

# Abstract Convexity and Cone-Vexing Abstractions

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## Agenda

This is an overview of some basic ideas of abstract convexity and a few vexing limitations on the range of abstraction in convexity. The idea of convexity feeds generation, separation, calculus, and approximation. Generation appears as duality; separation, as optimality; calculus, as representation; and approximation, as stability. Convexity is traceable from the remote ages and flourishes in functional analysis.

## Mathesis Universalis

Once upon a time mathematics was everything. It is not now but still carries the genome of *mathesis universalis*.

Abstraction is the mother of reason and the gist of mathematics. It enables us to collect the particular instances of any many with some property we observe or study. Abstraction entails generalization and proceeds by analogy. The latter lies behind the algebraic approach to idempotent functional analysis and, in particular, to the tropical theorems of Hahn–Banach type. Analogy is tricky and sometimes misleading. So, it is reasonable to overview the true origins of any instance of analogy from time to time. This talk deals with abstract convexity which is the modern residence of Hahn–Banach.

## *Elements*, Book I

Sometimes mathematics resembles linguistics and pays tribute to etymology, hence, history. Today's convexity is a centenarian, and abstract convexity is much younger. Vivid convexity is full of abstraction, but traces back to the idea of a solid figure which stems from Euclid. Book I of his *Elements* expounded plane geometry and defined a boundary and a figure as follows:

Definition 13. A boundary is that which is an extremity of anything.

Definition 14. A figure is that which is contained by any boundary or boundaries.

## *Elements, Book XI*

Narrating solid geometry in Book XI, Euclid travelled in the opposite direction from a solid to a surface:

Definition 1. A solid is that which has length, breadth, and depth.

Definition 2. An extremity of a solid is a surface.

He proceeded with the relations of similarity and equality for solids:

Definition 9. Similar solid figures *are those contained by similar planes equal in multitude.*

Definition 10. Equal and similar solid figures *are those contained by similar planes equal in multitude and magnitude.*

## Enter Convexity

Euclid's definitions seem vague, obscure, and even unreasonable if applied to the figures other than convex polyhedra. Euclid also introduced a formal concept of "cone" which has a well-known natural origin. However, convexity was ubiquitous in his geometry by default. The term "conic sections" was coined as long ago as 200 BCE by Apollonius of Perga. However, it was long before him that Plato had formulated his famous allegory of cave. The shadows on the wall are often convex.

Now we can see in Euclid's implicit definition of a convex solid body the intersection of half-spaces. However, the concept of intersection belongs to set theory which appeared only at the end of the nineteenth century. It is wiser to seek for the origins of the ideas of Euclid in his past rather than his future. Euclid was a scientist not a foreteller.

## Harpedonaptae

The predecessors of Euclid are the harpedonaptae of Egypt as often sounds at the lectures on the history of mathematics. The harpedonaptae or rope stretchers measured tracts of land in the capacity of surveyors. They administered *cadastral surveying* which gave rise to the notion of geometry. If anyone stretches a rope that surrounds however many stakes, he will distinguish a convex polyhedron, which is up to infinitesimals a typical compact convex set or abstract subdifferential of the present-day mathematics. The rope-stretchers discovered convexity experimentally by measurement. Hence, a few words are in order about these forefathers of their Hahn–Banach next of kin of today.

## The History of Herodotus

Herodotus wrote in Item 109 of Book II *Enerepre* as follows:

Egypt was cut up: and they said that this king distributed the land to all the Egyptians, giving an equal square portion to each man, and from this he made his revenue, having appointed them to pay a certain rent every year: and if the river should take away anything from any man's portion, he would come to the king and declare that which had happened, and the king used to send men to examine and to find out by measurement how much less the piece of land had become, in order that for the future the man might pay less, in proportion to the rent appointed: and I think that thus the art of geometry was found out and afterwards came into Hellas also.

## Sulva Sutras

Datta wrote:

... One who was well versed in that science was called in ancient India as samkhyajna (the expert of numbers), parimanajna (the expert in measuring), sama-sutra-niranchaka (uniform-rope-stretcher), Shulba-vid (the expert in Shulba) and Shulba-pari-prcchaka (the inquirer into the Shulba).

