SOME TRENDS IN MONADOLOGY

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Abstract. Monadology is treated as a field for interaction of the Boolean-valued and infinitesimal versions of nonstandard analysis.

Nonstandard methods of analysis split presently so as to form two main disciplines: *infinitesimal analysis* also known as *Robinsonian nonstandard analysis* and *Boolean-valued analysis*. The common feature of the disciplines is as follows: either executes comparative study of two interpretations of a mathematical claim or construction considered as a formal symbolic expression by means of two different set-theoretic models, one standard and the other nonstandard. For definitions and details see [1–3] and the literature cited therein.

It is sometimes beneficial to combine the theoretical and technical methods that are offered by the Boolean-valued and infinitesimal versions of nonstandard analysis which differ drastically in content and method. Therefore, at least two ways are open to their simultaneous application. One approach consists in studying a standard Boolean-valued model in the universe of Nelson's internal set theory (or in the universe of Kawai's external sets). Infinitesimals descend here from some outer world. Many applications require using the other approach that consists in discovering infinitesimals inside Boolean-valued universes.

In this talk the two combinations of nonstandard methods are presented in brief as regards monadology.

1. Boolean-valued modeling in a nonstandard universe.

Boolean-valued analysis introduces a new important class of mathematical objects — the class of structures with the *cyclic property* consisting in closure under mixing. These objects are the descents of the corresponding formations in the *(separated) Boolean-valued universe* $\mathbf{V}^{(B)}$ over a complete Boolean algebra B. As a matter of fact, the methodology developed by infinitesimal analysis is connected with inventing a special machinery for studying filters, monadology.

Let $\mathcal{F}$ be a standard filter, let ${}^{\circ}\mathcal{F}$ be its *standard core*, and let ${}^{a}\mathcal{F} := \mathcal{F} \setminus {}^{\circ}\mathcal{F}$ be the external set of *astray* or *remote elements* of $\mathcal{F}$. If

$$\mu(\mathcal{F}) := \bigcap {}^{\circ}\mathcal{F} = \bigcup {}^{a}\mathcal{F}$$

is the *monad* of $\mathcal{F}$ then $\mathcal{F} = {}^{*}\operatorname{fil}\,(\{\mu(\mathcal{F})\})$, with $\operatorname{fil}(A)$ standing for the collection of all supersets of A. The notion of monad is central to every external set theory. In this connection the development of combined methods, in particular simultaneous use of infinitesimals and ascents in K-space theory, requires adaptation of the notion of monad for filters and their images. In this section we pursue an approach in

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which the ordinary monadology is applied to the descents of the objects under study. An alternative, applying the conventional monadology inside $\mathbf{V}^{(B)}$ with further descending, will be considered in the next section.

1.1. We start with recalling some constructions from the theory of filters in $\mathbf{V}^{(B)}$.

Let $\mathcal{G}$ be a filterbase on X with $X \in \mathcal{P}(\mathbf{V}^{(B)})$. Put

$$\mathcal{G}' := \{F \in \mathcal{P}(X\uparrow)\downarrow : (\exists G \in \mathcal{G})\,[\![\, F \supset G\uparrow \,]\!] = \mathbf{1}\};$$
$$\mathcal{G}'' := \{G\uparrow : G \in \mathcal{G}\}.$$

Then $\mathcal{G}'\uparrow$ and $\mathcal{G}''\uparrow$ are bases of one and the same filter $\mathcal{G}^{\uparrow}$ on $X\uparrow$ inside $\mathbf{V}^{(B)}$. The filter $\mathcal{G}^{\uparrow}$ is called the *ascent* of $\mathcal{G}$. If $\operatorname{mix}(\mathcal{G})$ is the set of all mixings of nonempty families formed by elements of $\mathcal{G}$ and $\mathcal{G}$ consists of cyclic sets, then $\operatorname{mix}(\mathcal{G})$ is a filterbase on X and $\mathcal{G}^{\uparrow} = \operatorname{mix}(\mathcal{G})^{\uparrow}$.

