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SIMULTANEOUS LINEAR INEQUALITIES: YESTERDAY AND TODAY

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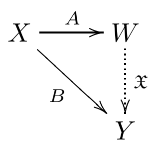
1. Agenda. Linear inequality implies linearity and order. When combined, the two produce an ordered vector space. Each linear inequality {in the simplest environment of the sort is some half-space. Simultaneity implies many instances and so yields intersections of half-spaces. This yields polyhedra as well as arbitrary convex sets, identifying the theory of linear inequalities with convexity.

Convexity stems from the remote ages and reigns in the federation of geometry, optimization, and functional analysis. Convexity feeds generation, separation, calculus, and approximation. Generation appears as duality; separation, as optimality; calculus, as representation; and approximation, as stability.

This talk addresses the origin and the state of the art of the relevant areas with a particular emphasis on the Farkas Lemma. Our aim is to demonstrate how Boolean valued analysis may be applied to simultaneous linear inequalities with operators. This particular theme is another illustration of the deep and powerful technique of "stratified validity" which is characteristic of Boolean valued analysis.

2. Environment. Assume that X is a real vector space, Y is a *Kantorovich space* also known as a complete vector lattice or a Dedekind complete Riesz space. Let $\mathbb{B} := \mathbb{B}(Y)$ be the *base* of Y, i. e., the complete Boolean algebras of positive projections in Y; and let $m(Y)$ be the universal completion of Y. Denote by $L(X,Y)$ the space of linear operators from X to Y. In case X is furnished with some Y-seminorm on X, by $L^{(m)}(X,Y)$ we mean the *space of dominated operators* from X to Y. As usual, $\{T \leqslant 0\} := \{x \in X : Tx \leqslant 0\}$; $\ker(T) = T^{-1}(0)$ for $T \in L(X,Y)$.

3. Kantorovich Theorem.



If W is ordered by W_+ and $A(X) - W_+ = W_+ - A(X) = W$, then

$$(\exists\, \mathfrak{X} \geqslant 0)\ \mathfrak{X}A = B \leftrightarrow \{A \leqslant 0\} \subset \{B \leqslant 0\}.$$

4. The Alternative. Let X be a Y-seminormed real vector space, with Y a Kantorovich space. Assume that $A_1, \dots, A_N$ and B belong to $L^{(m)}(X, Y)$.

Then one and only one of the following holds:

(1) There are $x \in X$ and $b, b' \in \mathbb{B}$ such that $b' \leqslant b$ and

$$b'Bx > 0,\ bA_1x \leqslant 0,\ \dots,\ bA_Nx \leqslant 0.$$

(2) There are $\alpha_1, \dots, \alpha_N \in \operatorname{Orth}(m(Y))_+$ such that $B = \sum_{k=1}^N \alpha_k A_k$.

5. Inhomogeneous Inequalities. Let X be a Y-seminormed real vector space, with Y a Kantorovich space. Assume given some dominated operators $A_1, \dots, A_N, B \in L^{(m)}(X, Y)$ and elements $u_1, \dots, u_N, v \in Y$. The following are equivalent:

(1) For all $b \in \mathbb{B}$ the inhomogeneous operator inequality $bBx \leqslant bv$ is a consequence of the consistent simultaneous inhomogeneous operator inequalities $bA_1x \leqslant bu_1, \dots, bA_Nx \leqslant bu_N$, i. e.,

$$\{bB \leqslant bv\} \supset \{bA_1 \leqslant bu_1\} \cap \ldots \cap \{bA_N \leqslant bu_N\}.$$

(2) There are positive orthomorphisms $\alpha_1, \dots, \alpha_N \in \operatorname{Orth}(m(Y))$ satisfying

$$B = \sum_{k=1}^{N} \alpha_k A_k; \quad v \geqslant \sum_{k=1}^{N} \alpha_k u_k.$$

6. Freedom and Inequality. Abstraction is the freedom of generalization. Freedom is the loftiest ideal and idea of man, but it is demanding, limited, and vexing. So is abstraction. So are its instances in convexity, hence, in simultaneous inequalities.

Freedom of set theory empowered us with the Boolean-valued models yielding a lot of surprising and unforeseen visualizations of the ingredients of mathematics. Many promising opportunities are open to modeling the powerful habits of reasoning and verification. Convexity, the theory of simultaneous linear inequalities in disguise, is a topical illustration of the wisdom and strength of mathematics, the ever fresh art and science of calculus.

Inequality paves way to freedom.