Shulba also written as Śulva or Sulva was in fact the geometry of vedic times as codified in *Śulva Sūtras*.

## Veda

Since “veda” means knowledge, the vedic epoch and literature are indispensable for understanding the origin and rise of mathematics. In 1978 Seidenberg wrote:

Old-Babylonia [1700 BC] got the theorem of Pythagoras from India or that both Old-Babylonia and India got it from a third source. Now the Sanskrit scholars do not give me a date so far back as 1700 B.C. Therefore I postulate a pre-Old-Babylonian (i.e., pre-1700 B.C.) source of the kind of geometric rituals we see preserved in the Sulvasutras, or at least for the mathematics involved in these rituals.

Some recent facts and evidence prompt us that the roots of rope-stretching spread in a much deeper past than we were accustomed to acknowledge.

## Vedic Epoch

The exact chronology still evades us and Kak commented on the Seidenberg paper:

That was before archaeological finds disproved the earlier assumption of a break in Indian civilization in the second millennium B.C.E.; it was this assumption of the Sanskritists that led Seidenberg to postulate a third earlier source. Now with our new knowledge, Seidenberg's conclusion of India being the source of the geometric and mathematical knowledge of the ancient world fits in with the new chronology of the texts.

...in the absence of conclusive evidence, it is prudent to take the most conservative of these dates, namely 2000 B.C.E. as the latest period to be associated with the Rigveda.

## Enter Abstract Convexity

Stretching a rope taut between two stakes produces a closed straight line segment which is the continuum in modern parlance. Rope stretching raised the problem of measuring the continuum. The continuum hypothesis of set theory is the shadow of the ancient problem of harpedonaptae. Rope stretching independent of the position of stakes is uniform with respect to direction in space. The mental experiment of uniform rope stretching yields a compact convex figure. The harpedonaptae were experts in convexity.

Convexity has found solid grounds in set theory. The Cantor paradise became an official residence of convexity. Abstraction becomes an axiom of set theory. The abstraction axiom enables us to reincarnate a property, in other words, to collect and to comprehend. The union of convexity and abstraction was inevitable. Their child is abstract convexity.

## Generation

Let  $\overline{E}$  be a boundedly complete lattice  $E$  with the adjoint top  $\top := +\infty$  and bottom  $\perp := -\infty$ . Unless otherwise stated,  $E$  is usually a *Kantorovich space* which is a Dedekind complete vector lattice in another terminology. Assume further that  $H$  is some subset of  $E$  which is by implication a (convex) cone in  $E$ , and so the bottom of  $E$  lies beyond  $H$ . A subset  $U$  of  $H$  is *convex relative to  $H$*  or  *$H$ -convex*, in symbols  $U \in \mathfrak{V}(H, \overline{E})$ , provided that  $U$  is the  *$H$ -support set*  $U_p^H := \{h \in H : h \leq p\}$  of some element  $p$  of  $\overline{E}$ .

## Minkowski Duality

Alongside the  $H$ -convex sets we consider the so-called  $H$ -convex elements. An element  $p \in \overline{E}$  is  $H$ -convex provided that  $p = \sup U_p^H$ ; i.e.,  $p$  represents the supremum of the  $H$ -support set of  $p$ . The  $H$ -convex elements comprise the cone which is denoted by  $\mathcal{C}(H, \overline{E})$ . We may omit the references to  $H$  when  $H$  is clear from the context. It is worth noting that convex elements and sets are “glued together” by the *Minkowski duality*  $\varphi : p \mapsto U_p^H$ . This duality enables us to study convex elements and sets simultaneously.

Since the classical results by Fenchel and Hörmander we know definitely that the most convenient and conventional classes of convex functions and sets are  $\mathcal{C}(A(X), \overline{\mathbb{R}^X})$  and  $\mathfrak{B}(X', \overline{\mathbb{R}^X})$ . Here  $X$  is a locally convex space,  $X'$  is the dual of  $X$ , and  $A(X)$  is the space of affine functions on  $X$  (isomorphic with  $X' \times \mathbb{R}$ ).

