Contents

Foreword
to the English Translation

I am deeply honored to introduce this great book of a great author to the English language reading community.

Denis Artem′evich Vladimirov (1929–1994) was a prominent representative of the Russian mathematical school in functional analysis which was founded by Leonid Vital′evich Kantorovich, a renowned mathematician and a Nobel Prize winner in economics.

This school comprises two affiliations in St. Petersburg and Novosibirsk which maintain intimate relations since the latter was set up by the former, so it is not astonishing that I enjoyed the wit and charm of Vladimirov for many years.

Our contacts were usually established through the students we supervised; he, in St. Petersburg and I, in Novosibirsk. I always tried to arrange matters so that my students spent some time near Vladimirov to master Boolean algebras and ordered vector spaces. Probably one of the results of this cooperation is the fact that there is now an active group in Boolean valued analysis in Novosibirsk. Unfortunately, the only possibility of continuing this practice is offered by the present book...

It was not long before Vladimirov's death when he and his friends had asked me to help with the publishing and editing of the English translation of the book. I agreed readily and soon Kluwer Academic Publishers decided to print the book.

The book was mostly translated by Professor A. E. Gutman and his students in Novosibirsk, all "descendants" of Vladimirov.

E. G. Taĭpale translated a few final sections and made the entire book more readable. I. I. Bazhenov, I. I. Kozhanova, Yu. N. Lovyagin, A. A. Samorodnitskiĭ, and Yu. V. Shergin helped me with the proofreading.

The translation took much more time than planned: the reasons behind this are understandable for anyone aware of the present standards of academic life in Russia. Unfortunately, capable mathematicians are not always experienced translators and knowledgeable grammarians. Therefore, the battle against solecism and mistranslation was partly lost in proofreading...

Vladimirov was unhappy that he had no opportunity to include a chapter on Boolean valued analysis in this edition of his book. At the publisher's request, I compiled a short appendix which is intended to serve as an introduction to this new and promising area for expansion and proliferation of Boolean algebras.

Denis Artem′evich Vladimirov was one of the giants of the past who bequeathed us his insight into part of the future with this book. I hope the reader will enjoy it.

S. S. Kutateladze

August, 2001

Denis Artem′evich Vladimirov
1929–1994

Vladimirov, the author of this book, died on August 2, 1994 after a fatal illness.

Vladimirov was born on February 7, 1929 in St. Petersburg (the former Leningrad). In the early 1940s he lived in the besieged city of St. Petersburg and shared the heavy burdens and oppressions of war. He left his reminiscences about the siege full of vivid observations and personal recollections together with interesting facts about the routine and realities of that time.

In 1950 he entered the Mechanics and Mathematics Department of St. Petersburg State University and graduated from the university in 1955 to become an assistant of the chair of mathematical analysis in his alma mater. His scientific preferences were formed under the influence of L. V. Kantorovich, G. P. Akilov, and A. G. Pinsker. He was a distinguished representative of the area of research in functional analysis which was charted by L. V. Kantorovich based on the concept of ordered vector space.

The scientific interests of Vladimirov encircled not only the general theory of ordered vector spaces but also measurable function spaces, invariants of measurable functions under metric isomorphisms of their domains of definition, the properties of integral operators, the theory of Boolean algebras and measure theory as well as their applications to general topology and probability.

The first publication of Vladimirov[1] [1] is a small masterpiece. It solves three difficult problems from the celebrated survey article of 1951

[1]Within this introductory obituary, the references are cited in accordance with the list at the end of the book.

by L. V. Kantorovich, B. Z. Vulikh, and A. G. Pinsker. The first part of [1] contains (negative) solutions to the problems of (o)-completeness and $(*)$-completeness (originally called (t)-completeness) of an arbitrary K-space: Vladimirov invented some ingenuous examples of $(*)$-incomplete and (o)-incomplete K-spaces (in the second case he used the hypothesis $2^{\aleph_0} < \aleph_\omega$), with $\omega = \{0, 1, 2, \dots\}$.

The second part of [1] is connected with the Pinsker Theorem (*"if the diagonal principle holds in some K-space then this space is regular"*). Proving this theorem, A. G. Pinsker made use of the continuum hypothesis $\mathbb{CH}$ and it was unclear whether some proof is possible without $\mathbb{CH}$. Vladimirov demonstrated that this is in fact impossible. Moreover, he found a set-theoretic proposition (known nowadays to be independent in $\mathbb{ZFC}$) equivalent to the Pinsker Theorem. This enabled Vladimirov to draw the following principal conclusion: *"use of set-theoretic hypotheses in a theory ... is not just a convenient proving tool but rather is deeply connected with the essence of this theory."* At that time this observation was new: exactly the opposite opinion had prevailed among mathematicians (in spite of the Gödel Theorem, many had been sure, for instance, that $\mathbb{CH}$ would be proved in near future whereas the Suslin hypothesis would be refuted). The two cardinals that were evoked by Vladimirov in [1] are very popular now in logic and general topology.

Continuing the investigations of the first part of [1], in 1973 A. I. Veksler, using $\mathbb{CH}$, has constructed a K-space in which $(*)$-incomplete are even the order intervals. The second part of [1] in the particular case of $X = \mathbb{R}^\tau$ was rediscovered 26 years later.

Vladimirov had completed his first research long before the article [1]. This research was devoted to his concept of strongly compact linear operator in a measurable function space. He published these results only in the late 1960s [6, 8]. His test for strong compactness in the weighted L_g^∞ space, with g a member of this space, was rediscovered by W. Schachermayer in 1986. We also mention the following important fact which was established by Vladimirov: each operator possessing an abstract norm in the sense of L. V. Kantorovich is integral and strongly compact. The ideas of these works of Vladimirov were further developed by his student A. V. Bukhvalov.

In the span of many years, Vladimirov returned periodically to the problem of the metric type of measurable functions, i.e., the complete system of invariants characterizing a measurable function up to mod 0 isomorphism of measure spaces. In the case of functions on a Lebesgue space, this problem was solved by V. A. Rokhlin in 1957. In the joint article [2] with A. A. Samorodnitskiĭ, Vladimirov indicated the class of measure spaces in which the distribution function not only serves as one

of the invariants of the metric type but rather determines it completely. The same article contains a system of invariants which enables us to discriminate between metric types over a wide class of spaces (furnished in general with a nonseparable measure).

In the joint article [9] with Yu. V. Shergin, Vladimirov gave an interesting intrinsic characterization of L^p spaces in the class of weakly rearrangement-invariant vector lattices by means of the groups of their isometric and order automorphisms.

The central topic of Vladimirov's research was the theory of Boolean algebras (BAs for short). A BA may be treated as an object of logic, algebra, topology, probability, analysis, and measure theory. Vladimirov was attracted by BAs as the objects of the last three disciplines. Many of his deep studies are devoted to the theory of BAs.

In the series of articles [3–5], Vladimirov addressed the problem of normability of a complete BA (recall that a BA is normable whenever it admits an essentially positive countably additive measure) as well as the problem of existence of such a measure satisfying an extra condition of invariance under a given group of automorphisms of the BA under study. This fundamental problem has attracted many authors (D. Maharam, A. G. Pinsker, J. Kelley, E. Hopf, A. Hajian and S. Kakutani, et al.). Vladimirov obtained many beautiful theorems which imply some results of these famous authors.

In 1965 Vladimirov obtained his Ph. D. thesis which was based on these articles (it also contained his other results in particular about the class of decomposable BAs he had introduced). The brilliant thesis gave rise to a vivid discussion. Only some formal reasons prevented the scientific council to award the applicant the highest doctor degree he deserved according to the prevailing opinion of the members of the council and audience.

In the articles [6, 8], Vladimirov addressed the following important and difficult problem. Let $\mathscr{X}$ be an NBA with measure μ, and let $\{\mathscr{X}_t\}$ be a (possibly, infinite) system of algebraically independent subalgebras of $\mathscr{X}$. Under which conditions does there exist a countably additive measure ν on $\mathscr{X}$ such that from

$$x_{t_1} \wedge x_{t_2} \wedge \dots \wedge x_{t_n} > \mathbf{0}$$

where $x_{t_i} \in \mathscr{X}_{t_i}$ and $t_i \neq t_k$ for $i \neq k$ it follows that

$$\nu(x_{t_1} \wedge x_{t_2} \wedge \dots \wedge x_{t_n}) = \nu(x_{t_1})\nu(x_{t_2})\dots\nu(x_{t_n})\,?$$

In this event, the subalgebras of the system $\{\mathscr{X}_t\}$ are called metrically ν-independent. If these subalgebras are generated by random functions

$\{f_t\}$ then the latter become independent with respect to the new measure. Note that in some "nice" cases ν is equivalent to μ. Vladimirov found a test for metric independence and exhibited some examples of its application.

In his last article (by the publication date), Vladimirov solved a very difficult problem of isomorphic classification of all pairs $(\mathscr{X}, \widetilde{\mathscr{X}})$, with $\mathscr{X}$ an NBA and $\widetilde{\mathscr{X}}$ a regular subalgebra of $\mathscr{X}$. He introduced two invariants that characterize such a pair completely (up to isomorphism). His result is a far-reading abstraction of the celebrated Maharam–Kolmogorov classification theorem.

Vladimirov also dealt with applications of the theory of BAs to general topology. The joint article [1] with B. A. Efimov is one of the first in the voluminous theory of cardinal invariants of topological spaces. In the article [1] joint with his student P. Zenf it is proved in particular that the regular subalgebras of the algebras of clopen subsets of a compact space coincide exactly with the subalgebras generating a partition continuous in the Kuratowski sense. These authors also constructed a new product of Boolean algebras that assigns to a pair of NBAs some *normed* BA recall that (the conventional product of NBAs may fail to be normable).

Vladimirov liked to work with young people. He supervised more than a dozen postgraduates. Two of his students achieved the degree of Sc. D.

Many generations of alumni of the Mathematics and Mechanics Department of Saint Petersburg State University remember Vladimirov as a gifted teacher. Methodological problems were always among the objects of his thought and concern. He happened to teach and lecture at various levels with very diverse members of the audience: he delivered many optional courses for future professional mathematicians as well as numerous compulsory courses not only for mathematicians; he spoke about the history of science and the methodology of education as well as teaching the pupils of secondary schools. He put much effort into his TV lectures for applicants to the university, which he delivered for more than 15 years. It is hard to overestimate this contribution to the education of the nation.

In 1969 Vladimirov published his monograph *Boolean Algebras.* This beautiful book may serve as an introduction to the state of the art for a mathematician of any speciality. At the same time, among the books on the general theory of BAs, this monograph is unique in exposing the theory from an analytical (and, partially, probabilistic) standpoint. Although the book exposes the traditional topics in sufficient detail, it is entirely original as regards its core. It also contains the extended versions of the author's contributions to the theory. Also, the intrinsic topologies of complete BAs are studied. In particular, the uniqueness

is established for a topology duly compatible with order. The book contains a classification of NBAs in the language of algebra and topology. Two translations of his book into German were printed.

During his last years, Vladimirov continued rewriting the book completely, his mortal illness notwithstanding. He managed to finish the task just before his death. This new book lies before the reader. Its scope is enlarged drastically; now the book includes the results of the articles [6, 8] (Chapter 10) and [7] (Chapter 9) in extended form; many new results on order topologies (Chapter 4), on homomorphisms of Boolean algebras (Chapter 5), and also the results of his students I. I. Bazhenov, A. V. Potepun, and A. A. Samorodnitskiĭ (Chapters 8 and 9).

A. V. Potepun undertook the editorial duties to finish the manuscript after Vladimirov's death. A tremendous help was offered by A. A. Samorodnitskiĭ who solely accomplished the compilation of the voluminous manuscript.

The publication of this book, crowning the scientific life and contributions of Denis Artem′evich Vladimirov, will be a valuable part of the memory of this broad-minded scholar and charming personality.

Friends and colleagues

Preface

This book consists of two parts. The first is devoted to the general theory of Boolean algebras. Chapters 0 and 1 collect the prerequisites that require no serious mathematical background. In particular, we exhibit the examples that enable the reader to see the possibilities of applying Boolean algebras to measure theory, probability, and functional analysis. The main content of the subsequent chapters comprises those sections of the theory of Boolean algebras which relate to these applications.

We mainly pay attention to complete Boolean algebras whose structure is examined in Chapter 2. The first part of the book also contains the extension theorems for continuous homomorphisms; we consider the topologies and uniformities related to order and expose the theory of lifting. Chapter 3 is entirely devoted to the realization of Boolean algebras; we consider Stone functors between the categories of Boolean algebras and totally disconnected spaces. Moreover, one of the chapters gives a sketch of the theory of vector lattices.

In the second part of the book we expose the metric theory of Boolean algebras. Here we study measure algebras. Alongside traditional matters such as the Lebesgue–Carathéodory Theorem, Radon–Nikodým Theorem, and Lyapunov Theorem on vector measures the reader will find the algebraic and metric classifications of probability algebras and their subalgebras, theorems about automorphism groups and invariant measures, etc. Much room is allotted to the problem of normability of a Boolean algebra, i.e., the problem of existence of an essentially positive totally additive measure. The closing chapter is devoted to the problem of algebraic and metric independence of subalgebras.

The general idea of the book compels us to pay less attention to some traditional questions (the applications to logic and cybernetics) in contrast to the prevailing tastes. We do not touch the problem of

foundation and always stand on the grounds of naive set theory, using the axiom of choice and its equivalents with no circumlocution.

The author tried to avoid any significant intersections with other books on the theory of Boolean algebras. Among the latter we must mention the celebrated monographs by R. Sikorski and P. Halmos as well as the huge recent treatise *Handbook of Boolean Algebras* edited by D. Monk and R. Bonnet. The existence of these books allows the author to concentrate on the metric aspects of the theory, especially in the concluding chapters.

Unfortunately, the so-called "Boolean valued analysis," a rapidly developed area of nonstandard analysis residing at the frontiers with algebra and model theory, falls completely beyond the limits of this book. However, it is beyond a doubt that more monographs and textbooks on Boolean valued analysis will appear soon, since this area of analysis is undergoing intensive study.

The author intended to make the book comprehensible to readers with diverse mathematical interests, that is why he abstained from attempts to make the exposition concise and to include the maximal number of results in a minimal volume. The main facts are usually furnished with complete proofs; however, we expect an active attitude from the reader. As was mentioned, the first two chapters of the book are comprehensible to a novice; the remaining part of the book presumes two years of the academic training that includes acquaintance with the basics of measure theory and general topology.

Formulating a theorem, we do not mention its author often; "mathematical folklore" occupies ample room in this book.

I used the content of my previous book published in 1969 under the title *Boolean Algebras* mainly in compiling the first part of the present edition.

D. A. Vladimirov

Introduction

We encounter many difficulties in attempting to define rigorously some general concepts rooted deeply in everyday scientific practice. For instance, the fundamental concept of "set" has no clear-cut definition, which, however, never prevents anyone from doing mathematics; it suffices to know the rules for handling the word "set" and master the corresponding symbolism.

In this book we focus our attention on another equally fundamental and seemingly far less definite concept of "event." Even the superficial analysis of situations in which we encounter the word "event" convinces us that it is hopeless to give this term a clear-cut definition. However, science never needs this definition; the interests of mathematics (and, first of all, probability theory) require some clearly stated axioms that describe the properties of systems of events. It is worth observing that we are actually dealing with SYSTEMS of events: no isolated event is possible in principle. Each event we discuss somehow somewhere sometimes is always surrounded by its next of kin, comprising a whole entity of related events.

The mathematical apparatus appropriate for describing systems of events has appeared first as a tool of symbolic logic. The creation of the "sentence algebra" or "proposition algebra" is customarily attributed to G. Boole (1815–1864); naturally, he had predecessors among which we must first of all mention Leibniz and the Bernoulli brothers. However, it is the Boole article of 1847[1] that has started the continuous flood of research which resulted in the flowering of mathematical logic which is one of the most characteristic features of mathematics of the twentieth century. It is Boole who indicated clearly in his volu-

[1] G. Boole [1].

minous monograph *An Investigation of the Laws of Thought on Which Are Founded the Mathematical Theories of Logic and Probabilities* the correlation of his calculus to the foundations of probability theory. This correlation rests on an analogy between "events" and "propositions" which allows a unique formal apparatus to serve to logic and probability theory. Roughly speaking, an "event" is that which can happen or not; whereas the "proposition" is that which can be true or false. There are certain and impossible events; propositions may be identically true or false. A causal relation is possible between events: one event may be a consequence of the other. Similarly, a logical relation is possible between propositions; one of them may follow from the other. To each event we may put in correspondence the proposition asserting that this event has happened. On the other hand, we may always interpret each proposition as asserting that some event has happened. This convinces us of the possibility of constructing a unique "calculus" that will, according to circumstances, be able to serve either as "calculus of propositions" or "calculus of events." It is this calculus that G. Boole created. During a half-century it then developed along "logical" lines. The first significant research into the axiomatics of probability theory appeared only in 1917; its author was S. N. Bernstein.[2]

Further research in this area is connected primarily with the contributions by A. N. Kolmogorov.[3] They have finally given solid grounds for probability theory and greatly influenced the related areas of mathematics and, in particular, measure theory.

This book is devoted to Boolean algebras. A Boolean algebra is an algebraic system that may be interpreted in various circumstances either as a system of events or as a system of propositions (other interpretations are also available). The axioms of Boolean algebra express the common features making the "events" and "propositions" relatives. The causal correlation between events or the logical correlation between propositions are described by formulas in the shape of inequalities. A Boolean algebra is an instance of a partially ordered set: The inequality $x \leq y$ expresses a "greater certainty" of the event y as compared with the event x or, if you please, a "greater plausibility" of the proposition y as compared with the proposition x. Each Boolean algebra must contain the greatest and least elements that correspond to "absolutely certain" and "absolutely impossible" events (or to "identically true" and "identically false" propositions). At last, each element must have a complement that may be interpreted as the "event opposite to a given event" or as the

[2] S. N. Bernstein [1].
[3] A. N. Kolmogorov [1].

"negation of a given proposition." We will not give the exact formulation right away: they will appear in due course.

The paucity of its axiomatics notwithstanding, the theory of Boolean algebras is rather meaningful. Therein we may find many difficult and deep problems many of which are still unsolved. These problems are rather diverse; they touch on logic and set theory as well as analysis and probability. The abundance of overlaps of the theory of Boolean algebras and surrounding mathematical disciplines reveals the former to be a relative of functional analysis to which it is intimately close also in the general mathematical style.

I

GENERAL THEORY OF BOOLEAN ALGEBRAS

Chapter 0

PRELIMINARIES ON BOOLEAN ALGEBRAS

Each Boolean algebra is a partially ordered set of a special form. Therefore, we start with some general facts and concepts relating to order.

1. Lattices

1.1 An introductory note

The main object of this book is a special poset,[1] a Boolean algebra. We usually denote a poset, which is by definition some underlying set with an order on it, by the same letter as the underlying set. Denoting inequalities, we always use the symbols

$$<,\ \leq,\ >,\ \geq$$

(as well as $\prec$ and $\succ$ on special occasions).

1.2 Classification of mappings

Let $\mathscr{X}$ and $\mathscr{Y}$ be two posets. A mapping Φ from $\mathscr{X}$ onto $\mathscr{Y}$ is an *(order) isomorphism* (or *isomorphic mapping*) if Φ is injective and preserves order, i.e. the inequalities

$$x \leq y \text{ and } \Phi(x) \leq \Phi(y)$$

are equivalent. Clearly, the inverse Φ^{-1} of an isomorphism Φ is an isomorphism too. If there is an isomorphism between two posets then the latter are called *isomorphic*. Isomorphic posets are often identified since

[1]The basics of the theory of partially ordered sets (= posets) are collected in Appendix A.

they are undistinguishable in regard to their order-theoretic properties. A mapping Φ is *isotonic* if from $x \leq y$ it follows that $\Phi(x) \leq \Phi(y)$.

Every isomorphism is always isotonic (the converse is false in general). An injective mapping Ψ from $\mathscr{X}$ onto $\mathscr{Y}$ is an *anti-isomorphism* or *dual isomorphism* if the inequalities $x \leq y$ and $\Psi(x) \geq \Psi(y)$ are equivalent. If there is an anti-isomorphism between two posets then the latter are called *anti-isomorphic* or *dually isomorphic*.

1.3 Bounds of a set

Let $\mathscr{X}$ be a poset and let E be a subset of $\mathscr{X}$. An element $y \in \mathscr{X}$ is an upper (lower) bound for E whenever the inequality $x \leq y$ $(x \geq y)$ holds for all $x \in \mathscr{X}$. The set of upper bounds for E is denoted by E^s, and the set of lower bounds for E is denoted by E^i. If E^s is nonempty then E is called *upper bounded* or *bounded above* or *bounded from above*. If E^i is nonempty then E is called *lower bounded* or *bounded below* or *bounded from below*. If an element z belongs to the intersection $E \cap E^s$ $(E \cap E^i)$ then z is the *greatest* (*least*) element of E. In expressions like $(E^s)^i$, we usually omit the parentheses and write E^{si}. The fact that the intersection $E^s \cap E^{si}$ $(E^i \cap E^{is})$ is nonempty means that there is a least (greatest) bound among all upper (lower) bounds for E; this bound is a *least upper bound* or a *supremum* (a *greatest lower bound* or *infimum*) for E. It is easy to show that the intersections $E^s \cap E^{si}$ and $E^i \cap E^{is}$ never contain more than a single element;[2] therefore, a least upper bound (a greatest lower bound), if existent, is necessarily unique.[3] The supremum of E is denoted by $\sup E$ and the infimum of E, by $\inf E$. If the elements of E are enumerated by some index set Ξ then we use the following notation:

$$\sup E \equiv \bigvee_{\xi \in \Xi} x_\xi, \qquad \inf E \equiv \bigwedge_{\xi \in \Xi} x_\xi.$$

Finally, if E consists of finitely many elements $x_1, x_2, \dots, x_n$ then we write[4]

$$\sup E \equiv x_1 \vee x_2 \vee \dots \vee x_n \qquad \text{or} \qquad \sup E \equiv \bigvee_{k=1}^{n} x_k,$$

[2]Indeed, assume for instance that $x, y \in E^s \cap E^{si}$. Then $x \leq y$ since $y \in E^s$ and $x \in E^{si}$. Analogously, we see that the reverse inequality $y \leq x$ holds. Hence, $x = y$.

[3]We thus use the definite article *the* speaking of the least upper bound and greatest lower bound or the supremum and infimum of a set. (S. S. Kutateladze)

[4]The terms "join" and "meet" are in common parlance for finite families. (S. S. Kutateladze)

$$\inf E \equiv x_1 \wedge x_2 \wedge \cdots \wedge x_n \qquad \text{or} \qquad \inf E \equiv \bigwedge_{k=1}^{n} x_k.$$

We now list the main properties of upper and lower bounds.

1°. *If $E_1 \subset E_2$ then $E_1^s \supset E_2^s$ and $E_1^i \supset E_2^i$.*

2°. *If $E_1 \subset E_2$ and the suprema $\sup E_1$ and $\sup E_2$ ($\inf E_1$ and $\inf E_2$) exist, then*

$$\sup E_1 \leq \sup E_2 \quad (\inf E_1 \geq \inf E_2).$$

3°. *The relations $x \leq y$, $x = x \wedge y$, and $y = x \vee y$ are equivalent.*

4°. *Let $\mathscr{E}$ be a nonempty class of subsets in $\mathscr{X}$, to each of which there is a supremum (infimum). Furthermore, suppose that there is a supremum (infimum) to the collection of these suprema (infima). Then the latter serves as the supremum (infimum) of the union $F \equiv \bigcup_{E \in \mathscr{E}} E$.*

The above property is called *associativity* of suprema and infima. This property can be expressed by the following formulas:

$$\sup F = \bigvee_{E \in \mathscr{E}} \sup E; \quad \inf F = \bigwedge_{E \in \mathscr{E}} \inf E,$$

assuming that the supremum and infimum on the right-hand sides exist.

Propositions 1°–3° are obvious. We only dwell upon proving associativity, restricting exposition to the case of least upper bounds. Given $E \in \mathscr{E}$, let $y_E \equiv \sup E$ and denote

$$y \equiv \bigvee_{E \in \mathscr{E}} y_E.$$

If x is an arbitrary element of F then we may find some $E \in \mathscr{E}$ such that $x \in E$. Therefore, $x \leq y_E \leq y$ and $y \in F^s$. Now, taking an arbitrary $z \in F^s$, we observe that, according to 1°, we have $z \in E^s$ for each $E \in \mathscr{E}$, i.e., $z \geq y_E$. We see that z is an upper bound for the set of all y_E's and, therefore, $z \geq \bigvee_{E \in \mathscr{E}} y_E \equiv y$. We have proved that y is the least of all upper bounds for F, i.e. the supremum of F, and, in particular, we have established the existence of this bound.[5]

The following obvious propositions speak about transformation of bounds under isomorphisms and dual isomorphisms.

5°. *If Φ is an isomorphism then*

$$\Phi(E^s) = [\Phi(E)]^s, \qquad \Phi(E^i) = [\Phi(E)]^i.$$

[5]The equality $y = \sup F$ means the CONJUNCTION: F has a supremum and this supremum is equal to y. The same is true for the equality $y = \inf F$.

6°. *If Φ is an isomorphism then*

$$\Phi(\sup E) = \sup \Phi(E), \qquad \Phi(\inf E) = \inf \Phi(E)$$

whenever at least one of the suprema (infima) in the equality exists.

7°. *If Ψ is a dual isomorphism then*

$$\Psi(E^s) = [\Psi(E)]^i, \qquad \Psi(E^i) = [\Psi(E)]^s.$$

8°. *If Ψ is a dual isomorphism then*

$$\Psi(\sup E) = \inf \Phi(E), \qquad \Phi(\inf E) = \sup \Phi(E)$$

with the same stipulation as in 6°.

In conclusion, we point out the obvious ISOTONICITY of the operations $\vee$ and $\wedge$.

9°. *If $x_1 \le y_1$, $x_2 \le y_2, \dots, x_n \le y_n$ then*

$$\bigvee_{k=1}^{n} x_k \le \bigvee_{k=1}^{n} y_k, \qquad \bigwedge_{k=1}^{n} x_k \le \bigwedge_{k=1}^{n} y_k.$$

1.4 The duality principle

Despite the evidence of Propositions 7° and 8° in 0.1.3, their role is very important. Let $\mathfrak{K}$ be a class of partially ordered sets that, together with each partially ordered set $\mathscr{X} \in \mathfrak{K}$, contains a set dually isomorphic to $\mathscr{X}$. Considering Propositions 7° and 8°, we may conclude the following: *if, given a proposition that concerns order properties and is valid for all $\mathscr{X} \in \mathfrak{K}$, we replace all inequalities by the reverse inequalities, least upper bounds by greatest lower bounds, etc.; then we obtain a proposition that is still valid for all $\mathscr{X} \in \mathfrak{K}$.* We call this the *general duality principle* for posets.[6]

1.5 Two important examples

Example A. Let $\mathscr{Q}$ be an arbitrary nonempty set and let Σ be a subset of its powerset. Introduce a partial order on Σ by assuming $e_1 \le e_2$ whenever $e_1 \subset e_2$. It is clear that the relation $\le$ is transitive and the inequalities $e_1 \ge e_2$ and $e_1 \le e_2$ imply the equality $e_1 = e_2$. The order just defined for an arbitrary system of sets is usually referred to as *natural* or *by inclusion*, and so this system is *inclusion-ordered.* If E is a class of subsets of Σ then each set e_0 that contains each $e \in E$ is an

[6]The term "reversal" is also in common parlance. (S. S. Kutateladze)

UPPER BOUND for E. The concept of a lower bound in this example is interpreted similarly. (In general, situations are possible in which upper or lower bounds do not exist at all.)

Example B. Let $\mathscr{Q}$ be again an arbitrary nonempty set. Consider a totality $\mathscr{S}$ of real functions defined on $\mathscr{Q}$. We introduce a partial order on $\mathscr{S}$ as follows: $f \le g$ whenever the inequality $f(q) \le g(q)$ holds for all $q \in \mathscr{Q}$. The axioms of a partial order are easily verified in much the same way as in the previous example. This order on a set of real functions is called *natural* or *pointwise*. It is clear that each common MAJORANT f_0 for a set E of functions in $\mathscr{S}$ is an upper bound (the inequality $f_0(q) \ge f(q)$ must hold for all $f \in E$ and $q \in Q$).

Inspecting these two examples, we point out the most important cases in which suprema and infima exist. Let a class of sets $E \subset \Sigma$ be such that the union $\overline{e} \equiv \bigcup_{e \in E} e$ (or the intersection $\underline{e} \equiv \bigcap_{e \in E} e$) belongs to Σ. Then $\overline{e} = \sup E$ (and $\underline{e} = \inf E$).

Indeed, it is clear that $\overline{e}$ ($\underline{e}$) is the least (greatest) set in the natural order which includes all $e \in E$ (lies in all $e \in E$). This means that $\overline{e}$ is the supremum of E (and $\underline{e}$ is the infimum of E).

In Example B, the supremum always exists for every set E such that the function f_0 defined by the equality

$$f_0(q) = \sup_{f \in E} f(q) \qquad (q \in \mathscr{Q})$$

belongs to $\mathscr{S}$. The function f_0 is the supremum of E. Analogously, the function

$$g_0(q) = \inf_{f \in E} f(q) \qquad (q \in \mathscr{Q})$$

is the infimum of E whenever it belongs to $\mathscr{S}$. Verification is left to the reader.

Examples A and B provide us with a good possibility of illustrating the notion of an isomorphism. Consider again an arbitrary system Σ of subsets in $\mathscr{Q}$ and take as $\mathscr{S}$ the system of all characteristic functions[7] χ_e, with $e \in \Sigma$. Denote by Φ the mapping that associates with each $e \in \Sigma$ the characteristic function χ_e of e. It is clear that Φ presents a bijection between Σ and $\mathscr{S}$. The inclusion $e_1 \subset e_2$ means that $\chi_{e_1} \le \chi_{e_2}$; therefore, the inequalities $e_1 \le e_2$ and $\Phi(e_1) \le \Phi(e_2)$ are equivalent. Thus, Φ is an isomorphism and Σ and $\mathscr{S}$ are isomorphic. Roughly

[7]The *characteristic function* or *indicator* of a set e is defined by the equality

$$\chi_e(q) = \begin{cases} 0, & q \notin e, \\ 1, & q \in e. \end{cases}$$

speaking, from an order-theoretic viewpoint, it does not matter whether we consider sets or their characteristic functions.

1.6 The concept of a lattice

A partially ordered set $\mathscr{X}$ is called a *lattice* if its every two-element subset $\{x, y\} \subset \mathscr{X}$ possesses the supremum or *join* $x \vee y$ and the infimum or *meet* $x \wedge y$ in $\mathscr{X}$.

Lemma 1. *Each nonempty finite subset of a lattice possesses the join and meet.*

The lemma can be easily proved by induction on using associativity of suprema and infima.

1.7 Distributivity

An important role in the theory of lattices is played by various conditions known as "distributive laws." We only present here the simplest of these laws: A lattice $\mathscr{X}$ is *distributive* if, for all x, y, z in $\mathscr{X}$, the following is satisfied:

$$(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z). \tag{1}$$

Note that the inequality

$$(x \vee y) \wedge z \geq (x \wedge z) \vee (y \wedge z) \tag{1*}$$

holds in every lattice. This is immediate from the inequalities

$$x \wedge z \leq (x \vee y) \wedge z, \qquad y \wedge z \leq (x \vee y) \wedge z,$$

each of which is obvious. Therefore, the proof of distributivity practically reduces to demonstrating the reverse inequality for (1*):

$$(x \vee y) \wedge z \leq (x \wedge z) \vee (y \wedge z).$$

The "dual" form of (1) is as follows:

$$(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z). \tag{2}$$

The proof of the following interesting assertion is left to the reader: *For a lattice $\mathscr{X}$ to be distributive, it is necessary and sufficient that* (1) *be valid for all $x, y, z \in \mathscr{X}$.*

In fact, we thus obtain another equivalent definition of distributivity.

1.8 Partially ordered sets with zero and unity

The least and greatest elements of a partially ordered set (if they exist) are called its *zero* and *unity*.

For instance, in Example A (see 0.1.5), we have considered a partially ordered set Σ consisting of subsets of some basic set $\mathscr{Q}$. If we additionally

require that the entire $\mathscr{Q}$ and the empty set $\varnothing$ belong[8] to Σ, then $\mathscr{Q}$ and $\varnothing$ obviously play the roles of zero and unity in Σ.

The zero and unity of a poset $\mathscr{X}$ are usually denoted by $\mathbf{0}$ and $\mathbf{1}$ and sometimes by $\mathbf{0}_{\mathscr{X}}$ and $\mathbf{1}_{\mathscr{X}}$. However, even if we simultaneously consider several posets, the same symbols $\mathbf{0}$ and $\mathbf{1}$ are applied to each of these posets.[9] Moreover, throughout the book, we conventionally denote by E^+ the set of all positive elements of E. In a poset with zero and unity, it is natural to assume that the supremum of the empty set equals $\mathbf{0}$ and the infimum of this set equals $\mathbf{1}$. (Of course, for a nonempty set E, we always have the inequality $\sup E \geq \inf E$.)

1.9 Examples

We turn again to Example A (see 0.1.5) and suppose that the basic set $\mathscr{Q}$ is an interval $[a, b]$ (with $a < b$) and the system Σ is constituted by all intervals lying in $\mathscr{Q}$ (open, half-open, and closed). The empty set also belongs to the system (as any interval of the form (p, p)). Denote the resulting partially ordered set by $\mathscr{I}$. Since the intersection of each family of intervals is again an interval, each (not necessarily finite) subset E of $\mathscr{I}$ possesses the infimum that coincides with the intersection of E. A union of intervals need not be an interval; however, this does not mean the absence of suprema: for each system $E \subset \mathscr{I}$, there is a least interval including all $e \in E$; this interval is the supremum of E. Therefore, the requirements of the definition of a lattice are redundantly satisfied in our case and $\mathscr{I}$ is a lattice. Obviously, $\mathscr{I}$ possesses zero and unity. However, this lattice is clearly not distributive.

Another example: the inclusion-ordered totality of all open sets in an arbitrary topological space $\mathscr{R}$ is a lattice. The supremum is now interpreted as the union of sets, while the infimum of a finite family is its intersection. As above, there are zero and unity, namely, the empty set $\varnothing$ and the entire $\mathscr{R}$. This lattice is obviously distributive. The closed sets, ordered by inclusion, constitute a lattice dually isomorphic to the former.

1.10 Disjoint elements. Complements

Let $\mathscr{X}$ be a partially ordered set with zero $\mathbf{0}$. Elements $x, y \in \mathscr{X}$ are called *disjoint*[10] if $x \wedge y = \mathbf{0}$. For disjointness, it is necessary and

[8] The Russian original uses the obsolete notation Λ instead of $\varnothing$. (S. S. Kutateladze)

[9] In the same manner, the symbols of inequalities $\leq$ and $\geq$, the signs of join $\vee$ and meet $\wedge$, etc., are used in this book for all posets (with rare exception).

[10] Such elements are sometimes called "orthogonal" or "inconsistent." The term "consistency" means the absence of disjointness.

sufficient that the pair x, y possess no nonzero lower bounds. In case x and y are disjoint, we write $x\,d\,y$. An element x is called *disjoint from a set* E if x is disjoint from every element of E; this fact is written as $x\,d\,E$. Finally, we say that a *set* E *is disjoint*, if its elements are pairwise disjoint: $x\,d\,y$ whenever $x, y \in E$ and $x \neq y$. If the system of sets Σ in Example A contains the empty set then every two elements of Σ with the empty intersection are necessarily disjoint.

A partially ordered set with zero $\mathbf{0}$ and unity $\mathbf{1}$ may contain pairs of disjoint elements with supremum unity. Each one element of such a pair is called a *complement* of the other. Thus, x is a complement of y (and y is a complement of x) if $x \wedge y = \mathbf{0}$ and, at the same time, $x \vee y = \mathbf{1}$.

As an example, we again consider the lattice $\mathscr{I}$ of all intervals included in $[a, b]$. It is clear that the elements $x = [a, c]$ and $y = (c, b]$ $(a < c < b)$ are complements of each other. We suggest the reader to prove that the element $x \equiv [c, d]$ $(a < c < d < b)$ has no complements.

2. Boolean algebras

2.1 Definition and the main properties

In this book, by a *Boolean algebra*[11] we mean a distributive lattice with distinct unity $\mathbf{1}$ and zero $\mathbf{0}$ whose every element possesses a complement. Thus, each BA always contains at least two elements. Every algebra containing only $\mathbf{0}$ and $\mathbf{1}$ is called *degenerate*. (The singletons are also regarded as Boolean algebras by some authors.)

Theorem 1. *Each element of a Boolean algebra possesses a unique complement.*

PROOF. Let y_1 and y_2 be complements of x. By distributivity, we then obtain

$$y_1 = y_1 \wedge \mathbf{1} = y_1 \wedge (x \vee y_2) = (y_1 \wedge x) \vee (y_1 \wedge y_2) = \mathbf{0} \vee (y_1 \wedge y_2) = y_1 \wedge y_2.$$

It follows that $y_1 \leq y_2$. The inequality $y_1 \geq y_2$ can be proved similarly. Therefore, $y_1 = y_2$, and the theorem is proved.

Thus, we may call a unique complement of an element x in a Boolean algebra, *the* complement of x, and denote it[12] by Cx (the denotations x', $-x$, $\overline{x}$, $\neg x$, and $\sim x$ are also common). The mapping C is thus defined from the BA $\mathscr{X}$ into itself which associates with each element $x \in \mathscr{X}$ its complement Cx.

We now indicate some important properties of the mapping C.

[11] Throughout this book we often use the abbreviation BA.

[12] This unfortunate choice of the Russian original freezes the popular letter C, which involves mostly innocent abusage in a few places, in particular, in Chapter 3. (S. S. Kutateladze)

1°. *For every x, the equality $C(Cx) = x$ holds.*

Indeed, if y is the complement of x then x is the complement of y. This yields 1°.

2°. *If $x \wedge y = \mathbf{0}$ then $y \leq Cx$.*

To this end, it is sufficient to establish the equality $Cx = Cx \vee y$. We have:

$$Cx = Cx \vee \mathbf{0} = Cx \vee (x \wedge y) = (Cx \vee x) \wedge (Cx \vee y)$$
$$= \mathbf{1} \wedge (Cx \vee y) = Cx \vee y.$$

(We have used distributivity here.) We may rephrase the property 2° by saying that *the complement of an element x is the greatest of the disjoint elements from x.*

3°. *The inequalities $x \leq y$ and $Cx \geq Cy$ are equivalent.*

It is sufficient to prove that the inequality $x \leq y$ implies $Cx \geq Cy$. Since $x \leq y$, we have $Cy \wedge x \leq Cy \wedge y = \mathbf{0}$. Clearly, $Cy\, d\, x$; hence, $Cy \leq Cx$ in view of 2°.

Proposition 1° implies that C *is an injection of the algebra $\mathscr{X}$ onto itself, and C coincides with its inverse C^{-1}.* Proposition 3° asserts that C *is a nontrivial dual isomorphism of the algebra $\mathscr{X}$ onto itself. Every Boolean algebra is thus dually isomorphic to itself.*

By Proposition 8° of 0.1.3, for every nonempty $E \subset \mathscr{X}$, the following hold:

$$C \bigvee_{x \in E} x = \bigwedge_{x \in E} Cx, \tag{3}$$

$$C \bigwedge_{x \in E} x = \bigvee_{x \in E} Cx. \tag{4}$$

These equalities are known as the *duality relations for Boolean algebras.* The exact meaning of the last equalities is as follows: if the supremum or infimum on one of the sides of either equality exists then the other side makes sense too and the corresponding relation holds. If E is finite then the above equalities hold without any stipulations. For the two-element set $E = \{x, y\}$, they take the following form:

$$C(x \vee y) = Cx \wedge Cy, \tag{5}$$

$$C(x \wedge y) = Cx \vee Cy. \tag{6}$$

2.2 The main Boolean operations

We already know three operations on elements of a Boolean algebra: the two operations $\vee$ and $\wedge$ and the operation C that is a dual isomorphism of the algebra onto itself. Operations f and g are conventionally

called *dual* to one another (or mutually dual) if they are connected by the identity

$$f(x_1, x_2, \ldots, x_n) = Cg(Cx_1, Cx_2, \ldots, Cx_n)$$

or, which is equivalent, by the identity

$$g(x_1, x_2, \ldots, x_n) = Cf(Cx_1, Cx_2, \ldots, Cx_n).$$

In view of the duality relations introduced above, the operations $\vee$ and $\wedge$ are mutually dual. The operations are of interest that are INVARIANT under the dual isomorphism C. An example of a "C-invariant" operation is presented by the binary operation of *symmetric difference* defined by the equality

$$|\, x - y \,| \equiv (x \wedge Cy) \vee (Cx \wedge y).$$

The symmetric difference of x and y is also denoted by $x \triangle y$ and, sometimes, by $x +_2 y$. The obvious identity

$$|\, Cx - Cy \,| \equiv |\, x - y \,|$$

expresses the "C-invariance" property of this operation. Endowed with the symmetric difference operation, an arbitrary BA becomes an abelian group whose every element is inverse to itself and the role of the neutral element is played by $\mathbf{0}$.

Another example of a C-invariant binary operation is presented by the dual of the previous operation, $\sim$ ("equivalence") which is defined by the equality $x \sim y \equiv C \,|\, x - y \,|$ or, in other words,

$$x \sim y \equiv (x \vee Cy) \wedge (Cx \vee y).$$

It is clear that

$$x \sim y = Cx \sim Cy.$$

This operation is often employed in logic.

The following theorem is obvious.

Theorem 2. *Each of the relations*

$$|\, x - y \,| = \mathbf{0},$$

$$x \sim y = \mathbf{1},$$

$$x = y$$

implies the other two.

Thus, each of the elements $| x - y |$ and $x \sim y$ can be regarded as some MEASURE OF PROXIMITY between x and y.

Another binary Boolean operation $\to$ (*implication*) is defined by the equality[13] $x \to y \equiv y \vee Cx$. It is easy to verify that the relations $x \leq y$ and $x \to y = \mathbf{1}$ are equivalent.

We mention the *Sheffer stroke*, $|$. It is introduced by the equality[14]

$$x \mid y \equiv Cx \wedge Cy$$

and is remarkable by the fact that the other operations $\vee$, $\wedge$, and C can be expressed through it. The corresponding formulas will be presented below.

We now define the operations of *addition* and *subtraction* for elements of a BA. Let E be an arbitrary DISJOINT set. If E possesses a supremum then the latter is called the *sum* or *disjoint sum* of E. In this case, we write

$$y = \sum_{x \in E} E$$

rather than $y = \sup E$. For finite disjoint sets, we use the notation

$$y = x_1 + x_2 + \cdots + x_n \quad \text{or} \quad y = \sum_{k=1}^{n} x_k.$$

Of course, this addition is commutative. The sense is also clear of the symbols

$$\sum_{\xi \in \Xi} x_\xi, \quad \sum_{k=1}^{\infty} x_k$$

which are applied to FAMILIES of elements. We point out that by using the signs $+$ and $\sum$ in the sequel, we thus imply that the elements of the set or family under consideration are PAIRWISE DISJOINT.

If $x \leq y$ then the *difference* $y - x$ is defined as the element $z = y \wedge Cx$. It is easy to see that this is a unique element satisfying the relation $x + y = z$.

These operations on elements lead naturally to the OPERATIONS ON SETS:

$$E_1 \vee E_2 \equiv \{y \mid \ y = x_1 \vee x_2,\ x_1 \in E_1,\ x_2 \in E_2\},$$
$$E_1 \wedge E_2 \equiv \{y \mid \ y = x_1 \wedge x_2,\ x_1 \in E_1,\ x_2 \in E_2\},$$
$$CE \equiv \{y \mid \ y = Cx,\ x \in E\}.$$

---

[13]The symbol $\Rightarrow$ is often used instead of $\to$.

[14]Cf. H. M. Sheffer [1]. Many authors apply the same term to the dual operation.

If one of the sets is a singleton then we write $u \vee E$ or $u \wedge E$ instead of $\{u\} \vee E$ or $\{u\} \wedge E$. The sense is also clear of the notations $E_1 \vee E_2 \vee \dots \vee E_n$, $E_1 \wedge E_2 \wedge \dots \wedge E_n$, etc.

2.3 The distributive law in a Boolean algebra

According to the main definition, the distributive law (1) (see 0.1.7) must be valid in every BA. A stronger assertion turns out to be true:

Theorem 3. *Let $\mathscr{X}$ be an arbitrary BA and let E be a subset of $\mathscr{X}$ that possesses a supremum. Then, for every element $x \in \mathscr{X}$, the following equality holds:*

$$x \wedge \bigvee_{y \in E} y = \bigvee_{y \in E} (x \wedge y), \tag{7}$$

where the supremum on the right-hand side exists. (Cf. 6° in 0.1.3.)

Using the notation at the end of 0.2.2, we can rewrite (7) as follows:

$$x \wedge \sup E = \sup(x \wedge E). \tag{7'}$$

PROOF. Since $x \wedge y \leq x \wedge \sup E$ for all $y \in E$; therefore,

$$\sup(x \wedge E) \leq x \wedge \sup E.$$

Suppose that $z \in (x \wedge E)^s$. Given $y \in E$, obtain

$$z \vee Cx \geq (x \wedge y) \vee Cx = (x \vee Cx) \wedge (y \vee Cx) = y \vee Cx \geq y.$$

Whence, $z \vee Cx \geq \sup E$ and

$$z = z \vee \mathbf{0} = z \vee (x \wedge Cx) = (z \vee x) \wedge (z \vee Cx)$$

$$\geq (z \vee x) \wedge \sup E \geq x \wedge \sup E.$$

Since z is an arbitrary element of $(x \wedge E)^s$, we conclude that

$$x \wedge \sup E = \sup(x \wedge E).$$

The proof of the theorem is complete.

By duality, from the above theorem we obtain the following

Corollary. *If E possesses an infimum then, for every $x \in \mathscr{X}$, the equalities hold:*

$$x \vee \bigwedge_{y \in E} y = \bigwedge_{y \in E} (x \vee y); \tag{8}$$

$$x \vee \inf E = \inf(x \vee E). \tag{8'}$$

We also present the following formula that expresses the finite distributive law in the most general form and is useful in various calculations:

$$\bigwedge_{i=1}^{n} \bigvee_{k=1}^{m_i} x_{ik} = \bigvee_{\substack{1 \le k_1 \le m_1 \\ 1 \le k_2 \le m_2 \\ \cdots\cdots\cdots\cdots \\ 1 \le k_n \le m_n}} (x_{1k_1} \wedge x_{2k_2} \wedge \cdots \wedge x_{nk_n}). \tag{7*}$$

The proof of the last identity can be obtained by consecutive application of (7). In the sequel, we will outline another, most practical, way of establishing similar identities by using set-theoretic concepts.

2.4 The simplest examples of Boolean algebras

Example 1. In order to obtain the first example of a Boolean algebra, we turn to Example A in 0.1.5. Take as Σ the system of ALL subsets of the basic set $\mathscr{Q}$. From what was said in 0.1.9, it follows that Σ is a lattice with zero and unity:

$$x \vee y = x \cup y, \quad x \wedge y = x \cap y, \quad \mathbf{0} = \varnothing, \quad \mathbf{1} = \mathscr{Q}.$$

This lattice is distributive, since the operations $\vee$ and $\wedge$ coincide in our case with the set-theoretic operations $\cup$ and $\cap$ whose distributivity is well known: the equality

$$x \vee (y \wedge z) = (x \wedge y) \vee (x \wedge z)$$

is equivalent to the obvious relation $x \cap (y \cup z) = (x \cap y) \cup (x \cap z)$. The COMPLEMENT Cx of an element x is its set-theoretic complement in our case, i.e. the difference $\mathscr{Q} \setminus x$. (Obviously, $x \cap (\mathscr{Q} \setminus x) = \varnothing$ and $x \cup (\mathscr{Q} \setminus x) = \mathscr{Q}$.) Finally, in view of the fact that $\mathscr{Q}$ is not empty, it is clear that the system Σ contains at least two different elements. Thus, *the totality of all subsets of an arbitrary nonempty set $\mathscr{Q}$, endowed with the natural order, is a Boolean algebra.*[15] We denote this BA by $2^{\mathscr{Q}}$.

We point out the following fact that is important for the future presentation: *each (not necessarily finite) totality of elements in the BA $2^{\mathscr{Q}}$ possesses a supremum and an infimum that coincide with the union and the intersection of all sets in the totality.*

Theorem 4. *For the Boolean algebras $2^{\mathscr{Q}_1}$ and $2^{\mathscr{Q}_2}$ to be isomorphic, it is necessary and sufficient that the sets $\mathscr{Q}_1$ and $\mathscr{Q}_2$ have the same cardinality.*

[15]This BA $2^{\mathscr{Q}}$ is called the *boolean* of $\mathscr{Q}$. (S. S. Kutateladze)

Proof. Necessity: Let Φ be an isomorphism from the BA $\mathscr{X}_1 \equiv 2^{\mathscr{Q}_1}$ onto $\mathscr{X}_2 \equiv 2^{\mathscr{Q}_2}$. It is clear that the mappings Φ and Φ^{-1} take singletons into singletons. Consequently, there exists a bijection between $\mathscr{Q}_1$ and $\mathscr{Q}_2$, i.e., the two sets are equivalent.

Sufficiency: If $\mathscr{Q}_1$ and $\mathscr{Q}_2$ have the same cardinality then there exists a bijection φ from $\mathscr{Q}_1$ onto $\mathscr{Q}_2$. Associate with each $e \subset \mathscr{Q}$ the set $\varphi(e)$ (the image of e). It is easy to see that we have thus established an isomorphism between the algebras $\mathscr{X}_1$ and $\mathscr{X}_2$.

Example 2. Let $\mathscr{Q}$ be again an arbitrary nonempty set. Consider the totality $X_{\mathscr{Q}}$ constituted by the characteristic functions of all subsets of $\mathscr{Q}$ (in other words, the totality of all functions with values 0 or 1). As in Example B of 0.1.5, we assume that this set of functions is endowed with the natural pointwise order. Then, as was noted at the end of Subsection 0.1.5, by associating with each set $e \subset \mathscr{Q}$ its characteristic function χ_e, we obtain an isomorphism of the two partially ordered sets, each of which is a Boolean algebra.

Thus, the Boolean algebras $2^{\mathscr{Q}}$ and $X_{\mathscr{Q}}$ are isomorphic. The latter of these algebras is of constant use as a mathematical model in formal logic and the theory of contact networks. Indeed, a "proposition" is that which can be true or false "depending on circumstances." A "contact" (an "on-off switch") is that which can transmit current or not. We abstract exposition from the context of propositions and are not interested in the technical structure of contacts. A push-button of a door bell and an oil switch designed for a kiloampere current and containing various relays and contacts stand equal before us. In both cases we deal with objects that can be only in two inconsistent states. The simplest mathematical analog of such an object is a function with only two values: 0 and 1. If we wish to construct a mathematical model for a system of propositions or a contact network then we should introduce a totality of two-valued functions defined on a distinguished set T or, which is the same, the totality of characteristic functions on T. In applied logic[16] as well as in the network theory, the set T is usually finite. The algebra X_T is the desired model. To illustrate the role of the set T, we can interpret T as a set of values for a parameter which is responsible for verity or falsity of the propositions (in the case of "logical" interpretation) or for the state of all contacts (in the case of "networks" interpretation). In the second case, it is more convenient to consider this parameter as time.

Suppose that a function $x \in X_T$ takes the value unity (zero) at a point $t = t_0$. On the "networks" language, we describe this fact by saying

[16]In essence, the "propositional variables" under consideration are "unary predicates" with T an "object domain."

that *the switch x is on (off)* at time t_0. Finally, in the "propositional language," we will say that *the proposition x is true (false)* for $t = t_0$. Translation to the set-theoretic language is left to the reader.

In conclusion, we present a dictionary that allows us to translate from one of these languages to the other. Table 1 below contains interpretations in different languages of some relations between elements of an algebra $\mathscr{X}$ isomorphic to the algebras 2^T and X_T. A set, a characteristic function, and a proposition that correspond to each other are denoted by the same letter. Interpretations in the language of the sentence algebra are indicated with the help of so-called "truth tables," where the symbols $\top$ and $\bot$ stand for "true" and "false." These tables are self-evident.

We present another "logical" interpretation of the duality relations (5) and (6):

$$C(x \vee y) = Cx \wedge Cy, \qquad C(x \wedge y) = Cx \vee Cy.$$

The last equalities express the rules of negation for disjunctions and conjunctions: *the disjunction $x \vee y$ is false if and only if both negations Cx and Cy are true; the conjunction $x \wedge y$ is false if and only if at least one of the negations Cx or Cy is true.* These rules are sometimes called "de Morgan laws" in logic.

We also note that the main identities $x \vee Cx = \mathbf{1}$ and $x \wedge Cx = \mathbf{0}$ are logically interpreted as THE LAW OF EXCLUDED MIDDLE or TERTIUM

NON DATUR: *one and only one of the propositions x and Cx is always true.* The logics without this principle are associated with more complicated partially ordered systems: the Brouwer and Heyting algebras took the place of Boolean algebras.[17]

Applications of Boolean algebras to logic and cybernetics are well elucidated in the literature.

Algebras of *Boolean functions* are used especially often. These are the algebras of the form

$$X_{X_Q} \equiv \mathscr{X}, \tag{9}$$

where $Q = \{1, 2, \dots, n\}$ is an interval of naturals. In this situation, X_Q consists of all n-tuples (binary numbers)

$$\chi = (\chi_1, \chi_2, \dots, \chi_n), \qquad \chi_i = 0, 1,$$

that are sometimes interpreted as "vertices of the n-dimensional cube." There are 2^n of them. The binary functions (with values 0 or 1) on X_Q are called *Boolean functions*. There are 2^{2^n} of them.

Algebras of Boolean functions are interesting in many aspects. They belong to the class of the so-called FREE BAs. We will consider them in more detail in the next chapter.

Consider several other examples of Boolean algebras.

Example 3.[18] Let $\mathscr{X}_p$ be the totality of all naturals of the form

$$n = 1 \cdot p_1 \cdot p_2 \cdots p_{k(n)},$$

where the factors p_i are prime, pairwise distinct, and not greater than p. Since there is a natural bijection between these numbers and subsets of the set P of all primes in the interval $[0, p]$, the totality $\mathscr{X}_p$ becomes a Boolean algebra isomorphic to 2^P. In this situation, the inequality $n \leq m$ means that m is a multiple of n. The product

$$\prod_{q \in P} q$$

is unity, while the number 1 is the zero element. The role of the supremum for a set of numbers is played by their least common multiple, whereas the infimum is the greatest common divisor.

Example 4. Consider an arbitrary complete orthonormal system $\mathfrak{E}$ of elements in a Hilbert space H.[19] By an $\mathfrak{E}$-subspace we mean each

[17] Cf. H. Rasiowa and R. Sikorski [1].

[18] Cf. S. N. Bernstein [1].

[19] The reader may assume H to be a finite-dimensional vector space, real or complex.

subspace of H spanned by some subset $\mathfrak{E}' \subset \mathfrak{E}$. The zero subspace is supposed to be spanned by the empty subset of $\mathfrak{E}$. There is a natural bijection from the system $\mathscr{L}_{\mathfrak{E}}$ of all $\mathfrak{E}$-subspaces onto the system of all subsets in $\mathfrak{E}$. This allows us to regard $\mathscr{L}_{\mathfrak{E}}$ as a Boolean algebra isomorphic to $2^{\mathfrak{E}}$. In this case, the inequality $L_1 \leq L_2$ is equivalent to the inclusion $L_1 \subset L_2$ and means that L_2 is representable as the orthogonal sum

$$L_2 = L_1 \oplus L,$$

with $L \in \mathscr{L}_{\mathfrak{E}}$. The role of unity is played by the entire H, while the zero of $\mathscr{L}_{\mathfrak{E}}$ is the zero subspace. The Boolean complement for an $L \in \mathscr{L}_{\mathfrak{E}}$ is the orthogonal complement $H \ominus L$.

REMARK. The inclusion-ordered system of ALL subspaces of a Hilbert space H also turns out to be a lattice with zero and unity. However, in this case, we do not obtain a Boolean algebra: the distributive law is not obeyed. This system plays an important role in quantum mechanics.[20]

Example 4*. The orthogonal system $\mathfrak{E}$ of the previous example could be constituted in particular by the eigenvectors of some compact selfadjoint operator A acting in the space H. Then the $\mathfrak{E}$-subspaces are nothing else but invariant subspaces of A. It can be shown that the inclusion-ordered system of all invariant subspaces of an arbitrary (not necessarily compact) selfadjoint operator in H is a Boolean algebra. The role of unity in this algebra is played, as above, by the entire H; the Boolean complementation coincides with the orthogonal complementation. This remains valid in the case when the operator is not bounded and is defined not on the entire space H but on a dense subspace of H.

We now continue acquaintance with the simplest examples of Boolean algebras. Let Q be again an arbitrary nonempty set (the "space"). We may consider not the class 2^Q of ALL subsets of Q but some nonempty PART $\mathscr{X}_0 \subset 2^Q$ that is also ordered by inclusion. For the resultant partially ordered set to be a Boolean algebra, we have to impose some additional requirements on $\mathscr{X}_0$. In most cases, the totality $\mathscr{X}_0$ is assumed to be a so-called "algebra of sets" (or "field of sets"). As is well known, this means that

1) if $e_1, e_2 \in \mathscr{X}_0$ then $e_1 \cup e_2 \in \mathscr{X}_0$;

1′) if $e_1, e_2 \in \mathscr{X}_0$ then $e_1 \cap e_2 \in \mathscr{X}_0$;

2) if $e \in \mathscr{X}_0$ then $Q \setminus e \in \mathscr{X}_0$.

[20] Cf. G. W. Mackey [1].

We note immediately that, as is easily seen from the definition, every algebra of sets must contain the entire Q and the empty set $\varnothing$. Furthermore, it is important to observe that the conditions presented in the definition are not independent: 1) and 2) imply 1′), and 1′) and 2) imply 1). This is clear from the following identities:

$$e_1 \cap e_2 = Q \setminus [(Q \setminus e_1) \cup (Q \setminus e_2)];$$

$$e_1 \cup e_2 = Q \setminus [(Q \setminus e_1) \cap (Q \setminus e_2)].$$

By way of usual induction, it is easy to verify that every algebra of sets contains the unions and intersections of its arbitrary finite systems of sets. An algebra of sets can be briefly described as a totality of subsets of the basic space Q which is closed under the main set-theoretic operations $\cup$, $\cap$, and $\setminus$ and includes Q.

The arguments, demonstrating that the system 2^Q of ALL subsets of Q is a Boolean algebra, can be applied word for word to an arbitrary algebra of sets. Consequently, each algebra of sets is a Boolean algebra with respect to the natural order. Such an algebra is automatically associated with the isomorphic Boolean algebra of the corresponding characteristic functions.

Example 5. Let Q be the interval $[0, 1]$ and let $\mathscr{X}_0$ be the system of LEBESGUE MEASURABLE subsets of Q. It is well known that the class of measurable sets is closed under all main set-theoretic operations applied to countable families of sets. Hence, $\mathscr{X}_0$ is an algebra of sets and thus a Boolean algebra (strictly smaller than 2^Q). In this example, every Lebesgue measurable set on the real axis can be taken as Q instead of $[0, 1]$.

Example 6. Similarly, we can consider the totality of all BOREL subsets of the interval

$$[0, 1] \equiv Q.$$

By analogy with the previous example, this is an algebra of sets and thus a Boolean algebra.

Example 7. Let Q be the square:

$$Q \equiv \{(s, t) \mid \ 0 \leq s, t \leq 1\}.$$

Construct $\mathscr{X}_0$ by comprising all "vertical cylinders," i.e. the sets constituted by vertical segments.

More precisely, the containment $e \in \mathscr{X}_0$ means that the relation $(s_0, t_0) \in e$ implies $(s_0, t) \in e$ for all t. As is easily seen, the totality of all cylinders is an algebra of sets. Consequently, $\mathscr{X}_0$ is a Boolean algebra with respect to the natural order. This algebra is isomorphic to the algebra of all subsets of the interval. For verifying existence of an

isomorphism, we may associate with each $e \in \mathscr{X}_0$ its projection to the axis of abscissas.

While discussing prerequisites, we have mentioned another important Boolean algebra.

In measure theory and the related areas of mathematics (for instance, in ergodic theory) there are many assertions concerning not individual measurable sets but rather the classes constituted by all "almost coincident" sets. In other words, the members of the same class can differ by a set of measure zero. We will see later that the system of such classes, endowed with an adequate order, turns out to be a Boolean algebra. We will be able to give a precise description of this algebra in the next chapter, after a preliminary acquaintance with the idea of FACTORIZATION. In the sequel, we permanently focus our attention on the algebra of these classes.

Example 8. Omitting details, we now adduce another example of a Boolean algebra important for the spectral theory of operators. Let $\mathscr{E}$ be an arbitrary σ-algebra of sets and let Σ be the class of all nonnegative countably additive set functions ("measures") defined on $\mathscr{E}$.

As is well known, two functions of this type are called equivalent if each of them is absolutely continuous with respect to the other. This equivalence defines some partition of Σ into cosets or disjoint classes of mutually equivalent functions.

Following A. I. Plesner,[21] we call these cosets *spectral types* or *Hellinger types.* A spectral type τ_1 is *subordinate* to a spectral type τ_2 whenever every function $\sigma_1 \in \tau_1$ is absolutely continuous with respect to an arbitrary function $\sigma_2 \in \tau_2$.

We denote the subordination relation by the symbol $\leq$. It is easy to see that the subordination is a partial order. We may prove that, endowed with this order, the system T of all spectral types turns out to be a distributive lattice with zero. An important property of this lattice consists in existence of the supremum and infimum of each bounded set.

If we distinguish an arbitrary nonzero type τ_0 and consider the set T_0 of all spectral types subordinate to τ_0 then we obtain a Boolean algebra. A proof of this fact can be found in the above-mentioned monograph[22] by A. I. Plesner or in the earlier article by A. I. Plesner and V. A. Rokhlin.[23] The role of unity in the Boolean algebra T_0 is played by the type τ_0, while the disjointness of types τ' and τ'' means the mutual singularity of arbitrary functions $\sigma' \in \tau'$ and $\sigma'' \in \tau''$.

---

[21] Cf. A. I. Plesner [1].
[22] A. I. Plesner [1].
[23] A. I. Plesner and V. A. Rokhlin [1].

The set T itself is not a Boolean algebra since T lacks unity. In order to obtain a Boolean algebra, we need to add some ideal elements to T, namely, all (possibly uncountable) formal sums of pairwise disjoint types. These sums are called *generalized spectral types*; the order relation extends to them in a natural way. In the resultant Boolean algebra, each set possesses a supremum and an infimum.

An inclusion-ordered system of sets can be a Boolean algebra BUT NOT AN ALGEBRA OF SETS. We present a very simple example.

Example 9. Consider the system of four sets

$$(0,1), \quad \Big(0,\frac{1}{2}\Big), \quad \Big(\frac{1}{2},1\Big), \quad \varnothing.$$

It is clear that this system is a Boolean algebra with respect to the natural order, with the interval $(0,1)$ playing the role of unity.

However, the system is not an algebra of sets: the join of the pair $\{(0,\frac{1}{2}),(\frac{1}{2},1)\}$ is the interval $(0,1)$, whereas the union $(0,\frac{1}{2})\cup(\frac{1}{2},1)$ does not belong to our system.

These examples notwithstanding, we may show that each BA is ISOMORPHIC to an algebra of sets. Namely, the following theorem holds:

Stone Theorem. *For each BA $\mathscr{X}$, there exist a set $\mathfrak{A}$ and an algebra of sets $\widetilde{\mathscr{X}} \subset 2^{\mathfrak{A}}$ such that $\mathscr{X}$ and $\widetilde{\mathscr{X}}$ are isomorphic as Boolean algebras.*

We have presented the Stone Theorem in the utmost simplicity. The complete statement and proof will appear below, in Chapter 3. Now, we only note that the representation of an arbitrary BA as an algebra of sets plays an important role in the theory of Boolean algebras and happens to be rather useful. In particular, the possibility of such a representation allows us to interpret each relation between finitely many elements of an algebra as a relation between sets. The proofs of various identities and inequalities thus reduce to establishing set-theoretic inclusions and can be practically replaced by considering rather general drafts ("Euler–Venn diagrams").

2.5 Some useful identities

We now list some relations that are valid in every BA. Their proofs can be easily carried out by the scheme outlined at the end of 0.2.4. However, it is also easy to check them directly without using the Stone theorem.

The following hold:

1°. $x = |\, y - |\, x - y \,||$;

2°. $|\, x \vee y - x \vee z \,| \ \le\ |\, y - z \,|$;

3°. $|\, x \wedge y - x \wedge z \,| \ \leq \ |\, y - z \,|$;

4°. $x \leq y \vee |\, x - y \,|$;

5°. $|\, x - y \,| \ = x \vee y - x \wedge y$;

6°. $|\, x - y \,| \ \leq \ |\, x - z| \vee \ |\, z - y \,|$;

7°. $(x \wedge Cy) \vee (Cx \wedge y) \vee (Cx \wedge Cy) = C(x \wedge y)$;

8°. $|\, x \vee y - z \vee u \,| \ \leq \ |\, x - z \,| \ \vee \ |\, y - u \,|$;

9°. $|\, x \wedge y - z \wedge u \,| \ \leq \ |\, x - z \,| \ \vee \ |\, y - u \,|$;

10°. $x \vee y = (x \mid y) \mid (x \mid y)$;

11°. $x \wedge y = (x \mid x) \mid (y \mid y)$;

12°. $Cx = x \mid x$.

Here $\mid$ stands for the Sheffer stroke introduced in 0.2.2. We have already mentioned that the remaining operations $\vee$, $\wedge$, and C can be expressed via this operation. The identities 10°–12° justify this fact. Such relations are sometimes useful in constructing electric circuits.

3. Additive functions on Boolean algebras. Measures. Relation to probability theory

3.1 The concept of an additive function

A real function φ on a BA $\mathscr{X}$ is called *additive* if, for every finite disjoint family E of elements in $\mathscr{X}$, the following holds:

$$\varphi\Big(\bigvee_{x \in E} x \Big) = \sum_{x \in E} \varphi(x). \tag{10}$$

We do not exclude the case in which some values of φ are equal to $+\infty$ or $-\infty$; however, we require that the sum in (10) make sense for all E. It is clear that, while proving additivity, we may restrict consideration to the case in which E consists of two elements, since the general case is easily settled by induction.

3.2 Quasimeasures

An additive function ψ on a BA $\mathscr{X}$ is called a *quasimeasure*[24] if all values of ψ are nonnegative. A quasimeasure is called finite if it never

[24] More frequently, such a function is called a "measure" in the literature. When using this term in the traditional meaning, we will usually place it within inverted commas.

attains $+\infty$. Every quasimeasure is isotonic: if $x \leq y$ then x and $y - x \equiv y \wedge Cx$ are disjoint; therefore,

$$\psi(y) = \psi(x) + \psi(y - x) \geq \psi(x).$$

Every BA possesses "sufficiently many" quasimeasures:

To a real number m and a nonzero element x of a Boolean algebra $\mathscr{X}$, there exists a finite quasimeasure ψ on $\mathscr{X}$ such that $\psi(x) = m$.

This fact follows from the Stone Theorem. Indeed, without loss of generality, we may assume that $\mathscr{X}$ is an algebra of subsets in some space Q (for instance, in the Stone space of $\mathscr{X}$). Since $x > \mathbf{0}$, there exists a point $q_0 \in x$. We now define a quasimeasure ψ by the following condition:

$$\psi(y) = \begin{cases} m & \text{if } q_0 \in y, \\ 0, & \text{if } q_0 \notin y. \end{cases}$$

It is clear that ψ is a finite quasimeasure and $\psi(x) = m$. We have used the possibility of representing a BA as an algebra of sets. As a matter of fact, the assertion that there are sufficiently many quasimeasures is itself a representation theorem in disguise.

The quasimeasure ψ of the above proof is said to be *supported* or *concentrated* at the point q_0 of the basic space. However, of the utmost interest are the quasimeasures of another kind which will be now considered. We first distinguish the class of *essentially positive* quasimeasures by collecting the quasimeasures that vanish only at zero: $\psi(x) = 0$ implies $x = \mathbf{0}$. The quasimeasures, supported at the points of Stone spaces, do not possess this property. Moreover, many Boolean algebras admit no essentially positive quasimeasure at all. We now indicate a class of Boolean algebras which contains every algebra with an essentially positive quasimeasure.

3.3 Boolean algebras with the countable chain condition

A BA $\mathscr{X}$ is said to satisfy the *countable chain condition* or to be of *countable type*[25] provided that $\mathscr{X}$ does not contain uncountable disjoint subsets.

Theorem 5. *If there is an essentially positive finite quasimeasure ψ on a BA $\mathscr{X}$ then $\mathscr{X}$ is a BA with the countable chain condition.*

Proof. Suppose that the claim is false. Then there exists an uncountable disjoint set E in $\mathscr{X}$. We may assume that $\mathbf{0} \notin E$. Define the

[25] In the literature of Russian provenance. (S. S. Kutateladze)

sets $E_1, E_2, \dots, E_n, \dots$ as follows:

$$E_n \equiv \left\{ x \in E \mid \ \psi(x) \geq \frac{1}{n} \right\} \qquad (n = 1, 2, \dots).$$

The equality $E = \bigcup_{n=1}^{\infty} E_n$ holds.[26] Therefore, at least one of the sets E_n is uncountable and, hence, infinite. Let this set be E_{n_0}. By choosing a natural m so that the inequality $m > n_0 \psi(\mathbf{1})$ be valid, we extract a subset E' from E_{n_0} which contains exactly m elements. Note the impossible inequality

$$\psi(\mathbf{1}) \geq \psi(\sup E') = \sum_{x \in E'} \psi(x) \geq \frac{m}{n_0} > \psi(\mathbf{1}).$$

This contradiction proves the theorem.

Theorem 5 provides only a necessary condition for existence of an essentially positive quasimeasure. It was in 1964 that H. Gaifman constructed a rather delicate example of a BA with the countable chain condition that does not possess an essentially positive measure.[27] The reader will easily prove that the algebra 2^Q satisfies the countable chain condition if and only if Q is countable.

3.4 Variations of an additive function

It is convenient to reduce the study of arbitrary additive functions to the case in which these functions are positive, i.e. quasimeasures. We describe a general scheme that is usual in such situations. Let φ be a finite real additive function on a BA $\mathscr{X}$. Associate with each $x \in \mathscr{X}$ the two numbers:

$$\varphi_+(x) \equiv \sup_{x' \leq x} \varphi(x'),$$

$$\varphi_-(x) \equiv -\inf_{x' \leq x} \varphi(x') = \sup_{x' \leq x} (-\varphi(x')).$$

Suppose that both numbers exist for all $x \in \mathscr{X}$. We thus obtain the two new functions φ_+ and φ_- on $\mathscr{X}$. They are called the *positive* and *negative variations* of φ. It is easy to verify validity of the following assertions:

1. φ_+ *and* φ_- *are additive and positive;*

2. $\varphi = \varphi_+ - \varphi_-$*;*

[26] This is the point at which we use the essential positivity of a quasimeasure.
[27] H. Gaifman [1].

3. *For each $x \in \mathscr{X}$, we have*

$$-\varphi_-(\mathbf{1}) \leq -\varphi_-(x) \leq \varphi(x) \leq \varphi_+(x) \leq \varphi_+(\mathbf{1}),$$

$$|\, \varphi(x) \,| \leq \varphi_+(x) + \varphi_-(x).$$

The sum $\varphi_+ + \varphi_-$ is called the *total variation* of φ and denoted by $|\, \varphi \,|$.

Thus, each additive real function with bounded image is representable as the difference of two quasimeasures. There are many such representations; that given by the equality 2 is called *Jordan* or "*Jordan decomposition.*" The following properties of Jordan decomposition are easy to verify:

4. *If ψ is a quasimeasure and $\varphi_+(x), \varphi_-(x) \geq \psi(x)$ for all x then $\psi(x) = 0$ for $x \in \mathscr{X}$;*

5. *If $\varphi = \psi_1 - \psi_2$ and ψ_1 and ψ_2 are quasimeasures then the inequalities $\psi_1(x) \geq \varphi_+(x)$ and $\psi_2(x) \geq \varphi_-(x)$ hold for all $x \in \mathscr{X}$.*

3.5 Totally additive and countably additive functions

A real function φ on $\mathscr{X}$ is called *totally additive* if the equality

$$\varphi\Big(\bigvee_{x \in E} x\Big) = \sum_{x \in E} \varphi(x)$$

holds for an ARBITRARY disjoint set of elements in $\mathscr{X}$ which possesses a supremum.[28] If we require that the equality be valid for every countable disjoint set E then we obtain the definition of a *countably additive* function. Of course, total additivity and countable additivity coincide for the algebras satisfying the countable chain condition.

3.6 The concept of a measure

Every finite totally additive essentially positive quasimeasure on a BA is called a *measure*. From Theorem 5 it follows that countable additivity of an essentially positive quasimeasure is sufficient for the quasimeasure to be a measure.

[28] The equality $s = \sum_{\alpha \in A} \varphi(\alpha)$ means that, for each $\varepsilon > 0$, there exists a finite set $A_\varepsilon \subset A$ such that the inclusions $A_\varepsilon \subset A' \subset A$, with A' finite, imply the inequality $|\, s - \sum_{\alpha \in A'} \varphi(\alpha) \,| < \varepsilon$. (The summation sign in the last inequality is understood as usual.) See Bourbaki's book [2] for details.

A *probability measure*[29] is a measure μ that satisfies the condition $\mu(\mathbf{1}) = 1$. If a measure is defined on a BA $\mathscr{X}$ then $\mathscr{X}$ is said to be a Boolean algebra with measure. Of course, such an algebra possesses a probability measure too.

An algebra with measure always satisfies the countable chain condition; hence, the algebra 2^Q can possess a measure only if Q is finite or countable. We show that some measure does exist in this case. To this end, we need to associate with each $q \in Q$ a strictly positive number μ_q so that the sum

$$\sum_{q \in Q} \mu_q$$

be finite. For instance, if $Q = \{q_1, q_2, \dots\}$ then we assign

$$\mu_{q_n} = \frac{1}{2^n} \quad (n = 1, 2, \dots).$$

Now it is easy to verify that by setting

$$\mu(e) \equiv \sum_{q \in e} \mu_q$$

for every $e \subset Q$, we obtain an essentially positive countably additive function, i.e. a measure.

Suppose that Q is finite in our case and let m be the number of its elements, the *size* of Q. Then it is natural to assign

$$\mu_{q_n} = \frac{1}{m}$$

for all $n = 1, 2, \dots, m$. We call the corresponding measure *basic*; this measure is the most important among the probability measures on 2^Q, where $card\, Q = m$.

As an example of an algebra without a measure, we can take each BA lacking the countable chain condition, for instance, the BA of Borel (or Lebesgue measurable) subsets of an interval. The examples of the algebras without any measure but satisfying the countable chain condition are of a more delicate character. We have already mentioned one of such examples which belongs to H. Gaifman. The question whether or not a given BA possesses a measure, is one of the most difficult and topical problems, since measure algebras form a class of Boolean algebras which is very important for applications. The bulk of this book is devoted exactly to this class.

---

[29] Also called a *probability*. (S. S. Kutateladze)

3.7 Boolean algebras and probability

The examples of this chapter demonstrate how numerous is the totality of interpretations and applications of Boolean algebras. We have adduced the most important as early as in the Introduction; this is the interpretation of a Boolean algebra as a family of events. Since each BA is isomorphic to some algebra of sets, we may imagine a "family of events" as a family of sets in a basic space Ω.

Furthermore, it is important for probability theory that the family be a σ-algebra;[30] roughly speaking, a probability is a countably additive measure on this σ-algebra.

Thus, the concepts of an "event" and a measurable set are identified as well as the concepts of "measure"[31] and "probability."

In the simplest cases, there are ample grounds for identifying events and sets. For instance, the fact that an event e happens or not can be conveniently interpreted as "hitting" or "missing" the set $e \subset \Omega$ by a random point $\omega \in e$. The inclusion $e_1 \subset e_2$ means that the event e_2 is a consequence of the event e_1. ("Hitting" e_1 implies "hitting" e_2.)

This approach to the basic concepts of probability theory became most popular after the monograph of A. N. Kolmogorov.[32] It opens a possibility of applying directly to probability theory the well-developed apparatus of measure and integration; in particular, the product of measures, the Radon–Nikodým Theorem (for defining conditional expectations), etc.

Moreover, probability theory is sometimes said to become an autonomous part of the general measure theory; however, the fact is not taken into account that measure theory itself has changed considerably in the last decades, becoming more and more "probabilistic."

However, the "set-theoretic" or "geometric" interpretation of probability theory has also some demerits noticed by many authors.[33]

For instance, the concept of an "elementary event" is an indispensable piece of it. As a rule, these events have probability zero, but they differ from the "impossible" event (the empty set). The probability is not a totally additive but only countably additive function. Pathological situations frequently occur that are connected not with the crux of the matter but rather with peculiarities of the space Ω containing "more" or "less" points than we need. The representation of real events as measurable subsets of some space might be unnatural.

---

[30]See below.
[31]We mean a probability "measure": $\mu\Omega = 1$.
[32]A. N. Kolmogorov [1].
[33]Cf. A. N. Kolmogorov [2] and D. Kaposs [1].

In the present book, we develop a viewpoint according to which it is an "abstract" Boolean algebra with a probability measure that provides an adequate mathematical model of the subject called a "family of events." Numerous representation theorems allow us, if need be, to switch to the terms of concrete interpretations by using the facts accumulated in the various areas of mathematics. In such cases, we avoid "privileged" representations and try to express the situation in a purely abstract language. The theory of Boolean algebras is now sufficiently developed for this. Its progress was greatly influenced by functional analysis and, in particular, by the theory of ordered vector spaces.

In conclusion, we present a table that contains the interpretations of the simplest relations between elements of a Boolean algebra in terms of events.

4. Automorphisms and invariant measures

4.1 Basic concepts

An *automorphism* of a BA $\mathscr{X}$ is an order automorphism of $\mathscr{X}$; i.e. an order isomorphism between $\mathscr{X}$ and $\mathscr{X}$. The totality of all automorphisms of a BA $\mathscr{X}$ is a group whose group operation ("multiplication") is defined as usual:

$$(AB)(x) \equiv A(B(x)).$$

This group is not commutative in general; the role of unity is played by the identity automorphism $I = \mathrm{Id}_{\mathscr{X}}$, i.e.,

$$I(x) = x, \quad x \in \mathscr{X}.$$

It may happen that a BA has no other automorphisms at all; the question about conditions for existence of nontrivial automorphisms is regarded

as rather difficult.[34] Alongside the (whole) automorphism group of $\mathscr{X}$ that comprises ALL automorphisms of a BA, we may consider various SUBGROUPS of this group; such a group is usually called an *automorphism group.*

An important role is played by CYCLIC GROUPS that consist of powers of a single automorphism.

We indicate an important class of automorphism groups. A group $\mathfrak{A}$ is called *ergodic* if for every $x > \mathbf{0}$ the following equality holds:

$$\bigvee_{A \in \mathfrak{A}} Ax = \mathbf{1}.$$

If an ergodic group consists of powers of a single automorphism A, then A is called an *ergodic automorphism.* The ergodicity condition has a simple meaning: every two elements $x, y > \mathbf{0}$ can be "linked" by an automorphism of the group, i.e., there is an $A \in \mathfrak{A}$ such that $Ax \wedge y > \mathbf{0}$.

As an example, consider the Boolean algebra 2^Q. We have already noted that every isomorphism and, in particular, every automorphism takes the singletons of Q into singletons.

Therefore, each automorphism of the BA under consideration generates a bijection from the basic set Q onto itself. The converse is also true: if φ is a bijection from Q onto itself then, by associating with each $e \subset Q$ its image $\varphi(e)$, we obtain an automorphism of 2^Q. Thus, each automorphism group $\mathfrak{A}$ on the algebra 2^Q can be identified with a group $\widetilde{\mathfrak{A}}$ of transformations of the set Q; the ergodicity of $\mathfrak{A}$ means that every two points $q_1, q_2 \in Q$ may be identified by a transformation in $\widetilde{\mathfrak{A}}$ (such groups of transformations are called "transitive"). *The automorphism group of 2^Q is necessarily ergodic.* The easy proof of this fact is left to the reader.

In general, in passing from a smaller subgroup to a larger subgroup, we see that the chances to meet ergodicity increase; however, even the whole automorphism group need not be ergodic.

Let $\mathfrak{A}$ be an automorphism group of a BA $\mathscr{X}$ and let φ be a quasimeasure on $\mathscr{X}$. We say that φ is an $\mathfrak{A}$-*invariant quasimeasure* if, for all $\mathfrak{A}$-congruent elements $x, y \in \mathscr{X}$, the equality $\varphi(x) = \varphi(y)$ holds ($\mathfrak{A}$-congruence of x and y means that $y = Ax$ for some $A \in \mathfrak{A}$). If $\mathfrak{A}$ is an ergodic automorphism group on the BA $\mathscr{X} = 2^Q$ and if φ is an $\mathfrak{A}$-invariant quasimeasure then all singletons Q are associated with the same value of φ. In this case, an essentially positive quasimeasure can be $\mathfrak{A}$-invariant only if Q is a finite set; the quasimeasure thus differs from the basic measure (considered in 0.3.6) by a numerical factor.

[34] Cf. B. Jónsson [1], L. S. Rieger [1], and B. Balcar and P. Štěpánek [1].

Therefore, *if $\mathfrak{A}$ is ergodic then an $\mathfrak{A}$-invariant probability measure on 2^Q is unique and coincides with the basic measure.* Moreover, this measure is obviously invariant under all automorphisms.

We will return to automorphism groups and invariant measures later. In particular, in the general case, we will prove uniqueness for a probability measure that is invariant under an ergodic automorphism group.

4.2 The traditional measure theory

The traditional measure theory[35] has a "geometric" character: it deals with the point sets of some space R. Under consideration are the algebras of such sets and quasimeasures on them. Of course, the reader has already noticed a difference in terminology: the object called "measure" in the traditional theory is called quasimeasure in this book. The reason behind this is that the "measure" of a point set, being nonnegative, can vanish even in the case when the set is nonempty. Furthermore, only countable (or even finite) additivity is assumed. We conventionally use the quotation marks if the word "measure" is employed in the traditional sense.

The bulk of the traditional theory is devoted to σ-algebras of sets and countably additive "measures" on them. "Measures" with infinite values are frequently considered.

The most important case is $R = \mathbb{R}^n$ (the Euclidean space of dimension n). The role of the basic Boolean algebra is often played by the σ-algebra $\mathscr{B}(\mathbb{R}^n)$ of all Borel subsets of $\mathbb{R}^n$.[36] This algebra is usually completed by adding sets of zero measure, and the completion process is always connected with a concrete "measure" on $\mathscr{B}(\mathbb{R}^n)$.

Which "measures" (quasimeasures) are most important?

The general theory is related to an arbitrary space R not endowed with any structure. But such a space is a rare event in mathematics and so in its applications.

Some measure usually appears not by itself but in connection with some structure on R and must agree with this structure. Here is the most typical situation: given a group Γ of transformations of a space R, we consider the "measures" that preserve their values under all transformations in Γ. Moreover, "measurability" is preserved; i.e., if $\mathscr{X}$ is the algebra of sets on which the "measure" is defined, then $\gamma(e) \in \mathscr{X}$ for all $\gamma \in \Gamma$ and $e \in \mathscr{X}$. Associate with each γ the bijection A_γ from

[35] The abundance of possible references makes detalization unnecessary. We refer the reader in particular to the books: A. N. Kolmogorov and S. V. Fomin [1]; K. R. Parthasarathy [1]; J. Neveau [1], and P. Halmos [1].

[36] The Borel σ-algebra of $\mathbb{R}^n$. (S. S. Kutateladze)

the BA $\mathscr{X}$ onto itself by the rule:

$$A_\gamma(e) \equiv \gamma(e), \quad e \in \mathscr{X}.$$

Denote by $\mathfrak{A}_\Gamma$ the group of these automorphisms. Among quasimeasures on the BA $\mathscr{X}$, we are interested primarily in those that are invariant under $\mathfrak{A}_\Gamma$. This interest is usually justified by a special role of the group Γ (or, equivalently, $\mathfrak{A}_\Gamma$) and its geometrical or physical sense.

We exhibit two important examples. Take $\mathbb{R}^n$ as R, confine exposition to the case $n = 2$ for simplicity, and let $\mathscr{X}$ be the Borel BA $\mathscr{B} \equiv \mathscr{B}(\mathbb{R}^n)$ mentioned above.

As is well known, the following are the most important groups of transformations in $\mathbb{R}^2$ (the groups in 2–4 are subgroups of the group in 1):

1. The group D of all isometric transformations.
2. The group Π of all translations.
3. The group Π^0 of all translations along the coordinate axes (a subgroup of the previous).
4. The group SO_2 of all rotations about the coordinate origin.

The group $\mathscr{B}$ is invariant under all these transformations; thus, we have the corresponding automorphism groups $\mathfrak{A}_D$, $\mathfrak{A}_\Pi$, $\mathfrak{A}_{\Pi^0}$, and $\mathfrak{A}_{SO_2}$.

Denote by Q the circle: $x_1^2 + x_2^2 \leq 1$. The following "folklore" theorem is well known:

Theorem. *There is a unique countably additive quasimeasure $l^{(2)}$ on the BA $\mathscr{B}$ invariant under the group Π^0 and satisfying the normalization condition $l^{(2)}Q = \pi$. This quasimeasure is also invariant under the entire group D.*

The quasimeasure $l^{(2)}$ is called Lebesgue "measure" on $\mathbb{R}^2$; note that this term oftener denotes the completion of $l^{(2)}$. (It seems reasonable to call $l^{(2)}$ "area.")

Lebesgue "measure" is invariant under the most important group in $\mathbb{R}^2$, the group of isometric transformations; this fact justifies the especial role of this measure.

We now turn to the group $\mathfrak{A}_{SO_2}$. Here, we discuss "measures" that are invariant under rotations. There are many such groups, even if we confine exposition to probability "measures" normalized by the condition $\mu\mathbb{R}^2 = 1$.

Which conditions can distinguish the main "measure" in this large totality?

One of the candidates is the representability of a given "measure" as a "square":

$$\mu = \nu \times \nu, \tag{11}$$

where ν is a Borel measure on the real axis.

Theorem. *To each $\alpha \in (0,1)$, there is a unique probability "measure" μ on $\mathscr{B}$ invariant under the group $\mathfrak{A}_{SO_2}$, representable as a square, and satisfying the normalization condition $\mu Q = \alpha$.*

This "measure" μ is defined by the formula

$$\mu(e) = \frac{1}{2\pi\sigma^2} \int\int_e e^{-\frac{x^2+y^2}{2\sigma^2}} dxdy,$$

where σ is a constant determined from α. This is the celebrated "Gauss measure."

In much the same way as the previous theorem the second is also available for a long time.[37]

So, among numerous probability "measures" on the Borel algebra of the plane $\mathbb{R}^2$, we mention the most important: first, Lebesgue "measure" $l^{(2)}$ (area) and, second, the family of Gauss measures.

The well-known probability interpretation of these measures deals with "shooting at the plane." The BA $\mathscr{B}$ is the algebra of events. When we say that "the event e happens," it means that a random shot hits a set e ("a random point hits e"). The shooting can be aimed or not aimed. In the latter case, only the entire $\mathbb{R}^2$ is hit CERTAINLY. The conditional expectation of hitting e given the event that a bigger set E is hit, is equal to

$$\frac{l^{(2)}(e)}{l^{(2)}(E)},$$

where $l^{(2)}$ is Lebesgue "measure."

If the shooting is aimed and the target is the origin, then the probability of hitting e is determined by the Gauss formula above.

As is well known, the parameter σ characterizes dispersion and is connected with the individual properties of the shot and the gun. The condition (11) means that random vertical and horizontal deviations are stochastically independent.

We have confined exposition to the two-dimensional case only for simplicity; the same is true for the space $\mathbb{R}^n$ of an arbitrary dimension n.

---

[37]For instance, see A. N. Krylov [1, pp. 365–368]; a more general theorem is presented in V. P. Skitovich [1] and G. Darmois [1].

So, the traditional measure theory deals with additive functions on concrete BAs, namely, on algebras of sets. In order to transform a traditional countably additive measure on a σ-algebra into a "genuine" measure (in the sense of this book), the initial σ-algebra must be "factorized" modulo negligible sets. A new BA will be created, a quotient of the initial algebra. In further chapters, we will consider this procedure in more detail.

Exercises for Chapter 0

1. Given a BA $\mathscr{X}$, prove the equivalence of the following propositions:

(a) $\mathscr{X}$ is finite;

(b) $\mathscr{X}$ is isomorphic to the boolean 2^Q, with Q a finite set;

(c) every family of pairwise disjoint elements in $\mathscr{X}$ is finite;

(d) every quasimeasure on $\mathscr{X}$ is totally additive.

2. Prove that the boolean 2^Q possesses ergodic automorphisms if and only if Q is countable.

3. Given a quasimeasure μ and arbitrary elements $x_1, x_2, \dots, x_n$, prove that

$$\mu(x_1 \vee x_2 \vee \cdots \vee x_n)$$
$$= \sum_{p=1}^{n} (-1)^{p-1} \sum_{1 \le i_1 < i_2 < \cdots < i_p \le n} \mu(x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_p}).$$

Chapter 1

THE BASIC APPARATUS

1. Subalgebras and generators

1.1 The concept of a subalgebra

A subset $\mathscr{X}_0$ of a Boolean algebra $\mathscr{X}$ is called a *subalgebra* of $\mathscr{X}$ if $\mathscr{X}_0$ contains $\mathbf{0}$ and $\mathbf{1}$ and is closed under the main Boolean operations $\vee$, $\wedge$, and C; i.e.,

$$x \vee y, x \wedge y, Cx, Cy \in \mathscr{X}_0$$

whenever $x, y \in \mathscr{X}_0$. The set $\mathscr{X}_0$, furnished with the induced order, is a Boolean algebra with the same zero and unity as in $\mathscr{X}$. By duality we easily show that *for a subset $\mathscr{X}_0$ to be a subalgebra it is sufficient that $\mathscr{X}_0$ be closed under the operations $\vee$ and C or under the operations $\wedge$ and C.* Finally, the induction demonstrates that each subalgebra contains the suprema and infima of its finite subsets.

Some examples illustrating the concept of a subalgebra can be found in the previous chapter. Namely, in Examples 5, 6, and 7 (but not 9!) of 0.2.4, we dealt with subalgebras of the Boolean algebra $\mathscr{X} = 2^Q$. In general, each algebra of sets is a subalgebra of 2^Q by definition. Two *trivial subalgebras* can be indicated in an arbitrary BA $\mathscr{X}$: the entire $\mathscr{X}$ and the degenerate subalgebra consisting only of $\mathbf{0}$ and $\mathbf{1}$. Moreover, we distinguish the important class of *simple subalgebras*; this term we use for the subalgebras consisting of exactly four different elements. Every simple subalgebra has the form

$$\mathscr{X}' = \{u, \ Cu, \ \mathbf{0}, \ \mathbf{1}\}.$$

It is uniquely determined by either of the elements u and Cu.

We meet subalgebras rather frequently on the various occasions of our life. Dealing with some totality of events, we usually select (consciously

or unconsciously) a part of the totality consisting of the events important for us.

For instance, on the eve of a drawing an owner of a government bond thinks about the events concerning this bond rather than all events of the drawing. Clearly, it stands to reason to choose a system of events that is a subalgebra since otherwise it would "lack" events.

1.2 Operations on subalgebras

The system of all subalgebras of a given BA $\mathscr{X}$ is naturally ordered by inclusion. We show that, with respect to this order, each class of subalgebras possesses a supremum and an infimum. The following important theorem is almost obvious.

Theorem 1. *If $\mathscr{E}$ is a nonempty class of subalgebras of an arbitrary BA $\mathscr{X}$ then the intersection*

$$\mathscr{Y}_0 \equiv \bigcap_{\mathscr{Y} \in \mathscr{E}} \mathscr{Y}$$

is a subalgebra too.

Indeed, the result of applying each of the Boolean operations $\vee$, $\wedge$, and C to elements of $\mathscr{Y}_0$ must belong to every subalgebra in the class $\mathscr{E}$ and, hence, to the intersection $\mathscr{Y}_0$. Moreover, $\mathbf{0}$ and $\mathbf{1}$ belong to $\mathscr{Y}_0$ obviously. Clearly, $\mathscr{Y}_0$ is a subalgebra. It is the GREATEST subalgebra (by inclusion) lying in each $\mathscr{Y} \in \mathscr{E}$; in other words, it is the infimum of $\mathscr{E}$.