If $\mathcal{F}$ is a filter on X inside $\mathbf{V}^{(B)}$, then put $\mathcal{F}^{\downarrow} := \operatorname{fil}\,(\{F\downarrow : F \in \mathcal{F}\downarrow\})$. The filter $\mathcal{F}^{\downarrow}$ on $X\downarrow$ is called the *descent* of $\mathcal{F}$. A filterbase $\mathcal{G}$ on $X\downarrow$ is called *extensional*, if there is a filter $\mathcal{F}$ on X such that $\operatorname{fil}(\mathcal{G}) = \mathcal{F}$. Finally, the descents of ultrafilters in X are called *proultrafilters* on $X\downarrow$. A filter having a base of cyclic sets is called *cyclic*. Proultrafilters are maximal cyclic filters.

1.2. Fix a standard complete Boolean algebra B and the corresponding Boolean-valued universe $\mathbf{V}^{(B)}$ thought of as being composed of internal sets. If A is an external set, then the *cyclic hull* $\operatorname{mix}(A)$ of A is introduced as follows. Say that an element $x \in \mathbf{V}^{(B)}$ belongs to $\operatorname{mix}(A)$, if there is an internal family $(a_\xi)_{\xi\in\Xi}$ of elements of A and an internal partition $(b_\xi)_{\xi\in\Xi}$ of unity in B such that x is the mixing of $(a_\xi)_{\xi\in\Xi}$ by $(b_\xi)_{\xi\in\Xi}$, i.e. $b_\xi x = b_\xi a_\xi$ for $\xi \in \Xi$ or, equivalently, $x = \operatorname{mix}_{\xi\in\Xi}(b_\xi a_\xi)$.

1.3. Theorem. *Given a filter $\mathcal{F}$ on $X\downarrow$, consider*

$$\mathcal{F}\uparrow\downarrow := \operatorname{fil}\,(\{F\uparrow\downarrow : F \in \mathcal{F}\}).$$

Then $\operatorname{mix}(\mu(\mathcal{F})) = \mu(\mathcal{F}\uparrow\downarrow)$, and $\mathcal{F}\uparrow\downarrow$ is the greatest cyclic filter coarser than $\mathcal{F}$.

In connection with this theorem, the monad of $\mathcal{F}$ is called *cyclic*, if $\mu(\mathcal{F}) = \operatorname{mix}(\mu(\mathcal{F}))$. Unfortunately, cyclicity of a monad is not completely responsible for extensionality of a filter. In this connection, the *cyclic monad hull* $\mu_c(U)$ of an external set U should be introduced. Namely, we are compelled to set

$$x \in \mu_c(U) \leftrightarrow (\forall^{\rm st}\ V = V\uparrow\downarrow) V \supset U \rightarrow x \in \mu(V).$$

In particular, if $B = \{0,1\}$, then $\mu_c(U)$ coincides with the monad of the standardization of the external filter of supersets of U, i.e. with the so-called (*discrete*)*monad hull* $\mu_d(U)$ (the word "monadic" is also employed).

1.4. *The cyclic monad hull of a set is the cyclic hull of its monad hull*

$$\mu_c(U) = \operatorname{mix}(\mu_d(U)).$$

A special role is played by the *essential points* of $X\downarrow$ constituting the external set ${}^eX$. By definition, ${}^eX$ comprises the elements of the monads of all proultrafilters in $X\downarrow$.

1.5. Criterion for a point to be essential. *A point is essential if and only if it can be separated by a standard set from every standard cyclic set not containing the point.*

Therefore, if the monad of an ultrafilter $\mathcal{F}$ contains an essential point then $\mu(\mathcal{F}) \subset {}^{e}X$ while, $\mathcal{F}{\uparrow}{\downarrow}$ is a proultrafilter.

The next claim is immediate from the above constructions and considerations.

1.6. Criterion for a filter to be extensional. *A filter is extensional if and only if its monad is the cyclic monad hull of the set of its own essential points.*

1.7. *A standard set is cyclic if and only if it is the cyclic monad hull of its own essential points.*

1.8. Nonstandard criterion for mixing filters. *Let $(\mathcal{F}_\xi)_{\xi\in\Xi}$ be a standard family of extensional filters, and let $(b_\xi)_{\xi\in\Xi}$ be a standard partition of unity. The filter $\mathcal{F}$ is the mixing of $(\mathcal{F}_\xi)_{\xi\in\Xi}$ by $(b_\xi)_{\xi\in\Xi}$ if and only if*

$$(\forall^{\mathrm{St}}\xi \in \Xi) b_\xi \mu(\mathcal{F}) = b_\xi \mu(\mathcal{F}_\xi).$$

A peculiarity of the approach presented exposes itself in applications to the descents of topological spaces through a special new role of essential points. In this connection, we note some properties of the latter.