## Young–Fenchel Transform

In the first case the Minkowski duality is the mapping  $f \mapsto \text{epi}(f^*)$  where

$$f^*(y) := \sup_{x \in X} (\langle y, x \rangle - f(x))$$

is the *Young–Fenchel transform* of  $f$  or the *conjugate function* of  $f$ .

In the second case we prefer to write down the inverse of the Minkowski duality which sends  $U$  in  $\mathfrak{B}(X', \overline{\mathbb{R}}^X)$  to the standard *support function*

$$\varphi^{-1}(U) : x \mapsto \sup_{y \in U} \langle y, x \rangle.$$

As usual,  $\langle \cdot, \cdot \rangle$  stands for the canonical pairing of  $X'$  and  $X$ .

## Enter Idempotents

This idea of abstract convexity lies behind many current objects of analysis and geometry.

Among them we list the “economical” sets with boundary points meeting the Pareto criterion, capacities, monotone seminorms, various classes of functions convex in some generalized sense, for instance, the Bauer convexity in Choquet theory, etc. There are ordered vector spaces consisting of the convex elements with respect to narrow cones with finite generators. To compute the meet or join of two reals is no harder than to compute their sum or product. This simple observation is one of the underlying ideas of supremal generation and idempotent analysis.

## Nonoblate Cones

Consider cones  $K_1$  and  $K_2$  in a topological vector space  $X$  and put  $\varkappa := (K_1, K_2)$ . Given a pair  $\varkappa$  define the correspondence  $\Phi_\varkappa$  from  $X^2$  into  $X$  by the formula

$$\Phi_\varkappa := \{(k_1, k_2, x) \in X^3 : x = k_1 - k_2 \in K_i\}.$$

Clearly,  $\Phi_\varkappa$  is a cone or, in other words, a conic correspondence.

The pair  $\varkappa$  is *nonoblate* whenever  $\Phi_\varkappa$  is open at the zero. Since  $\Phi_\varkappa(V) = V \cap K_1 - V \cap K_2$  for every  $V \subset X$ , the nonoblateness of  $\varkappa$  means that

$$\varkappa V := (V \cap K_1 - V \cap K_2) \cap (V \cap K_2 - V \cap K_1)$$

is a zero neighborhood for every zero neighborhood  $V \subset X$ .

## Open Correspondences

Since  $\varkappa V \subset V - V$ , the nonoblateness of  $\varkappa$  is equivalent to the fact that the system of sets  $\{\varkappa V\}$  serves as a filterbase of zero neighborhoods while  $V$  ranges over some base of the same filter.

Let  $\Delta_n : x \mapsto (x, \dots, x)$  be the embedding of  $X$  into the diagonal  $\Delta_n(X)$  of  $X^n$ . A pair of cones  $\varkappa := (K_1, K_2)$  is nonoblate if and only if  $\lambda := (K_1 \times K_2, \Delta_2(X))$  is nonoblate in  $X^2$ .

Cones  $K_1$  and  $K_2$  constitute a nonoblate pair if and only if the conic correspondence  $\Phi \subset X \times X^2$  defined as

$$\Phi := \{(h, x_1, x_2) \in X \times X^2 : x_i + h \in K_i \ (i := 1, 2)\}$$

is open at the zero.

## General Position of Cones

Cones  $K_1$  and  $K_2$  in a topological vector space  $X$  are *in general position* provided that

**(1)** the algebraic span of  $K_1$  and  $K_2$  is some subspace  $X_0 \subset X$ ; i.e.,  $X_0 = K_1 - K_2 = K_2 - K_1$ ;

**(2)** the subspace  $X_0$  is complemented; i.e., there exists a continuous projection  $P : X \rightarrow X$  such that  $P(X) = X_0$ ;

**(3)**  $K_1$  and  $K_2$  constitute a nonoblate pair in  $X_0$ .

## General Position of Operators

Let  $\sigma_n$  stand for the rearrangement of coordinates  $\sigma_n : ((x_1, y_1), \dots, (x_n, y_n)) \mapsto ((x_1, \dots, x_n), (y_1, \dots, y_n))$  which establishes an isomorphism between  $(X \times Y)^n$  and  $X^n \times Y^n$ .