Now, let E be an arbitrary set of elements in a Boolean algebra $\mathscr{X}$. There exist subalgebras including E; for instance, the BA $\mathscr{X}$ itself. In view of Theorem 1, there exists a least among such subalgebras; it is called the *subalgebra generated by E*. We denote this subalgebra by $\mathscr{X}\langle E\rangle$.

If $E = E_1 \cup E_2 \cup \dots \cup E_n$ then we sometimes write $\mathscr{X}\langle E_1, E_2, \dots, E_n\rangle$ instead of $\mathscr{X}\langle E\rangle$; in case of singletons $E_1 = \{u_1\}, \dots, E_s = \{u_s\}$, we write

$$\mathscr{X}\langle u_1, u_2, \dots, u_s, E_{s+1}, \dots, E_n\rangle.$$

If $\mathscr{X}_0 = \mathscr{X}\langle E\rangle$ then the set E is called a *system of generators* and its elements are called *generators* of $\mathscr{X}_0$.

Each simple subalgebra $\{u, Cu, \mathbf{0}, \mathbf{1}\}$ possesses a system of generators that consists of a single element u or Cu. Simple subalgebras can be completely characterized as nondegenerate subalgebras with one generator.

The above-mentioned owner of a bond was interested "in the first approximation" precisely in a simple subalgebra that contains, alongside

0 and **1** (the "certain" and "impossible" events), the events u, "the bond wins a nonzero sum," and Cu, "the bond loses."

1.3 Subalgebras and partitions

Every disjoint set τ_x with the supremum $x \in \mathscr{X}$ is conventionally called a *partition* of x. We are primarily interested in partitions of unity. They are denoted by the symbols $\tau, \tau', \ldots$, etc. without subscripts.

Each partition of unity τ is associated with some subalgebra $\mathscr{X}^\tau$. Namely, as $\mathscr{X}^\tau$ we take the totality of all suprema in $\mathscr{X}$ of subsets in τ.

Lemma 1. *The set $\mathscr{X}^\tau$ is a subalgebra of $\mathscr{X}$ isomorphic to the BA 2^τ.*

PROOF. Define the mapping φ from 2^τ to $\mathscr{X}^\tau$ by letting

$$\varphi(e) \equiv \sup e$$

for each $e \subset \tau$. (We recall that the supremum of the empty set is assumed to equal zero.)

By the definition of $\mathscr{X}^\tau$, each element of this set has the form $\varphi(e)$, where $e \subset \tau$. Consequently, φ maps 2^τ onto $\mathscr{X}^\tau$. It is clear that φ is isotonic: $e' \subset e$ always implies $\varphi(e') \leq \varphi(e)$. If, otherwise, $e' \not\subset e$, there is a nonzero element $x_0 \in e' \setminus e$. Then $x_0 \, d \, e$ and the relation $\sup e' \leq \sup e$ obviously fails. It is now clear that the inequality $\varphi(e') \leq \varphi(e)$ is equivalent to the inclusion $e' \subset e$; therefore, φ is an isomorphism. The proof of the lemma is complete.

The same can be proved for an infinite partition as well, assuming existence for the suprema of arbitrary subsets of τ; in this case, $\mathscr{X}^\tau$ consists of these suprema.

We say that the subalgebra $\mathscr{X}^\tau$ is *induced* or *generated by the partition* τ. It turns out that every finite subalgebra can be constructed in such a way (of course, in this case, $\mathscr{X}^\tau = \mathscr{X}\langle\tau\rangle$).

Lemma 2. *For each finite subalgebra $\mathscr{X}_0$ of a Boolean algebra $\mathscr{X}$, there is a finite partition of unity τ inducing this subalgebra: $\mathscr{X}_0 = \mathscr{X}^\tau = \mathscr{X}\langle\tau\rangle$.*

The proof of the lemma rests on the following obvious remark: since the subalgebra $\mathscr{X}_0$ is finite; therefore, given each nonzero element $x \in \mathscr{X}_0$, we can indicate an element $y \in \mathscr{X}_0$ that satisfies the inequality $\mathbf{0} < y \leq x$ and is such that the inequality $\mathbf{0} < z < y$ is impossible for $z \in \mathscr{X}_0$. We take as τ the set of all these y. It is clear that this is a disjoint set and each nonzero element of $\mathscr{X}_0$ (in particular, unity) is the supremum of some subset of τ.

From a "probabilistic" viewpoint, a partition of unity τ is a "complete collection of events"; it is usually connected with some "experiment" that

determines which of pairwise inconsistent events generating τ will actually happen. Moreover, the outcomes of all events in the subalgebra $\mathscr{X}^{\tau}$ are determined automatically. The remaining events are not affected by the experiment. We now see that the nonmathematical concept of "experiment" can be given a precise meaning. When speaking about an "experiment," we always deal mathematically with some subalgebra of events whose outcomes are all determined from the experiment. There is no other mathematical content in the concept of "experiment"; hence, it is natural to regard the notion of a subalgebra as a mathematical analog of this concept.[1]

It is easy to present an example of a subalgebra induced by an infinite partition. For instance, the Boolean algebra of Example 7 (see 0.2.4) is the subalgebra of the boolean of the square which is induced by the partition of the square into vertical segments: every union of these segments belongs to the subalgebra, and there are no other elements.

Suppose that we are given two partitions of unity τ and τ'' (finite or infinite). We say that τ'' is *finer* than τ' if each element of τ'' is dominated by some element of τ'. In this case, we write $\tau'' \succ \tau'$. It is clear that the totality of all partitions is thus endowed with a partial order. It is easy to verify that, for finite partitions τ' and τ'', the relations $\tau'' \succ \tau'$ and $\mathscr{X}^{\tau''} \supset \mathscr{X}^{\tau'}$ are equivalent; in this case, $\mathscr{X}^{\tau'}$ is a subalgebra of $\mathscr{X}^{\tau''}$.

Of course, it is not true that each subalgebra is generated by some partition, finite or infinite. We will extend the concept of a partition itself later.

In the algebra generated by a partition τ, the role of generators can be played by the elements of τ. If there are n of them then the algebra consists of 2^n elements. It is not difficult to show that the algebra cannot possess less than 2^n disjoint generators.

If an algebra of the form 2^Q consists of N elements then every disjoint system of its generators consists of $\log_2 N$ elements. However, we will see below that, without the disjointness requirement, it is possible to construct a system of generators for such an algebra which consists of $\log_2 \log_2 N$ elements.

By combining Lemmas 1 and 2 with Theorem 4 in 0.2.4, we obtain the following important

Theorem 2. *Every two equivalent finite Boolean algebras are isomorphic; each finite Boolean algebra is isomorphic to an algebra of the form 2^Q and the number of its elements has the form 2^n $(n = 1, 2, \ldots)$.*

---

[1] I. M. Gelfand, A. N. Kolmogorov, and A. M. Yaglom [1].

We describe a general set-theoretic construction that leads to partitions. Henceforth, $\mathscr{X} = 2^Q$.

Let Q be an arbitrary nonempty set and let $\mathscr{E}$ be a nonempty class of subsets of Q, i.e., $\mathscr{E} \subset 2^Q$. Consider the family of all pairs $\Delta_e = \{e, Ce\}$ $(e \in \mathscr{E})$. Construct the cartesian product

$$\Delta \equiv \prod_{e \in \mathscr{E}} \Delta_e.$$

The points of this product are families of the form $\delta = \{\delta_e\}_{e \in \mathscr{E}}$, where each δ_e is either e or Ce. Associate with each $\delta \in \Delta$ the intersection

$$z_\delta \equiv \bigcap_{e \in \mathscr{E}} \delta_e.$$

Some of these intersections may be empty; those nonempty form a partition of Q. We say that this partition is *generated* by the system of sets $\mathscr{E}$ and denote it by $\zeta(\mathscr{E})$ or simply ζ. (In the sequel, we use other letters as well: ξ, η, etc.) If Δ_0 stands for the set consisting of all δ for which $z_\delta \neq \varnothing$ then the mapping $\delta \longrightarrow z_\delta$ is a bijection from Δ_0 onto $\zeta(\mathscr{E})$.

Every set $E \subset Q$ constituted by whole elements of some partition ζ is called "saturated with respect to ζ" or a ζ*-set*.

We point out the following easily verified facts.

1°. *The system of all ζ-sets is closed under the possibly infinite operations $\cup$, $\cap$, and $\setminus$ and so it is an algebra of sets.* As before, we denote this algebra by $\mathscr{X}^{(\zeta)}$; it is a subalgebra of the BA $\mathscr{X} = 2^Q$.

By definition, a system $\mathscr{E}$ *separates the points* of Q or is point-separating if, for all points $q_1, q_2 \in Q$, there exist a set $e \in \mathscr{E}$ that contains exactly one of the points q_1 and q_2.

2°. Let $\zeta = \zeta(\mathscr{E})$. *For the equality $\mathscr{X} = \mathscr{X}^{(\zeta)}$ to hold, it is necessary and sufficient that the system $\mathscr{E}$ separate the points of Q. In this case, ζ is the partition into singletons.*

3°. *If $\zeta = \zeta(\mathscr{E})$ then $\mathscr{E} \subset \mathscr{X}^{(\zeta)}$ (all $e \in \mathscr{E}$ are ζ-sets).*

4°. *The subalgebra $\mathscr{X}\langle\mathscr{E}\rangle$ generated by a system $\mathscr{E}$ is contained in $\mathscr{X}^{(\zeta(\mathscr{E}))}$. (If Q is infinite then these subalgebras can differ.)*

5°. *For a system of sets $\mathscr{E}$ to generate a given partition ζ, it is necessary and sufficient that the following two conditions be satisfied:*

(a) *the system $\mathscr{E}$ separates the elements of ζ (i.e., for all $z', z'' \in \zeta$, there exists a set $e \in \mathscr{E}$ that contains exactly one of the two elements z' and z'');*

(b) *$\mathscr{E} \subset \mathscr{X}^{\zeta}$, i.e., all sets in $\mathscr{E}$ are saturated.*

We prove the last claim. Necessity is obvious, and so we check sufficiency. Let ζ^* be the partition generated by $\mathscr{E}$.

Take $z \in \zeta$ and $e \in \mathscr{E}$ arbitrarily. Assume that $z \not\subset Ce$. Then $z \cap e \neq \varnothing$ and, by (b), $z \subset e$. Hence, each $z \in \zeta$ belongs either to e or to Ce (for each $e \in \mathscr{E}$). Construct $\delta \in \Delta$ so that δ_e contains the entire z for all e. Then

$$z \subset \bigcap \delta_e = z_\delta.$$

The right-hand side includes a ζ-set. Hence, $z = z_\delta \in \zeta^*$. Thus, $\zeta \subset \zeta^*$. Assume now that there exists some $z^* \subset \zeta^* \setminus \zeta$. Then $z^* \cap z = \varnothing$ for all $z \in \zeta = \zeta \cap \zeta^*$ (the distinct elements of ζ^* are disjoint). But this is impossible since $\bigcup_{z\in\zeta} z = Q$. Therefore, $\zeta = \zeta^*$.

This proposition immediately implies

6°. *If* $\mathscr{E} \subset \mathscr{E}^* \subset \mathscr{X}^{(\zeta(\mathscr{E}))}$ *then* $\zeta(\mathscr{E}^*) = \zeta(\mathscr{E})$.

We recall that a partition ζ_1 is *finer* than ζ_2 (or ζ_2 is *coarser* than ζ_1) if each set $z_1 \in \zeta_1$ lies in some $z_2 \in \zeta_2$.

The following are obvious:

7°. *If* $\mathscr{E}_1 \subset \mathscr{E}_2$ *then* $\zeta(\mathscr{E}_2)$ *is finer than* $\zeta(\mathscr{E}_1)$.

8°. *If* ζ_1 *is finer than* ζ_2 *then* $\mathscr{X}^{(\zeta_1)} \supset \mathscr{X}^{(\zeta_2)}$.

The above construction is widely spread in measure theory in particular.

1.4 Structure of the subalgebra generated by a set

Let E be a nonempty set and let $\mathscr{X}_0 = \mathscr{X}\langle E\rangle$ be the subalgebra generated by this set. Which elements constitute this subalgebra? First of all, the subalgebra must certainly contain all elements of the form

$$y = \Big(\bigwedge_{u\in\Delta} u\Big) \wedge \Big(\bigwedge_{v\in\Delta'} Cv\Big), \tag{1}$$

where Δ and Δ' are finite subsets of E. The elements representable as (1) are called *elementary polynomials*; the set of all these elements is denoted by M_E. Besides elementary polynomials, the subalgebra contains the suprema of finite subsets of M_E, i.e. the elements of the form

$$z = \bigvee_{k=1}^{n(z)} \Big(\bigwedge_{u\in\Delta_k} u\Big) \wedge \Big(\bigwedge_{v\in\Delta'_k} Cv\Big). \tag{2}$$

They can be also written as

$$z = \bigvee_{k=1}^{n(z)} (u_{k1} \wedge u_{k2} \wedge \cdots \wedge u_{kp_k}) \wedge (Cv_{k1} \wedge Cv_{k2} \wedge \cdots \wedge Cv_{kq_k}), \tag{2'}$$

where

$$u_{ki}, v_{kj} \in E$$

for $1 \le i \le p_k$, $1 \le j \le q_k$, and $k = 1, 2, \ldots, n(z)$, or in the following simpler form:

$$z = \bigvee_{k=1}^{n(z)} (w_{k1} \wedge w_{k2} \wedge \cdots \wedge w_{km_k}), \tag{2''}$$

where $w_{ki} \in E \cup CE$ $(1 \le i \le m_k;\ k = 1, 2, \ldots, n(z))$.

The elements representable in the form (2), (2′), or (2″) are called *polynomials*[2] (or Boolean polynomials). The set of all these elements is conventionally denoted by P_E. The initial set E plays an essential role in the above definition; in order to reflect this role, we use the term "polynomials in elements of E." It turns out that such polynomials exhaust the entire subalgebra $\mathscr{X}_0$. This will become clear after proving the following:

Lemma 3. *The set P_E is a subalgebra.*

It is sufficient to show that P_E is closed under the operations $\wedge$ and C. This is easy to deduce from the following identities:

$$\Big[\bigvee_{k=1}^{n} (u_{k1} \wedge u_{k2} \wedge \cdots \wedge u_{kp_k})\Big] \wedge \Big[\bigvee_{i=1}^{m} (v_{i1} \wedge v_{i2} \wedge \cdots \wedge v_{iq_i})\Big]$$

$$= \bigvee_{\substack{1 \le k \le n \\ 1 \le i \le m}} (u_{k1} \wedge u_{k2} \wedge \cdots \wedge u_{kp_k} \wedge v_{i1} \wedge v_{i2} \wedge \cdots \wedge v_{iq_i})$$

and

$$C\Big[\bigvee_{k=1}^{n} (u_{k1} \wedge u_{k2} \wedge \cdots \wedge u_{kp_k})\Big]$$

$$= \bigvee_{\substack{1 \le k_1 \le p_1 \\ 1 \le k_2 \le p_2 \\ \cdots\cdots\cdots \\ 1 \le k_n \le p_n}} (Cu_{1k_1} \wedge Cu_{2k_2} \wedge \cdots \wedge Cu_{nk_n}).$$

If we take as u_{ks} and v_{ij} arbitrary elements of the set $E \cup CE$ in these identities then we obtain the formulas that express the infima and complements of elements in P_E as polynomials in the generators.

[2]The term "polynomial" is sometimes used for the operator determined by (2); however, we will never imply this meaning.

The lemma is proved. It immediately implies the theorem about the structure of the subalgebra generated by a given set E.

Theorem 3. *The subalgebra $\mathscr{X}_0 \equiv \mathscr{X}\langle E\rangle$ coincides with the set P_E of all polynomials of the form* (2).

Indeed, the fact that P_E is a subalgebra implies $P_E = \mathscr{X}_0$.

Theorem 3 obviously yields the following:

Corollary. *Every finitely generated subalgebra is finite.*

We have showed that each element of the subalgebra $\mathscr{X}\langle E\rangle$ is representable as the supremum of a finite set of elementary polynomials. This representation has a dual analog which can be easily found by the reader. We will illustrate the importance of such representations by considering the Boolean algebra X_Q of all characteristic functions of subsets of Q.

Recalling the interpretation of Boolean operations in X_Q of the previous chapter, note that the value of an arbitrary characteristic function $z \in \mathscr{X}\langle E\rangle$ at a point $q \in Q$ is uniquely determined from the values assumed at this point by the "basis" functions of E. This is shown by the formula (1). If we interpret elements of X_Q as propositions then the fact that a proposition z is "true" or "false" is uniquely determined from the truth-values of the "basis" propositions that constitute the set E.

In much the same way, in the "networks" interpretation, the problem of whether some switch z belonging to a subalgebra is on or off may be solved completely given the states of all "basis" switches of E.

Let E consist of m elements $e_1, e_2, \dots, e_m \in X_Q$. The values of these characteristic functions at a point $q_0 \in Q$ constitute a finite tuple of 0s and 1s

$$(e_1(q_0), e_2(q_0), \dots, e_m(q_0));$$

in other words, a binary number.

When communicating with a computer, we usually input information in the form of such numbers:[3] m binary digits are employed for this. Every binary digit contains information about the values of all functions $z \in X_Q\langle E\rangle$ at the point q_0; in order to extract this information, we need to express z as a polynomial.

Thus, the number of generators is, at the same time, the number of binary digits necessary for representing the complete information about the "state of our subalgebra at time q_0," i.e. about the values of all $z \in X_Q\langle E\rangle$ at $q = q_0$. Of course, we want to reduce the number of digits as much as possible for not overloading the computer's memory. Therefore, a practically important problem arises of finding the number of elements that can constitute a system of generators.

[3]The author wrote this at the dawn of the electronic computers. (S. S. Kutateladze)

1.5 Systems of generators in algebras of sets

We return to algebras of the form 2^Q. Assuming Q to be an arbitrary set, we pose the question: WHICH SYSTEM OF SUBSETS CAN BE A SYSTEM OF GENERATORS FOR THIS BA?

Theorem 4. *Let $\mathscr{X} = 2^Q$, let Q be a finite set, and let $\mathscr{E}$ be a system of subsets in Q. For $\mathscr{E}$ to be a system of generators for $\mathscr{X}$ it is necessary and sufficient that $\mathscr{E}$ separate the points of Q.*

The theorem follows from Proposition 2° (see 1.1.3). The necessity part is also true in the case when Q is infinite.

We obtain an important example of a point-separating system if we consider the BA $\mathscr{X} = 2^Q$, with Q, in turn, equal to the totality 2^P of all subsets of a set P; denote the cardinality of P by π (we do not assume it finite yet). Then Q consists of 2^π points and the cardinality of $\mathscr{X}$ is equal to 2^{2^π}. For instance, so are the algebras of Boolean functions mentioned in 0.2.4. It is clear that, for example, the system of all singletons of Q separates points; however, in this case, we can indicate a more "economical" point-separating system that consists not of 2^π, but only of π sets. Namely, associate with each $p \in P$ the subset $Q_p \subset Q$ constituted by all $q \subset P$ that contain the element p:

$$Q_p = \{q \mid \ p \in q\}.$$

Check that the system $\mathscr{E}$ of all subsets of the form Q_p, $p \in P$, separates the points of Q. This is clear, since points of Q are subsets of P; if $q_1, q_2 \in Q$ and $q_1 \neq q_2$ then there is a point p that belongs to one and only one of the sets q_1 and q_2.

We now assume that π is a FINITE cardinality. Then, as follows from Theorem 4, the system $\mathscr{E}$ generates $\mathscr{X}$. Recalling that, in view of Theorem 2, every BA consisting of 2^{2^π} elements is isomorphic to $\mathscr{X}$, we arrive at the following conclusion: *in each finite BA consisting of 2^{2^π} elements, there exists a system of π generators.* It is easy to establish that no system of less than π generators is available in our case. In particular, there exists a set of π Boolean functions, each π-place, such that every π-place Boolean function can be expressed via these functions with the help of the operations $\wedge$, $\vee$, and C.[4]

Now let $\mathscr{X}$ be an arbitrary finite BA and let n be the number of its elements. It is easy to show that the minimal number of generators for

[4] At the same time, by using the composition operation, we can express all Boolean functions via a single one, for instance, with the help of the "Sheffer stroke": $f(x,y) \equiv \min(1-x, 1-y)$, $x, y = 0, 1$. We mostly leave aside a sophisticated theory of Boolean functions. Among monographs dedicated to it, we mention the book by S. V. Yablonskiĭ, G. P. Gavrilov, and V. B. Kudryavtsev [1].

such an algebra is equal to either $\log_2 \log_2 n$ or $[\log_2 \log_2 n] + 1$. This is the answer to the question posed at the end of 1.1.4.

1.6 The canonical representation of an element of a subalgebra

The representation of 1.1.4 of an element of the subalgebra $\mathscr{X}\langle E\rangle$ as a polynomial in generators is not uniquely determined: for a single z, the choice of the sets Δ_k and Δ'_k necessary for defining the polynomial, as well as their numeration, can be performed in many ways. For finding a way out of the situation, it is natural to reduce the class of admissible representations first. The so-called "canonical representations" are most often employed. Let a set of generators be finite; then, as we know, the subalgebra $\mathscr{X}\langle E\rangle$ is generated by a finite partition of unity τ: each element $x \in \mathscr{X}\langle E\rangle$ is the sum of a set of elements in τ (see Lemma 2 and the corollary to Theorem 3). We will specify the elements that constitute this partition. While proving Lemma 2, we have characterized them as minimal positive elements of the subalgebra; the containment $y \in \tau$ means that $y \in \mathscr{X}\langle E\rangle$, $y > \mathbf{0}$, and that the inequality $\mathbf{0} < z < y$ is impossible for $z \in \mathscr{X}\langle E\rangle$. By Theorem 3, each $y \in \tau$ must be a polynomial of the form

$$y = \bigvee_{k=1}^{n} y_k,$$

where $y_1, y_2, \ldots, y_n$ are elementary polynomials. By minimality, only one of the elements $y_1, y_2, \ldots, y_n$ is nonzero; therefore, y is an elementary polynomial. By definition, this means that

$$y = \Big(\bigwedge_{u\in\Delta} u\Big) \wedge \Big(\bigwedge_{v\in\Delta'} Cv\Big), \tag{3}$$

where $\Delta, \Delta' \subset E$. It is clear that $\Delta \cap \Delta' = \varnothing$; otherwise, the element y would be zero. For all $w \in E$ (as well as for all elements of the subalgebra $\mathscr{X}\langle E\rangle$), exactly one of the following relations must hold: $y \le w$ or $y \le Cw$. In the first case, we include w into the set E_1, in the second case to the set E_2. These sets E_1 and E_2 possess the following properties:

(a) $E_1 \cap E_2 = \varnothing$;

(b) $E_1 \cup E_2 = E$;

(c) $\Delta \subset E_1$ and $\Delta \subset E_2$.

It is also clear that, replacing Δ by E_1, and Δ' by E_2 in (3), we obtain an element that, in any case, is not greater than the former one:

$$\Big(\bigwedge_{u\in E_1} u\Big) \wedge \Big(\bigwedge_{v\in E_2} Cv\Big) \le y.$$

But since the element y is minimal, the last formula is an equality. Before formulating a conclusion, we present the main definition.

Each element of the form

$$y_\Delta = \Big(\bigwedge_{u\in\Delta} u\Big) \wedge \Big(\bigwedge_{v\in\Delta'} Cv\Big),$$

where Δ and Δ' are disjoint sets with union E, is called an (E)-*canonical elementary polynomial*. Roughly speaking, this is an elementary polynomial that is formed by all elements of the system E; such polynomials are pairwise disjoint and their sum is equal to unity (sometimes, they are called "constituents of unity"). While proving Lemma 2, we have seen that these are the elements of the partition τ generating the subalgebra $\mathscr{X}\langle E\rangle$. We arrive at the following

Theorem 5. *Each element of the subalgebra generated by a finite set E is representable as a finite sum of (E)-canonical elementary polynomials.*

Every representation of an element as a sum of (E)-canonical elementary polynomials is conventionally called an (E)-canonical representation. The set E plays an essential role in this concept; if $E \subset E'$ then every $x \in \mathscr{X}\langle E\rangle$ possesses both (E)- and (E')-canonical representations.

If the elements of E are enumerated, $E = \{w_k\}_{k=1}^s$, then it is convenient to write an (E)-canonical representation as

$$z = \sum \omega_1^{\delta_1} \wedge \omega_2^{\delta_2} \wedge \cdots \wedge \omega_s^{\delta_s},$$

where

$$\omega_i^{\delta_i} = \begin{cases} \omega_i & \text{for} \quad \delta_i = 1, \\ C\omega_i & \text{for} \quad \delta_i = 0, \end{cases}$$

and the summation is extended to some class of binary numbers $\delta = (\delta_1, \delta_2, \dots, \delta_s)$ of length s independent of z. Summing up over ALL δ, we obtain an (E)-canonical representation of unity.

The dual version of the canonical representation is available which is not discussed here.

1.7 Independent systems

We have arrived at the concept of canonical representation, hoping to obtain, using additional restrictions, the unique representation of a given

element as polynomial in generators. However, even the simple examples show that we did not reach the aim. The reason behind this is the existence of (E)-canonical elementary polynomials that are equal to zero. Such polynomials cannot be regarded as elements of a generating partition since they can be formally added to an arbitrary sum without affecting the value of the latter. By forbidding the (E)-canonical polynomials to vanish, we obtain the definition of an INDEPENDENT SYSTEM OF GENERATORS. In this case, we remove the requirement of finiteness for the system.

Definition. A system of elements E is called *independent* provided that E is nonempty and for every pair of disjoint finite sets $\Delta, \Delta' \subset E$ the inequality holds

$$\Big(\bigwedge_{u\in\Delta} u\Big) \wedge \Big(\bigwedge_{v\in\Delta'} Cv\Big) > \mathbf{0}.$$

In other words, independence means that for all pairwise distinct $w_1, w_2, \dots, w_n \in E$ and every natural $p = 1, 2, \dots, n$ we have

$$w_1 \wedge w_2 \wedge \dots \wedge w_p \wedge Cw_{p+1} \wedge \dots \wedge Cw_n > \mathbf{0}.$$

For this definition we see that the independence of a system amounts to the independence of its every finite subsystem.

Theorem 6. *Let E be a nonempty set. In order that for all nonempty finite $E_1 \subset E$ there is a unique (E_1)-canonical representation of each element of the subalgebra $\mathscr{X}\langle E_1\rangle$ it is necessary and sufficient that E be an independent system.*

This theorem has already been proved in fact. We have mentioned that the uniqueness of (E_1)-canonical representations implies the absence of nonzero (E_1)-canonical elementary polynomials. However, the requirement that E is independent means exactly that for each nonempty finite $E_1 \subset E$ all these polynomials differ from zero while comprising the set that is a partition generating the subalgebra $\mathscr{X}\langle E\rangle$. This naturally implies the uniqueness of all canonical representations.

The representation of an element of a subalgebra as a sum of (E)-canonical elementary polynomials in independent variables, known as "perfect disjunctive normal form," is an established tool of research in logic and the theory of contact networks.

1.8 Existence of an independent system of generators

We have already found out that each finite algebra $\mathscr{X}$ is generated by some partition τ. Suppose that a given algebra has an independent

system E of n generators. This allows us to determine the cardinality of the partition: by independence, the elements of τ are all possible elementary (E)-canonical polynomials of the form

$$y_\Delta = \Big(\bigwedge_{x\in\Delta} x \Big) \wedge \Big(\bigwedge_{y\in E\setminus\Delta} Cy \Big).$$

There are as many these polynomials as there are the subsets of $\Delta \subset E$, i.e., 2^n. The algebra $\mathscr{X}$ is isomorphic to the boolean 2^τ, and so its cardinality is equal to 2^{2^n}. Therefore, not every Boolean algebra is generated by an independent system of generators.

Theorem 7. *For a finite BA $\mathscr{X}$ to include an independent subset of n generators it is necessary and sufficient that the cardinality of $\mathscr{X}$ equals 2^{2^n}. It particular, the BA of n-place Boolean functions possesses this property.*

Necessity was established a few lines above, and so we will prove sufficiency.

Assume first that $\mathscr{X}$ has the form 2^Q, with $Q = 2^P$, and the cardinality of P equals n. We know already that this algebra has a system of generators $E = \{Q_p\}_{p\in P}$ which contains precisely n elements. Check that E is independent. To this end, consider an element of the form

$$\overline{Q} \equiv \overline{Q}^{p_1\cdots p_k}_{p_{k+1}\cdots p_m} \equiv Q_{p_1} \wedge \dots \wedge Q_{p_k} \wedge CQ_{p_{k+1}} \wedge \dots \wedge CQ_{p_m},$$

where $p_1, p_2, \dots, p_m$ are arbitrary pairwise distinct points of P. Let us find out what is the set $\overline{Q}$. For a point $q \in Q$ (recall that q is a subset of P) to belong to $\overline{Q}$ it is necessary and sufficient that $p_i \in q$ for $1 \le i \le k$ whereas $p_i \notin q$ for $k+1 \le i \le m$.

Therefore,

$$\overline{Q} = \{q \mid \ p_1, p_2, \dots, p_k \in q; \ p_{k+1}, \dots, p_m \notin q\}.$$

Since all p_i are pairwise distinct, $\overline{Q}$ is nonempty by the obvious reason: the set consisting only of the points $p_1, p_2, \dots, p_k$ is an element of $\overline{Q}$.

To extend our arguments to a general algebra $\mathscr{X}$ of cardinality 2^{2^n} it suffices to note that since all equipollent finite algebras are isomorphic to one another (Theorem 2), there exists an isomorphism between $\mathscr{X}$ and some algebra 2^{2^R}, where R is an arbitrary set of n elements. Independence is clearly preserved under isomorphisms. The proof of the theorem is complete.

Remark 1. *The proof of the independence of E does not use the assumption that the algebra under study is finite.*

Remark 2. *It is easy to show that the cardinality of the set $\overline{Q}^{p_1\cdots p_k}_{p_{k+1}\cdots p_m}$ equals 2^{n-m}: there are just as many subsets of P containing all $p_1, p_2, \dots,$*

p_k *but none of the elements* $p_{k+1}, p_{k+2}, \dots, p_m$. We leave this easy exercise in combinatorics to the reader.

If $\mathscr{X} = 2^{\widetilde{Q}}$ and we know about the set $\widetilde{Q}$ only the fact that its cardinality equals 2^π then, mapping $\widetilde{Q}$ to the totality of subsets of some set of cardinality π in a whatever manner, we will automatically come to various systems of generators which are analogous to those we have constructed above.

2. The concepts of ideal, filter, and band

2.1 Ideals and filters

A nonempty set E is *embedded solidly* in a BA $\mathscr{X}$ provided that $y \in E$ and $x \le y$ imply $x \in E$. In this case, E is called a *solid*[5] *set.* A solid set containing the meet of its every finite subset is called an *ideal.*

Lemma 4. *The intersection of each set of ideals, if nonempty, is again an ideal.*

Proof. Let K be a class of ideals, while $I_0 \equiv \bigcap_{I \in K} I$, $I_0 \ne \varnothing$. If $x \le y \in I_0$ then x, together with y, belongs to every $I \in K$ and, therefore, to the intersection I_0. So, I_0 is a solid set. Similarly, the supremum of every finite subset of I_0 belongs to every ideal $I \in K$ and, consequently, to their intersection I_0. The proof of the lemma is complete.

Associating with an arbitrary nonempty set $E \subset \mathscr{X}$ the intersection of all ideals including E, we obviously obtain the least ideal including E; we denote the ideal by $\mathscr{I}\{E\}$ and call it the *ideal generated by the set* E.

In some cases, the ideal $\mathscr{I}\{E\}$ may be described "constructively."

Lemma 5. *If* E *is a solid set then the ideal* $\mathscr{I}\{E\}$ *consists of the joins of all finite subsets of* E.

Proof. Let E^* be the set of suprema of finite subsets of E. It is clear that $E^* \subset \mathscr{I}\{E\}$. The lemma will be proved if we establish that E^* is an ideal (we recall that, by definition, $\mathscr{I}\{E\}$ is the least ideal including E). As follows from the definition, the set E^* contains the supremum of its every finite subset; we are left with verifying the solidity of E^*. Let $x \le y \in E^*$, $y = y_1 \vee y_2 \vee \dots \vee y_m$, $y_1, y_2, \dots, y_m \in E$. Putting $x'_i \equiv x \wedge y_i$ $(i = 1, 2, \dots, m)$, by the solidity of E, we infer that all x'_i belong to E. Next,

$$x'_1 \vee x'_2 \vee \dots \vee x'_m = x \wedge (y_1 \vee y_2 \vee \dots \vee y_m) = x \wedge y = x,$$

[5]The term "normal" proliferates in the literature of Russian provenance. (S. S. Kutateladze)

whence $x \in E^*$. The proof of the lemma is complete.

The ideals other than $\mathscr{X}$ (i.e. not containing unity) are of the utmost interest; they are called *proper* ideals.

Lemma 6. *For each element $u \in \mathscr{X}$ other than unity there exists a proper ideal containing u.*

The proof consists in directly exhibiting the ideal. It is easily verified that the set[6]

$$\mathscr{X}_u \equiv [\mathbf{0}, u] \equiv \{x \mid \ x \leq y\}$$

satisfies each of the above requirements: $\mathscr{X}_u$ contains u and is a proper ideal. The ideals of the form $\mathscr{X}_u$ (we will use this notation henceforth) are called *principal ideals*. The Boolean algebra $\mathscr{X}$ is itself an ideal of this type (if regarded as the principal ideal $\mathscr{X}_{\mathbf{1}}$).

A SIMPLE EXAMPLE. In a Boolean algebra of the type 2^Q (as well as in every algebra of sets) the principal ideal $\mathscr{X}_u$ is the totality of all subsets of u.

We now find out conditions under which the ideal $\mathscr{I}\{E\}$ is proper, restricting exposition to the case in which E is solid.

Lemma 7. *Let E be solid and, for every finite subset $E' \subset E$, the ideal $\mathscr{I}\{E'\}$ is proper. Then the ideal $\mathscr{I}\{E\}$ is proper.*

This lemma is a direct corollary to Lemma 5. Indeed, if $\mathscr{I}\{E\}$ contains unity then there is a finite subset $E' \subset E$ with unity an upper bound. But this means that the ideal $\mathscr{I}\{E'\}$ is improper. Despite its simplicity, Lemma 7 will be very useful in the sequel.

A *filter* is a set dual to an ideal. More precisely, a set F is a filter whenever $I \equiv CF = \{x \mid \ Cx \in F\}$ is an ideal. Each theorem concerning ideals has a dual analog in the form of the corresponding theorem about filters and vice versa; the filter dual to a proper ideal is called proper or centered. Such a filter does not contain zero.

Closing this subsection, we consider the question of the structure of the SUBALGEBRA GENERATED BY AN IDEAL.

Consider an ideal I. What elements does $\mathscr{X}\langle I\rangle$ consist of? According to Theorem 3, the subalgebra $\mathscr{X}\langle I\rangle$ is the totality of all polynomials of the form

$$z = \bigvee_{k=1}^{n(z)} \Big(\Big(\bigwedge_{u \in \Delta_k} u \Big) \wedge \Big(\bigwedge_{v \in \Delta'_k} Cv \Big) \Big).$$

[6]In general, the notation $[a, b] \equiv \{x \mid \ a \leq x \leq b\}$ is often employed in the theory of ordered sets. This set may be called an "interval."

Using the main property of ideals and the duality formulas, it is easy to see that either z or the complement of z belongs to I. So,

$$\mathscr{X}\langle I\rangle = I \cup CI.$$

For example, in the Boolean algebra 2^Q, the subalgebra generated by the ideal of finite sets consists, first, of finite sets and, second, of cofinite sets (i.e. sets complementary to finite). This subalgebra does not contain any other sets. Later, we will see that each subalgebra is an intersection of subalgebras of the form $\mathscr{X}\langle I\rangle$.

2.2 Disjoint complements

Consider an arbitrary subset E of some Boolean algebra $\mathscr{X}$. The totality of all $x \in \mathscr{X}$ disjoint from E (i.e. disjoint from every $y \in E$) is called the *disjoint complement* of E and is denoted by E^d. If E^d consists only of zero then we say that E is *total* in $\mathscr{X}$. We now point out some important properties of disjoint complements. Below, we write E^{dd} instead of $(E^d)^d$.

1°. *The inclusion $E_1 \subset E_2$ implies $E_2^d \subset E_1^d$.*

2°. *The disjoint complement is always solid.*

3°. *The containments $y \in E^s$ and $Cy \in E^d$ are equivalent.*

4°. $E \subset E^{dd}$.

These properties are immediate.

5°. *If $\sup E$ exists and the element y belongs to E^d then $y\,d\,\sup E$.*

The proof is based on the strong distributive law. We have $y \wedge \sup E = \sup(y \wedge E) = \mathbf{0}$.

6°. *If $E_1 \subset E^d$ and $\sup E_1$ exists then $\sup E_1 \in E^d$.*

Indeed, if $E_1 \subset E^d$ then $E \subset E_1^d$ and, by 5°, each $x \in E$ is disjoint from $\sup E_1$. This means that $\sup E_1 \in E^d$.

7°. *If $x = \sup E$ then $Cx = \sup E^d$.*

Proof. Given $y \in E$, we have $Cx \wedge y \leq Cx \wedge x = \mathbf{0}$, whence $Cx \in E^d$. On the other hand, according to 6°, $x\,d\,E^d$ and the complement Cx is the greatest element disjoint from x; therefore, $Cx \in (E^d)^s \cap E^d$ and $Cx = \sup E^d$.

8°. *The disjoint complement of each set is an ideal.*

The assertion is almost obvious: if $x \leq y$ and $y\,d\,E$ then $x\,d\,E$. If $x, y\,d\,E$ then, by using the distributive law, it is easy to show that $(x \vee y)\,d\,E$.

9°. *The disjoint complement of a principal ideal $\mathscr{X}_u$ coincides with the principal ideal $\mathscr{X}_{Cu}$.*

Proof. Consider a principal ideal $\mathscr{X}_u$, with u the greatest element of the ideal. Every $y \in (\mathscr{X}_u)^d$ is disjoint from u and, therefore, is not

greater than Cy. This means that $(\mathscr{X}_u)^d \subset \mathscr{X}_{Cu}$. The reverse inclusion is obvious; thus, we have $(\mathscr{X}_u)^d = \mathscr{X}_{Cu}$, as claimed.

10°. *Let $\mathscr{E}$ be a class of subsets of $\mathscr{X}$. Then, the following equality holds:*

$$\bigcap_{E \in \mathscr{E}} E^d = \Big(\bigcup_{e \in \mathscr{E}} E\Big)^d.$$

PROOF. Each element disjoint from every $E \in \mathscr{E}$ is clearly disjoint from their union; hence, $\bigcap E^d \subset (\bigcup E)^d$. Moreover, each x that belongs to $(\bigcup E)^d$ belongs to every E^d as well and, therefore, to the intersection $\bigcap E^d$. Clearly, $(\bigcup E)^d \subset \bigcap E^d$. We are done on comparing the above inclusions.

We also present a lemma that is useful in proving equalities of the form $y = \sup E$.

Lemma 8. *For an element y to be the least upper bound of a set E it is necessary and sufficient that the relations $y\,d\,E^d$ and $Cy\,d\,E$ hold simultaneously.*

PROOF. NECESSITY follows directly from the above Propositions 5° and 6°. We prove SUFFICIENCY. If $Cy\,d\,E$ then, according to 3°, we have $y \in E^s$. Fix an element $z \in E^s$. The complement of z must belong to E^d (by the same Proposition 3°) and, since $y\,d\,E^d$, we have $y\,d\,Cz$ or $y \leq z$. This proves the equality $y = \sup E$.

2.3 The concept of a band

A set E is called a *band* whenever $E = E^{dd}$. For this, it is sufficient that inclusion $E \supset E^{dd}$ hold. The property of disjoint complements, stated in Proposition 8°, implies the following

Theorem 8. *Each band is an ideal.*

Indeed, by definition, each band is the disjoint complement of some set.

The converse theorem does not hold: not every ideal is a band. However, the following theorem is valid:

Theorem 9. *Each principal ideal is a band.*

The proof reduces to referring to Proposition 9°, according to which,

$$(\mathscr{X}_u)^d = \mathscr{X}_{Cu}, \quad (\mathscr{X}_u)^{dd} = \mathscr{X}_{CCu} = \mathscr{X}_u.$$

In the next chapter we will characterize the class of Boolean algebras in which the concept of a principal ideal and that of a band coincide. In particular, all Boolean algebras of the form 2^Q enjoy this property.

We now present a theorem that generalizes the preceding assertion.

Theorem 10. *The disjoint complement of each set is a band.*

PROOF. It is only necessary to prove that, for an arbitrary $E \subset \mathscr{X}$, the inclusion $E^d \supset E^{ddd}$ holds. Let $x \in E^{ddd}$. This means that $x\,d\,E^{dd}$. Since $E \subset E^{dd}$, we have $x\,d\,E$, i.e., $x \in E^d$. The proof of the theorem is complete.

Theorem 11. *The intersection of each nonempty class of bands is a band.*

PROOF. Let K be a class of bands and let $E_0 = \bigcap_{E \in K} E$. Since each $E \in K$ is a band, we have $E = E^{dd}$ and, using 10^0, we obtain

$$E_0 = \bigcap_{E \in K} E^{dd} = \Big(\bigcup_{E \in K} E^d\Big)^d.$$

According to the previous theorem, E_0 is a band.

Corollary. *For each set $E \subset \mathscr{X}$, there exists a least band that includes E.*

Indeed, the intersection of all bands including E is the desired band. It is called the *band generated by* E and is denoted by $\mathscr{X}_E$.[7]

We now demonstrate a way of constructing bands of the form $\mathscr{X}_E$. As the matter of fact, it is sufficient to be able to construct disjoint complements.

Theorem 12. *The equality $\mathscr{X}_E = E^{dd}$ is always true.*

PROOF. Since E^{dd} is a band including E, we obtain $\mathscr{X}_E = \mathscr{X}_E^{dd} \supset E^{dd}$. So, $\mathscr{X}_E = E^{dd}$.

We now find out conditions under which $\mathscr{X}_E$ is a principal ideal.

Theorem 13. *The following three relations are equivalent:*

$$1)\ u = \sup E, \quad 2)\ \mathscr{X}_u = \mathscr{X}_E, \quad 3)\ \mathscr{X}_{Cu} = E^d.$$

PROOF. Assume that 1) holds. According to Lemma 8, $u\,d\,E^d$ and $Cu\,d\,E$, i.e., $u \in E^{dd}$ and $Cu \in E^d$. The first containment implies $\mathscr{X}_u \subset E^{dd}$; the second, $\mathscr{X}_{Cu} \subset E^d$ or $\mathscr{X}_u = (\mathscr{X}_{Cu})^d \supset E^{dd}$. Thus, $\mathscr{X}_u = E^{dd} = \mathscr{X}_E$ and 2) is proved. From this equality, we derive 3). If $\mathscr{X}_u = \mathscr{X}_E$ then $\mathscr{X}_{Cu} = (\mathscr{X}_u)^d = (\mathscr{X}_E)^d = E^{ddd} = E^d$, i.e. we arrive at 3). Now, relying on 3), we prove 1). We have $Cu\,d\,E$, $u\,d\,E^d$ (since $\mathscr{X}_u = (\mathscr{X}_{Cu})^d = E^{dd}$). It only remains to refer to Lemma 8. So, 1) implies 2), 2) implies 3), and, finally, 3) implies 1). Consequently, the above three assertions are equivalent. The proof of the theorem is complete.

[7] Thus, $\mathscr{X}_u = \mathscr{X}_{\{u\}}$ whenever $u \in \mathscr{X}$.

REMARK. Speaking about disjoint complements and bands, we have assumed that the basic set $\mathscr{X}$ is a Boolean algebra. However, the concepts of "disjointness," "disjoint complement," and "band" make sense in the case when $\mathscr{X}$ is a lattice with zero $\mathbf{0}$. Therefore, the following question is in order: Which of the above propositions are valid in a more general situation? It is easy to show that, for an arbitrary lattice with zero, the above Propositions $1°, 2°, 4°, 10°$ as well as Theorems 10, 11 (with the Corollary), and 12 are valid. Propositions 5° and 6° require the infinite distributive law; Propositions $3°, 7°, 8°, 9°$, Lemma 8, Theorems 8, 9, and 13 make sense only for Boolean algebras.

2.4 A principal ideal as a free Boolean algebra

Consider the principal ideal $\mathscr{X}_u$ generated by a nonzero element u. Furnished with the induced order, $\mathscr{X}_u$ is a partially ordered set. Furthermore, it is clear that $\mathscr{X}_u$ is a distributive lattice with zero and unity; here, the role of unity is played by u (which is the greatest element of the band $\mathscr{X}_u$). In $\mathscr{X}_u$, each element x has a complement equal to the difference $u - x$. Indeed,

$$x \vee (u - x) = u, \;\; x \wedge (u - x) = \mathbf{0}.$$

So, $\mathscr{X}_u$ is a Boolean algebra if equipped with the order induced from $\mathscr{X}$. However, we emphasize that $\mathscr{X}_u$ is not a subalgebra of $\mathscr{X}$ (with the exception of the case $u = \mathbf{1}$).

2.5 A principal band projection

Each principal ideal $\mathscr{X}_u$ is connected with the operation P_u that associates with each $x \in \mathscr{X}$ the element $x \wedge u \in \mathscr{X}_u$. We call P_u the *principal band projection* onto $\mathscr{X}_u$. It is clear that P_u is an *isotonic mapping* of $\mathscr{X}$ onto $\mathscr{X}_u$. The following properties are also obvious:

1°. $P_u(x \wedge y) = P_u(x) \wedge P_u(y)$;
2°. $P_u(x \vee y) = P_u(x) \vee P_u(y)$;
3°. $P_u(Cx) = u \wedge Cx$;
4°. $P_u(\mathbf{1}) = u$;
5°. $P_u(\mathbf{0}) = \mathbf{0}$.

The image

$$P_u(E) = u \wedge E$$

of a set E is called the *projection* or the *trace* of E onto $\mathscr{X}_u$. In the sequel, we denote this image by $[E]_u$.

2.6 Decompositions and direct sums

Considering a Boolean algebra $\mathscr{X}$, distinguish some total[8] disjoint set U. Basing on Theorem 8, it is easy to show that the least upper bound of U exists and equals $\mathbf{1}$. Next, using the distributive law, we verify that, for each $x \in \mathscr{X}$, the following equality holds:

$$x = \sup(x \wedge U) = \bigvee_{u \in U} P_u(x).$$

In this case, the system of bands $\{\mathscr{X}_u\}$ is said to form a *decomposition* of the Boolean algebra $\mathscr{X}$; and the Boolean algebra $\mathscr{X}$ is said to be the *direct sum* of the bands $\mathscr{X}_u$; this fact is written by the formula:

$$\mathscr{X} = \bigoplus_{u \in U} \mathscr{X}_u$$

(the symbols $\mathbf{S}$, $\oplus\sum$, and the like are also used). Decompositions are often used for representing a given Boolean algebra as a direct sum of algebras of a simpler structure. As an example, the Boolean algebra 2^Q is the direct sum of the principal ideals corresponding to the individual points $q \in Q$. We now change the statement of the problem. Take an arbitrary set ξ of Boolean algebras. Consider the set-theoretic product $\mathscr{Z}$ of the algebras in ξ. This means that $\mathscr{Z}$ is the totality of functions z that are defined on ξ and possess the following property: for every $\mathscr{X} \in \xi$, the containment $z(\mathscr{X}) \in \mathscr{X}$ holds.

Next, endow $\mathscr{Z}$ with the "natural" pointwise order, assuming that $z_1 \leq z_2$ whenever the inequality $z_1(\mathscr{X}) \leq z_2(\mathscr{X})$ is satisfied for each $\mathscr{X} \in \xi$ in the sense of the order of $\mathscr{X}$. It can be easily verified that, furnished with this order, $\mathscr{Z}$ turns out to be a Boolean algebra in which the role of unity is played by the function $z_{\mathbf{1}}(\mathscr{X}) \equiv \mathbf{1}_{\mathscr{X}}$ and the role of zero is played by $z_{\mathbf{0}}(\mathscr{X}) \equiv \mathbf{0}_{\mathscr{X}}$. In $\mathscr{Z}$, each $\mathscr{X}_0 \in \xi$ corresponds to the principal ideal $\mathscr{Z}_{u_0}$, where

$$u_0(\mathscr{X}) = \begin{cases} \mathbf{0}_{\mathscr{X}}, & \text{if } \mathscr{X} \neq \mathscr{X}_0, \\ \mathbf{1}_{\mathscr{X}}, & \text{if } \mathscr{X} = \mathscr{X}_0. \end{cases}$$

If regarded as a free Boolean algebra, this band is obviously isomorphic to the algebra $\mathscr{X}_0$. The so-constructed Boolean algebra $\mathscr{Z}$ is called the *direct sum* of algebras in ξ. Clearly, the principal ideals that correspond to the algebras form a decomposition of $\mathscr{Z}$.

---

[8]See 2.2.

3. Factorization, homomorphisms, independence, and free Boolean algebras

3.1 Quotient algebras

Let $\mathscr{X}$ be a Boolean algebra and let I be an ideal other than $\mathscr{X}$. Endow $\mathscr{X}$ with the EQUIVALENCE generated by I; we write $x \overset{I}{\sim} y$ whenever $| x - y | \in I$. The elements x and y are called I-equivalent.[9]

This gives rise to a partition of $\mathscr{X}$ into disjoint cosets. We call them I-cosets. The fact that some elements x and y belong to the same coset means exactly that x and y are I-equivalent. Let $\widehat{\mathscr{X}}$ be the totality of all I-cosets. It turns out possible to make $\widehat{\mathscr{X}}$ into a Boolean algebra in a natural way.

Theorem 14. *Let $x \overset{I}{\sim} x'$ and $y \overset{I}{\sim} y'$. Then the following hold:*

a) $Cx \overset{I}{\sim} Cx'$;

b) $x \vee y \overset{I}{\sim} x' \vee y'$;

c) $x \wedge y \overset{I}{\sim} x' \wedge y'$.

PROOF. The property a) is obvious, since $|x - x'| = |Cx - Cx'|$. To prove b) and c), we use the inequalities 8° and 9° (p. 23):

$$|x \vee y - x' \vee y'| \leq |x - x'| \vee |y - y'|,$$

$$|x \wedge y - x' \wedge y'| \leq |x - x'| \vee |y - y'|.$$

If $| x - x' |$, $| y - y' | \in I$ then, taking the main property of ideals into account, we conclude that

$$|x \vee y - x' \vee y'|, |x \wedge y - x' \wedge y'| \in I,$$

and the theorem is proved.

Corollary 1. *If $x' \leq y \leq x''$ and $x' \overset{I}{\sim} x''$ then $y \overset{I}{\sim} x'$ and $y \overset{I}{\sim} x''$.*

Indeed, $y = y \vee x' \overset{I}{\sim} y \vee x'' = x''$. Analogously, $y \overset{I}{\sim} x'$.

Corollary 2. *If $x \leq y$ and $x' \overset{I}{\sim} x$ then there is an element $y' \geq x'$ I-equivalent to y.*

As such an element we can take $y' \equiv y \vee x'$. Analogously, we establish the following:

Corollary 3. *If $x \leq y$ and $y' \overset{I}{\sim} y$ then there exists an element $x' \leq y'$ that is I-equivalent to x.*

[9] The so-defined relation is indeed an equivalence: reflexivity and symmetry are obvious, while transitivity follows from the inequality $|x - y| \leq |x - z| \vee |z - y|$ implying that $x \overset{I}{\sim} z$ and $z \overset{I}{\sim} y$ yield $x \overset{I}{\sim} y$.

Furnish the set $\widehat{\mathscr{X}}$ of cosets with an order by assuming $\widehat{x} \le \widehat{y}$ whenever[10] there exist elements $x \in \widehat{x}$ and $y \in \widehat{y}$ satisfying the inequality $x \le y$. (We use the same symbol $\le$ for $\mathscr{X}$ and for $\widehat{\mathscr{X}}$ as this leads to no confusion.) By Corollaries 2 and 3, if $\widehat{x} \le \widehat{y}$ then, for every $x \in \widehat{x}$ (every $y \in \widehat{y}$), there is $y' \in \widehat{y}$ ($x' \in \widehat{x}$) satisfying the inequality $x \le y'$ ($x' \le y$).

We now prove that the relation $\le$ is an order. It is clear that, for $\widehat{x} = \widehat{y}$, we have $\widehat{x} \le \widehat{y}$ and $\widehat{x} \ge \widehat{y}$. Conversely, assume that the two inequalities hold. For each $x \in \widehat{x}$, there exist $y', y'' \in \widehat{y}$ satisfying the inequalities $y' \le x \le y''$. By Corollary 1, $x \in \widehat{y}$. Therefore, $\widehat{x} \subset \widehat{y}$. Analogously, $\widehat{x} \supset \widehat{y}$ and the cosets $\widehat{x}$ and $\widehat{y}$ coincide.

It remains to verify the transitivity of $\le$. Let $\widehat{x} \le \widehat{y}$ and $\widehat{y} \le \widehat{z}$. Taking an arbitrary $y \in \widehat{y}$ on using Corollaries 2 and 3, find elements $x \in \widehat{x}$ and $z \in \widehat{z}$ such that the inequalities $x \le y \le z$ hold. This proves that $\widehat{x} \le \widehat{y}$. So, the relation in $\widehat{\mathscr{X}}$ we introduced is indeed a partial order. In fact, we have even constructed a Boolean algebra as will be shown later. But first we point out a fact important for the future presentation.

Lemma 9. *Let $z \overset{I}{\sim} x \vee y$ ($z \overset{I}{\sim} x \wedge y$). Then there exist elements $x' \overset{I}{\sim} x$ and $y' \overset{I}{\sim} y$ such that $z = x' \vee y'$ ($z = x' \wedge y'$).*

We consider the case of join. The proof consists in producing the desired elements x' and y'. Namely, we put

$$x' \equiv x \wedge z, \ \ y' \equiv (y \wedge z) \vee (z \wedge C(x \vee y)).$$

It is clear that $x' \vee y' = z$; moreover, the relations

$$\mid x' - x \mid \ = x \wedge Cz \le (x \vee y) \wedge Cz \le \ \mid x \vee y - z \mid \ \in I,$$

$$\begin{aligned} \mid y' - y \mid \ &= (Cz \wedge y) \vee (z \wedge C(x \vee y)) \le (Cz \wedge (x \vee y)) \\ &\vee (z \wedge C(x \vee y)) = \ \mid x \vee y - z \mid \ \in I \end{aligned}$$

show the equivalence of x and x' as well as y and y'. The case of meet bound is settled by duality.

Theorem 15. *The set $\widehat{\mathscr{X}}$ is a Boolean algebra.*

PROOF. Note first that the cosets containing $\mathbf{0}$ and $\mathbf{1}$ are the least and greatest elements of $\widehat{\mathscr{X}}$, i.e. zero and unity. The former coincides with the ideal I, while the latter is dual to I. Second, we prove the existence of suprema and infima. Let $\widehat{x}, \ \widehat{y} \in \widehat{\mathscr{X}}$; and let $\widehat{z}$ be the I-coset formed by all elements of the type $x \vee y$, where $x \in \widehat{x}, \ y \in \widehat{y}$ ($\widehat{z}$ is an I-coset by Theorem 14 and Lemma 9). It is clear that $\widehat{z} \ge \widehat{x}, \widehat{y}$. Let now $\widehat{u}$ be an

[10] Here $\widehat{x}$ and $\widehat{y}$ are elements of $\widehat{\mathscr{X}}$, i.e. cosets in $\mathscr{X}$.

upper bound for the pair $\{\widehat{x}, \widehat{y}\}$. By choosing an arbitrary representative $u \in \widehat{u}$ and using Corollaries 2 and 3, we find elements $x \in \widehat{x}$ and $y \in \widehat{y}$ such that $x, y \leq u$. Then $x \vee y \leq u$ and $x \vee y \in \widehat{z}$, whence $\widehat{z} \leq \widehat{y}$, i.e., $\widehat{z} = \sup\{\widehat{x}, \widehat{y}\}$. The fact that the coset $\{z \mid z = x \wedge y, x \in \widehat{x}, y \in \widehat{y}\}$ is the infimum for the pair $\widehat{x}, \widehat{y}$ is verified analogously. We see that $\widehat{\mathscr{X}}$ is a lattice. It is easy to prove distributivity of $\widehat{\mathscr{X}}$: for all $\widehat{x}$, $\widehat{y}$, and $\widehat{z}$, the cosets $(\widehat{x} \vee \widehat{y}) \wedge \widehat{z}$ and $(\widehat{x} \wedge \widehat{z}) \vee (\widehat{y} \wedge \widehat{z})$ consist of all elements in $\mathscr{X}$ of the form

$$(x \wedge z) \vee (y \wedge z), \quad x \in \widehat{x}, y \in \widehat{y}, z \in \widehat{z},$$

and, consequently, coincide. It remains to check that complements exist. If $\widehat{x} \in \widehat{\mathscr{X}}$ then the totality of all Cx, with $x \in \widehat{x}$, is an I-coset by Theorem 14. Denoting it by $\widehat{x}'$, we see that $\widehat{x} \wedge \widehat{x}'$ and $\widehat{x} \vee \widehat{x}'$ are I-cosets that contain $\mathbf{0}$ and $\mathbf{1}$, i.e. they are the zero and unity of $\widehat{\mathscr{X}}$. The proof of the theorem is complete.

From the above proof, it is easy to grasp the meaning of the Boolean operations $\vee$, $\wedge$, and C in $\widehat{\mathscr{X}}$: the operations on COSETS are interpreted as the operations on SETS.

The Boolean algebra $\widehat{\mathscr{X}}$ is called the *quotient algebra*[11] of $\mathscr{X}$ by I and is denoted by $\mathscr{X}|_I$.

3.2 Homomorphisms

Let $\mathscr{X}$ and $\mathscr{Y}$ be two Boolean algebras. A mapping Φ from $\mathscr{X}$ into $\mathscr{Y}$ is said to be a *homomorphism* if Φ enjoys the following properties:

a) $\Phi(x \vee y) = \Phi(x) \vee \Phi(y)$;
b) $\Phi(x \wedge y) = \Phi(x) \wedge \Phi(y)$;
c) $\Phi(Cx) = C\Phi(x)$.

In other words, a homomorphism is a mapping permutable with all main Boolean operations. It is clear that if Φ is a homomorphism then, for every finite set $E \subset \mathscr{X}$, we have

$$\Phi(\sup E) = \sup \Phi(E), \quad \Phi(\inf E) = \inf \Phi(E).$$

The equalities $\Phi(\mathbf{0}) = \mathbf{0}$ and $\Phi(\mathbf{1}) = \mathbf{1}$ are also obvious. (Recall that we use the same symbols for the Boolean operations, zeros, and unities of $\mathscr{X}$ and $\mathscr{Y}$.)

The equalities

$$\Phi(x \vee y) = \Phi(C(Cx \wedge Cy)); \quad \Phi(x \wedge y) = \Phi(C(Cx \vee Cy))$$

show that one of the conditions a) and b) in the definition of homomorphism is superfluous. Therefore, to prove that a given mapping is

[11]The quotient algebra is said to result from $\mathscr{X}$ by "factorization by I."

a homomorphism, it is sufficient to establish for example that the equalities a) and c) hold.

One of the main properties of a homomorphism is the *isotonicity property*: if $x \leq y$ then $\Phi(x) = \Phi(x \wedge y) = \Phi(x) \wedge \Phi(y) \leq \Phi(y)$. (However, we note that not every isotonic mapping of a Boolean algebra is a homomorphism.)

The *kernel* of a homomorphism Φ is the set

$$\ker \Phi \equiv \{x \mid \ \Phi(x) = \mathbf{0}\}.$$

The following lemma is obvious:

Lemma 10. *The kernel of a homomorphism is an ideal.*

The kernel of a homomorphism Φ consists only of zero if and only if the mapping Φ is injective. Such a homomorphism is called a *monomorphism.* If $\Phi(\mathscr{X}) = \mathscr{Y}$ then Φ is said to be an *epimorphism.* (Obviously, an isomorphism is a mapping that is a monomorphism and epimorphism simultaneously.)

The concepts of homomorphism and quotient algebra are closely connected with each other. This connection is described by the following theorems of homomorphism.

Theorem 16. *Let $\mathscr{X}$ be a Boolean algebra and let I be an ideal in $\mathscr{X}$. The mapping that associates with each $x \in \mathscr{X}$ the I-coset $\widehat{x}$ containing x is an epimorphism from $\mathscr{X}$ onto the quotient algebra $\mathscr{X}|_I$.*

This homomorphism is called the *natural* (or *canonical) epimorphism* from $\mathscr{X}$ onto $\mathscr{X}|_I$.

Theorem 17. *Let $\mathscr{X}$ and $\mathscr{Y}$ be two Boolean algebras, let Φ be a homomorphism from $\mathscr{X}$ into $\mathscr{Y}$, and let I be the kernel of Φ. The homomorphic image $\Phi(\mathscr{X})$ is a subalgebra of $\mathscr{Y}$ isomorphic to the quotient algebra $\mathscr{X}|_I$.*

In fact, we have proved the former of the above theorems along with Theorem 15 when we found out the sense of the Boolean operations in $\widehat{\mathscr{X}} = \mathscr{X}|_I$. We dwell upon proving the latter of the above theorems. By the definition of a homomorphism, it is clear that $\Phi(\mathscr{X})$ is a subalgebra of $\mathscr{Y}$. It is also clear that the homomorphism Φ maps each I-coset $\widehat{x} \in \widehat{\mathscr{X}}$ into a SINGLE point $y \in \mathscr{Y}$; at the same time, the images of elements in different I-cosets cannot coincide.[12] We see that I-cosets are the inverse images of points in $\Phi(\mathscr{X})$. Consequently, the homomorphism Φ generates a one-to-one correspondence $\widetilde{\Phi}$ between $\mathscr{X}|_I$ and $\Phi(\mathscr{X})$.

[12] If $\Phi(x_1) = \Phi(x_2)$ then $\Phi(\,|\,x_1 - x_2\,|\,) = \mathbf{0}$ and $x_1 \overset{I}{\sim} x_2$.

If $\widehat{x} \leq \widehat{y}$ then there exist $x \in \widehat{x}$ and $y \in \widehat{y}$ such that $\Phi(x) \leq \Phi(y)$; that is why $\widetilde{\Phi}(\widehat{x}) \leq \widetilde{\Phi}(\widehat{y})$. Analogously, the inequality $\widehat{x} \geq \widehat{y}$ implies $\widetilde{\Phi}(\widehat{x}) \geq \widetilde{\Phi}(\widehat{y})$. We have proved that $\widetilde{\Phi}$ is an isomorphism from $\mathscr{X}|_I$ onto $\Phi(\mathscr{X})$. The proof of the theorem is complete.

This theorem is a special case of the general theorem of homomorphism for the so-called "universal algebras."

Theorem 16 shows that each ideal I of a Boolean algebra is related to some homomorphism, namely, to the natural homomorphism from the algebra onto the quotient algebra $\mathscr{X}|_I$. On the other hand, according to Theorem 17, each homomorphism may be regarded as a homomorphism onto some quotient algebra; the ideal corresponding to the quotient algebra is the kernel of the homomorphism. For this reason, the concepts of ideal, quotient algebra, and homomorphism are equivalent to each other in the sense that the study of one of these concepts may replace the study of the rest of them.

The question arises: What happens with the quotient algebra $\mathscr{X}|_I$ if we replace the ideal I by a wider ideal I^*? It is easy to see that the ideal I^* can be naturally identified with the ideal

$$I' \equiv \{\widehat{x} \in \mathscr{X}|_I \mid \widehat{x} \subset I^*\}$$

of $\mathscr{X}|_I$, and the quotient algebra $\mathscr{X}|_{I^*}$ with $(\mathscr{X}|_I)|_{I'}$. In what follows we consider the so-called "maximal" ideals that never admit a nontrivial extension; the quotient algebra by such an ideal consists only of the two elements: zero and unity.

In conclusion, consider the diagram:

$$\begin{array}{ccc} \mathscr{X} & \xrightarrow{\Phi} & \mathscr{Y} \\ \Psi\downarrow & & \downarrow\Omega \\ \widehat{\mathscr{X}} & \xrightarrow{\widehat{\Phi}} & \widehat{\mathscr{Y}} \end{array}$$

where Ψ and Ω are given epimorphisms; so, $\widehat{\mathscr{X}}$ and $\widehat{\mathscr{Y}}$ may be regarded as quotient algebras. The homomorphism Φ is also given. Does there exist a homomorphism $\widehat{\Phi}$ such that the diagram commutes? In other words, is it possible to "descend" the action of the homomorphism Φ from the algebras $\mathscr{X}$ and $\mathscr{Y}$ to the quotient algebras? It is easy to verify that for existence of $\widehat{\Phi}$ it is necessary and sufficient that the inclusion

$$\Phi(\ker\Psi) \subset \ker\Omega$$

hold. Indeed, let $\widehat{\Phi}$ exist. Then, if $x \in \ker\Psi$ and $y = \Phi(x)$, we have $\Omega(y) = \widehat{\Phi}(\widehat{x})$, where $\widehat{x} = \Psi(x)$; i.e., $\widehat{x} = \mathbf{0}$, $\Omega(y) = \widehat{\Phi}(\mathbf{0}) = \mathbf{0}$, and $y \in \ker\Omega$. We see that $\Phi(\ker\Psi) \subset \ker\Omega$.

Conversely, assume that the last inclusion holds. Fix an arbitrary $\widehat{x} \in \widehat{\mathscr{X}}$. For every $x', x'' \in \widehat{x}$, we have $|\, x' - x'' \,| \in \ker \Psi$. Therefore, $\Phi(\,|\, x' - x'' \,|\,) = |\, \Phi(x') - \Phi(x'') \,| \in \ker \Omega$. Consequently, all elements $\Phi(x)$, where $x \in \widehat{x}$, belong to a single coset of $\mathscr{Y}$ by $\ker \Omega$. We denote this coset by $\widehat{y} \in \widehat{\mathscr{Y}}$ and take it as $\widehat{\Phi}(\widehat{x})$. It is easy to verify that the so-constructed mapping $\widehat{\Phi} : \widehat{\mathscr{X}} \longrightarrow \widehat{\mathscr{Y}}$ is a homomorphism. We say that the homomorphism $\widehat{\Phi}$ is *induced* by the original homomorphism Φ. We point out a special case in which Ω is an ISOMORPHISM. Then for existence of an induced homomorphism it is necessary and sufficient that the equality $\Phi(\ker \Psi) = \{\mathbf{0}\}$ hold. Finally, consider the case in which Φ is an isomorphism. Then the induced homomorphism $\widehat{\Phi}$ is an isomorphism if and only if $\Phi(\ker \Psi) = \ker \Omega$.

3.3 Examples of quotient algebras

It is by means of factorization that many Boolean algebras are generated. We indicate the most important of them.

Let $\mathscr{X}$ be a Boolean algebra and let m be a quasimeasure on $\mathscr{X}$. Consider the set I of all elements of zero quasimeasure:

$$I \equiv \{x \mid \ m(x) = 0\}.$$

It is clear that I is an ideal. (The elements of I are called "m-negligible.") Arrange the quotient algebra $\widehat{\mathscr{X}} \equiv \mathscr{X}|_I$. Let x' and x'' be two I-equivalent elements, i.e., $m(|\, x' - x'' \,|) = 0$. The relations

$$m(x' \wedge x'') \leq m(x'), \ m(x'') \leq m(x' \vee x'')$$

$$= m(|\, x' - x'' \,|) + m(x' \wedge x'') = m(x' \wedge x'')$$

show that, in this case, $m(x') = m(x'')$. We define a real function $\widehat{m}$ on the quotient algebra $\widehat{\mathscr{X}}$ as follows. Given $\widehat{x} \in \widehat{\mathscr{X}}$, we put the value $\widehat{m}(\widehat{x})$ equal to the common value of all numbers $m(x)$, $x \in \widehat{x}$. Check that the so-defined function $\widehat{m}$ is an essentially positive quasimeasure on $\widehat{\mathscr{X}}$. First, we verify additivity of $\widehat{m}$. Let $\widehat{x}$ and $\widehat{y}$ be disjoint elements of $\widehat{\mathscr{X}}$ and let x and y be arbitrarily chosen representatives of the cosets. We have:

$$\widehat{m}(\widehat{x} \vee \widehat{y}) = m(x \vee y) = m[(x \wedge Cy) + (x \wedge y) + (y \wedge Cx)]$$

$$= m(x \wedge Cy) + m(x \wedge y) + m(y \wedge Cx).$$

By disjointness of $\widehat{x}$ and $\widehat{y}$, we have

$$m(x \wedge y) = 0, \quad m(x) = m(x \wedge Cy) + m(x \wedge y) = m(x \wedge Cy),$$

$$m(y) = m(y \wedge Cx) + m(x \wedge y) = m(y \wedge Cx),$$

and so

$$\widehat{m}(\widehat{x} \vee \widehat{y}) = m(x) + m(y) = \widehat{m}(\widehat{x}) + \widehat{m}(\widehat{y}).$$

Hence, $\widehat{m}$ is a quasimeasure. The equality $\widehat{m}(\widehat{x}) = 0$ means that $\widehat{x} = I$, i.e., $\widehat{x}$ coincides with the zero of $\widehat{\mathscr{X}}$. Therefore, the quasimeasure $\widehat{m}$ is essentially positive. We have established that *there is an essentially positive quasimeasure on the quotient algebra* $\widehat{\mathscr{X}} \equiv \mathscr{X}|_I$. The above implies that $\widehat{\mathscr{X}}$ *is a Boolean algebra with the countable chain condition.*

Example. Consider an arbitrary MEASURE SPACE which is by definition a triple $\{\Omega, \mathscr{E}, m\}$, where Ω is a set, $\mathscr{E}$ is a σ-algebra of subsets of Ω, and m is a countably additive nonnegative function ("measure") on $\mathscr{E}$. Applying the above construction to the algebra $\mathscr{X} = \mathscr{E}$, we come to the quotient algebra $\widehat{\mathscr{X}}$ and define the essentially positive quasimeasure $\widehat{m}$ on $\widehat{\mathscr{X}}$. It is easy to verify that the quasimeasure is indeed a measure in the sense of the definition of Section 3 of Chapter 0.

The quotient algebra $\widehat{\mathscr{X}}$, considered together with the measure $\widehat{m}$, is called the *metric structure of a measure space.*

3.4 Important examples of metric structures

Let I^n be the n-dimensional cube $\underbrace{[0,1] \times \cdots \times [0,1]}_{n \text{ times}}$, let $\mathscr{E}_l^{(n)}$ be the algebra of Lebesgue-measurable subsets in I, and let $l \equiv l^{(n)}$ be Lebesgue "measure" on I. We call the space $\{I^n, \mathscr{E}_l^{(n)}, l\}$, according to its dimension, the *Lebesgue interval, Lebesgue square,* etc. The algebra $\mathscr{E}_l^{(n)}$ is called the *Lebesgue algebra.* We denote the metric structure of the n-dimensional Lebesgue cube by E_0^n and write E_0 instead of E_0^1. All these Boolean algebras are called *Lebesgue algebras.*

At last, we consider an even more general class of measure spaces. Let Γ be a nonempty set and let Ω be the totality of all functions on Γ with values in $[0,1]$. The set Ω is the *product* of the intervals $I_\gamma \equiv [0,1]$, $\gamma \in \Gamma$; it is denoted by the symbol $\Omega \equiv \Omega_\Gamma \equiv \prod_{\gamma\in\Gamma} I_\gamma$. Every finite subset $\Gamma' \subset \Gamma$ is related with the "partial" product $\Omega_{\Gamma'} = \prod_{\gamma\in\Gamma'} I_\gamma$ which is a finite-dimensional unit cube. Distinguish a Lebesgue measurable subset A of the cube. Next, consider the CYLINDRICAL set C_A in Ω with "base" A: we define C_A by the equality

$$C_A \equiv A \times \prod_{\gamma\in\Gamma\setminus\Gamma'} I_\gamma = A \times \Omega_{\Gamma\setminus\Gamma'}.$$

This equality means that

$$C_A = \{\omega \in \Omega \mid \ \omega|_{\Gamma'} \in A\},$$

where $\omega|_{\Gamma'}$ is the restriction of ω to Γ'. It is easy to prove that the totality Z of all cylindrical subsets of Ω is an algebra of sets. A quasimeasure on this algebra is introduced by the relation

$$l'_{\Gamma'} = l_{\Gamma'}(C_A),$$

where $l_{\Gamma'}$ is Lebesgue "measure" on the finite-dimensional unit cube $\Omega_{\Gamma'}$. The definition is correct despite the easy fact that each cylindrical set has infinitely many representations C_A, $A \subset \Gamma'$, with various Γ'. It may be verified that the quasimeasure l' satisfies the conditions of the classical Lebesgue–Carathéodory Theorem.[13] There exist a σ-algebra $\mathscr{E}$ and a countably additive function l on $\mathscr{E}$; moreover, $\mathscr{E}$ contains Z, and the equality $lC_A = l'C_A$ holds for every cylindrical set.[14] Applying the above factorization process to the measure space $\{\Omega, \mathscr{E}, l\}$, we obtain a Boolean algebra. Denote this algebra by E^{Γ}. There is an essentially positive quasimeasure $\widehat{l}$ on E^{Γ}. Later, we will see that the quasimeasure is, in fact, a measure in the sense of the definition of Chapter 0. If the cardinality of Γ is equal to $\aleph_\alpha$ then we sometimes write E^{Γ}_{α} instead of E^{Γ}. We will also see later that the algebra E^{Γ}_{0} is isomorphic to the Lebesgue algebra E_0 for each Γ. For the "quotient" Lebesgue measure, we often use the notation λ instead of $\widehat{l}$.

3.5 Construction of a homomorphism given values at generators

Consider the following question. Let $\mathscr{X}$ and $\mathscr{Y}$ be two Boolean algebras, let E be a system of generators for $\mathscr{X}$, and let Φ_0 be a mapping on E with values in $\mathscr{Y}$. Under which conditions does there exist a homomorphism from $\mathscr{X}$ into $\mathscr{Y}$ that coincides with Φ_0 on E? The answer is given by the following theorem of R. Sikorski.[15]

We assume that, in the case $u \in E \cup CE$, the equality

$$\Phi_0(Cu) = C\Phi_0(u) \tag{4}$$

holds; so, we assume the mapping Φ_0 to be defined not only on E but also on $E \cup CE$. For the elements $u \in CE \setminus E$, we additionally define Φ_0 by (4).

---

[13]A proof of the theorem may be found in Chapter 7.
[14]See, for example, N. Dunford and J. T. Schwartz [1, pp. 218–229].
[15]R. Sikorski [1].

Theorem 18. *For existence of a homomorphism that coincides with* Φ_0 *on the set of generators* E*, it is necessary and sufficient that, for every finite subset* $\{u_1, u_2, \dots, u_m\}$ *of* $E \cup CE$ *satisfying the condition* $u_1 \wedge u_2 \wedge \dots \wedge u_m = \mathbf{0}$*, the equality*

$$\Phi_0(u_1) \wedge \Phi_0(u_2) \wedge \dots \wedge \Phi_0(u_m) = \mathbf{0}$$

holds.

PROOF. NECESSITY of our condition is obvious; we prove SUFFICIENCY.

Represent every element $z \in \mathscr{X}$ in the form of a polynomial in generators:

$$z = \bigvee_{k=1}^{n} (u_{k1} \wedge u_{k2} \wedge \dots \wedge u_{kp_k}), \tag{5}$$

$$u_{kj} \in E \cup CE \quad (1 \le j \le p_k,\ 1 \le k \le n).$$

Suppose that there exists another representation for the same element:

$$z = \bigvee_{i=1}^{m} (v_{i1} \wedge v_{i2} \wedge \dots \wedge v_{iq_i}), \tag{6}$$

$$v_{is} \in E \cup CE \quad (1 \le s \le q_i,\ 1 \le i \le m).$$

Using the identities from the proof of Lemma 3 in Section 1 of this chapter (see p. 41), we obtain:

$$\mathbf{0} = \mid z - z \mid = \Bigg\{ \Big[\bigvee_{k=1}^{n} (u_{k1} \wedge u_{k2} \wedge \dots \wedge u_{kp_k}) \Big]$$

$$\wedge\, C \Big[\bigvee_{i=1}^{m} (v_{i1} \wedge v_{i2} \wedge \dots \wedge v_{iq_i}) \Big] \Bigg\} + \Bigg\{ \Big[\bigvee_{i=1}^{m} (v_{i1} \wedge v_{i2} \wedge \dots \wedge v_{iq_i}) \Big]$$

$$\wedge\, C \Big[\bigvee_{k=1}^{n} (u_{k1} \wedge u_{k2} \wedge \dots \wedge u_{kp_k}) \Big] \Bigg\} = \Bigg\{ \Big[\bigvee_{k=1}^{n} (u_{k1} \wedge u_{k2} \wedge \dots \wedge u_{kp_k}) \Big]$$

$$\wedge \left[\bigvee_{\substack{1 \le i_1 \le q_1 \\ 1 \le i_2 \le q_2 \\ \cdots\cdots\cdots\cdots \\ 1 \le i_m \le q_m}} (Cv_{1i_1} \wedge Cv_{2i_2} \wedge \dots \wedge Cv_{mi_m}) \right] \Bigg\}$$

$$+ \Bigg\{ \Big[\bigvee_{i=1}^{m} (v_{i1} \wedge v_{i2} \wedge \dots \wedge v_{iq_i}) \Big]$$

$$\wedge \left[\bigvee_{\substack{1 \le k_1 \le p_1 \\ 1 \le k_2 \le p_2 \\ \cdots\cdots\cdots\cdots \\ 1 \le k_n \le p_n}} (Cu_{1k_1} \wedge Cu_{2k_2} \wedge \cdots \wedge Cu_{nk_n}) \right] \Bigg\}$$

$$= \bigvee_{\substack{k=1,2,\ldots,n \\ 1 \le i_1 \le q_1 \\ 1 \le i_2 \le q_2 \\ \cdots\cdots\cdots\cdots \\ 1 \le i_m \le q_m}} (u_{k1} \wedge u_{k2} \wedge \cdots \wedge u_{kp_k} \wedge Cv_{1i_1} \wedge Cv_{2i_2} \wedge \cdots \wedge Cv_{mi_m})$$

$$+ \bigvee_{\substack{i=1,2,\ldots,m \\ 1 \le k_1 \le p_1 \\ 1 \le k_2 \le p_2 \\ \cdots\cdots\cdots\cdots \\ 1 \le k_n \le p_n}} (v_{i1} \wedge v_{i2} \wedge \cdots \wedge v_{iq_i} \wedge Cu_{1k_1} \wedge Cu_{2k_2} \wedge \cdots \wedge Cu_{nk_n}).$$

We see that the expressions in parentheses under the last two supremum signs are equal to zero. According to the main hypothesis of the theorem, replacing all u_{kj} and v_{is} by $\Phi_0(u_{kj})$ and $\Phi_0(v_{is})$ in the expressions, we obtain zeros again. Next, transforming the symmetric difference

$$\Delta = \Big| \bigvee_{k=1}^{n} [\Phi_0(u_{k1}) \wedge \Phi_0(u_{k2}) \wedge \cdots \wedge \Phi_0(u_{kp_k})]$$

$$- \bigvee_{i=1}^{m} [\Phi_0(v_{i1}) \wedge \Phi_0(v_{i2}) \wedge \cdots \wedge \Phi_0(v_{iq_i})] \Big|$$

by analogy with what was done with the difference $|z - z|$, we reduce Δ to the form

$$\Delta = \bigvee_{\substack{k=1,2,\ldots,n \\ 1 \le i_1 \le q_1 \\ 1 \le i_2 \le q_2 \\ \cdots\cdots\cdots\cdots \\ 1 \le i_m \le q_m}} \begin{array}{l} [\Phi_0(u_{k1}) \wedge \Phi_0(u_{k2}) \wedge \cdots \wedge \Phi_0(u_{kp_k}) \\ \quad \wedge C\Phi_0(v_{1i_1}) \wedge C\Phi_0(v_{2i_2}) \wedge \cdots \wedge C\Phi_0(v_{mi_m})] \end{array}$$

$$+ \bigvee_{\substack{i=1,2,\ldots,m \\ 1 \le k_1 \le p_1 \\ 1 \le k_2 \le p_2 \\ \cdots\cdots\cdots\cdots \\ 1 \le k_n \le p_n}} \begin{array}{l}[\Phi_0(v_{i1}) \wedge \Phi_0(v_{i2}) \wedge \cdots \wedge \Phi_0(v_{iq_i}) \\ \quad \wedge C\Phi_0(u_{1k_1}) \wedge C\Phi_0(u_{2k_2}) \wedge \cdots \wedge C\Phi_0(u_{nk_n})],\end{array}$$

whence it is seen that $\Delta = \mathbf{0}$ and

$$\bigvee_{k=1}^{n} [\Phi_0(u_{k1}) \wedge \Phi_0(u_{k2}) \wedge \cdots \wedge \Phi_0(u_{kp_k})]$$

$$= \bigvee_{i=1}^{m} [\Phi_0(v_{i1}) \wedge \Phi_0(v_{i2}) \wedge \cdots \wedge \Phi_0(v_{iq_i})].$$

Therefore, we may define the desired mapping Φ by the equality

$$\Phi(z) = \bigvee_{k=1}^{n} [\Phi_0(u_{k1}) \wedge \Phi_0(u_{k2}) \wedge \cdots \wedge \Phi_0(u_{kp_k})]$$

using to this end EACH of the representations of z in the form (5) or (6). It may be easily verified that the so-constructed mapping is a homomorphism. Indeed, the equality $\Phi(z_1) \vee \Phi(z_2) = \Phi(z_1 \vee z_2)$ is obvious; the inequality (7*) in Chapter 0 shows that the equalities

$$Cz = \bigvee_{\substack{1 \le k_1 \le p_1 \\ 1 \le k_2 \le p_2 \\ \cdots\cdots\cdots\cdots \\ 1 \le k_n \le p_n}} [Cu_{1k_1} \wedge Cu_{2k_2} \wedge \cdots \wedge Cu_{nk_n}]$$

and

$$C\Phi(z) = \bigvee_{\substack{1 \le k_1 \le p_1 \\ 1 \le k_2 \le p_2 \\ \cdots\cdots\cdots\cdots \\ 1 \le k_n \le p_n}} [C\Phi(u_{1k_1}) \wedge C\Phi(u_{2k_2}) \wedge \cdots \wedge C\Phi(u_{nk_n})]$$

hold simultaneously, i.e., $\Phi(Cz) = C\Phi(z)$. Taking account of what was said on the page 57, we finish the verification. The proof of the theorem is complete.

REMARK. As is seen from the above proof, the homomorphism Φ is one-to-one if and only if the equality

$$\Phi_0(u_1) \wedge \Phi_0(u_2) \wedge \cdots \wedge \Phi_0(u_m) = \mathbf{0}$$

implies

$$u_1 \wedge u_2 \wedge \cdots \wedge u_m = \mathbf{0}.$$

We consider a "stronger" form of the Sikorski Theorem. Let $\{x_t\}_{t\in T}$ and $\{y_t\}_{t\in T}$, where $x_t \in \mathscr{X}$ and $y_t \in \mathscr{Y}$ $(t \in T)$, be two FAMILIES defined on the same arbitrary index set T. Let the set of all x_t generate the Boolean algebra $\mathscr{X}$.

Theorem 18*. *For existence of a homomorphism $\Phi : \mathscr{X} \longrightarrow \mathscr{Y}$ satisfying the condition $\Phi(x_t) = y_t$ $(t \in T)$, it is necessary and sufficient that the relation*

$$x_{t_1} \wedge x_{t_2} \wedge \cdots \wedge x_{t_s} \wedge Cx_{t_{s+1}} \wedge \cdots \wedge Cx_{t_m} = \mathbf{0}$$

imply

$$y_{t_1} \wedge y_{t_2} \wedge \cdots \wedge y_{t_s} \wedge Cy_{t_{s+1}} \wedge \cdots \wedge Cy_{t_m} = \mathbf{0}.$$

This theorem follows from the preceding. We only need to note that, if the condition of the theorem is satisfied, the equality

$$x_{t_1} = x_{t_2} \tag{1}$$

implies

$$y_{t_1} = y_{t_2}. \tag{2}$$

Indeed, (1) means that $x_{t_1} \wedge Cx_{t_2} = x_{t_2} \wedge Cx_{t_1} = \mathbf{0}$. Then $y_{t_1} \wedge Cy_{t_2} = y_{t_2} \wedge Cy_{t_1} = \mathbf{0}$, i.e., (2) holds. Now, we may introduce a mapping Φ_0 associating with each $x \in \mathscr{X}$ of the form $x = x_t$ the common value $\Phi_0(x)$ of all $y_{t'}$ for which $x_{t'} = x_t$. It is easy to verify that Φ_0 satisfies the conditions of the "basic" Sikorski Theorem and, therefore, permits an extension to a homomorphism Φ, which is desired. The necessity part of this theorem is as obvious as that of the preceding theorem.

It is convenient to demonstrate a way of application of the Sikorski Theorem by the example of a subalgebra generated by a LINEARLY ORDERED system of generators.

Let $E = \{e\}$ be a chain of elements in the Boolean algebra $\mathscr{X}$ and let Φ_0 be a mapping from E into some Boolean algebra $\mathscr{Y}$. It is clear that isotonicity of this mapping is NECESSARY for existence of a homomorphism from the subalgebra $\mathscr{X}\langle E\rangle$ into $\mathscr{Y}$ which is by definition an extension of Φ_0. We prove SUFFICIENCY. Suppose that the mapping Φ_0

is isotonic, i.e., $e_1 \leq e_2$ implies $\Phi_0(e_1) \leq \Phi_0(e_2)$. Suppose that some set of generators $e_1, e_2, \ldots, e_m \in E$ satisfies the equality

$$e_1 \wedge e_2 \wedge \cdots \wedge e_p \wedge Ce_{p+1} \wedge \cdots \wedge Ce_m = \mathbf{0},$$

where p is a number between 1 and m. Since E is a chain, we conclude that there is a least element among $e_1, e_2, \ldots, e_p$ and there is a greatest among $e_{p+1}, e_{p+2}, \ldots, e_m$. Denote these elements by $\underline{e}$ and $\overline{e}$. Then, obviously,

$$e_1 \wedge e_2 \wedge \cdots \wedge e_p = \underline{e}, \quad Ce_{p+1} \wedge Ce_{p+2} \wedge \cdots \wedge Ce_m = \overline{e}$$

and

$$\underline{e} \wedge C\overline{e} = \mathbf{0},$$

whence $\underline{e} \leq \overline{e}$. According to isotonicity of Φ_0, this inequality implies $\Phi_0(\underline{e}) \leq \Phi_0(\overline{e})$ and

$$\mathbf{0} = \Phi_0(\underline{e}) \wedge C\Phi_0(\overline{e}) \geq \Phi_0(e_1) \wedge \Phi_0(e_2)$$

$$\wedge \cdots \wedge \Phi_0(e_p) \wedge C\Phi_0(e_{p+1}) \wedge \cdots \wedge C\Phi_0(e_m).$$

Therefore,

$$\Phi_0(e_1) \wedge \Phi_0(e_2) \wedge \cdots \wedge \Phi_0(e_p) \wedge C\Phi_0(e_{p+1}) \wedge \cdots \wedge C\Phi_0(e_m) = \mathbf{0}$$

and the condition of the Sikorski Theorem is satisfied which guarantees existence of the desired homomorphism.

Later, we will apply Theorem 18 in the circumstances antipodal to the above situation. Namely, we will consider the case in which the mapping Φ_0 is defined on an INDEPENDENT system of generators. This system does not contain comparable elements; the exact definitions will be given below.

We note that, in the above reasoning, we have specified the structure of the subalgebra generated by a linearly ordered system E. In this case, elementary polynomials have the form $e' \wedge Ce''$ $(e', e'' \in E)$. Therefore, the general form of an element in $\mathscr{X}\langle E\rangle$ is given by the formula

$$x = (e_1' \wedge Ce_1'') \vee (e_2' \wedge Ce_2'') \vee \cdots \vee (e_m' \wedge Ce_m''),$$

where $e_1', e_1'', e_2', e_2'', \ldots, e_m', e_m''$ are elements of the system E.

We may assume that the elements $e_k' \wedge Ce_k''$ on the right-hand side are pairwise disjoint. Indeed, if

$$(e_k' \wedge Ce_k'') \wedge (e_i' \wedge Ce_i'') > \mathbf{0}$$

then, as is easily seen,

$$(e'_k \wedge Ce''_k) \vee (e'_i \wedge Ce''_i) = \overline{e} \wedge C\underline{e}$$

where[16]

$$\overline{e} \equiv \max\{e'_k, e'_i\}, \quad \underline{e} \equiv \min\{e''_k, e''_i\}.$$

Consequently, summing up all nondisjoint elements, we arrive in finitely many steps at the representation of the element x as a finite sum of pairwise disjoint elements of the form $\overline{e} \wedge C\underline{e}$, $\overline{e}, \underline{e} \in E$.

We present an important

Example. Let Q be a nonempty interval $\langle a, b\rangle$ (the case $Q = [-\infty, +\infty]$ is not excluded). Consider the Boolean algebra 2^Q and select such a subalgebra $\mathscr{R}_{\langle a,b\rangle}$ of it that is generated by the set E including all intervals of the form

$$\Delta_t^- \equiv Q \cap (-\infty, t), \quad \Delta_t^+ \equiv Q \cap (-\infty, t].$$

It is clear that the system E is linearly ordered by inclusion; indeed, if $t_1 < t_2$ then $\Delta_{t_1}^-, \Delta_{t_1}^+ \subset \Delta_{t_2}^-, \Delta_{t_2}^+$, and if $t_1 = t_2 = t$ then, obviously, $\Delta_t^- \subset \Delta_t^+$. Therefore, the Boolean algebra $\mathscr{R}_{\langle a,b\rangle}$ consists of all sets representable as finite sums of pairwise disjoint intervals. Sometimes, the subalgebras $\mathscr{R}^+_{\langle a,b\rangle}$ and $\mathscr{R}^-_{\langle a,b\rangle}$ generated by the sets $E^+ \equiv \{\Delta_t^+\}$ and $E^- \equiv \{\Delta_t^-\}$ are also considered. The form of elements in these subalgebras can be easily specified. In view of the above facts, we may assert that *each isotonic mapping from the system E (E^-, E^+) into an arbitrary Boolean algebra $\mathscr{X}$ can be extended to a homomorphism from the algebra $\mathscr{R}_{\langle a,b\rangle}$ $(\mathscr{R}^-_{\langle a,b\rangle}, \mathscr{R}^+_{\langle a,b\rangle})$ into $\mathscr{X}$*. Such a homomorphism, as well as all homomorphisms defined on algebras of sets, is often called a "Boolean measure."

3.6 Free Boolean algebras

We have showed that, in the case of a finite algebra, the question, if there exists an independent system of generators, is solved by a simple calculation. For infinite algebras, the situation is significantly more difficult. The algebras with an independent system of generators constitute a very important class: *free* Boolean algebras.

Theorem 19. *Let $\mathscr{X}$ be a free Boolean algebra with an independent system of generators E. Each mapping Φ_0 from E into an arbitrary Boolean algebra $\mathscr{Y}$ can be extended to a homomorphism from $\mathscr{X}$ into $\mathscr{Y}$.*

[16] Using the notations max..., min... instead of the usual $\vee, \wedge$, we emphasize that the system E is linearly ordered. That is why, for example, $\overline{e}$ is equal to e'_k or to e'_i.

This theorem is an obvious corollary to Theorem 18. In fact, it reduces to the assertion that every relation between "free generators" expressed in terms of the symbols $\vee, \wedge, C$ is identically valid for every Boolean algebra. In turn, the following corollary to Theorem 19 is immediate:

Theorem 20. *Every two free Boolean algebras $\mathscr{X}$ and $\widetilde{\mathscr{X}}$ possessing independent systems E and $\widetilde{E}$ of generators of the same cardinality are isomorphic. Moreover, each bijection from E onto $\widetilde{E}$ can be extended to an isomorphism from $\mathscr{X}$ onto $\widetilde{\mathscr{X}}$.*

The following theorem relates to the same range of questions.

