a *monad* (cp. [15]). In nonstandard analysis the monad $\mu(\mathscr{F})$ of a standard filter $\mathscr{F}$ is the intersection of all standard elements of $\mathscr{F}$.

Monadic Limits

- Let $F \subset X \times Y$ be an internal correspondence from a standard set X to a standard set Y. Assume given a standard filter $\mathcal{N}$ on X and a topology τ on Y. Put

$$\forall\forall(F) := {}^*\{y' \mid (\forall\, x \in \mu(\mathcal{N}) \cap \mathrm{dom}(F))(\forall\, y \approx y')(x, y) \in F\},$$
$$\exists\forall(F) := {}^*\{y' \mid (\exists\, x \in \mu(\mathcal{N}) \cap \mathrm{dom}(F))(\forall\, y \approx y')(x, y) \in F\},$$
$$\forall\exists(F) := {}^*\{y' \mid (\forall\, x \in \mu(\mathcal{N}) \cap \mathrm{dom}(F))(\exists\, y \approx y')(x, y) \in F\},$$
$$\exists\exists(F) := {}^*\{y' \mid (\exists\, x \in \mu(\mathcal{N}) \cap \mathrm{dom}(F))(\exists\, y \approx y')(x, y) \in F\},$$

  with * symbolizing standardization and $y \approx y'$ standing for the *infinite proxitity* between y and y' in τ, i.e. $y' \in \mu(\tau(y))$.
- Call $\mathrm{Q_1 Q_2(F)}$ the $\mathrm{Q_1 Q_2}$-*limit* of F (here $\mathrm{Q_k}$ ($k := 1, 2$) is one of the quantifiers $\forall$ or $\exists$).

Kuratowski–Painlevé Limits

- Assume for instance that F is a standard correspondence on some element of $\mathcal{N}$ and look at the $\exists\exists$-limit and the $\forall\exists$-limit. The former is the *limit superior* or *upper limit*; the latter is the *limit inferior* or *lower limit* of F along $\mathcal{N}$.
- **Theorem.** *If F is a standard correspondence then*

$$\exists\exists(F) = \bigcap_{U\in\mathcal{N}} \mathrm{cl}\Bigg(\bigcup_{x\in U} F(x)\Bigg);$$
$$\forall\exists(F) = \bigcap_{U\in\ddot{\mathcal{N}}} \mathrm{cl}\Bigg(\bigcup_{x\in U} F(x)\Bigg),$$

where $\ddot{\mathcal{N}}$ is the grill of a filter $\mathcal{N}$ on X, i.e., the family comprising all subsets of X meeting $\mu(\mathcal{N})$.

Hadamard, Clarke, and Bouligand Tangents

- $$\mathrm{Ha}(F,x') := \bigcup_{\substack{U\in\tau(x')\\ \alpha'}} \mathrm{int}_\tau \bigcap_{\substack{x\in F\cap U\\ 0<\alpha\le\alpha'}} \frac{F-x}{\alpha};$$
$$\mathrm{Cl}(F,x') := \bigcap_{V\in\mathcal{N}_\tau} \bigcup_{\substack{U\in\tau(x')\\ \alpha'}} \bigcap_{\substack{x\in F\cap U\\ 0<\alpha\le\alpha'}} \left(\frac{F-x}{\alpha}+V\right);$$
$$\mathrm{Bo}(f,x') := \bigcap_{\substack{U\in\tau(x')\\ \alpha'}} \mathrm{cl}_\tau \bigcup_{\substack{x\in F\cap U\\ 0<\alpha\le\alpha'}} \frac{F-x}{\alpha},$$
where, as usual, $\tau(x') := x' + \mathcal{N}_\tau$ and $\mathcal{N}_\tau$, the zero neighborhood filterbase of the topology τ. Obviously,
$$\mathrm{Ha}(F,x') \subset \mathrm{Cl}(F,x') \subset \mathrm{Bo}(F,x').$$

Infinitesimal Quantifiers

- Agree on notation for a ZFC formula φ and $x' \in F$:

$$(\forall^{\bullet} x)\varphi := (\forall x \approx_\tau x')\varphi := (\forall x)(x \in F \wedge x \approx_\tau x') \to \varphi,$$
$$(\forall^{\bullet} h)\varphi := (\forall h \approx_\tau h')\varphi := (\forall h)(h \in X \wedge h \approx_\tau h') \to \varphi,$$
$$(\forall^{\bullet} \alpha)\varphi := (\forall \alpha \approx 0)\varphi := (\forall \alpha)(\alpha > 0 \wedge \alpha \approx 0) \to \varphi.$$