Sublinear operators  $P_1, \dots, P_n : X \rightarrow E \cup \{+\infty\}$  are *in general position* if so are the cones  $\Delta_n(X) \times E^n$  and  $\sigma_n(\text{epi}(P_1) \times \dots \times \text{epi}(P_n))$ .

Given a cone  $K \subset X$ , put

$$\pi_E(K) := \{T \in \mathcal{L}(X, E) : Tk \leq 0 \ (k \in K)\}.$$

We readily see that  $\pi_E(K)$  is a cone in  $\mathcal{L}(X, E)$ .

**Theorem.** *Let  $K_1, \dots, K_n$  be cones in a topological vector space  $X$  and let  $E$  be a topological Kantorovich space. If  $K_1, \dots, K_n$  are in general position then*

$$\pi_E(K_1 \cap \dots \cap K_n) = \pi_E(K_1) + \dots + \pi_E(K_n).$$

This formula opens a way to various separation results.

## Separation

**Sandwich Theorem.** *Let  $P, Q : X \rightarrow E \cup \{+\infty\}$  be sublinear operators in general position. If  $P(x) + Q(x) \geq 0$  for all  $x \in X$  then there exists a continuous linear operator  $T : X \rightarrow E$  such that*

$$-Q(x) \leq Tx \leq P(x) \quad (x \in X).$$

Many efforts were made to abstract these results to a more general algebraic setting and, primarily, to semigroups.

## Canonical Operator

Consider a Kantorovich space  $E$  and an arbitrary nonempty set  $\mathfrak{A}$ . Denote by  $l_\infty(\mathfrak{A}, E)$  the set of all order bounded mappings from  $\mathfrak{A}$  into  $E$ ; i.e.,  $f \in l_\infty(\mathfrak{A}, E)$  if and only if  $f : \mathfrak{A} \rightarrow E$  and the set  $\{f(\alpha) : \alpha \in \mathfrak{A}\}$  is order bounded in  $E$ . It is easy to verify that  $l_\infty(\mathfrak{A}, E)$  becomes a Kantorovich space if endowed with the coordinatewise algebraic operations and order. The operator  $\varepsilon_{\mathfrak{A}, E}$  acting from  $l_\infty(\mathfrak{A}, E)$  into  $E$  by the rule

$$\varepsilon_{\mathfrak{A}, E} : f \mapsto \sup\{f(\alpha) : \alpha \in \mathfrak{A}\} \quad (f \in l_\infty(\mathfrak{A}, E))$$

is called the *canonical sublinear operator* given  $\mathfrak{A}$  and  $E$ . We often write  $\varepsilon_{\mathfrak{A}}$  instead of  $\varepsilon_{\mathfrak{A}, E}$  when it is clear from the context what Kantorovich space is meant. The notation  $\varepsilon_n$  is used when the cardinality of  $\mathfrak{A}$  equals  $n$  and we call the operator  $\varepsilon_n$  *finitely-generated*.

## Subdifferential

Let  $X$  and  $E$  be ordered vector spaces. An operator  $p : X \rightarrow E$  is called *increasing* or *isotonic* if for all  $x_1, x_2 \in X$  from  $x_1 \leq x_2$  it follows that  $p(x_1) \leq p(x_2)$ . An increasing linear operator is also called *positive*. As usual, the collection of all positive linear operators in the space  $L(X, E)$  of all linear operators is denoted by  $L^+(X, E)$ . Obviously, the positivity of a linear operator  $T$  amounts to the inclusion  $T(X^+) \subset E^+$ , where  $X^+ := \{x \in X : x \geq 0\}$  and  $E^+ := \{e \in E : e \geq 0\}$  are the *positive cones* in  $X$  and  $E$  respectively. Observe that every canonical operator is increasing and sublinear, while every finitely-generated canonical operator is order continuous.

Recall that  $\partial p := \partial p(0) = \{T \in L(X, E) : (\forall x \in X) Tx \leq p(x)\}$  is the *subdifferential* at the zero or *support set* of a sublinear operator  $p$ .