Theorem 21. *Each nonempty set E admits a bijection onto an independent system of generators for some Boolean algebra $\mathscr{X}$.*

To demonstrate, consider the Boolean algebra $\mathscr{Y} = 2^{2^E}$. According to Remark 1 on Theorem 7, the system $\mathscr{E}$ of subsets Q_e, with $e \in E$, constructed as in 1.1.8 is independent. It remains to put $\mathscr{X} = \mathscr{Y}\langle\mathscr{E}\rangle$.

Corollary. *There exist free Boolean algebras of each infinite cardinality.*

Of course, the independent system of generators has the same cardinality as the algebra in this event.

In fact, the general form of a finite free Boolean algebra is available. According to Theorem 7 of this chapter, a free Boolean algebra on n generators is an arbitrary Boolean algebra consisting of 2^{2^n} elements. In particular, so is the Boolean algebra of n-place Boolean functions; this is the simplest model of a free Boolean algebra. If we take the totality of all Boolean functions in arbitrarily many variables then we do not obtain a free Boolean algebra; it is necessary to restrict consideration to the functions "depending essentially " on finitely many arguments.

3.7 A Cantor discontinuum

The theorem has already been mentioned claiming that each Boolean algebra is isomorphic to an algebra of sets. We now have an opportunity to prove this fact in the case of a free Boolean algebra. The case of a finite free Boolean algebra was discussed above; now we consider a more general case. We return to Example 2 (see p. 16).

Let Ξ be a set of cardinality $\mathfrak{a}$. Let X be the cartesian Ξth power of $\{0,1\}$; i.e., the set-theoretic product

$$X \equiv X_\Xi \equiv \prod_{\xi\in\Xi} \delta_\xi,$$

where every δ_ξ is the two-point space $\{0,1\}$. In other words, X_Ξ is the totality of all functions defined on ξ with values 0 and 1: $X_\Xi \equiv \{0,1\}^\Xi$.

The points of X_Ξ are regarded as families and denoted as follows:

$$\chi \equiv \{\chi_\xi\}_{\xi\in\Xi} \quad (\chi_\xi = 0, 1).$$

To each ξ there corresponds the *projection* π_ξ that is the function defined by the equality $\pi_\xi(\chi) = \chi_\xi$. If $\Xi = \mathbb{N} = \{1, 2, \dots\}$ then we obtain the most important case in which the points of $X_\mathbb{N}$ are various binary sequences

$$\chi = (\chi_1, \chi_2, \dots) \quad (\chi_i = 0, 1).$$

The set X may be considered from various standpoints.

1) The elements $\chi \in X$ are CHARACTERISTIC FUNCTIONS of subsets in Ξ. Therefore, X is a *Boolean algebra* isomorphic to the boolean 2^Ξ. The cardinality of this algebra is equal to 2^a. We have already mentioned this class of algebras in Chapter 0.

2) The same set X may be regarded as an ABELIAN GROUP with respect to the operation $+_2$ defined as $\chi' +_2 \chi'' = \{\chi'_\xi +_2 \chi''_\xi\}$. The sign $+_2$ means the ADDITION MODULO 2:

$$0 +_2 0 = 0,\ \ 0 +_2 1 = 1 +_2 0 = 1,\ \ 1 +_2 1 = 0.$$

Zero is the family that consists only of 0s; the addition coincides with subtraction. We may write $|\chi' - \chi''|$ instead of $\chi' +_2 \chi''$. Adding modulo 2 the characteristic functions of two sets, we obtain the characteristic function of their symmetric difference.

3) The set X is a cartesian power of the two-point set $\{0, 1\}$ which may be regarded as a discrete compact topological space. Therefore, X is furnished with the Tychonoff topology, the weakest topology ensuring the continuity of all projections π_ξ. According to the celebrated Tychonoff Theorem, the space X is *compact*.

4) It may be easily verified that the group and topological structures described in 2) and 3) agree so that X becomes a *compact topological group*. This group is called the *Cantor discontinuum* (or *Cantor space*) of weight $\mathfrak{a}$. We apply this term to the set X too, neglecting the group-theoretic or topological structures.

5) Finally, the set X appears in many problems of probability theory. Imagine the following (generally, infinite) game: At "time" ξ, we toss a coin. Observing "heads" (the observe of the coin) we write 1, whereas observing "tails" (the reverse of the coin) we write 0. After the game is finished, we obtain the point $\chi = \{\chi_\xi\}$ of the Cantor space. The point χ is the history of the game. Various subsets of X may be interpreted as *events* that could happen depending on the fact which points χ were realized.

Consider the Boolean algebra 2^X of all subsets of X. The cardinality of this algebra is equal to 2^{2^a} but, for an infinite a, it is not a free Boolean algebra. However, of the utmost interest is one subalgebra of this algebra which will be described now.

Put

$$Q_\xi \equiv \pi_\xi^{-1}\{1\} \equiv \{\chi \in X \mid \chi_\xi = 1\}.$$

We have already met these sets in fact (see p. 47). While proving Theorem 7, we have seen that this system is independent. We recall that the intersection

$$Q_{\xi_1} \cap Q_{\xi_2} \cap \dots \cap Q_{\xi_k} \cap CQ\xi_{k+1} \cap \dots \cap CQ_{\xi_m} \tag{7}$$

is nonempty, since it contains each point χ such that $\chi_{\xi_1} = \dots = \chi_{\xi_k} = 1$, $\chi_{\xi_{k+1}} = \dots = \chi_{\xi_m} = 0$. The above property means the desired independence.

The cardinality of the set of all $\{Q_\xi\}$ is equal to $card\,\Xi$. Therefore, the *subalgebra generated by the system* $\{Q_\xi\}$ *is a free Boolean algebra on* $card\,\Xi$ *generators*. We denote this algebra by $\mathscr{D}_\Xi$ or simply by $\mathscr{D}$. Here, of importance is only the cardinality of Ξ. So, we obtain the following assertion: *Each free Boolean algebra* (regardless the number of generators) *is isomorphic to an algebra of sets.*

The elements of $\mathscr{D}$ are all finite sums of sets of the form (7):

$$\bigcup_{i=1}^{s} (Q_{\xi_1^i} \cap Q_{\xi_2^i} \cap \dots \cap Q_{\xi_{k_i}^i} \cap CQ\xi_{k_i+1}^i \cap \dots \cap CQ_{\xi_{m_i}^i}). \tag{8}$$

Denote by Ξ' the finite set of all indices appearing in the latter sum: $\Xi' \equiv \{\xi_1^1, \dots, \xi_{m_s}^s\}$. We may represent an arbitrary element of $\mathscr{D}$ as follows:

$$Q' \times \prod_{\xi \in \Xi \setminus \Xi'} \delta_\xi, \tag{9}$$

where Q' is a subset of the "partial" product $\prod_{\xi \in \Xi'} \delta_\xi$. It is natural to call this set a "finite-dimensional cylinder" with Q' the "base of the cylinder." All cylinders of this type are contained in $\mathscr{D}$. For this reason, we conclude that *the algebra* $\mathscr{D}_\Xi$ *of finite-dimensional cylinders in* X_Ξ *is a free Boolean algebra on some independent system of generators and the cardinality of this system is* $card\,\Xi$. The above construction is the main model of a free Boolean algebra. We may consider the characteristic functions of sets in $\mathscr{D}$. Then we obtain the Boolean functions "essentially dependent" on finitely many arguments. This model has already been discussed in the case when $card\,\Xi$ is a natural number.

Using the Tychonoff Theorem, it is easy to verify that, regarding X_Ξ as a topological power of the discrete two-point space, we obtain a TOTALLY DISCONNECTED compact Hausdorff space. Further, the base of the Tychonoff topology consists of all clopen (i.e. closed and open) sets that coincide with cylinders of the form (9). If $card\,\Xi = \aleph_0$ then it is well known[17] how to construct a homeomorphic mapping from X_Ξ onto the usual ("ternary") Cantor set in the interval $[0,1]$. Thus, we obtain a representation of a free Boolean algebra with a countable system of generators as an algebra of all clopen subsets of a "classical" Cantor discontinuum.

3.8 Metric independence

Let $\mathscr{X} = 2^Q$, where Q is a finite set that contains 2^n elements. We know that $\mathscr{X}$ has an independent system of generators $E = \{Q_k\}_{k=1}^n$ such that every intersection of the form

$$Q^{i_1 i_2 \dots i_k}_{i_{k+1}\dots i_m} \equiv Q_{i_1} \cap Q_{i_2} \cap \dots \cap Q_{i_k} \cap CQ_{i_{k+1}} \cap \dots \cap CQ_{i_m},$$

with i_s pairwise distinct, contains exactly 2^{n-m} points of Q. Along with $\mathscr{X}$, consider another free algebra $\mathscr{Y} = 2^{\widetilde{Q}}$ of the same cardinality. Let $\widetilde{E} = \{\widetilde{Q}_k\}_{k=1}^n$ be a system of generators for $\mathscr{Y}$. According to Theorem 17 the mapping $Q_k \longrightarrow \widetilde{Q}_k$ $(k = 1, 2, \dots, n)$ may be extended to an isomorphism from $\mathscr{X}$ onto $\mathscr{Y}$. It is clear that each isomorphism transforms the singletons of Q into the singletons of $\widetilde{Q}$. Therefore, the system $\widetilde{E}$ must possess the same property: every intersection

$$\widetilde{Q}^{i_1 i_2 \dots i_k}_{i_{k+1}\dots i_m} \equiv \widetilde{Q}_{i_1} \cap \widetilde{Q}_{i_2} \cap \dots \cap \widetilde{Q}_{i_k} \cap C\widetilde{Q}_{i_{k+1}} \cap \dots \cap C\widetilde{Q}_{i_m}$$

with pairwise distinct $i_1 i_2 \dots i_m$ contains exactly 2^{n-m} points of $\widetilde{Q}$. This property is common for all independent systems of generators in all free algebras of the form 2^Q.

Recalling now existence of the main "measure" μ_0 in each algebra of the form 2^Q (the measure of x is the number of points in x divided by 2^n), we see that, in the free algebras 2^Q, every independent system of generators E enjoys the following important property: for all pairwise distinct $e_1, e_2, \dots, e_m \in E$ and all $p = 1, 2, \dots, m$ the equality

$$\begin{aligned} \mu_0(e_1 \wedge e_2 \wedge \dots \wedge e_p \wedge Ce_{p+1} \wedge \dots \wedge Ce_m) &= \frac{1}{2^m} \\ &= \mu_0 e_1 \cdot \mu_0 e_2 \cdots \mu_0 e_p \cdot \mu_0 Ce_{p+1} \cdots \mu_0 Ce_m \end{aligned} \tag{10}$$

[17] See K. Kuratowski [2, p. 32].

holds.

We now let $\mathscr{X}$ be an arbitrary normed Boolean algebra endowed with a probability measure μ_0 (see 0.3). A system E is called μ_0-independent whenever for all pairwise distinct $e_1, e_2, \dots, e_m \in E$ and all $p = 1, 2, \dots, m$ the equality (10) holds. The above definition presents the concept of *metric independence* usual in probability theory. We see that it arises naturally from the earlier introduced concept of algebraic independence. *Every algebraically independent system of generators for a finite free algebra must also be metrically independent with respect to the "main" probability measure.* It is also easy to see that *metric independence always implies algebraic independence.*[18]

3.9 Construction of a quasimeasure given its values at generators

Let $\mathscr{X}$ be a Boolean algebra and let E be a system of generators for $\mathscr{X}$. Suppose, that there is a finite real function μ_0 defined on E. The question arises: Under which conditions is this function extendible to a quasimeasure on $\mathscr{X}$? First of all, we consider the case of an INDEPENDENT system E.

Theorem 22. *For a real function φ_0 on an independent system of generators E to be extendible to a quasimeasure φ on the algebra $\mathscr{X}$, it is necessary and sufficient that the values of φ_0 be nonnegative and bounded above.*

PROOF. The necessity of these conditions is obvious. We prove their sufficiency. Suppose that all values of φ_0 lie in the interval $[0, p]$. We may consider the algebra $\mathscr{Y}$ of all Lebesgue measurable subsets of $[0, p]$. Let l be the quasimeasure on this algebra that coincides with Lebesgue "measure." Consider the family $\{e_t\}_{0 \le t \le p}$ of all intervals of the form $[0, t]$; obviously, $l(e_t) = t$ and all values of φ_0 are listed among those of l. Send E to $\mathscr{Y}$ by assigning to each $x \in E$ the element $e_t \in \mathscr{Y}$ for which $l(e_t) = t = \varphi_0(x)$. By Theorem 10, this mapping may be extended to a homomorphism Γ from $\mathscr{X}$ into $\mathscr{Y}$. Given $x \in \mathscr{X}$, we now put

$$\varphi(x) = l(\Gamma(x)).$$

The additivity of φ follows from the general properties of a homomorphism and the additivity of l. It is also obvious that this function is nonnegative and extends the original function φ_0. Thus, the desired quasimeasure is constructed. Note that if we do not require the extension φ to be a quasimeasure and restrict consideration to constructing

[18]We recall that all measures are essentially positive by definition.

an additive real function then this is possible without any assumptions on φ: *each finite function* φ_0 *on an independent system of generators admits an additive extension on* $\mathscr{X}$ (see 3.11).

Assume now that the system E is linearly ordered. The following theorem is valid.

Theorem 23. *For a real function* φ_0 *on the linearly ordered system of generators* E *to be extendible to a quasimeasure* φ *on* $\mathscr{X}$, *it is necessary and sufficient that the values of* φ_0 *be nonnegative and bounded above and that the inequality* $x \leq y$ *imply* $\varphi_0(x) \leq \varphi_0(y)$ *for all* $x, y \in E$.

PROOF. Likewise in the preceding theorem, we need only to prove the sufficiency part.

Define the Boolean algebra $\mathscr{Y}$, the family $\{e_t\}$ of elements in $\mathscr{Y}$, and the mapping from E into $\mathscr{Y}$ as above. In our case, this mapping is clearly isotonic and so extendible to a homomorphism Γ from $\mathscr{X}$ into $\mathscr{Y}$. We are done on repeating the final part of the proof of Theorem 22.

3.10 A Boolean algebra as an algebraic system

Many constructions and concepts for Boolean algebras (ideal, homomorphism, factorization, etc.) belong to general algebra. We have defined the concept of Boolean algebra as a poset satisfying some axioms. Another approach is also possible in which the starting point is not an order relation but rather the system of the main Boolean operations whose properties are prescribed by the axioms. Now, a Boolean algebra is a ring of a special form. This gives an opportunity to apply the well-known theorems of the theory of rings to Boolean algebras. We start with the following definition.

A *Boolean ring* is a unital associative ring whose every element is idempotent. The operation of addition in a Boolean ring $\mathscr{X}$ we denote by the sign $+_2$ (the reasons for this choice of notation will become clear later); multiplication, as usual, is denoted by the dot which sometimes is omitted. The condition of idempotency means that the equalities $x^2 = x \cdot x = x$ hold for all $x \in \mathscr{X}$. It is easy to verify that every Boolean ring is always commutative and the equality $x +_2 x = \mathbf{0}$ holds for all $x \in \mathscr{X}$.

Our interest in Boolean rings is explained by the following theorem.

Theorem 24. *Each Boolean algebra* $\mathscr{X}$ *is a Boolean ring with respect to the operations* $+_2$ *and* $\cdot$ *defined as follows:*

$$x +_2 y \equiv |x - y|, \quad x \cdot y \equiv x \wedge y.$$

However, the zero and unity of this ring coincide with those of $\mathscr{X}$.

The proof reduces simply to verifying the axioms of a Boolean ring. This is not difficult and left to the reader. For instance, it is clear that the main condition of idempotency is satisfied since the equalities $x \cdot x = x \wedge x = x$ hold always.

There is a theorem converse to Theorem 24 in some sense. Namely, each Boolean ring $\mathscr{X}$ can be made into a partially ordered set with the order relation defined by the following condition: $x \leq y$ whenever $x \cdot y = x$. Endowed with this order, $\mathscr{X}$ is a Boolean algebra whose zero and unity coincide with those of the ring. The proof of this is not complicated.

Many facts will become clear if, using the Stone Theorem, we represent an algebra $\mathscr{X}$ as a system of subsets of some set $\mathfrak{A}$ and (as in 3.7) consider the totality $E(\mathscr{X})$ of all characteristic functions of these sets. To each $x \in \mathscr{X}$, we can uniquely put into correspondence the characteristic function $\chi_{e(x)}$ of some subset $e(x) \subset \mathfrak{A}$. This correspondence is an isomorphism if, as usual, we assume that the set of functions is naturally ordered. So, the Boolean algebra $E(\mathscr{X})$ of characteristic functions is isomorphic to the original algebra $\mathscr{X}$. We make the totality F of ALL real functions on $\mathfrak{A}$ into a ring: multiplication is defined in the routine fashion

$$(x \cdot y)(q) \equiv x(q)y(q), \quad q \in \mathfrak{A}, \tag{11}$$

and the operation of addition is the addition modulo 2:

$$(x +_2 y)(q) \equiv x(q) +_2 y(q) \tag{12}$$

(recall that the sum modulo 2 is the remainder of the usual sum divided by 2). Then $E(\mathscr{X})$ is a subring of the ring F; moreover, as is easily verified, the ring multiplication and addition coincide on $E(\mathscr{X})$ with the operations defined by (11) and (12).

Sometimes it is convenient to consider a Boolean algebra as an *algebra over the field* $\{0, 1\}$ (the residue field modulo 2). In other words, we add the structure of a vector space over this field to the structure of a ring.

Interpretation of the algebra of logic as an arithmetic of residues modulo 2 was given in 1927 by I. I. Zhegalkin.[19]

As might be noted by the reader, we let the same letter $\mathscr{X}$ stand for a Boolean ALGEBRA and the corresponding Boolean RING; moreover, $\mathscr{X}$ simultaneously signifies the underlying set. This usual inaccuracy of mathematics does not imply any inconveniences; moreover, it gives an opportunity to employ, depending on circumstances, either the "lattice" or "ring" terminology for studying $\mathscr{X}$. The reader may form a dictionary

[19] I. I. Zhegalkin [1].

for translation from one language into the other. For instance, the term "subalgebra" corresponds to the term "subring" in the "ring" language. Some words sound identically in both languages, for example, so is the word "ideal": it is easy to establish that each ideal in the sense of the definition on p. 48 is an ideal in the sense of ring theory and vice versa.

The interpretation of the main constructions and concepts related to Boolean algebras in terms of general algebra is established now within the modern theory of universal algebras and algebraic systems.[20]

3.11 Independence of subalgebras. Simultaneous extension of homomorphisms and quasimeasures. Bernoulli measures

A system of subalgebras $\{\mathscr{X}_t\}_{t\in T}$ of a Boolean algebra $\mathscr{X}$ is called *independent* whenever each "selection"

$$\{x_t\}_{t\in T},\ \ x_t \in \mathscr{X}_t^+ \quad (t\in T)$$

is independent, i.e. if $x_{t_1}\wedge x_{t_2}\wedge\cdots\wedge x_{t_m} > \mathbf{0}$ for all pairwise distinct $t_1, t_2, \ldots, t_m \in T$ and for all $x_{t_i}\in\mathscr{X}_{t_i}$.

Consider the following situation: the subalgebras $\mathscr{X}_t$ are independent (form an independent system) and $\mathscr{X} = \mathscr{X}\langle\bigcup_{t\in T}\mathscr{X}_t\rangle$. In this case, the Boolean algebra $\mathscr{X}$ consists of all elements of the form

$$x = u_1 + u_2 + \cdots + u_n, \quad n = n(x), \tag{13}$$

where

$$u_i = x_1^{(i)}\wedge x_2^{(i)}\wedge\cdots\wedge x_m^{(i)},\ \ x_k^{(i)}\in\mathscr{X}_{t_k} \quad (i=1,2,\ldots,n,\ \ k=1,2,\ldots,n)$$

and $t_1, t_2, \ldots, t_m$ are pairwise distinct indices. The elements u_i can be written as

$$u_i = \bigwedge_{t\in T} x_{ti}$$

assuming that $x_{ti}\in\mathscr{X}_t$ and all but finitely many x_{ti} are equal to $\mathbf{1}$. We call such an algebra $\mathscr{X}$ the *preproduct* of $\{\mathscr{X}_t\}$.

Assume given homomorphisms $\Phi_t:\ \mathscr{X}_t\longrightarrow\mathscr{Y}$, where $\mathscr{Y}$ is a Boolean algebra. Existence of simultaneous extensions for these homomorphisms is easily deduced from the Sikorski Theorem. In other words, there exists a homomorphism Φ such that $\Phi|_{\mathscr{X}_t} = \Phi_t$ for all $t\in T$. Clearly, that such an extension is unique since every element of the form (13) must

---

[20] G. Birkhoff [2] and A. G. Kurosh [1].

obey the following relation:

$$\Phi(x) = \sum_{i=1}^{n} \Phi_{t_1}(x_1^{(i)}) \wedge \cdots \wedge \Phi_{t_m}(x_m^{(i)}).$$

Of the utmost importance is the case in which all Φ_t are isomorphisms:

$$\mathscr{X}_t \xrightarrow{\Phi_t} \mathscr{Y}_t,$$

where $\mathscr{Y}_t$ are independent subalgebras of $\mathscr{Y}$ and, moreover,

$$\mathscr{Y} = \mathscr{Y}\langle \bigcup_{t \in T} \mathscr{Y}_t \rangle.$$

It is easy to understand that *in this case the simultaneous extension of* Φ *is an isomorphism from* $\mathscr{X}$ *onto* $\mathscr{Y}$.

We now pass from homomorphisms to real additive functions. For simplicity, we confine exposition to the case in which the system of subalgebras $\mathscr{X}_1, \mathscr{X}_2, \ldots, \mathscr{X}_m$ is finite and independent. Let $\varphi_1, \varphi_2, \ldots, \varphi_m$ be real additive functions on $\mathscr{X}_1, \mathscr{X}_2, \ldots, \mathscr{X}_m$ respectively. Their simultaneous extension φ on the entire Boolean algebra $\mathscr{X}$ may be constructed in many ways. We describe the most important of these constructions. Let $\varphi_i(\mathbf{1}) = 1$ $(i = 1, 2, \ldots, m)$.

First of all, we point out the following fact: if $x_1 \wedge x_2 \wedge \cdots \wedge x_m \le x_1' \wedge x_2' \wedge \cdots \wedge x_m'$, where $x_i, x_i' \in \mathscr{X}_i$, then $x_i \le x_i'$ for all i. Indeed, if $x_{i_0} \not\le x_{i_0}'$ then $x_{i_0} \wedge Cx_{i_0}' > \mathbf{0}$ and, by independence,

$$\begin{aligned} C(x_1' \wedge \cdots \wedge x_m') \wedge (x_1 \wedge \cdots \wedge x_m) &\ge Cx_{i_0}' \wedge (x_1 \wedge \cdots \wedge x_m) \\ &= x_1 \wedge x_2 \wedge \cdots \wedge x_{i_0-1} \wedge Cx_{i_0}' \wedge x_{i_0} \wedge x_{i_0+1} \wedge \cdots \wedge x_m > \mathbf{0}, \end{aligned}$$

which contradicts the original inequality. Hence, the element u *admits a unique representation in the form*

$$u = x_1 \wedge \cdots \wedge x_m. \tag{14}$$

Therefore, taking

$$\varphi(u) \equiv \varphi_1(x_1)\varphi_2(x_2)\ldots\varphi_m(x_m),$$

for every such u, we obtain a single-valued function. Show that this function is additive. The main case is $m = 2$. Let

$$u = \sum_{i=1}^{s} u_i, \quad u = x_1 \wedge x_2, \quad x_1 \in \mathscr{X}_1, \; x_2 \in \mathscr{X}_2,$$

$$u_i = x_1^i \wedge x_2^i, \quad x_k^i \in \mathscr{X}_k^+ \quad (k = 1, 2, \ i = 1, 2, \dots, s).$$

The subalgebras $\mathscr{X}\langle x_1^1, x_1^2, \dots, x_1^s\rangle$ and $\mathscr{X}\langle x_2^1, x_2^2, \dots, x_2^s\rangle$ are finite. Hence, they are induced by some partitions of unity

$$\tau_1 \equiv \{e_1^1, \dots, e_1^n\}, \quad \tau_2 \equiv \{e_2^1, \dots, e_2^n\}.$$

It is easy to see that

$$u \equiv x_1 \wedge x_2 = \sum_{p,q:\, e_1^p \le x_1,\, e_2^q \le x_2} e_1^p \wedge e_2^q = \sum_{p:\, e_1^p \le x_1} e_1^p \wedge \sum_{q:\, e_2^q \le x_2} e_2^q,$$

$$u_i \equiv x_1^i \wedge x_2^i = \sum_{p,q:\, e_1^p \le x_1^i,\, e_2^q \le x_2^i} e_1^p \wedge e_2^q = \sum_{p:\, e_1^p \le x_1^i} e_1^p \wedge \sum_{q:\, e_2^q \le x_2^i} e_2^q,$$

$$\varphi(u) = \varphi_1\Big(\sum_{p:\, e_1^p \le x_1} e_1^p\Big)\varphi_2\Big(\sum_{q:\, e_2^q \le x_2} e_2^q\Big) = \sum_{p,q:\, e_1^p \le x_1,\, e_2^q \le x_2} \varphi_1(e_1^p)\varphi_2(e_2^q),$$

$$\varphi(u_i) = \varphi_1\Big(\sum_{p:\, e_1^p \le x_1^i} e_1^p\Big)\varphi_2\Big(\sum_{q:\, e_2^q \le x_2^i} e_2^q\Big) = \sum_{p,q:\, e_1^p \le x_1^i,\, e_2^q \le x_2^i} \varphi_1(e_1^p)\varphi_2(e_2^q).$$

If $e_1^p \le x_1$ and $e_2^q \le x_2$ then the inequalities $e_1^p \le x_1^i$ and $e_2^q \le x_2^i$ hold for a unique i. Consequently,

$$\varphi(u) = \sum_{p,q:\, e_1^p \le x_1, e_2^q \le x_2} \varphi_1(e_1^p)\varphi_2(e_2^q)$$

$$= \sum_{i=1}^{s} \sum_{p,q:\, e_1^p \le x_1^i, e_2^q \le x_2^i} \varphi_1(e_1^p)\varphi_2(e_2^q) = \sum_{i=1}^{s} \varphi(u_i).$$

So, φ is additive. However, φ is defined only at the elements of the form (14). Extend φ to the entire algebra $\mathscr{X}$ which, as was noted, consists of the elements $u = \sum_{i=1}^n u_i$, where u_i are of the form (14). Given u, we put

$$\varphi(u) \equiv \sum_{i=1}^{n} \varphi(u_i).$$

It is necessary to prove the soundness of this definition. Assume that

$$u = \sum_{i=1}^{n} u_i = \sum_{j=1}^{m} v_j,$$

where u_i and v_j are elements of the form (14). Of the same form are the elements $u_i \wedge v_j \equiv w_{ij}$. However,

$$u_i = \sum_{j=1}^{m} w_{ij}, \quad v_j = \sum_{i=1}^{n} w_{ij}$$

and, as follows from above,

$$\varphi(u_i) = \sum_{j=1}^{m} \varphi(w_{ij}), \quad \varphi(v_j) = \sum_{i=1}^{n} \varphi(w_{ij}).$$

Therefore,

$$\varphi(u) \equiv \sum_{i=1}^{n} \varphi(u_i) = \sum_{i=1}^{n} \sum_{j=1}^{m} \varphi(w_{ij}) = \sum_{j=1}^{m} \varphi(v_j).$$

So, our definition is sound. Show that φ is additive on the entire algebra $\mathscr{X}$. If $u = \sum_{k=1}^{n} u_k$ and $u_k = \sum_{i=1}^{m_k} w_{ik}$, where w_{ik} are elements of the form (14); then

$$\sum_{k=1}^{n} \varphi(u_k) = \sum_{k=1}^{n} \sum_{i=1}^{m_k} \varphi(w_{ik}) = \varphi(u).$$

Passing to an arbitrary $m > 2$ is performed by usual induction, on recalling the independence of $\mathscr{X}_1$ and $\mathscr{X}\langle \mathscr{X}_2, \dots, \mathscr{X}_m \rangle$. It is clear that for $x \in \mathscr{X}_i$ the equality $\varphi(x) = \varphi_i(x)$ holds, and so φ is a simultaneous extension of $\varphi_1, \varphi_2, \dots, \varphi_m$.

The case of an infinite independent system reduces to the above-considered case by a standard application of the Kuratowski–Zorn Lemma. Thus, the following theorem is valid.

Theorem 25. *Let φ_t be real additive functions on the corresponding subalgebras $\mathscr{X}_t$ $(t \in T)$; let $\varphi_t(\mathbf{1}) = 1$ for all $t \in T$; and let $\{\mathscr{X}_t\}$ be an independent family generating $\mathscr{X}$. Then φ_t have a simultaneous additive extension φ on $\mathscr{X}$. This extension may be chosen so that the relation*

$$\varphi(x_{t_1} \wedge x_{t_2} \wedge \dots \wedge x_{t_m}) = \varphi_{t_1}(x_{t_1}) \dots \varphi_{t_m}(x_{t_m}) = \varphi(x_{t_1}) \dots \varphi(x_{t_m})$$

holds for all $t_1, t_2, \dots, t_m \in T$, $t_i \neq t_j$ $(i \neq j)$ and $x_{t_i} \in \mathscr{X}_{t_i}$ $(i = 1, 2, \dots, m)$.

This construction is well known in measure theory and probability, where it is accompanied with the additional requirement of "countable additivity." We consider these questions in the forthcoming chapters.

While introducing the Cantor space X_Ξ, we mentioned the possibility of some "probabilistic" interpretation of points and subsets in it. Each point is a "history of coin tossing"; Q_ξ is the event: "we observe heads at trial ξ." The player that made the bet for this event focuses his attention only on what happens at trial ξ; he is not interested in how the coin falls at other trials. Similarly, each element of $\mathscr{D}_\Xi$ can be interpreted as an event. For example, $Q_{\xi_1} \vee (Q_{\xi_2} \wedge CQ_{\xi_3})$ is the event: "we observe heads at trial ξ_1 or heads at trial ξ_2 and tails at trial ξ_3."

The genuine probability theory begins when we assign probabilities to events. In other words, the question is of how to define a probability measure. As regards the algebra $\mathscr{D}_\Xi$, such a definition can be given in many ways; we describe the most useful in which the events Q_ξ (independent generators for $\mathscr{D}_\Xi$) turn out to be metrically independent.

To each ξ, we assign the subalgebra $\mathscr{B}_\xi \equiv \{Q_\xi, CQ_\xi, \mathbf{0}, \mathbf{1}\}$ (here $\mathbf{0}$ is the empty set and $\mathbf{1}$ is the entire ξ). Given a pair of numbers p and q with the properties $p, q > 0$ and $p + q = 1$, we define the measure β_ξ in each subalgebra $\mathscr{B}_\xi$ as follows:

$$\beta_\xi(Q_\xi) \equiv p, \quad \beta_\xi(CQ_\xi) \equiv q, \quad \beta_\xi(\mathbf{0}) \equiv 0, \quad \beta_\xi(\mathbf{1}) \equiv 1.$$

This means that the probability of heads at every trial is p while the probability of tails is q. We always toss the same coin and so the numbers p and q reflect its individual properties (asymmetry). We now can define a simultaneous extension of all measures β_ξ, using the above-described scheme. Put

$$\beta(Q_{\xi_1} \wedge \cdots \wedge Q_{\xi_k} \wedge CQ_{\xi_{k+1}} \wedge \cdots \wedge CQ_{\xi_m}) = p^k q^{m-k}. \tag{15}$$

The measure extends to other elements of $\mathscr{D}_\Xi$ by additivity.

The function β is essentially positive and countably additive. The first property is obvious and the second follows from the fact that every $e \in \mathscr{D}_\Xi$ is simultaneously open and compact in the Tychonoff topology of X_Ξ. Therefore, in the equality of the form $e = e_1 + e_2 + \cdots$ $(e, e_i \in \mathscr{D}_\xi)$ the right-hand side always contains FINITELY many elements and $\varphi(e) = \varphi(e_1) + \varphi(e_2) + \cdots$.

This β is called the *Bernoulli measure*. We denote it also by β_{pq}. The main case is $p = q = \frac{1}{2}$ (the coin is absolutely symmetric). In relation to the space X, the term "Bernoulli space" is often used which implies a probability space with a Bernoulli measure. The metric independence of events that follows from the formula (15) is the "stochastic" independence which plays the central role in probability theory. Roughly speaking, the latter reflects the absence of any connection between the outcomes of particular trials.

It should be noted that the algebra $\mathscr{D}_\Xi$ is too small; it does not contain all events that might be of interest. This leads to the important problem of extending β onto a wider algebra of events. This problem is discussed in all textbooks on measure theory; we consider it in the second part of this book. We now make only one obvious remark. Irrespective of the way of extension for a measure, all singletons ("elementary events") must be negligible (if Ξ is infinite). Therefore, we may predict the history of an infinite game (i.e., distinguish a point $\chi \in X$) only with probability zero. The exceptions are the "degenerate" cases in which $p = 1, q = 0$ and $q = 1, p = 0$ (an extremely asymmetric coin that falls always on one side).

Exercises for Chapter 1

1. Prove that if I is an ideal then the set $I \cup CI$ is a subalgebra.

2. Prove that every free Boolean algebra satisfies the countable chain condition.

3. Prove that each Boolean algebra is isomorphic to a quotient algebra of some free algebra.

4. Let Q be a countable set and let I be the ideal of all finite subsets Q in the boolean $\mathscr{X} = 2^Q$. Prove that the quotient algebra $\mathscr{X}|_I$ contains continuum many disjoint elements.

5. Construct an example of a Boolean algebra that includes no nonzero bands with the countable chain condition.

Chapter 2

COMPLETE BOOLEAN ALGEBRAS

1. Complete Boolean algebras; their subalgebras and homomorphisms

1.1 Completeness of a Boolean algebra

By now, we have assumed only that in every algebra we encounter there are a supremum and an infimum of each finite subset (likewise in a general lattice). We now strengthen the requirements on an algebra.

Definition. A Boolean algebra is called *complete* whenever each subset of it possesses a supremum and an infimum.

Definition. A Boolean algebra is called *σ-complete* whenever each countable subset of it possesses a supremum and an infimum.[1]

Lemma 1. *The concepts of a principal ideal and a band coincide in every complete Boolean algebra.*

This lemma is an obvious corollary to Theorem 13 of the previous chapter.

When considering some examples of Boolean algebras in Chapter 0, we usually pointed out the cases in which we were able to prove existence of the supremum and infimum of an arbitrary subset of a given algebra. For instance, so is the algebra 2^Q, the boolean of Q (Example 1); therefore, this algebra is complete.

The algebras in Examples 2, 3, 4, 7, 8, and 9 are also complete. The proof of completeness (σ-completeness) of a Boolean algebra is often facilitated by the following lemma.

[1] Sometimes, the σ-complete Boolean algebras are called *Borel algebras*. However, it should be noted that the term "Borel algebra of a topological space" traditionally denotes the CONCRETE algebra that comprises all Borel sets.

Lemma 2. *For a Boolean algebra $\mathscr{X}$ to be complete (σ-complete), it is necessary and sufficient that each subset (respectively, each countable subset) of $\mathscr{X}$ possess an infimum.*

To prove, observe that the element $u \equiv C \inf CE$ is the supremum of E for all $E \subset \mathscr{X}$.

As usual, together with Lemma 2, the dual assertion is automatically proved whose formulation is left to the reader. We will refer to this assertion as to Lemma 2′.

Every complete algebra is obviously σ-complete. We will obtain more interesting examples of σ-complete Boolean algebras, considering the *σ-algebras of sets*; i.e., the totalities of sets which are closed under countable set-theoretic operations.

Some of these algebras are incomplete. For instance, so are the σ-algebra of all Borel subsets and the σ-algebra of all Lebesgue measurable subsets in an interval $\langle a, b\rangle$. We constantly deal with σ-algebras of sets in the classical measure theory.

We now interpret the elements of an algebra $\mathscr{X}$ as *events* and ask: What does completeness of $\mathscr{X}$ mean in practice?

As follows from Section 3 of Chapter 1, the completeness of a family of events $\mathscr{X}$ allows us to consider along with each nonempty totality of events $E \subset \mathscr{X}$ other two events y_1 and y_2 of the same family. The first consists in the simultaneous realization of all events $x \in E$ and the second, in the realization of at least one $x \in E$. The property of σ-completeness is interpreted analogously.

Interpretation of the concepts of completeness and σ-completeness in the propositional language is left to the reader.

The concept of a *σ-ideal* makes sense in a σ-complete algebra. A σ-ideal is an ideal that contains the least upper bound of its every countable subset. We will use this concept later.

In conclusion, we discuss the question of distributivity. In a complete Boolean algebra, the formulas (7)–(8′) are valid without any stipulation.

We recall that these formulas were deduced in Chapter 0 under the assumption that all necessary suprema and infima exist. Moreover, the formula

$$\bigwedge_{i=1}^{n} \bigvee_{k=1}^{\infty} x_{ik} = \bigvee_{k_1,k_2,\ldots,k_n \geq 1} (x_{1k_1} \wedge x_{2k_2} \wedge \cdots \wedge x_{nk_n}) \tag{1}$$

is valid (without additional assumptions either). The proof of the formula is not difficult.

1.2 Regular subalgebras

Let $\mathscr{X}_0$ be a subalgebra of a Boolean algebra $\mathscr{X}$. Then, for every FINITE subset $E \subset \mathscr{X}_0$, the supremum and infimum of E calculated in $\mathscr{X}$ must belong to $\mathscr{X}_0$. However, this may fail for an infinite subset, which corroborates the following definition.

Definition. A subalgebra $\mathscr{X}_0$ of a complete Boolean algebra $\mathscr{X}$ is called *regular (σ-regular)* whenever for every nonempty (countable) subset E the containments $\sup E \in \mathscr{X}_0$ and $\inf E \in \mathscr{X}_0$ hold.

REMARK. The duality relations (3), (4) in Chapter 0 show that every subalgebra closed under either of the operations sup or inf is regular. The situation is similar with σ-regularity.

Some examples of regular subalgebras may be found in Chapter 0. In particular, we met a regular subalgebra of the Boolean algebra 2^Q in Example 7.

The algebras of all Borel subsets or all Lebesgue measurable subsets of $Q \equiv \langle a, b\rangle$ are examples of the σ-regular (but not regular) subalgebras of 2^Q.

Frequently, the regular and σ-regular subalgebras are called "complete" and "σ-complete" in the literature. This terminology is confusing, since a subalgebra complete as a Boolean algebra in its own right may fail to be a regular subalgebra. The same is true for σ-completeness.

Theorem 1. *Let $\mathscr{X}$ be a complete (σ-complete) Boolean algebra and let E be a subset in $\mathscr{X}$. Then the intersection $\mathscr{Y}$ of all regular (σ-regular) subalgebras including E is a regular (σ-regular) subalgebra.*

The proof of the theorem repeats that of Theorem 11 in Chapter 1 almost word for word.

The subalgebra $\mathscr{Y}$ is clearly the least by inclusion among all regular subalgebras including E.

Thus, with each nonempty subset E of a complete algebra $\mathscr{X}$, the two subalgebras are naturally related: the least subalgebra and the least regular subalgebra including E. We denote the former by $\mathscr{X}\langle E\rangle$ and the latter, by $\overline{\mathscr{X}\langle E\rangle}$. It is clear that the inclusion $\mathscr{X}\langle E\rangle \subset \overline{\mathscr{X}\langle E\rangle}$ always holds. If E is finite then the subalgebras coincide. In the case when

$$E = E_1 \cup E_2 \cup \cdots \cup E_n,$$

we use the notation

$$\overline{\mathscr{X}\langle E_1, E_2, \cdots, E_n\rangle}$$

and in the case of singletons $E_1 = \{u_1\}, \ldots, E_s = \{u_s\}$ the notation

$$\overline{\mathscr{X}\langle u_1, u_2, \ldots, u_s, E_{s+1}, \ldots, E_n\rangle}.$$

The subalgebra $\overline{\mathscr{X}\langle E\rangle}$ is said to be the regular subalgebra fully generated by E.[2]

We know that the nonzero band $\mathscr{X}_u$ of a complete Boolean algebra may be considered as an independent Boolean algebra. Therefore, we may deal with regular subalgebras of $\mathscr{X}_u$. We call them *regular u-subalgebras* and use the notation

$$\overline{\mathscr{X}_u\langle E\rangle}.$$

Finally, the least σ-regular subalgebra that includes a set $E \subset \mathscr{X}$ is often considered (especially, in measure theory). Such a subalgebra is also called the *Borel subalgebra σ-generated by E.* We denote it by $\overset{\cdots\cdots}{\mathscr{X}\langle E\rangle}$. For many important algebras, $\overline{\mathscr{X}\langle E\rangle}$ and $\overset{\cdots\cdots}{\mathscr{X}\langle E\rangle}$ coincide for all E.

We present two simple but useful assertions.

Lemma 3. *Let $\mathscr{E}$ be a class of subsets $E \in \mathscr{X}$ such that $E = \bigcup_{e\in\mathscr{E}} e$ and let the subalgebras $\widetilde{\mathscr{X}_e}$ be defined as*

$$\widetilde{\mathscr{X}_e} \equiv \overline{\mathscr{X}\langle e\rangle}.$$

Then

$$\overline{\mathscr{X}\langle E\rangle} = \overline{\mathscr{X}\langle \bigcup_{e\in\mathscr{E}} \widetilde{\mathscr{X}_e}\rangle}.$$

Proof. It is clear that each subalgebra $\widetilde{\mathscr{X}_e}$ lies in $\overline{\mathscr{X}\langle E\rangle}$. Therefore, the regular subalgebra $\overline{\mathscr{X}\langle \bigcup_{e\in\mathscr{E}} \widetilde{\mathscr{X}_e}\rangle}$ generated by $\widetilde{\mathscr{X}_e}$ is in $\overline{\mathscr{X}\langle E\rangle}$. On the other side, $\overline{\mathscr{X}\langle \bigcup_{e\in\mathscr{E}} \widetilde{\mathscr{X}_e}\rangle} \supset E$; hence, the desired equality must hold by minimality of $\overline{\mathscr{X}\langle E\rangle}$. The proof of the lemma is complete.

Lemma 4. *If $\overline{\mathscr{X}\langle E\rangle} = \mathscr{X}$ then for every $u \neq \mathbf{0}$ the equality*[3]

$$\overline{\mathscr{X}_u\langle [E]_u\rangle} = \mathscr{X}_u$$

holds.

Proof. Arrange the set

$$\mathscr{Y} \equiv \{y \mid \ y \wedge u \in \overline{\mathscr{X}_u\langle [E]_u\rangle}\}.$$

[2]Sometimes, we simply say: "the regular (Borel) subalgebra generated by E." It is important to avoid the ambiguity that is connected with the various interpretations of the term "generated."

[3]Recall that $[E]_u = \{x \wedge u \mid \ x \in E\}$ denotes the trace of E on the band $\mathscr{X}_u$.

It is easy to verify that this set is a regular subalgebra of $\mathscr{X}$ which includes E. Therefore, $\mathscr{Y} = \mathscr{X}$ and

$$\overline{\mathscr{X}_u\langle [E]_u\rangle} = [\mathscr{Y}]_u = \mathscr{X}_u,$$

as desired.

So, E fully generates a complete Boolean algebra $\mathscr{X}$ (or a band $\mathscr{X}_u$) whenever $\overline{\mathscr{X}\langle E\rangle} = \mathscr{X}$ (respectively, $\overline{\mathscr{X}_u\langle E\rangle} = \mathscr{X}_u$). If $\overset{\cdots\cdots}{\mathscr{X}\langle E\rangle} = \mathscr{X}$ then E is called a σ-generating set for $\mathscr{X}$ (analogously, for a band).

Alongside complete and σ-complete algebras in the literature we meet the so-called $\mathfrak{m}$-complete Boolean algebras. Here $\mathfrak{m}$ is a cardinal, and $\mathfrak{m}$-completeness means that each sets of cardinality at most $\mathfrak{m}$ possesses a supremum and an infimum. The term "$\mathfrak{m}$-algebra" is used in the same sense. Also without additional explanations, the sense is clear of the terms "$\mathfrak{m}$-subalgebra" and "$\mathfrak{m}$-ideal." Many facts concerning "$\mathfrak{m}$-situations" are given, in particular, in the book of R. Sikorski. We mainly restrict exposition to the "σ-case" ($\mathfrak{m} = \aleph_0$).

1.3 Completion of a subalgebra

Let $\mathscr{X}_0$ be a subalgebra of a complete (σ-complete) Boolean algebra. What is the regular subalgebra $\overline{\mathscr{X}\langle\mathscr{X}_0\rangle}$ (the σ-regular subalgebra $\overset{\cdots\cdots}{\mathscr{X}\langle\mathscr{X}_0\rangle}$)? Generally speaking, the process of constructing these subalgebras is transfinite. We have to arrange the transfinite sequence

$$\mathscr{X}_0 \subset \mathscr{X}_1 \subset \dots \subset \mathscr{X}_\alpha \subset \dots,$$

where $\mathscr{X}_0$ is an original subalgebra and $\mathscr{X}_\alpha$, for every $\alpha > 0$, consists of the suprema and infima of all (countable) subsets of the union $\bigcup_{\beta<\alpha}\mathscr{X}_\beta$. This process terminates necessarily when the cardinality of α exceeds the cardinality of the original Boolean algebra. However, it may terminate earlier; sometimes, after the second step. We will discuss this question later.

Consider the following particular situation. Let $\mathscr{X}$ be a complete (σ-complete) Boolean algebra, let $\mathscr{X}_u \equiv [\mathbf{0}, u]$, $u > \mathbf{0}$, be a principal ideal, let $\mathscr{X}_0$ be a subalgebra in $\mathscr{X}$, and let $\mathscr{Y}_0 \equiv P_u(\mathscr{X}_0)$ be the *trace* of $\mathscr{X}_0$ on $\mathscr{X}_u$, i.e., the totality of all elements of the form $y = P_u x \equiv x \wedge u$, $x \in \mathscr{X}_0$. For brevity, we use the notations:

$$\mathscr{X}' \equiv \overline{\mathscr{X}\langle\mathscr{X}_0\rangle}, \quad \mathscr{X}'_\sigma \equiv \overset{\cdots\cdots}{\mathscr{X}\langle\mathscr{X}_0\rangle}, \quad \mathscr{Y}' \equiv \overline{\mathscr{X}_u\langle\mathscr{Y}_0\rangle}, \quad \mathscr{Y}'_\sigma \equiv \overset{\cdots\cdots}{\mathscr{X}_u\langle\mathscr{Y}_0\rangle}\,.$$

We establish that

$$\mathscr{Y}' = P_u(\mathscr{X}'), \tag{a}$$

$$\mathscr{Y}'_\sigma = P_u(\mathscr{X}'_\sigma). \tag{b}$$

In fact, both equalities have similar proofs; therefore, we consider only (b). First of all, it is clear that $P_u(\mathscr{X}'_\sigma)$ is a σ-regular u-subalgebra that includes $\mathscr{Y}_0$ and, therefore, $\mathscr{Y}'_\sigma$.

To prove the reverse inclusion, represent $\mathscr{X}'_\sigma$ as the union of the transfinite sequence $\{\mathscr{X}_\alpha\}$. It is sufficient to establish that

$$P_u(\mathscr{X}_\alpha) \subset \mathscr{Y}'_\sigma$$

for all α. We prove this by transfinite induction on α. For $\alpha = 0$ the inclusion $P_u(\mathscr{X}_\alpha) \subset \mathscr{Y}'_\sigma$ holds obviously. Suppose that it holds for all $\alpha < \beta$. Given an element $y \in P_u(\mathscr{X}_\beta)$, represent it as $y = u \wedge s$, with $s = \sup E$ or $s = \inf E$, where E is at most countable subset of the union

$$\bigcup_{\alpha<\beta} \mathscr{X}_\alpha.$$

Then y is a supremum or an infimum of some countable set of elements of the form $u \wedge x_\alpha,\ \alpha < \beta$. As we supposed, such elements belong to $\mathscr{Y}'_\sigma$ and, therefore, to $y \in \mathscr{Y}'_\sigma$. Thus, $P_u(\mathscr{X}_\beta) \subset \mathscr{Y}'_\sigma$ and the equality (b) is proved.

1.4 Continuous homomorphisms

Consider two complete (σ-complete) Boolean algebras $\mathscr{X}$ and $\mathscr{Y}$. A homomorphism Φ from $\mathscr{X}$ into $\mathscr{Y}$ is called *continuous* (respectively, *σ-continuous*) whenever for every set $E \subset \mathscr{X}$ (respectively, for every countable set) the following equalities are satisfied:

$$\Phi(\sup E) = \sup \Phi(E), \tag{2}$$

$$\Phi(\inf E) = \inf \Phi(E). \tag{3}$$

Clearly, it is sufficient in the above definitions that one of the equalities (2) and (3) holds for all E. If Φ is a continuous homomorphism then $\Phi(\mathscr{X})$ is a regular subalgebra. However, the converse assertion is not valid.

Continuous homomorphisms are called "complete" or "full." The so-called "$\mathfrak{m}$-homomorphisms" are also considered. Here $\mathfrak{m}$ is a cardinal number and the equalities (2) and (3) must hold for all sets of cardinality $\mathfrak{m}$.[4]

The term "continuous homomorphism" accepted in this book is justified by the fact that such homomorphisms prove to be continuous with

[4] See R. Sikorski [1].

respect to some topology that is studied later in Chapter 4. In conclusion, we point out an obvious but important fact. The kernel of a continuous (σ-continuous) homomorphism is a principal ideal (σ-ideal).

2. The exhaustion principle and the theorem of solid cores

2.1 The exhaustion principle

Let $\mathscr{X}_*$ be a band in a Boolean algebra $\mathscr{X}$. We will say that a set E *minorizes* $\mathscr{X}_*$ or is a *minorant* for $\mathscr{X}_*$ whenever for each $x \in \mathscr{X}_*^+$ there exists $y \in E$ such that $0 < y \leq x$. (Note that every set minorizes the zero band.) In the literature, in particular, in the monograph by R. Sikorski [1], the term "dense set" is used for the same purpose. In our book, the term "dense set" has another (topological) meaning.

Theorem 2. *Let M be a nonempty set of nonzero elements of a complete Boolean algebra $\mathscr{X}$ and let E be a minorant for the band $\mathscr{X}_* \equiv \mathscr{X}_M$. Then there exists a disjoint set $E' \subset E$ that possesses the properties*

1) $\sup E' = \sup M$;

2) *for each $x \in E'$ there exists an element $y \in M$ such that $y \geq x$.*

Proof. Consider the system D of all disjoint subsets $d \subset E$ such that each element x, $x \in d \in D$, is not greater than some $y \in M$. Endow D with the order by inclusion and check that we can apply the Kuratowski–Zorn Lemma to this system. Indeed, if D' is an arbitrary chain in D then, putting $d_0 \equiv \bigcup_{d \in D'} d$, we obtain, as is easy to understand, a disjoint subset of E belonging to the system D and including all $d \in D'$. We see that every chain in D is bounded from above and, therefore, there exists a maximal element $\overline{d}$ in D. It remains only to verify that $\sup \overline{d} = \sup M$. Suppose that $\sup \overline{d} < \sup M$. Then there is an element $y \in M$ such that $y \wedge C \sup \overline{d} > \mathbf{0}$. Since E is a minorant, there exists a nonzero element $x \in E$ satisfying the inequality $x \leq y \wedge C \sup \overline{d}$. Adjoining this element to $\overline{d}$, we obtain, as is easily verified, a disjoint set that belongs to the system D and is essentially wider than $\overline{d}$. We arrive at a contradiction with the maximality of $\overline{d}$. So, $\sup \overline{d} = \sup M$ and we can put $E' \equiv \overline{d}$. The proof of the theorem is complete.

Corollary (Exhaustion Principle). *If E minorizes a band $\mathscr{X}_*$ then each nonzero element $\mathscr{X}_*$ is the supremum of some disjoint subset of E.*

It is impossible to overestimate this principle ascending to Eudoxus and Archimedes. This principle holds also for an incomplete Boolean algebra.

We give here an important example of applications of the exhaustion principle. Consider a function φ on a complete Boolean algebra $\mathscr{X}$ with values in some partially ordered set; for instance, these values may be cardinal or ordinal numbers. Assume that this function is isotonic, i.e. $x_1 \geq x_2$ implies $\varphi(x_1) \geq \varphi(x_2)$. Agree to say that an element $x_0 \in \mathscr{X}^+$ is *φ-homogeneous* whenever the inequality $x_0 \geq x > \mathbf{0}$ implies $\varphi(x_0) = \varphi(x)$. In this case, the band $\mathscr{X}_{x_0}$ is also called φ-homogeneous. It is clear that φ-homogeneous elements comprise a minorant for $\mathscr{X}$. According to the exhaustion principle, we conclude that $\mathscr{X}$ can be represented as the direct sum of φ-homogeneous bands. We call a decomposition of $\mathscr{X}$ into φ-homogeneous bands a *φ-partition*.

We point out one more important fact that is immediate from Theorem 2.

Theorem 3. *Each nonempty set E in a Boolean algebra with the countable chain condition includes an at most countable subset E' whose supremum and infimum coincide with those of E.*

Theorem 3 implies in particular that *in a complete Boolean algebra with the countable chain condition the concepts of a regular and σ-regular subalgebras coincide.*

We call a set $E \subset \mathscr{X}$ *d-regular* whenever E contains the supremum of its every disjoint subset. So are, for example, all principal ideals; however, not each d-regular subset is an ideal.

Lemma 5. *For a d-regular subset F in a complete Boolean algebra $\mathscr{X}$ to be an ideal, it is necessary and sufficient that F be solid; in this case, F is a band.*

We need to prove only sufficiency. Let M be a nonempty subset of E. By Theorem 2 (the role of E is played by the whole $\mathscr{X}$), there exists such a disjoint set M' that a) $\sup M' = \sup M$ and b) for every $x \in M$ we can indicate $y \in M,\ y \geq x$.

According to the solidity of F, we have $M' \subset F$ and therefore, $\sup M = \sup M' \in F$. We see that F is an ideal. Taking as M the whole F, we obtain that $\sup F \in F$. Therefore, F is a principal ideal (a band). The lemma is proved.

2.2 Solid cores

Let A be a nonempty subset of a Boolean algebra $\mathscr{X}$. The set $A \cup \{\mathbf{0}\}$ is always nonempty and includes nonempty solid subsets (for example, the subset consisting of a sole zero). The following "Lemma of a Solid Core" is valid.

Lemma 6. *To each subset A of a Boolean algebra $\mathscr{X}$, there corresponds the greatest set among all solid subsets of $A \cup \{\mathbf{0}\}$; this greatest*

solid subset is defined by the equality:

$$A^0 \equiv \{x \mid \mathscr{X}_x \subset A \cup \{\mathbf{0}\}\}.$$

PROOF. It is clear that A^0 is a solid subset of $A \cup \{\mathbf{0}\}$ (if $x' \le x \le \sup A^0$ then $\mathscr{X}_{x'} \subset \mathscr{X}_x \subset A \cup \{\mathbf{0}\}$; therefore, $x' \in A \cup \{\mathbf{0}\}$). Let now B be a nonempty solid subset of $A \cup \{\mathbf{0}\}$. For every $x \in B$, the condition $x' \le x$ implies $x' \in B \subset A \cup \{\mathbf{0}\}$, i.e., $x' \in A \cup \{\mathbf{0}\}$. So, $x \in A^0$, whence $B \subset A^0$. The lemma is proved.

We call the set A^0 the *solid core* of A. We also employ the notation A^0 for the solid core of a set A in what follows.

We now prove a very important THEOREM OF SOLID CORES for sets complementing one another.

Theorem 4. *Let A be a subset of a complete Boolean algebra $\mathscr{X}$ and let $A' \equiv \mathscr{X} \setminus A$. Then*

1) *the solid cores A^0 and $(A')^0$ of A and A' are disjoint;*

2) *each one of the sets A and A' minorizes the disjoint complement of the solid core of the other;*

3) *if one of the sets A and A' is d-regular then the solid core of it is a band that coincides with the disjoint complement for the solid kernel of the other set;*

4) *if one of the sets A and A' is a band then it coincides with its solid core; moreover, the solid cores of both sets are disjoint bands.*

PROOF. 1) Let $x \in A^0$ and $x' \in (A')^0$. By solidity of sets A^0 and $(A')^0$, the following equality must hold:

$$x \wedge x' \in A^0 \cap (A')^0 \subset (A \cap A') \cup \{\mathbf{0}\} = \{\mathbf{0}\},$$

whence $x \wedge x' = \mathbf{0}$. Therefore, $A^0 \, d \, (A')^0$.

2) Distinguishing a nonzero element $x_0 \in (A^0)^d$, we note that the band $\mathscr{X}_{x_0}$ cannot be exhausted by elements of $A \cup \{\mathbf{0}\}$; otherwise, it would lie in the solid core A^0 whereas $x_0 \, d \, A^0$. That is why there exists nonzero element $x' \in \mathscr{X}_{x_0} \cap A'$. We see that A' minorizes $(A^0)^d$. Analogously, we see that A minorizes $[(A')^0]^d$.

3) For definiteness, suppose that the set A is d-regular. According to 2), this set minorizes $[(A')^0]^d$. By the exhaustion principle, we can conclude that $A \cup \{\mathbf{0}\} \supset [(A')^0]^d$. The set $[(A')^0]^d$ is solid, therefore, it must be contained in the greatest solid subset $A \cup \{\mathbf{0}\}$, i.e., in the core A^0. On the other hand, we know that $A^0 \, d \, (A')^0$. Therefore, $A^0 \subset [(A')^0]^d$. We see that $A^0 = [(A')^0]^d$, which implies the fact that A^0 is a band.

Finally, the validity of the assertion 4) is now obvious.

Corollary. *For each set $M \subset \mathscr{X}$, there exists a unique decomposition of $\mathscr{X}$ into two disjoint bands $\mathscr{X}_1$ and $\mathscr{X}_2$ such that M minorizes $\mathscr{X}_1$ and $\mathscr{X}_2$ does not contain any nonzero band for which M is a minorant.*

(It may be said that M "minorizes $\mathscr{X}_2$ nowhere.")

To prove, consider the set A of all $u \in \mathscr{X}$ such that M minorizes $\mathscr{X}_u$. It is clear that A is d-regular and solid, and so, $A^0 \equiv A$ is a band whose complement A^d coincides with $(A')^0$. Hence, we may put $\mathscr{X}_1 = A$ and $\mathscr{X}_2 = (A')^0$. Uniqueness of this decomposition is obvious.

Remark. If M is d-regular then $\mathscr{X}_1 \subset M$; if, moreover, M is solid then $\mathscr{X}_1 = M$. The first chain follows from the exhaustion principle and the second, from the fact that $\mathscr{X}_2$ cannot contain elements of M^+ in this case.

We will illustrate an application of Theorem 4 by an important example. Let $\mathfrak{A}$ be an ergodic automorphism group on a complete Boolean algebra $\mathscr{X}$ and let μ be a probability measure on $\mathscr{X}$ invariant under this automorphism group. We have already mentioned in Chapter 0 that such a measure is unique; we now prove this assertion. Let ν be another $\mathfrak{A}$-invariant measure. Define A by the equality $A \equiv \{x \mid \mu(x) > \nu(x)\}$. It is clear that both A and A' are d-regular; therefore, by Theorem 4 the solid cores A^0 and $(A')^0$ comprise a disjoint decomposition of $\mathscr{X}$. Suppose that both A^0 and $(A')^0$ are nonzero bands. Select nonzero elements $u \in A^0$ and $u' \in (A')^0$. Using the ergodicity of the group, we choose an automorphism $B \in \mathfrak{A}$ so that the equality $v' \equiv Bu \wedge u' > \mathbf{0}$ holds. Then elements $v' \in \mathscr{X}_{u'}$ and $v \equiv B^{-1}v' \in \mathscr{X}_u$ are $\mathfrak{A}$-congruent, and so

$$\mu(v) - \nu(v) = \mu(v') - \nu(v') > 0,$$

which is impossible. Hence, one of the bands A^0 and $(A')^0$ coincides with $\mathscr{X}$. Both μ and ν are probability measures; therefore, $\mathbf{1} \in A'$ and $\mathscr{X} = A' = (A')^0$. Consequently, $\mu(x) \le \nu(x)$ for all $x \in \mathscr{X}$. The reverse inequality can be proved analogously. Uniqueness of the invariant measure is established.

The conditions of Theorem 4 do not exclude the case in which one of the sets A and A' is empty. In this case, the solid core consists only of zero. The solid core of a set can be equal to zero even in the case when the set is nonempty. However, the following theorem is valid:

Theorem 5. *If A^+ is a nonempty set and A' is a d-regular set then $(A^0)^+$ is nonempty.*

Proof. By item 3) of the preceding theorem, $(A')^0 = (A^0)^d$. The fact that A^+ is nonempty means that $(A')^0 \neq \mathscr{X}$. Therefore, $(A^0)^d \neq \mathscr{X}$ and $(A^0)^+$ is nonempty.

2.3 Discrete and continuous algebras

Definition. A Boolean algebra $\mathscr{X}$ is called *discrete* or *atomic* whenever there exists a disjoint minorant A for $\mathscr{X}$.

It is clear that for each element $a \in A$ there is no element a' satisfying the inequality $\mathbf{0} < a' < a$; the elements of A are "indivisible." For each nonempty element $a \in A$ the band $\mathscr{X}_a$ is a trivial Boolean algebra consisting only of the two elements $\mathbf{0}$ and a. The nonzero elements with this property are called *atoms*. So, each discrete algebra is characterized by possessing a minorizing subset of atoms. As the exhaustion principle shows, each element in a discrete algebra is the supremum of some disjoint set of atoms; we may say that each element splits into atoms. As an example of a discrete algebra we may take a Boolean algebra of the form 2^Q; in this algebra, the atoms are all singletons of the basic set Q. It is easy to see that *the algebras of the form 2^Q in fact exhaust the class of all complete discrete Boolean algebras*: the elements of each complete atomic Boolean algebra $\mathscr{X}$ are in a natural one-to-one correspondence with the subsets of the set A of all atoms in $\mathscr{X}$. This correspondence establishes an isomorphism between $\mathscr{X}$ and 2^A. Each complete discrete Boolean algebra is a direct sum of degenerate ("two-point") algebras.[5]

It should be noted that discrete algebras are not interesting for us: they have too simple structure in regard to the problems considered in this book. The Boolean algebras without atoms (they are called *continuous*) are more important for us. In such algebras, for each nonzero element x there exist some x' satisfying the inequality $\mathbf{0} < x' < x$. An arbitrary Boolean algebra is not necessary continuous or discrete; however, the theorem is available that reduces the study of an arbitrary algebra to one of these cases.

Theorem 6. *Every complete Boolean algebra $\mathscr{X}$ is the direct sum of two bands: one, discrete and the other, continuous.*

Proof. Let A be the totality of all atoms in $\mathscr{X}$. Suppose that $A \neq \varnothing$. Put $\mathscr{X}' \equiv [(\mathscr{X} \setminus A)^0]^d$. The set $\mathscr{X}'$ is the band which A must minorize (see Theorem 4). Therefore, $\mathscr{X}'$, as an algebra in its own right, is a discrete Boolean algebra. The disjoint complement of x' does not contain atoms since $A \subset A^0 \subset [(\mathscr{X} \setminus A)^0]^d = \mathscr{X}'$ (see Theorem 4). So, the band $\mathscr{X}'' \equiv (\mathscr{X}')^d$ does not contain atoms and, therefore, it is a continuous Boolean algebra. The bands $\mathscr{X}'$ and $\mathscr{X}''$ comprise a decomposition of $\mathscr{X}$. This decomposition into the discrete and continuous bands is unique. The proof of this fact is left to the reader.

[5] That is why the complete discrete Boolean algebras are sometimes called "dyadic."

Closing this section, we note some important properties of continuous algebras.

1°. *If $u > \mathbf{0}$ is an element of a continuous Boolean algebra $\mathscr{X}$ then the band $\mathscr{X}_u$ includes a sequence $\{x_n\}_{n=1}^{\infty}$ whose entries are nonzero and pairwise disjoint.*

The proof of this assertion consists in constructing the elements x_1, x_2, ... consecutively. Namely, we first find x_1, $\mathbf{0} < x_1 < u$; then x_2, $\mathbf{0} < x_2 < u - x_1$; and so on.

2°. *If a continuous Boolean algebra $\mathscr{X}$ admits an essentially positive quasimeasure μ then for all $u > \mathbf{0}$ and for all $\varepsilon \in (0, \mu u)$ there exists an element $y \in \mathscr{X}_u^+$ such that $\mu y \leq \varepsilon$.*

Indeed, considering a sequence $\{x_n\}_{n=1}^{\infty}$ whose existence was established in 1°, we see that

$$\sum_{n=1}^{\infty} \mu x_n \leq \mu u.$$

Therefore, all sufficiently far entries of this sequence enjoy the required property.

3°. *If a continuous complete Boolean algebra $\mathscr{X}$ admits a measure μ then for all $u > \mathbf{0}$ and for all $\varepsilon \in (0, \mu u)$ there exists an element $y \in \mathscr{X}_u$ such that $\mu y = \varepsilon$.*

To this end, consider the naturally ordered system S_ε of all elements in $\mathscr{X}_u$ with measures not greater than ε. The set S_ε is a solid minorant for $\mathscr{X}_u$. Show, that we can apply the Kuratowski–Zorn Lemma to S_ε. Let $\mathbf{C} \subset S_\varepsilon$ be an arbitrary chain, $\overline{x} = \sup \mathbf{C}$. By Theorem 2, there exists a disjoint set $S' \subset S_\varepsilon$ with the following properties:

1) $\sup S' = \overline{x}$,

2) to each $x \in S'$ there is some $y \in \mathbf{C}$ satisfying $x \leq y$.

Since $\mathbf{C}$ is a chain, a stronger condition is satisfied:

2′) to each finite subset $\sigma \in S'$ there exists some $y \in \mathbf{C}$ such that $\sup \sigma \leq y$.

By the total additivity of μ, we have

$$\mu\overline{x} = \sum_{x \in S'} \mu x \leq \varepsilon,$$

i.e., $\overline{x} \in S_\varepsilon$. So, an arbitrary chosen chain in S_ε is bounded in S_ε. Therefore, we can apply the Kuratowski–Zorn Lemma and conclude that there is a maximal element y in S_ε. It is clear that $\mu y \leq \varepsilon$; if we suppose that $\mu y < \varepsilon$ then, putting $\varepsilon_1 \equiv \varepsilon - \mu y$, $u_1 \equiv u - y$, we can find an element

$y_1 > \mathbf{0}$ in the band $\mathscr{X}_{u_1}$ with measure at most ε_1. Hence, $y + y_1 \le u$ and $\mu(y + y_1) < \mu y + \varepsilon_1 \equiv \varepsilon$, i.e, $y + y_1 \in S_\varepsilon$ which clearly contradicts the maximality of y. Therefore, $\mu y = \varepsilon$ and y is a desired element. We note that this fact can also be proved without the Kuratowski–Zorn Lemma.

Proposition 3° is a special case of an important theorem of A. A. Lyapunov; we will consider this theorem in Chapter 7.

The terms "continuous" and "discrete" can also be applied to bands and subalgebras viewed as Boolean algebras in their own right. We describe a widely-spread method of constructing continuous subalgebras. Let $\mathscr{X}$ be a Boolean algebra and let T be a system of partitions of unity that enjoys the property: for every family $\{x_\tau\}_{\tau \in T}$ such that $x_\tau \in \tau$ for every τ (i.e., x_τ is an element of the partition τ), the equality $\bigwedge_{\tau \in T} x_\tau = \mathbf{0}$ holds. We prove that *every subalgebra that includes all elements of all partitions $\tau \in T$ is continuous*. Indeed, suppose by way of contradiction that there exists an atom $u > \mathbf{0}$ in the subalgebra $\mathscr{X}_0$ containing all members of partitions. Then it is easy to see that for each τ there is x_τ such that $x_\tau \ge u$, which is impossible.

3. Construction of complete Boolean algebras

3.1 Statement of the problem

Let $\mathscr{X}$ be a Boolean algebra, incomplete in general. We will obviate many obstacles that are connected with incompleteness of $\mathscr{X}$ if we find a way to embed a given BA into a complete BA. By the term "embedding," we mean an isomorphism from $\mathscr{X}$ onto some subalgebra $\mathscr{X}'$ of a complete Boolean algebra $\widehat{\mathscr{X}}$. The latter is called a *completion* of $\mathscr{X}$. Precisely speaking, an embedding $\mathscr{X}$ into $\widehat{\mathscr{X}}$ is a triple $\{\mathscr{X}, \Phi, \widehat{\mathscr{X}}\}$, where $\mathscr{X}$ and $\widehat{\mathscr{X}}$ are Boolean algebras, Φ is an order isomorphism from $\mathscr{X}$ onto some subalgebra $\mathscr{X}' \equiv \Phi(\mathscr{X})$ of $\widehat{\mathscr{X}}$. We assume that $\widehat{\mathscr{X}}$ is complete. The completions satisfying the following "minimality condition"

$$\overline{\widehat{\mathscr{X}}\langle \mathscr{X}' \rangle} = \widehat{\mathscr{X}} \tag{m}$$

are of the utmost interest.

In other words, the entire $\widehat{\mathscr{X}}$ is the least among regular subalgebras included in $\mathscr{X}'$. Another natural condition for us to require is the "preservation of suprema":

$$\text{If} \quad E \subset \mathscr{X} \quad \text{and} \quad x = \sup E \quad \text{in} \quad \mathscr{X},$$
$$\text{then} \quad \Phi(x) = \sup \Phi(E) \quad \text{in} \quad \widehat{\mathscr{X}}. \tag{b}$$

The preservation of suprema holds automatically for finite sets since $\mathscr{X}'$ is a subalgebra; in general case, it may fail.

There are various possibilities of constructing completions of algebras. We describe three constructions each enabling us to obtain a complete BA close to a given incomplete BA.

3.2 The Boolean algebra of bands

Let $\mathscr{X}$ be a lattice with zero $\mathbf{0}$. Denote by $\overline{\mathscr{X}}$ the set of all bands of $\mathscr{X}$ endowed with the natural order. Although $\mathscr{X}$ is not a Boolean algebra, we still may consider the set $\overline{\mathscr{X}}$. (We have mentioned this fact in the remark at the end of 1.2.3).

Theorem 7. *The partially ordered set $\overline{\mathscr{X}}$ is a complete Boolean algebra.*

Proof. It was established in the preceding chapter that each subset of the partially ordered set $\overline{\mathscr{X}}$ possesses a supremum and an infimum. Moreover, the greatest lower bound of a family of bands is the intersection of this family and the least upper bound is the band generated by the union of this family of bands (Theorem 1.11 and the corollary to it).

So, $\overline{\mathscr{X}}$ is a lattice. It is clear that $\overline{\mathscr{X}}$ possesses the distinct zero and unity: the zero of $\overline{\mathscr{X}}$ is the band consisting only of zero, while the unity of $\overline{\mathscr{X}}$ is the entire $\overline{\mathscr{X}}$.

Prove now that the disjoint complement E^d of an arbitrary band E is the complement of E in the sense of the order on $\overline{\mathscr{X}}$. Note first that the intersection $E \cap E^d$ is a sole zero. Recalling the interpretation of a greatest lower bound, we conclude that $E \cap E^d = E \wedge E^d = \mathbf{0}_{\overline{\mathscr{X}}}$. Further, it is easy to see that the system $\{E, E^d\}$ is complete in $\overline{\mathscr{X}}$;[6] therefore, the only band including the pair is $\mathscr{X} \equiv \mathbf{1}_{\overline{\mathscr{X}}}$. So, $E \wedge E^d = \mathbf{0}_{\overline{\mathscr{X}}}$ and $E \vee E^d = \mathbf{1}_{\overline{\mathscr{X}}}$; i.e., E^d is the complement of E. It remains to prove the distributivity of $\overline{\mathscr{X}}$. Let

$$A \equiv (E_1 \vee E_2) \wedge E_3, \quad B \equiv (E_1 \wedge E_3) \vee (E_2 \wedge E_3).$$

We know (see p. 8) that it suffices to prove the inequality $A \leq B$. If the relation fails then there is an element $u \in A \setminus B$. Show that there is a nonzero element $w \leq u$ belonging to B^d. Otherwise, all elements of the form $u \wedge z$, $z \in B^d$ must be equal to zero and, consequently, $u \, d \, B^d$, i.e., $u \in B^{dd} = B$, although $u \notin B$ by assumption. So, the required element w exists. Using Theorem 1.12 and the interpretation of suprema in $\overline{\mathscr{X}}$, obtain

$$B = [(E_1 \cap E_3) \cup (E_2 \cap E_3)]^{dd}, \quad B^d = [(E_1 \cap E_3) \cup (E_2 \cap E_3)]^d.$$

[6]An element disjoint from the sum $E \cup E^d$ is disjoint from E; therefore, it belongs to E^d and, simultaneously, is disjoint from E^d. Consequently, it is equal to zero.

Consequently,

$$w \in [(E_1 \cap E_3) \cup (E_2 \cap E_3)]^d = [(E_1 \cup E_2) \cap E_3]^d.$$

We know that $w \in A \equiv (E_1 \vee E_2) \wedge E_3$. Therefore, $w \in E_3$ and

$$w \in E_1 \vee E_2 = (E_1 \cup E_2)^{dd}.$$

The last containment means that $w \notin (E_1 \cup E_2)^d$ and there exists a nonzero $w' \leq w$ belonging to $E_1 \cup E_2$ (this set is solid). It is clear that $w' \in (E_1 \cup E_2) \cap E_3$ and, at the same time, $w' \in [(E_1 \cup E_2) \cap E_3]^d$ which is impossible since $w' > \mathbf{0}$. This contradiction proves the inequality $A \leq B$. Thus, the proof of Theorem 7 is complete.

3.3 Completion by cuts

We have established that, for every Boolean algebra $\mathscr{X}$, the partially ordered system of bands $\overline{\mathscr{X}}$ is a complete Boolean algebra. We may assign to each $x \in \mathscr{X}$ the principal ideal $\mathscr{X}_x \equiv [\mathbf{0}, x]$ which is always a band. Moreover, the correspondence $x \longleftrightarrow \mathscr{X}_x$ between elements and principal ideals is bijective; and the inequality $x_1 \leq x_2$ is equivalent to the inclusion $\mathscr{X}_{x_1} \subset \mathscr{X}_{x_2}$. Thus, the system $\mathscr{X}'$ of all principal ideals is a subalgebra of $\overline{\mathscr{X}}$ isomorphic to the original algebra $\mathscr{X}$. The isomorphism from $\mathscr{X}$ onto $\mathscr{X}'$ is an embedding of $\mathscr{X}$ into $\overline{\mathscr{X}}$ which is called the *canonical embedding*. In the sequel, we will identify $\mathscr{X}$ and $\mathscr{X}'$ and, taking convenient liberties, will refer to $\mathscr{X}$ as to a subalgebra of $\overline{\mathscr{X}}$.

In the theory of partially ordered sets, the idea of completion of a given partially ordered set by cuts is widely employed. This idea has originated with the research by R. Dedekind; it was applied to general partially ordered sets by MacNeille,[7] to vector structures by A. I. Yudin,[8] and to Boolean algebras by M. Stone and V. I. Glivenko. We first describe this method in general form.

Definition. A subset E of a partially ordered set M is said to be a $\mathfrak{D}$-*class* whenever the equality $E^{si} = E$ holds.

It is immediate from the definition that the containment $\sup F \in E$ holds for every subset $F \subset E$.

The totality of all $\mathfrak{D}$-classes for a partially ordered set M is always a complete lattice with respect to the order by inclusion. Here, we do not need the proof of this fact. We point out the following two assertions.

---

[7]H. MacNeille [1].

[8]A. I. Yudin [1].

Lemma 7. *For each $x \in M$, the set*

$$E_x \equiv \{y \mid y \leq x\} \equiv [\mathbf{0}, x]$$

is a $\mathfrak{D}$-class; moreover, the relations $x_1 \leq x_2$ and $E_{x_1} \subset E_{x_2}$ are equivalent.

PROOF. The inclusion $E_x \subset E_x^{si}$ is always true. Let now $y \in E_x^{si}$. Since $x \in E^s$, we have $y \leq x$, i.e., $y \in E_x$. Therefore, $E_x^{si} = E_x$ and the first part of the assertion is proved. Further, it is clear that, for $x_1 \leq x_2$, we have $E_{x_1} \subset E_{x_2}$; conversely, if the last inclusion holds then, since $x_1 \in E_{x_1}$, we have $x_1 \in E_{x_2}$ or $x_1 \leq x_2$.

Lemma 8. *If $x = \inf A$ then $E_x = \bigcap_{y \in A} E_y$; if $x = \sup A$ then E_x is the least $\mathfrak{D}$-class that includes E_y for all $y \in A$.*

PROOF. Let $z \in E_x$. Then, obviously, $z \in E_y$ for all $y \in A$ and, therefore, $z \in \bigcap_{y \in A} E_y$. If $z \in \bigcap_{y \in A} E_y$ then for every $y \in A$ we have $z \leq y$, whence $z \leq \inf A = x$ or $z \in E_x$. The first part of the lemma is proved. Suppose now that $x = \sup A$. Define a set B by the equality:

$$B \equiv \Big(\bigcup_{y \in A} E_y \Big)^{si}.$$

It is easy to see that

$$B^s = \Big(\bigcup_{y \in A} E_y \Big)^{sis} = \Big(\bigcup_{y \in A} E_y \Big)^{s}$$

and $B^{si} = B$. Therefore, B is a $\mathfrak{D}$-class that obviously includes all E_y, $y \in A$.

If we consider another $\mathfrak{D}$-class $\overline{B}$ including all E_y, we will have

$$\Big(\bigcup_{y \in A} E_y \Big)^{s} \supset \overline{B}^s, \quad B = \Big(\bigcup_{y \in A} E_y \Big)^{si} \subset \overline{B}^{si} = \overline{B}.$$

Therefore, B is the least $\mathfrak{D}$-class including all E_y. Show that $B = E_x$. Since B includes all E_y, $y \in A$, the set B must also include A. Hence, $\sup A = x \in B$ and $E_x \subset B$. On the other hand, E_x is a $\mathfrak{D}$-class including the entire A; by the minimality of B, we may conclude that $B \subset E_x$. The lemma is proved.

Definition. A partially ordered set N is said to be a *completion by cuts* or *Dedekind completion* of a partially ordered set M whenever N is isomorphic to the inclusion-ordered totality of all $\mathfrak{D}$-classes of M.

This terminology is justified, since there is an obvious isomorphism from M onto the set of all $\mathfrak{D}$-classes of the special form E_x, $x \in M$.

We will find out now what this completion is in the case of a Boolean algebra.

Lemma 9. *The concepts of a $\mathfrak{D}$-class and a band coincide in every Boolean algebra.*

PROOF. Let E be a $\mathfrak{D}$-class of a Boolean algebra $\mathscr{X}$. Show that $E^{dd} = E$. As we know, we need to prove only the inclusion $E^{dd} \subset E$.[9] Let $x \in E^{dd}$, $y \in E^s$. For every $z \in E$, we have $y \geq z$ or $Cy \, d \, z$. Consequently, $Cy \in E^d$. Then $x \, d \, Cy$ and $x \leq y$.

Since y was chosen arbitrarily, we may conclude that $x \in E^{si} = E$. So, $E^{dd} = E$ and E is a band. Now, consider a band E and prove that $E^{si} = E$.

The inclusion $E \subset E^{si}$ holds automatically; prove that $E^{si} \subset E$. Take arbitrary elements $x \in E^{si}$ and $y \in E^d$. It is clear that $Cy \in E^s$; therefore, $x \leq Cy$ or $x \, d \, y$. Consequently, $x \in E^{dd} = E$ and the desired inclusion is proved.

Theorem 8. *A Dedekind completion of a Boolean algebra $\mathscr{X}$ is a complete Boolean algebra isomorphic to the Boolean algebra $\overline{\mathscr{X}}$ of all bands in $\mathscr{X}$. This completion satisfies the conditions* (m) *and* (b).

This theorem is known as the Glivenko–Stone Theorem.[10] It has actually been proved in Lemmas 8 and 9. We are left with verifying only that the conditions (m) and (b) are satisfied. We first denote by Φ the canonical embedding of $\mathscr{X}$ into $\overline{\mathscr{X}}$ and describe its most important properties.

1°. *If a set $A \subset \mathscr{X}$ possesses a supremum (infimum) x in $\mathscr{X}$ then $\Phi(x)$ is the supremum (infimum) of $\Phi(A)$ in $\overline{\mathscr{X}}$. Moreover,*

$$\Phi(\mathbf{1}_{\mathscr{X}}) = \mathbf{1}_{\overline{\mathscr{X}}}, \quad \Phi(\mathbf{0}_{\mathscr{X}}) = \mathbf{0}_{\overline{\mathscr{X}}}.$$

This is immediate from Lemma 8. Property 1° means that the embedding of $\mathscr{X}$ into $\overline{\mathscr{X}}$ preserves suprema; i.e., the condition (b) is satisfied.

2°. *For the element $\overline{x} \in \overline{\mathscr{X}}$ to be nonzero, it is necessary and sufficient that there exist an element $x \in \mathscr{X}$, $x > \mathbf{0}_{\mathscr{X}}$, satisfying $\Phi(x) \leq \overline{x}$.*

Sufficiency of the above condition is obvious. In order to prove necessity, we note that nonexistence of the desired element x means that the band $\overline{x}$ consists only of the zero of $\mathscr{X}$, i.e., $\overline{x} = \mathbf{0}_{\overline{\mathscr{X}}}$.

3°. *Each element $\overline{x} \in \overline{\mathscr{X}}$ is the least upper bound of a subset of the form $\Phi(x)$, with $x \in \mathscr{X}$.*

[9] For every $\mathfrak{D}$-class E, the inequality $y \leq x$, $x \in E$, implies $y \in E$.

[10] M. Stone [1] and V. I. Glivenko [2].

We know that $\overline{x}$ is a band in $\mathscr{X}$; and it is a desired set: $\overline{x} = \sup \Phi(\overline{x})$.[11] (The least band that includes all principal ideals of the form $\mathscr{X}_y$, with $y \in \overline{x}$, is obviously the band $\overline{x}$.)

Corollary. *The set $\Phi(\mathscr{X})$ minorizes $\overline{\mathscr{X}}$.*

It is clear now that the condition (m) is also satisfied. The proof of the theorem is over.

REMARK 1. In the above proof, we have established an important property of a Dedekind completion: *the image $\Phi(\mathscr{X})$ minorizes $\overline{\mathscr{X}}$.*

REMARK 2. *If a completion $\widehat{\mathscr{X}}$ of a Boolean algebra $\mathscr{X}$ possesses the properties (m) and (b) or if the image $\Phi(\mathscr{X})$ minorizes $\widehat{\mathscr{X}}$ then $\widehat{\mathscr{X}}$ is a Dedekind completion of $\mathscr{X}$.* The proof of this fact may be found, for example, in the monograph by R. Sikorski.

As a rule, we will identify a completion of $\mathscr{X}$ with the algebra $\overline{\mathscr{X}}$ while identifying the original Boolean algebra $\mathscr{X}$ with its canonical image $\Phi(\mathscr{X})$; consequently, we will identify x and $\Phi(x)$.

The Glivenko–Stone Theorem allows us to focus our attention on complete algebras. If an algebra is incomplete, we may consider it as a subalgebra of a complete Boolean algebra constructed, for example, as above. However, it should be noted that a Dedekind completion may have pathological properties. Theorem 8 admits an important generalization. Namely, it remains valid if $\mathscr{X}$ is a partially ordered set that possesses the property: for all $x, y \in \mathscr{X}$ such that $x \not\leq y$, there exists some $z \leq y$ satisfying the condition

$$\{t \in \mathscr{X} \mid t \leq y\} \cap \{t \in \mathscr{X} \mid t \leq z\} = \varnothing.$$

After completing such a set by cuts, we also obtain a complete Boolean algebra. In particular, we may apply this procedure to the set of all nonzero elements of a Boolean algebra and so come to the Glivenko–Stone Theorem. This argument plays an important role in the theory of Boolean valued models of set theory (see T. Jech [1], and E. I. Gordon and S. F. Morozov [1]).

3.4 A test for completeness

We apply the Glivenko–Stone theorem for proving a convenient completeness test for Boolean algebras.

Theorem 9. *For a Boolean algebra $\mathscr{X}$ to be complete, it is necessary and sufficient that its every disjoint subset possess a least upper bound.*

[11] I.e., the image $\Phi(\overline{x})$ of $\overline{x} \subset \mathscr{X}$ possesses a supremum in the algebra $\overline{\mathscr{X}}$; this supremum is equal to the band $\overline{x}$ which is considered now as an element in $\overline{\mathscr{X}}$.

PROOF. Embed $\mathscr{X}$ into $\overline{\mathscr{X}}$, a Dedekind completion of $\mathscr{X}$. Consider a set E in $\mathscr{X}$. Let $\overline{x}$ be the least upper bound of E in $\overline{\mathscr{X}}$.

Since, as was noted, $\mathscr{X}$ minorizes $\overline{\mathscr{X}}$, by the exhaustion principle there is a disjoint subset $E' \subset \mathscr{X}$ such that the least upper bound of E' in $\mathscr{X}$ is equal to $\overline{x}$. By assumption, the least upper bound of E' in $\mathscr{X}$ is available and, by preservation of suprema, it must coincide with $\overline{x}$. The element $\overline{x}$ belongs to $\mathscr{X}$ and is the least upper bound of E in $\overline{\mathscr{X}}$; so it plays the same role also in $\mathscr{X}$. By Lemma $2'$, the Boolean algebra $\mathscr{X}$ is complete. The proof of the theorem is over.

Theorem 10. *Let $\mathfrak{a}$ be a cardinal number such that the cardinality of every disjoint subset in a Boolean algebra $\mathscr{X}$ is at most $\mathfrak{a}$. Then, for completeness of $\mathscr{X}$, it is necessary and sufficient that each subset of cardinality at most $\mathfrak{a}$ possess the least upper (greatest lower) bound in $\mathscr{X}$.*

This theorem is an obvious corollary to the preceding.

Corollary. *Every σ-complete algebra with the countable chain condition is complete.*

3.5 Quotients

Another widely spread method for constructing Boolean algebras consists in "factorization" of an original algebra. We consider only one of the relevant theorems.

Theorem 11. *Let $\mathscr{X}$ be a σ-complete Boolean algebra, and let I be a σ-ideal in $\mathscr{X}$ such that every disjoint subset in $\mathscr{X} \setminus I$ is at most countable. Then the quotient algebra $\widehat{\mathscr{X}} \equiv \mathscr{X}|_I$ is complete and satisfies the countable chain condition.*

PROOF. We show first that the quotient algebra is σ-complete. Take an arbitrary countable set $\widehat{E} \subset \widehat{\mathscr{X}}$.

Arrange a sequence

$$\{\widehat{x}_1, \widehat{x}_2, \dots\}$$

by enumerating the elements of $\widehat{E}$. These elements are I-cosets. So, arbitrarily choosing a "representative" x_i in each coset $\widehat{x}_i$, we obtain a sequence in $\mathscr{X}$. This sequence possesses the least upper bound x in $\mathscr{X}$. It is easy to verify that the I-coset $\widehat{x}$ containing x is the least upper bound of $\widehat{E}$ in $\widehat{\mathscr{X}}$.

Check that $\widehat{\mathscr{X}}$ satisfies the countable chain condition. Suppose the contrary. Then there is a disjoint subset $\widehat{D} \subset \widehat{\mathscr{X}}$ of cardinality $\aleph_1$. Arrange a transfinite sequence $\{\widehat{x}_\alpha\}_{\alpha<\omega_1}$, with ω_1 the least uncountable ordinal. As in the first part of the proof, choose "representatives" $x_\alpha \in$

$\mathscr{X}$ of the cosets $\widehat{x}_\alpha$. Since $\widehat{D}$ is disjoint in $\widehat{\mathscr{X}}$, all elements of the form $x_\alpha \wedge x_\beta$ $(\alpha \neq \beta)$ must belong to the ideal I. Given α, put

$$x'_\alpha \equiv x_\alpha \wedge C\Big(\bigvee_{\gamma<\alpha} x_\gamma\Big).$$

The least upper bound $\bigvee_{\gamma<\alpha} x_\gamma$ is calculated in $\mathscr{X}$. This supremum exists since the set $\{x_\gamma\}_{\gamma<\alpha}$ is at most countable and $\mathscr{X}$ is a σ-algebra.

Taking into account the fact that I is a σ-ideal, it is easy to verify the containment

$$|x_\alpha - x'_\alpha| \in I.$$

Therefore, the elements x'_α are representatives of the cosets $\widehat{x}_\alpha$. It is clear that they are pairwise disjoint. By the main property of an ideal, there exists such an "index" α_0 that $x_\alpha \in I$ for all $\alpha > \alpha_0$. This fact means that $\widehat{d}$ contains only countable set of elements, although its cardinality is $\aleph_1$ by supposition. So, $\widehat{\mathscr{X}}$ is an algebra with the countable chain condition. We now can conclude the proof of the theorem by applying the corollary to Theorem 10.

Remark. In the first part of the above proof, we have established that *the quotient algebra of a σ-complete Boolean algebra by a σ-ideal is always σ-complete.* This fact is of interest in its own right. Simultaneously, we determined the structure of the supremum and infimum of a countable set in $\mathscr{X}|_I$: each may be obtained by calculating the bound for a countable set of "representatives" in $\mathscr{X}$ and taking the coset of the result. We will consider the third construction in Chapter 4.

4. Important examples of complete Boolean algebras

4.1 The metric structure of a measure space

Let $\{\Omega, \mathscr{E}, m\}$ be a measure space. Arrange an algebra $\widehat{\mathscr{X}}$ as was described above (see p. 61). It is easy to prove that the ideal I participating in the construction of $\mathscr{X}$ (it consists of all x such that $m(x) = 0$) satisfies all conditions of Theorem 11. Therefore, the algebra $\widehat{\mathscr{X}}$ is complete. In particular, *each of the algebras E_0, $E_0^{(n)}$, and E^Γ is complete.*[12]

As was noted, the Boolean algebra resulting from factorization of $\mathscr{E}$ by the ideal I of negligible sets is called the "metric structure." Every metric structure is always complete despite the fact that the original algebra of sets $\mathscr{E}$ is assumed to be only σ-complete. This fact is of the utmost

[12]The definitions of these ("Lebesgue") Boolean algebras are given above on p. 62.

significance. We have already seen that each metric structure possesses a measure (i.e., an essentially positive totally additive quasimeasure). Later we will establish the converse: each complete Boolean algebra with measure may be represented as a metric structure of some measure space. This measure space may be chosen in many (essentially different) ways. Part II of this book is devoted to Boolean algebras with measure.

4.2 The Borel modulo meager algebra

Let R be a topological space, let $\mathscr{B}$ be the Borel σ-algebra of sets of R, and let I be the σ-ideal of meager[13] sets of R. As may be shown, the quotient algebra $\mathscr{B}|_I$ is complete. Note that this fact, established by G. Birkhoff and S. Ulam, is not covered by Theorem 11.

5. The Boolean algebra of regular open sets

Let R be a topological space. Consider the lattice $\mathfrak{J}$ of all open sets in this space (see p. 9). This lattice is not a Boolean algebra but, taking the system $\mathscr{E}$ of all bands in $\mathfrak{J}$, we come to a complete Boolean algebra (Theorem 7). We now select a part in $\mathfrak{J}$ that is order isomorphic to the Boolean algebra $\mathscr{E}$.

An open set G is called *regular* whenever G contains all interior points of its closure. Thus, every regular open set coincides with the interior of its closure. For example, each interval in $\mathbb{R}$ of the form (a, b) is a regular open set; however, the union $(a, b) \cup (b, c)$ is open but not regular. We point out some properties of regular open sets.

1°. *If G is open then the greatest open set disjoint from G; i.e., the set $R \setminus \overline{G}$ is regular.*

Indeed, put $G_1 \equiv R \setminus \overline{G}$ and take an arbitrary interior point x_0 of the closure $\overline{G}_1$. There is an open neighborhood U of x_0 included in $\overline{G}_1$. Clearly, $\overline{G}_1 \subset R \setminus G = \overline{R \setminus G}$. Therefore, $U \subset R \setminus G$ and $U \subset R \setminus G$. The set U is open and disjoint from G, and G_1 is the greatest among such sets. Hence, $U \subset G_1$ and $x_0 \in G_1$. So, G_1 is regular.