- The quantifiers $\exists^{\bullet} x$, $\exists^{\bullet} h$, $\exists^{\bullet} \alpha$ are defined in the natural way by duality on assuming that

$$(\exists^{\bullet} x)\varphi := (\exists x \approx_\tau x')\varphi := (\exists x)(x \in F \wedge x \approx_\tau x') \wedge \varphi,$$
$$(\exists^{\bullet} h)\varphi := (\exists h \approx_\tau h')\varphi := (\exists h)(h \in X \wedge h \approx_\tau h') \wedge \varphi,$$
$$(\exists^{\bullet} \alpha)\varphi := (\exists \alpha \approx 0)\varphi := (\exists \alpha)(\alpha > 0 \wedge \alpha \approx 0) \wedge \varphi.$$

Infinitesimal Representations

- *The Bouligand cone is the standardization of the* $\exists\exists\exists$*-cone; i.e., if* h' *is standard then*

$$h' \in \mathrm{Bo}(F, x') \leftrightarrow (\exists^{\bullet} x)(\exists^{\bullet} \alpha)(\exists^{\bullet} h)\, x + \alpha h \in F.$$

- *The Hadamard cone is the standardization of the* $\forall\forall\forall$*-cone:*

$$\mathrm{Ha}(F, x') = \forall\forall\forall(F, x'),$$

with $\mu(\mathbb{R}_{+})$ *the external set of positive infinitesimals.*

- *The Clarke cone is the standardization of the* $\forall\forall\exists$*-cone: i.e.,*

$$\mathrm{Cl}(F, x') = \forall\forall\exists(F, x').$$

In more detail,

$$h' \in \mathrm{Cl}(F, x') \leftrightarrow (\forall^{\bullet} x)(\forall^{\bullet} \alpha)(\exists^{\bullet} h)\, x + \alpha h \in F.$$

Convexity Is Stable

- Convexity of harpedonaptae was stable in the sense that no variation of stakes within the surrounding rope can ever spoil the convexity of the tract to be surveyed.
- Stability is often tested by perturbation or introducing various epsilons in appropriate places, which geometrically means that tangents travel. One of the earliest excursions in this direction is connected with the classical Hyers–Ulam stability theorem for ε-convex functions. Exact calculations with epsilons and sharp estimates are often bulky and slightly mysterious. Some alternatives are suggested by actual infinities, which is illustrated with the conception of *infinitesimal optimality*.

Enter Epsilon

- Assume given a convex operator $f : X \to E^{\bullet}$ and a point $\overline{x}$ in the effective domain $\operatorname{dom}(f) := \{x \in X \mid f(x) < +\infty\}$ of f.
- Given $\varepsilon \geq 0$ in the positive cone E_+ of E, by the ε-*subdifferential* of f at $\overline{x}$ we mean the set

$$\partial_\varepsilon f(\overline{x}) := \Big\{T \in L(X, E) \mid$$
$$(\forall x \in X)(Tx - f(x) \leq T\overline{x} - f(\overline{x}) + \varepsilon)\Big\}.$$

Topological Setting

- The usual subdifferential $\partial f(\overline{x})$ is the intersection of ε-subfifferentials:

$$\partial f(\overline{x}) := \bigcap_{\varepsilon \geq 0} \partial_\varepsilon f(\overline{x}).$$

In topological setting we use continuous operators, replacing $L(X, E)$ with $\mathscr{L}(X, E)$.

ε-Optimality

- **Theorem.** *Let $f_1 : X \times Y \to E^\bullet$ and $f_2 : Y \times Z \to E^\bullet$ be convex operators and $\delta, \varepsilon \in E^+$. Suppose that the convolution $f_2 \vartriangle f_1$ is δ-exact at some point (x, y, z); i.e., $\delta + (f_2 \vartriangle f_1)(x, y) = f_1(x, y) + f_2(y, z)$. If, moreover, the convex sets* $\operatorname{epi}(f_1, Z)$ *and* $\operatorname{epi}(X, f_2)$ *are in general position, then*

$$\partial_\varepsilon (f_2 \vartriangle f_1)(x, y) = \bigcup_{\substack{\varepsilon_1 \geq 0, \varepsilon_2 \geq 0, \\ \varepsilon_1 + \varepsilon_2 = \varepsilon + \delta}} \partial_{\varepsilon_2} f_2(y, z) \circ \partial_{\varepsilon_1} f_1(x, y).$$