## Support Hull

Consider a set  $\mathfrak{A}$  of linear operators acting from a vector space  $X$  into a Kantorovich space  $E$ . The set  $\mathfrak{A}$  is *weakly order bounded* if the set  $\{\alpha x : \alpha \in \mathfrak{A}\}$  is order bounded for every  $x \in X$ . We denote by  $\langle \mathfrak{A} \rangle x$  the mapping that assigns the element  $\alpha x \in E$  to each  $\alpha \in \mathfrak{A}$ , i.e.  $\langle \mathfrak{A} \rangle x : \alpha \mapsto \alpha x$ . If  $\mathfrak{A}$  is weakly order bounded then  $\langle \mathfrak{A} \rangle x \in l_\infty(\mathfrak{A}, E)$  for every fixed  $x \in X$ . Consequently, we obtain the linear operator  $\langle \mathfrak{A} \rangle : X \rightarrow l_\infty(\mathfrak{A}, E)$  that acts as  $\langle \mathfrak{A} \rangle : x \mapsto \langle \mathfrak{A} \rangle x$ . Associate with  $\mathfrak{A}$  one more operator

$$p_{\mathfrak{A}} : x \mapsto \sup\{\alpha x : \alpha \in \mathfrak{A}\} \quad (x \in X).$$

The operator  $p_{\mathfrak{A}}$  is sublinear. The support set  $\partial p_{\mathfrak{A}}$  is denoted by  $\text{cop}(\mathfrak{A})$  and referred to as the *support hull* of  $\mathfrak{A}$ .

## Enter Hahn–Banach

**Theorem.** *If  $p$  is a sublinear operator with  $\partial p = \text{cop}(\mathfrak{A})$  then  $P = \varepsilon_{\mathfrak{A}} \circ \langle \mathfrak{A} \rangle$ . Assume further that  $p_1 : X \rightarrow E$  is a sublinear operator and  $p_2 : E \rightarrow F$  is an increasing sublinear operator. Then*

$$\begin{aligned} & \partial(p_2 \circ p_1) \\ &= \{T \circ \langle \partial p_1 \rangle : T \in L^+(l_\infty(\partial p_1, E), F) \\ & \quad \wedge T \circ \Delta_{\partial p_1} \in \partial p_2\}. \end{aligned}$$

*Furthermore, if  $\partial p_1 = \text{cop}(\mathfrak{A}_1)$  and  $\partial p_2 = \text{cop}(\mathfrak{A}_2)$  then*

$$\begin{aligned} \partial(p_2 \circ p_1) &= \left\{ T \circ \langle \mathfrak{A}_1 \rangle : T \in L^+(l_\infty(\mathfrak{A}_1, E), F) \right. \\ & \quad \left. \wedge \left( \exists \alpha \in \partial \varepsilon_{\mathfrak{A}_2} \right) T \circ \Delta_{\mathfrak{A}_1} = \alpha \circ \langle \mathfrak{A}_2 \rangle \right\}. \end{aligned}$$

Hahn–Banach in the classical formulation is of course the simplest chain rule for removing any linear embedding from the subdifferential sign.

## Approximation

Convexity of harpedonaptae was stable in the sense that no variation of stakes within the surrounding rope can ever spoil the convexity of the tract to be surveyed.

Study of stability in abstract convexity is accomplished sometimes by introducing various epsilons in appropriate places. One of the earliest excursions in this direction is connected with the classical Hyers–Ulam stability theorem for  $\varepsilon$ -convex functions. Exact calculations with epsilons and sharp estimates are sometimes bulky and slightly mysterious. Some alternatives are suggested by actual infinities, which is illustrated with the conception of *infinitesimal optimality*.

## Enter Epsilon and Monad

Assume given a convex operator  $f : X \rightarrow E \cup \{+\infty\}$  and a point  $\bar{x}$  in the effective domain  $\text{dom}(f) := \{x \in X : f(x) < +\infty\}$  of  $f$ . Given  $\varepsilon \geq 0$  in the positive cone  $E_+$  of  $E$ , by the  $\varepsilon$ -subdifferential of  $f$  at  $\bar{x}$  we mean the set

$$\partial^\varepsilon f(\bar{x}) := \left\{ T \in L(X, E) : \right. \\ \left. (\forall x \in X)(Tx - Fx \leq T\bar{x} - f\bar{x} + \varepsilon) \right\},$$

with  $L(X, E)$  standing as usual for the space of linear operators from  $X$  to  $E$ .