2°. *If G is a regular open set then for every open set U the assertions*

$$U \subset G \qquad (*)$$

and

$$U \cap (R \setminus \overline{G}) = \varnothing \qquad (**)$$

are equivalent.

The fact that $(*)$ implies $(**)$ is obvious. If $(**)$ holds then $U \subset \overline{G}$ and, by the regularity of G, the relation $(*)$ holds. The property 2° can

[13] A meager set is also called a set of first Baire category. (S. S. Kutateladze)

be formulated as follows: *a regular open set G is the union of all open sets disjoint from $R \setminus \overline{G}$:*

$$G = \bigcup_{\substack{U \cap (R \setminus \overline{G}) = \varnothing, \\ U \text{ is open}}} U.$$

Assign now to each regular open set G the totality of all open subsets of G. Denote this totality by $f(G)$: in symbols,

$$f(G) \equiv \{U \mid U \subset G, \ U \text{ is open}\}.$$

We see that f is a mapping sending regular open sets to classes of open sets in the same space R. It is clear that we have

$$G = \bigcup_{U \in f(G)} U.$$

Thus, G can be uniquely reconstructed from its image and f.

Lemma 10. *If G is a regular open set then $f(G)$ is a band of $\mathfrak{J}$.*

Indeed, by 2°, the set $f(G)$ is the disjoint complement of some subset of $\mathfrak{J}$ (namely, of $R \setminus \overline{G}$). Therefore, $f(G)$ is a band.

Lemma 11. *For each band E of the lattice $\mathfrak{J}$, there exists a unique regular set G such that $f(G) = E$.*

Note that $E = E^{dd}$. Take

$$G \equiv \bigcup_{U \in E^{dd}} U \equiv \bigcup_{U \in E} U, \quad G_1 \equiv \bigcup_{U \in E^d} E.$$

It is clear that G is the greatest open set disjoint from G_1; therefore, G is regular by 1°. By the definition of G, we have $f(G) \supset E$. On the other hand, each open $U \subset G$ is disjoint from G_1 and, thus, is disjoint from every element in E^d. Therefore, such a U lies in $E^{dd} = E$, i.e., $f(G) \subset E$. So, $f(G) = E$. The uniqueness of G is clear since f is a one-to-one mapping.

The above-constructed mapping f preserves the natural order. It follows from the definition that the inclusions $G' \subset G''$ and $f(G') \subset f(G'')$ are equivalent. Moreover, f is bijective. Hence, f is an order isomorphism. We arrive at the following

Theorem 12. *For every topological space R, the inclusion-ordered system of all regular open sets in R is a complete Boolean algebra isomorphic to the Boolean algebra of all bands in the lattice of open sets.*

We denote this Boolean algebra by $\mathscr{O}(R)$. As a rule, the Boolean algebra $\mathscr{O}(R)$ is not an algebra of sets, although this system possesses

the natural order. The Boolean operations $\vee$ and C are not the set-theoretic operations of union and complementation.

For example, put $R \equiv (a,b)$ $(a < b)$, $G_1 \equiv (a,c)$, and $G_2 \equiv (c,b)$. Then G_1 and G_2 are the complements of one another in the "Boolean" sense, although $G_2 \neq R \setminus G_1$. The join of this pair is unity, i.e., the interval (a,b), however, the union $G_1 \cup G_2$ is not a regular open set.

It is clear that, in the case when R_1 and R_2 are homeomorphic topological spaces, the corresponding Boolean algebras $\mathscr{O}(R_1)$ and $\mathscr{O}(R_2)$ are isomorphic. The topological spaces with isomorphic Boolean algebras of regular open sets are called "coabsolute" in general topology.

The Boolean algebras of the type $\mathscr{O}(R)$ play an important role in the theory of Boolean algebras. Their properties differ essentially from those of metric structures. The Boolean algebra $\mathscr{O}(\mathbb{R})$, where $\mathbb{R}$ is the real axis, appears especially often in various examples.

We now state the two theorems that characterize this Boolean algebra.

Birkhoff Theorem. *The Boolean algebra $\mathscr{O}(\mathbb{R})$ is isomorphic to a Dedekind completion of a free Boolean algebra on countably many generators.*

Birkhoff–Ulam Theorem. *The Boolean algebra $\mathscr{O}(\mathbb{R})$ is isomorphic to the Borel modulo meager algebra of the real axis.*

The proofs of these theorems (in slightly more general formulations) can be found in Birkhoff's book [2].

We will repeatedly return to the algebra of regular open sets. Instead of $\mathscr{O}(\mathbb{R})$, it is often convenient to consider the isomorphic Boolean algebra $\mathscr{O}((0,1))$.

6. The type, weight, and cardinality of a complete Boolean algebra

6.1 The type of a Boolean algebra

We have already introduced the concept of a Boolean algebra satisfying the countable chain condition. Theorem 3 of this chapter asserts that, in such Boolean algebras, the supremum (as well as the infimum) of an arbitrary infinite set is attained at a countable subset. This fact prompts the following

Definition. The *type of a Boolean algebra* $\mathscr{X}$ is the least among the cardinal numbers $\mathfrak{a}$ such that there is no disjoint subsets in $\mathscr{X}$ of cardinality greater than $\mathfrak{a}$. The type of a band in a Boolean algebra $\mathscr{X}$ is defined analogously.

In other words, the type of a Boolean algebra is the least upper bound of the cardinalities of its disjoint subsets.

The above definition does not require completeness of a Boolean algebra. For a complete Boolean algebra, it is equivalent to the following:

Definition. The type of a Boolean algebra $\mathscr{X}$ (or of a band in $\mathscr{X}$) is the least among the cardinal numbers $\mathfrak{a}$ such that each set $E \subset \mathscr{X}$ contains a subset E' with the following properties:

$$card\, E' \leq \mathfrak{a},$$

$$\sup E' = \sup E.$$

It is not difficult to prove the equivalence of these two definitions (cf. the proof of Theorem 3). We will denote the type of a Boolean algebra $\mathscr{X}$ by $t(\mathscr{X})$.

Theorem 13. *Consider a strictly increasing transfinite sequence $\{x_\alpha\}_{\alpha<\alpha_0}$ of elements in a Boolean algebra $\mathscr{X}$. Then $card\,\alpha_0 \leq t(\mathscr{X})$.*

Proof. Define a new sequence $\{u_\alpha\}$ by putting $u_\alpha \equiv x_{\alpha+1} - x_\alpha \equiv x_{\alpha+1} \wedge Cx_\alpha$. All u_α are nonzero and pairwise disjoint, comprising some set of cardinality $card\,\alpha_0$. Therefore,

$$card\,\alpha_0 \leq t(\mathscr{X}).$$

This theorem is called the "stationary principle": a strictly increasing sequence terminates in at most $t(\mathscr{X})$ steps.

Theorem 13 does not require completeness of the Boolean algebra. The question arises whether we are always able to construct a disjoint set of cardinality $t(\mathscr{X})$ in a given Boolean algebra $\mathscr{X}$. In this situation, we face set-theoretic difficulties. Recall the definition of a "weakly inaccessible" cardinal. So we call each regular cardinality that does not have an immediate predecessor. (A cardinality is called regular whenever it is not the sum of a few strictly less cardinalities.) As is well known, the hypothesis of existence of an uncountable weakly inaccessible cardinal, likewise the opposite hypothesis, is consistent with the axioms of the "routine" set theory (for example, with the axioms of $\mathbb{ZFC}$).[14] We call a cardinal *accessible* whenever it is not weakly inaccessible.

Theorem 14. *If a cardinal $t(\mathscr{X})$ is either countable or accessible then there is a disjoint subset of cardinality $t(\mathscr{X})$ in $\mathscr{X}$.*

Proof. We consider only the case of a complete Boolean algebra $\mathscr{X}$ (for incomplete Boolean algebras the theorem can be proved analogously). Assume first that $t(\mathscr{X}) > \aleph_0$. We need to consider only the case in which $t \equiv t(\mathscr{X})$ is a limit cardinal. It is irregular, i.e., cofinal to

[14] T. Jech [1]; E. I. Gordon and S. F. Morozov [1].

a less cardinal t_0:

$$t \equiv t(\mathscr{X}) = \sup_{\beta<\overline{\beta}} \alpha_\beta = \sum_{\beta<\overline{\beta}} \alpha_\beta,$$

where $\overline{\beta}$ is the initial ordinal of cardinality t_0, while $\beta_1 < \beta_2 < \dots$ are ordinals and $\alpha_\beta < t$ $(\beta < \overline{\beta})$.

Let the Boolean algebra $\mathscr{X}$ be t-homogeneous. Since $t_0 < t(\mathscr{X})$, there exists a disjoint family $\{\mathscr{X}_{u_\beta}\}_{\beta<\overline{\beta}}$ of nonzero bands each of which has the same type $t(\mathscr{X})$. Choose a disjoint set of cardinality α_β in each band $\mathscr{X}_{u_\beta}$. Taking the union of all these sets, we obtain a desired disjoint system of cardinality $t(\mathscr{X})$.

If $\mathscr{X}$ is not a t-homogeneous Boolean algebra then $\mathscr{X}$ can be represented as a sum of homogeneous algebras:

$$\mathscr{X} = \bigoplus_{u\in U} \mathscr{X}_u,$$

where U is a set of pairwise disjoint nonzero elements. If $card\, U = t$ then we have nothing to prove. Assume that $card\, U \equiv t_1 < t$ and let $\overline{\gamma}$ be the initial ordinal of cardinality t_1. It is easy from the last inequality that

$$t = \sup_u t(\mathscr{X}_u). \tag{$*$}$$

If this supremum is attained we arrive at the above case: we have a disjoint system of cardinality t in one of the bands $\mathscr{X}_u$. Assume now that

$$t(\mathscr{X}) < t \quad (u \in U). \tag{$**$}$$

We may arrange the elements of U in some transfinite sequence $\{u_\gamma\}_{\gamma<\overline{\gamma}}$. Moreover, we require that the types increase:

$$t(\mathscr{X}_{u_1}) \le t(\mathscr{X}_{u_2}) \le \dots.$$

By $(*)$ and $(**)$, this sequence does not terminate. We rarefy it by removing some elements so as to obtain a transfinite sequence of pairwise disjoint nonzero bands

$$\{\mathscr{X}_{u'_\gamma}\}_{\gamma<\overline{\overline{\gamma}}}$$

with the properties $\overline{\overline{\gamma}} < \overline{\gamma}$ and $t(\mathscr{X}_{u'_1}) \le t(\mathscr{X}_{u'_2}) \le \dots$. This sequence does not terminate either. Clearly,

$$t = \sup_{\gamma<\overline{\overline{\gamma}}} t(\mathscr{X}_{u'_{\gamma+1}}).$$

In each band of the form $\mathscr{X}_{u'_{\gamma+1}}$, there is a disjoint set of cardinality $t(\mathscr{X}_{u'_\gamma})$. Taking the union of these bands, we obtain a desired disjoint system of cardinality t. The case in which $t = \aleph_0$ (an algebra with the countable chain condition) is simple. An infinite disjoint sequence exists in each infinite Boolean algebra (even in an incomplete algebra).

The above theorem was apparently proved by Erdös and Tarski [1]. We cannot refute the assumption on $t(\mathscr{X})$; otherwise, a disjoint system of cardinality $t(\mathscr{X})$ may fail to exist (B. A. Efimov [2]).

Another important cardinal characteristic of a complete Boolean algebra $\mathscr{X}$ is *weight*. So is called the least cardinality of a set that fully generates $\mathscr{X}$, i.e. such a set E that $\overline{\mathscr{X}\langle E\rangle} = \mathscr{X}$. We denote the weight of a Boolean algebra $\mathscr{X}$ by $\tau(\mathscr{X})$. A Boolean algebra $\mathscr{X}$ such that $\tau(\mathscr{X}) \leq \aleph_0$ is called *separable*. It is clear that the concept of weight makes sense not only for the entire Boolean algebra but also for subalgebras and bands. We will use this concept in the following subsection. Below, we connect the concept of weight with some topology on a Boolean algebra; and our terminology, in particular the concept of "separability," will sound more naturally.

In addition to the two characteristics t and τ, each Boolean algebra possesses one more characteristic, maybe the most important, namely, its CARDINALITY. An arbitrary cardinal may fail to be the cardinality of an infinite complete Boolean algebra. There is a necessary and sufficient condition for this given by the equality

$$\mathfrak{a}^{\aleph_0} = \mathfrak{a}.$$

Such and only such cardinal number is the cardinality of some complete Boolean algebra.[15]

Closing this section, we present a theorem about the three cardinal characteristics.

Theorem 15. *For every infinite complete Boolean algebra $\mathscr{X}$, the following holds:*

$$\mathit{card}\,\mathscr{X} \leq [\tau(\mathscr{X})]^{t(\mathscr{X})} \tag{4}$$

(recall that the symbol a^b denotes the cardinality of the set of all mappings from b into a).

Note that a finite Boolean algebra has cardinality $2^{t(\mathscr{X})}$.

We postpone the proof of this theorem to Chapter 4 in which we establish a more general result.

[15] R. S. Pierce [1].

The inequality (4) can be supplemented[16] with the lower estimate:

$$card\,\mathscr{X} \geq \max(2^{t(\mathscr{X})}, \tau^*),$$

where τ^* is the supremum of weights of all bands homogeneous by weight. If $t(\mathscr{X}) \geq \tau^*$ then we immediately obtain $card\,\mathscr{X} = 2^{t(\mathscr{X})}$.

We conclude this section with a few remarks.

1. *Each infinite cardinal can be the weight of a complete Boolean algebra.* We establish this fact in Chapter 4. Hence, the cardinality of an algebra with the countable chain condition can be as great as we choose.

2. *Each cardinal can be the type of a complete Boolean algebra.* It is clear since, given b, we may take some set Q of cardinality b and observe that the boolean 2^Q has the type b.

3. *The type (and, hence, the cardinality) of a separable Boolean algebra can be infinitely great.* This result was established by H. Gaifman.[17]

Above remarks show that the type and weight of a complete Boolean algebra are not related with each other: the type can be small while the weight is great, and vice versa. The situation changes drastically for the narrower class of discrete Boolean algebras. We will look at this situation closer.

As is well known, the LOGARITHM of a cardinal number $\mathfrak{a}$ is defined by the equality

$$\log \mathfrak{a} = \min\{\mathfrak{b} \mid 2^{\mathfrak{b}} \geq \mathfrak{a}\}.$$

We always have $\log \mathfrak{a} \leq \mathfrak{a}$; moreover, equality is possible (for example, in the case when $\mathfrak{a} = \aleph_0$). A characteristic property of the logarithm is described in the following lemma.

Lemma 12. *Let Q be an arbitrary set of cardinality $\mathfrak{a}$. The cardinal $\log \mathfrak{a}$ is the least cardinality of a set $\mathscr{E} \subset 2^Q$ that separates the points of Q.*

PROOF OF THE LEMMA. Let $\mathfrak{b} = \log \mathfrak{a}$. The two cases are possible:

1) $\mathfrak{a} = 2^{\mathfrak{b}}$. In this case, we already know how to construct a class of sets that has cardinality $\mathfrak{b}$ and separates the points of Q (see p. 43).

2) $\mathfrak{a} < 2^{\mathfrak{b}}$. Complete Q to a set $\overline{Q}$ of cardinality $2^{\mathfrak{b}}$ and construct a class $\overline{\mathscr{E}} \subset \overline{Q}$ that separates the points of $\overline{Q}$ and such that $card\,\overline{\mathscr{E}} = \mathfrak{b}$. The intersections of the form $e \cap Q$ ($e \in \overline{Q}$) comprise the system of cardinality at most $\mathfrak{b}$ which separates the points of Q. Thus, there exists a point-separating system of cardinality at most $\mathfrak{b}$.

[16] D. A. Vladimirov and B. A. Efimov [1].
[17] H. Gaifman [1].

Consider a class $\mathscr{E} \subset 2^Q$ that separates points. Let $\mathfrak{c} = card\,\mathscr{E}$. Arrange the set Δ of all families $\{\delta_e\}_{e\in\mathscr{E}}$, where $\delta_e \in \{e, Ce\}$. Each point $q \in Q$ generates the unique family $\delta(q) = \{\delta_e^{(q)}\}$ as follows:

$$\delta_e^{(q)} = \begin{cases} e, & \text{if } q \in e, \\ Ce, & \text{if } q \notin e. \end{cases}$$

Since the class $\mathscr{E}$ separates points, the mapping $q \longrightarrow \delta(q)$ is an embedding from Q into Δ. Therefore, $card\,\Delta = 2^{\mathfrak{c}} \geq \mathfrak{a}$, i.e., $\mathfrak{c} \geq \log \mathfrak{a}$. The proof of the lemma is complete.

The following lemma can be proved by the same scheme as Theorem 1.4.

Lemma 13. *Let $\mathscr{X} = 2^Q$. For a set $\mathscr{E} \subset \mathscr{X}$ to fully generate the Boolean algebra $\mathscr{X}$, it is necessary and sufficient that $\mathscr{E}$ separate the points of Q.*

The proof of this lemma is left to the reader.

The following theorem is immediate from the last two lemmas.

Theorem 16. *For every Boolean algebra $\mathscr{X}$ of the form 2^Q the equalities*

$$t(\mathscr{X}) = card\,Q, \quad \tau(\mathscr{X}) = \log t(\mathscr{X}), \quad card\,\mathscr{X} = 2^{t(\mathscr{X})} \tag{5}$$

hold.

For example, if $Q \equiv [0,1]$ then $\tau(\mathscr{X}) = \aleph_0$ and $t(\mathscr{X}) = 2^{\aleph_0}$. Thus, the boolean of the continuum is separable.

Another example: $Q \equiv \mathbb{N} \equiv \{1, 2, \ldots\}$. In this case,

$$t(\mathscr{X}) = \aleph_0, \quad card\,\mathscr{X} = 2^{\aleph_0}, \quad \tau(\mathscr{X}) = \aleph_0.$$

4. Finally, the cardinal presented in this section allows us to estimate the cardinality of the set of mappings. Namely, let $\mathscr{X}$ and $\mathscr{Y}$ be two complete Boolean algebras; suppose also that $\mathscr{X}$ is infinite. Denote by $\mathscr{C}[\mathscr{X} \longrightarrow \mathscr{Y}]$ the set of all continuous homomorphisms from $\mathscr{X}$ into $\mathscr{Y}$. There is a fully generating set of cardinality $\tau(\mathscr{X})$ in $\mathscr{X}$. Of the same cardinality is the algebra generated by this set; denote this algebra by $\mathscr{X}_0$. It is clear that every continuous homomorphism Φ is defined by its values on $\mathscr{X}_0$. Whence it follows that the cardinality $\mathscr{C}[\mathscr{X} \longrightarrow \mathscr{Y}]$ is at most

$$[card\,\mathscr{Y}]^{\tau(\mathscr{X})}.$$

This estimate is sharp in many important cases.

Denote by $a(\mathscr{X})$ the number of automorphisms of a complete Boolean algebra $\mathscr{X}$. All automorphisms are continuous homomorphisms, and so

the last estimate and Theorem 15 imply that

$$a(\mathscr{X}) \leq [card\,\mathscr{X}]^{\tau(\mathscr{X})} \leq [(\tau(\mathscr{X}))^{t(\mathscr{X})}]^{\tau(\mathscr{X})} = 2^{\max\{t(\mathscr{X}),\tau(\mathscr{X})\}}. \qquad (6)$$

In particular, a separable algebra with the countable chain condition ($\tau = t = \aleph_0$) has the cardinality of the set of automorphisms at most the cardinality of the continuum.

A Boolean algebra may lack nontrivial automorphisms; in this case, it is referred to as "rigid." The first examples of a rigid but incomplete Boolean algebra were constructed by B. Jónsson, M. Katětov, and L. Rieger.[18] Considerably later an example of a complete and rigid Boolean algebra was constructed.[19] This topic has been intensively developed since then.[20] The book[21] provides a detailed survey together with a complete bibliography.

7. Structure of a complete Boolean algebra

7.1 Decomposition of a complete Boolean algebra in a product of simple independent subalgebras

In this subsection, we start from the main theorems about existence and structure of independent complements to subalgebras. The problem under study may be called the "embedding problem." We want to find out how a given subalgebra can be embedded into the main Boolean algebra $\mathscr{X}$. The point is that two subalgebras may be isomorphic but differ significantly in the manner of their embedding into $\mathscr{X}$. Study of the most important cases is the main objective of this subsection.

We recall first that, according to the definition given in Chapter 1, a nonempty set $E \subset \mathscr{X}$ is *independent* whenever all inequalities of the following form are valid:

$$x_1 \wedge x_2 \wedge \cdots \wedge x_p \wedge Cx_{p+1} \wedge \cdots \wedge Cx_n > \mathbf{0},$$

with $x_1, x_2, \ldots, x_n$ pairwise distinct elements of E. We now need a more general concept of independence for CLASSES OF SETS.

Definition. A nonempty class $\mathscr{E}$ of nonempty sets is called *independent* whenever every finite set of nonzero elements chosen by one from pairwise distinct sets of $\mathscr{E}$ is independent. In this case, we also say that the *sets of the class* $\mathscr{E}$ *are independent.*

---

[18] B. Jónsson [1]; M. Katětov [1] and L. Rieger [1].
[19] K. McAlloon [1].
[20] S. Shelah [1]; B. Balcar and P. Štěpánek [1], et al.
[21] J. D. Monk and R. Bonnet [1].

Recalling the definition of a simple subalgebra (see 1.1), we see that independence of a nonempty set $E \subset \mathscr{X}$ is equivalent to independence of the system of simple subalgebras generated by the elements of E.

We now introduce the important concept of a product of subalgebras. Consider a class $\mathfrak{P}$ of subalgebras in a complete Boolean algebra $\mathscr{X}$. A subalgebra $\mathscr{X}_0$ of the algebra $\mathscr{X}$ is said to be the product[22] of the subalgebras of $\mathfrak{P}$ whenever the following conditions are satisfied:

1) $\mathfrak{P}$ is an independent class;

2) the equality

$$\mathscr{X}_0 = \overline{\mathscr{X}\Big\langle \bigcup_{\mathscr{Y}\in\mathfrak{P}} \mathscr{Y}\Big\rangle}$$

is valid.

In this case, we write

$$\mathscr{X}_0 = \prod_{\mathscr{Y}\in\mathfrak{P}} \mathscr{Y} \quad \text{or} \quad \mathscr{X}_0 = \mathscr{Y}_1 \times \mathscr{Y}_2 \times \cdots \times \mathscr{Y}_m,$$

if $\mathfrak{P} = \{\mathscr{Y}_1, \mathscr{Y}_2, \dots, \mathscr{Y}_m\}$. The sense is also clear of the notations

$$\prod_{\xi\in\Xi} \mathscr{Y}_\xi \quad \text{and} \quad \prod_{n=1}^{\infty} \mathscr{Y}_n,$$

applied to families of subalgebras.

Let $\mathscr{X} = \mathscr{X}_1 \times \mathscr{X}_2$, with $\mathscr{X}_1$ and $\mathscr{X}_2$ independent. In this case, we say that each one of these subalgebras is an independent complement of the other. We may formulate our main problem as follows: FIND OUT CONDITIONS UNDER WHICH A GIVEN SUBALGEBRA HAS AN INDEPENDENT COMPLEMENT.

Let $\widetilde{\mathscr{X}}$ be a regular subalgebra of $\mathscr{X}$ and let u be an element of $\mathscr{X}$. If $u \in \widetilde{\mathscr{X}}$ then the set

$$E \equiv P_u(\widetilde{\mathscr{X}}) \equiv \{x \mid x = u \wedge \widetilde{x},\ \widetilde{x} \in \widetilde{\mathscr{X}}\}$$

is a band in subalgebra $\widetilde{\mathscr{X}}$; we denote this band by $\widetilde{\mathscr{X}_u}$. If $u \notin \widetilde{\mathscr{X}}$ then E is not included in $\widetilde{\mathscr{X}}$; however, furnished with the natural order, E is a Boolean algebra with unity u. In accord with common parlance we call this algebra the *trace of $\widetilde{\mathscr{X}}$ on $\mathscr{X}_u$* and denote it by $[\widetilde{\mathscr{X}}]_u$.

[22]Sometimes, the concept of product is defined in another way (for example, see R. Sikorski [1] and D. Kappos [1]). Namely, the term "product" corresponds to our "preproduct" (cf. 1.3.10).

We now list some obvious properties of the "multiplication" operation for subalgebras.

1. *If Σ is the totality of disjoint subclasses of an independent class $\mathfrak{P}$ of subalgebras such that $\mathfrak{P} = \sum_{\sigma \in \Sigma} \sigma$ and the subalgebras*

$$\mathscr{Y}_\sigma \equiv \prod_{\mathscr{Y} \in \sigma} \mathscr{Y}, \quad \sigma \in \Sigma,$$

also comprise an independent system then

$$\prod_{\mathscr{Y} \in \mathfrak{P}} \mathscr{Y} = \prod_{\sigma \in \Sigma} \mathscr{Y}_\sigma. \tag{7}$$

The formula (7) expresses the ASSOCIATIVITY of multiplication.

2. *If $\widetilde{\mathscr{X}} = \prod_{\mathscr{Y} \in \mathfrak{P}} \mathscr{Y}$ then, for every $u > \mathbf{0}$, the least regular u-subalgebra including all traces $[\mathscr{Y}]_u$ coincides with $[\widetilde{\mathscr{X}}]_u$.* (However, $[\widetilde{\mathscr{X}}]_u$ is not the PRODUCT of these traces since they may fail to be dependent.)

Definition. Let $\mathscr{X}_u$ be a nonzero band in a Boolean algebra $\mathscr{X}$ and let $\widetilde{\mathscr{X}}$ be a regular subalgebra of $\mathscr{X}$. We say that $\widetilde{\mathscr{X}}$ *saturates* $\mathscr{X}_u$ whenever $[\widetilde{\mathscr{X}_u}] = \mathscr{X}_u$. In particular, $\widetilde{\mathscr{X}}$ saturates $\mathscr{X}_u$ provided that $\mathscr{X}_u = \widetilde{\mathscr{X}_u}$. If $\widetilde{\mathscr{X}}$ saturates only the zero band then we say that $\widetilde{\mathscr{X}}$ *does not saturate bands.*

In what follows, we will often deal with subalgebras not saturating bands. If $\widetilde{\mathscr{X}}$ is such a subalgebra then it is easy to see that every subalgebra generated by $\widetilde{\mathscr{X}}$ and some finite set does not saturate bands either. Notice that, in a discrete algebra, every subalgebra saturates bands.

We now give a lemma laying grounds for our further reasoning.

Lemma 14. *Let $\mathscr{X}$ be a complete Boolean algebra and let $\widetilde{\mathscr{X}}$ be a regular subalgebra of $\mathscr{X}$. There exists a decomposition of $\mathscr{X}$ into the two bands $\mathscr{X}_{u_0}$ and $\mathscr{X}_{v_0}$ with the properties:*

1) $u_0, v_0 \in \widetilde{\mathscr{X}}$;

2) $\widetilde{\mathscr{X}_{u_0}} = \mathscr{X}_{u_0}$;

3) *there is some $z \in \mathscr{X}_{v_0}$ satisfying*

$$\text{a) } z \wedge x > \mathbf{0}, \quad \text{b) } (v_0 - z) \wedge x > \mathbf{0}$$

for every nonzero $x \in \widetilde{\mathscr{X}_{v_0}}$.
(One of the elements u_0 and v_0 may be equal to zero).

Proof. Apply Theorem 4, taking the subalgebra $\widetilde{\mathscr{X}}$ as A. Each regular subalgebra is a d-regular set; hence, the solid core $(\widetilde{\mathscr{X}})^0$ is a band whose supremum is denoted by u_0. So, $(\widetilde{\mathscr{X}})^0 = \mathscr{X}_{u_0} = \widetilde{\mathscr{X}}_{u_0}$; $u_0 \in \widetilde{\mathscr{X}}$. We then put $v_0 \equiv Cu_0$ and try to produce a required element z. To this end, consider the set $V \subset \widetilde{\mathscr{X}}$ defined by the following condition: the containment $v \in V$ implies that $v \in \mathscr{X}_{v_0}$, $v_0 > \mathbf{0}$ and that there is an element $z_v \in (\mathscr{X} \setminus \widetilde{\mathscr{X}})^0$ such that $\mathbf{0} < z_v < v$ and, moreover, $v = \inf\{x \mid x \in \widetilde{\mathscr{X}},\ x \geq z_v\}$. Show that, for $v_0 > \mathbf{0}$, the set V is nonempty while minorizing $\widetilde{\mathscr{X}}_{v_0}$. Indeed, we know that the band $\mathscr{X}_{u_0}$ coincides with the disjoint complement to $(\mathscr{X} \setminus \widetilde{\mathscr{X}})^0$. Hence, the last set is complete in the band $(\mathscr{X}_{u_0})^d \equiv \mathscr{X}_{v_0}$. A complete and solid set is always a minorant; therefore, $(\mathscr{X} \setminus \widetilde{\mathscr{X}})^0$ minorizes $\widetilde{\mathscr{X}}_{v_0}$. For each nonzero $x_0 \in \widetilde{\mathscr{X}}_{v_0}$, there is a $z_0 \in (\mathscr{X} \setminus \widetilde{\mathscr{X}})^0$ satisfying the inequality $\mathbf{0} < z_0 < x_0$. Put $v \equiv \inf\{x \mid x \in \widetilde{\mathscr{X}},\ x \geq z_0\}$. It is clear that $v \in V$ and $\mathbf{0} < v \leq x_0$. We can take z_0 as z_v. Thus, V is a nonempty minorant for $\widetilde{\mathscr{X}}_{v_0}$. We can choose a disjoint subset V^* in V which is complete in the band $\mathscr{X}_{v_0}$. Define the element z by the equality

$$z \equiv \bigvee_{v \in V^*} z_v$$

and verify that z possesses the required properties. Let $x \in \widetilde{\mathscr{X}}_{v_0}$ and $x > \mathbf{0}$. We have

$$x \wedge z = \bigvee_{v \in V^*} (x \wedge z_v) = \bigvee_{v \in V^*} (x \wedge v \wedge z_v) = \bigvee_{v \in V^*} (x_v \wedge z_v),$$

where $x_v \equiv x \wedge v \in \widetilde{\mathscr{X}}$, and $v \in V^*$. Since $x > \mathbf{0}$ and V^* is complete in $\mathscr{X}_{v_0}$ we infer that

$$x = \bigvee_{v \in V^*} x_v$$

and, for some $\overline{v} \in V^*$, we have $x_{\overline{v}} > \mathbf{0}$. Hence, $\overline{v} - x_{\overline{v}} < \overline{v}$. Considering that $\overline{v} = \inf\{x \mid x \in \widetilde{\mathscr{X}},\ x \geq z_{\overline{v}}\}$ and $\overline{v} - x_{\overline{v}} \in \widetilde{\mathscr{X}}$, we obtain $z_{\overline{v}} \not\leq \overline{v} - x_{\overline{v}}$ or, which is the same, $x_{\overline{v}} \wedge z_{\overline{v}} > \mathbf{0}$. Therefore, $x \wedge z \geq x_{\overline{v}} \wedge z_{\overline{v}} > \mathbf{0}$.

Now, estimate the element $x \wedge (v_0 - z)$. With the previous notations, we have

$$x \wedge (v_0 - z) = \bigvee_{v \in V^*} x_v \wedge (v - z_v).$$

As above, let $x_{\overline{v}} > \mathbf{0}$. Then $x_{\overline{v}} \in \widetilde{\mathscr{X}}$ and $z_{\overline{v}} \in (\mathscr{X} \setminus \widetilde{\mathscr{X}})^0$. Therefore, $x_{\overline{v}} \notin \mathscr{X}_{z_{\overline{v}}}$. Hence, $(\overline{v} - z_{\overline{v}}) \wedge x_{\overline{v}} > \mathbf{0}$ and

$$x \wedge (v_0 - z) \geq x_{\overline{v}} \wedge (\overline{v} - z_{\overline{v}}) > \mathbf{0}.$$

Thus,

$$x \wedge z > \mathbf{0}, \quad x \wedge (v_0 - z) > \mathbf{0}$$

for every nonzero $x \in \widetilde{\mathscr{X}}_{v_0}$. The proof of the lemma is complete.

REMARK. It follows from the above proof that $\mathscr{X}_{u_0}$ coincides with the solid core of $\widetilde{\mathscr{X}}$. It is easy to see that there is no other decomposition of $\mathscr{X}$ into two disjoint bands obeying 1)–3). However, rejection of 1) will result in the loss of uniqueness.

The above lemma can certainly be applied not only to the algebra $\mathscr{X}$ but also to its every nonzero band $\mathscr{X}_y$. In this case, the trace $[\widetilde{\mathscr{X}}]_y$ of $\widetilde{\mathscr{X}}$ should be considered instead of the subalgebra. Mostly interesting is the case in which $\widetilde{\mathscr{X}}$ saturates no band $\mathscr{X}_{y'}$ with $y' \leq y$. We have then $u_0 = \mathbf{0}$ and $v_0 = y$. Consequently, there exists an element $z \in \mathscr{X}_y$ such that the inequalities

$$x \wedge z = x \wedge y \wedge z > \mathbf{0}, \quad x \wedge (y - z) = x \wedge y \wedge (y - z) > \mathbf{0} \tag{8}$$

hold whenever $x \in \widetilde{\mathscr{X}}$ and $x \wedge y > \mathbf{0}$. If we now take

$$\overline{x} = \inf\{x \mid x \in \widetilde{\mathscr{X}}, \ x \geq y\}, \tag{9}$$

then it is not difficult to note that, for every nonzero $x \in \mathscr{X}_{\overline{x}}$, we have $x \wedge y > \mathbf{0}$ which implies the inequalities (8). We arrive at the following

Lemma 15. *Let $\mathscr{X}$ be a complete Boolean algebra and let $\widetilde{\mathscr{X}}$ be a regular subalgebra of $\mathscr{X}$. Moreover, let $y \in \mathscr{X}$, $y > \mathbf{0}$ and let an element $\overline{x}$ be defined by the equality* (9). *If $\widetilde{\mathscr{X}}$ saturates no band $\mathscr{X}_{y'}$ with $y' \leq y$ then there is an element z_y with the properties:*

1) *$z_y \leq y$;*

2) *for every $x \in \widetilde{\mathscr{X}}_{\overline{x}}^{+}$, the following inequalities hold:*

$$z_y \wedge x > \mathbf{0}, \quad (y - z_y) \wedge x > \mathbf{0}.$$

If $y = \mathbf{1}$ then the conditions 1) *and* 2) *mean independence of the subalgebra $\widetilde{\mathscr{X}}$ and the simple*[23] *subalgebra generated by z.*

Lemma 16. *Let $\widetilde{\mathscr{X}}$ be a regular subalgebra of a complete Boolean algebra $\mathscr{X}$ which does not saturate bands, and let u be an element in $\mathscr{X}$. There exists an element $z \in \mathscr{X}$ such that*

[23]See Chapter 1, Section 1.

1) *the simple subalgebra* $\mathscr{X}^z \equiv \{z, Cz, \mathbf{0}, \mathbf{1}\}$ *and the subalgebra* $\widetilde{\mathscr{X}}$ *are independent;*

2) *the element* u *belongs to the subalgebra* $\mathscr{X}'$ *generated by the subalgebra* $\widetilde{\mathscr{X}}$ *and the element* z.

Proof. Put

$$\overline{u} \equiv \inf\{x \mid x \in \widetilde{\mathscr{X}},\ x \geq u\}, \quad \underline{u} \equiv \sup\{x \mid x \in \widetilde{\mathscr{X}},\ x \leq u\}, \quad v \equiv \overline{u} - \underline{u}.$$

If $v = \mathbf{1}$ then put $z = u$. If $v = \mathbf{0}$ then $u \in \widetilde{\mathscr{X}}$ and we can take any element satisfying 1) as z. Such an element certainly exists by Lemma 15. Let $v \neq \mathbf{0}, \mathbf{1}$. In this case, neither band lying in $\mathscr{X}_{Cv}$ is saturated by $\widetilde{\mathscr{X}}$; therefore, there is an element $z' \in \mathscr{X}_{Cv}$ satisfying the inequalities

$$x \wedge z' > \mathbf{0}, \quad x \wedge (Cv - z') > \mathbf{0}$$

for every nonzero $x \in \widetilde{\mathscr{X}_{Cv}}$. Putting $z \equiv z' + v \wedge u$ and noting that $x \wedge v \in \widetilde{\mathscr{X}}$ for $x \in \widetilde{\mathscr{X}}$, infer

$$x \wedge z = Cv \wedge x \wedge z' + x \wedge v \wedge u > \mathbf{0}$$

and

$$x \wedge Cz = Cv \wedge x \wedge Cz' + (v - v \wedge u) \wedge x > \mathbf{0}$$

for every nonzero $x \in \widetilde{(x)}$. Moreover, $u = \underline{u} + z \wedge v \in \mathscr{X}'$. The proof of the lemma is complete.

In the above reasoning, the hypothesis was important that $\widetilde{\mathscr{X}}$ does not saturate bands. To specify this hypothesis, we expose some method for evaluating the "degree of nonsaturation." Let $u \in \mathscr{X}^+$, let $\widetilde{\mathscr{X}}$ be a subalgebra of $\mathscr{X}$, and let $E \subset \mathscr{X}_u \setminus [\widetilde{\mathscr{X}}]_u$. We say that E *complements* $\widetilde{\mathscr{X}}$ *in the band* $\mathscr{X}_u$ whenever the regular u-subalgebra generated by $E \cup [\widetilde{\mathscr{X}}]_u$ coincides with $\mathscr{X}_u$. We denote the minimal cardinality of all such complementing sets by $\sigma(\widetilde{\mathscr{X}}, u)$ and call it the *degree of nonsaturation.* The degree of nonsaturation of a band is equal to zero if and only if $\widetilde{\mathscr{X}}$ saturates the band. If the equality

$$\sigma(\widetilde{\mathscr{X}}, u) = \sigma(\widetilde{\mathscr{X}}, u_0)$$

holds for every nonzero $u < u_0$ then we refer to both the element u and the corresponding band $\mathscr{X}_u$ as $\widetilde{\mathscr{X}}$*-homogeneous.* Therefore, defining φ by the rule

$$\varphi(u) = \sigma(\widetilde{\mathscr{X}}, u),$$

we (in accordance to what was said on p. 90) represent the Boolean algebra $\mathscr{X}$ as the direct sum of φ-homogeneous or, which is the same, $\widetilde{\mathscr{X}}$-homogeneous bands. Consequently, the following lemma is valid (under the usual assumption that the algebra we study is complete).

Lemma 17. *Each Boolean algebra $\mathscr{X}$ can be represented as the direct sum of $\widetilde{\mathscr{X}}$-homogeneous bands for every subalgebra $\widetilde{\mathscr{X}}$.*

Consider the case in which there is a countable set E in $\mathscr{X}$ satisfying $\overline{\mathscr{X}\langle E\rangle} = \mathscr{X}$. We call such an algebra separable. In this case, the degree of nonsaturation of an $\widetilde{\mathscr{X}}$-homogeneous band can equal $\aleph_0$ or 0. Therefore, the nonexistence of saturated bands in the separable case means that the entire algebra $\mathscr{X}$ is $\widetilde{\mathscr{X}}$-homogeneous and the equality $\sigma(\widetilde{\mathscr{X}}, \mathbf{1}) = \aleph_0$ is valid.

We now prove the main theorem on an independent complement of a regular subalgebra.

Theorem 17. *Let $\mathscr{X}$ be a complete Boolean algebra and let $\widetilde{\mathscr{X}}$ be a regular subalgebra other than $\mathscr{X}$. If $\mathscr{X}$ is $\widetilde{\mathscr{X}}$-homogeneous then there is an independent class $\mathfrak{P}$ of simple subalgebras which has cardinality $\sigma(\widetilde{\mathscr{X}}, \mathbf{1})$ and is such that the subalgebra*

$$\overline{\mathscr{X}\Big\langle \bigcup_{Z\in\mathfrak{P}} Z\Big\rangle}$$

is an independent complement of $\widetilde{\mathscr{X}}$.

Proof. Let E be the set of cardinality $\sigma(\widetilde{\mathscr{X}}, \mathbf{1})$ which complements $\widetilde{\mathscr{X}}$ to $\mathscr{X}$. Arrange the elements of E in some transfinite sequence $\{x_\alpha\}_{\alpha<\widetilde{\omega}}$, where $\widetilde{\omega}$ is an initial transfinite of cardinality $\sigma(\widetilde{\mathscr{X}}, \mathbf{1})$. Construct a transfinite sequence of subalgebras $\{\widetilde{\mathscr{X}}^{(\alpha)}\}_{0\le\alpha<\widetilde{\omega}}$ which possesses the following properties:

1) *the system $\{\widetilde{\mathscr{X}}^{(\alpha)}\}_{0\le\alpha<\widetilde{\omega}}$ is independent;*

2) $\widetilde{\mathscr{X}}^{(0)} = \widetilde{\mathscr{X}}$*; the subalgebra $\widetilde{\mathscr{X}}^{(\alpha)}$ is simple for $\alpha > 0$;*

3) *the containment*

$$x_\alpha \in \prod_{0\le\beta\le\alpha} \widetilde{\mathscr{X}}^{(\beta)}$$

holds for all $\alpha \ge 1$.

Lemma 16 plays a key role in constructing such a system of subalgebras. Indeed, the Boolean algebra $\mathscr{X}$ is $\widetilde{\mathscr{X}}$-homogeneous; therefore, it does not saturate bands. By Lemma 16 there is a simple subalgebra

$\widetilde{\mathscr{X}}^{(1)}$ (generated by a singleton) such that $\widetilde{\mathscr{X}}^{(0)} \equiv \widetilde{\mathscr{X}}$ while $\widetilde{\mathscr{X}}^{(1)}$ are independent and $x_1 \in \widetilde{\mathscr{X}}^{(0)} \times \widetilde{\mathscr{X}}^{(1)}$. The further reasoning is usual for such proofs. Let all $\widetilde{\mathscr{X}}^{(\beta)}$ $(\beta < \alpha_0 < \widetilde{\omega})$ comprising a system with the properties 1)–3) be constructed. To achieve the next subalgebra $\widetilde{\mathscr{X}}^{(\alpha_0)}$, take

$$\widetilde{\mathscr{Y}} \equiv \prod_{0 \le \beta < \alpha_0} \widetilde{\mathscr{X}}^{(\beta)}.$$

The cardinality of the system of factors is strictly less than $\sigma(\widetilde{\mathscr{X}}, \mathbf{1})$. Hence, it follows that the *regular subalgebra* $\widetilde{\mathscr{Y}}$ *does not saturate bands.*

Indeed, if $[\widetilde{\mathscr{Y}}]_v = \mathscr{X}_v$, $v > \mathbf{0}$, then the band $\mathscr{X}_v$ coincides with the least regular v-subalgebra containing all traces $[\widetilde{\mathscr{X}}^{(\beta)}]_v$. But then, taking the union

$$E_v \equiv \bigcup_{1 \le \beta < \alpha_0} [\widetilde{\mathscr{X}}^{(\beta)}]_v,$$

we obtain the set that complements $\widetilde{\mathscr{X}}$ to the band $\mathscr{X}_v$, which is impossible since the cardinality of E_v is insufficiently small. Now, we make one more step by Lemma 16, taking x_{α_0} as u. Denote the so-constructed simple subalgebra by $\widetilde{\mathscr{X}}^{(\alpha)}$. This process will not terminate while $\alpha_0 < \widetilde{\omega}$. So, the desired sequence of subalgebras is constructed inductively.

Denote by $\mathfrak{P}$ the class of all entries of this sequence with indices greater than $\sigma(\widetilde{\mathscr{X}}, \mathbf{1})$. Putting

$$\mathscr{X}' \equiv \mathscr{X}\Big\langle \bigcup_{Z \in \mathfrak{P}} Z \Big\rangle,$$

we obtain a desired complement to $\widetilde{\mathscr{X}}$ since, obviously, $\widetilde{\mathscr{X}}$ and $\mathscr{X}'$ are independent and $\overline{\mathscr{X}\langle \widetilde{\mathscr{X}}, \mathscr{X}' \rangle} = \mathscr{X}$. The proof of the theorem is complete.

Consider the DEGENERATE subalgebra $\widetilde{\mathscr{X}}$ consisting only of zero and unity. For this algebra, the value $\sigma(\widetilde{\mathscr{X}}, u)$ coincides with the minimal cardinality of E such that $\overline{\mathscr{X}_u\langle E \rangle} = \mathscr{X}_u$ (such a set is called *fully generating* $\mathscr{X}_u$). It is naturally to call such a cardinality the *weight of* $\mathscr{X}_u$. We denote it by $\tau(\mathscr{X}_u)$. The property of $\widetilde{\mathscr{X}}$-homogeneity of an algebra means in this case that all its nonzero bands has the same weight. We call complete a nondegenerate Boolean algebra *homogeneous*.[24] The

[24]In the literature, another use is encountered for the term "homogeneous algebra." The above definition was proposed by D. Maharam who formulated it for normed algebras in 1942 (see D. Maharam [1]).

proven theorem gives us an opportunity to describe the structure of all homogeneous algebras.

Theorem 18. *Each homogeneous algebra $\mathscr{X}$ is representable as the product*

$$\mathscr{X} = \prod_{\mathscr{Z} \in \mathfrak{P}} \mathscr{Z},$$

where $\mathfrak{P}$ is an independent class of simple subalgebras which has cardinality $\tau(\mathscr{X})$.

This theorem is an obvious corollary to the preceding; we take the degenerate subalgebra as $\widetilde{\mathscr{X}}$ and notice that

$$\mathscr{X} = \overline{\mathscr{Z}'} = \prod_{\mathscr{Z} \in \mathfrak{P}} \mathscr{Z}$$

in this case.

The above-proven theorem can be rephrased as follows: *each complete homogeneous Boolean algebra is fully generated by some free subalgebra of cardinality equal to the weight of the Boolean algebra.*

In the second part, we prove a "metric" theorem analogous to Theorem 18. The latter was historically the first and gave a model for Theorem 18 which seems to be first proved by the author of this book (see D. A. Vladimirov [6], [7]; D. A. Vladimirov and B. A. Efimov [1]).

As for Theorem 18, it admits a generalization.[25] Namely, *each infinite complete Boolean algebra is fully generated by some free subalgebra.*

We prove this fact in the special case of an algebra satisfying the countable chain condition.

Theorem 19. *Each infinite complete Boolean algebra $\mathscr{X}$ with the countable chain condition is fully generated by some free subalgebra of cardinality $\tau(\mathscr{X})$, i.e., $\mathscr{X}$ is representable as*

$$\mathscr{X} = \prod_{\mathscr{Z} \in \mathfrak{P}} \mathscr{Z},$$

where $\mathscr{P}$ is an independent class of simple subalgebras, and card $\mathscr{P} = \tau(\mathscr{X})$.

Proof.[26] We start with the following lemmas.

Lemma 18. *Let α be a limit transfinite number (ordinal) and let A be the set of all less ordinals.*[27] *Then we can split A into countably many cofinal subsets of A.*

[25]S. Koppelberg [1].

[26]We adduce the proof that was imparted to the author by S. V. Kislyakov.

[27]In the modern set theory, A and α are usually identified.

Indeed, each ordinal $\beta \geq \omega$ can be uniquely written as $\beta = \beta_0 + k$, where β_0 is a limit ordinal or zero and k is a nonnegative integer. Put $A_k = \{\beta < \alpha \mid \beta = \beta_0 + k\}$. These sets are pairwise disjoint; moreover, then A and are cofinal since α is a limit ordinal.

Lemma 19. *Let α and A be the same as in the preceding lemma. Then the Boolean algebra 2^A includes a countable fully generating independent system, i.e., it is generated by a countable free subalgebra. Moreover, each element of this free subalgebra is a cofinal subset of A.*

PROOF. Consider the Cantor discontinuum of a countable weight

$$X \equiv \{\chi = (\chi_1, \chi_2, \dots) \mid \chi_i = 0, 1\}.$$

Let $\mathscr{D}$ be the free Boolean algebra of all clopen (i.e., close and open) subsets in X; this algebra is countable. Enumerate elements of D into a sequence $d_1, d_2, \dots$. Choose a countable (infinite) set $T_n \equiv \{t_{n_1}, t_{n_2}, \dots\}$ in each d_n. Make all these sets pairwise disjoint. Denote their union by T. Put $d'_n \equiv d_n \cap T$. Each intersection contains T_n and, consequently, is nonempty. The system $\mathscr{D}'$ of all d'_n is obviously a free Boolean algebra isomorphic to $\mathscr{D}$. Split the set A as in Lemma 18: $A = \bigcup_k A_k$. There is a bijection from each A_k onto T_k which defines a bijection $\varphi : A \longrightarrow T$. The inverse images $\varphi^{-1}(d'_n)$ comprise a subalgebra that separates the points of A and, therefore, fully generates 2^A. This subalgebra is free and countable.

Lemma 20. *Let α and A be the same as in the preceding lemmas. Assume further that to each $\beta \in A$, there is a complete Boolean algebra $\mathscr{X}_\beta$ fully generated by a free subalgebra $\mathscr{Y}_\beta$. Suppose that $card\,\mathscr{Y}_1 \leq card\,\mathscr{Y}_2 \leq \dots \leq card\,\mathscr{Y}_\beta \leq \dots$. Then the Boolean algebra*

$$\mathscr{X} \equiv \bigoplus_{\beta<\alpha} \mathscr{X}_\beta$$

is fully generated by some free subalgebra of cardinality $\tau(\mathscr{X})$.

PROOF. We put $\Theta_\beta = card\,\mathscr{Y}_\beta$ and $\Theta = \sup_{\beta<\alpha} \Theta_\beta$; and let $\overline{\omega}_\beta$ and $\overline{\omega}$ be the initial ordinals of the cardinalities Θ_β and Θ respectively. Let $Q_\beta \equiv [1, \overline{\omega}_\beta]$ and $Q \equiv [1, \overline{\omega}]$ be ordinal intervals. We have $Q_1 \subset Q_2 \subset \dots$. Given β, consider an independent system $\{v_{\beta q}\}_{q \in Q_\beta}$ of generators for the subalgebra $\mathscr{Y}_\beta$ (its cardinality is $\Theta_\beta = card\,Q_\beta$).

Put

$$\overline{v}_{\beta q} \equiv \begin{cases} v_{\beta q}, & \text{if} \quad q \leq \overline{\omega}_\beta, \\ v_{\beta \overline{\omega}_\beta}, & \text{if} \quad q > \overline{\omega}_\beta, \end{cases}$$

$$\overline{v}_q \equiv \bigvee_\beta \overline{v}_{\beta q}.$$

Denote the unity of $\mathscr{X}_\beta$ by e_β and the totality of all $\overline{v}_q$ $(q \in Q)$ by V.

By Lemma 19, the algebra 2^A possesses a fully generating countable free subalgebra $\mathbf{C}$ which consists of cofinal subsets of A. To each set $c \in \mathbf{C}'$, where $\mathbf{C}'$ is an independent system of generators in $\mathbf{C}$, there corresponds the element

$$u_c \equiv \bigvee_{\beta \in c} e_\beta \in \mathscr{X}.$$

It is clear that the system $U \equiv \{u_c\}_{c \in \mathbf{C}'}$ of all these elements is independent.

Consider the set $U \cup V \equiv W$. Show that W is independent. Let $x = \widetilde{w}_1 \wedge \widetilde{w}_2 \wedge \cdots \wedge \widetilde{w}_s$, where each $\widetilde{w}_i$ is either w_i or $Cw_i \quad (w_i \in W)$. We need to prove that $x > \mathbf{0}$. Let $s = p + q$, $\widetilde{w}_1, \dots, \widetilde{w}_p \in V \cup CV$ and $\widetilde{w}_{p+1}, \dots, \widetilde{w}_{p+q} \in U \cup CU$. The elements $\widetilde{w}_{p=1}, \dots, \widetilde{w}_{p+q}$ are of the form

$$e = \bigvee_{\beta \in \overline{c}} e_\beta,$$

where $\overline{c} \in \mathbf{C}$ is a cofinal subset of A. For $i \leq p$ we have either $\widetilde{w}_i = \overline{v}_{q_i}$ or $\widetilde{w}_i = C\overline{v}_{q_i} \quad (q_i \in Q)$. The cofinal set $\overline{c}$ certainly contains the ordinal β^* for which $\overline{w}_{\beta^*} > q_1, \dots, q_p$ and, thus, all $\overline{v}_{\beta^* q_1}, \dots, \overline{v}_{\beta^* q_p}$ are pairwise disjoint elements of an independent system. Therefore,

$$x = \widetilde{w}_1 \wedge \cdots \wedge \widetilde{w}_p \wedge e \geq \widetilde{w}_1 \wedge \cdots \wedge \widetilde{w}_p \wedge e_{\beta^*}$$

$$= (\widetilde{w}_1 \wedge E_{\beta^*}) \wedge \cdots \wedge (\widetilde{w}_p \wedge e_{\beta^*}) > \mathbf{0},$$

since each element $\widetilde{w}_i \wedge e_{\beta^*}$ is either $\overline{v}_{\beta^* q_i}$ or its complement to e_{β^*}.

Thus, the system W is independent indeed. Each element $x \in \mathbf{C}$ can be represented as

$$x = \bigvee_\beta (x \wedge e_\beta).$$

Each element $x \wedge e_\beta$ belongs to the subalgebra fully generated by the system $\{\overline{v}_{\beta q} \wedge e_\beta\}$ and the elements of this system are obviously contained in the subalgebra generated by W. Consequently, this set, as well as the free subalgebra generated by it, fully generates the entire Boolean algebra.

It remains only to find out the cardinality of this free subalgebra. It coincides with the cardinal

$$\mathit{card}\, W = \mathit{card}\, V + \mathit{card}\, U = \mathit{card}\, V + \aleph_0 = \mathit{card}\, Q + \aleph_0$$

$$= \sup_\beta \mathit{card}\, Q_\beta + \aleph_0.$$

If all Q_β are finite (i.e., $\mathscr{X}_\beta$ are finite) then

$$card\,W = \aleph_0 = \tau(\mathscr{X}).$$

Otherwise,

$$card\,W = \sup_\beta card\,Q_\beta = \sup_\beta \tau(\mathscr{X}_\beta) \leq \tau(\mathscr{X}).$$

On the other hand, the cardinality of a fully generating system is always at least $\tau(\mathscr{X})$. So, $card\,W = \tau(\mathscr{X})$. The proof of the lemma is complete.

Finally, consider the general case in which $\mathscr{X}$ is a complete Boolean algebra with the countable chain condition. Then, as usual, it can be represented as the sum of bands homogeneous by weight:

$$\mathscr{X} = \bigoplus_{\beta<\alpha} \mathscr{X}_\beta,$$

where $\tau(\mathscr{X}_1) \leq \tau(\mathscr{X}_2) \leq \dots$ and α is an arbitrary countable ordinal. Represent α in the form $\alpha = \alpha_0 + n$, where n is finite and α_0 is a limit ordinal or zero. First, consider the case in which $\alpha_0 > 0$. Then we may write

$$\mathscr{X} = \mathscr{X}^{(0)} \oplus \mathscr{X}^1 \oplus \mathscr{X}^2 \oplus \dots \oplus \mathscr{X}^n, \quad \mathscr{X}^i \equiv \mathscr{X}_{\alpha_0+i+1},$$

where

$$\mathscr{X}^{(0)} \equiv \bigoplus_{\beta<\alpha_0} \mathscr{X}_\beta$$

is an algebra of the same type as in the last lemma. Such a subalgebra is fully generated by a free subalgebra of weight

$$\tau(\mathscr{X}^{(0)}) = \sup_{\beta<\alpha_0} \tau(\mathscr{X}_\beta).$$

The other subalgebras has weights $\tau(\mathscr{X}^1), \dots, \tau(\mathscr{X}^n)$ and

$$\tau(\mathscr{X}^{(0)}) \leq \tau(\mathscr{X}^1) \leq \tau(\mathscr{X}^2) \leq \dots \leq \tau(\mathscr{X}^n).$$

The band $\mathscr{X}^n$ is a homogeneous Boolean algebra. Such an algebra is always representable as the sum of countably many homogeneous bands of the same weight. Thus, we have the representation

$$\mathscr{X} = \mathscr{X}^{(0)} \oplus \mathscr{X}^1 \oplus \mathscr{X}^2 \oplus \dots \oplus \mathscr{X}^{n-1} \oplus \mathscr{Z} \oplus \mathscr{Z} \oplus \dots.$$

We arrive at the decomposition of the same type as in Lemma 20. Namely, $\alpha = \omega$ is a limit ordinal and

$$\tau(\mathscr{X}^{(0)}) \leq \dots \leq \tau(\mathscr{X}^i) \leq \dots \leq \tau(\mathscr{Z}) \leq \dots.$$

Since the bands $\mathscr{X}^{(0)}, \mathscr{X}^{1}, \ldots$ are homogeneous by weight, they are generated by free subalgebras of cardinalities equal to weights. Therefore, the Boolean algebra $\mathscr{X}$ is generated by a free subalgebra of cardinality $\tau(\mathscr{X})$.

We will use Theorem 19 later.

The proof of the general theorem (for a Boolean algebra of uncountable type) is somewhat more complicated but can be carried out by the same scheme. This result is significantly strengthened now.

Compare the two main constructions: preproduct and product. The preproduct is a Boolean algebra associated uniquely (up to isomorphism) with an arbitrary family of subalgebras of the original Boolean algebra; as a rule, this preproduct is an incomplete algebra. The preproduct does not change if all "factors" are replaced by isomorphic algebras. We discuss this and similar constructions in Chapter 3, devoted to the categories of Boolean algebras.

As for the product, it results from an independent family of subalgebras in some complete Boolean algebra $\mathscr{X}$ and is a regular subalgebra in $\mathscr{X}$. Accordingly, the product is always a complete Boolean algebra. An attempt to study this construction for arbitrary families of algebras not embedded into some algebra causes troubles. Indeed, in view of the Koppelberg Theorem, each complete Boolean algebra is a product of simple algebras; so, the different products of the same system of Boolean algebras are not isomorphic necessarily. We might assume that each product is a Dedekind completion of a preproduct; however, such a definition will not be very fruitful in this book.

7.2 Free subalgebras of a complete Boolean algebra

A free subalgebra is a subalgebra generated by an independent system of generators, i.e., is (itself) a free Boolean algebra. It is clear that an infinite complete Boolean algebra $\mathscr{X}$ always possesses an independent system (or, which is equivalent, a free subalgebra) of cardinality $\tau(\mathscr{X})$. However, an independent system of a greater cardinality (for example, of cardinality $card\, \mathscr{X}$) is needed in some cases. G. M. Fikhtengol′ts and L. V. Kantorovich faced this problem for the first time. In the article [1] they proved that, in the Boolean algebra 2^T, where $card\, T = \aleph_0$, there exists an independent system of cardinality $2^{\aleph_0} = 2^{card\, T} = card\, 2^T$. Soon, F. Hausdorff [1] generalized this result to the case of the arbitrary infinite algebras of the form 2^T. Using Theorem 18 in 1970, D. A. Vladimirov and B. A. Efimov [1] found a few cases in which a complete (continuous) Boolean algebra $\mathscr{X}$ has an independent sys-

tem of cardinality $card\,\mathscr{X}$. It became clear that such a situation is typical. In 1973, S. V. Kislyakov [1] proved existence of a free subalgebra of cardinality $card\,\mathscr{X}$ for every complete Boolean algebra $\mathscr{X}$ whose cardinality is less than the first weakly inaccessible cardinal. Later, even the last restriction was renounced by B. Balcar and F. Franěk [1]. The final solution to the problem was given by R. McKenzie: For each infinite complete Boolean algebra $\mathscr{X}$ and for each cardinal $\lambda \in [\tau(\mathscr{X}), card\,\mathscr{X}]$, there exists an independent system of cardinality λ which fully generates $\mathscr{X}$. The proof of this theorem basing on the Kislyakov–Balcar–Franěk Theorem can be found in the book.[28] The McKenzie Theorem generalizes the above-mentioned theorem as well as the Koppelberg Theorem.

Exercises for Chapter 2

1. Prove that an infinite free Boolean algebra cannot be complete.

2. Prove that every complete continuous Boolean algebra possessing a countable minorant is isomorphic to the algebra $\mathscr{O}((0,1))$ of regular open subsets of $(0,1)$.

[28] J. D. Monk and R. Bonnet [1].

Chapter 3

REPRESENTATION OF BOOLEAN ALGEBRAS

Each Boolean algebra is isomorphic to an algebra of sets. This fact was already mentioned in the preceding chapters. Below, we give an exact formulation and a complete proof of the celebrated Stone Theorem and discuss the problems that arise in connection with this theorem.

1. The Stone Theorem

1.1 Maximal ideals and ultrafilters

We got acquaintance with the concepts of ideal and filter in Chapter 0. In particular, it was proved that each element $x \neq \mathbf{1}$ belongs to some proper ideal. Analogously, each $x \neq \mathbf{0}$ lies in a proper filter.

Among proper ideals, a specific role is played by those not included in any strictly greater proper ideal; these ideals are called *maximal*. Note some important properties of maximal ideals.

Lemma 1. *To each proper ideal I, there is a maximal ideal including I.*

The proof bases on the Kuratowski–Zorn Lemma cited in the above chapters. Consider a linearly inclusion-ordered set of proper ideals. It is clear that the set-theoretic union of ideals is a proper ideal. We see that every chain is bounded above in the inclusion-ordered set of all proper ideals. By the Kuratowski–Zorn Lemma, each ideal is then included in a maximal ideal.

This obviously yields

Lemma 2. *Each element $x \neq \mathbf{1}$ belongs to a maximal ideal.*

Lemma 3. *If I is a maximal ideal and u is an element then one of the elements u and Cu belongs to I.*

PROOF. Assume that $u \notin I$ and $Cu \notin I$. Then $u < \mathbf{1}$ and $Cu < \mathbf{1}$. Consider the principal ideal $I^* = \mathscr{X}_u$ and introduce the set $K \equiv I \vee I^*$ that comprises all joins of the form $x \vee x^*$ with $x \in I$ and $x^* \in I^*$. Show that K is an ideal. We first verify solidity. Let $z \leq w \in K$. The element w can be represented as

$$w = y \vee y^*, \quad y \in I, \;\; y^* \in I^*.$$

Then the elements $x \equiv z \wedge y$ and $x^* \equiv z \wedge y^*$ belong respectively to the ideals I and I^* (since the last two sets are solid). Simultaneously, we have

$$z = z \wedge w = z \wedge (y \vee y^*) = (z \wedge y) \vee (z \wedge y^*) = x \vee x^*,$$

whence it follows that $z \in K$. With solidity established, show now that K contains the meets of all pairs of its elements. If $v, w \in K$ then

$$v = x \vee x^*, \quad w = y \vee y^*; \quad x, y \in I, \quad x^*, y^* \in I^*,$$

$$v \vee w = (x \vee x^*) \vee (y \vee y^*) = (x \vee y) \vee (x^* \vee y^*) \in K.$$

So, K is an ideal. If we suppose that $\mathbf{1} \in K$ then there are such elements $v' \in I$ and $w' \in I^*$ that $v' \vee w' = \mathbf{1}$. However, $w' \leq u$; therefore, $v' \geq Cu$ and Cu must belong to I despite the supposition. Thus, K is a proper ideal. Moreover, it is strictly greater than I which is impossible by the maximality of the latter. The lemma is proved.

Corollary 1. *If a maximal ideal I includes the intersection of two ideals I_1 and I_2 then I includes at least one of these ideals.*

Otherwise, there are elements $x_1 \in I_1 \setminus I$ and $x_2 \in I_2 \setminus I$. By the above lemma, $Cx_1 \in I$ and $Cx_2 \in I$. Moreover, $x_1 \wedge x_2 \in I_1 \cap I_2 \subset I$. Then $\mathbf{1} = Cx_1 \vee Cx_2 \vee (x_1 \wedge x_2) \in I$ which is impossible since I is a proper ideal.

This corollary is obviously valid for finitely many ideals.

Corollary 2. *The quotient algebra by a maximal ideal consists only of the two elements: zero and unity.*

Lemma 4. *The elements u and Cu cannot belong simultaneously to a maximal ideal.*

This lemma is obvious.

A set dual to a maximal ideal is called a *maximal filter* or *ultrafilter*.

An ultrafilter is a proper filter not included in any proper filter. Lemmas 1–4 have obvious dual analogs about ultrafilters. The reader can easily find the respective formulations.

1.2 Totally disconnected topological spaces

Consider a topological space R. Let $\mathfrak{G}$ be a totality of open sets in R. Can this totality be an algebra of sets? To ensure this, it is necessary at least that the totality contains the set-theoretic complements of sets in it. This is possible only if all $G \in \mathfrak{G}$ are closed and open simultaneously. Such sets are called *clopen*. The following theorem is obvious.

Theorem 1. *The system of all clopen subsets of an arbitrary topological space is an algebra of sets.*

However, in the commonest topological spaces (particularly, as regards analysis) there are few clopen sets; as a rule, there are only two such sets: the entire space and empty set. For the clopen algebra to be nontrivial, it is necessary to impose some additional restrictions on the space R. We say that R is *totally disconnected* (or "of measure zero") whenever all clopen sets form a base for the topology of R (i.e., each open set is a union of clopen sets). The condition of total disconnectedness ensures the presence of "sufficiently" many clopen sets in R. In particular, we arrive at the important fact: *the totality of all clopen sets separates the points of a totally disconnected space in which every two points are separated by an open set.*

We will show further that each Boolean algebra is isomorphic to the clopen algebra of a topological space R. In Chapter 1, this fact was established in the case of a free Boolean algebra; the role of R was played by a Cantor discontinuum X_Ξ.

1.3 The Stone Theorem

We now prove the main Stone Theorem on representation of a Boolean algebra.

Theorem 2. *For each Boolean algebra $\mathscr{X}$, there is a totally disconnected compact space $\mathfrak{Q}$ whose clopen algebra is isomorphic to $\mathscr{X}$.*

Proof. Let the set $\mathfrak{Q}$ comprise all maximal ideals of $\mathscr{X}$. Select a class of subsets of $\mathfrak{Q}$ related with the ideals of the original algebra. Namely, to each ideal I, we assign the set $\mathfrak{M}(I)$ of all maximal ideals including I. We first establish some important properties of these sets.

1°. *The equality*

$$\bigcap_{I \in \mathscr{E}} \mathfrak{M}(I) = \mathfrak{M}\Big(\mathfrak{I}\Big\{ \bigcup_{I \in \mathscr{E}} I \Big\}\Big) \tag{1}$$

holds for every set $\mathscr{E}$ of ideals.

For brevity, denote by P and Q respectively the left-hand and right-hand sides of the above equality. Assume first that a maximal ideal q

belongs to P. This means that $q \supset \bigcup_{I\in\mathscr{E}} I$; therefore, q also includes the least ideal that contains all $I \in \mathscr{E}$, i.e., the ideal $\mathfrak{I}\{\bigcup_{I\in\mathscr{E}} I\}$. In other words, $q \in \mathfrak{M}(\mathfrak{I}\{\bigcup_{I\in\mathscr{E}}\}) = Q$ and $P \subset Q$. Take $q \in Q$. This containment means that

$$q \supset \bigcup_{I\in\mathscr{E}} I, \quad q \in \bigcap_{I\in\mathscr{E}} \mathfrak{M}(I) = P.$$

Thus, $Q \subset P$, and 1° is proved.

2°. *Given two ideals I_1 and I_2, we have*

$$\mathfrak{M}(I_1) \cup \mathfrak{M}(I_2) = \mathfrak{M}(I_1 \cap I_2). \tag{2}$$

As in the above reasoning, denote the sides of this equality by P and Q. Each $q \in P$ includes either I_1 or I_2 and, therefore, includes their intersection. That is why $q \in Q$ and $P \subset Q$. On the other hand, if $q \in Q$ then $I_1 \cap I_2 \subset q$. By the corollary to Lemma 3, one of the ideals I_1 and I_2 must lie in q. This implies that $q \in \mathfrak{M}(I_1) \cup \mathfrak{M}(I_2) = P$. So, $Q \subset P$ and the equality is proved.

Note that, in the case of a proper ideal I, the set $\mathfrak{M}(I)$ is nonempty by Lemma 1. If $I = \mathscr{X}$ then obviously $\mathfrak{M}(I) = \varnothing$.

Among the sets of the form $\mathfrak{M}(I)$, of the utmost interest are those that correspond to the principal ideals or, which is the same, to the elements of $\mathscr{X}$. Introduce the notations:

$$G_u \equiv \mathfrak{M}(\mathscr{X}_u), \tag{3}$$

$$G'_u \equiv \mathfrak{Q} \setminus G_u. \tag{4}$$

By Lemmas 3 and 4, the equality

$$G'_u = G_{Cu} \tag{5}$$

holds for every $u \in \mathscr{X}$. This equality shows that the sets of the type (4) comprise the same class as the sets of the type (5). We denote this class by Γ and call a set in it a *basis set*. Show that Γ is *an algebra of sets isomorphic to $\mathscr{X}$*.

Define the mapping Φ_0 from $\mathscr{X}$ onto the naturally ordered system of sets Γ by the formula:

$$\Phi_0(x) \equiv G'_x \tag{5'}$$

and prove that this mapping is an isomorphism.

We first notice that Γ is obviously exhausted by elements of the form $\Phi_0(x)$. Further, by the solidity of an ideal, we conclude that the inequality $x \le y$ implies the inclusion $G'_x \subset G'_y$. On the other hand, if $v = Cy \wedge x > \mathbf{0}$ then, by Lemmas 2 and 4, there is a maximal ideal q

including Cv but not Cy.[1] Then $G'_x \supset G'_v$ and $G'_v \not\subset G'_y$, i.e., $G'_x \not\subset G'_y$. We can conclude that the inclusion $G'_x \subset G'_y$, in turn, implies the inequality $x \leq y$. So, *these two relations are equivalent and the mapping* Φ_0 *preserves order and is bijective.* We have proved that the inclusion-ordered system Γ is isomorphic to the Boolean algebra $\mathscr{X}$. Show that this system is an algebra of sets (the above implies only the fact that Γ is a Boolean algebra). It follows from the equalities (5) and (5′) that

$$\Phi_0(Cx) = G'_{Cx} = \mathfrak{Q} \setminus G'_x = \mathfrak{Q} \setminus \Phi_0(x);$$

therefore, Γ contains the complement of each set in it. Establish the identity

$$\Phi_0(u) \cap \Phi_0(v) = \Phi_0(u \wedge v) \tag{$*$}$$

which can be written as

$$G'_u \cap G'_v = G'_{u \wedge v}$$

or

$$\mathfrak{M}(\mathscr{X}_{Cu}) \cap \mathfrak{M}(\mathscr{X}_{Cv}) = \mathfrak{M}(\mathscr{X}_{Cu \vee Cv}).$$

We see that $(*)$ ensues from 1° since $\mathscr{X}_{Cu \vee Cv} = \mathfrak{I}\{\mathscr{X}_{Cu} \cup \mathscr{X}_{Cv}\}$. The equality $(*)$ shows that the system Γ is closed under intersections. This suffices to assert that Γ is an algebra of sets.

Now, introduce a topology on $\mathfrak{Q}$ assuming that the closed sets are precisely those of the form $\mathfrak{M}(I)$. From 1° and 2° it follows that the totality of all such sets is closed under all intersections and finite unions. These two properties are sufficient for defining a topology (see Appendix A). Moreover, the basis sets turn out to be not only closed but also open in view of (5). It is easy to see that the system Γ is a base for our topology. Indeed, each open set has the form $\mathfrak{Q} \setminus \mathfrak{M}(I)$; it consists of all maximal ideals that do not include the ideal I. Moreover, the maximal ideals that do not include I are precisely those ideals that include the complement of at least one of the elements in I (Lemmas 3 and 4). Therefore,

$$\mathfrak{Q} \setminus \mathfrak{M}(I) = \bigcup_{u \in I} G_{Cu}.$$

We see that each open set is a union of basis sets and the topology we have introduced is actually generated by the system Γ; this fact, by the way, justifies the term "basis set." Since the base for the topology consists of clopen sets, $\mathfrak{Q}$ is totally disconnected.

[1] As q, we can take a maximal ideal that contains Cv.

So, we have established that *the original Boolean algebra $\mathscr{X}$ admits a one-to-one isotonic mapping onto the clopen algebra of a totally disconnected topological space $\mathfrak{Q}$.*

Show that $\mathfrak{Q}$ is a compact space under the above-introduced topology. We need to verify that the two properties hold: the Hausdorff axiom and compactness. Let q_1 and q_2 be distinct points in $\mathfrak{Q}$, i.e. distinct maximal ideals. There exists an element $u \in \mathscr{X}$ such that $u \in q_1 \setminus q_2$. Then, by Lemmas 3 and 4, $Cu \in q_2 \setminus q_1$. In other words, $q_1 \in G_u$ and $q_2 \in G_{Cu}$. By Lemma 4, the sets G_u and G_{Cu} cannot have common elements; therefore, they separate the points q_1 and q_2. Thus, the Hausdorff axiom holds in $\mathfrak{Q}$. Establish the compactness of $\mathfrak{Q}$.

Let π be a system of closed sets whose every finite subsystem has a nonempty intersection. Show that the intersection of the entire system π is nonempty; this fact means the compactness of $\mathfrak{Q}$. In accordance with 1°, we have

$$\bigcap_{f \in \pi} F = \mathfrak{M}\Big(\mathfrak{I}\Big\{\bigcup_{I \in \pi^*} I\Big\}\Big),$$

where π^* is the totality of all ideals corresponding to the sets of π. If we suppose that the intersection in question is empty then the ideal $\mathfrak{I}\{\bigcup_{I \in \pi^*} I\}$ is improper. Hence there is a finite set of ideals $I_1, I_2, \dots, I_m \in \pi^*$ whose union also generates an improper ideal by Lemma 7 in Chapter 1. Applying 1° once again, we see that the intersection

$$\bigcap_{k=1}^{m} \mathfrak{M}(I_k)$$

is empty, which is impossible since $\mathfrak{M}(I_k) \in \pi$ $(k = 1, 2, \dots, m)$.

The proof will be complete if we show that there is no other clopen sets except the sets G_x in $\mathfrak{Q}$. But this is obvious since each clopen set is a union of finitely many basis sets of the form G_x. The proof of the theorem is finished.

The clopen algebra of the compact space $\mathfrak{Q}$ is said to be a *representation* for the original Boolean algebra $\mathscr{X}$. As usual, we may consider a "function" representation as the algebra of the CHARACTERISTIC FUNCTIONS of clopen subsets in $\mathfrak{Q}$. Note that all such functions are continuous. Each maximal ideal $q \in \mathfrak{Q}$ corresponds to the set of functions vanishing on q.

1.4 Stone spaces

Every totally disconnected compact space $\mathfrak{Q}$ with the clopen algebra isomorphic to a BA $\mathscr{X}$ is called a *Stone space* of $\mathscr{X}$. Theorem 2 shows a way to constructing such a compact space; the points of it are maximal

ideals. It is possible to construct a Stone space with ULTRAFILTERS of the Boolean algebra $\mathscr{X}$ as points. In this case, the basis sets can be introduced more easily: to each $x \in \mathscr{X}$, we assign the set $\Phi_0'(x)$ of all ultrafilters containing this element. Both ways to constructing a Stone space are equivalent; we will see that there is a unique Stone space of a Boolean algebra $\mathscr{X}$ up to isomorphism. Grounding on this fact, we will often denote each of these compact spaces by the same symbol $\mathfrak{Q}(\mathscr{X})$. In the rare cases when the concrete nature of a Stone space is material, we supply the letter $\mathfrak{Q}$ with an index. So, we denote the compact space of maximal ideals as in Theorem 2 by the symbol $\mathfrak{Q}_i(\mathscr{X})$; and the compact space of ultrafilters by $\mathfrak{Q}_f(\mathscr{X})$. If $\mathfrak{Q}$ is a distinguished Stone space of a Boolean algebra $\mathscr{X}$ then the clopen algebra of $\mathfrak{Q}$ is denoted by $\mathscr{X}^0$ or, more precisely, by $\mathscr{X}^0_{\mathfrak{Q}}$. The algebra of characteristic functions of clopen sets is denoted by $\mathscr{X}^1$ ($\mathscr{X}^1_{\mathfrak{Q}}$). Both algebras are obviously isomorphic to $\mathscr{X}$ and we call them the *Stone representations of* $\mathscr{X}$. It is often convenient to identify a Boolean algebra $\mathscr{X}$ and one of its representations, i.e., to assume that $\mathscr{X} = \mathscr{X}^0$ or $\mathscr{X} = \mathscr{X}^1$. Sometimes, a totally disconnected compact space $\mathfrak{Q}$ is given without regard to any Boolean algebra. In this case, we denote its algebra by $\mathscr{C}O(\mathfrak{Q})$. Thus, $\mathscr{C}O(\mathfrak{Q}) \equiv [\mathscr{C}O(\mathfrak{Q})]^0$. The least σ-algebra including $\mathscr{C}O(\mathfrak{Q})$ ("Baire algebra") is denoted by $\mathscr{B}_0(\mathfrak{Q})$ or $\mathscr{B}_0$; the algebra of all Borel sets ("Borel algebra") by $\mathscr{B}(\mathfrak{Q})$ or $\mathscr{B}$.

Note some important properties of every Stone space $\mathfrak{Q} = \mathfrak{Q}(\mathscr{X})$. Throughout this subsection, we denote by Φ an arbitrary isomorphism:

$$\mathscr{X} \xrightarrow{\Phi} \mathscr{X}^0_{\mathfrak{Q}} \equiv \mathscr{C}O(\mathfrak{Q}).$$

(For instance, Φ may coincide with one of the above isomorphisms Φ_0 or Φ_0'.) Assign to each closed set $E \subset \mathfrak{Q}$ the two following sets in $\mathscr{X}$:

$$I\langle E\rangle \equiv \{x \in \mathscr{X} \mid \Phi(x) \cap E = \varnothing\},$$

$$F\langle E\rangle \equiv \{x \in \mathscr{X} \mid \Phi(x) \supset E\}.$$

1°. *Each $I\langle E\rangle$ is an ideal and each $F\langle E\rangle$ is a filter on $\mathscr{X}$. There is no other ideals and filters on $\mathscr{X}$.*

2°. *The correspondences $E \longleftrightarrow I\langle E\rangle$ and $E \longleftrightarrow F\langle E\rangle$ are one-to-one.*

3°. *$I\langle E\rangle$ is a proper ideal if and only if $E \neq \varnothing$, while $I\langle E\rangle$ is a maximal ideal if and only if E is a singleton.*

The claim of 3° has an obvious dual analog for filters.

Prove 1°–3° for ideals.

1°. It is clear that each set of the form $I\langle E\rangle$ is an ideal. Show that all ideals have this form. Take an ideal I and consider the open set

$$G \equiv \bigcup_{x\in I} \Phi(x).$$

Put $E = \mathfrak{Q} \setminus G$ and show that $I = I\langle E\rangle$. Take $x \in I$. Then, obviously, $\Phi(x) \subset G$ and $\Phi(x) \cap E = \varnothing$, i.e., $x \in I\langle E\rangle$. Now take $x \in I\langle E\rangle$. This means that $\Phi(x) \subset G$. Each point of the image $\Phi(x)$ belongs to some set of the form $\Phi(u)$, where $u \in I$. These sets comprise an open cover

$$\Phi(x) \subset \Phi(u_1) \cup \Phi(u_2) \cup \dots \cup \Phi(u_m),$$

where $u_1, u_2, \dots, u_m \in I$. But $\Phi(u_1) \cup \Phi(u_2) \cup \dots \cup \Phi(u_m) = \Phi(u_1 \vee u_2 \vee \dots \vee u_m)$ and $u_1 \vee u_2 \vee \dots \vee u_m \in I$ since I is an ideal. Thus, $\Phi(x) \subset \Phi(u),\ u \in I$, whence $x \le u$ and $x \in I$. Therefore, $I = I\langle E\rangle$.

2°. Let $E_1 \neq E_2$. Show that $I\langle E_1\rangle \neq I\langle E_2\rangle$. Take, for example, $q \in E_1 \setminus E_2$. There exists a clopen set e separating q from E_2, i.e., $q \in e$ and $e \cap E_2 = \varnothing$. Set $x \equiv \Phi^{-1}(e)$. The element x belongs to $I\langle E_2\rangle$ but not to $I\langle E_1\rangle$; therefore, $I\langle E_1\rangle \neq I\langle E_2\rangle$.

3°. If $E = \varnothing$ then $I\langle E\rangle = \mathscr{X}$, i.e., this ideal is improper. By 2°, the ideal is proper for $E \neq \varnothing$.

Let $E = \{q\},\ q \in Q$. Then, for all $x \in \mathscr{X}$, we have either $\Phi(x) \ni q$ or $\mathfrak{Q} \setminus \Phi(x) = \Phi(Cx) \ni q$. In other words, we always have either $x \in I\langle E\rangle$ or $Cx \in I\langle E\rangle$. Such an ideal does not admit a proper extension and therefore it is maximal. Consider a maximal ideal I now. Take the set

$$G \equiv \bigcup_{x\in I} \Phi(x)$$

again. We have established that $I\langle E\rangle = I$, where $E = \mathfrak{Q} \setminus G$. The set E is nonempty a fortiori. Show that it contains at most a single point. Indeed, if suppose that $q_1, q_2 \in E$ and $q_1 \neq q_2$ then these two points are separated by a clopen set e:

$$q_1 \in e, \quad q_2 \in Ce \equiv \mathfrak{Q} \setminus e.$$

Put $x \equiv \Phi^{-1}(e)$. Then $Cx = \Phi^{-1}(Ce)$ and at least one of the elements x or Cx must belong to a maximal ideal I. But this is impossible since

$$\Phi(x) \cap E = e \cap E \neq \varnothing, \quad \Phi(Cx) \cap E = Ce \cap E \neq \varnothing.$$

We dwell on some topological properties of Stone spaces. Each compact space is a normal topological space. The totally disconnected compact spaces we meet in the theory of Boolean algebras are normal in a strengthened sense:

Theorem 3. 1) *Every two disjoint closed sets in a totally disconnected compact space $\mathfrak{Q}$ are separated by disjoint clopen sets.*

2) *Every algebra of clopen sets which separates the points of a totally disconnected compact space contains all clopen sets.*

Proof. 1) Let F_1 and F_2 be disjoint closed sets: $F_1 \cap F_2 = \varnothing$. Given $q \in F_2$ and $p \in F_1$, find a clopen set v_p such that $p \in v_p$, $q \notin v_p$. We can refine a finite cover of F_1 from all sets v_p: $F_1 \subset v_{p_1} \cup v_{p_2} \cup \cdots \cup v_{p_n}$. The complement $u_q \equiv \mathfrak{Q} \setminus (v_{p_1} \cup v_{p_2} \cup \cdots \cup v_{p_n})$ contains the point q and lies in $\mathfrak{Q} \setminus F_1$. From this cover of F_2, we can refine a finite cover of F_2:

$$F_2 \subset u_{q_1} \cup u_{q_2} \cup \cdots \cup u_{q_m}.$$

Taking $G \equiv u_{q_1} \cup u_{q_2} \cup \cdots \cup u_{q_m}$, we see that G and $\mathfrak{Q} \setminus G \equiv CG$ are clopen, disjoint, while $F_2 \subset G$ and $F_1 \subset CG$.

2) Let U_0 be a subalgebra of the clopen algebra $U \equiv \mathscr{C}O(\mathfrak{Q})$, and assume that U_0 separates points. Take an arbitrary $u \in U$. Each point $q \in u$ is separated from the closed set Cu by a clopen set u_q in U_0: $q \in u_q \subset u$. We have $u = \bigcup_{q \in u} u_q$. But u is compact and all u_q are open. Therefore, there exist a finite cover $u = u_{q_1} \cup u_{q_2} \cup \cdots \cup u_{q_s}$. It is clear now that $u \in U_0$. So, $U_0 = U$. The proof of the theorem is complete.

Corollary. *Every closed G_δ-subset of a totally disconnected compact space can be represented as the intersection of countably many clopen sets.*

Indeed, let a closed set have the form $F = \bigcap G_n$, where all G_n are open. By the first claim of Theorem 3, the sets F and $\mathfrak{Q} \setminus G_n$ are separable by some clopen sets E_n:

$$F \subset E_n, \;\; \mathfrak{Q} \setminus G_n \subset \mathfrak{Q} \setminus E_n, \;\; E_n \subset G_n.$$

We see that $F \subset \bigcap E_n \subset \bigcap G_n = F$, i.e., $F = \bigcap E_n$.

We now pass to arbitrary closed sets.

Theorem 4. *Every closed set $E \subset \mathfrak{Q}$ is the Stone space of the quotient algebra $\mathscr{X}|_{I\langle E\rangle}$.*

Proof. The cosets in the quotient algebra consist of the inverse images of clopen sets which have the same intersection with E. (We mean the inverse image under Φ.) To each coset of this intersection, we obtain a clopen subset of E; it is easy to verify that the system of all these subsets is an algebra isomorphic to the quotient algebra. It remains to show that each clopen set $A \subset E$ can be obtained as an intersection of the form $E \cap G$, where G is clopen in $\mathfrak{Q}$. Let A be a clopen set in E. There exists an open set $\widetilde{G}$ in $\mathfrak{Q}$ such that $\widetilde{G} \cap E = A$.

The clopen sets make a base for $\mathfrak{Q}$; hence, $\widetilde{G}$ may be written as follows:

$$\widetilde{G} = \bigcup_{\xi\in\Xi} G_\xi,$$

with G_ξ clopen in $\mathfrak{Q}$. We have

$$A = \bigcup_{\xi\in\Xi} (G_\xi \cap E).$$

The set A is closed and the sets $G_\xi \cap E$ are open in E. So, there exists a finite system $(\xi_1, \xi_2, \dots, \xi_m)$ such that

$$A = (G_{\xi_1} \cap E) \cup (G_{\xi_2} \cap E) \cup \dots \cup (G_{\xi_m} \cap E).$$

Putting $G = G_{\xi_1} \cup G_{\xi_2} \cup \dots \cup G_{\xi_m}$, we obtain a clopen set in $\mathfrak{Q}$ such that $A = E \cap G$. The proof of the theorem is complete.

Thus, the closed sets in $\mathfrak{Q}$ correspond uniquely to ideals or, which is equivalent, to quotient algebras. Moreover, it is clear that clopen sets correspond to principal ideals. If E is a closed G_δ-set then, by the corollary to Theorem 3, the corresponding ideal is the union of countably many principal ideals.

The idea behind the Stone Theorem turns out rather fruitful; it plays an important role in the modern mathematics; in particular, in the theory of Banach algebras and in the theory of compactification of topological spaces. Sometimes, the representation theorem serves as the main tool for studying Boolean algebras. In this book, we pursue another approach. In our opinion, the Stone representation is often not illuminative in much the same way as other set-theoretic representations. The concept of "abstract" Boolean algebra is more convenient when we deal with the most principal and difficult problems of the theory. However, in many cases, the representation of a Boolean algebra as an algebra of sets is rather helpful.

If a Boolean algebra is finite and has the form 2^Q then the Stone space has a simple structure. This compact space is the same set Q furnished with the discrete topology. "To visualize" an infinite Stone space is almost always impossible. In fact, the only opportunity of the sort is the case of a free Boolean algebra. In other cases, this space does not admit a simple and visual description. (We recall that the theorem on existence of maximal ideals or ultrafilters is equivalent to the axiom of choice.)

1.5 The Stone space of an algebra of sets

Consider a nonempty set Q and an algebra of sets $\mathscr{X} \subset 2^Q$ which separates the points of Q. Each point q generates the ultrafilter F_q and

the maximal ideal I_q:

$$F_q \equiv \{x \in \mathscr{X} \mid q \in x\}, \quad I_q \equiv \{x \in \mathscr{X} \mid q \notin x\}.$$

Since $\mathscr{X}$ separates points, the correspondence $q \longleftrightarrow F_q$ is one-to-one as well as the correspondence $q \longleftrightarrow I_q$. The ultrafilters of the type F_q and the maximal ideals of the type I_q, related to the points of Q, are called *trivial*. For definiteness, we will consider only ultrafilters and raise the question: When does the algebra of sets $\mathscr{X} \subset 2^Q$ fail to have a nontrivial ultrafilter? A necessary and sufficient condition for this can be easily found: it consists in the COMPACTNESS of $\mathscr{X}$. An algebra of sets $\mathscr{X}$ is called *compact* whenever each centered system of its elements (in particular, each ultrafilter) has a nonempty intersection. In this case, the mapping P that assigns to each ultrafilter the unique point of its intersection is a bijection:

$$q = P(F) = \bigcap_{x \in F} x \quad \longleftrightarrow \quad F = F_q.$$

In such a situation, we IDENTIFY Q and the Stone space $\mathfrak{Q}_f(\mathscr{X})$.

It is clear that the clopen algebra of each totally disconnected compact space is compact. The boolean 2^Q is compact only if Q is finite. Otherwise, this algebra always has nontrivial ultrafilters; however, their existence can be proved only on using the axiom of choice.

We return to the general situation in which $\mathscr{X}$ is an arbitrary (generally, not compact) algebra of subsets of Q. To each set $x \in \mathscr{X}$, we assign the set $\overline{x}$ of all ultrafilters containing x. Respectively, we write $\overline{Q}$ instead of $\mathfrak{Q}(\mathscr{X})$. The points of Q are identified with the principal ultrafilters; therefore, we always have $x \subset \overline{x}$. It is easy to verify that the following are equivalent:

1) Every $\overline{x}$ is the closure of x (in the topology of the Stone space $\overline{Q}$). In particular, the closure of Q is $\overline{Q}$ (the "canonical $\mathscr{X}$-extension of Q").

2) For all $x \in \mathscr{X}$, the equality $x = Q \cap \overline{x}$ holds (i.e., $q \in x \longleftrightarrow x \in F_q$).

3) The set $\overline{\mathscr{X}} \equiv \{\overline{x} \mid x \in \mathscr{X}\}$ is a compact algebra of sets of $\overline{Q}$.

4) The mapping $x \longleftrightarrow \overline{x}$ is bijective and, moreover, an isomorphism of $\mathscr{X}$ onto $\overline{\mathscr{X}}$.

Suppose that there is another algebra of sets $\mathscr{X}_0$ which is compact and separates the points of Q; let $\mathscr{X}_0 \subset \mathscr{X}$. The ultrafilters of this algebra are in a one-to-one correspondence with the points of Q; therefore, we can identify Q and $\mathfrak{Q}(\mathscr{X}_0)$. Take an element $\overline{x} \in \overline{\mathscr{X}}$. It is an ultrafilter on $\mathscr{X}$ and we can consider the intersection of $\overline{x}$ with $\mathscr{X}_0$; clearly, the intersection is an ultrafilter in $\mathscr{X}_0$. Thus, the mapping $r : \overline{Q} \longrightarrow Q$ is defined. It is clear that if $q \in Q$ then $r(q) = q$. We may introduce

a new topology on $\overline{Q}$ taking as a base all sets of the form $r^{-1}(x)$, with $x \in \mathscr{X}_0$. Then the mapping r is continuous as acting from $\overline{Q}$ onto Q ("retraction").

Assume now that $\mathscr{X} = 2^Q$. In this case, the space $\overline{Q}$ is called the "Stone–Čech compactification of a discrete topological space Q." It is usually denoted by βQ. The set $\beta Q \setminus Q$ is called the "excrescence" of Q; the points of it correspond to nontrivial ultrafilters. These ultrafilters play a key role in model theory, in particular, in the so-called "nonstandard analysis" invented by A. Robinson. If we consider the original set Q as a part of βQ then the Stone space of 2^Q is determined as follows: Each finite e represents itself: $\Phi(e) = e$. If e is infinite then $\Phi(e)$ is the closure of e in βQ; it consists of the points of e and some points of the "excrescence" of Q. This closure is an open set in βQ (we will discuss this topic in detail later in Section 2). The correspondence between sets $e \subset Q$ and their closures in βQ is one-to-one and possesses the properties of a Boolean isomorphism. In connection with the Stone Theorem, the series of natural questions arises: How are the familiar "Boolean" concepts (such as meet, join, completeness, homomorphism, and so on) interpreted in the "Stone" language? The following section addresses this series of questions.

2. Interpretation of the basic notions of the theory of Boolean algebras in the language of Stone spaces

Throughout this section, $\mathfrak{Q}$ is a totally disconnected compact space; $\mathscr{X} \equiv \mathscr{C}O(\mathfrak{Q})$. Passage to the general case of an arbitrary Boolean algebra $\mathscr{X}$ and $\mathfrak{Q} = \mathfrak{Q}(\mathscr{X})$ is not difficult.