1.9. *The following are true:*

(**1**) *the image of an essential point under an extensional mapping is an essential point of the image;*

(**2**) *let E be a standard set, and let X be a standard element of $\mathbf{V}^{(B)}$. Consider the product $X^{E^\wedge}$ inside $\mathbf{V}^{(B)}$, where $X^{E^\wedge}$ is the standard name of E in $\mathbf{V}^{(B)}$. If x is an essential point of $X^{E^\wedge}{\downarrow}$, then for every standard $e \in E$ the point $x{\downarrow}(e)$ is essential in $X{\downarrow}$;*

(**3**) *let $\mathcal{F}$ be a cyclic filter in $X{\downarrow}$, and let ${}^{e}\mu(\mathcal{F}) := \mu(\mathcal{F}) \cap {}^{e}X$ be the set of essential points of its monad. Then*

$${}^{e}\mu(\mathcal{F}) = {}^{e}\mu(\mathcal{F}^{\uparrow\downarrow}).$$

Let $(X,\mathcal{U})$ be a uniform space inside $\mathbf{V}^{(B)}$. The uniform space $(X{\downarrow},\mathcal{U}^{\downarrow})$ is called *procompact* or *cyclically compact*, if $(X,\mathcal{U})$ is compact inside $\mathbf{V}^{(B)}$. A similar sense is implied in the notion of *pro-total-boundedness* and so on.

1.10. Nonstandard criterion for procompactness. *Every essential point of $X{\downarrow}$ is nearstandard (i.e. infinitesimally close to a standard point) if and only if $X{\downarrow}$ is procompact.*

It is easily seen from 1.10 that the Boolean-valued criterion for procompactness differs from the usual one: "A compact space is a space composed of nearstandard points." The existence of many procompact but noncompact spaces provides the variety of examples of nonessential points. We note here that the combined application of 1.10 and 1.9 (2), of course, allows us to produce some nonstandard proof of the natural analog of Tychonoff's theorem for product of procompact spaces — "the descent of Tychonoff's theorem in $\mathbf{V}^{(B)}$."

1.11. Nonstandard criterion for proprecompactness. *A standard space is the descent of a totally bounded uniform space if and only if its every essential point is prenearstandard (i.e. lies in the monad of a Cauchy filter).*

We will apply the above approach to describing o-convergence in a K-space Y. To save room, we restrict ourselves to the consideration of filters containing order intervals (or, equivalently, filters with bounded monads). In accordance with the same end, the K-space Y is further assumed universally complete. By Gordon's theorem, the space Y can be viewed as the descent $\mathcal{R}{\downarrow}$ of the element $\mathcal{R}$ representing the field of reals $\mathbb{R}$ in the Boolean-valued universe $\mathbf{V}^{(B)}$ constructed over the base B of the space Y. We denote by $\mathcal{E}$ the filter of order units in Y, i.e. the set $\mathcal{E} := \{\varepsilon \in Y_+ : [\![\varepsilon = 0]\!] = \mathbf{0}\}$. We write $x \approx y$ whenever elements $x, y \in Y$ are infinitely close with respect to the descent of the natural topology of $\mathcal{R}$ in $\mathbf{V}^{(B)}$, i.e. $x \approx y \leftrightarrow (\forall^{\mathrm{St}}\varepsilon \in \mathcal{E})|x - y| < \varepsilon$. For $a, b \in Y$, we write $a < b$ if $[\![a < b]\!] = \mathbf{1}$, in other words, $a > b \leftrightarrow a - b \in \mathcal{E}$. Observe that this deviates from the routine of the theory of ordered vector spaces. Of course, our choice is necessitated if we follow the consistency principles for introducing notations as regards descents and ascents. Let ${}^{\approx}Y$ be the *nearstandard part* of Y. For $y \in {}^{\approx}Y$, denote by ${}^{\circ}y$ (or by st (y)) the *standard part* of y, i.e. the unique standard element infinitely close to y.