Enter Monad

- Distinguish some downward-filtered subset $\mathcal{E}$ of E that is composed of positive elements. Assuming E and $\mathcal{E}$ standard, define the *monad* $\mu(\mathcal{E})$ of $\mathcal{E}$ as $\mu(\mathcal{E}) := \bigcap\{[0,\varepsilon] \mid \varepsilon \in {}^{\circ}\mathcal{E}\}$. The members of $\mu(\mathcal{E})$ are *positive infinitesimals* with respect to $\mathcal{E}$. As usual, ${}^{\circ}\mathcal{E}$ denotes the external set of all standard members of E, the *standard part* of $\mathcal{E}$.
- Assume that the monad $\mu(\mathcal{E})$ is an external cone over ${}^{\circ}\mathbb{R}$ and, moreover, $\mu(\mathcal{E}) \cap {}^{\circ}E = 0$. In application, $\mathcal{E}$ is usually the filter of order-units of E. The relation of *infinite proximity* or *infinite closeness* between the members of E is introduced as follows:

$$e_1 \approx e_2 \leftrightarrow e_1 - e_2 \in \mu(\mathcal{E}) \ \& \ e_2 - e_1 \in \mu(\mathcal{E}).$$

Infinitesimal Subdifferential

- Now
$$Df(\overline{x}) := \bigcap_{\varepsilon \in {}^{\circ}\mathcal{E}} \partial_{\varepsilon} f(\overline{x}) = \bigcup_{\varepsilon \in \mu(\mathcal{E})} \partial_{\varepsilon} f(\overline{x}),$$
which is the *infinitesimal subdifferential* of f at $\overline{x}$. The elements of $Df(\overline{x})$ are *infinitesimal subgradients* of f at $\overline{x}$.

Infinitesimal Solution

- Assume that there exists a limited value $e := \inf_{x\in C} f(x)$ of some program (C, f). A feasible point x_0 is called an *infinitesimal solution* if $f(x_0) \approx e$, i.e., if $f(x_0) \leq f(x) + \varepsilon$ for every $x \in C$ and every standard $\varepsilon \in \mathcal{E}$.
- *A point $x_0 \in X$ is an infinitesimal solution of the unconstrained problem $f(x) \to \inf$ if and only if $0 \in Df(x_0)$.*

Exeunt Epsilon

- **Theorem.** *Let* $f_1 : X \times Y \to E^\bullet$ *and* $f_2 : Y \times Z \to E^\bullet$ *be convex operators. Suppose that the convolution* $f_2 \vartriangle f_1$ *is infinitesimally exact at some point* (x, y, z)*; i.e.,* $(f_2 \vartriangle f_1)(x, y) \approx f_1(x, y) + f_2(y, z)$. *If, moreover, the convex sets* $\operatorname{epi}(f_1, Z)$ *and* $\operatorname{epi}(X, f_2)$ *are in general position then*

$$D(f_2 \vartriangle f_1)(x, y) = Df_2(y, z) \circ Df_1(x, y).$$

References I

Clarke F.,
"Nonsmooth Analysis in Systems and Control Theory"
In: Encyclopedia of Complexity and Control Theory. Berlin: Springer-Verlag, 2009, 6271–6184.

Kusraev A. G. and Kutateladze S. S.,
Subdifferential Calculus: Theory and Applications.
Moscow: Nauka, 2007.

Bell J. L.,
Set Theory: Boolean Valued Models and Independence Proofs.
Oxford: Clarendon Press, 2005.

Kusraev A. G. and Kutateladze S. S.,
Introduction to Boolean Valued Analysis.
Moscow: Nauka, 2005.

References II

Kanovei V. and Reeken M.,
Nonstandard Analysis: Axiomatically.
Berlin: Springer-Verlag, 2004.

Gordon E. I., Kusraev A. G., and Kutateladze S. S.,
Infinitesimal Analysis: Selected Topics.
Moscow: Nauka, 2011.

Ariew R.,
G. W. Leibniz and Samuel Clarke Correspondence.
Indianopolis: Hackett Publishing Company, 2000.

Ekeland I.,
The Best of All Possible Worlds: Mathematics and Destiny.
Chicago and London: The University of Chicago Press, 2006.

References III

Kusraev A. G. and Kutateladze S. S.,
"Boolean Methods in Positivity,"
J. Appl. Indust. Math., **2**:1, 81–99 (2008).

Kutateladze S. S.,
"Mathematics and Economics of Leonid Kantorovich,"
Siberian Math. J., **53**:1, 1–12 (2012).

Kutateladze S. S.,
"Harpedonaptae and Abstract Convexity,"
J. Appl. Indust. Math., **2**:1, 215–221 (2008).

Kutateladze S. S.,
"Boolean Trends in Linear Inequalities,"
J. Appl. Indust. Math., **4**:3, 340–348 (2010).

References IV

Kutateladze S. S.,
"Nonstandard tools of nonsmooth analysis,"
J. Appl. Indust. Math., **6**:3, 332–338 (2012).

Scott D.,
"Boolean Models and Nonstandard Analysis,"'
Applications of Model Theory to Algebra, Analysis, and Probability,
Holt, Rinehart, and Winston, 1969, pp. 87–92.

Kutateladze S. S.,
"Leibnizian, Robinsonian, and Boolean Valued Monads,"
J. Appl. Indust. Math., **5**:3, 365–373 (2011).