2.1 Interpretation of suprema and infima. A test for completeness of a Boolean algebra

The Boolean operations over finite sets have natural interpretation since $\mathscr{X}$ is an algebra of sets. The least upper bound amounts to set-theoretic union; whereas the greatest lower bound, to intersection. As for infinite sets, they may have neither supremum nor infimum. Let E be a subset in $\mathscr{X}$.

Lemma 5. *For existence of a supremum for E in $\mathscr{X}$, it is necessary and sufficient that the closure*

$$u \equiv \overline{\bigcup_{x \in E} x}$$

be open; this closure is the supremum of E.

Proof. Suppose first that u is open. It is clear that $u \in E^s$. Consider an upper bound for E, i.e., a clopen set v containing the entire $x \in E$. Then $v \supset \bigcup_{x\in E} x$. Since v is closed, we have $v \supset \overline{\bigcup_{x\in E} x} \equiv u$. In other words, $v \geq u$. Thus, u is the least upper bound.

Suppose now that there exists $u_0 \equiv \sup E$. Then, obviously,

$$u_0 \supset \bigcup_{x\in E} x \quad \text{and} \quad u_0 \supset \overline{\bigcup_{x\in E} x} \equiv u.$$

Suppose towards a contradiction that the difference $u_0 \setminus u$ is nonempty. Then it must be open; and there exists a nonempty clopen set $v \subset u_0 \setminus u$ (the compact space is totally disconnected). Put $w = u_0 \setminus v$. The element w is the supremum of E in our Boolean algebra, although $w < u_0$ and $u_0 = \sup E$. This is impossible. So, $u_0 \setminus u = \varnothing$; the set u is open and presents the supremum of E in $\mathscr{X}$. The lemma is proved.

It has a dual analog:

Lemma 5′. *For existence of an infimum for E in $\mathscr{X}$, it is necessary and sufficient that the interior of the intersection $\bigcap_{x\in E} x$ be closed; this interior is the infimum of E.*

Corollary. *The equality $\bigwedge_{x\in E} x = \mathbf{0}$ means that the intersection $\bigcap_{x\in E} x$ is rare. In general, the sets $\bigcap_{x\in E} x \setminus \bigwedge_{x\in E} x$ and $\bigvee_{x\in E} x \setminus \bigcup_{x\in E} x$ are always rare.*

We now obtain an important test for completeness.

Definition. A compact space $\mathfrak{Q}$ is called *extremally disconnected* or, briefly, *extremal* whenever the closure of every open set in it is open.

Theorem 5. *For completeness of a Boolean algebra $\mathscr{X}$, it is necessary and sufficient that the Stone space $\mathfrak{Q}$ of $\mathscr{X}$ be extremal.*

This theorem follows immediately from the lemma. We only need to notice that each open set in $\mathfrak{Q}$ is the union of a totality of clopen sets.

In essence, Theorem 5 was established by M. Stone [3]. It was formulated in the modern terms by T. Ogasawara [1] and, independently, by B. Z. Vulikh [3]. The concept of extremally disconnected topological space originated with P. S. Urysohn.

We suggest the reader to verify that a Cantor discontinuum does not possess the property of extremality, i.e., neither of the infinite free Boolean algebras is complete.

It is easy to produce a necessary and sufficient condition of σ-completeness for a Boolean algebra. It is the so-called condition of "quasiextremality": the closure of every open F_σ-set must be open. The proofs of this and other similar assertions can be found in the books by R. Sikorski [1] and B. Z. Vulikh [1].

For brevity and consistency, we call a compact space that represents a complete Boolean algebra *extremally disconnected* or simply *extremal.*

In conclusion, we rivet the reader's attention on the following important fact: the infinite clopen algebra is never a σ-algebra of sets (which does not contradict the σ-completeness of this Boolean algebra).

2.2 Interpretation of homomorphisms

Let $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$ be two totally disconnected compact spaces; their clopen algebras are denoted by $\mathscr{X}_1$ and $\mathscr{X}_2$. The Boolean operations $\vee$, $\wedge$, and C are the usual set-theoretic operations of union, intersection, and complementation.

We say that a homomorphism $\Phi : \mathscr{X}_2 \longrightarrow \mathscr{X}_1$ and a mapping $\varphi : \mathfrak{Q}_1 \longrightarrow \mathfrak{Q}_2$ are *canonically related* (or Φ *is induced by* φ) whenever

$$\Phi(x) = \varphi^{-1}(x) \quad \text{for all} \quad x \in \mathscr{X}_2. \tag{6}$$

We note first that a mapping canonically related to a homomorphism must be continuous (the preimage of an open set is always open). Moreover, it is closed as a continuous mapping from a compact space into a Hausdorff space.

Theorem 6. 1) *For each continuous mapping $\varphi : \mathfrak{Q}_1 \longrightarrow \mathfrak{Q}_2$, there exists a canonically related homomorphism $\Phi : \mathscr{X}_2 \longrightarrow \mathscr{X}_1$.*

2) *For each homomorphism $\Phi : \mathscr{X}_2 \longrightarrow \mathscr{X}_1$, there exists a unique canonically related continuous mapping $\varphi : \mathfrak{Q}_1 \longrightarrow \mathfrak{Q}_2$.*

PROOF. 1) It is clear that, by the continuity of φ, the inverse images $\varphi^{-1}(x)$ of clopen sets are also clopen. Therefore, the mapping Φ, defined by the formula (6), acts from $\mathscr{X}_2$ into $\mathscr{X}_1$. The operation of inverse image preserves the operations $\cup, \cap$, and C; therefore, Φ is a homomorphism from $\mathscr{X}_2$ into $\mathscr{X}_1$.

2) Let now a homomorphism $\Phi : \mathscr{X}_2 \longrightarrow \mathscr{X}_1$ be defined. Given $q \in \mathfrak{Q}_1$, consider the set

$$e(q) \equiv \bigcap_{x \in \mathscr{X}_2, \Phi(x) \ni q} x. \tag{7}$$

It is easy to see that this set is nonempty (as the intersection of a centered system of compact sets) and consists only of one point (since the algebra $\mathscr{X}_2$ separates the points of $\mathfrak{Q}_2$).

We denote the only element of $e(q)$ by $\varphi(q)$. It remains only to verify that φ is a continuous mapping canonically related to Φ.

Show first that the equality (6) holds. Take $q \in \varphi^{-1}(x)$. We have $\varphi(q) \in x$, $q \in \Phi(x)$ (if $q \notin \Phi(x)$ then $q \in C\Phi(x) = \Phi(Cx)$, i.e., $\varphi(q) \in Cx$). So, $\varphi^{-1}(x) \subset \Phi(x)$. On the other hand, if $q \in \Phi(x)$ then $\varphi(q) \in$

$e(q) \subset x$ (cf. (7)) and $\varphi(q) \in x,\ q \in \varphi^{-1}(x)$. Thus, $\Phi(x) \subset \varphi^{-1}(x)$ and (6) is proved.

Continuity of φ was established above in connection with (6). The proof of the theorem is complete.

We now pass to the joint study of a homomorphism Φ and the canonically related mapping φ.

We first consider the KERNEL of this homomorphism. This kernel $\ker\Phi$ is the ideal of clopen subsets of $\mathfrak{Q}_2$ which go to the zero of $\mathscr{X}_1$ under Φ; these sets have the empty preimage $\varphi^{-1}(x) \equiv \Phi(x)$. Arrange the union

$$G \equiv \bigcup_{x \in \ker\Phi} x.$$

This set is open. We list some properties of G.

1) Every clopen subset $x \subset G$ belongs to the ideal $\ker\Phi$. Indeed, x is compact and covered by the open sets of the ideal. Consequently, there is a finite subcover:

$$x \subset x_1 \vee x_2 \vee \cdots \vee x_m, \quad x_i \in \ker\Phi \ \ (i = 1, 2, \dots, m).$$

We see that x belongs to the ideal.

2) G does not contain the values of φ. This follows from the fact that each point $q \in G$ belongs to an element of the ideal; i.e., to a set with the empty preimage.

Now consider the complement $F \equiv \mathfrak{Q}_2 \setminus G$. It is closed and, in accord with what was said on p. 132,[2] generates the ideal $\ker\Phi$. In our notations: $\ker\Phi \equiv I\langle E\rangle$. From 2) it follows that $\varphi(\mathfrak{Q}_1) \subset F$. Since φ is closed, $\varphi(\mathfrak{Q}_1)$ is closed and each point beyond $\varphi(\mathfrak{Q}_1)$ is separated from $\varphi(\mathfrak{Q}_1)$ by a clopen set with the empty preimage. Therefore, this clopen set belongs to $\ker\Phi$ and lies in G. So,

$$\mathfrak{Q}_2 \setminus \varphi(\mathfrak{Q}_1) \subset G = \mathfrak{Q}_2 \setminus F \quad \text{or} \quad F \subset \varphi(\mathfrak{Q}_1).$$

Thus, $\varphi(\mathfrak{Q}_1) = F$.

As we saw (Theorem 4), the set F is the Stone space of the quotient algebra $\mathscr{X}_2|_{\ker\Phi}$. The equality $\varphi(\mathfrak{Q}_1) = \mathfrak{Q}_2$ means exactly that $\ker\Phi$ consists only of zero; i.e., that Φ is a monomorphism.

We have thus proved

Theorem 7. *Let Φ and φ be canonically related.*

1) *For φ to be a surjection, it is necessary and sufficient that Φ be a monomorphism.*

---

[2]The mapping, denoted by Φ on that page, is the identity mapping in our case ($\mathscr{X}_2 \equiv \mathscr{C}O(\mathfrak{Q}_2)$). HERE the letter Φ is the symbol of an ARBITRARY homomorphism.

2) *The image* $\varphi(\mathfrak{Q}_1)$ *is the Stone space of the quotient algebra* $\mathscr{X}_2|_{\ker\Phi}$.

Consider the totality of all LEVEL SETS of the mapping φ. Each of them is closed and they form a partition of $\mathfrak{Q}_1$. By (6), it is clear that every clopen set of the form $\Phi(x)$ consists of the whole elements of the partition (is "saturated" with respect to this partition). It is not difficult to see that these sets separate the elements of the partition. Indeed, for every two level sets

$$e_1 \equiv \{q \in \mathfrak{Q}_1 \mid \varphi(q) = q_1\}, \quad q_1 \in \mathfrak{Q}_2,$$

$$e_2 \equiv \{q \in \mathfrak{Q}_1 \mid \varphi(q) = q_2\}, \quad q_2 \in \mathfrak{Q}_2,$$

there exists $x \in \mathscr{X}_2$ such that $q_1 \in x$, $q_2 \in Cx$. Then the preimages $\varphi^{-1}(x) \equiv \Phi(x)$ and $\varphi^{-1}(Cx) \equiv \Phi(Cx)$ separate e_1 and e_2. By Theorem 3, a subalgebra separating the points of $\mathfrak{Q}_1$ must coincide with the entire algebra $\mathscr{X}_1$. Thus, if all level sets are singletons, i.e., if φ is injective; then $\Phi(\mathscr{X}_2)$ coincides with $\mathscr{X}_1$ and Φ is an epimorphism. Conversely, if Φ is an epimorphism then the elements of the form $\Phi(x)$ separate every two points of $\mathfrak{Q}_2$. In other words, every two points of $\mathfrak{Q}_2$ belong to different preimages and cannot belong to a common level set. Consequently, φ is an injection. Thus, we have proved

Theorem 8. *For the mapping φ to be injective, it is necessary and sufficient that the homomorphism Φ canonically related to φ be an epimorphism.*

An essential part of Theorems 7 and 8 is of a "category-theoretic" nature and may be comprehended in more general terms. We return to these questions later.

The two following important theorems ensue from Theorem 8 as simple corollaries.

Theorem 9. *A homomorphism Φ is an isomorphism if and only if φ is a homeomorphism.*

Theorem 10. *The Stone space of a Boolean algebra is unique up to homeomorphism.*

We have already mentioned this fact.

Assuming that $\mathscr{X}$ and $\mathscr{Y}$ are complete Boolean algebras, we consider one more question concerning the canonically related mappings φ and Φ. Namely: what properties must a continuous mapping φ have for continuity of Φ? (The definition of a continuous or "complete" homomorphism is given in 2.1.4.) An answer to this question is provided by the following

Theorem 11. *For continuity of a homomorphism Φ from a complete Boolean algebra $\mathscr{X}$ into a complete Boolean algebra $\mathscr{Y}$, it is necessary*

and sufficient that the canonically related mapping to Φ, *acting from* $\mathfrak{Q}(\mathscr{Y})$ *into* $\mathfrak{Q}(\mathscr{X})$, *be open.*

Here, as usual, $\mathscr{X} = \mathscr{X}^0$ is the clopen algebra of $\mathfrak{Q}(\mathscr{X})$ and y is the same algebra for $\mathfrak{Q}(\mathscr{Y})$. Recall that openness of a mapping means that the image of every open set is open.

Another equivalent definition: a mapping φ is open whenever the closure of the preimage of each set e includes the preimage of the closure:

$$\overline{\varphi^{-1}(e)} \supset \varphi^{-1}(\overline{e}).$$

Proof of the theorem. Let φ be an open (and, obviously, continuous) mapping and let Φ be a homomorphism canonically related to φ. Prove that the equality

$$\sup \Phi(M) = \Phi(\sup M)$$

holds for every set $M \subset \mathscr{X}$. This fact means the continuity of Φ. Using the second definition, we have

$$\sup \Phi(M) = \overline{\bigcup_{x \in M} \Phi(x)} = \overline{\bigcup_{x \in M} \varphi^{-1}(x)} \supset \varphi^{-1}\Big(\overline{\bigcup_{x \in M} x} \Big) = \Phi(\sup M).$$

The inequality $\sup \Phi(M) \le \Phi(\sup M)$ holds obviously for every isotonic mapping.

Next, let Φ be a continuous homomorphism. Show that φ is open. It is sufficient to verify the openness of the image $\varphi(e)$, where e is a clopen set in $\mathfrak{Q}(\mathscr{Y})$. Suppose that this image is not open for some $e \neq \varnothing$. Anyway, $\varphi(e)$ is a closed set (φ is always closed); and it is easy to reduce our reasoning to the case in which $\varphi(e)$ is rare.[3] Let $G \equiv \mathfrak{Q}_2 \setminus \varphi(e)$. This set is everywhere dense and open. Consider the system $\mathscr{E}$ of all clopen subsets of G. It is clear that $\sup \mathscr{E} = \overline{G} = \mathfrak{Q}_2 = \mathbf{1}_{\mathscr{X}_2}$. By continuity of Φ,

$$\sup \Phi(\mathscr{E}) = \Phi(\sup \mathscr{E}) = \Phi(\mathbf{1}_{\mathscr{X}_2}) = \mathbf{1}_{\mathscr{X}_1} = \mathfrak{Q}_1. \qquad (*)$$

However, for all $x \in \mathscr{E}$, we have

$$e \cap \Phi(x) = \varphi^{-1}(x) \cap e \subset \varphi^{-1}(x) \cap \varphi^{-1}(\varphi(e))$$
$$= \varphi^{-1}(x \cap \varphi(e)) = \varphi^{-1}(\varnothing) = \varnothing.$$

Therefore, every element of $\Phi(\mathscr{E})$ is disjoint from e and $\sup \Phi(\mathscr{E}) \wedge e = \mathbf{0}$. This contradicts the equality $(*)$. The proof of the theorem is complete.

[3] We need to replace e by $e \setminus \varphi^{-1}(E)$, where E is the greatest clopen subset of $\varphi(e)$ (such a subset always exists in an extremally disconnected compact space).

Consider homomorphisms of discrete algebras individually. As was noted while closing Section 1, the Stone space of $\mathscr{X} = 2^Q$ is the Stone–Čech compactification $\overline{Q}$ of the discrete topological space Q. It can be represented as the union of Q and the "excrescence" $\beta Q \setminus Q$. The points of Q comprise the discrete part of the compact space; the corresponding singletons are clopen as well as all finite subsets of Q; there are no clopen sets in the excrescence; the latter is closed and rare. If Q is finite then there is no excrescence at all. This case is not interesting.

Let $\mathscr{X}_1 = 2^{Q_1}$ and $\mathscr{X}_2 = 2^{Q_2}$ be two algebras. For a homomorphism $\Psi : \mathscr{X}_1 \longrightarrow \mathscr{X}_2$, there is a continuous canonically related mapping $\psi : \beta(Q_2) \longrightarrow \beta(Q_1)$. In the general case, the image $\psi(Q_2)$ may have a nonempty intersection with Q_1 and $\beta(Q_1) \setminus Q_1$ and even can consists of a singleton of the excrescence. However, the situation changes drastically if Ψ is a CONTINUOUS homomorphism. In this case, ψ is open (Theorem 11) and the image $\psi(\{q\}) \equiv \psi(q)$ of each point $q \in Q_2$ is an open set which consists of a single point. Such sets can be only in Q_1 and, therefore, $\psi(Q_2) \subset Q_1$ in this case. Denoting by ψ_0 the restriction of ψ onto Q_2, we obtain a mapping from Q_2 into Q_1. This mapping is related to the homomorphism Ψ by the formula:

$$\Psi(e) = \psi_0^{-1}(e), \qquad (**)$$

where e is an element in $\mathscr{X}_1$, i.e., an arbitrary subset in Q_1. The formula $\Psi^0(e^0) \equiv \psi^{-1}(e^0)$ defines the corresponding isomorphism of the Stone spaces $\mathscr{X}_1^0$ and $\mathscr{X}_2^0$. If the isomorphism $\Phi : \mathscr{X} \longrightarrow \mathscr{X}^0$ is defined by the formula $\Phi(e) = e^0 \equiv clos\, e$ then $\Psi^0 = \Psi \circ \Phi$.

Conversely, take a mapping $\psi_0 : Q_2 \longrightarrow Q_1$. It generates a homomorphism Ψ by the formula $(**)$. Since the equalities

$$\Psi\Big(\bigcup_{t\in T} e_t\Big) = \psi_0^{-1}\Big(\bigcup_{t\in T} e_t\Big) = \bigcup_{t\in T} \psi_0^{-1}(e_t) = \bigcup_{t\in T} \Psi(e_t)$$

are valid for all T and since the operations $\vee$ and $\cup$ coincide in the algebras of the form 2^Q, this homomorphism is continuous. Hence, in particular, we see that an arbitrary mapping ψ_0 from Q_2 into Q_1 can be uniquely extended to a continuous and open mapping ψ from the Stone–Čech compactification βQ_2 into βQ_1. Namely, this extension is the mapping ψ that corresponds to the homomorphism Ψ by Theorem 11. It is clear that $\psi|_{Q_2} = \psi_0$. We arrive at the theorem supplementing Theorem 11.

Theorem 12. *The general form of a continuous homomorphism from a Boolean algebra 2^{Q_1} into a Boolean algebra 2^{Q_2} is given by the formula $\Psi(e) = \psi_0^{-1}(e)$ $(e \subset Q_1)$, where ψ_0 is an arbitrary mapping from*

Q_2 into Q_1. This formula defines a one-to-one correspondence between mappings and continuous homomorphisms.

2.3 Subalgebras and partitions of a Stone space

In Chapter 1, we have considered a class of subalgebras of a Boolean algebra $\mathscr{X}$. These subalgebras were related to partitions of unity of the original Boolean algebra into disjoint addends. However, not every subalgebra has such a structure. We now pass to partitions of another type; namely, we consider partitions of a STONE SPACE (see 1.1.3). Generally speaking, elements of such a partition are not the elements of the given Boolean algebra $\mathscr{X}$.

Let ξ be a partition of the compact space $\mathfrak{Q}$ into closed sets and let $\mathfrak{Q}|_\xi$ be the corresponding QUOTIENT SPACE whose elements are the elements of the partition ξ and the topology is the quotient topology, i.e., the finest topology in $\mathfrak{Q}|_\xi$ ensuring continuity of the projection[4]

$$\mathscr{P} : \mathfrak{Q} \longrightarrow \mathfrak{Q}|_\xi.$$

Each ξ-set has the form $x = \mathscr{P}^{-1}(\mathscr{P}(x))$; it is open (closed) simultaneously with its image $\mathscr{P}(x)$. The quotient space is always compact with respect to this topology.

If the projection $\mathscr{P}$ is a closed mapping then the partition ξ is called *upper semicontinuous*; if the projection is closed then ξ is *lower semicontinuous* by definition. An upper and lower semicontinuous partition is called continuous.[5]

Using the term "partition" up to the end of this section, we will imply a partition of a totally disconnected compact space $\mathfrak{Q}$ into closed subsets; by the term "subalgebra," a subalgebra of the clopen algebra $\mathscr{X} \equiv \mathscr{C}O(\mathfrak{Q})$ of the compact space.

Some partitions of $\mathfrak{Q}$ are related to subalgebras of the Boolean algebra $\mathscr{X} = \mathscr{C}O(\mathfrak{Q})$. Namely, we say that ξ is a *Boolean* partition whenever it is generated by a subalgebra $\mathscr{X}_0$ in the sense of the definition of 1.1.3. Recall that, by this definition, all elements z of the Boolean partition are various intersections of the form

$$\bigcap_{x \in \mathscr{X}_0} \widetilde{x}, \tag{$*$}$$

where each $\widetilde{x}$ is either x or Cx.

[4]The projection $\mathscr{P}$ related to ξ assigns to each $q \in \mathfrak{Q}$ the element of ξ containing q.
[5]K. Kuratowski [2, Theorem 1, p. 194].

For instance, the trivial subalgebra $\{\mathbf{0}, \mathbf{1}\}$ corresponds to the partition with the single element $\mathfrak{Q}$; the entire Boolean algebra $\mathscr{X}$ generates the partition into separate points. We consider other examples later. But now we pass to a systematic study of Boolean partitions.

We start with the question on uniqueness of a generating subalgebra.

Theorem 13. *For each Boolean partition ξ, there is a unique subalgebra generating this partition; namely, the algebra of all clopen ξ-sets.*

This theorem rests on the following

Lemma 6. *Every subalgebra $\mathscr{X}_0$ separating the points of a partition ξ contains all clopen ξ-sets.*

This lemma generalizes the second claim of Theorem 3 and can be proved by a similar argument.

Theorem 13 can be proved by simply combining Proposition 5° of 1.1.3 and Lemma 6. Certainly, this procedure can be applied only to Boolean partitions. An example of "non-Boolean" partition into closed sets will be adduced later.

The unique subalgebra of $\mathscr{X}$, generating a partition ξ, is denoted further by $\mathscr{X}_\xi$.

Theorem 14. *Every Boolean partition is upper semicontinuous.*

Proof. From Proposition 5° of 1.1.3, it follows that if a partition ξ is Boolean then the quotient space is Hausdorff. We know that every continuous mapping $\mathscr{P}$ from a compact space into a Hausdorff space is always closed (we have already used this fact in the above reasoning).

Theorem 15. *A partition is Boolean if and only if the corresponding quotient space is totally disconnected; this quotient space is the Stone space of the subalgebra generating the partition.*

Proof. If a partition ξ is Boolean then it is generated by a subalgebra $\mathscr{X}_\xi$. The sets in $\mathscr{X}_\xi$ separate the elements of the partition; they are clopen in $\mathfrak{Q}$ and also ξ-sets. Therefore, the "canonical" images of these sets are clopen in $\mathfrak{Q}|_\xi$. Hence, the space $\mathfrak{Q}|_\xi$ is totally disconnected and the clopen sets of it separate the points, i.e., separate the elements of the partition ξ. The inverse images of these sets under the canonical mapping $\mathscr{P}$ comprise the generating subalgebra $\mathscr{X}_\xi \subset \mathscr{X}$. Thus, the partition with a totally disconnected quotient space is Boolean.

The second claim of the theorem is clear since the images of elements of $\mathscr{X}_\xi$ exhaust all clopen sets of the quotient space.

Now let $\mathfrak{Q}$ be an extremally disconnected compact space. Then the clopen algebra is complete. Which partitions of a compact space do the regular subalgebras correspond to? It turns out that the regular subal-

gebras are generated exactly by continuous partitions. More precisely, the following theorem is valid.[6]

Theorem 16. 1) *If a subalgebra $\mathscr{X}_\xi$ is regular then the partition ξ is continuous;*

2) *If ξ is a continuous partition then ξ is Boolean and the subalgebra $\mathscr{X}_\xi$ is regular. (In this theorem, the compact space $\mathfrak{Q}$ is extremal.)*

Proof. 1) Let $\mathscr{X}_\xi$ be a regular subalgebra. To prove continuity of the partition, it is sufficient to establish that the corresponding projection $\mathscr{P}$ is an open mapping. To this end, we must show that, for every clopen $x \subset \mathfrak{Q}$, its image $\mathscr{P}(x)$ is open in $\mathfrak{Q}|_\xi$. Put

$$\overline{x} \equiv \inf\{x' \mid x' \in \mathscr{X}_\xi,\ x' \geq x\},$$

$$\overline{\overline{x}} \equiv \mathscr{P}^{-1}(\mathscr{P}(x)).$$

It is sufficient to show the openness of $\overline{\overline{x}}$ (this set must be closed since the projection $\mathscr{P}$ is closed). By the regularity of $\mathscr{X}_\xi$, we have $\overline{x} \in \mathscr{X}_\xi$. Observe that $\overline{x} = \overline{\overline{x}}$. It is clear that $\overline{x} \geq \overline{\overline{x}}$. Suppose that $\overline{x} > \overline{\overline{x}}$. Then, there is an element $z \in \xi$ such that $z \subset \overline{x}$ and $z \cap \overline{\overline{x}} = \varnothing$. Applying Theorems 3 and 15, we find a clopen set $\widetilde{x} \in \mathscr{X}_\xi$ separating z and $\overline{\overline{x}}$, i.e., $\widetilde{x} \supset \overline{\overline{x}}$ and $\widetilde{x} \cap z = \varnothing$. In the "Boolean" notations, we have:

$$\overline{x} > \overline{x} \wedge \widetilde{x} \geq x, \quad \overline{x} \wedge \widetilde{x} \in \mathscr{X}_\xi$$

which is inconsistent with the definition of $\overline{x}$. So, $\overline{x} = \overline{\overline{x}}$ and $\overline{\overline{x}}$ is open, whence $\mathscr{P}(x)$ is open too. Thus, we have proved that the projection $\mathscr{P}$ is open and, therefore, ξ is a continuous partition.

2) Conversely, let ξ be a continuous partition. Show first that this partition is Boolean. The projection $\mathscr{P}$ is open and closed. Therefore, the image $\mathscr{P}(x)$ is clopen for all $x \in \mathscr{X}$. Denote by $\mathscr{Y}$ the system of all clopen sets of the form $\mathscr{P}^{-1}(\mathscr{P}(x))$ $(x \in \mathscr{X})$. It is easy to see that $\mathscr{Y}$ is a subalgebra. Observe that this subalgebra generates our partition. Since $\mathscr{P}$ is closed for all $z_1, z_2 \in \xi \quad (z_1 \neq z_2)$ (they are closed), there is an open ξ-set G which separates z_1 and z_2 (the quotient topology is normal).[7] The set G is a union of clopen sets

$$G = \bigcup_\alpha x_\alpha, \quad x_\alpha \in \mathscr{X}.$$

Let $q \in z_1$; there exists an index α_0 such that $q \in x_{\alpha_0}$. Then

$$G \supset \mathscr{P}^{-1}(\mathscr{P}(x_{\alpha_0})) \supset z_1,\ \mathscr{P}^{-1}(\mathscr{P}(x_{\alpha_0})) \cap z_2 = \varnothing,\ \mathscr{P}^{-1}(\mathscr{P}(x_{\alpha_0})) \in \mathscr{Y}.$$

---

[6] D. A. Vladimirov and P. Zenf [1].

[7] K. Kuratowski [2, pp. 194–195].

So, the elements of $\mathscr{Y}$ are ξ-sets and separate the elements of the partition. Therefore, the subalgebra $\mathscr{Y}$ generates ξ, so we can write $\mathscr{Y} = \mathscr{X}_\xi$.

We now prove the regularity of $\mathscr{X}_\xi$. Given an arbitrary set $\mathscr{E} \subset \mathscr{X}_\xi$, put

$$F \equiv \overline{\bigcup_{x\in\mathscr{E}} x}.$$

By the extremality of $\mathfrak{Q}$, the set F is clopen and obviously is the supremum of $\mathscr{E}$ IN THE ALGEBRA $\mathscr{X}$. To complete the proof, it remains to establish that F is a ξ-set. This fact follows from

Lemma 7. *If a partition ξ is continuous then the closure of a ξ-set in $\mathfrak{Q}$ is a ξ-set.*

PROOF. Let E be a ξ-set and let $\overline{E}$ be its closure. For an arbitrary $z \in \xi$, show that if $\overline{E} \cap z = \varnothing$ then z is entirely included in $\overline{E}$, which means that $\overline{E}$ is a ξ-set. Let q_0 be in $\overline{E} \cap z$; and consider a point $q \in z$. If $U(q)$ is a neighborhood of q then $\mathscr{P}^{-1}(\mathscr{P}(U(q)))$ is a neighborhood of the set z and, therefore, of the point q_0. It follows from the fact that $\mathscr{P}$ is an open mapping. Moreover, since $q_0 \in \overline{E}$, the intersection $E \cap \mathscr{P}^{-1}(\mathscr{P}(U(q)))$ is a nonempty ξ-set. Consequently, there exists an element z_0 of the partition which is included in the intersection $E \cap \mathscr{P}^{-1}(\mathscr{P}(U(q)))$. For this z_0, we have

$$z_0 \subset \mathscr{P}^{-1}(\mathscr{P}(U(q))), \quad z_0 \cap U(q) \neq \varnothing$$

and, therefore, $E \cap U(q) \neq \varnothing$. Thus, $q \in \overline{E}$. The lemma is proved, which completes the proof of the theorem.

REMARK. While proving the above theorem, we have established in particular that every continuous partition of a totally disconnected compact space (not necessarily extremal) is Boolean.

We now adduce an example of a "non-Boolean" partition of the utmost importance. Let f be a continuous function on $\mathfrak{Q}$ with values in $[A, B]$ $(A < B)$. It is easy to construct such a function on an infinite totally disconnected compact space. Consider a partition ξ, the elements of which are all LEVEL SETS of the function f. It is not difficult to see that these sets are closed and the partition ξ is upper semicontinuous. However, ξ is not Boolean. Indeed, let E be a nontrivial clopen ξ-set. So is its complement $\mathfrak{Q} \setminus E$. But then their images $f(E)$ and $f(\mathfrak{Q} \setminus E) = [A, B] \setminus f(\mathfrak{Q})$ are clopen in $[A, B]$, which is impossible. Thus, there exist only trivial clopen ξ-sets which separate the points of the partition.

The situation changes drastically in the case of the partition into the level sets of a continuous mapping from one Stone space into another. We have met this situation already; and we now discuss it in detail.

Let $\varphi : \mathfrak{Q}_1 \longrightarrow \mathfrak{Q}_2$ be a continuous mapping; let $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$ be, as usual, totally disconnected compact spaces; and let $\mathscr{X}_1$ and $\mathscr{X}_2$ be the two corresponding clopen algebras. Denote by ξ the partition $\mathfrak{Q}_1$ into the level sets of the form

$$C_r \equiv \{q \mid \varphi(q) = r\} \quad (r \in \mathfrak{Q}_2).$$

Such a partition is always Boolean. It corresponds to the subalgebra $\mathscr{X}_{1\xi}$ that consists of all images $\Phi(x)$ where Φ is the homomorphism canonically related to φ. This subalgebra, in turn, is isomorphic to the quotient algebra $\mathscr{X}_2|_{\ker\Phi}$ and the quotient space $\mathfrak{Q}_1|_\xi$ is the Stone space for $\mathscr{X}_{1\xi}$ and, therefore, for the isomorphic algebra $\mathscr{X}_2|_{\ker\Phi}$. So, we have two representations of the same Stone space: the first is $\mathfrak{Q}_1|_\xi$ and the second is the closed subset of $\mathfrak{Q}_2$ complementary to the union of sets in the ideal $\ker\Phi$. This set is the image $\varphi(\mathfrak{Q}_1)$. The two spaces are homeomorphic; moreover, such homeomorphism ψ can be defined naturally:

$$\psi(r) = C_r \quad (r \in \varphi(\mathfrak{Q}_1)).$$

This homeomorphism is canonically related to the canonical isomorphism $\Psi : \mathscr{X}_1 \longrightarrow \mathscr{X}_2|_{\ker\Phi}$.

2.4 The countable chain condition

Let $\mathscr{X}$ be a complete Boolean algebra and let $\mathfrak{Q} \equiv \mathfrak{Q}(\mathscr{X})$ be the corresponding extremally disconnected compact space. How can we interpret the countable chain condition in $\mathscr{X}$ in the "Stone" language?

Theorem 17.[8] *For a Boolean algebra $\mathscr{X}$ to satisfy the countable chain condition it is necessary and sufficient that each rare set $E \subset \mathfrak{Q}$ admit an embedding into a closed rare G_δ-set.*

Proof. We first prove necessity. Let $\mathscr{X}$ be a complete algebra with the countable chain condition. Take an arbitrary rare set $E \subset \mathfrak{Q}$. We can assume that E is closed (for otherwise, we will take the closure of E). For each point $q \notin E$, there is a clopen set x_q such that $E \subset x_q,\ q \notin x_q$. It is clear that $\bigwedge_{q\in E} x_q = \mathbf{0}$ (otherwise, the interior of E is nonempty). Next, using the countable chain condition, obtain a sequence $\{x_{q_n}\}_{n=1}^\infty$ such that $\bigwedge_{n=1}^\infty x_{q_n} = \mathbf{0}$. Put $\mathscr{D} = \bigcap_{n=1}^\infty x_{q_n}$. This D is a rare G_δ-set including E.

We now prove sufficiency. It is enough to establish the following fact: if $x = \sup\mathscr{E}\quad(\mathscr{E} \subset \mathscr{X})$ then $\mathscr{E}$ includes a sequence of elements $\{x_n\}_{n=1}^\infty$ such that $\bigvee_{n=1}^\infty x_n = x$. Consider the difference of sets $E \equiv x \setminus \bigcup_{y\in\mathscr{E}} y$. This set is rare. By hypothesis, there exists a rare G_δ-set

[8] Z. T. Dikanova [1].

$\mathscr{D}$ such that $E \subset \mathscr{D}$. The set $\mathscr{D}$ can be represented as $\mathscr{D} = \bigcap_{n=1}^{\infty} G_n$, where G_n are open. The complements $F_n \equiv \mathfrak{Q} \setminus G_n$ are compact and each of them is included in $Cx \cup \bigcup_{y \in \mathscr{E}} y$. Therefore, for each n there is a finite system $\{y_1^{(n)}, \dots, y_{k_n}^{(n)}\}$ consisting of elements of $\mathscr{E}$ such that $F_n \subset Cx \cup y_1^{(n)} \cup \dots \cup y_{k_n}^{(n)}$. It is not difficult to verify that

$$\bigvee_{n=1}^{\infty} (y_1^{(n)} \cup \dots \cup y_{k_n}^{(n)}) = x.$$

It remains to re-enumerate the elements $y_i^{(n)}$, making them into a desired simple sequence $\{x_1, x_2, \dots\}$.

This theorem is easily generalized to complete Boolean algebras of an arbitrary type t. We should only replace G_δ-sets by sets representable as an intersection of t many open sets in the above reasoning.

2.5 Cardinal characteristics of a Stone space

We start with the most difficult question on cardinality. What is the cardinality of an extremally disconnected compact space $\mathfrak{Q}$ equal to? In essence, the answer is given by a nontrivial theorem in the preceding chapter according to which an infinite complete Boolean algebra $\mathscr{X}$ always includes an independent system of cardinality $card\, \mathscr{X}$.

Let $\mathscr{E}$ be such a system. Using the apparatus of 1.1.3, introduce the sets $\Delta_e \equiv \{e, Ce\}$ $(e \in \mathscr{E})$ and their cartesian product Δ which consists of all families $\delta = \{\delta_e\}$, where $\delta_e = e, Ce$. Assume that $\mathscr{X} = \mathscr{C}O(\mathfrak{Q})$. By independence of $\mathscr{E}$ and compactness of δ_e, all intersections

$$\mathfrak{Q}_\delta \equiv \bigcap_{e \in \mathscr{E}} \delta_e$$

are nonempty. Moreover, it is obvious that the distinct $\mathfrak{Q}_\delta$ do not have common points. Selecting a point in each $\mathfrak{Q}_\delta$, obtain an embedding from Δ into $\mathfrak{Q}$.

The cardinality of Δ is $2^{card \mathscr{E}} \equiv 2^{card \mathscr{X}}$; consequently, the cardinality of $\mathfrak{Q}$ is not less than this number.

Repeat the above procedure, now replacing $\mathscr{E}$ by the entire $\mathscr{X}$. Instead of Δ, consider the product

$$\Gamma \equiv \prod_{x \in \mathscr{X}} \Gamma_x$$

where $\Gamma_x \in \{x, Cx\}$. The cardinality of Γ is $2^{card \mathscr{X}}$. In this case, the sets P_γ, with

$$P_\gamma \equiv \bigcap_{x \in \mathscr{X}} \gamma_x, \quad \gamma = \{\gamma_x\}_{x \in \mathscr{X}} \in \Gamma,$$

separate the points of $\mathfrak{Q}$. Assigning to each point $q \in \mathfrak{Q}$, the unique set P_γ that contains q (and does not contain points other than q), it is easy to construct an embedding of the compact space $\mathfrak{Q}$ into Γ. Thus, $card\,\mathfrak{Q} \leq 2^{card \mathscr{X}}$.

We arrive at the conclusion: *the cardinality of the Stone space of a complete Boolean algebra* $\mathscr{X}$ *is always equal to* $2^{card \mathscr{X}}$. The same is true for every incomplete Boolean algebra $\mathscr{X}$ that includes an independent system of cardinality $card\,\mathscr{X}$. For instance, the claim is valid in the case of an infinite free Boolean algebra.

We now pass to other cardinal characteristics.

The topological weight of $\mathfrak{Q}$ *coincides with the cardinality of a Boolean algebra* $\mathscr{X}$. Indeed, the system of sets $\mathscr{X}$ is a base for the topology on $\mathfrak{Q}$. So, the topological weight of $\mathfrak{Q}$ is not greater than $card\,\mathscr{X}$. On the other hand, consider another base U consisting of open sets. Then every clopen set x is compact and, therefore, x can be represented as a finite union of sets in U. Hence, $card\,\mathscr{X} \leq card\,U$. Thus, $card\,\mathscr{X}$ is the least possible cardinality of a base, i.e. the topological weight of $\mathfrak{Q}$. Notice that the Boolean algebra $\mathscr{X}$ can also be incomplete.

The type of a Boolean algebra $\mathscr{X}$ *coincides with the "Suslin number" (the supremum of cardinalities of disjoint systems of open sets) of the compact space* $\mathfrak{Q}$. This fact is obvious. Here, we do not require completeness of $\mathscr{X}$ either.

The weight of a complete Boolean algebra also admits some topological interpretation, although more complicated. Namely, $\tau(\mathscr{X})$ is the least cardinality of a system of clopen sets possessing the following property: if ξ is a Boolean partition generated by this system then the only continuous partition finer than ξ is the partition into the separate points.

As we have observed (see p. 111), the cardinality of the set

$$\mathscr{C}[\mathscr{X} \longrightarrow \mathscr{Y}]$$

of all continuous homomorphisms of a complete Boolean algebra $\mathscr{Y}$ meets the estimate:

$$card\,\mathscr{C}[\mathscr{X} \longrightarrow \mathscr{Y}] \leq [card\,\mathscr{Y}]^{\tau(\mathscr{X})}.$$

This is valid for the cardinality of the set of continuous homomorphisms or, which is equivalent, of open mappings from $\mathfrak{Q}(\mathscr{Y})$ into $\mathfrak{Q}(\mathscr{X})$. In particular, we obtain that an extremally disconnected compact space admits at most $[card\,\mathscr{X}]^{\tau(\mathscr{X})}$ homeomorphisms (onto itself). For example, let $\mathscr{X}$ be a separable Boolean algebra, i.e., $\tau(\mathscr{X}) = \aleph_0$. Then

$$[card\,\mathscr{X}]^{\aleph_0} = card\,\mathscr{X},$$

and the cardinality of $\mathfrak{Q}(\mathscr{X})$ is $2^{card \mathscr{X}} > card\,\mathscr{X}$.

Hence, it is clear that not every two points of $\mathfrak{Q}(\mathscr{X})$ can be transformed into each other by a homeomorphism (and even by an open continuous mapping). This fact means that the Stone space of a complete separable Boolean algebra is not topologically homogeneous. (However, it is not an obstacle to the ergodicity of the automorphism group.) Actually, no infinite extremal compact space is homogeneous; however, the proof of this fact requires a rather sophisticate technique.

3. Stone functors

3.1 Categories and functors

This book is written on the basis of Cantor set theory. This approach reigns now in mathematical research. However, the opinion appeared in the last years that the "Cantor epoch" comes to an end. We abstain from discussing this topic.

In fact, it is unclear what will be a substitute for the "Cantorian paradise." Nevertheless, there is an opinion that the set-theoretic mentality will be replaced by the "categorical" one.

In the above sections, we have already met the situations in which the categorical language was preferable. We now try to explicate the categorical sense of the most important facts and constructions of the theory of Boolean algebras. Throughout this section, we will have to distinguish between the concepts of a "set" and "class," assuming the Gödel–Bernays axiomatics implicitly.

Certainly, we will not expound category theory,[9] restricting exposition to the most basic concepts. A category consists of *objects* and *morphisms*. The objects of a category $\mathbb{C}$ constitute a class (not a set in general) which is denoted by $Ob\,\mathbb{C} \equiv Ob$.[10] The morphisms also constitute a class which is denoted by $Mor\,\mathbb{C} \equiv Mor$. This class has the following structure:

$$Mor\,\mathbb{C} = \bigcup_{(A,B)} Mor_{\mathbb{C}}(A, B),$$

where the right-hand side is the disjoint union of the classes

$$Mor_{\mathbb{C}}(A, B) \equiv Mor\,(A, B)$$

extended to all ordered pairs of objects (A, B), with $A, B \in Ob\,\mathbb{C}$. Thus, each morphism $f \in Mor$ always belongs exactly to one of the classes

[9] For instance, see H. Schubert [1]. Semadeni's book [1] contains an exposition of the basics of category theory which is oriented to functional analysis.

[10] Each set is a class but not vice versa.

$Mor\,(A,B)$; moreover, it is sometimes said that A is the *origin* and B is the *target* of a morphism f. Some of the classes $Mor\,(A,B)$ can be empty. Mostly, it is supposed that all $Mor\,(A,B)$ are sets.

However, a category cannot be defined only by indicating the classes Ob and Mor; we need to define additionally a COMPOSITION LAW that assigns to each pair $f \in Mor\,(A,B)$, $g \in Mor\,(B,C)$ the unique morphism $h \in Mor\,(A,C)$; this morphism is called the *composition* of f and g and is denoted by the symbol $g \circ f$, i.e., $h \equiv g \circ f$. Moreover, composition obeys the following axioms:

(A) *The composition of morphisms is associative, i.e.,* $f \circ (g \circ h) = (f \circ g) \circ h$ *for all* $h \in Mor\,(A,B)$, $g \in Mor\,(B,C)$, *and* $f \in Mor\,(C,D)$, *where* $A, B, C, D \in Ob$.

(E) *Each class* $Mor\,(A,A)$ *is nonempty and contains a "unity" morphism* i_A *that possesses the property* $i_A \circ f = f$, *and* $g \circ i_A = g$ *for all* $f \in Mor\,(C,A)$ *and* $g \in Mor\,(A,B)$, *where* $C, B \in Ob$. It is easy to establish the uniqueness of the morphism i_A.

Most assertions of category theory are stated with the aid of DIAGRAMS. Accordingly, the axioms (A) and (E) are expressed by the diagrams:

The axiom (A) asserts that the first diagram commutes and the axiom (E), that so does the second.

Example. The category $\mathbb{E}\mathrm{ns}$ of all sets. The objects of this category are sets; the morphisms are mappings; and the composition is the ordinary composition of mappings. In this book, the categories under study are SUBCATEGORIES of $\mathbb{E}\mathrm{ns}$ in most cases.

Here is the general definition: a category $\mathbb{D}$ is called a *subcategory* of $\mathbb{C}$ if 1) $Ob\,\mathbb{D} \subset Ob\,\mathbb{C}$ and 2) $Mor_{\mathbb{D}}(A,B) \subset Mor_{\mathbb{C}}(A,B)$ for all $A, B \in Ob\,\mathbb{D}$.

If the condition 2) is valid with the inclusion sign replaced by "=" then $\mathbb{D}$ is referred to as a *complete subcategory*.

The subcategories of $\mathbb{E}\mathrm{ns}$ are sometimes called "concrete." The examples of such categories are as follows:

1) The category $\mathbb{G}$ of all groups, with the morphisms homomorphisms;

2) The category $\mathbb{T}\mathbb{o}\mathbb{p}$ of all topological spaces, with the morphisms continuous mappings.

We pay attention mainly to the next two categories:

3) The category $\mathbb{B}\mathbb{o}\mathbb{o}\mathbb{l}\mathbb{e}$ whose objects are Boolean algebras and morphisms, Boolean homomorphisms;

4) The category $\mathbb{S}\mathbb{t}$ of all totally disconnected compact spaces, with the morphisms continuous mappings.

The last two categories are concrete; i.e., subcategories of $\mathbb{E}\mathbb{n}\mathbb{s}$; moreover, $\mathbb{S}\mathbb{t}$ is a complete subcategory of $\mathbb{T}\mathbb{o}\mathbb{p}$.

The central concept in category theory is that of a FUNCTOR.

A mapping $\mathbb{F}$ transforming $Ob\,\mathbb{C}$ into $Ob\,\mathbb{D}$ and $Mor\,\mathbb{C}$ into $Mor\,\mathbb{D}$ is called a *covariant functor* from a category $\mathbb{C}$ into a category $\mathbb{D}$ whenever its action on morphisms is naturally related to the action on objects by the following conditions:

1) $\mathbb{F}(i_A) = i_{\mathbb{F}(A)}$ for all $A \in Ob\,\mathbb{C}$;

2) $\mathbb{F}(f \circ g) = \mathbb{F}(f) \circ \mathbb{F}(g)$ for all $f, g \in Mor\,\mathbb{C}$ such that the composition $f \circ g$ is defined.

Let $f \in Mor_{\mathbb{C}}(A, B)$. Then

$$f = i_B \circ f \circ i_A;$$

$$\mathbb{F}(f) = \mathbb{F}(i_B) \circ \mathbb{F}(f) \circ \mathbb{F}(i_A)$$
$$= i_{\mathbb{F}(A)} \circ \mathbb{F}(f) \circ i_{\mathbb{F}(A)} \in Mor_{\mathbb{D}}(\mathbb{F}(A), \mathbb{F}(B)).$$

The sense is clear of the *identity functor* $\mathrm{Id}_{\mathbb{C}}$; this concept can be introduced for every category $\mathbb{C}$. It is also clear how to define the composition of some functors $\mathbb{F}_1$ and $\mathbb{F}_2$. This composition is denoted by $\mathbb{F}_2 \circ \mathbb{F}_1$.

If we replace the condition 2) in the above definition of a covariant functor by the condition

2*) $\mathbb{F}(f \circ g) = \mathbb{F}(g) \circ \mathbb{F}(f)$

for all $f, g \in Mor\,\mathbb{C}$, then we arrive at the concept of *contravariant* functor; i.e., with the condition 2*) valid, $\mathbb{F}$ become a contravariant functor from $\mathbb{C}$ to $\mathbb{D}$.

In this case, the following relations hold for $f \in Mor_{\mathbb{C}}(A, B)$:

$$\mathbb{F}(f) = \mathbb{F}(i_B \circ f \circ i_A) = \mathbb{F}(i_A) \circ \mathbb{F}(f) \circ \mathbb{F}(i_B)$$

$$= i_{\mathbb{F}(A)} \circ \mathbb{F}(f) \circ i_{\mathbb{F}(B)} \in Mor_{\mathbb{D}}(\mathbb{F}(B), \mathbb{F}(A)).$$

If $f \in Mor_{\mathbb{C}}(A,B)$ and $g \in Mor_{\mathbb{C}}(B,C)$ then, in the case when $\mathbb{F}$ is covariant, the following diagram commutes:

and, in the case when $\mathbb{F}$ is contravariant, another diagram commutes:

The first diagram expresses the condition 2) and the second, the condition 2*).

It can be said that a covariant functor preserves the direction of arrows in diagrams whereas a contravariant functor "reverses" these arrows.

In closing this subsection, we present an example of a "nonconcrete" category which is absolutely not similar to $\mathbb{E}\mathrm{ns}$. Namely, each Boolean algebra $\mathscr{X}$ (or each ordered set) can be regarded as a category whose objects are all elements $x \in \mathscr{X}$ and, for all $x, y \in \mathscr{X}$, the set $Mor\,(x,y)$ is empty for $x \not\leq y$ and consists of the pair (x,y) for $x \leq y$. The composition law holds obviously: if $x \leq y$ and $y \leq z$ then the unique morphism from x into z is $(x,y) \circ (y,z) = (x,z)$. (In fact, such a unique element of $Mor\,(x,y)$ is actually immaterial incase these sets are disjoint.)

To each category $\mathbb{C}$, there corresponds the *dual* category $\mathbb{C}^*$ with the same objects whose morphisms are obtained from morphisms of $\mathbb{C}$ by "reversal of the arrows." More precisely, $Ob\,\mathbb{C} \equiv Ob\,\mathbb{C}^*$ and $Mor_{\mathbb{C}^*}(A,B) \equiv Mor_{\mathbb{C}}(B,A)$. It is convenient to denote by f^* the morphism $f \in Mor\,\mathbb{C}$ regarded as an element of $Mor\,\mathbb{C}^*$; an object A referred to as an object of $\mathbb{C}^*$ is denoted by the symbol A^*. The composition law for the dual category is as follows: $f^* \circ g^* = (g \circ f)^*$. Hence, $i_{A^*} = (i_A)^*$.

A functor acting by the rule $A \longrightarrow A^*$, $f \longrightarrow f^*$ is obviously a contravariant functor from $\mathbb{C}$ into $\mathbb{C}^*$. We denote it by $(\ )^*_{\mathbb{C}}$ or simply by $(\ )^*$, always writing $(X)^*$ instead of $(\ )^*(X)$.

3.2 Classification of morphisms

Each morphism $f \in Mor\,(A, B)$ defines the mappings

$$h \longrightarrow f \circ h \text{ and } h \longrightarrow h \circ f$$

from $Mor\,(X, A)$ into $Mor\,(X, B)$ and from $Mor\,(B, Y)$ into $Mor\,(A, Y)$ for all $X, Y \in Ob\,\mathbb{C}$. If the first mapping is bijective then f is called a *monomorphism* and if the second mapping is bijective then f is called an *epimorphism*. In other words, a monomorphism can be "cancelled" from the left:

$$f \circ h_1 = f \circ h_2 \quad \longrightarrow \quad h_1 = h_2$$

and an epimorphism, from the right:

$$h_1 \circ f = h_2 \circ f \quad \longrightarrow \quad h_1 = h_2.$$

It is clear that the functor $(\ )^*$, described in 3.3.1, transforms monomorphisms to epimorphisms and vice versa. The terms "epimorphism" and "monomorphism" have been already employed for the category $\mathbb{B}\mathrm{oole}$. The new definition is consistent with the former. The reason is that, as is easy to verify, an epimorphism is a surjective mapping and a monomorphism is an injective mapping in every concrete category. An *isomorphism* is such a morphism $f \in Mor\,(A, B)$ that there is a morphism $g \in Mor\,(B, A)$ satisfying $f \circ g = I_B$ and $g \circ f = I_A$. Each isomorphism is a monomorphism and epimorphism simultaneously. (The converse is not always true!) The morphism g, which was mentioned in the definition, is unique and presents an isomorphism; it is natural to call g the inverse isomorphism of f and denote it by f^{-1}. An isomorphism $f \in Mor\,(A, A)$ is called an *automorphism*. Objects A and B are called *isomorphic* whenever there exists an isomorphism between them. The binary relation of "being isomorphic" is an equivalence; it generates a partition of $Ob\,\mathbb{C}$ into equivalence classes. Selecting an object in each of these classes, we thus define a complete subcategory $\mathbb{C}'$ in $\mathbb{C}$ which is called a *skeleton* of $\mathbb{C}$. For all $A \in Ob\,\mathbb{C}$, a coset $Ob\,\mathbb{C}'$ contains one and only one object A' isomorphic to A.

The main constructions of category theory do not "discriminate" between isomorphic objects and refer, in essence, to a skeleton wherein the terms "isomorphic" and "coinciding" are equivalent. The principle of identifying isomorphic objects lies in the basis of the "categorical" mentality. It should be noted that, the inflexible implementation of this

principle would be inconvenient in our book as well as in any application-oriented research. In our study, the Boolean algebras are not merely the objects of the category $\mathbb{B}\mathbb{o}\mathbb{o}\mathbb{l}\mathbb{e}$ but entities with some specific properties reflecting their nature and applications. For instance, there is no reason to always identify a Boolean algebra and its Stone representation or identify the Borel algebras of an interval and a square although they are isomorphic. Nevertheless, the "pure" theory of Boolean algebras in which algebras are considered up to isomorphism can be written in the language of category theory.

3.3 Isomorphism and equivalence of categories

By definition some categories $\mathbb{C}$ and $\mathbb{D}$ are *isomorphic* whenever there are covariant functors $\mathbb{F}$ and $\mathbb{G}$ satisfying the conditions:

$$\mathbb{F} \circ \mathbb{G} = \mathrm{Id}_{\mathbb{D}}, \quad \mathbb{G} \circ \mathbb{F} = \mathrm{Id}_{\mathbb{C}}.$$

This means that there is a one-to-one correspondence between the objects and morphisms of these two categories which allows us to identify the categories in many cases. Isomorphic categories are related too tightly; the concept of an *equivalence of categories* is more useful and will be described now.

We start with introducing the concept of an *equivalence of functors*. Functors $\mathbb{F}_1$ and $\mathbb{F}_2$ acting from a category $\mathbb{C}$ into a category $\mathbb{D}$ are called equivalent (or "naturally equivalent") whenever there is a mapping h assigning to each $A \in Ob\,\mathbb{C}$ an ISOMORPHISM $h(A) \in Mor(\mathbb{F}_1(A), \mathbb{F}_2(A))$ that makes the diagram

commutative for all A", $A'' \in Ob\,\mathbb{C}$, and $f \in Mor_{\mathbb{C}}(A', A'')$. (A particular case of a more general concept of a "functor morphism.")

In this event, speaking about "equivalence" we imply the mapping h itself. If $\mathbb{F}_1$ and $\mathbb{F}_2$ are equivalent then we denote this fact as follows: $\mathbb{F}_1 \sim \mathbb{F}_2$.

We distinguish the most important case in which one of the functors is the identity (and $\mathbb{C} = \mathbb{D}$). A functor $\mathbb{F}$ from $\mathbb{C}$ into $\mathbb{C}$ is equivalent to $\mathrm{Id}_{\mathbb{C}}$ whenever there exists a mapping h assigning to each $A \in Ob\,\mathbb{C}$

an isomorphism $h(A) \in Mor\,(A, \mathbb{F}(A))$ such that all diagrams of the following form are equivalent:

Now we are in a position to present the concept of EQUIVALENCE OF CATEGORIES. This can be defined by each of the following three equivalent conditions:

1°. There exist covariant functors $\mathbb{F}$ and $\mathbb{G}$ satisfying $\mathbb{F} \circ \mathbb{G} \sim \mathrm{Id}_{\mathbb{D}}$ and $\mathbb{G} \circ \mathbb{F} \sim \mathrm{Id}_{\mathbb{C}}$.

2°. $\mathbb{C}$ and $\mathbb{D}$ have isomorphic skeletons.

3°. There is a functor $\mathbb{F}$ from $\mathbb{C}$ into $\mathbb{D}$ mapping bijectively each class $Mor_{\mathbb{C}}(A, B)$ onto $Mor_{\mathbb{D}}(\mathbb{F}(A), \mathbb{F}(B))$ and such that each object $P \in Ob\,\mathbb{D}$ is isomorphic to some $\mathbb{F}(A) \quad (A \in Ob\,\mathbb{C})$.

If one of the above conditions holds then the categories $\mathbb{C}$ and $\mathbb{D}$ are called *equivalent*. Each of the functors, mentioned in the conditions 1° and 3°, is called an equivalence of categories.

In general, functors $\mathbb{F}$ and $\mathbb{G}$ (covariant or contravariant) are called *quasi-inverse* to one another whenever $\mathbb{F} \circ \mathbb{G} \sim \mathrm{Id}_{\mathbb{D}}$ and $\mathbb{G} \circ \mathbb{F} \sim \mathrm{Id}_{\mathbb{C}}$. For instance, so are the functors from 1°. Equivalent categories, which are not isomorphic in general, can be considered as "the same."

3.4 Some important constructions: cones, cocones, products, and sums

Take a family $\{P, P_1, P_2, \dots, P_m\}$ of objects in a category $\mathbb{C}$ and a family of morphisms $\{f_1, f_2, \dots, f_m\}$, where $f_i \in Mor\,(P, P_i)$ $(i = 1, 2, \dots, m)$.

We call such a pair a *cone with vertex P*. (Infinite cones are also considered.)

Actually, a cone is defined only by the family $\{f_1, f_2, \dots, f_m\}$; it should be only stipulated that all f_i have the same domain: the vertex P. However, we denote the cone as follows:

$$\{P; P_1, P_2, \dots, P_m; f_1, f_2, \dots, f_m\}.$$

Two cones

$$\{P'; P_1, P_2, \dots, P_m; f_1', f_2', \dots, f_m'\}, \{P''; P_1, P_2, \dots, P_m; f_1'', f_2'', \dots, f_m''\}$$

are *isomorphic* whenever there is an isomorphism $\varphi \in Mor\,(P', P'')$ making the following diagram commutative:

The relation of isomorphism is an equivalence over cones. Reversing all arrows in the above definitions, we arrive at the dual concepts of *cocone* and *isomorphism of cocones.* If

$$\{P; P_1, P_2, \dots, P_m; f_1, f_2, \dots, f_m\}$$

is a cone in $\mathbb{C}$ then

$$\{P^*; P_1^*, P_2^*, \dots, P_m^*; f_1^*, f_2^*, \dots, f_m^*\}$$

is a cocone in $\mathbb{C}^*$ and vice versa. It is clear how to define an isomorphism of cocones.

A *product* of $A, B \in Ob\,\mathbb{C}$ is a cone

$$\{P; A, B; f_A, f_B\}$$

such that there is a unique morphism $k \in Mor\,(X, P)$ making the diagram

commutative for all $X \in Ob\,\mathbb{C}$, $g \in Mor\,(X, A)$, and $h \in Mor\,(X, B)$.

The dual concept is a *sum* or, in other words, *coproduct* of A and B. So we call a cocone $\{P; A, B; f'_A, f'_B\}$ whose dual cone

$$\{P^*; A^*, B^*; f'^*_A, f'^*_B\}$$

is a product of the objects A^* and B^* in the dual category $\mathbb{C}^*$. (We recall that $P^*, f'^*, \ldots$ are the same $P, f', \ldots$ regarded as objects and morphisms of the dual category.) In the full form, the definition of sum requires that, for all $X \in Ob\,\mathbb{C}$, $g' \in Mor\,(A, X)$, and $h' \in Mor\,(B, X)$, there is a unique morphism $k \in Mor\,(P, X)$ which makes the diagram

commutative. The terms "product" and "sum" are sometimes used for P; however, this terminology may leads to confusion. It is also said that the object P is "representable" as a product or sum.

The results of the above-described operations are determined uniquely up to isomorphism. More precisely, a cone (cocone) that is isomorphic to a product (sum) is a product (sum) too. Every two products (two sums) are always isomorphic. (We mean an isomorphism of cones and cocones, of course.) Since all products of two objects A and B fulfill an entire class of equivalent cones, we can refer the term "product" to this class or to a distinguished representative of the class. The same applies to the term "sum."

We point out in addition that the representability of an object in the form of a sum or product is preserved under passing to isomorphic objects. If P is representable as a sum (product) of A and B and if there are isomorphisms $a \in Mor\,(A, A')$, $b \in Mor\,(B, B')$, and $p \in Mor\,(P, P')$ then the object P' is representable as a sum (product) of A' and B'. For instance, let the cone

$$\{P; A, B; f_A, f_B\}$$

be a product of A and B. Then the cone

$$\{P'; A', B'; a \circ f_A \circ p^{-1}, b \circ f_B \circ p^{-1}\}$$

is a product of A' and B'.

Here is a diagram that will help the reader to grasp the situation:

(given g' and h', put $g \equiv a^{-1} \circ g'$ and $h \equiv b^{-1} \circ h'$ and find a unique morphism k; then let $k' \equiv p \circ k$).

In the category $\mathbb{E}\mathrm{ns}$, the products and sums are cartesian products and disjoint sums. For instance, assume given two disjoint sets A and B. Putting $P \equiv A \cup B$ and taking the identical embeddings of A and B into P as f_A and f_B, we obtain a sum of the objects A and B. If A and B are not disjoint then, to construct a sum, we first need to find some disjoint A' and B' isomorphic (i.e., equipollent) to A and B respectively. If $a \in Mor\,(A'A')$ and $b \in Mor\,(B, B')$ are isomorphisms then the cocone

$$\{A' \cup B'; A, B; f_{A'} \circ a, f_{B'} \circ b\}$$

is, as is easy to see, a sum of A and B. (Here, $f'_{A'}$ and $f'_{B'}$ are the identical embeddings of A' and B' into $A' \cup B'$.)

As for the product of sets A and B (considered as objects of $\mathbb{E}\mathrm{ns}$), it can be represented as

$$\{A \times B; A, B; P_A, P_B\},$$

where P_A and P_B are the projections defined on the cartesian product $A \times B$ by the formulas:

$$P_A(x, y) = x, \quad P_B(x, y) = y \quad (x \in A, y \in B).$$

The categorical concepts of product and sum of objects stem from a desire to give an abstract characterization for the operations of disjoint sum and cartesian product of sets; a characterization that uses only arrows and diagrams and avoid ordered pairs or the symbols $\in, \cap, \cup$. There are other examples of abstract interpretation of set-theoretic notions and constructions. (For instance, recall the discussion of the concepts of monomorphism and epimorphism in 3.3.2.)

3.5 A subobject and a quotient object

Consider the following simple cone $\{P; Q; j\}$, where j is a monomorphism:

$$P \xrightarrow{j} Q.$$

(In fact, we simply consider a monomorphism j or, in other words, an "object P together with an embedding of P into Q.") The fact that two such cones $\{\widetilde{P}; Q; \widetilde{j}\}$ and $\{P; Q; j\}$ are isomorphic means, as we know, that there is an isomorphism $\varphi \in Mor\,(\widetilde{P}, P)$ making the diagram

commutative, i.e., $\widetilde{j} = j \circ \varphi$ and $j = \widetilde{j} \circ \varphi^{-1}$. We refer to the cosets corresponding to this isomorphism as *subobjects of the object* Q. So, to define a subobject is to select an object P and a class of monomorphisms $j \circ \varphi \in Mor\,(\widetilde{P}, Q)$, where $\widetilde{P}$ is an object isomorphic to P and φ is an isomorphism in $Mor\,(\widetilde{P}, P)$. It is clear that this class does not depend on the choice of j; however, it is convenient to distinguish some monomorphism j as "canonical." For instance, in the category $\mathbb{E}\mathrm{ns}$, the set P is usually a subset of Q and j is the identical embedding: $j(x) = x$. Specifying a "canonical" monomorphism j,[11] we uniquely determine some subobject that will be denoted further by $\langle P, j, Q\rangle$. Certainly, it can be one object for different j and P. Sometimes, we apply the term "subobject" abusing the language to the object P in the general situation. Such a terminology is admissible only in the case when there is a reason to select some privileged morphism as canonical; this morphism is usually presupposed when P is referred to as a subobject of Q.[12] For instance, in concrete categories, it is usually assumed that j is the identical embedding and the term "subobject" simply means a SUBSET P_Q. (For the above assumption to be valid, it is important that $P \in Ob\,\mathbb{C}$ and the identical embedding belongs to $Mor\,\mathbb{C}$.) We employ the abbreviation $\langle P, Q\rangle$.

---

[11] Together with j, we distinguish P by the condition $j \in Mor\,(P, Q)$.

[12] Choosing j and P, we prefer to select the main "name" for the subobject and forget about other names.

We now address the following important problem: In which sense must we speak about isomorphism of subobjects?

Consider the two subobjects: $\langle P, j, Q\rangle$ and $\langle P', j', Q'\rangle$. They are subobjects of two generally different objects Q and Q'. Suppose that there are isomorphisms making the following diagram commutative:

$$\begin{tikzcd} Q \arrow[r, "\Psi"] & Q' \\ P \arrow[u, "j"] \arrow[r, "\psi"] & P' \arrow[u, "j'"'] \end{tikzcd} \tag{8}$$

Obviously, this property (existence of isomorphisms) characterizes SUBOBJECTS; it does not depend on the choice of a "name" and is preserved under replacement of j by $\widetilde{j} = j \circ \varphi$ and j' by $\widetilde{j}' = j' \circ \varphi'$, where φ and φ' are arbitrary isomorphisms: $\varphi \in Mor\,(\widetilde{P}, P)$ and $\varphi' \in Mor\,(\widetilde{P}, P')$. In this case, the following diagram commutes:

$$\begin{tikzcd} Q \arrow[r, "\Psi"] & Q' \\ \widetilde{P} \arrow[u, "\widetilde{j}"] \arrow[r, "\widetilde{\psi}"] & \widetilde{P}' \arrow[u, "\widetilde{j}'"'] \end{tikzcd} \tag{9}$$

where $\widetilde{\psi} \equiv (\varphi')^{-1} \circ \psi \circ \varphi$.

Let $\mathbb{C}$ be a concrete category (in particular, $\mathbb{B}\mathbb{o}\mathbb{o}\mathbb{l}\mathbb{e}$ or $\mathbb{S}\mathbb{t}$) and let j and j' be the identical embeddings of subsets $P \subset Q$ and $P' \subset Q'$. The diagram (8) claims that the bijective mapping $\Psi \in Mor\,\mathbb{C}$ from Q into Q' simultaneously transforms P into P'. In such a situation, it is said that Ψ can be "descended" to the bijection $\psi \in Mor\,(P, P')$. The diagram (9) shows that the mapping can be "descended" after replacement of the identical embeddings j and j' with $\widetilde{j} = j \circ \varphi$ and $\widetilde{j}' = j' \circ \varphi'$, where φ, φ' are arbitrary bijections presenting morphisms of our category.

Now, we are able to formulate the following:

Definition. Subobjects $\langle P, j, Q\rangle$ and $\langle P', j', Q'\rangle$ of objects $Q, Q' \in Ob\,\mathbb{C}$ are *isomorphic* whenever there exist isomorphisms $\psi \in Mor\,(P, P')$ and $\Psi \in Mor\,(Q, Q')$ making the diagram (8) commutative (and, therefore, each of the diagrams (9) commutes under a proper choice for $\widetilde{\psi}$).

The concept of a *quotient object* is dual to the concept of a subobject. Consider a cocone $\{P; Q; j\}$; $P \xleftarrow{j} Q$, where j is an epimorphism.

We have already introduced the equivalence for cocones: $\{P; Q; j\}$ and $\{\widetilde{P}; Q; \widetilde{j}\}$ are *isomorphic* whenever $\widetilde{j} = \varphi \circ j$, where $\varphi \in Mor\,(P, P')$ is an arbitrary isomorphism. The cosets corresponding to this relation are called *quotient objects* of Q. For defining a quotient object, it is sufficient to distinguish the objects P and Q and a "canonical" epimorphism $j \in Mor\,(Q, P)$. For a quotient object defined by this data, we use the notation $\rangle P; j; Q\langle$. Consider two quotient objects $\rangle P; j; Q\langle$ and $\rangle P'; j'; Q'\langle$. If there are isomorphisms $\Psi \in Mor\,(Q, Q')$ and $\psi \in Mor\,(P, P')$ making the diagram

$$\begin{tikzcd} Q \arrow[r, "\Psi"] \arrow[d, "j"'] & Q' \arrow[d, "j'"] \\ P \arrow[r, "\psi"] & P' \end{tikzcd}$$

commutative then such a pair of isomorphisms exists also after passing from j to $\widetilde{j} = \varphi \circ j$ and from j' to $\widetilde{j}' = \varphi' \circ j'$ (where φ and φ' are isomorphisms); namely, the following diagram commutes:

$$\begin{tikzcd} Q \arrow[r, "\Psi"] \arrow[d, "\widetilde{j}"'] & Q' \arrow[d, "\widetilde{j}'"] \\ \widetilde{P} \arrow[r, "\widetilde{\psi}"] & \widetilde{P}' \end{tikzcd}$$

where $\widetilde{\psi} = \varphi' \circ \psi \circ \varphi^{-1}$.

The quotient objects $\rangle P; j; Q\langle$ and $\rangle P'; j'; Q'\langle$ are called in this case *isomorphic*.

3.6 Retracts; injective and projective objects

A retraction is a morphism $f \in Mor\,(A, B)$ such that there is a mapping $g \in Mor\,(B, A)$ satisfying the relation $f \circ g = i_B$. If $g' \circ f = i_A$ holds for some $g' \in Mor\,(B, A)$ then f is called a *coretraction* or *section*. It is clear that a retraction is always an epimorphism and a coretraction is a monomorphism. If there is a retraction $f \in Mor\,(A, B)$ ("retraction from A onto B") then the object B is called a *retract* of A; and A is called a *coretract* of B. Here is an example of a retract in the category $\mathbb{E}\mathrm{ns}$: let $B \subset A$ and let f be a mapping from A onto B. If we assign to g some right inverse to f from B into A then we obtain $f \circ g = \mathrm{Id}_B$.

This means that f is a retraction, g is a coretraction, and B is a retract of A. Each subset $B \subset A$ is a retract of A[13] (as well as every set of cardinality at most the cardinality of A). The situation is more complicated in other categories. It is clear that the property "to be a retract" is preserved under isomorphisms; if B is a retract of A and A' is isomorphic to A while B' is isomorphic to B, then B' is a retract of A'.

An object $C \in Ob\,\mathbb{C}$ is referred to as *injective* if, for all $B, B_0 \in Ob\,\mathbb{C}$, $j \in Mor\,(B_0, B)$, and $f \in Mor\,(B_0, C)$ (where j is a monomorphism), there exists $F \in Mor\,(B, C)$ such that the diagram

is commutative.

We present an example that refers to the category $\mathbb{E}\mathrm{ns}$. Let $B_0 \subset B$, let j be the identical embedding, and let f be a mapping from B_0 into C. The commutativity of the last diagram means that F is an EXTENSION of f from the subset B_0 to the entire set B. In the category $\mathbb{E}\mathrm{ns}$, such an extension is always possible; hence, every object in $\mathbb{E}\mathrm{ns}$ is injective. The situation is rather different for subcategories of the category $\mathbb{E}\mathrm{ns}$. In those cases, the morphisms we want to extend are to comply with some additional requirements; and the question on extendibility becomes less trivial.

---

[13] In particular, a retraction f can be identical on B. This case is very important (for instance, in topology).

Proposition. *A retract of an injective object is injective.*

Indeed, let C_1 be a retract of C:

(β is a retraction and α is a coretraction). Further, let j be a monomorphism ("embedding") from B_0 into B and let f_1 be a morphism from B_0 to C_1:

Complete this diagram, by assigning $f \equiv \alpha \circ f_1$ and deriving F from f by injectivity: $f = F \circ j$:

Take $F_1 = \beta \circ F$. We have

$$F_1 \circ j = \beta \circ F \circ j = \beta \circ f = \beta \circ \alpha \circ f_1 = f_1.$$

Thus, we are able to "extend" f_1 and, therefore, C_1 is an injective object.

An object $A \in Ob\,\mathbb{C}$ is referred to as *projective* whenever A is injective in the dual category $\mathbb{C}^*$, i.e., A^* is injective. This fact means the following: for all $B, B_0 \in Ob\,\mathbb{C}$, $j \in Mor\,(B, B_0)$, and $f \in Mor\,(A, B_0)$,

there exists a morphism $F \in Mor\,(A, B)$ such that the following diagram commutes:

Proposition. *A retract of a projective object is projective.*

The proof is left to the reader.

3.7 Duality

We have already met the examples of mutually dual notions and constructions: monomorphism and epimorphism, product and sum, cone and cocone, subobject and quotient object, injective and projective objects; this list can be continued. The basis for this duality is the concept of a dual category. Under passing from $\mathbb{C}$ to $\mathbb{C}^*$ with the aid of the contravariant functor $(\ )^*$, the directions of arrows in diagrams and the order of morphisms in compositions are reversed; we leave unaltered all the rest. Hence it follows in particular that a monomorphism transforms to an epimorphism (in the reverse direction), a product to a sum, etc. "Selfdual" are the concepts of retract and coretract: if B is a retract of A then B^* is a retract of A^*.

Let $R \equiv R(u, v, w, \ldots)$ be a proposition about morphisms and objects of a category. Here $u, v, w, \ldots$ are the names of the morphisms and objects; and the validity of R depends on the choice of these names. In other words, we deal with a PROPERTY that is true or false for the family $(u, v, w, \ldots)$. We assume that the property makes sense in every category. In particular, we can consider the proposition R in each of the two mutually dual categories $\mathbb{C}$ and $\mathbb{C}^*$. Together with R, we can consider the *dual* proposition $R^*(u, v, w, \ldots)$ which is true in $\mathbb{C}$ if and only if the proposition $R(u^*, v^*, w^*, \ldots)$ is true in $\mathbb{C}^*$ (the property of the family $(u, v, w, \ldots)$ which consists in the fact that the family $(u^*, v^*, w^*, \ldots)$ possesses the property R).

All definitions in this section reduce to a description of properties that must hold for an object, a morphism, or a family of objects and morphisms. Replacing the properties in each formulation by their dual analogs, we arrive at the DUAL DEFINITION, to the definition of a DUAL

CONCEPT. For instance, so was introduced the concept of a quotient object.

Let R be a theorem valid in every category. Then, for every category $\mathbb{C}$, the theorem R is valid in $\mathbb{C}^*$ and, therefore, the theorem R^* is valid in $\mathbb{C}$. So, the fact that some proposition R is true in all categories implies that so is the dual proposition R^*.

We have thus formulated the DUALITY PRINCIPLE; it is clear that this principle is valid for every class of categories which contains the dual category to its every category. In particular, this implies the duality principle for ordered sets which was mentioned in Chapter 0 (we recall that each ordered set can be regarded as a category).

The above requires elaboration. When talking about a "proposition" or "property" R, we actually have in mind a formula of a special logical-mathematical language, the language of category theory.

The alphabet of this language contains the symbols of the first order predicate calculus with equality and, moreover, the special symbols: $\in Ob$, $\in Mor\,(\dots)$, and $\dots \circ \dots$.

The reader familiar with the basic concepts of logic knows the rules for constructing the formulas of this language. These rules are such that each formula $R(u, v, w, \dots)$ can be uniquely deciphered as an encoded proposition about morphisms and objects; this proposition can be true or false depending on the fact what objects and morphisms are taken as $u, v, w, \dots$. (In practice, formulas are usually written in a rather reduced form on using diagrams.)

In order to apply the duality reasons practically, we should be able to determine a structure of the dual formula

$$R^*(u, v, w, \dots)$$

to a formula

$$R(u, v, w, \dots).$$

We will not try to describe a general procedure of transforming R to R^*; we only present a simple rule that is possibly somewhat unclear but applicable to practice (at least, to uncomplicated formulas). The rule is as follows: "reverse the direction of all arrows in diagrams and the order of morphisms in compositions." We have already applied this rule, for instance, when defining some dual concepts.

3.8 The main categories and functors in the theory of Boolean algebras

First of all, we list some categories.

1°. The category $\mathbb{Boole}$ which we have considered above. The objects of $\mathbb{Boole}$ are all Boolean algebras and the morphisms are Boolean homomorphisms.

2°. Denote by $\mathbb{Boole}_0$ the complete subcategory of $\mathbb{Boole}$ whose objects are all algebras of the form $\mathscr{C}O(\mathfrak{Q})$, with $\mathfrak{Q}$ a totally disconnected compact space.

The Stone Theorem ensures the equivalence of the categories $\mathbb{Boole}$ and $\mathbb{Boole}_0$. This equivalence allows us to assume each Boolean algebra under study to be an algebra of the form $\mathscr{C}O(\mathfrak{Q})$.

3°. The category $\mathbb{St}$ of all totally disconnected compact spaces. The morphisms of $\mathbb{St}$ are all continuous mappings between these compact spaces.

There are other categories: of complete algebras, normed algebras, etc. However, we will not introduce them here right away. We now pass to functors. Here are the main of them:

4°. The functor $\mathscr{R}$ (the "first Stone functor"). It assigns to each Boolean algebra $\mathscr{X} \in Ob\,\mathbb{Boole}$ the totally disconnected compact space $\mathfrak{Q}_i(\mathscr{X})$ whose points are maximal ideals of the Boolean algebra $\mathscr{X}$. As for morphisms, this functor acts on them as follows. Proving the Stone Theorem, we have introduced the canonical isomorphism Φ_0; we now denote it by $\Phi_{\mathscr{X}}$. This isomorphism sends each $x \in \mathscr{X}$ to the set $\Phi_{\mathscr{X}}(x) \in \mathscr{C}O(\mathfrak{Q})$ consisting of all maximal ideals not containing x:

$$\Phi_{\mathscr{X}}(x) \equiv \{q \in \mathfrak{Q}_i(\mathscr{X}) \mid q \not\ni x\}.$$

Consider now arbitrary Boolean algebras $\mathscr{X}', \mathscr{X}'' \in Ob\,\mathbb{Boole}$ and a homomorphism $\Psi \in Mor\,(\mathscr{X}', \mathscr{X}'')$. The latter corresponds uniquely to $\Psi_i \equiv \Phi_{\mathscr{X}''} \circ \Psi \circ \Phi_{\mathscr{X}'}^{-1}$:

We have already seen that the homomorphism Ψ_i is canonically related to a continuous mapping ψ_i from $\mathfrak{Q}_i(\mathscr{X}'')$ into $\mathfrak{Q}_i(\mathscr{X}')$:

$$\Psi_i(e) = \psi_i^{-1}(e) \quad (e \in \mathscr{C}O(\mathfrak{Q}_i(\mathscr{X}'))),$$

$$\psi_i(q) = \bigcap_{q \in \Psi_i(e);\, e \in \mathscr{C}O(\mathfrak{Q}_i(\mathscr{X}'))} e \qquad (q \in \mathfrak{Q}_i(\mathscr{X}'')).$$

We take this mapping as $\mathscr{R}(\Psi)$. It is easy to verify that we really define a functor from $\mathbb{Boole}$ to $\mathbb{S}\mathfrak{t}$ and this functor is covariant.

We now separately consider the important case in which the Boolean algebra $\mathscr{X}$ has the form $\mathscr{C}O(\mathfrak{Q})$, where $\mathfrak{Q}$ is a totally disconnected compact space. For brevity, denote by $\mathfrak{Q}_i$ the compact space $\mathfrak{Q}_i(\mathscr{C}O(\mathfrak{Q}))$ that consists of maximal ideals of $\mathscr{X}$. In this special case, we denote the isomorphism $\Phi_{\mathscr{X}}$ by $\Theta \equiv \Theta_{\mathfrak{Q}}$ and by $\Omega \equiv \Omega_{\mathfrak{Q}}$ the canonically related homomorphism. How does the mapping Ω act? It is easy to understand that, for every maximal ideal $p \in \mathfrak{Q}_i$, the point $\Omega(p) \in \mathfrak{Q}$ is defined as the intersection of all nonempty $y \notin p$:

$$\Omega(p) = \bigcap_{Cy \in p} y$$

(recall that p is a maximal ideal and, therefore, the relations $y \notin p$ and $Cy \in p$ are equivalent).

Now, given a continuous mapping $\psi : \mathfrak{Q}'' \longrightarrow \mathfrak{Q}'$ and the canonically related homomorphism Ψ, find out the form of the mappings Ψ_i and ψ_i. If the diagram

is commutative given Ψ_i then, obviously, the dual diagram

is commutative given ψ_i. Thus, $\psi_i = \Omega_{\mathfrak{Q}'}^{-1} \circ \psi \circ \Omega_{\mathfrak{Q}''}$.

5°. The "second Stone functor" assigns to each compact space $\mathfrak{Q} \in Ob\,\mathbb{S}\mathfrak{t}$ the Boolean algebra $\mathscr{C}O(\mathfrak{Q})$ and to each continuous mapping $\psi \in Mor_{\mathbb{S}\mathfrak{t}}(\mathfrak{Q}', \mathfrak{Q}'')$ the homomorphism Ψ canonically related to it. We

denote this functor by the symbol $\mathbb{S}$. So,

$$\mathbb{S}(\mathfrak{Q}) \equiv \mathscr{C}O(\mathfrak{Q}), \quad \mathbb{S}(\psi) \equiv \Psi,$$

where $\Psi(e) = \psi^{-1}(e)$.

This functor is contravariant. Verification is left to the reader.

We note an important property of Stone functors. The functor $\mathbb{S}$ acts into the complete subcategory $\mathbb{Boole}_0$ smaller than $\mathbb{Boole}$. Moreover, it is a bijection from the class $Ob\,\mathbb{St} \cup Mor\,\mathbb{St}$ onto $Ob\,\mathbb{Boole}_0 \cup Mor\,\mathbb{Boole}_0$. There is a left inverse functor $\mathbb{S}^{-1}$ from $\mathbb{Boole}_0$ into $\mathbb{St}$. Analogously, the functor $\mathscr{R}$ acts actually into a complete subcategory of $\mathbb{St}$. We denote it by $\mathbb{St}_0$. The objects of this subcategory are compact spaces of the form $\mathfrak{Q}_i(\mathscr{X})$, where $\mathscr{X}$ is a Boolean algebra. There is a left inverse functor $\mathscr{R}^{-1}$ as well. It is clear that the categories $\mathbb{Boole}_0$ and $\mathbb{St}$ are not merely equivalent, they are isomorphic as well as the categories $\mathbb{Boole}$ and $\mathbb{St}_0$.

Now, consider two important covariant functors.

6°. Denote by $\mathscr{I}$ the functor that acts from $\mathbb{Boole}$ into $\mathbb{Boole}_0$ and assigns to Boolean algebras and homeomorphism their "Stone representations":

$$\mathscr{I}(\mathscr{X}) \equiv \mathscr{C}O(\mathfrak{Q}_i(\mathscr{X})), \quad \mathscr{I}(\Psi) \equiv \Phi_{\mathscr{X}''} \circ \Psi \circ \Phi_{\mathscr{X}'}^{-1} \equiv \Psi_i$$

$$(\mathscr{X}, \mathscr{X}', \mathscr{X}'' \in Ob\,\mathbb{Boole}, \quad \Psi \in Mor_{\mathbb{Boole}}(\mathscr{X}', \mathscr{X}'')).$$

It is easy to verify that $\mathscr{I}$ is a covariant functor.

7°. Finally, let $\mathbb{K}$ be a covariant functor acting from $\mathbb{Boole}_0$ into $\mathbb{Boole}$ as the identical embedding:

$$\mathbb{K}(\mathscr{X}) \equiv \mathscr{X}, \ \mathbb{K}(\Psi) \equiv \Psi \quad (\Psi \in Mor_{\mathbb{Boole}_0}(\mathscr{X}', \mathscr{X}'')).$$

We have already claimed that the categories $\mathbb{Boole}$ and $\mathbb{Boole}_0$ are equivalent. We now prove this claim. It is most convenient to refer to the third definition of equivalence of categories; we can take the above defined functor $\mathscr{I}$ as the functor $\mathbb{F}$ in that definition. Show now that *the functors $\mathscr{I}$ and $\mathbb{K}$ form a pair of mutually quasi-inverse functors of the first definition.* We need to show that

$$\mathscr{I} \circ \mathbb{K} \sim \mathrm{Id}_{\mathbb{Boole}_0}, \quad \mathbb{K} \circ \mathscr{I} \sim \mathrm{Id}_{\mathbb{Boole}}.$$

Both functors $\mathscr{I} \circ \mathbb{K}$ and $\mathbb{K} \circ \mathscr{I}$ act in the same way; they differ only by their domains:

$$(\mathscr{I} \circ \mathbb{K})(\mathscr{X}) = \mathscr{C}O(\mathfrak{Q}_i(\mathscr{X})), \quad (\mathscr{I} \circ \mathbb{K})(\Psi) = \Psi_i = \Phi_{\mathscr{X}''} \circ \Psi \circ \Phi_{\mathscr{X}'}^{-1}$$

$$(\mathscr{X}, \mathscr{X}', \mathscr{X}'' \in Ob\,\mathbb{Boole}_0, \quad \Psi \in Mor_{\mathbb{Boole}_0}(\mathscr{X}', \mathscr{X}''))$$

and

$$(\mathbb{K}\circ\mathscr{I})(\mathscr{X})=\mathscr{C}O(\mathfrak{Q}_i(\mathscr{X})),\quad (\mathbb{K}\circ\mathscr{I})(\Psi)=\Psi_i=\Phi_{\mathscr{X}''}\circ\Psi\circ\Phi_{\mathscr{X}'}^{-1}.$$

Therefore, the familiar diagram

which commutes for all $\mathscr{X}', \mathscr{X}'' \in Ob_{\mathbb{Boole}}$ allows us to conclude that $\mathscr{I}\circ\mathbb{K}\sim \mathrm{Id}_{\mathbb{Boole}_0}$ and $\mathbb{K}\circ\mathscr{I}\sim\mathrm{Id}_{\mathbb{Boole}}$. (Recall that $\Phi_{\mathscr{X}'}$ and $\Phi_{\mathscr{X}''}$ are isomorphisms.)

Analogously, *the Stone functors* $\mathbb{S}$ *and* $\mathscr{R}$ *are mutually quasi-inverse*. Indeed, $\mathbb{S}\circ\mathscr{R}$ is the same functor as $\mathbb{K}\circ\mathscr{I}$, and we have established that it is equivalent to $\mathrm{Id}_{\mathbb{Boole}}$. As for the functor $\mathscr{R}\circ\mathbb{S}$, it acts as follows:

$$(\mathscr{R}\circ\mathbb{S})(\mathfrak{Q})=\mathfrak{Q}_i(\mathscr{C}O(\mathfrak{Q})),\quad (\mathscr{R}\circ\mathbb{S})(\psi)=\psi_i,$$

where $\mathfrak{Q}$ is a totally disconnected compact space, ψ is a continuous mapping from one compact space into the other. The homeomorphisms $\Omega_{\mathfrak{Q}}$ are isomorphisms in the category $\mathbb{St}$; therefore, the above-observed commutativity of all diagrams (9) reveals the equivalence of the functors $\mathscr{R}\circ\mathbb{S}$ and $\mathrm{Id}_{\mathbb{St}}$. So, the categories $\mathbb{Boole}$ and $\mathbb{St}$ are related by the pair of mutually quasi-inverse Stone functors $\mathscr{R}$ and $\mathbb{S}$.

Consider such a situation in a general form. Let two categories $\mathbb{C}$ and $\mathbb{D}$ be related by mutually quasi-inverse contravariant functors $\mathbb{F}:\mathbb{C}\longrightarrow\mathbb{D}$ and $\mathbb{G}:\mathbb{D}\longrightarrow\mathbb{C}$. Show that in this case the categories $\mathbb{C}$ and $\mathbb{D}^*$ are equivalent. Define the functors $\widetilde{\mathbb{F}}\equiv(\ )^*_{\mathbb{D}}\circ\mathbb{F}$ and $\widetilde{\mathbb{G}}\equiv\mathbb{G}\circ(\ )^*_{\mathbb{D}^*}$; they are covariant as compositions of contravariant functors. It is easy to see that

$$\widetilde{\mathbb{F}}\circ\widetilde{\mathbb{G}}=(\ )^*_{\mathbb{D}}\circ\mathbb{F}\circ\mathbb{G}\circ(\ )^*_{\mathbb{D}^*}\sim(\ )^*_{\mathbb{D}}\circ\mathrm{Id}_{\mathbb{D}}\circ(\ )^*_{\mathbb{D}^*}=(\ )^*_{\mathbb{D}}\circ(\ )^*_{\mathbb{D}^*}=\mathrm{Id}_{\mathbb{D}^*},$$

$$\widetilde{\mathbb{G}}\circ\widetilde{\mathbb{F}}=\mathbb{G}\circ(\ )^*_{\mathbb{D}^*}\circ(\ )^*_{\mathbb{D}}\circ\mathbb{F}=\mathbb{G}\circ\mathbb{F}\sim\mathrm{Id}_{\mathbb{C}}.$$

We have proved the equivalence of the categories $\mathbb{C}$ and $\mathbb{D}^*$. The equivalence of $\mathbb{C}^*$ and $\mathbb{D}$ can be established analogously. Returning to the Stone functors, we draw the following important conclusion: *the category* $\mathbb{St}$ *is equivalent to the category* $\mathbb{Boole}^*$ *and* $\mathbb{Boole}$ *is equivalent to*

$\mathbb{St}^*$. Roughly speaking, we may "view" the category $\mathbb{St}$ as the dual of the category of Boolean algebras; similarly, the latter is "almost dual" to $\mathbb{St}$. This fact opens an opportunity for translating various propositions from one of the categories $\mathbb{Boole}$ or $\mathbb{St}$ to the other. This translation is performed by duality.

3.9 The main category-theoretic constructions in $\mathbb{Boole}$ and $\mathbb{St}$

The concepts of sum and product are available an arbitrary category. However, not every pair of objects must have a sum or product in the general case. In the categories $\mathbb{Boole}$ and $\mathbb{St}$, these constructions are plain and always realizable.

We start with the category $\mathbb{St}$. In this category (as well as in a wider category of topological spaces), the PRODUCT is the Tychonoff product of compact spaces. In the case when $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$ are totally disconnected compact spaces, their product $\mathfrak{Q}_1 \times \mathfrak{Q}_2$ is a compact space of the same type. (A base for the topology consists of the sets $e = e_1 \times e_2$, where $e_i \in \mathscr{C}O(\mathfrak{Q}_i)$, $i = 1, 2$.)

The morphisms $f_{\mathfrak{Q}_1}$ and $f_{\mathfrak{Q}_2}$ (see the definition in 3.3.4) are the projections sending the point (q_1, q_2) to its coordinates q_1 and q_2. In this construction, $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$ can be replaced by arbitrary homeomorphic compact spaces. The products of the category $\mathbb{St}$ correspond to coproducts (sums) in the dual category $\mathbb{St}^*$.

Thus, the products are defined in the category $\mathbb{Boole}_0$ isomorphic to $\mathbb{St}$ and equivalent to $\mathbb{Boole}$.

The COPRODUCT of the Boolean algebras $\mathscr{X}_1 \equiv \mathscr{C}O(\mathfrak{Q}_1)$ and $\mathscr{X}_2 \equiv \mathscr{C}O(\mathfrak{Q}_2)$ can be defined as the image of the product $\mathfrak{Q} \equiv \mathfrak{Q}_1 \times \mathfrak{Q}_2$ under the functor $\mathbb{S}$, i.e., as the Boolean algebra

$$\mathscr{X} \equiv \mathscr{C}O(\mathfrak{Q}).$$

The morphisms $f'_{\mathscr{X}_1}$ and $f'_{\mathscr{X}_2}$ (from the definition in 3.3.4) are also obtained from $f_{\mathfrak{Q}_1}$ and $f_{\mathfrak{Q}_2}$ with the aid of the functor $\mathbb{S}$; they are Boolean homomorphisms from $\mathscr{X}_1$ and $\mathscr{X}_2$ into $\mathscr{X}$ canonically related to the projections $f_{\mathfrak{Q}_1}$ and $f_{\mathfrak{Q}_2}$, i.e.,

$$f'_{\mathscr{X}_1}(x) = f^{-1}_{\mathfrak{Q}_1}(x), \quad f'_{\mathscr{X}_2}(x) = f^{-1}_{\mathfrak{Q}_2}(x).$$

Since the projections $f_{\mathfrak{Q}_1}$ and $f_{\mathfrak{Q}_2}$ are epimorphisms; therefore, $f_{\mathscr{X}_1}$ and $f_{\mathscr{X}_2}$ are monomorphisms.

It is clear that, using the equivalence of the categories $\mathbb{Boole}_0$ and $\mathbb{Boole}$, we may give a definition of coproduct for all Boolean algebras without assuming them to have the form $\mathscr{C}O(\mathfrak{Q})$.

In this book, the algebra $\mathscr{X}$ is called neither the "coproduct" nor "sum" but rather *preproduct* of $\mathscr{X}_1$ and $\mathscr{X}_2$ (cf. 2.3.10).[14] Obviously, this construction can be generalized to an arbitrary finite system of algebras.