1.12. Theorem. *For a standard filter $\mathcal{F}$ in Y and a standard $z \in Y$, the following are true:*

(**1**) $\inf_{F\in\mathcal{F}} \sup F \le z \leftrightarrow (\forall y \in {}^{\cdot}\mu(\mathcal{F}{\uparrow}{\downarrow}))\,{}^{\circ}y \le z \leftrightarrow (\forall y \in {}^{e}\mu(\mathcal{F}{\uparrow}{\downarrow}))\,{}^{\circ}y \le z$;
(**2**) $\sup_{F\in\mathcal{F}} \inf F \ge z \leftrightarrow (\forall y \in {}^{\cdot}\mu(\mathcal{F}{\uparrow}{\downarrow}))\,{}^{\circ}y \ge z \leftrightarrow (\forall y \in {}^{e}\mu(\mathcal{F}{\uparrow}{\downarrow}))\,{}^{\circ}y \ge z$;
(**3**) $\inf_{F\in\mathcal{F}} \sup F \ge z \leftrightarrow (\exists y \in {}^{\cdot}\mu(\mathcal{F}{\uparrow}{\downarrow}))\,{}^{\circ}y \ge z \leftrightarrow (\exists y \in {}^{e}\mu(\mathcal{F}{\uparrow}{\downarrow}))\,{}^{\circ}y \ge z$;
(**4**) $\sup_{F\in\mathcal{F}} \inf F \le z \leftrightarrow (\exists y \in {}^{\cdot}\mu(\mathcal{F}{\uparrow}{\downarrow}))\,{}^{\circ}y \le z \leftrightarrow (\exists y \in {}^{e}\mu(\mathcal{F}{\uparrow}{\downarrow}))\,{}^{\circ}y \le z$;
(**5**) $\mathcal{F} \xrightarrow{(o)} z \leftrightarrow (\forall y \in {}^{e}\mu(\mathcal{F}{\uparrow}{\downarrow}))y \approx z \leftrightarrow (\forall y \in \mu(\mathcal{F}^{\uparrow\downarrow}))y \approx z$.

Here ${}^{\cdot}\mu(\mathcal{F}{\uparrow}{\downarrow}) := \mu(\mathcal{F}{\uparrow}{\downarrow}) \cap {}^{\approx}Y$, and, as usual, ${}^{e}\mu(\mathcal{F}{\uparrow}{\downarrow})$ is the set of essential points of the monad $\mu(\mathcal{F}{\uparrow}{\downarrow})$, i.e. ${}^{e}\mu(\mathcal{F}{\uparrow}{\downarrow}) = \mu(\mathcal{F}{\uparrow}{\downarrow}) \cap {}^{e}\mathcal{R}$.

Proof. By way of illustration, we prove (3).

Suppose first that in the greater set ${}^{\cdot}\mu(\mathcal{F}{\uparrow}{\downarrow})$ there is an element y such that ${}^{\circ}y \ge z$. For every standard $F \in \mathcal{F}$ we have $y \in F{\uparrow}{\downarrow}$. Hence, if $\varepsilon \in {}^{\circ}\mathcal{E}$ then $y > z - \varepsilon$ and $\sup F = \sup F{\uparrow}{\downarrow} > z - \varepsilon$. By Leibniz's principle we obtain: $(\forall^{\mathrm{st}}\ F \in \mathcal{F})\,(\forall^{\mathrm{st}}\ \varepsilon > 0)\, \sup F \ge z$, i.e. $(\forall F \in \mathcal{F}) \sup F \ge z$ and $\inf_{F\in\mathcal{F}} \sup F \ge z$.