Distinguish some downward-filtered subset  $\mathcal{E}$  of  $E$  that is composed of positive elements. Assuming  $E$  and  $\mathcal{E}$  standard, define the *monad*  $\mu(\mathcal{E})$  of  $\mathcal{E}$  as  $\mu(\mathcal{E}) := \bigcap \{[0, \varepsilon] : \varepsilon \in {}^\circ\mathcal{E}\}$ . The members of  $\mu(\mathcal{E})$  are *positive infinitesimals* with respect to  $\mathcal{E}$ . As usual,  ${}^\circ\mathcal{E}$  denotes the external set of all standard members of  $E$ , the *standard part* of  $\mathcal{E}$ .

## Infinitesimal Subdifferentials

Assume that the monad  $\mu(\mathcal{E})$  is an external cone over  ${}^\circ\mathbb{R}$  and, moreover,  $\mu(\mathcal{E}) \cap {}^\circ E = 0$ . In application,  $\mathcal{E}$  is usually the filter of order-units of  $E$ . The relation of *infinite proximity* or *infinite closeness* between the members of  $E$  is introduced as follows:

$$e_1 \approx e_2 \leftrightarrow e_1 - e_2 \in \mu(\mathcal{E}) \wedge e_2 - e_1 \in \mu(\mathcal{E}).$$

Now

$$Df(\bar{x}) := \bigcap_{\varepsilon \in {}^\circ\mathcal{E}} \partial_\varepsilon f(\bar{x}) = \bigcup_{\varepsilon \in \mu(\mathcal{E})} \partial_\varepsilon f(\bar{x}),$$

which is the *infinitesimal subdifferential* of  $f$  at  $\bar{x}$ . The elements of  $Df(\bar{x})$  are *infinitesimal subgradients* of  $f$  at  $\bar{x}$ .

## Exeunt Epsilons

**Theorem.** *Let  $f_1 : X \times Y \rightarrow E \cup +\infty$  and  $f_2 : Y \times Z \rightarrow E \cup +\infty$  be convex operators. Suppose that the convolution  $f_2 \Delta f_1$  is infinitesimally exact at some point  $(x, y, z)$ ; i.e.,  $(f_2 \Delta f_1)(x, y) \approx f_1(x, y) + f_2(y, z)$ . If, moreover, the convex sets  $\text{epi}(f_1, Z)$  and  $\text{epi}(X, f_2)$  are in general position then*

$$D(f_2 \Delta f_1)(x, y) = Df_2(y, z) \circ Df_1(x, y).$$

## Enter Boole

The essence of mathematics resides in freedom, and abstraction is the freedom of generalization. Freedom is the loftiest ideal and idea of man, but it is demanding, limited, and vexing. So is abstraction. So are its instances in convexity. Abstract convexity starts with repudiating the heritage of harpedonaptae, which is annoying but may turn out rewarding.

Freedom of set theory empowered us with the Boolean valued models yielding various realizations of the continuum with idempotents galore. Many instances of Hahn–Banach in modules and semimodules are just the descents or Boolean interpretations of their classical analogs. The celebrated Hahn–Banach–Kantorovich theorem is simple Hahn–Banach in a Boolean disguise.

## Abstraction Vexing or Not

We know now that many sets with idempotents are indistinguishable from their standard analogs in the paradigm of distant modeling. Boolean valued analysis has greatly changed the appearance of abstract convexity, while demonstrating that many seemingly new results are just canny interpretations of harpedonap-tae or Hahn–Banach.

“Scholastic” differs from “scholar.” Abstraction is limited by taste, tradition, and common sense. The challenge of abstraction is alike the call of freedom. But no freedom is exercised in solitude. The holy gift of abstraction coexists with gratitude and respect to the legacy of our predecessors who collected the gems of reason and saved them in the treasure-trove of mathematics.