We now return to the category $\mathbb{St}$ to discuss the second main construction, coproduct, for compact spaces. The coproduct or sum of two totally disconnected compact spaces $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$ is the so-called topological sum of them, i.e., a topological space $\mathfrak{Q}$ representable a disjoint union of two clopen sets $\mathfrak{Q}_1'$ and $\mathfrak{Q}_2'$ homeomorphic to $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$. The corresponding homeomorphisms are $f'_{\mathfrak{Q}_1}$ and $f'_{\mathfrak{Q}_2}$ of the definition in 3.3.4. The topology on $\mathfrak{Q}$ is defined uniquely as the finest topology ensuring continuity of $f'_{\mathfrak{Q}_1}$ and $f'_{\mathfrak{Q}_2}$; the space $\mathfrak{Q}$ is a totally disconnected compact space with respect to this topology.

Passing again to Boolean algebras, consider the dual construction of "product." The "product" of Boolean algebras of the form $\mathscr{C}O(\mathfrak{Q})$ can be obtained from the topological sums of compact spaces under the action of the Stone functor $\mathbb{S}$. We have already met products (they were called "direct sums," cf. 1.2.6).

Recall that the direct sum $\mathscr{X}$ of two Boolean algebras $\mathscr{X}_1$ and $\mathscr{X}_2$ is the algebra whose elements are the following pairs

$$x = (x_1, x_2) \quad (x_i \in \mathscr{X}_i, i = 1, 2).$$

The projections $f_{\mathscr{X}_1}$ and $f_{\mathscr{X}_2}$ are defined by the formulas

$$f_{\mathscr{X}_1}(x) = x_1, \;\; f_{\mathscr{X}_2}(x) = x_2;$$

they obviously are epimorphisms. Below, we mainly use the term "direct sum." As was noted, direct summation may be applied to every family of Boolean algebras.

In conclusion, we forewarn the reader of a possible ambiguity: the "Boolean product" is the category-theoretic "sum" while the "direct sum" is the "product" in the category $\mathbb{Boole}$.

Consider one more pair of dual concepts: a SUBOBJECT and QUOTIENT OBJECT.

We start with the category $\mathbb{Boole}$. Let $\mathscr{X}$ be a Boolean algebra. A SUBOBJECT of $\mathscr{X}$ (as an object of $\mathbb{Boole}$) can be defined as follows: take a subalgebra $\mathscr{X}_0 \subset \mathscr{X}$ and consider the entire class of algebras $\mathscr{X}_0'$ isomorphic to $\mathscr{X}_0$. If j_0 is an identical embedding of $\mathscr{X}_0$ into $\mathscr{X}$ then each algebra $\mathscr{X}_0'$ is "embedded" into $\mathscr{X}$ by means of the monomorphism $j' = j_0 \circ \varphi$, where φ is an isomorphism from $\mathscr{X}_0'$ onto $\mathscr{X}_0$.

[14]The term "Boolean product" is also employed.

Thus, the totality of all cones $\{\mathscr{X}_0'; \mathscr{X}; j'\}$ is a subobject of $\mathscr{X}$ uniquely determined by the choice of the subalgebra $\mathscr{X}_0$. This subobject can be written as $\langle \mathscr{X}_0, \mathscr{X}, j_0\rangle$.

Since j_0 is a definite (identical) embedding, we may briefly write: $\langle \mathscr{X}_0, \mathscr{X}\rangle$.

Roughly speaking, "a subobject is a subalgebra." The following fact is worth noting. Even in the case when a subalgebra $\mathscr{X}_1 \subset \mathscr{X}$ is isomorphic (as a Boolean algebra) to the subalgebra $\mathscr{X}_0 \subset \mathscr{X}$, the corresponding subobjects can be different. In order to treat a subalgebra $\mathscr{X}_0$ as a subobject, it is not sufficient to know $\mathscr{X}_0$ "from inside" as a Boolean algebra; we should also know how $\mathscr{X}_0$ is embedded into $\mathscr{X}$. (For instance, $\mathscr{X}_0$ may coincide or may fail to coincide with $\mathscr{X}$.)

In the sequel, when talking about an isomorphism of subalgebras, we always mean some isomorphism of the corresponding subobjects. In particular, if $\mathscr{X}_0'$ and $\mathscr{X}_0''$ are subalgebras of Boolean algebras $\mathscr{Y}_0$ and $\mathscr{Y}_1$ then the fact that they are isomorphic (i.e., the subobjects $\langle \mathscr{X}_0', \mathscr{X}'\rangle$ and $\langle \mathscr{X}_0'', \mathscr{X}''\rangle$ are isomorphic) means the following: there exist isomorphisms $\Phi \in Mor\,(\mathscr{X}', \mathscr{X}'')$ and $\Phi_0 \in Mor\,(\mathscr{X}_0', \mathscr{X}_0'')$ for which the diagram commutes:

Here j' and j'' are identical embeddings; so, this means that there is an isomorphism from $\mathscr{X}'$ onto $\mathscr{X}''$ which transforms $\mathscr{X}_0'$ into $\mathscr{X}_0''$.

The dual concept is the concept of QUOTIENT OBJECT. In the category $\mathbb{Boole}$, a quotient object of a Boolean algebra $\mathscr{X}$ is usually defined as follows: a Boolean algebra $\widehat{\mathscr{X}}$ is distinguished representable as the homomorphic image $\Phi(\mathscr{X})$, where Φ is an epimorphism. (Most often the role of $\widehat{\mathscr{X}}$ is played by the quotient algebra of $\mathscr{X}$ by some ideal I, and the problem reduces to choosing this ideal.)

Next, the totality of all Boolean algebras $\widehat{\mathscr{X}'}$ isomorphic to $\widehat{\mathscr{X}}$ is considered. Each of these algebras has the form $\widehat{\mathscr{X}'} = j(\mathscr{X})$, where j is an epimorphism of the form $\psi \circ \Phi$ and ψ is an isomorphism. The cocones $\{\widehat{\mathscr{X}'}; \mathscr{X}; j\}$ comprise the quotient object that is denoted by $\rangle \widehat{\mathscr{X}}, \Phi, \mathscr{X}\langle$.

Actually, it is defined by indicating either the epimorphism Φ or the ideal I.

However, it is important to understand that distinguishing an algebra $\widehat{\mathscr{X}}$ is not sufficient for defining a quotient object. The Boolean algebra $\widehat{\mathscr{X}}$ can be represented as a homomorphic image of $\mathscr{X}$ in essentially different ways: in some situations, there are epimorphisms Φ_0 and Φ_1 from $\mathscr{X}$ onto $\widehat{\mathscr{X}}$ which are not related to each other by any equality of the form $\Phi_1 = \psi \circ \Phi_0$, where ψ is an isomorphism.

Nevertheless, the rough formulation "a quotient object is a quotient algebra" is possible; however, in this event we presume the quotient algebra by a particular ideal and the "CANONICAL" homomorphism defined in Chapter 1. (In much the same way as the formulation "a subobject is a subalgebra" presumes the canonical, namely, identical, embedding.)

Let $\widehat{\mathscr{X}_0} = \mathscr{X}_0|_{I_0}$ and $\widehat{\mathscr{X}_1} = \mathscr{X}_1|_{I_1}$ be two quotient algebras. What should we imply by asserting that they are isomorphic? Of course, we could bear in mind the isomorphism of the Boolean algebras $\widehat{\mathscr{X}_0}$ and $\widehat{\mathscr{X}_1}$. However, when talking about ISOMORPHISM OF QUOTIENT ALGEBRAS, it is more sound to imply the isomorphism of the corresponding quotient objects. That is, the phrase: "the quotient algebras $\mathscr{X}_0|_{I_0}$ and $\mathscr{X}_1|_{I_1}$ are isomorphic" means the existence of some isomorphisms Ψ and ψ making the following diagram commutative:

where j_0 and j_1 are the canonical epimorphisms. In other words, the isomorphism Ψ transforms the ideal I_0 into the ideal I_1.

We adduce an example that clarifies the difference. Take as $\mathscr{X}_0$ the σ-algebra of all Lebesgue measurable subsets of an interval; and as $\mathscr{X}_1$ the σ-algebra of all Borel subsets of the same interval. Let I_0 be the ideal of all sets of measure zero and put $I_1 = I_0 \cap \mathscr{X}_1$. The quotient algebras $\mathscr{X}_0|_{I_0}$ and $\mathscr{X}_1|_{I_1}$ are isomorphic as Boolean algebras in their own right ("individually isomorphic") but not isomorphic as quotient objects since $\mathscr{X}_0$ and $\mathscr{X}_1$ are not isomorphic (they have different cardinality).

Consider now the same constructions in the category $\mathbb{St}$. Let $\mathfrak{Q}$ be a totally disconnected compact space. For defining a SUBOBJECT of $\mathfrak{Q}$,

we can choose a compact subset $F \subset \mathfrak{Q}$ and, next, consider the totality of all compact spaces F' homeomorphic to F.

Denote by j_0 the identical embedding of F into $\mathfrak{Q}$. Then, each compact space F' is embedded into $\mathfrak{Q}$ by the mapping $j' = j_0 \circ \varphi$, where φ is a homeomorphism. All cones

$$\{F'; \mathfrak{Q}; j'\}$$

comprise a subobject that is uniquely determined by F. It can be said that "a subobject is a compact subset," assuming as usual that we deal with a set CANONICALLY EMBEDDED into $\mathfrak{Q}$. It can be another set F_1 in $\mathfrak{Q}$ which is homeomorphic to F but embedded into $\mathfrak{Q}$ in some other way and defining another subobject.

Choosing F, we thus distinguish an ideal I in the algebra $\mathscr{C}O(\mathfrak{Q})$ such that $F = \mathfrak{M}(I)$ (see p. 127). So, the quotient object for this algebra is defined that is dual for the subobject. It can be said that the concepts of a closed subspace in $\mathfrak{Q}$ and a quotient algebra in $\mathscr{C}O(\mathfrak{Q})$ are dual to each other.

Finally, we discuss the QUOTIENT OBJECTS in $\mathbb{St}$. Each of those quotient objects of the compact space $\mathfrak{Q}$ is defined by a continuous mapping j_0 from $\mathfrak{Q}$ onto some compact space $\mathfrak{Q}' \equiv j_0(\mathfrak{Q})$. The totality of all compact spaces homeomorphic to $\mathfrak{Q}$ and the mappings $\varphi \circ j_0$ (φ is a homeomorphism) define a quotient object. To realize this quotient object, it is sufficient to choose the Boolean partition ξ of $\mathfrak{Q}$ into the level sets of the mapping j_0. This partition corresponds to a SUBALGEBRA in $\mathscr{C}O(\mathfrak{Q})$ and, thus, to a subobject of the Boolean algebra $\mathscr{C}O(\mathfrak{Q})$ which is dual to our quotient object. We already know the structure of this subalgebra. It consists of all clopen sets constituted by whole elements of the partition ξ.

At the same time, these elements can be regarded as points of a totally disconnected compact space. This space is the quotient space; it is the Stone space of our subalgebra. So, one more "rough" formulation: "a quotient object of a compact space is a quotient space." Certainly, we assume a totally disconnected compact quotient space.

As for isomorphism of subspaces and quotient spaces of a compact space, the approach is reasonable (in the spirit of the "categorical" mentality) in which the subspaces and quotient spaces are regarded as subobjects and quotient objects.

Moreover, each isomorphism of two subspaces or quotient spaces is equivalent to some isomorphism of the corresponding subalgebras or quotient algebras in the category $\mathbb{Boole}$. (Certainly, here we mean the correspondence by the Stone functor $\mathbb{S}$.) Practically, this means that

two subspaces or quotient spaces of some compact spaces $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$ are isomorphic whenever there exists a homeomorphism from $\mathfrak{Q}_1$ onto $\mathfrak{Q}_2$.

3.10 Retracts and coretracts; injectivity and projectivity in the categories $\mathbb{Boole}$ and $\mathbb{St}$

The concepts of retract and coretract are selfdual. A Boolean algebra $\mathscr{Y}$ is a retract of a Boolean algebra $\mathscr{X}$ if and only if $\mathfrak{Q}(\mathscr{Y})$ is a retract of the compact space $\mathfrak{Q}(\mathscr{X})$ or, which is the same, if $\mathfrak{Q}(\mathscr{X})$ is a coretract for $\mathfrak{Q}(\mathscr{Y})$. We can decipher this in two ways:

(a) There are homomorphisms $\Phi : \mathscr{X} \longrightarrow \mathscr{Y}$ and $\Psi : \mathscr{Y} \longrightarrow \mathscr{X}$ such that $\Phi \circ \Psi = \mathrm{Id}_{\mathscr{Y}}$.

(b) There are continuous mappings $\varphi : \mathfrak{Q}(\mathscr{Y}) \longrightarrow \mathfrak{Q}(\mathscr{X})$ and $\psi : \mathfrak{Q}(\mathscr{X}) \longrightarrow \mathfrak{Q}(\mathscr{Y})$ such that

$$\psi \circ \varphi = \mathrm{Id}_{\mathfrak{Q}(\mathscr{Y})}.$$

These two formulations are equivalent; moreover, Φ and φ as well as Ψ and ψ are canonically related in the sense of Section 2. In this case, Φ is an epimorphism and φ is a continuous injection, while Ψ is a monomorphism and ψ is a continuous surjection. A retract of a Boolean algebra can be considered both as a quotient algebra and as a subalgebra.

The category-theoretic concepts of injectivity and projectivity are connected with the important properties of Boolean algebras.

In accordance with the general definition (cf. 3.3.6), a Boolean algebra $\mathscr{Y}$ is *injective* whenever each homomorphism $\Phi_0 : \mathscr{X}_0 \longrightarrow \mathscr{Y}$ can be extended to a homomorphism $\Phi : \mathscr{X} \longrightarrow \mathscr{Y}$. (Here $\mathscr{X}_0 \subset \mathscr{X}$ is a subalgebra of $\mathscr{Y}$.) The following diagram describes this situation:

In this diagram, j is an embedding of $\mathscr{X}_0$ into $\mathscr{X}$. As was proved in 3.3.6, the injectivity property is preserved under retractions. The name "injectivity" conceals an important property of a Boolean algebra. Namely, *a Boolean algebra is injective if and only if it is complete.* We present a detailed proof of this important theorem in Chapter 5.

Now we only establish the completeness of every injective algebra. Let $\mathscr{Y}$ be an injective Boolean algebra and let $\overline{\mathscr{Y}}$ be a Dedekind completion of $\mathscr{Y}$ (see p. 98).

Consider an arbitrary nonempty set $E \subset \mathscr{Y}$ and denote by $\overline{y}$ its supremum calculated in $\overline{\mathscr{Y}}$. If we prove that $\overline{y} \in \mathscr{Y}$ then we see that this element is the supremum of E in $\mathscr{Y}$. Denote by E_1 the set $\{y \in \mathscr{Y} \mid y \geq \overline{y}\}$. According to the main property of Dedekind completion,

$$\overline{y} = \sup E = \inf E_1$$

(all suprema and infima are calculated in $\overline{\mathscr{Y}}$).

Let Φ_0 be an identical embedding from $\mathscr{Y}$ into $\overline{\mathscr{Y}}$. In view of injectivity of $\mathscr{Y}$, there is a homomorphism $\Phi : \overline{\mathscr{Y}} \longrightarrow \mathscr{Y}$ extending Φ_0. It is clear that

$$\overline{y} = \sup E = \sup \Phi_0(E) \leq \Phi(\overline{y}) \leq \inf \Phi_0(E_1) = \inf E_1 = \overline{y}.$$

We have $\Phi(\overline{y}) \in \mathscr{Y}$ and $\Phi(\overline{y}) = \overline{y}$; therefore, $\overline{y} \in \mathscr{Y}$. We see that $\sup E \in \mathscr{Y}$ for all $E \subset \mathscr{Y}$; this proves the completeness of $\mathscr{Y}$.

The duality argument shows that the injectivity of a Boolean algebra $\mathscr{Y}$ is equivalent to the projectivity of its Stone space $\mathfrak{Q}(\mathscr{Y})$ (in the category $\mathbb{S}\mathfrak{t}$).

In accordance with the general definition, it means the following: for all totally disconnected compact spaces $\mathfrak{Q}_1$ and $\mathfrak{Q}_0$, each continuous surjection $j : \mathfrak{Q}_1 \longrightarrow \mathfrak{Q}_0$, and each continuous mapping $f : \mathfrak{Q} \longrightarrow \mathfrak{Q}_0$, there is a continuous mapping $F : \mathfrak{Q} \longrightarrow \mathfrak{Q}_1$ making the diagram

commutative. Having the general form of an injective Boolean algebra available, we thus found out that *all projective totally disconnected compact spaces are extremal compact spaces.* As regards, the terms "projective" and "extremal" are synonyms.

The following two facts are now trivial:

1. *A retract of a complete Boolean algebra is a complete Boolean algebra.* It is not difficult to show that each complete Boolean algebra $\mathscr{X}$ is a retract of the algebra $2^{\mathfrak{Q}}$, with $\mathfrak{Q} = \mathfrak{Q}(\mathscr{X})$.

2. *A retract of an extremally disconnected space is extremally disconnected.* (We have established the fact that the projectivity property is preserved under retractions in 3.3.6.)

We now proceed to PROJECTIVE BOOLEAN ALGEBRAS (in the category $\mathbb{Boole}$).

The projectivity property of a Boolean algebra $\mathscr{X}$ means that, for all Boolean algebras $\mathscr{X}_1$ and $\mathscr{X}_2$, each epimorphism $f : \mathscr{X}_1 \longrightarrow \mathscr{X}_2$, and each homomorphism $h : \mathscr{X} \longrightarrow \mathscr{X}_2$, there is a homomorphism g making the following diagram commutative:

This property is equivalent to the injectivity of the compact space $\mathfrak{Q} \equiv \mathfrak{Q}(\mathscr{X})$.

Namely, for all Stone spaces $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$, each continuous surjection $\varphi : \mathfrak{Q}_2 \longrightarrow \mathfrak{Q}_1$, and each continuous mapping $\eta : \mathfrak{Q}_2 \longrightarrow \mathfrak{Q}$, there is a continuous mapping γ making the following diagram commutative:

In other words, each continuous mapping from a "subspace" $\mathfrak{Q}_2$ into $\mathfrak{Q}$ is extendible to a mapping from $\mathfrak{Q}_1$ into $\mathfrak{Q}$. The general form of a projective Boolean algebra is also available: all retracts of free Boolean algebras, and only them, are projective (P. Halmos [2]). The proof of this theorem is not difficult; and it implies in particular that the retracts of Cantor discontinua are exactly the injective Stone spaces.

Let $\mathscr{X}$ be a complete Boolean algebra and let $\widehat{\mathscr{X}}$ be the quotient algebra of $\mathscr{X}$ by an ideal I; i.e.,

$$\widehat{\mathscr{X}} = \mathscr{X}|_I.$$

We have discussed the question on completeness of this quotient algebra in Chapter 2, and so the completeness takes place provided that the canonical homomorphism

$$\Psi : \mathscr{X} \longrightarrow \widehat{\mathscr{X}}$$

is a retraction. In this case, the corresponding CORETRACTION ρ is called a *lifting*. The same name applies to the endomorphism $\widetilde{\rho}$ defined by the diagram

(the "ascent" of the canonical homomorphism Ψ). The question on existence of a lifting is not trivial; we discuss it in Chapter 5 for measure algebras (not assuming the completeness of $\mathscr{X}$).

In terms of category theory are useful in pondering over those properties of mathematical objects which speak about their relations with other "analogous" objects. Moreover, even some purely intrinsic properties (for instance, the completeness of a Boolean algebra) acquire an adequate description. The main definition of completeness of a Boolean algebra refers to a sole algebra; it says nothing about the others. However, completeness is equivalent to injectivity; and the definition of the latter is formulated only with the aid of arrows not involving the intrinsic structure of an algebra. Furthermore, the "logical level" of the two definitions is different: we have the first order language in one of the cases and the second order, in the other. Namely, in the "category-theoretic" language we talk about ALL OBJECTS B and B_0; and in the main definition, about EVERY SET OF ELEMENTS. At the same time, the "object domain" (i.e. the range of quantifiers) looks much more illimited in the "category-theoretic" case: the class *Ob* is even not a set. Most likely, this is a reason behind the wonderful transformation of "intrinsic" properties to "extrinsic" and vice versa which is so typical of category theory.

Exercises to Chapter 3

1. Show that an infinite extremally disconnected compact space is not metrizable.

2. Show that the clopen algebra of the Stone space of an infinite BA is not an algebra of sets.

Chapter 4

TOPOLOGIES ON BOOLEAN ALGEBRAS

1. Directed sets and generalized sequences

1.1 Generalized sequences

A partially ordered set A is said to be *directed upward* (*downward*) whenever for every two elements $\alpha_1, \alpha_2 \in A$ there is an element $\alpha \in A$ such that the following relations hold simultaneously:[1] $\alpha \succ \alpha_1$ and $\alpha \succ \alpha_2$ ($\alpha \prec \alpha_1$ and $\alpha \prec \alpha_2$).

A directed set is also called a *direction*. A classical example of a directed set is the set of positive integers $\{1, 2, \ldots\}$ with the natural order. As is well known, every function on this set is called a *sequence*. Accordingly, a *generalized sequence* or *net* is a function defined on a directed set. Such a function is usually viewed as a family:

$$\{x_\alpha\}_{\alpha \in A}.$$

An element x_α is called an *entry* or *term* of the generalized sequence. The word "generalized" is sometimes omitted.

Let A be a directed upward (downward) set; we say that a subset $A' \subset A$ is *cofinal* if, for each $\alpha \in A$, there is an element $\alpha' \in A'$ such that $\alpha' \succ \alpha$ ($\alpha' \prec \alpha$). It is clear that such a set is also directed. If $\xi \equiv \{x_\alpha\}_{\alpha \in A}$ is generalized sequence on A then its each restriction of the form $\xi' \equiv \xi|_{A'}$, where $A' \subset A$ is a cofinal subset of A, is called a *cofinal* subsequence. In other words,

$$\xi' \equiv \{x_\alpha\}_{\alpha \in A'}.$$

[1] The sign $\succ$ amounts to the sign $\geq$.

The following "Alternative Theorem" is immediate from the definition.

Theorem 1. *Let $\xi = \{x_\alpha\}_{\alpha \in A}$ be a generalized sequence with entries in a set $\mathscr{X}$, let $\mathscr{X}_1$ be a subset of $\mathscr{X}$, and let A be an upward-directed set. Then there is an index α_0 such that $x_\alpha \in \mathscr{X}_1$ for every $\alpha \succ \alpha_0$ or there exists a cofinal sequence ξ' whose all members lie in $\mathscr{X} \setminus \mathscr{X}_1$.*

In the sequel, we consider only generalized sequences with upward-directed index sets (for downward-directed index sets, all results are analogous).

Directed sets and generalized sequences were introduced by S. O. Shatunovskiĭ [1] as well as by E. H. Moore and H. L. Smith [1] for the purposes of the theory of limits. Namely, the concept of a *limit* makes sense for generalized sequences with entries in a topological space $\mathscr{X}$: an element $x \in \mathscr{X}$ is called a *limit of a generalized sequence* $\xi \equiv \{x_\alpha\}_{\alpha \in A}$ whenever for each neighborhood V about x there is an index $\alpha_V \in A$ such that $x_\alpha \in V$ for all $\alpha \succ \alpha_V$. This fact is written in symbols as $x = \lim \xi$ or $x = \lim_\alpha x_\alpha$. The following theorem demonstrates the usefulness of these concepts.

Theorem 2. *A point x belongs to the closure $\overline{E}$ of a nonempty set E if and only if there exists a generalized sequence ξ whose all entries belong to E and such that $\lim \xi = x$.*

Proof. It is clear that all limits of generalized sequences in E must belong to the closure $\overline{E}$. Suppose now that $x \in \overline{E}$. Arrange the directed set $\mathfrak{V}$ that consists of all neighborhoods of x; the inequality $V_1 \succ V_2$ means the inclusion $V_1 \subset V_2$. In view of the main topological axioms, $\mathfrak{V}$ is a direction. Choosing a point x_V in each V, we obtain a generalized sequence having x as a limit. This theorem was proved by G. Birkhoff.

Theorem 2 shows that a topology τ_1 is finer than another topology τ_2 if and only if each generalized sequence that has a limit in τ_1 has the same limit in τ_2.

Note that a generalized sequence may have more than one limit in an arbitrary topological space. *Uniqueness of the limit is equivalent to the separatedness of $\mathscr{X}$; namely, $\mathscr{X}$ must satisfy the Hausdorff axiom* according to which every two distinct points are separated by disjoint neighborhoods.

Two generalized sequences $\{x_\alpha\}_{\alpha \in A}$ and $\{y_\alpha\}_{\alpha \in A}$ with the same index set A are called *similar* or *homogeneous*. If A is the set of natural numbers then the sequence is called *simple*.

It is convenient to formulate the definition of continuity of a mapping at a point with the aid of generalized sequences: a mapping f from a topological space $\mathscr{X}$ into a topological space $\mathscr{Y}$ is continuous at a point

$x_0 \in \mathscr{X}$ whenever $\lim_\alpha x_\alpha = x_0$ implies $\lim_\alpha f(x_\alpha) = f(x_0)$. This definition of continuity is equivalent to the other in this book. Verification of this fact is left to the reader.

In the cases when a directed set A is included in some partially ordered set $\mathscr{X}$, it is usually assumed that the order on A either coincides with the order induced from x or is reverse to it. In such cases, we say that A is directed *upward* or, respectively, *downward*.

Let $\mathscr{X}$ be a Boolean algebra (or even a lattice). With each nonempty set $S \subset \mathscr{X}$ we associate the two directed sets $S^{\uparrow}$ and $S^{\downarrow}$. The former set contains the suprema of all finite subsets of S and the latter, the infima. Moreover, the relation $\succ$ is interpreted as $\geq$ in the first case; and as $\leq$, in the second. We say that $S^{\uparrow}$ and $S^{\downarrow}$ are directions *associated* with S. The former is directed upward and the latter, downward. Further, we may arrange two generalized sequences by assigning to each element $S^{\uparrow}$ ($S^{\downarrow}$) (as to an index) this element itself. These two sequences are monotone. We refer to them as to the *monotone sequences associated with* S. As an example, we consider the case in which S is a subalgebra, ideal, or filter; it is obvious that, in this situation, the sets S, $S^{\uparrow}$, and $S^{\downarrow}$ coincide and both generalized sequences "consist" of the same entries (which, however, does not give grounds for identifying these sequences).

Let S be a disjoint set, let $\{x_\alpha\}$ be the increasing sequence associated with S, and let φ be an additive real function. Then the equality

$$\lim_\alpha \varphi(x_\alpha) = L \tag{1}$$

can be written as

$$\sum_{x \in S} \varphi(x) = L. \tag{1$'$}$$

Indeed, if (1) holds then, for a given $\varepsilon > 0$, there is a finite subset $S' \subset S$ such that $\sup S'' \geq \sup S'$ or, which is the same, $S'' \supset S'$ implies

$$|\, L - \sum_{x \in S''} \varphi(x)\,| < \varepsilon.$$

Therefore, (1$'$) holds. Analogously, (1) ensues from (1$'$).

1.2 Topology through convergence

Let ξ be the set of generalized sequences in a set R. Suppose that, to each $\xi \in \Xi$ there corresponds some element $x_\xi \in R$. It is clear that there are topologies of R with respect to which each sequence ξ has the element x_ξ as a limit. Let T be a class of all such topologies. It is not difficult to see that *the weakest topology dominating the class T must itself belong to T*. Indeed, in such a topology, each neighborhood of the

point x_ξ is an intersection of the form $V_1 \cap V_2 \cap \dots \cap V_m$, where each V_i is a neighborhood of x_ξ in some topology of the class T. Therefore, all "sufficiently far" members of the sequence ξ must belong to this intersection and the point x_ξ is a limit of ξ with respect to the topology in question. Thus, the latter belongs to the class T. We see that *there is a finest topology in the class* T, intending to use this claim in the next section.

In conclusion, we point out two classes of directions that are particularly important for us.

1) The intervals of the transfinite series (ordinals)

$$1, 2, \dots, \omega, \dots .$$

The most important case is the set of naturals.

2) Let $\mathfrak{a}$ be a cardinal and let T be an arbitrary set of cardinality $\mathfrak{a}$. Denote by $A_\mathfrak{a}$ the system of all finite subsets of T endowed with the natural order; $\alpha \prec \beta$ means that $\alpha \subset \beta$ (in other words, $A_\mathfrak{a}$ is the ideal of all finite subsets of T). It is clear that $A_\mathfrak{a}$ is a direction. The choice of the set T does not play an essential role; of importance is only the cardinality of this set. Sometimes, it is convenient to consider the filter dual to the ideal $A_\mathfrak{a}$. Of particular importance is the case in which $\mathfrak{a}$ is the TYPE of a Boolean algebra $\mathscr{X}$. In this case, we call the direction $A_\mathfrak{a}$ *standard* (for this Boolean algebra). The generalized sequences, indexed by the elements of $A_\mathfrak{a}$, are sometimes called standard too.

2. Various topologies on Boolean algebras

2.1 Preliminary notes

By now, when studying Boolean algebras we were interested only in those properties that are immediately connected with order. However, in addition to order, the set of elements of a Boolean algebra can be furnished with various topologies. Thus, a Boolean algebra should be studied not only as a partially ordered set but also as a topological space. Certainly, among all possible topologies we are interested primarily in those that are reasonably compatible with order.

Throughout this section, we assume $\mathscr{X}$ a complete BA. This requirement is sometimes superfluous; the most important cases in which it can be weakened will be pointed out.

2.2 Order topologies

The connection between topology and order can be described by various axioms. First, we consider the topologies that satisfying one of the following conditions (o) or (os).

(*o*) *Let three similar generalized sequences*

$$\{x_\alpha\}_{\alpha\in A},\quad \{y_\alpha\}_{\alpha\in A},\quad \{z_\alpha\}_{\alpha\in A}$$

satisfy the inequality $y_\alpha \le x_\alpha \le z_\alpha$ *for all* $\alpha \in A$. *If* $\{y_\alpha\}$ *increases,* $\{z_\alpha\}$ *decreases, and*

$$\bigvee_{\alpha\in A} y_\alpha = \bigwedge_{\alpha\in A} z_\alpha = x$$

then $\{x_\alpha\}$ *converges topologically to* x.

A topology may fail to satisfy the condition (*o*) but still have a weaker property:

(*os*) *Let three usual sequences*

$$\{x_n\}_{n=1}^{\infty},\quad \{y_n\}_{n=1}^{\infty},\quad \{z_n\}_{n=1}^{\infty}$$

satisfy the inequality $y_n \le x_n \le z_n$ *for all* $n = 1, 2, \ldots$. *If* $\{y_n\}$ *increases,* $\{z_n\}$ *decreases, and*

$$\bigvee_{n=1}^{\infty} y_n = \bigwedge_{n=1}^{\infty} z_n = x$$

then $\{x_n\}$ *converges topologically to* x.

There is a unique finest topology among those satisfying the condition (*o*) ((*os*)).

Definition. The finest topology satisfying the condition (*o*) is called the (*o*)-*topology*.

Definition. The finest topology satisfying the condition (*os*) is called the (*os*)-*topology*.

These two topologies are called *topologies of ordering* or *order topologies*. Since the condition (*o*) is stronger than (*os*), the (*o*)-topology is obviously weaker than the (*os*)-topology. As we will see, these topologies coincide in some important cases.

The conditions (*o*) and (*os*) are tightly related with the concept of (*o*)-convergence.

Definition. A generalized sequence $\xi \equiv \{x_\alpha\}_{\alpha\in A}$ is called (*o*)-*convergent to* x whenever there are generalized sequences $\{y_\alpha\}_{\alpha\in A}$ and $\{z_\alpha\}_{\alpha\in A}$ satisfying the inequality $y_\alpha \le x_\alpha \le z_\alpha$ for all $\alpha \in A$ and such that $\{y_\alpha\}$ increases, $\{z_\alpha\}$ decreases, and

$$\bigvee_{\alpha\in A} y_\alpha = \bigwedge_{\alpha\in A} z_\alpha = x.$$

In this case, we write $x_\alpha \xrightarrow{(o)} x$ or $(o)\text{-}\lim_\alpha x_\alpha = x$ or simply $(o)\text{-}\lim \xi = x$. If A is the set of naturals (the case of a simple sequence) then we write $x_n \xrightarrow{(o)} x$ or $(o)\text{-}\lim x_n = x$.

We can say now that *the* (o)*-topology is the finest among those topologies in which* (o)*-convergence implies topological convergence. As will be shown later, an* (o)*-convergent generalized sequence has exactly one topological limit.*

Note that the definitions of the (o)-topology and (o)-convergence do not require completeness of the ambient algebra.

We point out the fact that the (o)-convergence is, as a rule, essentially stronger than the convergence with respect to the (o)- or (os)-topology; these convergences coincide only in degenerative cases. We denote the topological convergence (convergence in the (o)-topology) simply by the arrow $\longrightarrow$. The generalized sequences $\{y_\alpha\}$ and $\{z_\alpha\}$ in the definition of (o)-convergence are usually called *contracting* for $\{x_\alpha\}$.

It is clear that the inequalities

$$y_\alpha \le \bigwedge_{\beta \succ \alpha} x_\beta \le x_\alpha \le \bigvee_{\beta \succ \alpha} x_\beta \le z_\alpha \tag{2}$$

hold for all α. Therefore, taking

$$\overline{y}_\alpha \equiv \bigwedge_{\beta \succ \alpha} x_\beta, \quad \overline{z}_\alpha \equiv \bigvee_{\beta \succ \alpha} x_\beta, \tag{3}$$

we obtain two generalized sequences that contract $\{x_\alpha\}$ most tightly. It follows immediately from the inequalities (2) that

$$\bigwedge_\alpha \overline{z}_\alpha = \bigvee_\alpha \overline{y}_\alpha = (o)\text{-}\lim_\alpha x_\alpha,$$

whenever the last limit exists. Note now that the formulas (3) make sense for every generalized sequence $\xi \equiv \{x_\alpha\}$. The generalized sequences $\{y_\alpha\}$ and $\{z_\alpha\}$ given by these formulas are monotonic (the first increases and the second decreases).

Definition. The elements

$$y \equiv \bigvee_\alpha \overline{y}_\alpha, \quad z \equiv \bigwedge_\alpha \overline{z}_\alpha \tag{4}$$

are called the *lower* and *upper limits*[2] of a generalized sequence $\xi \equiv \{x_\alpha\}$. They are denoted as follows:

$$y \equiv \underline{\lim}_\alpha x_\alpha, \quad z \equiv \overline{\lim}_\alpha x_\alpha$$

[2]This definition assumes the completeness of the ambient Boolean algebra. In the case of a simple sequence, it is sufficient to assume σ-completeness.

or briefly:

$$y \equiv \underline{\lim}\, \xi, \quad z \equiv \overline{\lim}\, \xi.$$

It is clear that $\underline{\lim}\, \xi \leq \overline{\lim}\, \xi$. The following theorem is obvious.

Theorem 3. *For a generalized sequence $\{x_\alpha\}$ to (o)-converge to an element x, it is necessary and sufficient that the following equality holds:*

$$\underline{\lim}_\alpha x_\alpha = \overline{\lim}_\alpha x_\alpha = x. \tag{5}$$

This theorem implies, in particular, the *uniqueness* of an (o)-limit.

The formula (5) can be written as

$$\bigvee_{\beta \in A} \bigwedge_{\alpha \succ \beta} x_\alpha = \bigwedge_{\beta \in A} \bigvee_{\alpha \succ \beta} x_\alpha. \tag{5'}$$

For an increasing or decreasing sequence we respectively have

$$\bigvee_{\alpha \in A} x_\alpha = x \quad \text{or} \quad \bigwedge_{\alpha \in A} x_\alpha = x.$$

In this case, we write $x_\alpha \uparrow x$ or $x_\alpha \downarrow x$.

Let $\xi \equiv \{x_\alpha\}_{\alpha \in A}$ be a generalized sequence. Consider an arbitrary cofinal subdirection A' of the direction A and the corresponding cofinal subsequence $\xi' \equiv \{x_\alpha\}_{\alpha \in A'}$. Since

$$\{x_\alpha \mid \alpha \in A', \ \alpha \succ \alpha_0\} \subset \{x_\alpha \mid \alpha \in A, \ \alpha \succ \alpha_0\}$$

for all $\alpha_0 \in A'$, we deduce that the supremum of the first set is not greater than the supremum of the second. Consequently (here we use the cofinality of A),

$$\overline{\lim}\, \xi' \leq \overline{\lim}\, \xi.$$

Analogously, $\underline{\lim}\, \xi' \geq \underline{\lim}\, \xi$.

So, *after passing to a cofinal subsequence, the upper limit can only decrease and the lower, can only increase.* This, in particular, implies the following

Theorem 4. *A cofinal subsequence of an (o)-convergent generalized sequence (o)-converges to the same limit.*

Indeed, if $x = (o)\text{-}\lim \xi$ and ξ' is a cofinal subsequence of ξ then

$$x = \underline{\lim}\, \xi \leq \underline{\lim}\, \xi' \leq \overline{\lim}\, \xi' \leq \overline{\lim}\, x\xi = x,$$

i.e., $x = (o)\text{-}\lim \xi'$.

Theorem 4 is still valid for an incomplete Boolean algebra. This case requires a direct proof which also is not complicated.

An example of an (o)-convergent sequence is delivered by each infinite generalized sequence $\{x_\alpha\}$ whose entries comprise a disjoint set. Indeed, it is clear that $\underline{\lim} x_\alpha = \mathbf{0}$. Moreover, for $\alpha \succ \alpha_0$ and $\alpha \neq \alpha_0$, we have $(\bigvee_{\beta \succ \alpha} x_\beta)\, d\, x_{\alpha_0}$, whence it follows that the element $y \equiv \overline{\lim}\, x_\alpha$ is disjoint from each x_{α_0}. Then $(\bigvee_{\alpha \in A} x_\alpha)\, d\, y$, although it is obvious that $y \leq \bigvee_{\alpha \in A} x_\alpha$. The latter can be only if $y = \mathbf{0}$. So, $\underline{\lim}\, x_\alpha = \overline{\lim}\, x_\alpha = \mathbf{0}$, i.e., $(o)\text{-}\lim x_\alpha = \mathbf{0}$.

We now present the main facts concerning the above concepts. Note that all generalized sequences in (a)–(f) are assumed to be indexed by the same directed set A. The Boolean algebra $\mathscr{X}$ is assumed complete.

(a) If $y_\alpha = Cx_\alpha$ then

$$\overline{\lim}\, y_\alpha = C(\underline{\lim}\, x_\alpha), \quad \underline{\lim}\, y_\alpha = C(\overline{\lim}\, x_\alpha).$$

Thus, under the dual isomorphism of the Boolean algebra $\mathscr{X}$ onto itself we have described in Chapter 0, the upper limit corresponds to the lower and vice versa.

(b) If $y_\alpha = Cx_\alpha$ and $x_\alpha \xrightarrow{(o)} x$ then $y_\alpha \xrightarrow{(o)} Cx$.

(c) If $x_\alpha \leq y_\alpha$ for all $\alpha \succ \alpha_0 \in A$ then

$$\overline{\lim}\, x_\alpha \leq \overline{\lim}\, y_\alpha, \quad \underline{\lim}\, x_\alpha \leq \underline{\lim}\, y_\alpha.$$

Hence it ensues, in particular, the possibility of passing to the (o)-limit in all inequalities of the form $a \leq x_\alpha \leq b$.[3]

(d) If $x_\alpha \leq y_\alpha \leq z_\alpha$ for all $\alpha \succ \alpha_0 \in A$ and if

$$(o)\text{-}\lim_\alpha x_\alpha = (o)\text{-}\lim_\alpha z_\alpha = x$$

then $y_\alpha \xrightarrow{(o)} x$.

(e) For an arbitrary finite family of similar generalized sequences $\{x_\alpha^{(k)}\}_{\alpha \in A}$ $(k = 1, 2, \dots, n)$, we have

$$\overline{\lim}_\alpha \bigvee_{k=1}^{n} x_\alpha^{(k)} = \bigvee_{k=1}^{n} \Big(\overline{\lim}_\alpha\, x_\alpha^{(k)}\Big),$$

$$\underline{\lim}_\alpha \bigvee_{k=1}^{n} x_\alpha^{(k)} \geq \bigvee_{k=1}^{n} \underline{\lim}_\alpha\, x_\alpha^{(k)}.$$

[3]This fact can easily be proved straightforwardly even without assuming the completeness of the algebra.

(f) For every finite family of similar generalized sequences $\{x_\alpha^{(k)}\}_{\alpha\in A}$ $(k = 1, 2, \ldots, n)$, the relations

$$x_\alpha^{(k)} \xrightarrow{(o)} x^{(k)} \quad (k = 1, 2, \ldots, n)$$

imply that

$$\bigvee_{k=1}^{n} x_\alpha^{(k)} \xrightarrow{(o)} \bigvee_{k=1}^{n} x^{(k)} \quad \text{and} \quad \bigwedge_{k=1}^{n} x_\alpha^{(k)} \xrightarrow{(o)} \bigwedge_{k=1}^{n} x^{(k)}.$$

The assertions (b) and (d) claim that the operations C, $\vee$, and $\wedge$ are "(o)-continuous." However, we cannot assert the continuity of these operations with respect to the (o)-*topology* since this continuity does not take place as we will see later.

As for the proofs of the above assertions, they are usually rather simple. Namely, (a) can be proved by comparing the definitions (4) and the duality formulas in Chapter 1; (b) follows directly from (a); (c) is an obvious corollary of the definitions (4); and (d) ensues from (c). In order to prove (e), we apply the distributive law. After several simple transformations, we find

$$\bigvee_{k=1}^{n} \left(\overline{\lim}_\alpha\, x_\alpha^{(k)}\right) = \bigvee_{k=1}^{n} \bigwedge_{\alpha\in A} \bigvee_{\beta\succ\alpha} x_\beta^{(k)}$$

$$= \bigwedge_{\alpha_1,\alpha_2,\ldots,\alpha_n\in A} \Big[\Big(\bigvee_{\beta\succ\alpha_1} x_\beta^{(1)}\Big) \vee \cdots \vee \Big(\bigvee_{\beta\succ\alpha_n} x_\beta^{(n)}\Big)\Big]$$

$$= \bigwedge_{\alpha\in A} \bigvee_{k=1}^{n} \bigvee_{\beta\succ\alpha} x_\beta^{(k)} = \bigwedge_{\alpha\in A} \bigvee_{\beta\succ\alpha} \Big(\bigvee_{k=1}^{n} x_\beta^{(k)}\Big) = \overline{\lim}_\alpha \bigvee_{k=1}^{n} x_\alpha^{(k)}$$

which proves the first equality in (e). Analogously,

$$\bigvee_{k=1}^{n} \left(\underline{\lim}_\alpha\, x_\alpha^{(k)}\right) = \bigvee_{k=1}^{n} \bigvee_{\alpha\in A} \bigwedge_{\beta\succ\alpha} x_\beta^{(k)} = \bigvee_{\alpha\in A} \Big(\bigvee_{k=1}^{n} \bigwedge_{\beta\succ\alpha} x_\beta^{(k)}\Big)$$

$$= \bigvee_{\alpha\in A} \Big(\bigwedge_{\beta_1,\beta_2,\ldots,\beta_n\succ\alpha} \Big[x_{\beta_1}^{(1)} \vee \cdots \vee x_{\beta_n}^{(n)}\Big]\Big)$$

$$\leq \bigvee_{\alpha\in A} \bigwedge_{\beta\succ\alpha} (x_\beta^{(1)} \vee \cdots \vee x_\beta^{(n)}) = \underline{\lim}_\alpha \bigvee_{k=1}^{n} x_\alpha^{(k)}.$$

In the first relation, the equality holds since the generalized sequences $z_\alpha^{(k)} = \bigvee_{\beta\succ\alpha} x_\beta^{(k)}$ decrease. Taking $\alpha \succ \alpha_1 \succ \dots \succ \alpha_n$, we obtain

$$\Big(\bigvee_{\beta\succ\alpha} x_\beta^{(1)}\Big) \vee \dots \vee \Big(\bigvee_{\beta\succ\alpha} x_\beta^{(n)}\Big) \le \Big(\bigvee_{\beta\succ\alpha_1} x_\beta^{(1)}\Big) \vee \dots \vee \Big(\bigvee_{\beta\succ\alpha_n} x_\beta^{(n)}\Big),$$

$$\bigwedge_{\alpha\in A} \bigvee_{k=1}^{n} \bigvee_{\beta\succ\alpha} x_\beta^{(k)} \le \bigwedge_{\alpha_1,\alpha_2,\dots,\alpha_n\in A} \Big[\Big(\bigvee_{\beta\succ\alpha_1} x_\beta^{(1)}\Big) \vee \dots \vee \Big(\bigvee_{\beta\succ\alpha_n} x_\beta^{(n)}\Big)\Big].$$

The inequality

$$\bigwedge_{\alpha_1,\alpha_2,\dots,\alpha_n\in A} \Big[\Big(\bigvee_{\beta\succ\alpha_1} x_\beta^{(1)}\Big) \vee \dots \vee \Big(\bigvee_{\beta\succ\alpha_n} x_\beta^{(n)}\Big)\Big] \le \bigwedge_{\alpha\in A} \bigvee_{k=1}^{n} \bigvee_{\beta\succ\alpha} x_\beta^{(k)}$$

is obvious.

Prove (f). In view of (e), we have

$$\begin{aligned}\bigvee_{k=1}^{n} x^{(k)} &= \bigvee_{k=1}^{n} \underline{\lim}_\alpha\, x_\alpha^{(k)} \le \underline{\lim} \bigvee_{k=1}^{n} x_\alpha^{n} \le \overline{\lim}_\alpha \bigvee_{k=1}^{n} x_\alpha^{(k)} \\ &= \bigvee_{k=1}^{n} \overline{\lim}_\alpha\, x_\alpha^{(k)} = \bigvee_{k=1}^{n} x^{(k)},\end{aligned}$$

i.e.,

$$\underline{\lim}_\alpha \bigvee_{k=1}^{n} x_\alpha^{(k)} = \overline{\lim}_\alpha \bigvee_{k=1}^{n} x_\alpha^{(k)} = \bigvee_{k=1}^{n} x^{(k)}.$$

The second claim of (f) follows from the relations dual to (e).

Definition. Let $A = \{\alpha\}$ and $B = \{\beta\}$ be arbitrary directions, and let $\xi = \{x_\alpha\}_{\alpha\in A}$ and $\eta = \{y_\beta\}_{\beta\in B}$ be generalized sequences indexed by A and B. We say that ξ *majorizes (minorizes)* η whenever for all $\beta \in B$ there is an $\alpha_0 \in A$ such that $\alpha \succ \alpha_0$ follows $x_\alpha \ge y_\beta$ $(x_\alpha \le y_\beta)$. We use the following notations: $\xi \vdash \eta$ if ξ majorizes η; and $\xi \lhd \eta$ if ξ minorizes η.

We propose to the reader to guess why we do not use the sign $\dashv$ in the second case.

Lemma 1. *If* $\xi \vdash \eta$ $(\xi \lhd \eta)$ *then*

$$\overline{\lim}\, \eta \le \underline{\lim}\, \xi \quad (\underline{\lim}\, \eta \ge \overline{\lim}\, \xi).$$

PROOF. Let $\xi \vdash \eta$. Then, taking an arbitrary $\beta \in B$, we have $x_\alpha \ge y_\beta$ for all $\alpha \succ \alpha_0(\beta)$. Consequently, $\bigwedge_{\alpha\succ\alpha_0(\beta)} x_\alpha \ge y_\beta$ whence it follows that

$$\underline{\lim}\, \xi = \bigvee_{\alpha_0\in A} \bigwedge_{\alpha\succ\alpha_0(\beta)} x_\alpha \ge y_\beta.$$

Moreover, since β is arbitrary, we obtain $\underline{\lim}\,\xi \geq \bigvee_{\beta' \succ \beta} y_{\beta'}$. Finally,

$$\underline{\lim}\,\xi \geq \bigwedge_{\beta \in B} \bigvee_{\beta' \succ \beta} y_{\beta'} = \overline{\lim}\,\eta.$$

Thus, $\underline{\lim}\,\xi \geq \overline{\lim}\,\eta$. The inequality $\underline{\lim}\,\eta \geq \overline{\lim}\,\xi$ can be proved analogously in the case when $\xi \lhd \eta$.

Lemma 2. *For a generalized sequence $\{x_\alpha\}$ to converge to an element x, it is necessary and sufficient that the symmetric difference $|x_\alpha - x|$ (o)-converge to zero.*

In the necessity part, the proof reduces to application of the identity

$$|x_\alpha - x| = (x_\alpha \wedge Cx) \vee (x \wedge Cx_\alpha),$$

while in the sufficiency part we use the identity

$$x_\alpha = |x - |x_\alpha - x|| = [x \wedge C|x_\alpha - x|] \vee [|x_\alpha - x| \wedge Cx]$$

and the main properties (b) and (f).

We also present a useful formula for the difference of the upper and lower limits of a simple sequence. The ambient algebra is assumed σ-complete.

Lemma 3. *The equality*

$$\overline{\lim}\,x_n - \underline{\lim}\,x_n = \overline{\lim}\,|x_n - x_{n+1}|.$$

holds for every sequence $\{x_n\}_{n=1}^{\infty}$ in a σ-complete Boolean algebra.

We first show that the identity

$$\bigvee_{k=n}^{\infty} x_k = \bigvee_{k=n}^{\infty} |x_k - x_{k+1}| \vee x_n \tag{6}$$

is valid for all $n = 1, 2, \dots$.

To prove this, denote by a and b the left-hand and right-hand sides of (6). Next, assigning

$$x'_n = x_n,\ x'_{n+1} = x_{n+1} \wedge Cx_n, \dots,\ x'_k = x_k \wedge C(x_n \vee x_{n+1} \vee \dots \vee x_{k-1}), \dots,$$

obtain

$$\bigvee_{k=n}^{\infty} x'_k = \bigvee_{k=n}^{\infty} x_k = a.$$

We have

$$x'_k = x_k \wedge (Cx_n \wedge Cx_{n+1} \wedge \dots \wedge Cx_{k-1}) \leq x_k \wedge Cx_{k-1} \leq |x_k - x_{k-1}| \leq b$$

for all $k \geq n+1$. Hence it follows that

$$a = \Big(\bigvee_{k=n+1}^{\infty} x_k\Big) \vee x_n \leq b \vee x_n = b.$$

This inequality, together with the obvious inequality $b \leq a$, proves the identity (6).

Using the identity (6), we then write

$$\bigwedge_{k=n}^{\infty} x_k = C \bigvee_{k=n}^{\infty} Cx_k = C\Big\{\bigvee_{k=n}^{\infty} |Cx_k - Cx_{k+1}| \vee Cx_n\Big\}.$$

Taking it into account that $|Cx_k - Cx_{k+1}| = |x_k - x_{k+1}|$, we derive one more equality

$$C \bigwedge_{k=n}^{\infty} x_k = \bigvee_{k=n}^{\infty} |x_k - x_{k+1}| \vee Cx_n. \tag{7}$$

We now have

$$\Big(\bigvee_{k=n}^{\infty} x_k\Big) - \Big(\bigwedge_{k=n}^{\infty} x_k\Big) = \Big(\bigvee_{k=n}^{\infty} x_k\Big) \wedge \Big(C \bigwedge_{k=n}^{\infty} x_k\Big)$$

$$= \Big\{\bigvee_{k=n}^{\infty} |x_k - x_{k+1}| \vee x_n\Big\} \wedge \Big\{\bigvee_{k=n}^{\infty} |x_k - x_{k+1}| \vee Cx_n\Big\} = \bigvee_{k=n}^{\infty} |x_k - x_{k+1}|.$$

Finally,

$$\overline{\lim}\, x_n - \underline{\lim}\, x_n = \Big((o)\text{-}\lim \bigvee_{k=n}^{\infty} x_k\Big) - \Big((o)\text{-}\lim \bigwedge_{k=n}^{\infty} x_k\Big)$$

$$= (o)\text{-}\lim_n \Big[\bigvee_{k=n}^{\infty} x_k - \bigwedge_{k=n}^{\infty} x_k\Big] = (o)\text{-}\lim \bigvee_{k=n}^{\infty} |x_k - x_{k+1}| = \overline{\lim}_k\, |x_k - x_{k+1}|.$$

The lemma is proved.

Corollary. *If $|x_n - x_{n+1}| \xrightarrow{(o)} \mathbf{0}$ then the sequence $\{x_n\}_{n=1}^{\infty}$ (o)-converges.*

We now describe the class of sets closed with respect to the (o)-topology. The Boolean algebra $\mathscr{X}$ in the formulation of the next theorem can be incomplete.

Theorem 5. *For a set $E \subset \mathscr{X}$ to be closed in the (o)-topology, ((os)-topology), it is necessary and sufficient that E contain the (o)-limits of all (o)-convergent generalized (simple) sequences lying in E.*

PROOF. The necessity of the condition is obvious. To prove sufficiency, suppose that there is a set F containing all (o)-limits of generalized sequences in E and failing to be (o)-closed. Define a new topology on $\mathscr{X}$. We declare closed the three sets: F, the entire $\mathscr{X}$, and the empty set. This topology is obviously not weaker than the order topology. However, it does not satisfy the condition (o). Indeed, let $x_\alpha \xrightarrow{(o)} x$. Only the two sets $\mathscr{X}$ and $\mathscr{X} \setminus F$ (the only nonempty open sets) can be open neighborhoods of the point x. Let Ω be one of these sets and $x \in \Omega$. By the "Alternative Theorem" (Theorem 1), there are two possibilities: either Ω contains all sufficiently far entries of the sequence $\{x_\alpha\}$ or there is a cofinal subsequence whose every entry does not belong to Ω, i.e., belongs to F. In the last case, we obtain a subsequence that (o)-converges to x; therefore, the element x must belong to F and cannot belong to Ω in contradiction to the supposition. Thus, the neighborhood Ω must contain all sufficiently far entries x_α; and so, we have the topological convergence. The condition (o) is satisfied, which proves the theorem in the case of the (o)-topology. The proof is analogous in the case of the (os)-topology.

The closed sets in the (o)- or (os)-topology are called (o)-closed or (os)-closed.

Corollary 1. *Let functions f_1, f_2, and f_3 be defined by the equalities*

$$f_1(x) \equiv x \vee x_0, \quad f_2(x) \equiv x \wedge x_0, \quad f_3(x) \equiv Cx,$$

where x_0 is an element of $\mathscr{X}$. Then each of these functions is continuous in the (o)-topology.

It is sufficient to verify that the preimage $f_i^{-1}(F)$ is closed for all $i = 1, 2, 3$ and every subset $F \subset \mathscr{X}$ closed in the (o)-topology. Take an arbitrary generalized sequence $\{x_\alpha\}$ that consists of the elements $f_i^{-1}(F)$ and (o)-converges to some x. By the properties (b) and (f) of the (o)-convergence, we have

$$f_i(x) = (o)\text{-}\lim_{\alpha} f_i(x_\alpha).$$

Since F is closed, we may conclude that $f_i(x) \in F$ and $x \in f_i^{-1}(F)$. We have thus established the closure of $f_i^{-1}(F)$ and thereby proved Corollary 1. In particular, the continuity of the projection $f_2 \equiv P_{x_0}$ is proved.

Corollary 2. *If the inequalities $y \leq x_\alpha \leq z$ hold for all $\alpha \in A$ and $x_\alpha \longrightarrow x$ then $y \leq x \leq z$. In other words, every interval $[y, z]$ is closed with respect to the (o)-topology.*

As was shown, we can pass to the (o)-limit in such inequalities, which means the (o)-closure of the interval.

Corollary 3. *If* $x_\alpha \longrightarrow x$ *then* $\underline{\lim}\, x_\alpha \le x \le \overline{\lim}\, x_\alpha$.

We can prove this fact by passing to the limit on β in the obvious inequalities $\overline{y}_\alpha \le x_\beta \le \overline{z}_\alpha \quad (\beta \succ \alpha)$ (see p. 186).

Corollary 4. *The concept of a regular subalgebra coincides with that of a closed subalgebra with respect to the* (o)*-topology.*

It is clear that each regular subalgebra must contain the upper and lower limits of all generalized sequences of its elements. Consequently, it contains all (o)-limits of such sequences and is closed with respect to the (o)-topology by Theorem 4. On the other hand, the least upper bound of every set $E \subset \mathscr{X}_0$ is the (o)-limit of the increasing sequence associated with E (see p. 183); therefore, it must belong to $\mathscr{X}_0$.

Corollary 5. *For every two distinct points* x *and* y *there is an open set in the* (o)*-topology* $((os)$*-topology*$)$ *containing* x *but not* y.

This $G \equiv \mathscr{X} \setminus \{y\}$ is a desired set: its complement is closed since the (o)-limit of the stationary generalized sequence $\{y_\alpha\}_{\alpha \in A}$ with $y_\alpha = y$, can be equal only to y.

Corollary 5 means that $\mathscr{X}$ *is a* T_1*-space with respect to each of these topologies.* However, the Hausdorff axiom may fail; in general, the (o)-topology is not separated. In particular, we cannot assert the uniqueness of the topological limit. However, if

$$x_\alpha \xrightarrow{(o)} x$$

then Corollary 3 implies that the topological limit is unique and equal to x. We do not need the completeness of $\mathscr{X}$ in Corollaries 1, 2, and 5.

In Chapter 2 there were introduced some important classes of continuous and σ-continuous homomorphisms. It is not difficult to verify now that this terminology is justified: a homomorphism is continuous (σ-continuous) if and only if it is continuous with respect to the (o)-topologies ((os)-topologies) of the algebras $\mathscr{X}$ and $\mathscr{Y}$.

We now address the following question: What is the closure $\overline{E}$ of a set E in the (o)-topology? It is clear that $\overline{E}$ must contain all (o)-limits of generalized sequences in E. However, by simply adjoining these limits to E, we may fail even in obtaining a closed set. In order to construct the closure, we need to arrange the transfinite sequence of $E \subset E_0 \subset E_1 \subset \ldots \subset E_\alpha \subset \ldots$ by assembling in each $E_\alpha \quad (\alpha \ge 1)$ all limits of generalized sequences in $\bigcup_{\beta<\alpha} E_\beta$. If $\mathfrak{a}$ is the cardinality of the algebra $\mathscr{X}$ and $\overline{\omega}$ is the initial transfinite of cardinality at least $\mathfrak{a}$ then, as is easy to verify, the union $\bigcup_{\alpha<\overline{\omega}} E_\alpha$ is the closure of E with respect to the (o)-topology. The closure with respect to the (os)-topology can be described analogously; the only difference is that we use usual (rather

than generalized) sequences in this case. We call the so-constructed closures respectively the (o)- and (os)-*closures*.

The "constructive" description of a closure given above makes it possible to establish that *the (o)-closure of each subalgebra $\mathscr{X}_0$ is a subalgebra*. By Corollary 4, this closure coincides with the least regular subalgebra $\overline{\mathscr{X}\langle\mathscr{X}_0\rangle}$ including $\mathscr{X}_0$. It is clear that, for all $E \subset \mathscr{X}$, the (o)-closure of a subalgebra $\mathscr{X}\langle E\rangle$ is the regular subalgebra $\overline{\mathscr{X}\langle E\rangle}$ (which justifies the choice of notation on p. 85). In the sequel, the regular subalgebra $\overline{\mathscr{X}\langle\mathscr{X}_0\rangle}$ generated by a subalgebra $\mathscr{X}_0$ will be denoted by $\overline{\mathscr{X}_0}$.

The above construction of a closure can be clarified and simplified. It turns out that all problems concerning the order topology can be solved by using only STANDARD sequences indexed by a standard direction A_t, where t is the type of a complete Boolean algebra $\mathscr{X}$. Up to the end of this subsection we will assume that $\mathscr{X}$ is an infinite BA.

Lemma 4. *For each sequence $\{x_\gamma\}_{\gamma\in\Gamma}$ indexed by an arbitrary direction Γ and such that*

$$x_\gamma \xrightarrow{(o)} x,$$

there exists a generalized sequence of the form $\{x_{\gamma_\alpha}\}_{\alpha\in A_t}$ with the same (o)-limit.

PROOF. Put $u_\gamma \equiv \bigvee_{\delta\succ\gamma}|x - x_\delta|$. It is clear that $u_\gamma \downarrow \mathbf{0}$. Choose a subset Γ' in Γ of cardinality at most t so that the relation $\bigwedge_{\gamma\in\Gamma'} u_\gamma = \mathbf{0}$ holds. Let φ be a mapping from T onto Γ'. For each $\alpha \in A_t$ there is an index γ_α such that $\gamma_\alpha \succ \varphi(t)$ for all $t \in \alpha$ (we recall that α is a finite subset of T). Assign $\widetilde{u}_\alpha \equiv u_{\gamma_\alpha}$ for $\alpha \in A_t$. The inequality

$$\widetilde{u}_\alpha \le \bigwedge_{t\in\alpha} u_{\varphi(t)}$$

shows that $\widetilde{u}_\alpha \xrightarrow{(o)} \mathbf{0}$. Since $|x - x_{\gamma_\alpha}| \le \widetilde{u}_\alpha$ we have $x_{\gamma_\alpha} \xrightarrow{(o)} x$. The lemma is proved.

Corollary 1. *The (o)-closure $\overline{E}$ of an arbitrary set $E \subset \mathscr{X}$ can be represented as*

$$\overline{E} = \bigcup_{\xi\prec\eta} E_\xi,$$

where η is the least ordinal of cardinality at least $t(\mathscr{X})$, while $E_0 \equiv E$ and E_ξ consists of the (o)-limits of all sequences of the form

$$\{x_\alpha\}_{\alpha\in A_t}, \quad x_\alpha \in \bigcup_{\eta\prec\xi} E_\eta.$$

PROOF. Consider an arbitrary (o)-convergent sequence of the form $\{x_\alpha\}_{\alpha\in A_t}$ consisting of the elements of $B \equiv \bigcup_{\xi\prec\eta} E_\xi$. The entries of this sequence comprise a set of cardinality at most $t(\mathscr{X})$ but strictly less than η. Therefore, this set of entries lies in the union of the form $\bigcup_{\xi\prec\xi_0} E_\xi$ for some $\xi_0 \prec \eta$ (the cardinal $card\,\eta$ is regular). Hence (o)-$\lim x_\alpha \in E_{\xi_0} \subset B$. Thus, B is closed and $B \supset \overline{E}$. On the other hand, it is clear that $\overline{E} \supset B$. The corollary is proved.

Corollary 2. *In the same context,*

$$card\,\overline{E} \leq (card\,E)^{t(\mathscr{X})}.$$

Indeed, let $\mathfrak{a} \equiv card\,E$ and let t^+ be the successor of t. Show that

$$card\,E_\gamma \leq \mathfrak{a}^{t(\mathscr{X})}$$

for all $\xi \prec \eta$. This inequality holds for $\xi = 0$. Suppose that it is valid for all $\xi \prec \xi_0 \prec \eta$. Then

$$card\Big(\bigcup_{\eta\prec\xi_0} E_\eta\Big) \leq t\mathfrak{a}^t \leq \mathfrak{a}^t\mathfrak{a}^t = \mathfrak{a}^t.$$

Next,

$$card\,E_{\xi_0} \leq \Big[card\Big(\bigcup_{\eta\prec\xi_0} E_\eta\Big)\Big]^{card\,A_t} \leq (\mathfrak{a}^t)^t = \mathfrak{a}^{t^2} = \mathfrak{a}^t.$$

Therefore,

$$card\,\overline{E} \leq t^+\mathfrak{a}^t \leq 2^t\mathfrak{a}^t \leq \mathfrak{a}^t\mathfrak{a}^t = \mathfrak{a}^t.$$

We have obtained an estimate for the cardinality of the closure. Thereby (considering Corollary 4 to Theorem 5), we have proved Theorem 15 of Chapter 2.[4]

2.3 Interplay between the (o)- and (os)-topologies. The scale of order topologies

We have already noted that the (o)-topology is weaker than the (os)-topology. However these topologies may coincide. The full description of all cases is given by the following

Theorem 6. *For the* (o)- *and* (os)-*topologies to coincide in a complete Boolean algebra* $\mathscr{X}$, *it is necessary and sufficient that* $\mathscr{X}$ *be an algebra with the countable chain condition.*

---

[4]The weight of an INFINITE complete Boolean algebra is the least of the cardinalities of everywhere dense sets with respect to the (o)-topology. The proof of this fact is not difficult.

PROOF. NECESSITY. Assume that $\mathscr{X}$ does not satisfy the countable chain condition. Then there exists an uncountable disjoint set M in $\mathscr{X}$. Enumerate the elements of M by ordinal numbers, so obtaining the transfinite sequence $\{x_\alpha\}_{\alpha<\overline{\alpha}}$. Next, we put

$$y_\alpha \equiv \bigvee_{\beta \le \alpha} x_\beta.$$

Clearly, we find an increasing generalized sequence; the least upper bound (and, simultaneously, the (o)-limit) of this sequence we denote by y. Show that the set of entries of this sequence is closed with respect to the (os)-topology. Consider an arbitrary simple sequence $\{y_{\alpha_n}\}_{n=1}^\infty$ that (o)-converges to some $x \in \mathscr{X}$. It is clear that we can extract an INCREASING subsequence $\{\overline{y}_k \equiv y_{\alpha_{n_k}}\}_{k=1}^\infty$ with the same (o)-limit x. Denote by α_0 the successor of α_{n_k}. It is easy to establish that $x = y_{\alpha_0}$, which proves the closure of the set of entries with respect to the (os)-topology. The element y is not an entry of the generalized sequence $\{y_\alpha\}$, but is the (o)-limit of it. This means that the set of members is not closed with respect to the (o)-topology. So, in the case when the Boolean algebra $\mathscr{X}$ is not an algebra with the countable chain condition, the (o)-topology is essentially weaker.

SUFFICIENCY. Let $\mathscr{X}$ be a Boolean algebra with the countable chain condition and let F be a set closed with respect to the (os)-topology. Show that F is also closed with respect to the (o)-topology. Take an arbitrary generalized sequence $\{x_\alpha\}_{\alpha\in A}$ in F whose (o)-limits is x. As usual, we put

$$\overline{y}_\alpha \equiv \bigwedge_{\beta \succ \alpha} x_\beta, \quad \overline{z}_\alpha \equiv \bigvee_{\beta \succ \alpha} x_\beta.$$

Since the algebra satisfies the countable chain condition, there are two countable sequences of indices $\{\alpha'_n\}_{n=1}^\infty$ and $\{\alpha''_n\}_{n=1}^\infty$ such that

$$\bigvee_{n=1}^\infty y_{\alpha'_n} = \bigwedge_{n=1}^\infty z_{\alpha''_n} = x.$$

Arrange a sequence of indices $\alpha_1 \prec \alpha_2 \prec \dots$ so that

$$\alpha_n \succ \alpha'_1, \dots, \alpha'_n, \alpha''_1, \dots, \alpha''_n.$$

It is clear that

$$\bigwedge_{n=1}^\infty y_{\alpha_n} = \bigvee_{n=1}^\infty z_{\alpha_n} = x,$$

which implies that the SIMPLE sequence $\{x_{\alpha_n}\}_{n=1}^\infty$ has the (o)-limit x. Since F is closed with respect to the (os)-topology, x must belong to F.

We see that F is closed with respect to the (o)-topology too. The proof of the theorem is complete.

Remark. When proving the above theorem, we have established the following important fact: *in a complete Boolean algebra with the countable chain condition, to each (o)-convergent generalized sequence $\{x_\alpha\}_{\alpha\in A}$ there is an increasing sequence of indices $\{\alpha_n\}_{n=1}^\infty$ such that*

$$(o)\text{-}\lim_n x_{\alpha_n} = (o)\text{-}\lim_\alpha x_\alpha.$$

In the case of Boolean algebras with the countable chain condition, we saw that a key role is played by simple sequences. In this connection, we are interested in the theorem describing the important class of simple sequences having a topological limit. In the following theorem, $\mathscr{X}$ is an arbitrary Boolean algebra.

Theorem 7. *For a simple sequence $\{x_n\}_{n=1}^\infty$ to converge in the (os)-topology to an element $x \in \mathscr{X}$, it is necessary and sufficient that its each partial sequence*

$$\{x_{n_k}\}_{k=1}^\infty \quad (n_1 < n_2 < \dots < n_k < \dots)$$

contain a subsequence $\{x_{n_{k_i}}\}_{i=1}^\infty$ (o)-convergent to x.

Proof. Necessity. Let $x_n \longrightarrow x$; then, obviously, $x_{n_k} \longrightarrow x$. We may assume that $x_{n_k} \neq x$ for all k. Note first that we may extract an (o)-convergent subsequence from $\{x_{n_k}\}$. Otherwise the set of all x_{n_k} will be closed and its complement G will be a neighborhood of x that fails to contain infinitely many entries of the original sequence x_n, which is impossible. Let now $x_{n_{k_i}} \xrightarrow{(o)} y$, $k_1 < k_2 < \dots$. If we suppose that $x \neq y$ then the union of all $x_{n_{k_i}}$ and y will be (os)-closed and its complement G will be a neighborhood of x that fails to contain infinitely many entries x_n, which is impossible.

Sufficiency. Let G be an arbitrary neighborhood of x. Suppose that there is an infinite subsequence $\{x_{n_k}\}_{k=1}^\infty$ of G, where $n_1 < n_2 < \dots$. Extract a partial sequence $\{x_{n_{k_i}}\}_{i=1}^\infty$ $(k_1 < k_2 < \dots)$ (o)-convergent to x from this subsequence. The entries of the sequence $\{x_{n_{k_i}}\}_{i=1}^\infty$ with sufficiently large indices must belong to G, which contradicts our supposition. Thus, $x_n \longrightarrow x$. The proof of the theorem is complete.

Proceeding with a useful test for divergence of a simple sequence, we agree on writing $\overline{\lim}\, abs\, x_n = y$ whenever $\overline{\lim}\, x_{n_k} = y$ for every subsequence $\{x_{n_k}\}$, $n_1 < n_2 < \dots$ (D. A. Vladimirov [5]).

Lemma 5. *Let $\mathscr{X}$ be a complete Boolean algebra with the countable chain condition. For a sequence $\{x_n\}_{n=1}^\infty$ to fail to vanish in the order topology, it is necessary and sufficient that there exist a subsequence*

$\{x_{n_k}\}_{k=1}^{\infty}$ $(n_1 < n_2 < \ldots)$ *such that*

$$\overline{\lim}\, abs\, x_{n_k} > \mathbf{0}.$$

PROOF. Let $\{y_n\}$ be a sequence. Introduce an order on the set of all strictly increasing sequences of naturals $\tau = \{n_k^\tau\}_k$ by assuming $\tau' \gg \tau''$ whenever there exists $n_0 = n_0(\tau', \tau'')$ such that

$$\{n_k^{\tau'}\}_{k>n_0} \subset \{n_k^{\tau''}\}$$

and simultaneously

$$\overline{\lim}_k\, y_{n_k^{\tau'}} < \overline{\lim}_k\, y_{n_k^{\tau''}}.$$

Consider an arbitrary linearly ordered set T of such sequences. Put

$$z = \bigwedge_{\tau \in T} \overline{\lim}_k\, y_{n_k^\tau}.$$

Since the algebra $\mathscr{X}$ satisfies the countable chain condition, there are $\tau_1, \tau_2, \ldots, \tau_m, \ldots$ such that $\tau_m \in T$ and

$$z = \bigwedge_m \overline{\lim}\, y_{n_k^{\tau_m}}.$$

Since T is linearly ordered, for all m we can find the "greatest" sequence among $\tau_1, \tau_2, \ldots, \tau_m$ in the sense of our order. Denote this sequence by τ_m^* and assign

$$n_m \equiv n_m^{\tau_m^*} \quad (m = 1, 2, \ldots), \quad \tau_0 \equiv \{n_m\}_{m=1}^{\infty}.$$

It is easy to verify that either $\tau_0 \gg \tau$ for all $\tau \in T$ or there is a greatest element in T with respect to the order introduced. By the Kuratowski–Zorn Lemma, there exists a maximal element $\tau^* \equiv \{n_k^*\}$ in the class of all sequences. In particular, this means that

$$\overline{\lim}_k\, y_{n_k^*} = \overline{\lim}_s\, y_{n_{k_s}^*}$$

for all $k_1 < k_2 < \ldots$.

Consider now an arbitrary sequence $\{x_n\}$ that does not converge to $\mathbf{0}$. In this case, there exists a partial subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ from which it is impossible to extract a subsequence (o)-convergent to zero. Applying to $\{x_{n_k}\}_{k=1}^{\infty}$ the above reasoning, we obtain the desired subsequence $\{x_{n_{k_i}}\}$ for which $\overline{\lim}\, abs\, x_{n_{k_i}} > \mathbf{0}$. Thus we have proved the necessity of the condition; its sufficiency is obvious.

Lemma 4 shows that, in the formulation of the condition (o), we may imply only the standard generalized sequences indexed by a standard

direction A_t. We recall once again that the elements of A_t are all finite subsets of a set T of cardinality $t \equiv t(\mathscr{X})$. However, we can consider other cardinalities between $\aleph_0$ and $t(\mathscr{X})$ (if they exist). We thus obtain some scale of analogs of the condition (o) and the corresponding scale of order topologies, the weakest among which is the (o)-topology and the strongest, the (os)-topology. It is easy to understand that the last topology corresponds to the cardinality $\aleph_0$. This scale degenerates in the case of Boolean algebras with the countable chain condition which is of the utmost significance for us. For that reason, we will not continue studying the scale in question.

2.4 Topologies on bands. Examples

We now turn to the connection between the topologies on a Boolean algebra $\mathscr{X}$ and on the bands of $\mathscr{X}$. Let $\mathscr{X}_u$ stand for the principal ideal [5] generated by some $u \in \mathscr{X}^+$, on assuming $\mathscr{X}$ complete. This principal ideal is a Boolean algebra with unity u under the order induced from $\mathscr{X}$. We denote the order topology of this Boolean algebra by $(o)_u$. It is clear that $\mathscr{X}_u$ is a sublattice of $\mathscr{X}$ and, for every $E \subset \mathscr{X}_u$, all Boolean operations calculated in $\mathscr{X}_u$ coincide with those calculated in $\mathscr{X}$. This means in practice that the signs of the lattice operations ($\vee$, $\wedge$, $\overline{\lim}\ldots$, $\underline{\lim}\ldots$, $(o)\text{-}\lim\ldots$) can be used for the elements of $\mathscr{X}_u$ irrespectively of whether we imply the topologies of $\mathscr{X}$ or $\mathscr{X}_u$. Show that for each $u \in \mathscr{X}^+$ *the band* $\mathscr{X}_u$ *is always a closed subset of* $\mathscr{X}$ *with respect to the* (o)*-topology. Moreover, the topology* $(o)_u$ *on* $\mathscr{X}_u$ *coincides with the topology induced by the* (o)*-topology* τ *on* $\mathscr{X}$.

PROOF. The closure of $\mathscr{X}_u$ follows immediately from the Corollary 2 to Theorem 5. We note next that, since $\mathscr{X}_u$ is closed in $\mathscr{X}$, the τ-closure of $E \subset \mathscr{X}_u$ is equivalent to the closure of E in $\mathscr{X}$ with respect to the (o)-topology. For the (o)-closure of E, it is necessary and sufficient that E contain the (o)-limits of all generalized sequences in E. The same condition is necessary and sufficient for the closure of E with respect to the $(o)_u$-topology. Thus, the totalities of closed sets in $\mathscr{X}_u$ with respect to the $(o)_u$- and τ-topologies are the same; therefore, these topologies coincide. An analogous assertion is valid for the (os)-topology.

In the above presentation (see p. 53), we associate with each band $\mathscr{X}_u$ the band projection P_u. We have already proved (see p. 193) that P_u is continuous as an operator from $\mathscr{X}$ into $\mathscr{X}$. Now we can assert that it is continuous as an operator from $\mathscr{X}$ into $\mathscr{X}_u$ (we mean continuity with respect to the order topology).

[5] We recall that the concepts of band and principal ideal coincide in a complete Boolean algebra.

Consider a simple example. Let $\mathscr{X} = 2^Q$, where $Q \neq \varnothing$. It is easy to verify that the (o)-convergence and convergence with respect to the (o)-topology coincide and have the following simple meaning: a sequence $\{e_\alpha\}$ converges to e if and only if the corresponding characteristic functions converge at every point ($\chi_{e_\alpha}(q) \longrightarrow \chi_e(q)$ for all $q \in Q$). In fact, the (o)-topology of this Boolean algebra is a Tychonoff topology. Namely, $\mathscr{X}$ is isomorphic to the Boolean algebra of characteristic functions and the latter is the cartesian Qth power of the discrete two-point space. The Tychonoff topology of this power is the (o)-topology. Therefore, the Tychonoff topology is compact (such a situation is rather rare and is explained by the fact that $\mathscr{X}$ is a discrete Boolean algebra).

We leave to the reader the proof of the following proposition: if $\mathscr{X}_0$ is an algebra of subsets of Q which separates the points of Q then $\overline{\mathscr{X}_0} = \mathscr{X} \equiv 2^Q$.

In conclusion, we consider a more general situation. Let a complete Boolean algebra $\mathscr{X}$ be the DIRECT SUM of the form

$$\mathscr{X} = \bigoplus_{u \in U} \mathscr{X}_u,$$

where U is an arbitrary disjoint set of nonzero elements and the bands $\mathscr{X}_u$ form a disjoint decomposition of $\mathscr{X}$. Induce the $(o)_u$-topology in each band $\mathscr{X}_u$. By "taking the product" of these bands, we can introduce the Tychonoff topology τ on $\mathscr{X}$. The convergence of a generalized sequence $\{x_\alpha\}$ with respect to this topology means "convergence in each band." In other words, $x_\alpha \longrightarrow x$ with respect to the topology τ if and only if

$$u \wedge x_\alpha \longrightarrow u \wedge x$$

for all $u \in U$ (with respect to the $(o)_u$-topology).

The topology τ is always weaker than the (o)-topology of the Boolean algebra $\mathscr{X}$. It can be essentially weaker in the nondiscrete case even if $U = \{u_1, u_2\}$. The corresponding example was exhibited by A. V. Potepun [3] (the Boolean algebra $\mathscr{O}$ of regular open sets).

2.5 Various order compatible topologies

In addition to the above introduced two topologies, some other topologies can be considered on Boolean algebras. We now show that, for the algebras of the utmost importance, *only the (o)-topology is "sensible" as regards compatibility with order.*

We start with outlining the class of topologies which can be naturally regarded as well compatible with order.

We have already noted that each Boolean algebra $\mathscr{X}$ is an abelian group under the binary operation $x, y \longrightarrow |x - y|$. It is natural to

become interested first in those topologies on $\mathscr{X}$ under which the group operation is continuous or, in other words, with respect to which $\mathscr{X}$ is a *topological group.* The identity

$$|x-y| \equiv [Cx \wedge y] \vee [x \wedge Cy]$$

shows that to this end it suffices to require the continuity of the main Boolean operations $\vee, \wedge$, and C. Each topology on $\mathscr{X}$ making the operations $\vee, \wedge$, and C continuous[6] is called a *uniform topology.*

The choice of this term is connected with the fact that in this case all topological facts can be described in terms of the neighborhoods of ZERO; each neighborhood of an arbitrary element is obtained (as always in groups) from a neighborhood of zero by translation. The system $\mathfrak{V}$ of all neighborhoods of zero generates a "uniform structure" on $\mathscr{X}$ in the sense of the definition by A. Weyl. In order to obtain Weyl's "entourages" or "encirclements of the diagonal," we must consider all sets of the form

$$E_V \equiv \{(x,y) \mid \ |x-y| \in V\},$$

where $V \in \mathfrak{V}$ is a neighborhood of zero.

Notice that the (os)-topology usually fails to be uniform (S. A. Malyugin [1]).

Another important feature of a topology is its MONOTONICITY. We call a uniform topology on a Boolean algebra *monotone* whenever for each neighborhood V of a point $x \in \mathscr{X}$ there exists a neighborhood $W \subset V$ of the same point with the following property: for all $x' \in W$, the inequality $|x''-x| \leq |x'-x|$ implies $x'' \in W$. To define a monotone topology on $\mathscr{X}$ means to select a class $\mathfrak{V}$ of nonempty sets with the properties:

1) *every* $V \in \mathfrak{V}$ *is solid;*
2) *for all* $V_1, V_2 \in \mathfrak{V}$ *there is* $V \in \mathfrak{V}$ *such that* $V \subset V_1 \cap V_2$*;*
3) *for each* $V \in \mathfrak{V}$ *there exists* $V' \in \mathfrak{V}$ *such that* $V' \vee V' \subset V$.

The system $\mathfrak{V}$ is a base for the filter of neighborhoods about zero; a base for the neighborhoods about another point x is constituted by the sets of the form $x +_2 V$, with $V \in \mathfrak{V}$.[7] The resulting topology is indeed monotone; the verification of the axioms I and II (see Appendix A) is trivial.

The formulas

$$|x-y| = |Cx - Cy|, \quad |x \wedge y - x' \wedge y'| \leq |x-x'| \vee |y-y'|,$$

[6] Clearly, it suffices to require continuity of $\vee, C$ or $\wedge, C$.

[7] We recall that the sign $+_2$ is another notation for the symmetric difference. The symbol $x +_2 V$ denotes the set that contains all elements of the form $x +_2 y \equiv |x-y|$, with $y \in V$.

$$|x \vee y - x' \vee y'| \le |x - x'| \vee |y - y'|$$

show the continuity of operations $\vee$, $\wedge$, and C with respect to the topology defined by the system $\mathfrak{V}$. Therefore, this topology is uniform. Moreover, the condition 3) together with the above inequalities ensures the uniform continuity of these operations in the sense of the theory of uniform spaces.

Finally, wishing to obtain a SEPARATED topology, we must require that the following condition holds:

4) $\bigcap_{V \in \mathfrak{V}} V = \{\mathbf{0}\}$.

The condition 4) does ensure the separatedness of the topology. It is sufficient to establish that $\mathbf{0}$ and $u > \mathbf{0}$ can always be separated. Let $V \in \mathfrak{V}$, $u \notin V$ (such a neighborhood V always exists by the condition 4)). Applying 3), choose a neighborhood $V' \in \mathfrak{V}$ such that $V' \vee V' \subset V$ and put $W \equiv u +_2 V'$. If we suppose that $x \in V' \cap W$ then $u \le |u - x| \vee x \in V$ because $x \in V'$ and the containment $x \in W$ means that $|u - v| \in V'$. However, V is solid; therefore, $u \in V$ despite the supposition. Thus, we have constructed some neighborhoods of $\mathbf{0}$ and u that satisfy the Hausdorff axiom.

We formulate one more condition that is enjoyed by the bases of many important topologies:

5) *each set $V \in \mathfrak{V}$ contains the (o)-limits of all increasing sequences*[8] ("monotone closure" of a neighborhood of zero). Sometimes, this condition is used in a weakened "sequential" form:

5′) *each set $V \in \mathfrak{V}$ contains the (o)-limits of all increasing simple sequences.*

A system $\mathfrak{V}$ satisfying the conditions 1)–4) will be called a *uniformity base*; and the topology of $\mathscr{X}$ generated by the base $\mathfrak{V}$ will be called the $(\mathfrak{V})$-*topology*. A uniform topology with the properties 1)–4) can be considered in an incomplete Boolean algebra. The properties 5) and 5′) assume respectively the completeness and σ-completeness of a Boolean algebra.

In the sequel, the term "uniform topology" is mostly referred to a $(\mathfrak{V})$-topology with the properties 1)–4) (but not necessarily 5)). We point out a few general facts related to the topologies. Convergence with respect to $(\mathfrak{V})$-topology we denote by the symbols $\xrightarrow{(\mathfrak{V})}$ and $(\mathfrak{V})\text{-}\lim$.

1°. We first note that the *continuity of lattice operations implies the closure of disjoint complements (bands) and principal ideals and the*

[8]We mean GENERALIZED sequences. We could consider only transfinite sequences; however, there is no use in this restriction in the case of SOLID sets $V \in \mathfrak{V}$.

possibility of passage to the limit in the relation of the form

$$x_\alpha \le y_\alpha, \quad y_\alpha = Cx_\alpha, \quad y_\alpha \, d \, x_\alpha.$$

2°. Let $\mathscr{X}_u$ be a principal ideal. Then the sets $V_u \equiv \mathscr{X}_u \cap V \quad (V \in \mathfrak{V})$ make a uniformity base on $\mathscr{X}_u$ that inherits the properties 1)–4) and 5) (if it is valid) from $(\mathfrak{V})$. Thus, $\mathscr{X}_u$ is endowed with the induced uniform topology. A base of the induced topology is denoted by $\mathfrak{V}_u$. In this case, $V_u = u \wedge V = \{u \wedge x \mid x \in V\}$.

Consider the band projection P_u, i.e., $P_u(x) \equiv x \wedge u$. It is clear that P_u is continuous both as an operator from $\mathscr{X}$ into $\mathscr{X}$ and as an operator from $\mathscr{X}$ into $\mathscr{X}_u$ (with respect to the corresponding uniform topologies).

3°. We may furnish the disjoint decomposition

$$\mathscr{X} = \bigoplus_{u \in U} \mathscr{X}_u \qquad \Bigl(\sum_{u \in U} u = \mathbf{1}\Bigr),$$

with the "Tychonoff" uniformity yielding the weakest topology with respect to which all band projections P_u are continuous if the bands $\mathscr{X}_u$ are equipped with $(\mathfrak{V}_u)$-topologies. As a base of the Tychonoff uniformity we take the system of all sets of the form

$$V' = \bigcap_{k=1}^{m} P_{u_k}^{-1}\Bigl(V_{u_k}^{(k)}\Bigr) \equiv \{x \mid x \wedge u_1 \in V_{u_1}^{(1)}, \dots, x \wedge u_m \in V_{u_m}^{(m)}\},$$

where

$$m = m(V'); \quad V^{(1)}, V^{(2)}, \dots, V^{(m)} \in \mathfrak{V}; \quad u_1, u_2, \dots, u_m \in U.$$

This uniform topology is related to $\mathfrak{V}$ and U; we denote the above base of it by the symbol $\mathfrak{V}'_{[U]}$ or simply by $\mathfrak{V}'$ if no ambiguity is possible. Comparing the $(\mathfrak{V})$- and $(\mathfrak{V}')$-topologies, we notice first that *the former is always stronger than the latter*. If U is finite then both topologies coincide. In terms of convergence the situation can be described as follows: the $(\mathfrak{V}')$-convergence is a "local" $(\mathfrak{V})$-convergence. The relation $x_\alpha \xrightarrow{(\mathfrak{V}')} x$ means exactly that, for all $u \in U$, we have

$$u \wedge x_\alpha \xrightarrow{(\mathfrak{V})} u \wedge x.$$

This convergence follows from the $(\mathfrak{V})$-convergence; the converse may fail in general.

Consider an EXAMPLE. Let a complete Boolean algebra be of the form

$$\mathscr{X} = \bigoplus_{u \in U} \mathscr{X}_u$$

and let each band $\mathscr{X}_u$ be equipped with a probability measure μ_u.[9] Further, let m be a finite nonzero quasimeasure on the algebra 2^U of all subsets of U. The function m is called a "measure" [10] in the traditional terminology; every bounded real function can be integrated with respect to m. Define a uniformity base as follows. We first introduce two types of neighborhoods of zero.

1) Let $\mathfrak{V}_1$ be the system of all sets of the form

$$\{x \mid \mu_{u_1}(x \wedge u_1) < \varepsilon, \mu_{u_2}(x \wedge u_2) < \varepsilon, \dots, \mu_{u_k}(x \wedge u_k) < \varepsilon\}$$
$$(u_1, u_2, \dots, u_k \in U,\ \varepsilon > 0).$$

2) Let $\mathfrak{V}_2$ be the system of all sets of the form

$$\Big\{x \mid \int\limits_U \mu_u(x \wedge u)\, dm_u < \delta\Big\} \quad (\delta > 0).$$

Constitute a uniformity base $\mathfrak{V}$ by all sets of the form $V_1 \cap V_2$, where $V_1 \in \mathfrak{V}_1$ and $V_2 \in \mathfrak{V}_2$.

Verification of the properties 1)–4) is easy; and so we have indeed constructed a uniformity base. Much depends on the properties of the quasimeasure m. We point out several important special cases.

1) If the quasimeasure m is concentrated in a finite subset $U' \subset U$ then the base can be constituted only by the sets of the first type: $\mathfrak{V}_1$ and $\mathfrak{V}$ generate the same uniform topology that also coincide with the $(\mathfrak{V}')$-topology.

2) Consider a more interesting case in which m is equal to zero at every finite (in particular, one-point) subset of U. Then for all $u \in U$ and $V \in \mathfrak{V}_2$ we have

$$V_u = P_u(V) = \mathscr{X}_u, \quad P_u^{-1}(V_u) = \mathscr{X}.$$

That is why the $(\mathfrak{V}')$-topology coincides in fact with the $(\mathfrak{V}_1)$-topology; the neighborhoods of the second type are not used in its construction.

Let A be the totality of all finite subsets of U which is naturally ordered and directed upward. Consider a generalized sequence

$$x_\alpha \equiv \bigvee_{u \in U \setminus \alpha} u \quad (\alpha \in A).$$

It is clear that

$$u \wedge x_\alpha \downarrow \mathbf{0}, \quad \mu_u(u \wedge x_\alpha) \to 0$$

---

[9] I.e., $\mu_u u = 1$.

[10] Not necessarily a countably additive function.

for all $u \in U$, i.e., $(\mathfrak{V}')$- $\lim x_\alpha = \mathbf{0}$.

At the same time,

$$\int\limits_U \mu_u(x_\alpha \wedge u)\, dm_u = \int\limits_{U\setminus\alpha} \mu_u(u)\, dm_u = m(U \setminus \alpha) = mU > \mathbf{0},$$

which means that $x_\alpha \overset{(\mathfrak{V})}{\not\longrightarrow} \mathbf{0}$. We see that the $(\mathfrak{V})$-topology is essentially stronger than the "Tychonoff" topology.

As proven in the general theory of uniform spaces, each uniformity can be determined from a set of pseudometrics.[11] We now establish this fact in a form adapted to Boolean algebras. So, let $\mathfrak{V}$ be a uniformity base with the properties 1)–4) in a Boolean algebra $\mathscr{X}$. We say that a sequence $\{V_n\}_{n=0}^{\infty}$ of neighborhoods of zero in $\mathfrak{V}$ is "main" whenever it possesses the following properties:

1) $V_0 = \mathscr{X}$; 2) $V_{n+1} \vee V_{n+1} \vee V_{n+1} \subset V_n$ for all $n = 0, 1, \dots$.

Such a sequence exists in view of the property 3). We point out some obvious properties of these sequences:

1°. $V_0 \supset V_1 \supset \dots$;

2°. For each neighborhood of zero $V \in \mathfrak{V}$, there is a main sequence $\{V_n\}$ satisfying $V_1 \subset V$.

A finite real function θ on $\mathscr{X}$ is called an *outer quasimeasure* whenever θ satisfies the following conditions for all $x, y \in \mathscr{X}$:

$\langle 1\rangle$ $\theta(x) \geq 0$; $\theta(\mathbf{0}) = 0$;

$\langle 2\rangle$ if $x \leq y$ then $\theta(x) \leq \theta(y)$;

$\langle 3\rangle$ $\theta(x \vee y) \leq \theta(x) + \theta(y)$.

We call a system $\mathfrak{M}$ of outer quasimeasures *sufficient* whenever $\mathfrak{M}$ possesses the properties:

a) $\sup_{\theta\in\mathfrak{M}} \theta(x) > 0$ for all $x > \mathbf{0}$,

b) for all $\theta_1, \theta_2 \in \mathfrak{M}$, there are $\theta \in \mathfrak{M}$ and numbers $\alpha_1, \alpha_2 > 0$ such that

$$\theta(x) \geq \alpha_1\theta_1(x) + \alpha_2\theta_2(x) \quad (x \in \mathscr{X}).$$

As is easy to see, a sufficient system of outer quasimeasures generates a uniformity on $\mathscr{X}$ whose base is constituted by the "balls" of the form

$$K_{\theta,r} \equiv \{x \mid \theta(x) \leq r\} \quad (r > 0).$$

[11] For instance, see J. Kelley [2].