To complete the proof, begin with observing that by properties of the upper limit in $\mathbb{R}$ and by the transfer principle of Boolean-valued analysis we have

$$[\![\,(\exists\mathcal{G})\,(\mathcal{G} \text{ is an ultrafilter in } \mathcal{R} \wedge \mathcal{G} \supset \mathcal{F}^{\uparrow} \wedge \inf_{G\in\mathcal{G}} \sup G \ge z)\,]\!] = \mathbf{1}.$$

According to the maximum principle, there is a proultrafilter $\mathcal{G}$ such that $\mathcal{G} \supset \mathcal{F}^{\uparrow\downarrow}$ and $\inf_{G\in\mathcal{G}} \sup G \ge z$. Using the transfer and idealization principles, we successively obtain

$$(\forall^{\mathrm{st}}\ G \in \mathcal{G}) \sup G \ge z \leftrightarrow (\forall^{\mathrm{st}}\ G \in \mathcal{G})\,[\![\sup(G{\uparrow}) = z\,]\!] = \mathbf{1}$$

$$\leftrightarrow (\forall^{\mathrm{st}}\ G \in \mathcal{G})\,[\![\,(\forall\varepsilon > 0)\,(\exists g \in G{\uparrow})\, g > z - \varepsilon\,]\!] = \mathbf{1}$$

$$\leftrightarrow (\forall^{\mathrm{st}}\ G \in \mathcal{G})\,(\forall\varepsilon > 0)\,(\exists g \in G{\uparrow}{\downarrow}) g > z - \varepsilon$$

$$\leftrightarrow (\forall^{\mathrm{st}}\ G \in \mathcal{G})\,(\forall^{\mathrm{st}}\ \varepsilon > 0)\,(\exists g \in G{\uparrow}{\downarrow}) g > z - \varepsilon$$

$$\leftrightarrow (\forall^{\text{st fin}} \mathcal{G}_0 \supset \mathcal{G})\,(\forall^{\text{st fin}} \mathcal{E}_0 \subset \mathcal{E})\,(\exists g)$$

$$(\forall G \in \mathcal{G}_0)\,(\forall \varepsilon \in \mathcal{E}_0)\,(g \in G{\uparrow}{\downarrow} \wedge g > z - \varepsilon)$$

$$\leftrightarrow (\exists g)\,(\forall^{\text{st}}\ G \in \mathcal{G})\,(\forall^{\text{st}}\ \varepsilon > 0)\,(g \in G{\uparrow}{\downarrow} \wedge g > z - \varepsilon)$$

$$\leftrightarrow (\exists g \in \mu(\mathcal{G}^{\uparrow\downarrow}))\,{}^{\circ}g \geq z \leftrightarrow (\exists g \in \mu(\mathcal{G}))^{\circ}g = z.$$

The observation,

$$\mu(\mathcal{G}) \subset {}^{e}\mu(\mathcal{F}^{\uparrow\downarrow}) = {}^{e}\mu(\mathcal{F}{\uparrow}{\downarrow}) \subset {}^{\circ}\mu(\mathcal{F}{\uparrow}{\downarrow}).$$

completes the proof.

2. Infinitesimal modeling inside a Boolean-valued universe.

In this section we assume fixed some complete Boolean algebra B and the (separated) Boolean-valued universe $\mathbf{V}^{(B)}$ over B.

When applying methods of infinitesimal analysis, we adopt A. Robinson's classical approach which is realized inside $\mathbf{V}^{(B)}$. In other words, in concrete situations we treat the classical and internal universes and the corresponding *-map (Robinsonian standardization) as elements of $\mathbf{V}^{(B)}$. Moreover, the nonstandard world is supposed to be saturated so as required.

2.1. The descent of the *-map will be called the *descent standardization*. Side by side with the term "descent standardization" we also use the expressions "B-standardization," "prostandardization," etc. Furthermore, for the Robinsonian standardization of a B-set A the symbol ${}^*A$ is used. Respectively, the *descent standardization* of a set A with B-structure (i.e. a subset of $\mathbf{V}^{(B)}$) is defined as $({}^*(A{\uparrow})){\downarrow}$ and is denoted by the symbol ${}_*A$ (it is meant here that $A{\uparrow}$ is an element of the standard world located in $\mathbf{V}^{(B)}$). Thus, ${}^*a \in {}_*A \leftrightarrow a \in A{\uparrow}{\downarrow}$. The descent standardization ${}_*\Phi$ of an extensional correspondence Φ is also defined in a natural way. While considering descent standardizations of the standard names of elements of the von Neumann universe $\mathbf{V}$, for convenience we use abbreviations, writing ${}^*x := {}^*(x^\wedge)$ and ${}_*x := ({}^*x){\downarrow}$ for $x \in \mathbf{V}$. The rules for placing and omitting asterisks (by default) in using descent standardization are also assumed so liberal as those for the Robinsonian $*$-map.

2.2. Transfer principle. *Let $\varphi = \varphi(x, y)$ be a formula of the Zermelo–Fraenkel theory (without any free variables except x and y). For a nonempty element F in $\mathbf{V}^{(B)}$ and for every z we have:*

$$(\exists x \in {}_*F)\,[\![\,\varphi(x, {}^*z)\,]\!] = \mathbf{1} \leftrightarrow (\exists x \in F{\downarrow})\,[\![\,\varphi(x, z)\,]\!] = \mathbf{1};$$

$$(\forall x \in {}_*F)\,[\![\,\varphi(x, {}^*z)\,]\!] = \mathbf{1} \leftrightarrow (\forall x \in F{\downarrow})\,[\![\,\varphi(x, z)\,]\!] = \mathbf{1}.$$

If G is a subset of $\mathbf{V}^{(B)}$, then the following are true:

$$(\exists x \in {}_*G)[\![\,\varphi(x, {}^*z)\,]\!] = \mathbf{1} \leftrightarrow (\exists x \in G{\uparrow}{\downarrow})\,[\![\,\varphi(x, z)\,]\!] = \mathbf{1};$$

$$(\forall x \in {}_*G)[\![\,\varphi(x, {}^*z)\,]\!] = \mathbf{1} \leftrightarrow (\forall x \in G{\uparrow}{\downarrow})\,[\![\,\varphi(x, z)\,]\!] = \mathbf{1}.$$

2.3. Idealization principle. *Let $X\!\uparrow$ and Y be classical elements of $\mathbf{V}^{(B)}$ and let $\varphi = \varphi(x, y, z)$ be a formula of the Zermelo–Fraenkel theory. For an internal element z in $\mathbf{V}^{(B)}$ we have:*

$$(\forall^{\mathrm{fin}} A \subset X)\,(\exists y \in {}_*Y)\,(\forall x \in A)\,[\![\,\varphi({}^*x, y, z)\,]\!] = \mathbf{1}$$

$$\leftrightarrow (\exists y \in {}_*Y)\,(\forall x \in X)\,[\![\,\varphi({}^*x, y, z)\,]\!] = \mathbf{1}.$$

For a filter $\mathcal{F}$ of sets with B-structure, its *descent monad* $m(\mathcal{F})$ is defined as follows:

$$m(\mathcal{F}) := \bigcap_{F \in \mathcal{F}} {}_*F.$$

2.4. Theorem. *Let $\mathcal{S}$ be a set of filters and let $\mathcal{S}^{\uparrow} := \{\mathcal{F}^{\uparrow} : \mathcal{F} \in \mathcal{S}\}$ be its ascent in $\mathbf{V}^{(B)}$. The following are equivalent:*

(**1**) *the set of cyclic hulls $\mathcal{S}\!\uparrow\downarrow := \{\mathcal{F}\!\uparrow\downarrow : \mathcal{F} \in \mathcal{S}\}$, is bounded above;*

(**2**) *the set $\mathcal{S}^{\uparrow}$ is bounded above inside $\mathbf{V}^{(B)}$;*

(**3**) $\bigcap\{m(\mathcal{F}) : \mathcal{F} \in \mathcal{S}\} \neq \varnothing$.

Moreover, if the conditions (1)–(3) *are satisfied, then*

$$m(\sup \mathcal{S}\!\uparrow\downarrow) = \bigcap\{m(\mathcal{F}) : \mathcal{F} \in \mathcal{S}\};$$

$$\sup \mathcal{S}^{\uparrow} = (\sup \mathcal{S})^{\uparrow}.$$

It is useful to note that for an infinite set of descent monads, its union (and even the cyclic hull of this union) is not a descent monad in general. The situation here is the same as for ordinary monads.