Obviously, each outer quasimeasure $\theta \in \mathfrak{M}$ is continuous with respect to this uniformity (and even uniformly continuous). Moreover, the balls $K_{\theta,\frac{1}{3^n}}$ constitute a main sequence of basis neighborhoods of zero. Now, we return to the original base $\mathfrak{V}$ and extract from it an arbitrary main sequence $\{V_n\}_{n=0}^{\infty}$. Show that *there exists an outer quasimeasure θ on $\mathscr{X}$ related to the sequence $\{V_n\}_{n=0}^{\infty}$ by the conditions*

$$V_n \subset K_{\theta,\frac{1}{2^n}} \subset V_{n-1} \quad (n = 1, 2, \ldots). \tag{8}$$

Introduce the real function h as follows:

$$h(x) = \begin{cases} \dfrac{1}{2^n}, & \text{if} \quad x \in V_n \setminus V_{n+1} \quad (n = 0, 1, 2, \ldots), \\ 0, & \text{if} \quad x \in \bigcap\limits_{n=0}^{\infty} V_n. \end{cases}$$

It is clear that the function is defined everywhere on $\mathscr{X}$, finite, positive, and monotone: if $x \le y$ then $h(x) \le h(y)$. The latter ensues from the solidity of the neighborhoods V_n.

We now put

$$\theta(x) \equiv \inf \sum_{k=1}^{m} h(x_k),$$

where the infimum is taken over all finite sets $\{x_1, \ldots, x_m\}$ such that $x \le \bigvee_{k=1}^{m} x_k$. It is clear that $\theta(\mathbf{0}) = 0$.

Show first that θ is an outer quasimeasure. Positivity and monotonicity are obvious. Let $z = x \vee y$. Given an arbitrary $\varepsilon > 0$, find $u_1, \ldots, u_p, v_1, \ldots, v_q$ so that

$$x \le \bigvee u_i, \;\; y \le \bigvee v_j, \;\; \sum h(u_i) < \theta(x) + \varepsilon, \;\; \sum h(v_j) < \theta(y) + \varepsilon.$$

It is clear that $z \le u_1 \vee \cdots \vee u_p \vee v_1 \vee \cdots \vee v_q$. Therefore,

$$\theta(z) \le h(u_1) + \cdots + h(v_q) < \theta(x) + \theta(y) + 2\varepsilon.$$

Since ε is arbitrary, we conclude that $\theta(z) \le \theta(x) + \theta(y)$. Thus, θ is an outer quasimeasure. We now proceed to the main part of the proof.

We need to prove the inclusions (8). To this end, we establish the inequality

$$h(x_1 \vee x_2 \vee \cdots \vee x_m) \le 2 \sum_{i=1}^{m} h(x_i) \quad (m = 1, 2, \ldots). \tag{9}$$

We proceed by induction. The case $m = 1$ is clear. Assume the inequality (9) proven for all $k < m$ (i.e., for every set $\{x_1, x_2, \ldots, x_k\}$,

$k < m$). Taking $\{x_1, x_2, \ldots, x_m\}$ arbitrarily, put $s \equiv \sum_{i=1}^{m} h(x_i)$ and choose i_0 so that

$$\sum_{i=1}^{i_0-1} h(x_i) \le \frac{s}{2}, \quad \sum_{i=i_0+1}^{m} h(x_i) \le \frac{s}{2}.$$

By the inductive hypothesis we have

$$\begin{aligned} h(x_1 \vee x_2 \vee \cdots \vee x_{i_0-1}) &\le 2\sum_{i=1}^{i_0-1} h(x_i) \le 2\frac{s}{2} = s, \\ h(x_{i_0+1} \vee \cdots \vee x_m) &\le 2\sum_{i=i_0+1}^{m} h(x_i) \le 2\frac{s}{2} = s, \\ h(x_{i_0}) &\le s. \end{aligned} \tag{10}$$

Denote by n a natural number with the property $\frac{1}{2^n} \le s < \frac{1}{2^{n-1}}$. The inequalities (10) show that

$$h(x_1 \vee x_2 \vee \cdots \vee x_{i_0-1}) < \frac{1}{2^{n-1}}, \; h(x_{i_0+1} \vee \cdots \vee x_m) < \frac{1}{2^{n-1}}, h(x_{i_0}) < \frac{1}{2^{n-1}},$$

whence

$$h(x_1 \vee x_2 \vee \cdots \vee x_{i_0-1}) \le \frac{1}{2^n}, \quad h(x_{i_0+1} \vee \cdots \vee x_m) \le \frac{1}{2^n}, \quad h(x_{i_0}) \le \frac{1}{2^n}$$

and

$$x_1 \vee x_2 \vee \cdots \vee x_{i_0-1} \in V_n, \quad x_{i_0+1} \vee \cdots \vee x_m \in V_n, \quad x_1 \vee \cdots \vee x_m \in V_{n-1}.$$

Then

$$h(x_1 \vee \cdots \vee x_m) \le \frac{1}{2^{n-1}} = \frac{2}{2^n} \le 2s,$$

which is what we need. Next, let $\theta(x) \le \frac{1}{2^n}$. For each $\varepsilon > 0$, there is a set $\{x_1, \ldots, x_m\}$ with the properties

$$x \le x_1 \vee \cdots \vee x_m, \quad \sum_{i=1}^{m} h(x_i) < \frac{1}{2^n} + \varepsilon.$$

Then

$$h(x_1 \vee \cdots \vee x_m) \le 2\sum_{i=1}^{r} h(x_i) < \frac{1}{2^{n-1}} + 2\varepsilon$$

and $h(x) \le \frac{1}{2^{n-1}}$, i.e., $x \in V_{n-1}$. We have the inclusion

$$K_{\theta, \frac{1}{2^n}} \subset V_{n-1}.$$

On the other hand, it is clear from the definition of θ that $\theta(x) \leq h(x)$. Therefore, if $x \in V_n$ then

$$h(x) \leq \frac{1}{2^n}, \quad \theta(x) \leq \frac{1}{2^n}, \quad x \in K_{\theta,\frac{1}{2^n}}.$$

Thus, the inclusions (8) are proven.

So, each main sequence $\gamma = \{V_n\}$ ($V_n \in \mathfrak{V}$) generates an outer quasimeasure $\theta \equiv \theta_\gamma$ related to the sequence by the conditions (8). Thus, a sufficient system of outer quasimeasures appears; moreover, the balls $K_{\theta,r}$ constitute a new base of the ($\mathfrak{V}$)-topology. The system of these balls obviously possesses the properties 1)–4). Moreover, the balls are closed with respect to the uniform topology. If $\mathfrak{V}$ possesses also the property 5) then the balls have this property in part. Namely, if $\theta = \theta_\gamma$, then the (o)-limit of each increasing sequence in the ball $K_{\theta,\frac{1}{2^n}}$ always belongs to $K_{\theta,\frac{1}{2^{n-1}}}$ for all $n \geq 1$. This fact follows from the inclusions

$$K_{\theta,\frac{1}{2^n}} \subset V_{n-1} \subset K_{\theta,\frac{1}{2^{n-1}}}.$$

(We recall that V_{n-1} are monotonically closed.)

We can, if need be, replace the original base $\mathfrak{V}$ with the new base constituted by the balls $K_{\theta,\frac{1}{2^n}}$. However, we prefer using both new and old sets. The overall conclusion matters:

Each uniform topology on a Boolean algebra $\mathscr{X}$ is generated by a family of outer quasimeasures (this fact is also true for incomplete Boolean algebras).

Let $\gamma \equiv \{V_n\}$ be a main sequence and let $\theta \equiv \theta_\gamma$ be the corresponding outer quasimeasure. The inclusions

$$\ldots \subset V_n \subset K_{\theta,\frac{1}{2^n}} \subset V_{n-1} \subset \ldots$$

show that the intersection $\bigcap_n K_{\theta,\frac{1}{2^n}}$ coincides with the intersection $\bigcap_n V_n$. Denote the former by $\Omega \equiv \Omega_\gamma$ and list a few main properties of this set.

1°. Ω is a ($\mathfrak{V}$)-*closed set* (as an intersection of closed balls).

2°. Ω is an *ideal*. Indeed, the set is solid as an intersection of solid sets; and if $x, y \in \Omega$ then $x, y \in V_{m+1}$ and $x \vee y \in V_m$ for all $m = 1, 2, \ldots$. Therefore, $x \vee y \in \Omega$ and Ω is an ideal.

3°. *If the algebra $\mathscr{X}$ is complete and the base $\mathfrak{V}$ possesses the property* 5) *then Ω is a principal ideal.*

Proof. Let $v = \sup \Omega$. Show that $v \in \Omega$. Enumerate all elements of Ω, which yields a transfinite sequence $\{x_\alpha\}_{\alpha \leq \overline{\alpha}}$, where $\overline{\alpha}$ is an ordinal of a proper cardinality. Put $v_\alpha \equiv \bigvee_{\beta \leq \alpha} x_\beta$. Suppose that $v = v_{\overline{\alpha}} \notin \Omega$.

Then the set $\{\alpha \mid v_\alpha \notin \Omega\}$ is nonempty; denote the least element of it by $\overline{\overline{\alpha}}$. The following alternative is open:

a) $\overline{\overline{\alpha}}$ is a nonlimit ordinal of the form $\overline{\overline{\alpha}} = \alpha' + 1$. Then $v_{\overline{\overline{\alpha}}} = v_{\alpha'} \vee x_{\alpha'+1} \in \Omega$, because Ω is an ideal. Therefore, the case a) is impossible.

b) $\overline{\overline{\alpha}}$ is a limit ordinal. Put

$$\widetilde{v} \equiv \bigvee_{\alpha < \overline{\overline{\alpha}}} x_\alpha = \bigvee_{\alpha < \overline{\overline{\alpha}}} v_\alpha.$$

The sequence $\{v_\alpha\}_{\alpha < \overline{\overline{\alpha}}}$ increases; its members belong to each V_n ($n = 1, 2, \ldots$). According to the property 5), we have $\widetilde{v} \in V_n$ ($n = 1, 2, \ldots$) and $\widetilde{v} \in \Omega$; but then $v_{\overline{\overline{\alpha}}} = \widetilde{v} \vee x_{\overline{\overline{\alpha}}} \in \Omega$. So, the case b) is also impossible, which means that $v \in \Omega$. Therefore, Ω is a principal ideal.

4°. Another case in which Ω is a principal ideal in a complete Boolean algebra can be obtained by replacing the property 5) by the condition (o): *if the* $(\mathfrak{V})$*-topology satisfies the condition* (o) *then the set* Ω *is a principal ideal.* Indeed, if $x_\alpha \uparrow v$ then this generalized sequence is also $(\mathfrak{V})$-convergent. Since Ω is closed with respect to the $(\mathfrak{V})$-convergence, we conclude that $v \in \Omega$. We also note that the condition (o) can be replaced by the condition (os) in a Boolean algebra with the countable chain condition.

The case is most important and interesting in which Ω_γ consists only of zero: $\Omega_\gamma = \{\mathbf{0}\}$. Such a situation occurs when the sets V_n comprising the main sequence γ separate the elements of $\mathscr{X}^+$ from zero. The same situation can be described by the following phrase: θ_γ is an *essentially positive outer quasimeasure* i.e., $x > \mathbf{0}$ implies $\theta_\gamma(x) > 0$. We consider this case in more detail.

Let $\mathscr{X}$ be a σ-complete Boolean algebra; let $\mathfrak{V}$ be a uniformity base with the properties 1)–4); let Γ be the set of all main sequences $\gamma = \{V_n\}$ ($V_n \in \mathfrak{V}$); and let $\{\theta_\gamma\}$ be the family of the corresponding outer quasimeasures (we saw that this family generates the $(\mathfrak{V})$-topology). Under the above assumptions the following theorem is valid.

Theorem 8. I. *If the condition* 5′) *holds and there is an essentially positive outer quasimeasure among the outer quasimeasures* θ_γ *(i.e., there exists a main sequence* γ_0 *with zero intersection) then the* $(\mathfrak{V})$*-topology of* $\mathscr{X}$ *is stronger than the* (os)*-topology.*

II. *If, moreover, the condition* (os) *holds then the* $(\mathfrak{V})$-, (o)- *and* (os)-*topologies coincide and the Boolean algebra* $\mathscr{X}$ *satisfies the countable chain condition (and therefore* $\mathscr{X}$ *is complete).*

PROOF. Let $\gamma_0 = \{V_n\}_{n=0}^\infty$ be a main sequence with zero intersection: $\bigcap_n V_n = \{\mathbf{0}\}$. It is not difficult to understand that this sequence (as well

as every main sequence) possesses the property

$$V_{m+1} \vee V_{m+2} \vee \cdots \vee V_{m+k} \subset V_m$$

for all m and k. It follows that *if $x_m \in V_m$ for all sufficiently large m then $x_m \xrightarrow{(o)} \mathbf{0}$*. Indeed, we have

$$\bigvee_{i=m+1}^{\infty} x_i = (o)\text{-}\lim \bigvee_{i=M=1}^{n} x_i \in V_m$$

by the condition 5′). Next, using the solidity of V_m, we obtain

$$\overline{\lim}\, x_i = \bigwedge_m \bigvee_{m+1}^{\infty} x_i \in V_m,$$

which holds for all m. Therefore, $\overline{\lim}\, x_n = \mathbf{0}$ and $x_n \xrightarrow{(o)} \mathbf{0}$.

Take now an arbitrary set F closed with respect to the (os)-topology. Let $x \equiv (\mathfrak{V})\text{-}\lim_\alpha x_\alpha$, where $x_\alpha \in F$. There are indices $\alpha_1, \alpha_2, \dots$ such that $|x - x_\alpha| \in V_n$. By the above remark, $x_{\alpha_n} \xrightarrow{(o)} x$. Therefore, $x \in F$ in view of the (os)-closure of F. Thus, F contains the $(\mathfrak{V})$-limits of all generalized sequences in F. Therefore, F is $(\mathfrak{V})$-closed. The $(\mathfrak{V})$-topology has more closed sets than the (os)-topology; consequently, it is stronger. Part I of the theorem is proved.

II. Validity of the condition (os), together with I, implies that the $(\mathfrak{V})$- and (os)-topologies coincide. We now prove that this algebra satisfies the countable chain condition. This will imply coincidence of both order topologies as well as the completeness of the algebra.

Consider an arbitrary transfinite disjoint sequence $\{x_\alpha\}_{\alpha<\omega_1}$, where ω_1 is the first uncountable ordinal. Suppose that there are uncountably many nonzero entries among x_α. All nonzero elements of $\mathscr{X}$ belong to the sets $\mathscr{X} \setminus V_n$ $(n = 1, 2, \dots)$; therefore, there is an n_0 such that V_{n_0} does not contain infinitely many entries of the sequence. Take a simple sequence $\{x_{\alpha_k}\}_{k=1}^{\infty}$ of those entries. Since $\{x_{\alpha_k}\}_{k=1}^{\infty}$ is disjoint, it must (o)-converge to zero. Then, by the property (os), the entries of the sequence must belong to V_{n_0}, contradicting the supposition. Therefore, the sequence $\{x_\alpha\}$ cannot have uncountably many nonzero entries. So $\mathscr{X}$ satisfies the countable chain condition.

Remark 1. In part I of the theorem, we have proved more than claimed. Namely, it was proved that *the (os)-topology is majorized by the topology with base γ_0* which is weaker than the $(\mathfrak{V})$-topology in general.

REMARK 2. *In part* II *of the theorem, the condition* $5'$) *is superfluous.* Indeed, we can employ only the condition (os) as follows. Instead of V_n, we may use the balls $K_{\theta_\gamma, \frac{1}{2^n}}$ which are closed with respect to the $(\mathfrak{V})$-topology. By the condition (os), these balls are monotonically closed, i.e., possess the property $5'$).

REMARK 3. *The countable chain condition in part II is proved with the aid of the weak form of the condition* (os)*: "every simple disjoint sequence converges to zero."*

An outer quasimeasure θ will be called an *outer measure* whenever θ is essentially positive and (o)-continuous: $\theta(x) > 0$ for $x > \mathbf{0}$ and $x_n \downarrow \mathbf{0}$ implies $\theta(x_n) \longrightarrow 0$.

Thus, the outer quasimeasure in part II of Theorem 8 is an outer measure. In fact, we have proved the following fact: *If the family* $\{\theta_\gamma\}$ *corresponding to the base* $\mathfrak{V}$ *contains at least one "genuine" outer measure* $\theta^0 \equiv \theta_{\gamma_0}$ *and the condition* (os) *is valid then* θ_{γ_0} *defines the* $(\mathfrak{V})$*-topology that coincides with the* (os)*-topology and is metrizable by the metric*

$$\rho : \quad \rho(x, y) \equiv \theta^0(|x - y|).$$

The main sequence $\gamma_0 = \{V_n^0\}$ in this case is a base of the $(\mathfrak{V})$-topology; this base is countable. The remaining elements of the base $\mathfrak{V}$ and the remaining quasimeasures of the family $\{\theta_\gamma\}$ are not needed for defining this topology. In particular, *if there is at least one outer measure* θ^0 *on a complete*[12] *Boolean algebra* $\mathscr{X}$ *then this algebra satisfies the countable chain condition, while the order topology is metrizable and coincides with the* $(\mathfrak{V})$*-topology defined by* θ^0*. The base of the order topology consists of the balls* $K_{\theta^0, r}$*. Every two outer measures on a complete Boolean algebra are topologically equivalent; i.e., they define the same metric topology that coincides with the order topology.* In fact, we reformulated part II of Theorem 8 with Remark 2 taken into account.

In the above reasoning, we assumed that the family $\{\theta_\gamma\}$ contains an essentially positive outer quasimeasure. Below, we discuss the situation in which this assumption is not valid. Let $\mathscr{X}$ be a complete Boolean algebra and suppose that one of the conditions is satisfied under which all sets of the form Ω_γ (the intersections of main sequences) are principal ideals. We have already exhibited the examples of such conditions: the property 5) of the base of $\mathfrak{V}$ and the condition (o). Consider an outer quasimeasure $\theta \equiv \theta_\gamma$ and put

$$\Omega \equiv \Omega_\gamma = \{x \mid \theta_\gamma(x) = 0\}.$$

[12] It suffices only to assume σ-completeness. Completeness follows from the countable chain condition.

By assumption, Ω is a principal ideal. Put $u \equiv \sup\Omega$, i.e., $\Omega = \mathscr{X}_u$. Consider the complementary band $\mathscr{X}_{Cu} \equiv \mathscr{X}^*$. Let $\theta^* \equiv \theta|_{\mathscr{X}^*}$ be the restriction of θ. The outer quasimeasure θ^* is essentially positive on the band $\mathscr{X}^*$. Obviously, the same is true for every band included in $\mathscr{X}^*$. For each nonzero $z \in \mathscr{X}$, there exists a main sequence γ such that $z \notin \Omega_\gamma$.

Assigning

$$u \equiv \sup\Omega_\gamma, \ \ v \equiv Cu \wedge z,$$

we obtain the band $\mathscr{X}_v \subset \mathscr{X}_z$ such that the restriction $\theta_\gamma|_{\mathscr{X}_v}$ is essentially positive. Thus, we arrive at the situation described in Theorem 8. We see that the set of all v such that the outer quasimeasure $\theta_\gamma|_{\mathscr{X}_v}$ is essentially positive for some $\gamma = \gamma(v)$ minorizes $\mathscr{X}$. Therefore, the Boolean algebra $\mathscr{X}$ decomposes into the disjoint band $\mathscr{Y}$ of this form. More precisely,

$$\mathscr{X} = \bigoplus_{v\in V} \mathscr{X}_v, \tag{11}$$

where V is a set of pairwise disjoint elements, $\sup V = \mathbf{1}$, and for each $v \in V$ there exists a main sequence $\gamma \equiv \gamma(v)$ such that the restriction $\theta_\gamma|_{\mathscr{X}_v}$ is an essentially positive outer quasimeasure. Thus, the assertion I of Theorem 8 holds for every band $\mathscr{X}_v$. If, moreover, the condition (o) is satisfied then the assertion II and Remarks 2)–3) are valid. We proceed to the next

Theorem 9. I. *If a $(\mathfrak{V})$-topology of a complete Boolean algebra $\mathscr{X}$ possesses the properties* 1)–5) *then there exists a decomposition* (11) *into the bands $\mathscr{X}_v$ such that the $(\mathfrak{V}_v)$-topology is stronger than the (os)-topology of those bands.*

II. *If a $(\mathfrak{V})$-topology satisfies the conditions (o) and* 1)–4) *then there exists a decomposition of the same type* (11) *such that the $(\mathfrak{V}_v)$-topology of $\mathscr{X}_v$ is metrizable and coincides with the (o)- and (os)-topologies. Moreover, all bands of the decomposition satisfy the countable chain condition.*

The decomposition (11) may consist of countably many bands. Obviously, this situation is possible if and only if $\mathscr{X}$ is a Boolean algebra with the countable chain condition. In this case, we arrive at the conditions of Theorem 8. More precisely, the following theorem is valid.

Theorem 10. *Let $\mathscr{X}$ be a complete Boolean algebra with the countable chain condition and let $\mathfrak{V}$ be a uniformity base of $\mathscr{X}$ which possesses the properties* 1)–4). *If the $(\mathfrak{V})$-topology satisfies the condition (o) then it coincides with the (o)-topology and is metrizable. Moreover, there is an outer measure on $\mathscr{X}$ determining the $(\mathfrak{V})$-topology.*

Proof. The property (o) allows us to conclude that all sets Ω_γ are principal ideals. Therefore, there exists a (countable) decomposition

$$\mathscr{X} = \bigoplus_{n=1}^{\infty} \mathscr{X}_{u_n},$$

where $u_1 + u_2 + \cdots = \mathbf{1}$; and there is an outer measure θ_n on each band $\mathscr{X}_{u_n}$ which defines the $(\mathfrak{V}_{u_n})$-topology that coincides with the $(o)_u$-topology. Define a new outer measure θ by assigning

$$\theta(x) \equiv \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{1}{\theta_n(u_n)} \theta_n(u_n \wedge x).$$

It is not difficult to verify that θ possesses the properties of an outer measure including (o)-continuity. The presence of an outer measure ensures validity of the remaining assertions of the theorem.

Remark. If the property (o) is not assumed but 5) holds then the function θ is an essentially positive quasimeasure on $\mathscr{X}$; and the metric topology defined by θ is intermediate between the $(\mathfrak{V})$-topology and (o)-topology; θ is stronger that the (o)-topology and weaker that the $(\mathfrak{V})$-topology.

We recall that, in this case, the (o)- and (os)-topologies coincide in view of the countable chain condition.

The conditions 1)–5) and (o), characterizing a $(\mathfrak{V})$-topology, reveal the good relations between the topology and order (the "Boolean structure"). We have shown that a complete Boolean algebra with the countable chain condition may have only one such a topology, namely, the order topology. The algebras admitting such a topology were studied as "topological Boolean algebras."[13]

Thus, the topological Boolean algebras are complete Boolean algebras that have "rather good" (o)-topologies; they constitute the most natural class of Boolean algebras as regards functional analysis and probability theory.

An example of an algebra with a "bad" order topology is provided by the algebra $\mathscr{O}((0,1)) \equiv \mathscr{O}$ of regular open sets (see Theorem 12 of Chapter 2). Some pathological properties of this algebra were indicated by E. Floyd [1]. The algebra $\mathscr{O}$ satisfies the countable chain condition; hence, the (o)- and (os)-topologies coincide.

Let us show that the topology of $\mathscr{O}$ does not satisfy the Hausdorff axiom. Consider arbitrary open sets Ω_0 and Ω_1 containing zero and

[13] T. A. Sarymsakov et al. [1].

unity respectively. Show that these sets have a nonempty intersection. To this end, we enumerate all rational numbers of the interval $(0,1)$ into a simple sequence $\{r_1, r_2, \dots\}$ and put

$$x_{nm} \equiv \Bigl(r_n - \frac{1}{m}, r_n + \frac{1}{m}\Bigr) \cap (0,1) \quad (n, m = 1, 2, \dots).$$

It is clear that $x_{nm} \downarrow \mathbf{0}$ for all $n = 1, 2, \dots$.

There is an index m_1 such that the containment $x_{1m_1} \in \Omega_0$ holds. Since $x_{1m_1} \vee x_{2m_2} \xrightarrow{(o)} x_{1m_1}$ we have

$$x_{1m_1} \vee x_{2m_2} \in \Omega_0$$

for some m_2.

Repeating the above reasoning, we find a sequence of indices $\{m_1, m_2, \dots\}$ such that

$$x_{1m_1} \vee x_{2m_2} \vee \dots \vee x_{nm_n} \in \Omega_0$$

for all $n = 1, 2, \dots$. At the same time, the sequence

$$y_n \equiv x_{1m_1} \vee x_{2m_2} \vee \dots \vee x_{nm_n}$$

increases. The least upper bound of all y_n is equal to unity, since the least regular open subset $(0,1)$ containing all rational points is the interval $(0,1)$ itself. Thus, $y_n \xrightarrow{(o)} \mathbf{1}$ and $y_n \in \Omega_1 \cap \Omega_0$ for n sufficiently large.

We have proved that the intersection $\Omega_1 \cap \Omega_0$ is nonempty, i.e., zero and unity cannot be separated by disjoint neighborhoods. Therefore, the order topology is not separated. Hence it follows that a generalized sequence in $\mathscr{O}((0,1))$ may converge to more than one limit.

Basing on Theorem 9, we can show that *the Boolean operations $\vee$ and $\wedge$ are not continuous in $\mathscr{O}((0,1))$ with respect to the order topology. Therefore, the algebra $\mathscr{O}((0,1))$ is not a topological group.*

The problem of existence of a base of solid neighborhoods for the order topology is worth considering. Namely, does each (o)-neighborhood of zero include a solid subneighborhood? Or, which is equivalent, is the solid core of an (o)-neighborhood a neighborhood? It should be noted that some solid neighborhoods, obviously, exist. In particular, if V is an (o)-neighborhood of zero then the "solid hull"

$$\widetilde{V} = \bigcup_{u \in V} [0, u]$$

of V is a neighborhood. However, these neighborhoods might be insufficient for defining the (o)-topology. It is not difficult to establish a simple test for existence of a base of solid (o)-neighborhoods of zero. Namely,

this test is as follows: the conditions $x_\alpha \leq y_\alpha, \quad y_\alpha \longrightarrow \mathbf{0}$ must imply that $x_\alpha \longrightarrow \mathbf{0}$. Here, $\{x_\alpha\}$ and $\{y_\alpha\}$ are generalized sequences and the arrow designates the topological convergence.

A. V. Potepun [3] gave a nontrivial example demonstrating that there is no base of solid neighborhoods in the Boolean algebra $\mathscr{O}((0,1))$. Theorem 10 shows that this Boolean algebra has no uniform topology duly compatible with order at all. We may say that the Boolean algebra $\mathscr{O}((0,1))$ fails to comply with the principle "the less, the closer to zero." We return to the question of solid (o)-neighborhoods in the last section of this chapter.

2.6 Uniform completion of a Boolean algebra

The concept of completeness with respect to a uniformity plays an important role in the theory of uniform spaces (as well as beyond it). In our situation, the main definition is as follows: a Boolean algebra $\mathscr{X}$ with a uniformity base $\mathfrak{V}$ is called *uniformly complete* whenever every fundamental generalized sequence $\{x_\alpha\}_{\alpha\in A}$ converges (with respect to the uniform topology). Being fundamental means in our case that, for all $V \in \mathfrak{V}$ there exists an index $\alpha_V \in A$ such that $|x_{\alpha'} - x_{\alpha''}| \in V$ holds for all $\alpha', \alpha'' \succ \alpha_V$. It is not difficult, for instance, to prove uniform completeness in the conditions of Theorem 10.

Thus, both types of completeness, uniform and order, are often correlated. We now return to the question mentioned in the preceding chapter; namely, to the question of the order completion of a Boolean algebra. It turns out that completion with respect to a uniformity may lead to completeness with respect to order. This manner of constructing a complete Boolean algebra seems to be the most important among all considered so far.

Every uniformity on a Boolean algebra is naturally translated to an isomorphic algebra. For instance, let $\mathfrak{V}$ be a uniformity base of a Boolean algebra $\mathscr{X}$ and let Φ be an isomorphism from $\mathscr{X}$ onto $\mathscr{Y}$. The sets of the form $\Phi(V)$ $(V \in \mathfrak{V})$ constitute a uniformity base on $\mathscr{Y}$ which will be denoted by $\Phi(\mathfrak{V})$. It is clear that the $(\Phi(\mathfrak{V}))$-topology of $\mathscr{Y}$ does not, in fact, differ from the $(\mathfrak{V})$-topology of $\mathscr{X}$. The two algebras are not only algebraically (order) but also uniformly isomorphic, which is a reason for identifying $\mathscr{X}$ and $\mathscr{Y}$. The above-described construction can be generalized of course; however, we need not such a generalization here.

Theorem 11. *If a uniformity base $\mathfrak{V}$ in a (not necessarily complete) Boolean algebra $\mathscr{X}$ possesses the property:*

$(*)$ *every disjoint simple sequence (o)-converges to zero*

then $\mathscr{X}$ has a completion in the sense of 2.3. *Namely, there exist a complete Boolean algebra $\widehat{\mathscr{X}}$, a uniformity base $\widehat{\mathfrak{V}}$ in $\widehat{\mathscr{X}}$, and a monomorphism $\Phi : \mathscr{X} \longrightarrow \widehat{\mathscr{X}}$ such that*

1) *the subalgebra $\mathscr{X}' = \Phi(\mathscr{X})$ is $(\widehat{\mathfrak{V}})$-dense in $\widehat{\mathscr{X}}$;*

2) *the uniform topology induced on $\mathscr{X}'$ by the $(\widehat{\mathfrak{V}})$-topology coincides with the $\Phi(\mathfrak{V})$-topology;*

3) *the algebra $\widehat{\mathscr{X}}$ is complete with respect to uniformity $((\widehat{\mathfrak{V}})$-complete);*

4) *the $(\widehat{\mathfrak{V}})$-topology of $\widehat{\mathscr{X}}$ satisfies the conditions* 1)–5) *and* (os).

We recall that the uniformity base $\mathfrak{V}$ possesses the properties 1)–4). The property 5) is not assumed.

Proof. According to the well-known theorem of the theory of uniform spaces, the set $\mathscr{X}$ can be identified with a dense subset $\mathscr{X}'$ of a uniform Hausdorff space $\widehat{\mathscr{X}}$; moreover, the uniform topology of $\mathscr{X}'$ is induced by the uniform topology of $\widehat{\mathscr{X}}$.[14]

For the sake of simplicity, we will identify $\mathscr{X}$ and $\mathscr{X}'$ on assuming that $\mathscr{X}$ is a dense subset of $\widehat{\mathscr{X}}$. Thus, each $x \in \widehat{\mathscr{X}}$ can be represented as the limit

$$x = \lim_{\alpha} x_\alpha, \quad x_\alpha \in \mathscr{X}.$$

It is also well known that each uniformly continuous mapping from $\mathscr{X} \times \mathscr{X}$ into $\mathscr{X}$ can be extended to a uniformly continuous mapping from $\widehat{\mathscr{X}} \times \widehat{\mathscr{X}}$ into $\widehat{\mathscr{X}}$.

We now extend the order onto $\widehat{\mathscr{X}}$ and show that $\widehat{\mathscr{X}}$ is a complete Boolean algebra with respect to the extended order. We outline the most important steps.

1°. We have already seen that the mappings

$$\begin{aligned} S : &\quad S(x,y) \equiv x \vee y, \\ I : &\quad I(x,y) \equiv x \wedge y, \\ C : &\quad C(x) \equiv Cx \end{aligned}$$

are uniformly continuous. Therefore, they admit the unique uniformly continuous extensions to $\widehat{\mathscr{X}} \times \widehat{\mathscr{X}}$ (or to $\widehat{\mathscr{X}}$).

2°. By the continuity of these functions on $\widehat{\mathscr{X}}$, the main relations are valid that characterize $\vee$, $\wedge$, and C as operations in a Boolean ring. Namely, we have associativity and distributivity, as well as the identities $x \wedge Cx = \mathbf{0}$ and $x \vee Cx = \mathbf{1}$.

[14]See N. Bourbaki [1, pp. 232–235]; J. Kelley [2, Chapter 6]. The problem of uniform completion of a general poset was fully settled by T. G. Kiselëva [1]. Also see H. Weber [1].

3°. We can introduce the order on $\widehat{\mathscr{X}}$ (as on every Boolean ring) by the condition

$$x \leq y \quad \longleftrightarrow \quad x \vee y = y \quad \longleftrightarrow \quad x \wedge y = x.$$

Thus, we obtain a Boolean algebra. This construction was mentioned in Chapter 1.

We note that the order on $\widehat{\mathscr{X}}$ is related to the original order on $\mathscr{X}$ (as well as the Boolean operations).

4°. The base $\widehat{\mathfrak{V}}$ of the uniform topology of $\widehat{\mathscr{X}}$ consists of all closures (with respect to the uniformity) of the sets of the original base:

$$\widehat{\mathfrak{V}} = \{\overline{V} \mid V \in \mathfrak{V}\}.$$

All above assertions are simple consequences of the definitions and main facts of the theory of uniform spaces.

5°. We are left with proving the main fact: the completeness of the Boolean algebra $\widehat{\mathscr{X}}$. We proceed by transfinite induction. Assume that the existence of the supremum is proved for every set $E \subset \widehat{\mathscr{X}}$ such that $card\, E < \mathfrak{a} \leq card\, \widehat{\mathscr{X}}$. Show that the least upper bound exists also for $card\, E = \mathfrak{a}$. Denote by ω^* the initial ordinal of cardinality $\mathfrak{a}$ and enumerate E by the ordinals less than $\omega^*: \ x_1, x_2, \dots$. Introduce the elements $\overline{x}_\beta = \bigvee_{t<\beta} x_t$. They constitute an increasing transfinite sequence whose supremum, if existent, is the least upper bound of E. Thus, we need to prove the existence of $\sup \overline{x}_\beta$. To this end, we show that the sequence $\{\overline{x}_\beta\}$ has a limit in the sense of uniformity. It is sufficient to establish that this sequence is fundamental.

Suppose to the contrary that the sequence is not fundamental. Then there exist $V_0 \in \widehat{\mathfrak{V}}$ and an infinite (simple) sequence of pairs of indices $\{\beta_i', \beta_i''\}$ such that $\beta_{i+1}'' > \beta_{i+1}' > \beta_i''$ (for all i) and $|x_{\beta_i''} - x_{\beta_i'}| \notin V_0$. Put

$$u_i \equiv |\overline{x}_{\beta_i''} - \overline{x}_{\beta_i'}| \equiv \overline{x}_{\beta_i''} \wedge C\overline{x}_{\beta_i'}.$$

This sequence is disjoint. It is easy to verify that the property 3) of a uniformity base is valid in our case. According to this property, we arrange the family of basis sets $\{W_V\}_{V \in \widehat{\mathfrak{V}}}$ with the following properties:

$$1)\ W_V \in \widehat{\mathfrak{V}}, \qquad 2)\ W_V \vee W_V \subset V$$

for all $V \in \widehat{\mathfrak{V}}$. We now assign $V_1 \equiv W_{V_0}$, $V_2 \equiv W_{V_1}, \dots$ and choose $u_1', u_2', \dots$ such that

$$1)\ u_i' \in \mathscr{X}; \qquad 2)\ |u_i - U_i'| \in V_{i+2}$$

hold for all $i = 1, 2, \dots$. Next, we take

$$u_1'' \equiv u_1', \ \ u_2'' \equiv u_2' \wedge Cu_1' \equiv u_2' \wedge Cu_1'',$$

$$\dots, u_i'' \equiv u_i' \wedge C(u_1' \vee \dots \vee u_{i-1}') \equiv u_i' \wedge C(u_1'' \vee \dots \vee u_{i-1}''), \dots$$

(a well-known way of arranging a disjoint sequence). Using the formulas of 0.2.5, we obtain

$$|u_1''-u_1'| \in V_2, \ \ |u_2''-u_2'| = u_2' \wedge u_1' = |u_2 \wedge u_1 - u_2' \wedge u_1'| \le |u_2-u_2'| \vee |u_1-u_1'|,$$

$$|u_3'' - u_3'| = |u_3' \wedge (u_1' \vee u_2')| = |u_3 \wedge (u_1 \vee u_2) - u_3 \wedge (u_1' \vee u_2')|$$
$$\le |u_3 - u_3'| \vee |u_2 - u_2'| \vee |u_1 - u_1'|,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$|u_m'' - u_m'| \le |u_m - u_m'| \vee |u_{m-1} - u_{m-1}'| \vee \dots \vee |u_1 - u_1'|,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

It is clear that

$$|u_m'' - u_m'| \in V_{m+2} \vee V_{m+1} \vee \dots \vee V_3 \subset V_2.$$

The sequence $\{u_m''\}$ is disjoint and consists of the elements of $\mathscr{X}$. Therefore, we have $u_m'' \in V_1$ for sufficiently large m (we use here the main hypothesis of the theorem). Since the set $V_0 \in \widehat{\mathfrak{V}}$ is obviously solid, we derive

$$u_m \le |u_m - u_m'| \vee |u_m' - u_m''| \vee u_m'' \in V_{m+2} \vee V_2 \vee V_1 \subset V_1 \vee V_1 \subset V_0,$$

which contradicts our supposition.

We have thus proved that the sequence $\{\overline{x}_\beta\}$ converges:

$$\overline{x}_\beta \longrightarrow \overline{x} \in \widehat{\mathscr{X}}.$$

Finally, we show that $\overline{x} = \sup \overline{x}_\beta$. We have

$$\overline{x}_\alpha \le \overline{x}_\beta$$

for all $\alpha, \beta, \ \alpha < \beta < \omega^*$. Passing to the limit with respect to β in the above inequalities, we obtain

$$\overline{x}_\alpha \le \overline{x},$$

i.e., $\overline{x}$ is an upper bound for $\overline{x}_\beta$. On the other hand, if $\overline{x}_\beta \le \overline{y}$ for all β then, after passage to the limit, we must have $\overline{x} \le \overline{y}$. (The possibility of passage to the limit ensues from the uniform continuity of the operations

$\vee$ or $\wedge$.) Thus, $\overline{x}$ is the supremum of E, which completes the proof by induction. (The existence of infima follows by duality.) So, $\widehat{\mathscr{X}}$ is a complete Boolean algebra.

Find out the properties of the $(\widehat{\mathfrak{V}})$-topology. It follows from the proof that each monotone transfinite sequence in $\widehat{\mathscr{X}}$ has a limit in the sense of the $(\widehat{\mathfrak{V}})$-topology and this limit coincides with the supremum or infimum of the sequence. This implies the property (os). The properties 1)–4) hold for every uniform topology; they are still valid after completion, since the base $\widehat{\mathfrak{V}}$ consists of the closures of the sets in $\mathfrak{V}$. The condition 5) also holds: each transfinite increasing sequence $\{x_\alpha\}$ has a $(\widehat{\mathfrak{V}})$-limit equal to $\bigvee_\alpha x_\alpha$; if all x_α belong to $V \in \widehat{\mathfrak{V}}$ then this limit belongs to V by the $(\widehat{\mathfrak{V}})$-closure of the neighborhood V. Using the solidity of $V \in \widehat{\mathfrak{V}}$, we can prove the same fact for an arbitrary generalized sequence. The proof of the theorem is over.

By Theorem 9, the above-constructed Boolean algebra $\widehat{\mathscr{X}}$ admits the disjoint decomposition

$$\widehat{\mathscr{X}} = \bigoplus_{u \in U} \widehat{\mathscr{X}_u},$$

where each band $\widehat{\mathscr{X}_u}$ satisfies the countable chain condition and has an outer measure θ_u. This outer measure defines the $(\widehat{\mathfrak{V}}_u)$-topology that is, thereby, metrizable and coincides with the order topology of the band.

When proving the last theorem, we simplified the situation by assuming that the original Boolean algebra $\mathscr{X}$ is a part of a new Boolean algebra $\widehat{\mathscr{X}}$. Even oftener, we assume that $\widehat{\mathscr{X}}$ includes a uniformly and order isomorphic IMAGE of $\mathscr{X}$:

$$\mathscr{X} \xrightarrow{i} \widehat{\mathscr{X}}.$$

Here i is an injective mapping that preserves the order and uniform structures. It is not a great inaccuracy to identify $\mathscr{X}$ and $i(\mathscr{X})$. We point out however that there are a lot of uniform completions $\widehat{\mathscr{X}}$ that contain an isomorphic image of $\mathscr{X}$ as a dense subset. Nevertheless, all images are order and uniformly isomorphic.

Theorem 11 implies an important

Corollary. *Let a* $(\mathfrak{V})$*-topology of a Boolean algebra* $\mathscr{X}$ *satisfy the conditions of Theorem* 11. *If* $\mathscr{X}$ *is uniformly complete then* $\mathscr{X}$ *is order complete.*

Indeed, the process of "uniform" completion when applied to such an algebra $\mathscr{X}$ yields the same algebra.

The case is of the utmost importance in which $\widehat{\mathscr{X}}$ is an algebra with the countable chain condition. This case is interesting in particular

because the (o)- and $(\mathfrak{V})$-topologies of $\widehat{\mathscr{X}}$ must coincide by Theorem 10; and the algebra $\mathscr{X}$ (as a subalgebra of $\widehat{\mathscr{X}}$) fully generates $\widehat{\mathscr{X}}$ (i.e., $\widehat{\mathscr{X}}$ is the (o)-closure of $\mathscr{X}$). Therefore, the so-constructed completion satisfies the condition (m) (see p. 95).

We point out the circumstances in which the completion satisfies the countable chain condition.

As above, let $\mathscr{X}$ be a Boolean algebra with a uniformity base $\mathfrak{V}$ that satisfies the conditions of Theorem 11, let the complete Boolean algebra $\widehat{\mathscr{X}}$ be a uniform completion of $\mathscr{X}$ as described in Theorem 11, and let $\widehat{\mathfrak{V}}$ be the corresponding uniformity base in $\widehat{\mathscr{X}}$ (we assume $\mathscr{X}$ to be a subalgebra of $\widehat{\mathscr{X}}$).

Theorem 12. *For the Boolean algebra $\widehat{\mathscr{X}}$ to satisfy the countable chain condition, it is necessary and sufficient that the $(\mathfrak{V})$-topology of $\mathscr{X}$ be generated by an essentially positive outer quasimeasure θ; in this case, the $(\widehat{\mathfrak{V}})$-topology of $\widehat{\mathscr{X}}$ is generated by an outer measure $\widehat{\theta}$ whose restriction is θ.*

Proof. Let $\widehat{\mathscr{X}}$ satisfy the countable chain condition. According to Theorem 10, the $(\widehat{\mathfrak{V}})$-topology is generated by an outer measure $\widehat{\theta}$. Taking $\theta = \widehat{\theta}|_{\mathscr{X}}$, we obtain an essentially positive outer quasimeasure. We need to prove that the quasimeasure generates the original uniform topology. Since $\theta = \widehat{\theta}|_{\mathscr{X}}$ and the $(\mathfrak{V})$-topology is included by the $(\widehat{\mathfrak{V}})$-topology, we have

$$\theta(x_\alpha) \to 0 \Leftrightarrow \widehat{\theta}(x_\alpha) \to 0 \Leftrightarrow (\widehat{\mathfrak{V}})\text{-}\lim x_\alpha = \mathbf{0} \Leftrightarrow (\mathfrak{V})\text{-}\lim x_\alpha = \mathbf{0}$$

for every generalized sequence $\{x_\alpha\}$ in $\mathscr{X}$. The necessity part of the theorem is proved.

Sufficiency. Suppose that the $(\mathfrak{V})$-topology is generated by an essentially positive outer quasimeasure θ. Thereby θ is $(\mathfrak{V})$-continuous. There exists a continuous extension $\widehat{\theta}$ of θ to $\widehat{\mathscr{X}}$. Obviously, this extension is a $(\widehat{\mathfrak{V}})$-continuous outer quasimeasure. Show that it is essentially positive. Suppose to the contrary that the equality $\widehat{\theta}(x) = 0$ holds for some $x > \mathbf{0}$, $x \in \widehat{\mathscr{X}}$. The element x can be represented as

$$x = (\mathfrak{V})\text{-}\lim x_\alpha, \quad x_\alpha \in \mathscr{X}.$$

Since the $(\widehat{\mathfrak{V}})$-topology is separated, there is $V \in \mathfrak{V}$ such that $x \notin \overline{V}$ (the base $\widehat{\mathfrak{V}}$ consists of the closures of the sets $V \in \mathfrak{V}$). The equality $\widehat{\theta}(x) = 0$ follows that $\widehat{\theta}(x_\alpha) = \theta(x_\alpha) \to 0$. The $(\mathfrak{V})$-topology is generated by the function θ; therefore, $(\mathfrak{V})\text{-}\lim x_\alpha = \mathbf{0}$. But then we have $x_\alpha \in V$ for $\alpha \succ \alpha_0$; consequently, $x \in V$, which contradicts the choice of V.

We have showed that $\widehat{\theta}(x) > 0$ for all $x > \mathbf{0}$. The presence of an essentially positive outer quasimeasure together with the condition (os) (this condition is valid in $\widehat{\mathscr{X}}$ by Theorem 11) implies the countable chain condition (see part II of Theorem 8). The proof of the theorem is complete.

Examples. Consider an arbitrary convex set M of probability quasimeasures on a Boolean algebra $\mathscr{X}$. This set is assumed to be "total": if $m(x) = 0$ for all $m \in M$ then $x = \mathbf{0}$. The set M is sufficient for defining the uniformity on $\mathscr{X}$ whose base $\mathfrak{V}$ consists of the sets

$$V_\varepsilon^m \equiv \{x \mid m(x) < \varepsilon\}, \quad m \in M, \quad \varepsilon > 0.$$

It is easy to verify that the conditions 1)–4) are valid. Moreover, every disjoint sequence $\{x_n\}$ must $(\mathfrak{V})$-converge to zero. The last assertion follows from the convergence of every series

$$\sum_{n=1}^{\infty} m(x_n) \quad (m \in M)$$

whose partial sums are at most $m(\mathbf{1}) = 1$.

Thus, the process of uniform completion of the Boolean algebra $\mathscr{X}$ leads in this case to a complete Boolean algebra $\widehat{\mathscr{X}}$. The quasimeasures $m \in M$ are uniformly continuous and can be extended from $\mathscr{X}$ to $\widehat{\mathscr{X}}$; their extensions are, as it is easy to check, probability quasimeasures constituting a total set $\widehat{M}$ with respect to $\widehat{\mathscr{X}}$. This set defines the base $\widehat{\mathfrak{V}}$ in much the same way as the original set M defines $\mathfrak{V}$. This $\widehat{\mathscr{X}}$ splits into disjoint bands each of which satisfies the countable chain condition:

$$\widehat{\mathscr{X}} = \bigoplus_{u \in U} \widehat{\mathscr{X}_u}.$$

The quasimeasures $\widehat{m} \in \widehat{M}$ are $(\widehat{\mathfrak{V}})$-continuous. In view of the fact that the $(\widehat{\mathfrak{V}}_u)$-topologies coincide with the order topologies, the traces of the quasimeasures on the bands $\widehat{\mathscr{X}_u}$ are totally additive. Each quasimeasure $\widehat{m}$ can be nonzero only on at most countably many bands $\widehat{\mathscr{X}_u}$; and, therefore, is totally additive. To each quasimeasure $\widehat{m}$, there is a partition of unity $\mathbf{1} = u' + u''$ such that the quasimeasure is equal to zero on u' and is essentially positive on u''; the supremum of all u'''s is equal to unity. So, the bands $\widehat{\mathscr{X}_u}$ of the disjoint decomposition can be chosen in such a way that, for each u, one of the quasimeasures is essentially positive on the band $\widehat{\mathscr{X}_u}$, i.e., is a genuine measure. Therefore, the Boolean algebra *admits decomposition into normed bands.*

We describe two most important applications of the just-outlined scheme.

I. The set M consists of a SOLE essentially positive quasimeasure m. We arrive at the conditions of Theorem 12. The algebra $\widehat{\mathscr{X}}$ is normable in this case; the measure on $\widehat{\mathscr{X}}$ is $\widehat{m}$, the result of extension of m from $\mathscr{X}$ onto $\widehat{\mathscr{X}}$.

II. Let $\mathscr{X}$ be the clopen Boolean algebra of a compact space $\mathfrak{Q}$. To each point $q \in \mathfrak{Q}$, we assign the two-valued quasimeasure that is concentrated at this point:

$$m_q(x) = \begin{cases} 1, & q \in x, \\ 0, & q \notin x. \end{cases}$$

We take the convex hull of all these quasimeasures as M. It is not difficult to see that $\widehat{\mathscr{X}}$ is naturally identified in this case with the boolean $2^{\mathfrak{Q}}$ of $\mathfrak{Q}$.

3. Regular Boolean algebras. Various forms of distributivity

3.1 Regular Boolean algebras

Boolean algebras with a measure (or "normed") are of the utmost interest for applications. We have already adduced a few examples of such Boolean algebras. Some important properties of a normed Boolean algebra are not directly related to a particular measure but ensues already from existence of a measure. The most important among these properties is *regularity* which was defined by L. V. Kantorovich.

The following assertion will be called the *diagonal principle*:

For each double sequence $\{x_{nm}\}_{n,m=1}^{\infty}$ in a Boolean algebra $\mathscr{X}$ satisfying the condition

$$x_{nm} \downarrow \mathbf{0} \text{ when } m \to \infty \quad (n = 1, 2, \ldots), \tag{12}$$

there exists a "diagonal" sequence $\{x_{nm_n}\}_{n=1}^{\infty}$ such that

$$x_{nm_n} \xrightarrow{(o)} \mathbf{0}.$$

We may always assume that $m_1 < m_2 < \ldots$.

Now the definition of regularity is in order. A *regular Boolean algebra* is a complete Boolean algebra with the countable chain condition which obeys the diagonal principle. An example of a regular algebra is provided by every complete Boolean algebra $\mathscr{X}$ with an outer measure θ. Indeed, we have already established that such an algebra satisfies the countable chain condition (Theorem 12). Show that the diagonal principle also

holds. Consider a double sequence $\{x_{nm}\}$. Let $x_{nm} \downarrow \mathbf{0}$ when $m \to \infty$ for all $n = 1, 2, \ldots$. Given $m = 1, 2, \ldots$, select an index m_n such that $x_{nm_n} \in K_{\theta, \frac{1}{3^n}}$ (we use the (o)-continuity of an outer measure). The sequence of the balls $\{K_{\theta, \frac{1}{3^n}}\}$ is main; therefore, $x_{nm_n} \xrightarrow{(o)} \mathbf{0}$.[15] Thus, we have constructed a required diagonal sequence.

Theorem 13 (General Diagonal Principle). Let $\mathscr{X}$ be a regular Boolean algebra. If

$$x_{ik} \xrightarrow{(o)} x_i \text{ on } k, \quad x_i \xrightarrow{(o)} x$$

then there exists a sequence of indices $k_1 < k_2 < \ldots$ such that

$$x_{ik_i} \xrightarrow{(o)} x.$$

Proof. Put

$$y_{ik} = \bigvee_{j \geq k} |x_{ij} - x_i| \quad (k, i = 1, 2, \ldots).$$

Then $y_{ik} \downarrow \mathbf{0}$ as $k \to \infty$ for all $i = 1, 2, \ldots$. By the diagonal principle, there exists a diagonal sequence $\{y_{ik_i}\}$ $(k_1 < k_2 < \ldots)$ such that $y_{ik_i} \xrightarrow{(o)} \mathbf{0}$. The inequality $|x - x_{ik_i}| \leq |x - x_i| \vee y_{ik_i}$ holds for all i; whence $x_{ik_i} \xrightarrow{(o)} x$. The proof of the theorem is complete.

We dwell on some properties of the order topology of a regular Boolean algebra. The countable chain condition implies that the (o)-topology and (os)-topology coincide. Therefore, a set E is closed in the order topology whenever E contains the (o)-limits of all sequences in E. It is easy to describe the structure of the closure of a set in a regular Boolean algebra.

Theorem 14. *For an element x to belong to the closure of a set E, it is necessary and sufficient that there exist a simple sequence in E whose (o)-limit is equal to x.*

Proof. We need only to prove necessity. Denote by $\overline{E}$ the closure of E and by $\widetilde{E}$, the union of E and the (o)-limits of all simple sequences in E. It is clear that $\widetilde{E} \subset \overline{E}$. The equality $\widetilde{E} = \overline{E}$ (and thereby, the theorem) will be proved if we verify the closure of $\widetilde{E}$. Let

$$x_n \xrightarrow{(o)} x, \quad x_n \in \widetilde{E} \quad (n = 1, 2, \ldots).$$

For each n, there exists a sequence x_{nm} in E (o)-convergent to x_n. By the above theorem, there exists a diagonal sequence $\{x_{nm_n}\}$ such that

[15] See the proof of Theorem 8.

$x_{nm_n} \xrightarrow{(o)} x$. Therefore, $x \in \widetilde{E}$ and $\widetilde{E}$ is closed. The proof of the theorem is complete.

Theorem 15. *The (o)-topology of a regular Boolean algebra $\mathscr{X}$ has a base of solid neighborhoods of zero.*

Proof. Suppose that, some regular Boolean algebra $\mathscr{X}$ has no base of solid neighborhoods of zero. As was established, in this case there exist a neighborhood of zero V and two generalized sequences $\{x_\alpha\}$ and $\{y_\alpha\}$ such that $\mathbf{0} < x_\alpha \le y_\alpha$, $y_\alpha \longrightarrow \mathbf{0}$, and $x_\alpha \notin V$ for all α. Consider the set Y of all entries of the sequence $\{y_\alpha\}$. The point $\mathbf{0}$ belongs to the closure of Y; therefore, according to Theorem 14, there exists a simple sequence $\{\alpha_n\}$ such that $y_{\alpha_n} \xrightarrow{(o)} \mathbf{0}$. But then, obviously, $x_{\alpha_n} \xrightarrow{(o)} \mathbf{0}$, although all x_{α_n} do not belong to V and even the topological convergence is impossible. This contradiction proves the theorem.

The above theorem is a particular case of a more general theorem by A. V. Potepun (see [2, 3]). A regular Boolean algebra complies therefore with the principle "the less, the closer to zero."

We present an example of a nonregular algebra. Such an example is provided by our acquaintance, the Boolean algebra $\mathscr{O}((0,1))$ of all regular open subsets of $(0,1)$. Arrange all rational numbers of the interval into some sequence $\{r_n\}_{n=1}^{\infty}$ and assign to each n the sequence of intervals

$$x_{nm} \equiv \left(r_n - \frac{1}{m}, r_n + \frac{1}{m}\right) \cap (0,1).$$

Each of the intervals is a regular set and belongs therefore to the Boolean algebra $\mathscr{O}((0,1))$. For a fixed n, we have a decreasing sequence $\{x_{nm}\}$ whose (o)-limit is equal to zero. On the other hand, for every diagonal sequence $\{x_{nm_n}\}$ and for every k, the union $\bigcup_{n=k}^{\infty} x_{nm_n}$ contains "almost" all rational numbers and is dense in $(0,1)$. Therefore, the least regular open subset of the interval which includes the union must be $(0,1)$ itself. In other words,

$$\overline{\lim}\, x_{nm_n} = \mathbf{1}$$

and $\mathscr{O}((0,1))$ fails to obey the diagonal principle. Thus, $\mathscr{O}((0,1))$ is nonregular algebra, and therefore it admits no norm. However, $\mathscr{O}((0,1))$ is obviously an algebra with the countable chain condition. So, this example shows that the countable chain condition does not imply the diagonal principle. As for the converse implication, the situation is more complicated. It turns out that the countable chain condition can be deduced from the diagonal principle by using some set-theoretic hypotheses that are weaker than the continuum hypothesis. We only formulate the corresponding result (A. G. Pinsker and D. A. Vladimirov).

Consider the class T of all unbounded increasing sequences

$$\tau = (n_1^{(\tau)}, n_2^{(\tau)}, \dots, n_k^{(\tau)}, \dots)$$

of naturals. Introduce a preorder on T by the following condition: $\tau' \succ \tau''$ whenever $n_k^{(\tau')} \geq n_k^{(\tau'')}$ for all $k \geq k_0(\tau', \tau'')$. It is clear that all countable subsets of T are always bounded with respect to this preorder. Denote by $\mathfrak{a}$ *the least cardinality of an unbounded above set*. Clearly, $2^{\aleph_0} \geq \mathfrak{a} \geq \aleph_1 > \aleph_0$. If we assume the continuum hypothesis then $\mathfrak{a} = 2^{\aleph_0} = \aleph_1$.

Theorem. *The following assertions are equivalent:*

(A) *A complete Boolean algebra that complies with the diagonal principle satisfies the countable chain condition;*

(B) $\mathfrak{a} = \aleph_1$.

The proof of the above theorem belongs to the theory of vector lattices and is not given here.[16] It is clear that the assertions (A) and (B) are consistent with the usual axioms of set theory.

3.2 Instances of distributivity. Weak countable distributivity

We return to the formulas (7) and (7*) of Chapter 0. They express the distributivity property of a Boolean algebra. If an algebra is complete then the formula (1) of Chapter 2 is valid. The most general formula of this type is as follows:

$$\bigwedge_{\alpha \in A} \bigvee_{\beta \in B_\alpha} x_{\alpha\beta} = \bigvee_{\varphi \in \Phi} \bigwedge_{\alpha \in A} x_{\alpha\varphi(\alpha)}, \tag{13}$$

where $\Phi \equiv \prod_\alpha B_\alpha$ is the cartesian product of some sets B_α each of which (as well as the set A) has an arbitrary cardinality. The dual analog of the above formula also can be written:

$$\bigvee_{\alpha \in A} \bigwedge_{\beta \in B_\alpha} x_{\alpha\beta} = \bigwedge_{\varphi \in \Phi} \bigvee_{\alpha \in A} x_{\alpha\varphi(\alpha)}. \tag{13$'$}$$

A Boolean algebra is called *totally distributive* whenever the above identities are valid (either of them is enough). *A complete Boolean algebra is totally distributive if and only if it is discrete; i.e., it is some*

[16]See D. A. Vladimirov [1]. Earlier, A. G. Pinsker proved that the continuum hypothesis implies (A) (L. V. Kantorovich, B. Z. Vulikh, and A. G. Pinsker [1]). See also Yu. F. Sirvint [1].

boolean 2^Q.[17] Thus, the formulas (13) and (13′) are not valid in the most interesting cases. Therefore, the various weakened versions of the above relations are of interest. For instance, some restrictions on the cardinalities of the sets A and B_α can be imposed; the corresponding theory is presented in the book by R. Sikorski. We leave these questions aside and dwell upon only one case of the utmost importance. The Boolean algebra $\mathscr{X}$ is assumed to be complete.

Let $\{M_i\}_{i=1}^\infty$ be a countable family of countable downward directed subsets of $\mathscr{X}$. Consider the totality

$$V \equiv \prod_{i=1}^\infty M_i$$

of all "sections" i.e., the sequences $v = \{x_i\}_{i=1}^\infty$, where $x_i \in M_i$. For brevity, put $\sup v \equiv \bigvee_{i=1}^\infty x_i$.

If the equality

$$\bigwedge_{v\in V} \sup v = \bigvee_{i=1}^\infty \inf M_i, \tag{14}$$

holds for each of these families then we say that the complete Boolean algebra $\mathscr{X}$ obeys the *weak σ-distributive law.*[18]

Theorem 16.

1) *A regular Boolean algebra always obeys the weak σ-distributive law;*

2) *A weak σ-distributive Boolean algebra with the countable chain condition is regular.*

Thus, the situation can be described in terms of the following relations: "regularity = the diagonal principle + the countable chain condition = the weak σ-distributive law + the countable chain condition."

It should be noted that the diagonal principle is close to but not equivalent to the weak σ-distributive law. A discrete Boolean algebra without the countable chain condition obeys the weak σ-distributive law but fails to possess the diagonal principle.

Proof of the theorem. 1) Let $\mathscr{X}$ be a regular Boolean algebra and let the family $\{M_i\}$ satisfy the above-stated conditions. Prove that

$$\bigwedge_{v\in V} \sup v = \bigvee_{i=1}^\infty \inf M_i.$$

[17] A. Tarski [1].

[18] Or the "weak countably distributive law."

Clearly,

$$\bigwedge_{v\in V} \sup v \geq \bigvee_{i=1}^{\infty} \inf M_i.$$

Therefore, we need to prove only the reverse inequality.

Since $\{M_i\}$ is countable and directed downward, we can select a decreasing sequence in M_i with the same greatest lower bound:

$$M_i' = \{x_m^i\}_{m=1}^{\infty} \subset M_i, \ \inf M_i' = \inf M_i.$$

It is clear that

$$\bigwedge_{v\in V} \sup v \leq \bigwedge_{v\in V'} \sup v \quad \Bigl(V' = \prod_{i=1}^{\infty} M_i'\Bigr)$$

and so it suffices to prove the inequality

$$\bigwedge_{v\in V'} \sup v \leq \bigvee_{i=1}^{\infty} \inf M_i'.$$

In the sequel, we assume that M_i are decreasing sequences.

Set

$$y_i \equiv \inf M_i, \quad y \equiv \bigvee_{i=1}^{\infty} y_i, \quad z_k \equiv \bigvee_{i=1}^{k} y_i.$$

Put the elements of each set M_i into a simple sequence $\{x_m^i\}_{m=1}^{\infty}$ and assign

$$z_k^s \equiv \bigwedge_{m=1}^{s} (x_m^1 \vee x_m^2 \vee \cdots \vee x_m^k).$$

It is not difficult to see that

$$z_k = \bigwedge_{s=1}^{\infty} z_k^s, \quad z_k^s \xrightarrow{(o)} z_k \text{ on } s, \quad z_k \xrightarrow{(o)} y.$$

Using the diagonal principle, find a sequence of indices $\{s_k\}_{k=1}^{\infty}$ such that the equality

$$(o)\text{-}\lim_k z_k^{s_k} = y$$

holds. Since the sets M_i are directed, it is easy to verify that the intersection $M_i \cap \mathscr{X}_{z_q^p}$ is nonempty for all i and $p, q \geq i$. Let $\overline{x}_i^m \in M_i$ and $\overline{x}_i^m \leq z_{i+m}^{s_{i+m}}$. Consider the selections $v_m = \{\overline{x}_i{}^m\}_{i=1}^{\infty}$. We see that

$$\bigwedge_{v\in V} \sup v \leq \bigwedge_{m=1}^{\infty} \sup v_m \leq \bigwedge_{m=1}^{\infty} \bigvee_{i=1}^{\infty} z_{i+m}^{s_{i+m}}$$

$$= \bigwedge_{m=1}^{\infty} \bigvee_{k=m+1}^{\infty} z_k^{s_k} = \overline{\lim}\, z_k^{s_k} = (o)\text{-}\lim_k z_k^{s_k} = y.$$

2) Let $\mathscr{X}$ satisfy the countable chain condition and obey the weak σ-distributive law. Prove regularity. We only need to establish the validity of the diagonal principle. Consider an arbitrary double sequence $\{x_{nm}\}$ such that

$$x_{nm} \downarrow \mathbf{0} \quad (m \longrightarrow \infty;\ n = 1, 2, \ldots).$$

We should find a diagonal sequence $\{x_{nm_n}\}$ (o)-converging to zero.

Let M_n be the set of entries of the sequence $\{x_{nm}\}_{m=1}^{\infty}$. Obviously, this set is directed downward and $\inf M_n = \mathbf{0}$ $(n = 1, 2, \ldots)$. Using the weak σ-distributive law and the countable chain condition, arrange the sequence of selections

$$v_k \equiv \{x_{1m_1^{(k)}}, x_{2m_2^{(k)}}, \ldots\}$$

such that $\bigwedge_k \sup v_k = \mathbf{0}$. The sequence $\{x_{nm}\}_{m=1}^{\infty}$ decreases monotonically. Therefore, by increasing $m_1^{(k)}, m_2^{(k)}, \ldots$, if need be, we arrive at the inequalities

$$m_i^{(k)} \le m_i^{(k+1)}, \quad x_{im_i^{(k)}} \ge x_{im_i^{(k+1)}} \quad (i, k = 1, 2, \ldots).$$

Then, for all k, we have

$$x_{km_k^{(k)}} \vee x_{k+1, m_{k+1}^{(k+1)}} \vee \cdots \le x_{km_k^{(k)}} \vee x_{km_{k+1}^{(k)}} \vee \cdots \le \sup v_k.$$

It is clear that $\overline{\lim}\, x_{km_k^{(k)}} = \mathbf{0}$, i.e., $x_{km_k^{(k)}} \xrightarrow{(o)} \mathbf{0}$. The proof of the theorem is complete.

3.3 The Stone space of a regular and weak σ-distributive algebra

How is the regularity of a Boolean algebra reflected in the properties of its Stone space? Chapter 3 contains Theorem 16 which characterizes the algebras with the countable chain condition in the "Stone" language. We will translate to this language the conditions of regularity and weak σ-distributivity. Throughout this section $\mathscr{X}$ is a complete Boolean algebra and the extremally disconnected compact space $\mathfrak{Q} \equiv \mathfrak{Q}(\mathscr{X})$ is the Stone space of $\mathscr{X}$.

Theorem 17.[19] *A Boolean algebra $\mathscr{X}$ is regular if and only if each countable set of closed rare sets in $\mathfrak{Q}$ admits an embedding into a rare G_δ-set.*

---

[19]Z. T. Dikanova [1].

Theorem 18.[20] *A Boolean algebra $\mathscr{X}$ obeys the weak σ-distributive law if and only if each countable set of closed rare G_δ sets in $\mathfrak{Q}$ has a rare union.*

Prove Theorem 18 first. Let the conditions of the theorem be satisfied. Consider the sequence of downward-directed sets $\{M_i\}_{i=1}^{\infty}$. Prove the inequality (14) in the particular case when $\inf M_i = \mathbf{0}$. The general case can easily be reduced to this situation. All intersections

$$e_i \equiv \bigcap_{x \in M_i} x$$

are closed rare G_δ-sets. Therefore, the union $e \equiv \bigcup_i e_i$ is also rare. Take a nonempty clopen set u disjoint from e. For each i, the sets $\{y \cap u\}_{y \in M_i}$ comprise a downward-directed (hence, centered) system of compact sets with empty intersection. Consequently, there exists $y_i \in M_i$ such that $y_i \cap u = \varnothing$. Consider the "selection" $v = \{y_i\}$. It is clear that $\sup v \wedge u = \mathbf{0}$ and, certainly,

$$\Big(\bigwedge_{v \in V} \sup v\Big) \wedge u = \mathbf{0}.$$

Since $u \subset \mathfrak{Q} \setminus e$ was chosen arbitrarily, and e is rare, we infer:

$$\bigwedge_{v \in V} \sup v = \mathbf{0} = \bigvee_{i=1}^{\infty} \inf M_i.$$

The equality (14) is proved.

Assume now that $\mathscr{X}$ is a Boolean algebra obeying the weak σ-distributive law, and let $e_i \subset \mathfrak{Q}$ be closed rare G_δ-sets. By the corollary to Theorem 3 in Chapter 3, for all $i = 1, 2, \dots$ there exists a sequence of clopen sets $\{x_s^i\}_{s=1}^{\infty}$ such that

$$\bigcap_s x_s^i = e_i.$$

Obviously, we can assume that $x_1^i \supset x_2^i \supset \dots$. Then the set M_i of all x_s^i $(s = 1, 2, \dots)$ is directed downward. Put $e \equiv \bigcup_i e_i$ and show that e is rare. Indeed, the closure of e would otherwise include a nonempty clopen u. It is clear that $\bigwedge_s x_s^i = \mathbf{0}$. Therefore, $\bigwedge \sup v = \mathbf{0}$, although $u \le \sup v$ for all selections v. This contradiction shows that e is rare. The proof of the theorem is complete.

Now, we can easily prove Theorem 17. If $\mathscr{X}$ is a regular Boolean algebra then $\mathscr{X}$ satisfies the countable chain condition and obeys the

[20] R. Sikorski [1].

weak σ-distributive law. If the sets e_i $(i = 1, 2, \ldots)$ are closed and rare then they can be replaced by greater G_δ-sets. By Theorem 18, the union of those G_δ-sets is rare, which allows us to embed the union into a G_δ-set with the aid of Theorem 17 in Chapter 3.

Assume finally that the condition of Theorem 17 is valid. This condition is obviously stronger then that of Theorem 18; therefore, $\mathscr{X}$ is a Boolean algebra obeying the weak σ-distributive law. Using Theorem 17 in Chapter 3, we infer that the countable chain condition is satisfied; and so, by Theorem 16 we conclude that $\mathscr{X}$ is regular.

3.4 Concluding remarks

The concept of weak σ-distributivity can be generalized to the case of an arbitrary cardinal $\mathfrak{m}$ ("weak $\mathfrak{m}$-distributivity"). In this case the analogs of Theorem 17 are valid. We refer the reader to the book by R. Sikorski [1]. Several regularity-type conditions were introduced by K. Matthes [1]. Theorem 15 on a base of solid neighborhoods of zero is generalized to the case of algebras in which the (o)-closure of a set can be obtained in "one step," i.e., by adjoining to the set the limits of all (o)-convergent generalized sequences (A. V. Potepun [2, 3]).

Exercises for Chapter 4

1. Prove that the coincidence of the (o)-convergence and convergence in the (o)-topology means that the algebra under study is discrete.

2. Using the continuum hypothesis, construct a complete Boolean algebra that includes a double sequence with the following properties:

a) $x_{nm} \downarrow \mathbf{0}$ as $m \longrightarrow \infty$ for all $n = 1, 2, \ldots$;

b) $x_{nm_n} \xrightarrow{(o)} \mathbf{1}$ for every increasing sequence $\{m_n\}$.

Chapter 5

HOMOMORPHISMS

This chapter is devoted, primarily, to constructing homomorphisms or, equivalently, continuous mappings of Stone spaces. The following extension problem is central here: Let $\mathscr{X}_0$ be a subalgebra of a Boolean algebra $\mathscr{X}$ and let Φ_0 be a homomorphism from $\mathscr{X}_0$ into a Boolean algebra $\mathscr{Y}$. Does Φ_0 admit an extension to a homomorphism Φ from $\mathscr{X}$ into $\mathscr{Y}$? We mean the existence of a homomorphism Φ such that the following diagram commutes:

Here, as usual, i is the embedding of $\mathscr{X}_0$ into $\mathscr{X}$. Clearly, the role of i can be played by each monomorphism and we need not assume that $\mathscr{X}_0$ is a part of $\mathscr{X}$. We have already discussed this problem in Chapter 3. Those Boolean algebras $\mathscr{Y}$ for which the answer to the question is positive (regardless of the choice of Φ_0, $\mathscr{X}_0$, and $\mathscr{X}$) are injective objects in the category of Boolean algebras. In this chapter we will give a complete characterization for these algebras by proving the Sikorski Theorem mentioned before. If Φ_0 is continuous in some particular sense then the corresponding question arises about existence of a continuous extension to the entire $\mathscr{X}$ or at least to the closure of $\mathscr{X}_0$ in the corresponding topology. The problem will be considered in this chapter too (in the spirit of the articles by K. Mattes, J. D. M. Wright, and the author).

A special class of homomorphisms (which has already been mentioned in Chapter 3) is formed by *liftings*. Here, we consider them in more detail with attention paid mostly to the existence problem. This problem stems from A. Haar; the main results about liftings belong to J. von Neumann, D. Maharam, and A. and K. Ionescu-Tulcea. We will present some of them.

After developing a technique for constructing homomorphisms and, in particular, isomorphisms, we will be able to return once again to the representation problem. This time, we will discuss representation of a Boolean algebra as a quotient algebra of some algebra of sets. Such representations do not resemble Stone spaces; they often seem more "palpable" and are closer to the applications and origins of the theory of Boolean algebras.

1. Extension of homomorphisms

1.1 Transfinite recursion

The process of extending a homomorphism initially defined on some subalgebra resembles the classical proof of the Hahn–Banach Theorem: it is a transfinite chain of "elementary steps." We begin with studying a typical "elementary" situation.

Let $\mathscr{X}_0$ be a subalgebra of a Boolean algebra $\mathscr{X}$ and let h_0 be a homomorphism from $\mathscr{X}_0$ into some Boolean algebra $\mathscr{Y}$. Take an arbitrary element $u \in \mathscr{X} \setminus \mathscr{X}_0$ and consider the subalgebra $\mathscr{X}_1$ generated by $\mathscr{X}_0$ together with this element:

$$\mathscr{X}_1 \equiv \mathscr{X}\langle u, \mathscr{X}_0 \rangle.$$

Is it possible to extend h_0 from $\mathscr{X}_0$ onto $\mathscr{X}_1$? Let us dwell upon this problem. Each element $z \in \mathscr{X}_1$ has the form

$$z = (u \wedge x) \vee (Cu \wedge y), \tag{1}$$

where x and y are some elements of $\mathscr{X}_0$ not uniquely determined in general. It is easy to verify that the equality

$$(u \wedge x) \vee (Cu \wedge y) = (u \wedge x_1) \vee (Cu \wedge y_1) \quad (x, x_1, y, y_1 \in \mathscr{X}_0)$$

is equivalent to the following system of inequalities:

$$\begin{cases} |x - x_1| \leq Cu, \\ |y - y_1| \leq u. \end{cases} \tag{2}$$

Therefore, if (2) is valid then, replacing x and y in (1) by x_1 and y_1, we obtain another representation of z.

Let v be some element of $\mathscr{Y}$ satisfying all inequalities of the form

$$h_0(x') \le v \le h_0(x'') \quad (x', x'' \in \mathscr{X}_0,\ x' \le u \le x''). \tag{3}$$

Put $h(z) \equiv (v \wedge h_0(x)) \vee (Cv \wedge h_0(y))$ whenever z is represented as (1). This definition is sound, since the representation of z as

$$z = (u \wedge x_1) \vee (Cu \wedge y_1) \quad (x_1, y_1 \in \mathscr{X}_0)$$

yields the same value of $h(z)$. Indeed, in this case we have $|x - x_1| \le Cu$ and $|y - y_1| \le u$, whence, taking $x'' \equiv C|x - x_1|$ and $x' \equiv |y - y_1|$, we obtain:

$$|h_0(y_1) - h_0(y)| = h_0(|y - y_1|) \le v \le h_0(C|x - x_1|) = Ch_0(|x - x_1|),$$

$$Cv \ge h_0(|x - x_1|) = |h_0(x) - h_0(x_1)|.$$

Hence, in view of the above remark,

$$(v \wedge h_0(x)) \vee (Cv \wedge h_0(y)) = (v \wedge h_0(x_1)) \vee (Cv \wedge h_0(y_1)).$$

Thus, we obtain a mapping $h : \mathscr{X}_1 \longrightarrow \mathscr{Y}$. It is easy to verify that h is a homomorphism:

$$h(Cz) = h[C((u \wedge x) \vee (Cu \wedge y))] = h[(u \wedge Cx) \vee (Cu \wedge Cy)]$$

$$= (v \wedge h_0(Cx)) \vee (Cv \wedge h_0(Cy)) = C[(v \wedge h_0(x)) \vee (Cv \wedge h_0(y))] = Ch(z);$$

$$h(z_1 \vee z_2) = h[(u \wedge x_1) \vee (Cu \wedge y_1) \vee (u \wedge x_2) \vee (Cu \wedge y_2)]$$

$$= h[(u \wedge (x_1 \vee x_2)) \vee (Cu \wedge (y_1 \vee y_2))] = [v \wedge h_0(x_1 \vee x_2)] \vee [Cv \wedge h_0(y_1 \vee y_2)]$$

$$= [v \wedge (h_0(x_1) \vee h_0(x_2))] \vee [Cv \wedge (h_0(y_1) \vee h_0(y_2))]$$

$$= [(v \wedge h_0(x_1)) \vee (Cv \wedge h_0(y_1))]$$

$$\vee [(v \wedge h_0(x_2)) \vee (Cv \wedge h_0(y_2))] = h(z_1) \vee h(z_2).$$

It is also clear that $h(u) = v$ and the "new" homomorphism h acts at the elements of $\mathscr{X}_0$ in the same way as the initial, thus presenting an extension of the latter: $h_0 = h|_{\mathscr{X}_0}$. Of course, this construction remains valid if $u \in \mathscr{X}_0$; in this case, $h = h_0$.

The choice of an element v that satisfies all inequalities (1) is a key point of the above construction. If such a choice is possible then we say that the subalgebra $\mathscr{X}_0$ and the element u *admit an elementary extension of* h. (Otherwise, if there is no v then the extension is impossible, since, in view of the isotonicity of a homomorphism, all inequalities (3) have to be valid for $v = h(u)$.) Our goal is to extend the homomorphism h_0 to a subalgebra as wide as we can, possibly to the entire $\mathscr{X}$. We can execute

the process transfinitely, "step by step," enumerating the elements of $\mathscr{X} \setminus \mathscr{X}_0$ by ordinals thus making a sequence

$$\{x_\alpha\}_{\alpha<\bar{\alpha}}$$

satisfying (3) and defining homomorphisms h_α on the subalgebras $\mathscr{X}_\alpha$ generated by $\mathscr{X}_0$ together with the elements x_β $(\beta < \alpha)$ so that, at each step, the homomorphism $h_{\alpha+1}$ results from the elementary extension of h_α onto the subalgebra $\mathscr{X}_{\alpha+1} \equiv \mathscr{X}\langle x_\alpha, \mathscr{X}_\alpha\rangle$. The main point here is to provide that, at each step, $\mathscr{X}_\alpha$ and x_α admit such an extension.[1] For this it is sufficient that, for all α and all $E \subset \mathscr{X}_\alpha$, the image $h_\alpha(E)$ have a supremum and an infimum in $\mathscr{Y}$. Then the inequalities (3) are satisfied by each

$$v \in \Big[\bigvee_{x' \in \mathscr{X}_0, x' \le u} h(x'), \bigwedge_{x'' \in \mathscr{X}_0, x'' \ge u} h(x'') \Big].$$

We indicate some obvious simplest cases in which the outlined program can be carried out:

(1) the algebra $\mathscr{Y}$ is complete;

(2) *card* $\mathscr{X}_0 <$ *card* $\mathscr{X}$ and each subset of $\mathscr{Y}$ of cardinality less than card $\mathscr{X}$ has a supremum and an infimum in $\mathscr{Y}$.

Case (1) immediately yields an important theorem:[2]

Theorem 1. *A Boolean algebra $\mathscr{Y}$ is injective* (*i.e., $\mathscr{Y}$ is an injective object in the category* $\mathbb{B}\mathbb{o}\mathbb{o}\mathbb{l}\mathbb{e}$) *if and only if $\mathscr{Y}$ is complete.*

The "if" part of the theorem ensues from the previous reasoning, the "only if" part is already known (cf. 3.3).

1.2 Tychonoff's method

While constructing homomorphisms, another general method is sometimes useful that rests on the celebrated Tychonoff Theorem. This theorem asserts the compactness (with respect to the Tychonoff topology) of the space U^W of all mappings from W into U whenever U is a compact Hausdorff space and W is an arbitrary set.

Here, we consider the case in which $U = [a, b]$. Let F be a continuous function of n real variables. Take a family of n-tuples

$$\Omega = \{\omega^\xi = (\omega_1^\xi, \omega_2^\xi, \dots, \omega_n^\xi)\}_{\xi\in\Xi}$$

[1] If α is a limit ordinal then h_α is (as it is routine in such arguments) the naturally combined extension of all homomorphisms h_β $(\beta < \alpha)$ defined earlier on the subalgebras $\mathscr{X}_\beta$. These subalgebras constitute a chain whose union is $\mathscr{X}_\alpha$.

[2] R. Sikorski [1].

(Ξ is arbitrary and $\omega_1^\xi, \dots, \omega_n^\xi \in W$).

The system of equalities

$$F(f(\omega_1^\xi), \dots, f(\omega_n^\xi)) = 0 \quad (\xi \in \Xi) \tag{4}$$

describes some compact subset $\mathscr{F}$ of U^W that is constituted by all functions f satisfying equalities (4).

The system (4) is infinite in general; to each finite subset $\Xi_0 \subset \Xi$ there corresponds a finite subsystem of equalities which defines a compact set $\mathscr{F}_{\Xi_0} \supset \mathscr{F}$. These sets constitute a downward-directed system; their simultaneous nonemptiness guarantees nonemptiness of $\mathscr{F}$; i.e., solvability of (4) for f. The same remains true in a more general case in which a single function F is replaced by a family $\{F_\alpha\}_{\alpha \in \mathrm{A}}$. The index set A can be constituted in particular by the elements of W or their n-tuples. We will not present the most general statements but rather apply this scheme of a general mathematical nature to constructing a homomorphism from some Boolean algebra $\mathscr{X}$ into the Boolean algebra $\mathscr{Y} \equiv X_T \equiv [0,1]^T$, where T is an arbitrary nonempty set.

Take $U = [0,1]$ and $W = \mathscr{X} \times T$. If f is a function on W and $\omega = (x,t)$ then we sometimes write $f(x,t)$ instead of $f(\omega)$. Put

$$H(u_1, u_2, u_3, u_4, u_5) \equiv [u_1^2 - u_1]^2 + [u_2 + u_1 - 1]^2 + [u_3 - u_4 u_5]^2.$$

We take as Ξ the system of all ordered quadruples $\xi = (x, x_1, x_2, t)$ and take Ω in which

$$\omega^\xi = ((x,t), (Cx,t), (x_1 \wedge x_2, t), (x_1, t), (x_2, t)).$$

Consider the set $\mathscr{F}$ defined in this case by the system of the form (4) for $F = H$. It is just the set of functions $f \in [0,1]^W$ that satisfy all equalities of the form

$$\begin{aligned}&[f^2(x,t) - f(x,t)]^2 + [f(Cx,t) + f(x,t) - 1]^2 \\ &+ [f(x_1 \wedge x_2, t) - f(x_1,t) f(x_2,t)]^2 = 0.\end{aligned} \tag{5}$$

It is easy to understand that $\mathscr{F}$ is the set of all homomorphisms from $\mathscr{X}$ into $\mathscr{Y} = X_T$. More precisely, each function f satisfying (5) generates the homomorphism

$$\Phi_f : \quad \Phi_f(x)(t) = f(x,t),$$

and each homomorphism from $\mathscr{X}$ into X_T is of this provenance.

In the given example, solvability of (5) is beyond any doubt (homomorphisms do exist). However, we are usually interested in the question

of existence of homomorphisms with additional properties. If the properties can also be given by the equations of the type (4) then a new positive summand (corresponding to the desired properties) appears in (5) and we arrive at a problem of the same type. We illustrate this with an example. Suppose that we are looking for an extension of some concrete homomorphism Φ_0 defined on a subalgebra $\mathscr{X}_0$. Then the set of the extensions is given by the following system:

$$\begin{aligned} &H(f(x,t), f(Cx,t), f(x_1 \wedge x_2, t), f(x_1,t), f(x_2,t)) \\ &+ [f(x_0,t) - \Phi_0(x_0)(t)]^2 = 0 \quad (x, x_1, x_2 \in \mathscr{X}, x_0 \in \mathscr{X}_0, t \in T). \end{aligned} \tag{6}$$

In order to prove solvability of this system, acting in accordance with a general recommendation, we could consider all finite subsystems of (6). However, we need not do this, since the Boolean algebra $\mathscr{Y} = X_T$ is complete and the Sikorski Theorem is applicable.

In the next section we will give a more interesting example of applying Tychonoff's method.

2. Lifting

First, we recall the definition. Let $\widehat{\mathscr{X}} = \mathscr{X}|_I$ and let Ψ be the canonical homomorphism from $\mathscr{X}$ onto $\widehat{\mathscr{X}}$. A homomorphism $\rho : \widehat{\mathscr{X}} \longrightarrow \mathscr{X}$ is called a *lifting* whenever ρ satisfies the condition

$$\Psi \circ \rho = Id_{\widehat{\mathscr{X}}}$$

(is a coretraction).

Of the utmost interest are the liftings of metric structures. We have already considered such structures. For every "measure space" $\{\Omega, \mathscr{E}, m\}$ and every ideal $I = \{e \in \mathscr{E} \mid me = 0\}$, the quotient algebra $\widehat{\mathscr{E}} = \mathscr{E}|_I$ is the metric structure of this space. A lifting $\rho : \widehat{\mathscr{E}} \longrightarrow \mathscr{E}$ (if it exists) chooses "representatives" from the cosets in the algebra $\widehat{\mathscr{E}}$; the Boolean operations $\vee$, $\wedge$, and C in $\widehat{\mathscr{E}}$ correspond to the usual set-theoretic operations of union, intersection, and complementation. It is well known that in integration theory and functional analysis as well as in many other areas of mathematics, it is the cosets or the so-called "functions $mod\,0$" that are of the utmost importance. The existence of a lifting makes it possible to consider "usual" functions instead of cosets. Such a possibility is important, for instance, in the theory of integral operators. The problem of existence of a lifting was first raised by A. Haar [1] in the case of a Lebesgue interval. In 1931, a solution to the problem was given by J. von Neumann [2]. We present this result later. First we prove a simple theorem that follows from the results of the above

section. Let us agree to denote by $\mathfrak{m}'$ the successor of a cardinal $\mathfrak{m}$. In the sequel, $\mathscr{X}$ is a Boolean algebra and $\widehat{\mathscr{X}} = \mathscr{X}|_I$.

Theorem 2. *Let $\mathscr{X}$ be an $\mathfrak{m}$-complete Boolean algebra and let I be an $\mathfrak{m}$-ideal in $\mathscr{X}$. If card $\widehat{\mathscr{X}} \leq \mathfrak{m}'$ then there exists a lifting*

$$\rho : \widehat{\mathscr{X}} \longrightarrow \mathscr{X}.$$

PROOF. We reason by the scheme of Section 1. Arrange the elements of $\widehat{\mathscr{X}}$ into the transfinite sequence

$$x_0, x_1, \dots, x_\alpha, \dots; \quad \alpha < \overline{\alpha},$$

where $\overline{\alpha}$ is an initial ordinal of the cardinality *card* $\widehat{\mathscr{X}}$. Put

$$\widehat{\mathscr{X}_0} \equiv \{\mathbf{0}_{\widehat{\mathscr{X}}}, \mathbf{1}_{\widehat{\mathscr{X}}}\}; \quad \rho_0(\mathbf{0}_{\widehat{\mathscr{X}}}) \equiv \mathbf{0}_{\mathscr{X}}, \quad \rho_0(\mathbf{1}_{\widehat{\mathscr{X}}}) \equiv \mathbf{1}_{\mathscr{X}}.$$

Suppose that all $\widehat{\mathscr{X}_\beta}$ $(\beta < \alpha)$ and homomorphisms $\rho_\beta : \widehat{\mathscr{X}} \longrightarrow \mathscr{X}$ are defined so that

a) the subalgebras $\widehat{\mathscr{X}_\beta} \subset \widehat{\mathscr{X}}$ constitute a chain such that $\widehat{\mathscr{X}_{\beta+1}} = \widehat{\mathscr{X}}\langle \widehat{\mathscr{X}_\beta}, x_\beta \rangle$;

b) the homomorphism ρ_β is an extension of ρ_γ if $\beta > \gamma$;

c) the element $\rho_\beta(x)$ belongs to x for all $\beta < \alpha,\ x \in \widehat{\mathscr{X}_\beta}$.

If α is a limit ordinal then, as in Section 1, we put $\widehat{\mathscr{X}_\alpha} \equiv \bigcup_{\beta<\alpha} \widehat{\mathscr{X}_\beta}$ and take ρ_α as a common extension of all ρ_β $(\beta < \alpha)$. For the case in which $\alpha = \gamma + 1$, we show that the subalgebra $\widehat{\mathscr{X}_\gamma}$ and the element x_γ admit an elementary extension ρ_α of ρ_γ and this extension satisfies the condition c) for $\beta = \alpha$.

Put $E^+ \equiv \{x \in \widehat{\mathscr{X}_\gamma} \mid x \geq x_\gamma\}$ and $E^- \equiv \{x \in \widehat{\mathscr{X}_\gamma} \mid x \leq x_\gamma\}$. Both sets are of cardinality at most $\mathfrak{m}$; therefore, there exist

$$a \equiv \bigvee_{x \in E^-} \rho_\gamma(x), \quad b \equiv \bigwedge_{x \in E^+} \rho_\gamma(x)$$

in $\mathscr{X}$ such that $a \leq b$. This ensures the possibility of extending the homomorphism. However, we should guarantee the condition c). Let w be an arbitrary representative of the coset x_γ. Put

$$v \equiv (w \vee a) \wedge b = (w \wedge b) \vee a.$$

It is clear that $a \leq v \leq b$. Show that the elements $v \wedge Cw$ and $w \wedge Cv$ belong to the ideal I. We have

$$v \wedge Cw = a \wedge Cw, \quad w \wedge Cv = w \wedge Cb.$$

Next,

$$a \wedge Cw = \bigvee_{x \in E^-} \rho_\gamma(x) \wedge Cw.$$

The set under the supremum sign has cardinality at most $\mathfrak{m}$; and I is an $\mathfrak{m}$-ideal. Therefore, to prove the desired inclusion it suffices to establish that

$$\rho_\gamma(x) \wedge Cw \in I$$

for all $x \in E^-$. The element $\rho_\gamma(x)$ is a representative of some $x \leq x_\gamma$; and Cw represents Cx_γ. Hence, the infimum $\rho_\gamma(x) \wedge Cw$ calculated in $\mathscr{X}$ represents the element $x \wedge Cx_\gamma = \mathbf{0}_{\widehat{\mathscr{X}}}$. This means that $\rho_\gamma(x) \wedge Cw \in I$, which implies that $v \wedge Cw \in I$. Then $|v - w| \in I$, i.e., $v \in [a, b]$ is a representative of $x_\gamma \in \widehat{\mathscr{X}}$. In accordance with the general scheme of the preceding section, we can put $\rho_\alpha(x_\gamma) = v$ and define ρ_α on the entire subalgebra $\widehat{\mathscr{X}_\alpha}$. It is easy to verify the validity of the condition c). Since all elements of $\widehat{\mathscr{X}}$ are listed in the sequence $\{x_\alpha\}$, the above process will lead to the mapping ρ on $\widehat{\mathscr{X}}$ which is a common extension of all ρ_α. This mapping is a desired lifting.

We apply the last theorem in the case when $\mathfrak{m} = \aleph_0$, $\mathfrak{m}' = \aleph_1$. Here, we should accept the continuum hypothesis: $2^{\aleph_0} = \aleph_1$. Otherwise, there is no σ-complete algebra of cardinality $\aleph_1$. If we accept the continuum hypothesis, Theorem 2 can be applied, in particular, to the case in which $\mathscr{X}$ is the Borel σ-algebra of a complete separable metric space (for definiteness, we can take an interval, since all such sets are isomorphic). As an ideal I we can take the σ-ideal of negligible sets with respect to a countably additive "measure" m on $\mathscr{X}$. (In the case of an interval, Lebesgue measure can be taken.) The quotient algebra $\mathscr{X}|_I$ is the metric structure of cardinality $2^{\aleph_0} = \aleph_1$. Theorem 2 implies existence of a lifting that chooses a Borel representative in each "class 0" of measurable sets. Thus, we have obtained a proof of the above-mentioned theorem of J. von Neumann. This proof is rather simple but uses the continuum hypothesis. We now show how to prove the theorem without appealing to the additional axioms of set theory.

Lemma 1. *Let $\widehat{\mathscr{X}}$ be the metric structure of a measure space $\{T, \mathscr{X}, m\}$ and let the "measure" m be countably additive, finite, and complete. Suppose that there is a mapping $\underline{\rho} : \widehat{\mathscr{X}} \longrightarrow \mathscr{X}$ on $\widehat{\mathscr{X}}$ possessing the properties:*

1) *The set $\underline{\rho}(x)$ is a representative of x (i.e., belongs to x) for all $x \in \widehat{\mathscr{X}}$.*

2) *The equality*

$$\underline{\rho}(x_1 \wedge x_2) = \underline{\rho}(x_1) \cap \underline{\rho}(x_2)$$

holds for all $x_1, x_2 \in \widehat{\mathscr{X}}$.

3) $\underline{\rho}(\mathbf{0}_{\widehat{\mathscr{X}}}) = \varnothing$.

Denote $\overline{\rho}(x) \equiv T \setminus \underline{\rho}(Cx)$. *There exists a lifting* $\rho : \widehat{\mathscr{X}} \longrightarrow \mathscr{X}$ *satisfying the following condition:*

$$\underline{\rho}(x) \subset \rho(x) \subset \overline{\rho}(x)$$

for all $x \in \widehat{\mathscr{X}}$.

The mappings $\underline{\rho}$ and $\overline{\rho}$ are called "lower density" and "upper density" respectively. It is easy to verify that

$$\overline{\rho}(x_1 \vee x_2) = \overline{\rho}(x_1) \cup \overline{\rho}(x_2), \quad \overline{\rho}(\mathbf{1}_{\widehat{\mathscr{X}}}) = T.$$

However, $\underline{\rho}$ and $\overline{\rho}$ are not homomorphisms in general.