2.5. Nonstandard criteria for a proultrafilter. *The following are equivalent:*

(**1**) *$\mathcal{U}$ is a proultrafilter;*

(**2**) *$\mathcal{U}$ is an extensional filter whose descent monad is inclusion-minimal;*

(**3**) *the representation $\mathcal{U} = (x)^{\downarrow} := \mathrm{fil}\,(\{U\!\uparrow\downarrow : x \in {}_*A\})$ holds for each point x of the descent monad $m(\mathcal{U})$;*

(**4**) *$\mathcal{U}$ is an extensional filter such that its descent monad can be easily caught by a cyclic set, i.e. for every $U = U\!\uparrow\downarrow$ we have either $m(\mathcal{U}) \subset {}_*U$, or $m(\mathcal{U}) \subset {}_*(X \backslash U)$;*

(**5**) *$\mathcal{U}$ is a cyclic filter satisfying the condition: for every cyclic U, if ${}_*U \cap m(\mathcal{A}) \neq \varnothing$ then $U \in \mathcal{U}$.*

2.6. Nonstandard criterion for mixing filters. *Let $(\mathcal{F}_\xi)_{\xi\in\Xi}$ be a family of filters, let $(b_\xi)_{\xi\in\Xi}$ be a partition of unity, and let $\mathcal{F} = \mathrm{mix}_{\xi\in\Xi}(b_\xi \mathcal{F}_\xi^{\uparrow})$ be the mixing of $(\mathcal{F}_\xi^{\uparrow})_{\xi\in\Xi}$ by $(b_\xi)_{\xi\in\Xi}$. Then*

$$m(\mathcal{F}^{\downarrow}) = \mathop{\mathrm{mix}}_{\xi\in\Xi}(b_\xi m(\mathcal{F}_\xi)).$$

It is useful to compare 2.6 with 1.8.

A point y of the set ${}_*X$ is called *descent-nearstandard* (or simply *nearstandard* when there is no danger of misunderstanding) provided that for some $x \in X\!\downarrow$ one has ${}^*x \approx y$ (i.e. $(x, y) \in m(\mathcal{U}^{\downarrow})$, with $\mathcal{U}$ denoting the uniformity on X).

2.7. Nonstandard criterion for procompactness. *The set $A{\uparrow}{\downarrow}$ is procompact if and only if every point of ${}_*A$ is descent-nearstandard.*

It is reasonable to compare 2.7 with 1.10.

In closing, we formulate certain general principles for descent standardization.

2.8. *Let $\varphi = \varphi(x)$ be a formula of the Zermelo–Fraenkel theory. The truth value of φ is constant on the descent monad of every proultrafilter $\mathcal{A}$, i.e.*

$$(\forall x, y \in m(\mathcal{A}))\, [\![\, \varphi(x) \,]\!] = [\![\, \varphi(y) \,]\!].$$

2.9. Theorem. *Let $\varphi = \varphi(x, y, z)$ be a formula of the Zermelo–Fraenkel theory and let $\mathcal{F}, \mathcal{G}$ be filters of sets with B-structure. The following quantification rules are valid (for internal y, z in $\mathbf{V}^{(B)}$):*

(**1**) $(\exists x \in m(\mathcal{F}))\, [\![\, \varphi(x, y, z) \,]\!] = \mathbf{1} \leftrightarrow (\forall F \in \mathcal{F})\, (\exists x \in {}^*F)\, [\![\, \varphi(x, y, z) \,]\!] = \mathbf{1}$;

(**2**) $(\forall x \in m(\mathcal{F}))\, [\![\, \varphi(x, y, z) \,]\!] = \mathbf{1} \leftrightarrow (\exists F \in \mathcal{F}^{\uparrow\downarrow})(\forall x \in {}_*F)\, [\![\, \varphi(x, y, z) \,]\!] = \mathbf{1}$;

(**3**) $(\forall x \in m(\mathcal{F}))\, (\exists y \in m(\mathcal{G}))[\![\, \varphi(x, y, z) \,]\!] = \mathbf{1} \leftrightarrow (\forall G \in \mathcal{G})\, (\exists F \in \mathcal{F}^{\uparrow\downarrow})$
$(\forall x \in {}^*F)\, (\exists y \in {}^*G)[\![\, \varphi(x, y, z) \,]\!] = \mathbf{1}$;

(**4**) $(\exists x \in m(\mathcal{F}))\, (\forall y \in m(\mathcal{G}))\, [\![\, \varphi(x, y, z) \,]\!] = \mathbf{1} \leftrightarrow (\exists G \in \mathcal{G}^{\uparrow\downarrow})\, (\forall F \in \mathcal{F})$
$(\exists x \in {}^*F)\, (\forall y \in {}^*G)[\![\, \varphi(x, y, z) \,]\!] = \mathbf{1}$.

Moreover, for standardized free variables, we have:

(**1′**) $(\exists x \in m(\mathcal{F}))[\![\, \varphi(x, {}^*y, {}^*z) \,]\!] = \mathbf{1} \leftrightarrow (\forall F \in \mathcal{F})(\exists x \in F{\uparrow}{\downarrow})[\![\, \varphi(x, y, z) \,]\!] = \mathbf{1}$;

(**2′**) $(\forall x \in m(\mathcal{F}))[\![\, \varphi(x, {}^*y, {}^*z) \,]\!] = \mathbf{1} \leftrightarrow (\exists F \in \mathcal{F}^{\uparrow\downarrow})(\forall x \in F)[\![\, \varphi(x, y, z) \,]\!] = \mathbf{1}$;

(**3′**) $(\forall x \in m(\mathcal{F}))(\exists y \in m(\mathcal{G}))[\![\, \varphi(x, y, {}^*z) \,]\!] = \mathbf{1} \leftrightarrow (\forall G \in \mathcal{G})(\exists F \in \mathcal{F}^{\uparrow\downarrow})$
$(\forall x \in F)(\exists y \in G{\uparrow}{\downarrow})[\![\, \varphi(x, y, z) \,]\!] = \mathbf{1}$;

(**4′**) $(\exists x \in m(\mathcal{F}))(\forall y \in m(\mathcal{G}))[\![\, \varphi(x, y, {}^*z) \,]\!] = \mathbf{1} \leftrightarrow (\exists G \in \mathcal{G}^{\uparrow\downarrow})(\forall F \in \mathcal{F})$
$(\exists x \in F{\uparrow}{\downarrow})(\forall y \in G)[\![\, \varphi(x, y, z) \,]\!] = \mathbf{1}$.

By way of illustration, we give a formula for calculating *fragments* or *components* of a positive linear operator. Recall that a positive y is a *fragment* of a positive x, with x and y some members of a vector lattice, whenever y is *disjoint* from $x - y$, i.e. $y \wedge (x - y) = 0$. There are many reasons for describing fragments, which is a relatively easy matter in the case of functionals but an intriguing problem in the case of operators (cf. [4]).

Let Y be a Kantorovich space with base B and assume that ${}_*Y$ is some B-standardization of Y. Assume further that X is a vector lattice while S, T and R are three positive operators: $S, T, R \in L_+(X, Y)$. Suppose also that R is the band projection of S to the principal component generated by T in $L_*(X, Y)$. The symbols $\approx$ and $^\circ$ play the usual role, signifying the infinite closeness and standard part operation in ${}_*Y$.

Infinitesimally interpreting an obvious scalar formula for components of a positive functional, we arrive at the desired representation of components of a positive operator.

2.10. Theorem. *The following is an exact formula, i.e., the infimum is attainable:*

$$Rx = \inf \left\{ \pi\, {}^\circ Sy + \pi^d Sx : \pi T(x - y) \approx 0, \quad 0 \le y \le x \right\},$$

with π standing for an element of the base of Y, and π^d denoting the complement of π, and x a member of X.

Observe also that the routine practice omits the symbols of the Robinsonian B-standardization since this leads to no confusion.

References

1. Kusraev A. G. and Kutateladze S. S., *Nonstandard Methods of Analysis*, Kluwer Academic Publishers, Dordrecht, 1994.
2. Kusraev A. G. and Kutateladze S. S., *Boolean Valued Analysis*, Kluwer Academic Publishers, Dordrecht, 1999.
3. Kutateladze S. S. (Ed.), *Nonstandard Analysis and Vector Lattices*, Kluwer Academic Publishers, Dordrecht, 2000.
4. Kutateladze S. S. (Ed.), *Vector Lattices and Positive Operators*, Kluwer Academic Publishers, Dordrecht, 1996.

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