In the proof below, it will be more convenient to consider not the algebra of sets $\mathscr{X}$ but the isomorphic Boolean algebra X, a part of the Boolean algebra of characteristic functions X_T which consists of all MEASURABLE two-valued functions. Instead of $\underline{\rho}$ and $\overline{\rho}$, we introduce the corresponding mappings $\underline{R}$ and $\overline{R}$:

$$\underline{R}(x) \equiv \chi_{\underline{\rho}(x)}, \quad \overline{R}(x) \equiv \chi_{\overline{\rho}(x)};$$

and instead of the desired lifting, we first construct the corresponding homomorphism $\Phi : \widehat{\mathscr{X}} \longrightarrow X$. This homomorphism should satisfy the condition

$$\underline{R}(x) \leq \Phi(x) \leq \overline{R}(x)$$

or, which is equivalent,

$$\underline{R}(x)(t) \leq \Phi(x)(t) \leq \overline{R}(x)(t)$$

for all $t \in T$.

We now prefer to use the Tychonoff method. Let H be the function introduced in the end of the preceding section. The main system of identities in our case is as follows:

$$H(f(x,t), f(Cx,t), f(x_1 \wedge x_2, t), f(x_1,t), f(x_2,t)) + [|\overline{R}(x)(t) - f(x,t)|$$
$$- (\overline{R}(x)(t) - f(x,t))]^2 + [|f(x,t) - \underline{R}(x)(t)| - (f(x,t) - \underline{R}(x)(t))]^2. \quad (7)$$

The additional summands satisfy the condition

$$\underline{R}(x)(t) \leq f(x,t) \leq \overline{R}(x)(t).$$

In order to prove consistency of this system, we verify the solvability (for f) of its every finite subsystem. Moreover, we can restrict exposition to subsystems of the form

$$F(x, x_1, x_2, t) = 0 \tag{7'}$$

$$(x, x_1, x_2 \in \widehat{\mathscr{X}_0}, t \in T' \subset T,\ \mathit{card}\, T' < \aleph_0),$$

where $\widehat{\mathscr{X}_0}$ is an arbitrary finite subalgebra of the algebra $\widehat{\mathscr{X}}$; and $F(x, x_1, x_2, t)$ denotes the left-hand side of (7). Consistency of (7′) will be established if we construct a homomorphism $\Phi_0 : \widehat{\mathscr{X}_0} \longrightarrow X_T$ that satisfies the conditions

$$\underline{R}(x)(t) \le \Phi_0(x)(t) \le \overline{R}(x)(t)$$

for all $x \in \widehat{\mathscr{X}_0}, t \in T'$. In this case, the function f, $f(x,t) \equiv \Phi_0(x)(t)$, is the solution of (7′). Describe the construction of the homomorphism Φ_0. Since the subalgebra $\widehat{\mathscr{X}_0}$ is finite, it is generated by a partition of unity $\{x_i\}_{i=1}^{\infty}$. To each $i = 1, 2, \dots, k$, we assign the set $T_i' \subset T'$ by the rule

$$T_i' \equiv \{t \in T' \mid \overline{R}(x_i)(t) = 1\}.$$

Since $\bigvee_i \overline{R}(x_i) = \mathbf{1}$, we have $\max \overline{R}(x_i)(t) = 1$ for all $t \in T$; therefore, $\bigcup_i T_i' = T'$. Put

$$T_1'' \equiv T_1',\ T_2'' \equiv T_2' \setminus T_1'', \dots, T_k'' \equiv T_k' \setminus (T_1'' \cup T_2'' \cup \dots \cup T_{k-1}'').$$

These sets are pairwise disjoint and $\bigcup_i T_i'' = T'$. For all $t \in T_i''$, the equality $\overline{R}(x_i)(t) = 1$ holds; therefore,

$$\chi_{T_i''}(t) \le \overline{R}(x_i)(t)$$

for all $t \in T$. Arrange a partition $\{e_1, e_2, \dots, e_k\}$ of the set T such that the relations $T_i'' \subset e_i$ $(i = 1, 2, \dots, k)$ be valid. Define the desired homomorphism first at the elements x_i by the equality

$$\Phi_0(x_i) \equiv \chi_{e_i};$$

and next at every $x \in \widehat{\mathscr{X}_0}$ in a natural manner: if $x = x_{i_1} \vee \dots \vee x_{i_s}$ then

$$\Phi_0(x) \equiv \chi_{e_{i_1} \cup e_{i_2} \cup \dots \cup e_{i_s}} \equiv \chi_{e_{i_1}} \vee \dots \vee \chi_{e_{i_s}}.$$

It is clear that we have really defined a homomorphism. For all $x \in \widehat{\mathscr{X}_0}$, $x = x_{i_1} \vee \dots \vee x_{i_s}$, and for all $t \in T$ we have

$$\overline{R}(x)(t) = \overline{R}(x_{i_1} \vee \dots \vee x_{i_s})(t) = \max\{\overline{R}(x_{i_1})(t), \dots, \overline{R}(x_{i_s})(t)\},$$

$$\begin{aligned}\Phi_0(x)(t) &= \max\{\chi_{e_{i_1}}(t), \ldots, \chi_{e_{i_s}}(t)\} \\ &= \max\{\chi_{T''_{i_1}}(t), \ldots, \chi_{T''_{i_s}}(t)\} \\ &\le \max\{\overline{R}(x_{i_1})(t), \ldots, \overline{R}(x_{i_s})(t)\} = \overline{R}(x)(t).\end{aligned}$$

On the other hand,

$$\Phi_0(x)(t) = 1 - \Phi_0(Cx)(t) \ge 1 - \overline{R}(Cx)(t) = \underline{R}(x)(t)$$

for all $t \in T'$.

Thus, we have reached our objective. The system (7) is consistent; if f is a function satisfying the system then the equality

$$\Phi(x)(t) \equiv f(x,t) \quad (x \in \widehat{\mathscr{X}}, t \in T)$$

determines a homomorphism from $\widehat{\mathscr{X}}$ into X_T which satisfies

$$\underline{R}(x) \le \Phi(x) \le \overline{R}(x)$$

for all $x \in \widehat{\mathscr{X}}$. We show now that, in fact, $\Phi(\widehat{\mathscr{X}}) \subset X$.

Recall that $\underline{R}(x)$ and $\overline{R}(x)$ are the characteristic functions of the equivalent *mod* 0 measurable sets $\underline{\rho}(x)$ and $\overline{\rho}(x)$. The value $\Phi(x)$ is also of the form χ_e, where $e \subset T$. The last inequality shows that

$$\underline{\rho}(x) \subset e \subset \overline{\rho}(x).$$

Hence it follows that $e \in \mathscr{X}$ $\Phi(x) \equiv \chi_e \in X$, and e is a set of the same coset x. (Here we used the completeness of the "measure" m.) Put $\rho(x) \equiv e$. It is clear that a desired lifting is constructed.

Using the last lemma, we can prove existence of a lifting for every metric structure endowed with a lower (and also upper) density. The most important, and first historically, example of such a situation is provided by the Lebesgue interval and its metric structure E_0. As is well known, for each Lebesgue set e, there is an equivalent *mod* 0 set of DENSITY POINTS (see, for instance, I. P. Natanson [1]). Such a set is the same for all sets of a coset. Therefore, we can introduce the mapping $\underline{\rho}$ that assigns to each $x \in \mathscr{X}$ the set $\underline{\rho}(x)$ of density points common for all representatives of x. It is easy to verify that $\underline{\rho}$ has all properties of a "lower density," so we arrive at the conditions of Lemma 1.

Thus, we have proved the following

Theorem 3. *Let $\mathscr{X}$ be the σ-algebra of all Lebesgue measurable sets of the nondegenerate interval $[0,1]$; let I be the σ-ideal of negligible sets; and let $\widehat{\mathscr{X}}$ be the corresponding metric structure, i.e., $\widehat{\mathscr{X}} \equiv \mathscr{X}|_I = E_0$. Then there exists a lifting $\rho : \widehat{\mathscr{X}} \longrightarrow \mathscr{X}$.*

As was said above, this theorem was proved first by J. von Neumann. In 1958, D. Maharam [4] extended this result to the metric structures of the type E^{Γ} (see p. 61). Her method consists in constructing a lower density for subsets of an infinite-dimensional cube. At present, the existence of a lifting is established for all measure spaces whose metric structure is order complete[3] (A. and K. Ionescu–Tulcea [1]).

It follows from the theorem of lifting that, in the space T, there exists an algebra of sets isomorphic to the metric structure. We emphasize however that such an algebra is not a σ-algebra[4] as a rule; and such a lifting is not continuous nor even σ-continuous. A lifting is just a monomorphism.

3. Extension of continuous homomorphisms

3.1 Continuous extensions

We now consider the problem of finding a continuous extension of a given homomorphism Φ_0 from $\mathscr{X}_0$ to a wider REGULAR subalgebra. Both by the formulation and method of solution, this problem is similar to that of measure theory.

Let $\mathscr{X}$ and $\mathscr{Y}$ be complete algebras. With each $x \in \mathscr{X}$, we associate the system S_x of all sets $e \subset \mathscr{X}_0$ such that $x \leq \sup e$; these sets will be called a *cover* of x. Next, define a function Φ^* on $\mathscr{X}$ with values in $\mathscr{Y}$ by the equality

$$\Phi^*(x) \equiv \bigwedge_{e \in S_x} \bigvee_{y \in e} \Phi_0(y).$$

Note the most important properties of Φ^*.

1°. *If $x_1 \leq x_2$ then $\Phi^*(x_1) \leq \Phi^*(x_2)$.*

This fact follows from the obvious inclusion $S_{x_1} \supset S_{x_2}$ and implies immediately that

2°. *If $x = \sup E$ then*

$$\bigvee_{y \in E} \Phi^*(y) \leq \Phi^*(x).$$

3°. *If $x = \sup E$ and E is finite then*

$$\bigvee_{y \in E} \Phi^*(y) = \Phi^*(x).$$

---

[3] Finiteness of the measure is not supposed.

[4] However, it is a complete Boolean algebra as an isomorphic image of a complete algebra $\widehat{\mathscr{X}}$.

In view of 2°, it suffices to establish the inequality

$$\Phi^*(x) \le \bigvee_{y \in E} \Phi^*(y).$$

First, we consider the case in which e contains two elements: $E = \{y_1, y_2\}$. Choose an arbitrary cover $e_1 \in S_{y_1}$, $e_2 \in S_{y_2}$ and note that the system $\overline{e}$ of all elements of the form $u \vee v$, where $u \in e_1$ and $v \in e_2$, is a cover of x. We have

$$\Phi^*(x) = \bigwedge_{e \in S_x} \bigvee_{y \in e} \Phi_0(y) \le \bigvee_{y \in \overline{e}} \Phi_0(y)$$

$$= \bigvee_{u \in e_1, v \in e_2} \Phi_0(u \vee v) = \Big(\bigvee_{u \in e_1} \Phi_0(u) \Big) \vee \Big(\bigvee_{v \in e_2} \Phi_0(v) \Big).$$

Hence, by the distributive law, we obtain

$$\Phi^*(x) \le \bigwedge_{e_1 \in S_{y_1}, e_2 \in S_{y_2}} \Big[\Big(\bigvee_{u \in e_1} \Phi_0(u) \Big) \vee \Big(\bigvee_{v \in e_2} \Phi_0(v) \Big) \Big]$$

$$= \bigwedge_{e_1 \in S_{y_1}} \Big(\bigvee_{u \in e_1} \Phi_0(u) \Big) \vee \bigwedge_{e_2 \in S_{y_2}} \Big(\bigvee_{v \in e_2} \Phi_0(v) \Big) = \Phi^*(y_1) \vee \Phi^*(y_2).$$

Passage to the general case in which the set E consists of n elements is performed by induction.

4°. *If* $x = \inf E$ *then*

$$\Phi^*(x) \le \bigwedge_{y \in E} \Phi^*(y).$$

Indeed, we have $\Phi^*(x) \le \Phi^*(y)$ for all $y \in E$ by 1°.

5°. *If* $x \in \mathscr{X}_0$ *then* $\Phi^*(x) \le \Phi_0(x)$.

Indeed, the singleton $\{x\}$ belongs to the system S_x.

The property 5° immediately implies

6°. $\Phi^*(\mathbf{0}) = \mathbf{0}$. (We denote the zeros and unities of the two algebras by the same symbols.)

The equality $\Phi^*(\mathbf{1}) = \mathbf{1}$ may fail in general. However, it is valid under the following additional hypothesis:

(*E*) *If* $M \subset \mathscr{X}_0$ *and the supremum of* M *calculated in* $\mathscr{X}$ *belongs to* $\mathscr{X}_0$ *then*

$$\Phi_0(\sup M) = \bigvee_{y \in M} \Phi_0(y).$$

We refer to (E) as to the "*extension condition.*"

Lemma 2. *If the condition (E) holds then $\Phi^*(x) = \Phi_0(x)$ for all $x \in \mathscr{X}_0$.*

Proof. It is sufficient to establish the inequality $\Phi^*(x) \geq \Phi_0(x)$ for $x \in \mathscr{X}_0$. For every cover $e \in S_x$, all elements of the form $y' = x \wedge y$, $y \in e$, belong to the subalgebra $\mathscr{X}_0$ and their supremum is equal to x. Therefore, from (E) we derive

$$\bigvee_{y \in e} \Phi_0(y) \geq \bigvee_{y \in e} \Phi_0(x \wedge y) = \Phi_0(x).$$

Thus, for every cover $e \in S_x$ we have

$$\bigvee_{y \in e} \Phi_0(y) \geq \Phi_0(x),$$

yielding the desired inequality $\Phi^*(x) \geq \Phi_0(x)$.

Corollary 1. *The condition (E) implies $\Phi^*(\mathbf{1}) = \mathbf{1}$.*

Corollary 2. *If (E) is valid then the inequality*

$$C\Phi^*(x) \leq \Phi^*(Cx)$$

holds for all $x \in \mathscr{X}$.

Indeed, this follows from the equality

$$\Phi^*(x) \vee \Phi^*(Cx) = \Phi^*(x \vee Cx) = \mathbf{1}.$$

Lemma 2 shows that, whenever the condition (E) is valid, the restriction $\Phi^*|_{\mathscr{X}_0}$ coincides with Φ_0. Thus, Φ^* is an extension of Φ_0 in this case. However, the mapping Φ^* is not a homomorphism in general. We dwell upon the problem of constructing such a regular subalgebra $\widetilde{\mathscr{X}}$ that includes $\mathscr{X}_0$ and for which the restriction $\Phi^*|_{\widetilde{\mathscr{X}}}$ is a continuous homomorphism. From now on, we assume that (E) is valid. Moreover, we also formulate one more condition that is similar but not equivalent to (E).

(E^*) *For every $M \subset \mathscr{X}$, the equality*

$$\Phi^*(\sup M) = \bigvee_{y \in M} \Phi^*(y)$$

holds. We will refer to (E^*) as to the "*continuous extension condition.*"

Denote by $\widetilde{\mathscr{X}}$ the set of all elements $x \in \mathscr{X}$ such that

$$\Phi^*(Cx) = C\Phi^*(x).$$

The restriction $\Phi^*|_{\widetilde{\mathscr{X}}}$ will be denoted by Φ.

Theorem 4. *The set $\widetilde{\mathscr{X}}$ is a subalgebra of the Boolean algebra $\mathscr{X}$ which includes $\mathscr{X}_0$; and the mapping Φ is a homomorphism from $\widetilde{\mathscr{X}}$ into $\mathscr{Y}$. If, moreover, the condition (E^*) holds then $\widetilde{\mathscr{X}}$ is a regular subalgebra and Φ is a continuous homomorphism.*

Proof. First of all, we prove the validity of the relation

$$\Phi^*(\inf M) = \bigwedge_{y \in M} \Phi^*(y), \tag{8}$$

where M is a finite (an arbitrary whether (E^*) holds) subset of $\widetilde{\mathscr{X}}$. The property 4° makes it possible to prove only the inequality

$$\Phi^*(\inf M) \geq \bigwedge_{y \in M} \Phi^*(y).$$

We have

$$\Phi^*(\inf M) = \Phi^*\Big[C \bigvee_{y \in M} Cy\Big].$$

From Corollary 2 to Lemma 2, we obtain the following lower estimate:

$$\Phi^*\Big[C \bigvee_{y \in M} Cy\Big] \geq C\Phi^*\Big(\bigvee_{y \in M} Cy\Big).$$

Next, we make use of the property 3° if M is finite or the condition (E^*) if M is infinite:

$$C\Phi^*\Big(\bigvee_{y \in M} Cy\Big) = C \bigvee_{y \in M} [\Phi^*(Cy)].$$

Since all y belongs $\widetilde{\mathscr{X}}$, we conclude that $\Phi^*(Cy) = C\Phi^*(y)$ and

$$C \bigvee_{y \in M} [\Phi^*(Cy)] = C \bigvee_{y \in M} [C\Phi^*(y)] = \bigwedge_{y \in M} \Phi^*(y).$$

Combining these relations, we derive the desired estimate and the equality (8).

It is clear from the definition of $\widetilde{\mathscr{X}}$ that it contains complements of its elements. Using (8), we establish that the least upper bound of every finite subset $M \subset \widetilde{\mathscr{X}}$ (an arbitrary subset, if (E^*) holds) also belongs to $\widetilde{\mathscr{X}}$:

$$\Phi^*(C \sup M) = \Phi^*\Big(\bigwedge_{y \in M} Cy\Big) = \bigwedge_{y \in M} \Phi^*(Cy)$$

$$= \bigwedge_{y \in M} \Big[C\Phi^*(y) \Big] = C \bigvee_{y \in M} \Phi^*(y) = C\Phi^*(\sup M).$$

The proven equality means that $\sup M \in \widetilde{\mathscr{X}}$. Thus, we see that $\widetilde{\mathscr{X}}$ *is a subalgebra* (*a regular subalgebra* if (E^*) holds). Next, the inclusion $\mathscr{X}_0 \subset \widetilde{\mathscr{X}}$ ensues from the equality

$$\Phi^*(Cx) = \Phi_0(Cx) = C\Phi_0(x) = C\Phi^*(x)$$

valid for all $x \in \mathscr{X}_0$. Finally, the definition of $\mathscr{X}$ together with 3° and (8) implies that Φ is a homomorphism. The condition (E^*) means the continuity of the homomorphism. The proof of the theorem is complete.

REMARK 1. *Let (E^*) hold. The subalgebra $\widetilde{\mathscr{X}}$ contains all elements that satisfy the condition $\Phi^*(x) = \mathbf{0}$.*

This is obvious from the equality

$$\Phi^*(Cx) \geq C\Phi^*(x) = \mathbf{1}$$

yielding

$$\Phi^*(Cx) = C\Phi^*(x).$$

We see that the equalities $\Phi(x) = \mathbf{0}$ and $\Phi^*(x) = \mathbf{0}$ are equivalent. Therefore, the set

$$\ker \Phi \equiv \{x \mid x \in \mathscr{X},\ \Phi(x) = \mathbf{0}\}$$

is a principal ideal not only of the subalgebra $\widetilde{\mathscr{X}}$ but also of the entire algebra $\mathscr{X}$. We denote this ideal by $\mathscr{X}^*$.

REMARK 2. *Let (E^*) hold. Each element $x \in \widetilde{\mathscr{X}}$ can be represented as $x = \overline{x} - x^*$, where $\overline{x} \in \overline{\mathscr{X}}_0$ and $x^* \in \mathscr{X}^*$.*

Indeed, put

$$\overline{x} \equiv \bigwedge_{e \in S_x} \sup e, \quad x^* \equiv \overline{x} - x. \tag{9}$$

Each $e \in S_x$ is contained in $\overline{\mathscr{X}}_0$; therefore, $\overline{x} \in \overline{\mathscr{X}}_0$. By continuity,

$$\Phi(\overline{x}) = \bigwedge_{e \in S_x} \bigvee_{y \in e} \Phi(y) = \bigwedge_{e \in S_x} \bigvee_{y \in e} \Phi_0(y) = \Phi^*(x) = \Phi(x)$$

and

$$\Phi(x^*) = \Phi(\overline{x}) - \Phi(x) = \mathbf{0},$$

i.e., $x^* \in \mathscr{X}^*$.

The notation $\overline{x}$ for the element defined by the formula (9) is also used in the sequel.

3.2 σ-Continuous extensions

Construction of a σ-continuous extension is carried out mainly by the same scheme. Suppose that the Boolean algebra $\mathscr{X}$ is σ-complete; the algebra $\mathscr{Y}$ is still assumed complete. To each $x \in \mathscr{X}$, we assign the system S_x^σ of all AT MOST COUNTABLE covers of x. Let

$$\Phi_\sigma^*(x) \equiv \bigwedge_{e \in S_x^\sigma} \bigvee_{y \in e} \Phi_0(y).$$

The mapping Φ_σ^* possesses the properties $1°$–$6°$; the proof is exactly the same as above. In order to prove Lemma 2, its corollaries, and Theorem 4 in the "countable" case, we have to modify the formulations of the main conditions, replacing (E) by the condition

(E_σ) *If a set $M \subset \mathscr{X}_0$ is countable and $\sup M \in \mathscr{X}_0$ then*

$$\Phi_0(\sup M) = \bigvee_{y \in M} \Phi_0(y);$$

while replacing (E^*) by the condition

(E_σ^*) *If $M \subset \mathscr{X}$ is countable then*

$$\Phi_\sigma^*(\sup M) = \bigvee_{y \in M} \Phi_\sigma^*(y).$$

Let (E_σ^*) hold. Put

$$\widetilde{\mathscr{X}_\sigma} \equiv \{x \mid x \in \mathscr{X},\ \Phi_\sigma^*(Cx) = C\Phi_\sigma^*(x)\}.$$

The set $\widetilde{\mathscr{X}_\sigma}$ is a subalgebra including $\mathscr{X}_0$; it also contains all elements for which $\Phi_\sigma^*(x) = \mathbf{0}$. If (E_σ^*) is valid then $\widetilde{\mathscr{X}_\sigma}$ is a σ-complete subalgebra. The restriction $\Phi_\sigma \equiv \Phi_\sigma^*|_{\widetilde{\mathscr{X}_\sigma}}$ is a homomorphism from $\widetilde{\mathscr{X}_\sigma}$ into $\mathscr{Y}$; the condition (E_σ^*) ensures its σ-continuity. The homomorphism Φ_σ coincides on $\mathscr{X}_0$ with Φ_0. Finally, the kernel of Φ_σ is a σ-ideal of the algebra $\mathscr{X}$ (but not necessarily a principal ideal).

We now consider Φ^* and Φ_σ^* simultaneously. Let Φ_0 satisfy the condition (E). Then a weaker condition (E_σ) is valid. It follows from the inclusion $S_x^\sigma \subset S_x$ that $\Phi^*(x) \leq \Phi_\sigma^*(x)$. Applying Corollary 2 to Lemma 2, we replace this relation by the two-sided estimate:

$$C\Phi_\sigma^*(Cx) \leq C\Phi^*(Cx) \leq \Phi^*(x) \leq \Phi_\sigma^*(x).$$

Hence we derive that, first, $\widetilde{\mathscr{X}_\sigma} \subset \widetilde{\mathscr{X}}$ and, second, the homomorphisms Φ and Φ_σ coincide on $\widetilde{\mathscr{X}_\sigma}$: $\Phi^*|_{\widetilde{\mathscr{X}_\sigma}} = \Phi_\sigma^*|_{\widetilde{\mathscr{X}_\sigma}}$, or, in other words $\Phi_\sigma = \Phi|_{\widetilde{\mathscr{X}_\sigma}}$. The condition (E^*) implies the continuity and so the σ-continuity of Φ on $\widetilde{\mathscr{X}}$; in this case Φ_σ is also σ-continuous on $\widetilde{\mathscr{X}_\sigma}$. Under the condition (E^*), the subalgebra $\widetilde{\mathscr{X}}$ is regular and, moreover, σ-regular.

3.3 Applications

The method presented in this subsection can be applied to the problem of constructing an ISOMORPHISM. Suppose that the subalgebra $\mathscr{X}_0$ and its image $\Phi_0(\mathscr{X}_0) \equiv \mathscr{Y}_0$ are dense in the corresponding algebras $\mathscr{X}$ and $\mathscr{Y}$ (with respect to the (o)-topologies). We also suppose that the conditions (E) and (E^*) are satisfied. Then, obviously, $\widetilde{\mathscr{X}} = \mathscr{X}$, $\Phi(\widetilde{\mathscr{X}}) = \mathscr{Y}$, and Φ is a continuous epimorphism from $\mathscr{X}$ onto $\mathscr{Y}$. If Φ_0 is bijective and Φ_0^{-1} in turn satisfies the conditions (E) and (E^*) then there exists a continuous epimorphism Ψ from $\mathscr{Y}$ onto $\mathscr{X}$ that coincides with Φ_0^{-1} on $\mathscr{Y}_0$. If $x \in \mathscr{X}_0$ and $y \in \mathscr{Y}_0$ then

$$\Psi(\Phi(x)) = x, \quad \Phi(\Psi(y)) = y.$$

By the continuity of Φ and Ψ these equalities are valid for all $x \in \mathscr{X}$ and $y \in \mathscr{Y}$. Therefore, $\Psi = \Phi^{-1}$ and each mapping Φ and Ψ is an isomorphism.

We now consider an important example related to Theorem 4. Let $\mathscr{Y}$ be a complete Boolean algebra, let $\mathscr{X}$ be the boolean of the Stone space $\mathfrak{Q}(\mathscr{Y})$, and let $\mathscr{Y}^0$ be the clopen algebra (i.e., $\mathscr{X} = 2^{\mathfrak{Q}(\mathscr{Y})}$ and $\mathscr{Y}^0 = \mathscr{C}O(\mathfrak{Q}(\mathscr{Y}))$). We know that there exists an isomorphism from the subalgebra $\mathscr{Y}^0$ onto $\mathscr{Y}$; denote this isomorphism by Ψ_0. It is clear that the condition (E) holds obviously (since $\mathfrak{Q}$ is compact). Therefore, there is a homomorphism Ψ on a subalgebra $\widetilde{\mathscr{X}} \supset \mathscr{Y}^0$ coinciding with Ψ_0 on $\mathscr{Y}^0$ which is constructed in Theorem 4.

Lemma 3. *The kernel of Ψ consists of all rare subsets of $\mathfrak{Q}$.*

PROOF. Let M be a set in the kernel of Ψ. Then

$$\Psi(M) = \Psi^*(M) = \mathbf{0}.$$

For each nonempty clopen set G^0, there exists a family e of clopen sets which covers M and for which the element

$$y \equiv \Psi_0(G^0) \wedge C\Big(\bigvee_{G \in e} \Psi_0(G)\Big)$$

is nonzero. The clopen set $G^1 \equiv \Psi_0^{-1}(y)$ is nonempty and disjoint from each $G \in e$ and, therefore, G^1 is disjoint from M. Moreover, $G^1 \subset G^0$. Since G^0 is arbitrary, we conclude that M is rare.

We now let M be an arbitrary rare set. Denote by Σ the totality of all elements of the form $\Psi_0(G)$, where $G \in \mathscr{Y}^0$ and $G \cap M = \varnothing$. It is clear that Σ minorizes $\mathscr{Y}$ and the supremum of Σ is equal to unity. Consider all complements of the elements of Σ. Their greatest lower

bound is equal to zero and the corresponding clopen set includes M. Hence $\Psi^*(M) = \mathbf{0}$; i.e., M belongs to $\widetilde{\mathscr{X}}$ and to the kernel of Ψ.

Corollary 1. *If the algebra $\mathscr{Y}$ is infinite then the subalgebra $\widetilde{\mathscr{X}}$ is essentially wider than $\mathscr{Y}^0$.*

Indeed, it is easy to see that some nonempty rare set exists in each infinite compact space $\mathfrak{Q}$.

Alongside Ψ, we can consider the homomorphism Ψ_σ defined in 5.3.2. It is clear from the proof of Lemma 3 that this lemma is still valid for Ψ_σ. Thus, the homomorphisms Ψ and Ψ_σ have the common kernel consisting of all rare sets; moreover, it can be proved that these homomorphisms coincide. They are both defined on the algebra $\widetilde{\mathscr{X}}$ consisting of all sets of the form $a = e +_2 q$, where e is clopen and q is rare. Moreover, $\Psi(a) = \Psi_\sigma(a) = \Psi_0(e)$. The question on the degree of continuity for the homomorphism is related to that of the properties (E^*) or (E^*_σ). The same relates to the question of the closure of $\widetilde{\mathscr{X}}$.

Corollary 2. *If the condition (E^*_σ) holds in the above-described situation then the concepts of a meager set and a rare set coincide in the compact space $\mathfrak{Q}$.*

Corollary 3. *If the condition (E^*) holds in the same situation then the algebra $\mathscr{Y}$ is discrete.*

Indeed, let Ψ be a continuous homomorphism extending Ψ_0. The kernel of Ψ is a principal ideal in the Boolean algebra $\mathscr{X} = 2^{\mathfrak{Q}}$. The ideal is the boolean of some set $R \subset \mathfrak{Q}$. The set R cannot include any clopen set e, since $\Psi(e) = \Psi_0(e)$. At the same time, R obviously contains all rare sets.

Every singleton $\{q_0\}$ lying beyond R must be clopen; such a set corresponds to an atom of the algebra $\mathscr{Y}$. Since R cannot include any nonempty clopen sets, there is no nonzero elements disjoint from every atom in $\mathscr{Y}$. Therefore, $\mathscr{Y}$ consists only of the discrete band and has the form $\mathscr{Y} = 2^Q$; $\mathfrak{Q} = \beta Q$ (the Stone–Čech compactification), $R = \mathfrak{Q} \setminus Q$ (the "excrescence") $\mathscr{X} = \widetilde{\mathscr{X}} = 2^{\mathfrak{Q}}$. The homomorphism Ψ coincides with Ψ^* and acts by the formula

$$\Psi(x) = \Psi_0(x \cap Q).$$

If $\mathscr{Y} = \mathscr{Y}^0$ then $\Psi_0(x) \equiv x$ and $\Psi(x) = x \cap Q$.

Remark. For Corollary 3 and the above-made conclusions to be valid, of importance is not the property (E^*) but some secondary fact: existence of a continuous homomorphism extending Ψ_0 to a regular subalgebra. Corollary 3 can be conversed in the certain sense: if $\mathscr{Y}$ is a complete discrete Boolean algebra then, as is easy to verify, the con-

ditions (E^*) and (E^*_σ) always hold irrespectively of $\mathscr{X}$, $\mathscr{X}_0$, and Φ_0. In particular, Ψ^* in our example possesses the property (E^*).

The last remark shows that, in the context of the last example, if the homomorphism Ψ_0 admits a continuous extension Ψ to the regular subalgebra $\overline{\mathscr{Y}^0}$ of the Boolean algebra $\mathscr{X} = 2^{\mathfrak{Q}}$ then such an extension is always realized by the scheme of 5.3.1 the subalgebra $\overline{\mathscr{Y}^0}$ coincides in fact with $\mathscr{X} = 2^{\mathfrak{Q}}$, and

$$\Psi(x) = \Psi_0(x \cap Q)$$

for all $x \subset \mathfrak{Q} = \beta Q$.

The situation is not so trivial if we pass to σ-continuous extensions. There are extensions not embraced by the scheme of 5.3.1 and 5.3.2. We present an example.

Suppose that the quasiextremally disconnected compact space $\mathfrak{Q}(\mathscr{Y})$ has a σ-ideal I containing no nonempty $x \in \mathscr{Y}^0$ but containing all rare sets. Then we can easily verify that the system $\widetilde{\mathscr{X}} \equiv \mathscr{Y}^0 +_2 I$ of the sets of the form

$$x +_2 \omega \quad (x \in \mathscr{Y}^0,\ \omega \in I)$$

is a σ-regular subalgebra in $\mathscr{X}$, $\mathscr{Y}^0 \subset \widetilde{\mathscr{X}}$, and the mapping $\widetilde{\Psi}$ assigning to each $e = x +_2 \omega$ $(\omega \in I)$ the (unique) element $\Psi_0(x) \in \mathscr{Y}$ is a σ-continuous homomorphism from $\widetilde{\mathscr{X}}$ onto $\mathscr{Y}$ which extends Ψ_0. Such a σ-ideal exists in every compact space; we can take, for instance, the ideal of meager sets.[5] Thus, the homomorphism Ψ_0 always has a σ-continuous extension[6] irrespectively of the properties of $\mathfrak{Q}$. Suppose now that the ideal of meager sets contains at least one nonrare set E. Then by Corollary 2, the condition (E^*_σ) may fail. The reason for this failure lies, of course, in existence of such a set E. It exists if $\mathscr{Y}$ is a weakly σ-distributive Boolean algebra; if the Boolean algebra $\mathscr{Y}$ is regular then such a construction is impossible. We consider these questions in more detail in the next subsection.

3.4 The Birkhoff–Ulam homomorphism

Let $\mathfrak{Q}$ be a quasiextremal compact space. In this subsection, we deal with two important subalgebras of $2^{\mathfrak{Q}}$, each of which is constituted by the sets of the form $a = x +_2 \omega$, where x is clopen and ω belongs to some ideal I. This ideal consists of all rare sets in the first case and of meager

---

[5] It is not difficult to see that a clopen subset of a compact space cannot be meager (an analog of the celebrated Baire Category Theorem).

[6] We recall, to make sure, that σ-continuity of $\widetilde{\Psi}$ means that $\widetilde{\Psi}(\bigcup_1^\infty e_n) = \bigvee_1^\infty \widetilde{\Psi}(e_n) = \overline{\bigcup_1^\infty \Psi(e_n)}$.

sets in the second. We will denote the first subalgebra by $\Sigma(\mathfrak{Q})$ and the second, by $\Sigma'(\mathfrak{Q})$ (or, briefly, Σ and Σ'). These subalgebras coincide if $\mathscr{C}O(\mathfrak{Q})$ is a regular Boolean algebra. The subalgebra Σ' is an algebra of sets. It is easy to verify that all Baire sets and, in case $\mathfrak{Q}$ is extremally disconnected, all Borel sets belong to Σ'. (In this case, each open set can be made into a clopen set by adjoining the rare set of boundary points.)

In 5.3.3 we have pointed out the following important fact: *for each quasiextremal compact space $\mathfrak{Q}$, there exists a σ-continuous epimorphism (retraction)*

$$U_{\mathfrak{Q}} : \Sigma' \longrightarrow \mathscr{C}O(\mathfrak{Q})$$

identical at the clopen sets. This homomorphism is called the "*Birkhoff–Ulam homomorphism.*" It is defined, in particular, on the Baire σ-algebra $\mathscr{B}_0(\mathfrak{Q})$ and, if the compact space is extremal, on the Borel σ-algebra $\mathscr{B}(\mathfrak{Q})$ which is wider. The kernel $U_{\mathfrak{Q}}$ of this homomorphism is, as was mentioned, the σ-ideal of meager sets.

3.5 Algebras with the homomorphism extension property

Let $\mathscr{Y}$ be a complete Boolean algebra, let $\mathscr{X}_0$ be a subalgebra of a complete (or a σ-complete) Boolean algebra $\mathscr{X}$, and let Φ_0 be a homomorphism from $\mathscr{X}_0$ into $\mathscr{Y}$ satisfying the condition (E_σ). Does there exist a σ-continuous extension Φ of Φ_0 to a σ-regular subalgebra that includes $\mathscr{X}_0$? A positive answer to this question is ensured by the condition (E^*_σ). However, we saw that this condition does not always hold. Moreover, an extension may exist even when (E^*_σ) fails (see the last example in 5.3.3).

Let us agree to say that a complete Boolean algebra $\mathscr{Y}$ *possesses the σ-continuous extension property* whenever for all σ-complete Boolean algebra $\mathscr{X}$, an arbitrary subalgebra $\mathscr{X}_0$ and a homomorphism Φ_0 from $\mathscr{X}_0$ into $\mathscr{Y}$ satisfying the condition (E_σ) there exist:

a) a σ-regular subalgebra $\widetilde{\mathscr{X}}^\sigma \subset \mathscr{X}$ that includes $\mathscr{X}_0$;

b) a σ-continuous homomorphism $\widetilde{\Phi}$ from $\widetilde{\mathscr{X}}^\sigma$ into $\mathscr{Y}$ whose restriction to $\mathscr{X}_0$ coincides with Φ_0.

Theorem 5. *For a complete Boolean algebra to possess the σ-continuous extension property, it is necessary and sufficient that the algebra be weakly σ-distributive.*

We start with SUFFICIENCY. For brevity, we omit the letter σ and write Φ^* instead of Φ^*_σ. Moreover, we introduce one more mapping Φ_*:

$$\Phi_*(x) \equiv C\Phi^*(Cx) \quad (x \in \widetilde{\mathscr{X}}).$$

Since $\Phi^*(\mathbf{1}) = \mathbf{1}$, we immediately conclude that $\Phi_*(x) \leq \Phi^*(x)$. Let us agree to call an *upper table* every family $\{Q_i\}_{i\in I}$, where I is a countable set of "indices" and Q_i are countable downward-directed subsets of $\mathscr{X}_0$; moreover, we assume that $\mathbf{0}, \mathbf{1} \in Q_i$ for all $i \in I$. Analogously, we introduce the concept of a *lower table*, a countable family $\{P_j\}_{j\in J}$ of countable upward-directed sets of subsets of $\mathscr{X}_0$. As above, we assume that $\mathbf{0}, \mathbf{1} \in P_j$ for all $j \in J$.

The sets Q_i and P_j are called *columns* of the corresponding table. Let V be the cartesian product of the columns of a table. The elements of the product are "selections," i.e., families

$$v = \{x_i\}_{i\in I}, \quad x_i \in Q_i, \quad \text{or} \quad v = \{x_j\}_{j\in J}, \quad x_j \in P_j.$$

The set of the members of a selection v will be denoted by $|v|$.

Consider an arbitrary element $x \in \mathscr{X}$ and an upper table. Among various selections, we choose the part

$$V_x \equiv \{v \in V \mid \ \sup |v| \geq x\}.$$

For a lower table, we analogously put

$$V^x \equiv \{v \in V \mid \ \inf |v| \leq x\}.$$

It is clear that we always have $V_x, V^x \neq \varnothing$, and

$$\Phi^*(x) \leq \bigwedge_{v\in V_x} \bigvee_{y\in|v|} \Phi_0(y),$$

$$\Phi_*(x) \geq \bigvee_{v\in V^x} \bigwedge_{y\in|v|} \Phi_0(y).$$

If the above two relations are still valid when the inequality signs are replaced by the sign $=$ and, moreover, $\Phi_*(x) = \Phi^*(x)$ then we say that both tables *belong to the element* x.

The *dual* table is obtained by replacing the elements in the columns of the original table by their complements. It is clear that, under this transformation, an upper table becomes a lower table and vice versa.

Introduce the subset $\widetilde{\mathscr{X}^\sigma}$ of $\mathscr{X}$ of all x such that there exists a pair of tables belonging to x. We study some properties of $\widetilde{\mathscr{X}^\sigma}$.

1°. $\mathscr{X}_0 \subset \widetilde{\mathscr{X}^\sigma}$. Indeed, the table whose columns consist only of $x \in \mathscr{X}_0$, $\mathbf{0}$, and $\mathbf{1}$ is simultaneously upper and lower and, obviously, belongs to x.

2°. *If* $x \in \widetilde{\mathscr{X}^\sigma}$ *then* $Cx \in \widetilde{\mathscr{X}^\sigma}$. Indeed, if there are two tables that belong to x then the dual tables belong to Cx.

3°. *Let $\{x_n\}_1^\infty$ be a countable sequence in $\widetilde{\mathscr{X}^\sigma}$. Show that*

$$x \equiv \bigvee_{n=1}^{\infty} x_n \in \widetilde{\mathscr{X}^\sigma}.$$

For each n, there are two tables that belong to x_n. Let $\{Q_i^{(n)}\}_{i\in I_n}$ be an upper table and let $\{P_j^{(n)}\}_{j\in J_n}$ be a lower table. Construct some upper and lower tables for x.

An UPPER TABLE is obtained by adjoining all columns Q_i. More precisely, it can be described as follows. We assume that all index sets I_n are pairwise disjoint (clearly, we can achieve it.) Define the new index set by the equality

$$I \equiv \bigcup_{n=1}^{\infty} I_n$$

and put $\overline{Q}_i = Q_i^{(n)}$ whenever $i \in I_n$. Prove that the family $\{\overline{Q}_i\}$ is a desired table. Introduce the sets of selections $\overline{V}_x$ and V_{x_n} (the meaning of these notations is clear). Distinguish the part $\widetilde{V}_x$ in $\overline{V}_x$ that consists of all selections $v = \{x_i\}_{i\in I}$ such that the restrictions

$$v_n \equiv v|_{I_n} \equiv \{x_i\}_{i\in I_n}$$

are selections in V_{x_n}.

Let

$$m(v) \equiv \bigvee_{y\in|v|} \Phi_0(y), \quad m(v_n) \equiv \bigvee_{y\in|v|} \Phi_0(y) \quad (v \in \widetilde{V}_x).$$

It is clear that

$$m(v) = \bigvee_{n=1}^{\infty} m(v_n)$$

and the set M_n of all $m(v_n)$ $(v \in \widetilde{V}_x)$ is directed downward for all $n = 1, 2, \ldots$. Moreover, $\inf M_n = \Phi^*(x_n)$. Using the σ-distributive law, we obtain

$$\Phi^*(x) \le \bigwedge_{v\in\overline{V}_x} m(v) \le \bigwedge_{v\in\widetilde{V}_x} \bigvee_{n=1}^{\infty} m(v_n) = \bigvee_{n=1}^{\infty} \inf M_n = \bigvee_{n=1}^{\infty} \Phi^*(x_n).$$

The inequality $\Phi^*(x) \ge \bigvee_{n=1}^{\infty} \Phi^*(x_n)$ is always valid; therefore,

$$\Phi^*(x) = \bigvee_{n=1}^{\infty} \Phi^*(x_n) = \bigwedge_{v\in\overline{V}_x} m(v) = \bigwedge_{v\in\widetilde{V}_x} m(v).$$

Moreover, since $x_n \in \widetilde{\mathscr{X}^\sigma}$ for $n = 1, 2, \dots$, we have $\Phi^*(x_n) = \Phi_*(x_n)$. Whence,

$$\bigvee_{n=1}^{\infty} \Phi^*(x_n) = \bigvee_{n=1}^{\infty} \Phi_*(x_n) \le \Phi_*(x) \le \Phi^*(x) = \bigvee_{n=1}^{\infty} \Phi^*(x_n).$$

Thus, we conclude that

$$\Phi_*(x) = \Phi^*(x) = \bigwedge_{v \in \overline{V}_x} \bigvee_{y \in |v|} \Phi_0(y) = \bigvee_{n=1}^{\infty} \Phi_*(x_n) = \bigvee_{n=1}^{\infty} \Phi^*(x_n).$$

It remains to construct a lower table $\{\overline{P}_j\}_{j \in \overline{J}}$. It is defined by analogy to the upper table. Suppose again that $J_n \cap J_m = \varnothing$ $(n \ne m)$. Put

$$\overline{J} \equiv \bigcup_{n=1}^{\infty} J_n, \quad \overline{P}_j \equiv P_j^{(n)} \quad (j \in J_n). \tag{10}$$

We introduce the sets $\overline{V}^x$ and V^{x_k} in an obvious way. For an arbitrary $k = 1, 2, \dots$, we arrange the selection $w \in V^{x_k}$ as follows:

$$w = \{w_j\}_{j \in J_k}, \quad w_l \in P_j^{(k)}.$$

Assign to this selection w the new selection

$$v \equiv v(w) \in \overline{V}^x$$

as follows. Put

1) if $j \in J_k$ then $v_j \equiv w_j$;

2) $v_j \equiv \mathbf{1}$, otherwise.

It is clear that

$$\inf |w| = \inf |v(w)|, \quad v(w) \in \overline{V}^x, \tag{11}$$

$$\bigwedge_{y \in |w|} \Phi_0(y) = \bigwedge_{y \in |v(w)|} \Phi_0(y),$$

$$\Phi_*(x_k) = \bigvee_{w \in V^{x_k}} \bigwedge_{y \in |v(w)|} \Phi_0(y) \le \bigvee_{v \in \overline{V}^x} \bigwedge_{y \in |v|} \Phi_0(y)$$

$(k = 1, 2, \dots)$. Hence

$$\Phi_*(x) = \bigvee_{k=1}^{\infty} \Phi_*(x_k) \le \bigvee_{v \in \overline{V}^x} \bigwedge_{y \in |v|} \Phi_0(y). \tag{12}$$

Actually, the equality takes place in the above relation.

We have thus proved that $\Phi_*(x) = \Phi^*(x)$ and there exists a pair of tables belonging to x; therefore, $x \in \widetilde{\mathscr{X}^\sigma}$.

The assertions 1°, 2°, and 3° mean together that $\widetilde{\mathscr{X}^\sigma}$ is a *σ-regular subalgebra of $\mathscr{X}$ including $\mathscr{X}_0$*. The equality $\Phi_*(x) = \Phi^*(x)$ is valid on this subalgebra and we can introduce the mapping $\widetilde{\Phi} : \widetilde{\mathscr{X}^\sigma} \longrightarrow \mathscr{Y}$ by putting

$$\widetilde{\Phi}(x) \equiv \Phi_*(x) \equiv \Phi^*(x) \quad (x \in \widetilde{\mathscr{X}^\sigma}).$$

The formula (12) shows that $\widetilde{\Phi}$ is a σ-continuous homomorphism extending Φ_0 (we have already seen that under the condition (E) the equality $\Phi_0 = \Phi^*|_{\mathscr{X}_0}$ holds).

Thus, the sufficiency part of the theorem is proved.

Prove NECESSITY. Suppose that the Boolean algebra $\mathscr{Y}$ is not weakly σ-distributive. Show that there exists an "inextendible" homomorphism. According to Theorem 18 in Chapter 4, there exists a sequence of rare G_δ-sets $\{F_n\}$ in $\mathfrak{Q} \equiv \mathfrak{Q}(\mathscr{Y})$ whose union $F \equiv \bigcup_{n=1}^\infty F_n$ is dense in a clopen set $e_0 \neq \varnothing$. Without loss of generality we may assume that $e_0 = \mathfrak{Q}$. As above, we denote by $\mathscr{Y}^0$ the clopen algebra of $\mathfrak{Q}$ isomorphic to $\mathscr{Y}$; and by $\mathscr{Y}'^0$ the system of sets of the form $e' = e \cap F$ $(e \in \mathscr{Y}^0)$. Since F is everywhere dense, the correspondence $e \longleftrightarrow e'$ is bijective and preserves the natural order. Thus, $\mathscr{Y}'^0$ is isomorphic to $\mathscr{Y}$ and the mapping

$$\Phi_0 : \quad \Phi_0(e') = e$$

is a homomorphism (and even an isomorphism) from $\mathscr{Y}'^0$ onto $\mathscr{Y}^0$. We regard $\mathscr{Y}'^0$ as a subalgebra of the Boolean algebra $\mathscr{X} \equiv 2^F$. Show that the condition (E) (and therefore, (E_σ)) holds.

Let $e' \equiv \bigvee e'_t$ in $\mathscr{X}$, $e' = e \cap F$ and $e'_t = e_t \cap F$, where $e, e_t \in \mathscr{Y}^0$. This means that $e' = \bigcup e'_t$. (In a discrete algebra, each supremum is a union.)

Since F is everywhere dense, we have $\overline{F \cap G} = \overline{G}$ for every open $G \subset \mathfrak{Q}$. Applying this equality to the open set $\bigcup e_t$ and to the clopen e, we obtain

$$e = \overline{e} = \overline{F \cap e} = \overline{f \cap \bigcup e_t} = \overline{\bigcup e_t} = \bigvee e_t.$$

On the right-hand side we have the supremum calculated in $\mathscr{Y}^0$. In other words,

$$\Phi_0(e') = \bigvee \Phi_0(e'_t).$$

The condition (E) is obtained.

All F_n are closed rare G_δ-sets. They can be represented[7] as the countable intersections

$$F_n = \bigcap_k e_{nk} = \bigcap_k (e_{nk} \cap F),$$

with e_{nk} clopen and $\bigwedge_k e_{nk} = \mathbf{0}$ (in $\mathscr{Y}^0$).

It follows that:

1) Each F_n belongs to the Boolean algebra $\widetilde{\mathscr{X}}$ that is the least σ-regular subalgebra in $\mathscr{X}$ including $\mathscr{Y}'^0$.

2) If Φ_0 admits a countably continuous extension $\widetilde{\Phi}$ to $\widetilde{\mathscr{X}}$ then

$$\widetilde{\Phi}(F_n) = \mathbf{0}_{\mathscr{Y}^0} \text{ and } \widetilde{\Phi}\Big(\bigcup F_n\Big) = \widetilde{\Phi}(F) = \mathbf{0}_{\mathscr{Y}^0}.$$

On the other hand, $\widetilde{\Phi}(F)$ is certainly equal to $\Phi_0(F) = \mathbf{1}_{\mathscr{Y}^0} = \mathfrak{Q}$. Therefore, there is no σ-continuous extension.

The theorem is proved completely. It has a rather long[8] history. A complete proof of the theorem was given for the first time by K. Matthes (sufficiency) and J. D. M. Wright (necessity). Note that these authors considered the so-called "$\mathfrak{m}$-homomorphisms" related to an arbitrary cardinal $\mathfrak{m}$. We studied the case in which $\mathfrak{m} = \aleph_0$; passage to an arbitrary $\mathfrak{m}$ does not meet difficulties.

In our book we are mostly interested in the algebras with the countable chain condition. For these algebras, weak σ-distributivity coincides with regularity. Therefore, Theorem 5 implies

Corollary. *A complete Boolean algebra with the countable chain condition possesses the σ-continuous extension property if and only if it is regular in the sense of L. V. Kantorovich.*

In the case when $\mathscr{Y}$ is a regular algebra, the proof of Theorem 5 can be modified in the sufficiency part: it is easy to verify that the condition (E_σ^*) is valid. Indeed, let M be a countable subset of $\mathscr{X}$. Assign to each $y \in M$ the downward-directed set $\bigvee_{z \in e} \Phi_0(z)$, where $e \in S_y^\sigma$.

In view of the countable chain condition, this set has a countable downward-directed part M_y with the same greatest lower bound. Then $\Phi_\sigma^*(y) = \inf M_y$ for all $y \in M$.

Take a selection $v = \{v_y\}$, where $v_y \in M_y$. This selection presents a countable cover of the element $\sup M$; therefore, $\Phi_\sigma^*(\sup M) \le \sup v$. On the other hand, a regular Boolean algebra is weakly σ-distributive, whence

$$\bigwedge_v \sup v = \bigvee_{y \in M} \inf M_y = \bigvee_{y \in M} \Phi_\sigma^*(y).$$

[7] The corollary to Theorem 3 in Chapter 3.

[8] D. A. Vladimirov [2], R. Sikorski [1, pp. 147–150], and J. D. M. Wright [1].

Thus, for every countable set M we have

$$\Phi^*_\sigma(\sup M) \leq \bigvee_{y\in M} \Phi^*_\sigma(y),$$

which is equivalent to the condition (E^*_σ). We have thus proved the possibility of the σ-continuous extension of a homomorphism.

This proof is shorter than that adduced above. Moreover, we now established the universality of the method of 5.3.2: if a Boolean algebra $\mathscr{Y}$ is regular then a σ-continuous extension Φ of the homomorphism Φ_0 that satisfies the condition (E_σ) can always be constructed by the scheme of 5.3.2.

If weak σ-distributivity takes place without the countable chain condition (regularity does not hold) then Theorem 5 acts. The condition (E^*_σ) may fail; nevertheless, Φ coincides with Φ^* on the σ-algebra $\widetilde{\mathscr{X}}^\sigma$ that is narrower than $\widetilde{\mathscr{X}_\sigma}$.

As for continuous extensions, they were actually studied in the preceding subsection; it remains only to formulate the corresponding result. But first we agree on the terminology: we say that a complete Boolean algebra $\mathscr{Y}$ *possesses the continuous extension property* whenever every homomorphism Φ_0 that satisfies the condition (E) and acts from a subalgebra $\mathscr{X}_0$ of an arbitrary complete Boolean algebra $\mathscr{X}$ into the algebra $\mathscr{Y}$ can be extended to a continuous homomorphism $\Phi : \overline{\mathscr{X}_0} \longrightarrow \mathscr{Y}$.

Theorem 6. *A complete Boolean algebra $\mathscr{Y}$ possesses the continuous extension property if and only if $\mathscr{Y}$ is discrete.*

This theorem ensues from the results of 5.3.3. Indeed, a complete discrete Boolean algebra is an algebra of the form 2^Q. We have already noted that, in this case, the condition (E^*) is valid that implies the continuous extension property. On the other hand, if $\mathscr{Y}$ possesses the continuous extension property then the homomorphism Ψ_0 in the example on the p. 250 can be extended. As we saw, the algebra $\mathscr{Y}$ is then discrete.

The last theorem refers in fact to a very plain situation. The point is that a complete Boolean algebra $\mathscr{X}$ admitting a continuous homomorphism Φ into the Boolean algebra $\mathscr{Y} = 2^Q$ must have a very simple structure. Namely, it consists of two bands, one is discrete and the other is the kernel of Φ. The atoms of $\mathscr{X}$ are the elements of the form

$$x_q \equiv \bigwedge_{\Phi(x')\ni q} x'.$$

Such elements may coincide, but those DISTINCT are necessarily disjoint; their common disjoint complement coincides with the kernel of the homomorphism. In this connection, we recall Theorem 12 in Chapter 3

that gives a complete description of continuous homomorphisms of discrete algebras.

3.6 Another method of extending a homomorphism

The construction of a σ-continuous extension of a homomorphism can be made in two ways. We may first impose some restrictions only on the Boolean algebra $\mathscr{Y}$. This way was studied in the preceding subsection; it leads to Theorem 5. Another approach[9] is based on using the special properties of the original algebra $\mathscr{X}_0$ and of the homomorphism Φ_0. In this case, the assumptions on $\mathscr{Y}$ may be superfluous. We start with the following lemma.

Lemma 4. *Let $\mathscr{X}$ be a σ-complete Boolean algebra; let $\mathscr{X}_0$ be a subalgebra of $\mathscr{X}$; let $\mathscr{Y}$ be a σ-complete Boolean algebra; let $\mathfrak{Q} \equiv \mathfrak{Q}(\mathscr{Y})$; and let Φ_0 and Ψ_0 be homomorphisms on $\mathscr{X}_0$ with values in $\mathscr{Y}^0$ and $\Sigma'(\mathfrak{Q})$ respectively. Suppose that these homomorphisms are related by the following condition: for every $x \in \mathscr{X}_0$, the symmetric difference of $\Phi_0(x)$ and $\Psi_0(x)$ is a meager set in $\mathfrak{Q}$. If, moreover, the homomorphism Ψ_0 satisfies the condition (E_σ) then there exists a σ-regular subalgebra $\mathscr{X}_1 \subset \mathscr{X}$, $\mathscr{X}_1 \supset \mathscr{X}_0$, and a σ-continuous homomorphism from this subalgebra into $\mathscr{Y}^0$ which extends Φ_0.*

Here, $\Sigma'(\mathfrak{Q})$ is considered as a σ-subalgebra of $\mathfrak{Q}$ and $\mathscr{Y}^0 \equiv \mathscr{C}O(\mathfrak{Q})$.

Proof. The values of the homomorphism Ψ_0 belong to the discrete subalgebra $2^{\mathfrak{Q}}$. Therefore, there exists a σ-continuous extension $\Psi : \mathscr{X}_1 \longrightarrow 2^{\mathfrak{Q}}$ constructed by the scheme of 5.3.5, where $\mathscr{X}_1$ is a σ-regular subalgebra of $\mathscr{X}$ that includes $\mathscr{X}_0$. We assume that $\mathscr{X}_1 = \mathscr{X}_\sigma\langle\mathscr{X}_0\rangle$. It is clear that $\Psi(\mathscr{X}_1) \subset \Sigma'(\mathfrak{Q})$ (since it is true for $\Psi(\mathscr{X}_0)$). Introduce the mapping $\Phi \equiv U \circ \Psi$, where U is the Birkhoff–Ulam homomorphism. We have a σ-continuous homomorphism from $\mathscr{X}_1$ into $\mathscr{Y}^0$. Given $x \in \mathscr{X}_0$, infer that

$$\Phi(x) = U(\Psi(x)) = U(\Psi_0(x)) = U(\Phi_0(x)) = \Phi_0(x),$$

since the meager sets constitute the kernel of U and the set $\Phi_0(x)$ is clopen. Therefore, $\Phi|_{\mathscr{X}_0} = \Phi_0$ and the desired extension is constructed. If need be, we can extend the algebra $\mathscr{X}_1$ by adjoining all elements dominated by the elements in $\ker \Phi$. So we obtain a wider σ-regular subalgebra to which Φ extends uniquely. The lemma is proved.

The kernel of the above-constructed homomorphism Φ contains those and only those $x \in \mathscr{X}$ whose image $\Psi(x)$ is a meager set.

[9] A. G. Kusraev and S. G. Malyugin [1]; J. D. M. Wright [1].

We now let $\mathscr{A}$ be an algebra of sets of a compact space R. Suppose that this algebra separates the points of R in a strengthened form:

(∗) *for every two distinct points $t_1, t_2 \in R$ there are sets $A_1, A_2 \in \mathscr{A}$ with the properties:*

$$t_1 \in \mathit{Int}\, A_1, \quad t_2 \in \mathit{Int}\, A_2, \quad \mathit{clos}\, A_1 \cap \mathit{clos}\, A_2 = \varnothing.$$

We point out some facts easily deductible from the property (∗).

1°. *If F is compact, G is open, and $F \subset G$ then there exists $A \in \mathscr{A}$ such that $F \subset \mathit{Int}\, A \subset \mathit{clos}\, A \subset G$.*

2°. *If F_1 and F_2 are disjoint compact spaces then there exist $A_1, A_2 \in \mathscr{A}$ such that*

$$F_1 \subset \mathit{Int}\, A_1, \quad F_2 \subset \mathit{Int}\, A_2, \quad \mathit{clos}\, A_1 \cap \mathit{clos}\, A_2 = \varnothing.$$

Let us agree to assign

$$\Sigma_E^+ \equiv \{A \in \mathscr{A} \mid \mathit{Int}\, A \supset E\}, \quad \Sigma_E^- \equiv \{A \in \mathscr{A} \mid \mathit{clos}\, A \subset E\}.$$

3°. *If F is compact and G is open then*

$$F = \bigcap_{A \in \Sigma_F^+} A, \qquad G = \bigcup_{A' \in \Sigma_G^-} A'.$$

Suppose that, on some algebra $\mathscr{A}$ with the property (∗), there is a homomorphism Φ_0 with values in an arbitrary algebra $\mathscr{Y}$. We identify the last algebra with its Stone space $\mathscr{Y}^0 \equiv \mathscr{C}O(\mathfrak{Q})$, where $\mathfrak{Q} \equiv \mathfrak{Q}(\mathscr{Y})$, and regard it as a subalgebra of the discrete algebra $2^{\mathfrak{Q}}$. We now construct a new homomorphism Ψ with values in $2^{\mathfrak{Q}}$ which is generated in a certain sense by the homomorphism Φ_0. We define it consecutively, starting with closed sets.

If F is compact then we put

$$\Psi_0(F) \equiv \bigcap_{A \in \Sigma_F^+} \Phi_0(A).$$

It is clear that $\Psi_0(\varnothing) = \varnothing$. Note some properties of the mapping Ψ_0.

1) *Monotonicity:* $F_1 \subset F_2$ *implies* $\Psi_0(F_1) \subset \Psi_0(F_2)$. This is obvious.

2) *If $F = F_1 \cap F_2 \cap \cdots \cap F_m$, where F_i are compact, then*

$$\Psi_0(F) = \Psi_0(F_1) \cap \Psi_0(F_2) \cap \cdots \cap \Psi_0(F_m).$$

It is sufficient to prove this for $m = 2$. Let $F = F_1 \cap F_2$, where F_1 and F_2 are compact. Take an arbitrary $A \in \Sigma_F^+$ and consider the compact sets

$$\widetilde{F}_1 \equiv F_1 \setminus Int\, A, \quad \widetilde{F}_2 \equiv F_2 \setminus Int\, A.$$

They are disjoint. Therefore, by 2°, we can choose $\widetilde{A}_1, \widetilde{A}_2 \in \mathscr{A}$ such that the conditions

$$\widetilde{F}_1 \subset Int\, \widetilde{A}_1, \quad \widetilde{F}_2 \subset Int\, \widetilde{A}_2, \quad clos\, \widetilde{A}_1 \cap clos\, \widetilde{A}_2 = \varnothing$$

are satisfied. Certainly, $\widetilde{A}_1 \cap \widetilde{A}_2 = \varnothing$. Put $A_1 \equiv A \cup \widetilde{A}_1$ and $A_2 \equiv A \cup \widetilde{A}_2$. It is clear that

$$A_1, A_2 \in \mathscr{A}, \;\; A_1 \cap A_2 = A, \;\; \Phi_0(A) = \Phi_0(A_1) \cap \Phi_0(A_2) \supset \Psi_0(F_1) \cap \Psi_0(F_2).$$

Since A is arbitrary, we conclude that

$$\Psi_0(F) \supset \Psi_0(F_1) \cap \Psi_0(F_2).$$

The reverse inclusion is obvious.

To each $q \in \mathfrak{Q}$, we now assign the system of compact sets

$$T_q \equiv \{F \mid \Psi_0(F) \ni q\}$$

(here and in the sequel, the letter F always denotes a compact subset of R). The system T_q is nonempty, directed, and centered. Indeed, first, $R \in T_q$ and, second, every intersection $F_1 \cap F_2 \cap \cdots \cap F_m$, where $F_i \in T_q$, is nonempty and belongs to T_q, since by 2),

$$\Psi_0(F_1 \cap \cdots \cap F_m) = \Psi_0(F_1) \cap \cdots \cap \Psi_0(F_m) \ni q$$

and $\Psi_0(\varnothing) = \varnothing$. Denote by S_q the intersection of T_q: $S_q \equiv \bigcap_{F \in T_q} F$. This set is compact and nonempty for every $q \in \mathfrak{Q}$. Moreover, it possesses the following properties:

a) *If $\Phi_0(A) \not\ni q$ and $A \in \mathscr{A}$ then $Int\, A \cap S_q = \varnothing$.* Indeed, consider the compact set $B \equiv R \setminus Int\, A$. It is clear that $B \supset R \setminus A$ and for all $A' \in \mathscr{A}$ and $A' \supset B$ we have $\Phi_0(A') \supset \mathfrak{Q} \setminus \Phi_0(A) \ni q$. Hence, $\Psi_0(B) \ni q$; i.e., $B \in T_q$ and $S_q \subset B$. The assertion is proved.

b) *The set S_q contains only one point for every $q \in \mathfrak{Q}$.* We have already seen that $S_q \neq \varnothing$. If it contained two distinct points t_1 and t_2; then, using $(*)$, we could find $A_1, A_2 \in \mathscr{A}$ with the properties

$$Int\, A_1 \ni t_1, Int\, A_2 \ni t_2, A_1 \cap A_2 = \varnothing.$$

But then at least one of the sets $\Phi_0(A_1)$ and $\Phi_0(A_2)$ does not contain the point q; therefore, t_1 and t_2 cannot belong to S_q simultaneously.

Denote the unique point of S_q by $\psi(q)$. Thus, we have defined some mapping $\psi : \mathfrak{Q} \longrightarrow R$. Together with it, the homomorphism $\Psi : 2^R \longrightarrow 2^{\mathfrak{Q}}$ appears that is defined by the formula

$$\Psi(e) \equiv \psi^{-1}(e), \quad e \subset R.$$

This homomorphism is continuous by Theorem 3.12; i.e., the condition (E_σ) holds.

Check now that the homomorphism Ψ coincides with Ψ_0 at closed sets. In other words, prove the equality

$$\psi^{-1}(F) = \Psi_0(F),$$

where F is compact.

Take $q \in \Psi_0(F)$ first. Recall that the set S_q that consists of the unique element $\psi(q)$ is included in every compact set F' such that $q \in \Psi_0(F')$. Since $q \in \Psi_0(F)$ we have $\psi(q) \in F$ or, which is the same, $q \in \psi^{-1}(F)$. Thus, $\Psi_0(F) \in \psi^{-1}(F)$. Let now $q \in \psi^{-1}(F)$, i.e., $\psi(q) \in F$. This containment means that $F \supset S_q \equiv \{\psi(q)\}$. Take an arbitrary $A \in \mathscr{A}$ such that $Int\, A \supset F \supset S_q$. We have $\Phi_0(A) \ni q$ (the property a)). Then $\Psi_0(F) \ni q$. We have proved the inclusion $\psi^{-1}(F) \subset \Psi_0(F)$ and, thereby, the desired equality.

We formulate an intermediate result: *if an algebra of sets $\mathscr{A}$ of a compact space R possesses the property $(*)$ then, for every homomorphism Φ_0 from this algebra into an algebra $\mathscr{Y} = \mathscr{C}O(\mathfrak{Q})$, there exists a continuous homomorphism $\Psi : 2^R \longrightarrow 2^{\mathfrak{Q}}$ related to Φ_0 by the condition: for every compact set $F \subset R$, the equality holds*

$$\Psi(F) = \bigcap_{F \subset Int\, A,\, A \in \mathscr{A}} \Phi_0(A).$$

We should observe that the "new" homomorphism Ψ is not an extension of the "old" Φ_0 in general. However, such an extension can be constructed with the aid of Ψ. To be able to do this, we need some additional assumptions. First of all, we assume the algebra $\mathscr{Y}$ to be σ-complete. Next, denoting by the symbol $\mathscr{A}_0$ the subalgebra of $\mathscr{A}$ generated by the closed sets in $\mathscr{A}$, we suppose that the following conditions hold:

$(**)$ *Each $A \in \mathscr{A}$ can be represented as*

$$A = \bigcup_{n=1}^{\infty} A_n^0, \quad A_1^0 \subset A_2^0 \subset \ldots; \quad A_n^0 \in \mathscr{A}_0 \quad (n = 1, 2, \ldots).$$

$(***)$ *For each closed set F in $\mathscr{A}$, there exists a sequence $\{A_n\}$ of sets in Σ_F^+ such that every $A \in \Sigma_F^+$ includes some A_n.*

Notice that the condition $(***)$ holds certainly if R is a metrizable compact space (for every algebra $\mathscr{A}$ with the property $(*)$).

Finally, suppose that the original homomorphism Φ_0 possesses the property (E_σ) (for Ψ, this property follows from continuity).

Let F be a closed set in $\mathscr{A}$ and let $\{A_n\}$ be a sequence of sets from the condition $(***)$. By the property (E_σ), we have

$$\Phi_0(F) = \bigwedge_{n=1}^{\infty} \Phi_0(A_n) = Int \bigcap_{n=1}^{\infty} \Phi_0(A_n),$$

and $(***)$ ensures that

$$\Psi(F) = \bigcap_{n=1}^{\infty} \Phi_0(A_n).$$

Therefore, the difference $\Psi(F) \setminus \Phi_0(F)$ is rare. The same is obviously true for every open $G \in \mathscr{A}$. Hence and from the conditions $(**)$ and (E_σ), it follows that the symmetric difference

$$(\Psi(A) \setminus \Phi_0(A)) \cup (\Phi_0(A) \setminus \Psi(A))$$

is a meager set in $\mathfrak{Q}$ for all $A \in \mathscr{A}$.

Finally, we notice that all values $\Psi(A)$ $(A \in \mathscr{A})$ lie in the Baire algebra $\mathscr{B}_0 \subset \Sigma'(\mathfrak{Q})$. This ensues from the continuity of Ψ, the condition $(**)$, and from the fact that the values $\Psi(F)$ for closed sets F are calculated by the formula

$$\Psi(F) = \bigcap_{n=1}^{\infty} \Phi_0(A_n),$$

where all sets $\Phi_0(A_n)$ are clopen.

We see that Lemma 4 is applicable to the pair of homomorphisms Φ_0 and $\Psi|_{\mathscr{A}}$. We have thus proved the following

Theorem 7. *Let $\mathscr{A}$ be an algebra of sets of a compact space R which satisfies the conditions $(*)$, $(**)$, and $(***)$; let $\mathscr{Y}$ be a complete Boolean algebra; and let Φ_0 be a homomorphism from $\mathscr{A}$ into $\mathscr{Y}$ possessing the property (E_σ). There exist a σ-algebra $\widetilde{\mathscr{A}}$ including $\mathscr{A}$ and a σ-continuous homomorphism $\Phi : \widetilde{\mathscr{A}} \longrightarrow \mathscr{Y}$ extending Φ_0.*

The above theorem can be applied even when the scheme of 5.3.2 and Theorem 5 fail to provide a countably additive extension. (It is possible that the extension of Theorem 7 differs from Φ^* on each σ-regular subalgebra including $\mathscr{A}$.) We now present an important example of such an application.

3.7 Free Borel algebras

Let $\mathfrak{a}$ be an arbitrary cardinal. A *free Borel algebra*[10] on $\mathfrak{a}$ generators is such a σ-complete Boolean algebra $\mathscr{X}$ that includes a subset Γ with the following properties:

1) $card\,\Gamma = \mathfrak{a}$;

2) $\overset{\cdots\cdots\cdots}{\mathscr{X}_\sigma\langle\Gamma\rangle} = \mathscr{X}$ (the set Γ "σ-generates" $\mathscr{X}$);

3) each mapping from Γ into an arbitrary σ-complete Boolean algebra $\mathscr{Y}$ is extendible to a σ-continuous homomorphism from $\mathscr{X}$ into $\mathscr{Y}$.

Theorem 7 allows us to indicate such an algebra.

Consider the Cantor discontinuum X_T of weight $\mathfrak{a}$ and the system comprising the sets

$$Q_t \equiv \{\chi \in X_T \mid \chi_t = 1\}, \quad t \in T, \quad card\,T = \mathfrak{a}.$$

(Recall that the points of X_T are "binary" families of the following form: $\chi = \{\chi_t\}_{t\in T}$, $\chi_t = 0, 1$.) The system $\{Q_t\}$ is independent. The algebra $\mathscr{D} \equiv \mathscr{D}_T \equiv \mathscr{C}O(X_T)$ generated by $\{Q_t\}$ is free on $\mathfrak{a}$ generators. Consider the Baire algebra $\mathscr{B}_0 \equiv \mathscr{B}_0(X_T)$, the least σ-algebra that includes $\mathscr{D}$. Show that $\mathscr{B}_0$ *is a "free Borel" algebra on* $\mathfrak{a}$ *generators.* Indeed, let $\mathscr{Y}$ be a σ-complete Boolean algebra and let Φ_{00} be a mapping from the system $\{Q_t\}$ into $\mathscr{Y}$. First, using independence of the system, we extend Φ_{00} to a homomorphism Φ_0 that acts from $\mathscr{D}$ into $\mathscr{Y}$. Now we apply Theorem 7 taking as $\mathscr{A}$ the algebra $\mathscr{D}$. The latter algebra consists of all clopen subsets of the compact space X_T; therefore, Φ_0 possesses the property (E_σ). As for the conditions $(*)$, $(**)$, and $(***)$, they are trivially hold. Whence, there exists a σ-continuous homomorphism $\Phi : \mathscr{B}_0 \longrightarrow \mathscr{Y}$ extending Φ_0 and Φ_{00}. This means that the Baire algebra $\mathscr{B}_0$ is a free Borel algebra. (This fact was established for the first time by R. Sikorski and L. Rieger.)

It is easy to understand that all free Borel algebras on the systems of generators of the same cardinality are isomorphic. K. Kuratowski actually proved that the Borel σ-algebra of every infinite complete separable metric space is a free Borel algebra on countably many generators.[11]

---

[10] In this context, the term "Borel algebra" ITSELF means "σ-complete algebra." However, the term "free Borel algebra" cannot be divided: it is formed in accordance with the "category-theoretic" mentality. We could introduce the category of Borel algebras; then a free object of this category would be a "free Borel algebra" (see M. Sh. Tsalenko and E. G. Shul′geĭfer [1, p. 27]). Such an algebra is not free in the usual sense. Recall that the term "Borel algebra of a topological space" is used for the σ-algebra of Borel sets which is wider in general than the Baire σ-algebra $\mathscr{B}_0$.

[11] K. Kuratowski [2, p. 462] and G. Birkhoff [3].

The Borel algebras of an interval, square, etc., are isomorphic Boolean algebras.

By now we have considered only the class of free Borel (σ-complete) algebras. A wider class of "free algebras" for every cardinal $\mathfrak{a}$ and every cardinality $\mathfrak{b}$ of the system of generators can be considered too. L. Rieger proved existence of such algebras for all $\mathfrak{a}$ and $\mathfrak{b}$.[12] It might appear reasonable to introduce the concept of a "free complete Boolean algebra," i.e., a complete Boolean algebra $\mathscr{X}$ that has a fully generating system Γ with the property: each mapping Γ into each complete Boolean algebra $\mathscr{Y}$ is extendible to a continuous homomorphism from $\mathscr{X}$ into $\mathscr{Y}$. However, H. Gaifman and A. Hales proved simultaneously that these algebras do not exist.[13]

3.8 Homomorphisms of Boolean products

Consider complete Boolean algebras $\mathscr{X}$ and $\mathscr{Y}$ represented as products:

$$\mathscr{X} = \prod_{t \in T} \mathscr{X}_t, \quad \mathscr{Y} = \prod_{t \in T} \mathscr{Y}_t.$$

Here, $\{\mathscr{X}_t\}$ and $\{\mathscr{Y}_t\}$ are independent systems of regular subalgebras indexed by the same index set. Suppose that there are continuous (σ-continuous) homomorphisms $\Phi_t : \mathscr{X}_t \longrightarrow \mathscr{Y}_t$. We know that these homomorphisms have a common extension $\widetilde{\Phi}$ defined on the subalgebra

$$\mathscr{X}_0 = \mathscr{X}\Big\langle \bigcup_t \mathscr{X}_t \Big\rangle.$$

Usually, we wish to extend this homomorphism to the entire Boolean algebra $\mathscr{X} = \overline{\mathscr{X}_0}$ without loss of continuity. We restrict exposition to the simplest and most important case. Let $\mathscr{X}$ and $\mathscr{Y}$ be *regular* Boolean algebras. Then, applying Theorem 5, we arrive at the following conclusion: *a common continuous extension of continuous homomorphisms Φ_t $(t \in T)$ to the entire $\mathscr{X}$ exists if and only if the condition (E_σ) holds which reduces in our case to the following: if*

$$u_0 = \sum_1^\infty u_k,$$

where

$$u_i = x_1^i \wedge \cdots \wedge x_{n_i}^i, \;\; x_j^i \in \mathscr{X}_{t_j} \quad (i = 0, 1, \dots; j = 1, 2, \dots, n_i),$$

[12] R. Sikorski [1, p. 213].
[13] H. Gaifman [2] and A. Hales [1].

and $t_1, t_2, \ldots$ are pairwise disjoint indices; then

$$\Phi_{t_1}(x_1^0) \wedge \cdots \wedge \Phi_{t_{n_0}}(x_{n_0}^0) = \sum_k \Phi_{t_1}(x_1^k)^0 \wedge \cdots \wedge \Phi_{t_{n_k}}(x_{n_k}^k)^0$$

(the "countable additivity of $\widetilde{\Phi}$"). Such a form of the condition (E_σ) will be denoted by (E'_σ).

In particular, *if all Φ_t are isomorphisms from the regular subalgebras $\mathscr{X}_t$ onto the corresponding regular independent subalgebras $\mathscr{Y}_t$, $\mathscr{Y} = \prod_t \mathscr{Y}_t$, satisfying the condition (E'_σ) and if, moreover, the inverse isomorphisms Φ_t^{-1} satisfy the condition (E'_σ) too then the algebras $\mathscr{X}$ and $\mathscr{Y}$ are isomorphic; moreover, the common extension Φ of the isomorphisms Φ_t is such an isomorphism.* The common extension is obviously unique, and we will call it the *product* of Φ_t. Certainly, such a product does not always exist. To denote this isomorphism, we use the symbols

$$\prod_{t \in T} \Phi_t, \quad \Phi_1 \times \Phi_2 \times \cdots \times \Phi_m \quad (\text{if } T = \{1, 2, \ldots, m\}).$$

4. Again on representation of a Boolean algebra

In this section, we return to the question on representation of a Boolean algebra. We know that each Boolean algebra can be represented as an algebra of sets. The Stone space we use in this representation has a very large cardinality. In concrete mathematical situations, a Boolean algebra appears often as a homomorphic image (a quotient algebra) of a σ-algebra of sets. Of such type, for instance, are all metric structures. In such cases, the cardinality of the original space is not so large as in the "Stone case." The theorem of lifting makes it possible to obtain a representation in the form of an algebra of sets of that (more palpable) space.

4.1 The Sikorski–Loomis Theorem on representation of a σ-complete Boolean algebra

Theorem 8. *For each σ-complete Boolean algebra $\mathscr{X}$, there exist a space R, a σ-algebra $\mathscr{E}$ of subsets of R, and a σ-ideal I of the σ-algebra $\mathscr{E}$ such that $\mathscr{X}$ is isomorphic to the quotient algebra $\mathscr{E}|_I$.*

(In other words, *there exists an epimorphism from $\mathscr{E}$ onto $\mathscr{X}$.*)

Let us agree to call R the *representation space of $\mathscr{X}$*.

In fact, this theorem has already been proved. We saw that the Birkhoff–Ulam homomorphism of 5.3.4 is σ-continuous and maps the σ-algebra $\Sigma'(\mathfrak{Q})$ onto $\mathscr{C}O(\mathfrak{Q})$. If $\mathfrak{Q} = \mathfrak{Q}(\mathscr{X})$ then, with the isomorphism between $\mathscr{X}$ and $\mathscr{C}O(\mathfrak{Q})$ taken into account, we can construct

a desired σ-continuous homomorphism from $\Sigma'(\mathfrak{Q})$ onto $\mathscr{X}$. Thus, the representation space is the compact space $\mathfrak{Q}(\mathscr{X})$, $\mathscr{E} = \Sigma'(\mathfrak{Q})$, and I is the σ-ideal of meager sets in $\mathfrak{Q}$.

The Sikorski–Loomis Theorem has essentially "countable" character. It does not hold for greater cardinalities. If $\mathfrak{m} > \aleph_0$ then not every $\mathfrak{m}$-algebra is an $\mathfrak{m}$-homomorphic image of an $\mathfrak{m}$-algebra of sets.[14]

The above-presented proof of the Sikorski–Loomis Theorem has two shortcomings. First, it leads to the representation space of a too great cardinality. Second, the representation space itself (the Stone space) is defined by the algebra $\mathscr{X}$ and is not the same for distinct algebras. A universal representation space can be adduced for every σ-complete Boolean algebras of a fixed cardinality. Let a cardinal $\mathfrak{b}$ be the cardinality of a σ-complete Boolean algebra $\mathscr{X}$, and let T be a set of cardinality $\mathfrak{b}$. Consider the Cantor discontinuum X_T. As we know, this compact space includes an independent system of sets $\{Q_t\}_{t\in T}$ that σ-generates the Baire algebra $\mathscr{B}_0 = \mathscr{B}_0(X_T)$. Let Φ_{00} be an arbitrary surjection of this system onto $\mathscr{X}$. There exists a σ-continuous homomorphism Φ that maps $\mathscr{B}_0$ onto $\mathscr{X}$ (see 5.3.7). If the Boolean algebra $\mathscr{X}$ is complete then the homomorphism Φ can be extended to a homomorphism from a Dedekind completion $\widehat{\mathscr{D}}$ of the free Boolean algebra $\mathscr{D}(X_T)$ onto $\mathscr{X}$. This time, the Boolean algebra $\mathscr{X}$ is a homomorphic (but generally not σ-homomorphic) image of the complete Boolean algebra $\widehat{\mathscr{B}}$. Both above-described constructions will be also valid in the case when $card\,\mathscr{X} < \mathfrak{b}$; it should only be required that all values of the mapping Φ_{00} belong to the subfamily $\{Q_t\}_{t\in T'}$, where $card\,T' = card\,\mathscr{X}$. This remark shows "universality." All σ-complete Boolean algebras are listed among the σ-homomorphic images of the σ-algebra $\mathscr{B}_0$; and all complete Boolean algebras of the same cardinality are listed among the quotient algebras of the complete Boolean algebra $\widehat{\mathscr{D}}$. Note that if $\mathfrak{b}$ is infinite then the Dedekind completion $\widehat{\mathscr{D}}$ has the same cardinality $\mathfrak{b}$. The "universality" of $\mathscr{D}$ was discovered for the first time by B. A. Efimov. At present, we know that all complete Boolean algebras possess this property.

We supposed that $card\,T = \mathfrak{b}$. In this case,

$$\mathscr{X} = \Phi(\mathscr{B}_0(X_T)), \quad card\,T = card\,\mathscr{X}.$$

We now let $card\,T = \tau(\mathscr{X})$. Suppose that the Boolean algebra $\mathscr{X}$ is complete and select a fully generating set E of cardinality $\tau \equiv \tau(\mathscr{X})$ in $\mathscr{X}$. This time, we map the system $\{Q_t\}_{t\in T}$ onto E. The corresponding surjection will be denoted by Φ_{00}. There exists a σ-continuous

[14] C. R. Karp [1]; R. Sikorski [1]; L. H. Loomis [1], and G. Birkhoff [3].

homomorphism Φ that maps the σ-algebra $\mathscr{B}_0 \equiv \mathscr{B}_0(X_T)$ into $\mathscr{X}$ and extends Φ_{00}. The image $\Phi(\mathscr{B}_0)$ is a σ-regular subalgebra fully generating $\mathscr{X}$. Suppose now that $\mathscr{X}$ is an infinite complete Boolean algebra with the countable chain condition. Then the subalgebra $\Phi(\mathscr{B}_0)$ is regular and so it must coincide with $\mathscr{X}$. Thus, in the case of a complete algebra with the countable chain condition the cardinality of the representation space can be diminished: we have 2^τ instead of $2^{card\,\mathscr{X}}$. In view of the theorem by S. Koppelberg (Chapter 2, Theorem 19), we can take an independent system as E; in this case, no nonempty clopen set belongs to the kernel of Φ. And so the image $\Phi(\mathscr{C}O(X_T))$ is a free Boolean algebra on τ generators isomorphic to $\mathscr{C}O(X_T))$. We formulate these results in the following

Theorem 9. 1. *Let $\mathscr{X}$ be a complete Boolean algebra and let X_T be a Cantor discontinuum of weight $card\,\mathscr{X}$. There exists a σ-continuous epimorphism from the Baire algebra $\mathscr{B}_0(X_T)$ onto $\mathscr{X}$.*

2. *Let $\mathscr{X}$ be an infinite complete Boolean algebra of weight τ satisfying the countable chain condition and let X_T be a Cantor discontinuum of weight τ (i.e., $card\,T = \tau$). There exists a σ-continuous epimorphism Φ from the Baire algebra $\mathscr{B}_0(X_T)$ onto $\mathscr{X}$ which possesses the following properties:*

1) $\Phi(e) \neq \mathbf{0}$ *for every* $e \in \mathscr{C}O(X_T)$, $e \neq \varnothing$;

2) *the image $\Phi(\mathscr{C}O(X_T))$ is a free subalgebra fully generating $\mathscr{X}$.*

Thus, each complete algebra is a σ-homomorphic image of a free Borel algebra: up to an isomorphism, it can be regarded as the quotient algebra of the Borel algebra by a σ-ideal.

Theorem 9 provides a standard description of all infinite complete Boolean algebras with the countable chain condition. Each of these algebras is determined up to an isomorphism by a σ-ideal I of the Baire algebra X_T, where $card\,T = card\,X_T = \tau$. The ideal must satisfy the two conditions:

(α) *every system of pairwise disjoint sets not included in I is at most countable;*

(β) *no nonempty open set belongs to I.*

The quotient algebra of the Baire algebra by each σ-ideal with the properties (α) and (β) is always complete and satisfies the countable chain condition.[15] This is ensured by the condition (α); and due to (β), each clopen set in X_T corresponds under the canonical isomorphism to

[15]Theorem 11 in Chapter 2.

exactly one element of the quotient algebra. Moreover, these elements comprise a free fully generating subalgebra of the quotient algebra.

All main "Boolean" properties of the quotient algebra are encoded, as in an embryo, in the properties of the ideal I. Thus, the theory of complete Boolean algebras with the countable chain condition is the theory of the Baire σ-ideals of Cantor spaces with the properties (α) and (β). In the last form, however, this theory is not elaborated except for some topics concerning regular and normed Boolean algebras. We address this question at the end of the chapter.

Theorem 9 on representation of a Boolean algebra with the countable chain condition has "parallel" variants differing by the choice of the representation space R. Consider, for instance, the case of a separable (i.e., $\tau(\mathscr{X}) = \aleph_0$) Boolean algebra $\mathscr{X}$. In this case, we have $R = X_{\mathbb{N}}$ in Theorem 9. The easiest way to obtain a new space R_1 is to take a homomorphic set or even a set of the same cardinality. For instance, we can take the classical Cantor set in the interval $[0,1]$ which is homomorphic to $X_{\mathbb{N}}$. However, such a representation is alike the above considered. It is more interesting to take as a representation space the entire interval $[0,1]$ and try to use it "completely." The easiest way to show that this construction is possible consists in applying the result of K. Kuratowski presented in Section 5.3: the Baire algebra $\mathscr{B}_0$ of the Cantor space $X_{\mathbb{N}}$ is isomorphic to the Borel σ-algebra of an interval, square, etc. Hence, from Theorem 9 we immediately obtain

Theorem 10. *A separable complete Boolean algebra with the countable chain condition is a σ-isomorphic image of the Borel algebra $\mathscr{B}([0,1])$ of the interval $[0,1]$ (or of every infinite complete separable metric space).*

We now sketch the direct proof of this theorem not appealing to the result of K. Kuratowski and demonstrate the mechanism of representation.

Let $\mathscr{X}$ be a CONTINUOUS Boolean algebra. Consider the mapping

$$\psi : \quad \psi(\chi) = \sum_{k=1}^{\infty} \frac{\chi_k}{2^k} \quad (\chi = (\chi_1, \chi_2, \ldots),\ \chi_i = 0, 1)$$

from the Cantor space $X_{\mathbb{N}}$ onto the interval $[0,1]$ and the totality of sets $b \subset [0,1]$ such that $\psi^{-1}(b) \in \mathscr{B}_0(X_{\mathbb{N}})$. It is not difficult to understand that this totality is exactly the Borel σ-algebra $\mathscr{B}([0,1])$. Therefore, the mapping defined by the equality $\Psi(e) \equiv \psi^{-1}(e)$ is a σ-continuous homomorphism from $\mathscr{B}([0,1])$ into $\mathscr{B}_0(X_{\mathbb{N}})$. It is clear that Ψ is not an epimorphism. It is well known that ψ is "not absolutely bijective": the binary rational numbers (fractions with the denominator some power of

two) and only them have more than one preimage. Each binary rational y corresponds to exactly two points in $X_{\mathbb{N}}$ of the following form:

$$\chi' = (\chi_1, \chi_2, \ldots, \chi_k, 1, 0, 0, \ldots), \quad \chi'' = (\chi_1, \chi_2, \ldots, \chi_k, 0, 1, 1, \ldots).$$

Let Q be the set of all these points. This set Q is countable and lies in the kernel of the epimorphism Φ constructed in Theorem 9, which ensues from the continuity of $\mathscr{X}$. Taking an arbitrary $x \in \mathscr{X}$, for each set $e \in \Phi^{-1}(\{x\})$ we have the containment $\psi^{-1}(\psi(e)) \in \Phi^{-1}(\{x\})$, since the difference $\psi^{-1}(\psi(e)) \setminus e$ may consists only of points of Q and is included in the kernel of Φ. Therefore, $\Phi(\Psi(\psi(e))) = x$. Thus, the mapping $\theta \equiv \Phi \circ \Psi$ is an epimorphism from $\mathscr{B}([0,1])$ onto $\mathscr{X}$. It is clear that this epimorphism is σ-continuous.

If the algebra $\mathscr{X}$ is DISCRETE then to construct the epimorphism θ is even more simple: selecting a countable (or finite) set S, we map S bijectively onto the set of atoms of the Boolean algebra $\mathscr{X}$, and assign to the Borel sets disjoint from S the zero of the algebra. It is clear that these conditions uniquely determine the σ-continuous epimorphism θ. The general case (in which the algebra $\mathscr{X}$ has both discrete and continuous bands) reduces easily to the above-considered.

The proven theorem is generalized to the inseparable case. This time, as the representation space we should take not the interval $[0,1]$ but the product of $\tau(\mathscr{X})$ intervals: the unit cube of dimension $\tau(\mathscr{X})$. The scheme of the proof remains the same in essence; we leave details to the reader and formulate the final result.

Theorem 10′. *Each complete Boolean algebra of uncountable weight with the countable chain condition is a σ-homomorphic image of some σ-algebra $\widetilde{\mathscr{B}}$ of the space $\widetilde{R} \equiv \prod_{s \in S} R_s$, where $R_s = [0,1]$ and $\operatorname{card} S = \tau(\mathscr{X})$. Such a σ-algebra $\widetilde{\mathscr{B}}$ contains all cylindrical sets $\prod_{s \in S} E_s$, where E_s are Borel subsets of $R_s = [0,1]$ such that only finitely many of them differ from R_s.*

Theorems 9, 10, and 10′ give a general form of a complete Boolean algebra with the countable chain condition. In this case, all information about the algebra lies also in the kernel of the homomorphism. We may say that the theory of these Boolean algebras is the theory of σ-ideals in Cantor spaces.

REMARK. When using Theorems 9, 10, and 10′, the corresponding algebra $\mathscr{B}_0$, $\mathscr{B}$, or $\widetilde{\mathscr{B}}$ is replaced usually by the wider σ-algebra obtained by adjoining the sets that are dominated by the elements of the kernel of the main homomorphism. This process of "completion" is well known in measure theory where, for instance, the system of Lebesgue measurable subsets of an interval is considered instead of the Borel σ-algebra. The role of I goes to the σ-ideal $I' \equiv \{e' \mid e' \subset e \in I\}$.

4.2 Representation of a regular algebra

Regular algebras are of the utmost interest: they were discussed in Theorems 9, 10, and 10′. We dwell only upon the case of a separable algebra. Let $\mathscr{R}$ be some of the spaces $X_{\mathbb{N}}, [0,1]$ and let $\mathscr{B}$ be some of the σ-algebras $\mathscr{B}_0(X_{\mathbb{N}})$ and $\mathscr{B}([0,1])$. Moreover, we denote by $\mathscr{E}_0$ either the clopen algebra of $X_{\mathbb{N}}$ or (another variant) the algebra generated by "binary-rational" intervals in $[0,1]$. In both cases, $\mathscr{B}$ is the least σ-algebra including $\mathscr{E}_0$. Finally, denote by Φ a σ-continuous homomorphism that acts from the σ-algebra $\mathscr{E}$ of the space $\mathscr{R}$ into a complete regular separable Boolean algebra $\mathscr{X}$. Let $I \equiv \ker \Phi$. In the regular case, the extension of a homomorphism implemented in the proofs of Theorems 9 and 10 can be constructed by the scheme of 5.3.2 (we noted this in connection with Theorem 5). Assume that Φ and $\mathscr{E}$ are constructed by that scheme; so, $\mathscr{E}_0 \subset \mathscr{B} \subset \mathscr{E}$,

$$\mathscr{E} = \{E \in \mathscr{R} \mid C\Phi^*(E) = \Phi^*(CE)\}, \quad I = \{E \mid \Phi^*(E) = \mathbf{0}\},$$

$$\Phi^*(E) = \bigwedge_{\sigma \in S_E} \bigvee_{e \in \sigma} \Phi_0(e),$$

where $\Phi_0 = \Phi|_{\mathscr{E}_0}$. Thus, $I \equiv I'$ is a σ-ideal of the boolean $2^{\mathscr{R}}$.

Each $\sigma \in S_E$ is a countable family of sets in $\mathscr{E}_0$ whose union includes E. Let us agree to call the countable unions of sets in $\mathscr{E}_0$ by *G^+-sets* and their complements by *F^+-sets.* The countable intersections of G^+-sets we call *G^+_δ-sets* and the countable unions of F^+-sets, *F^+_σ-sets.* Without any risk of confusion, the G^+-sets are sometimes referred to as *σ-open.*

In the case when $\mathscr{R} = X_{\mathbb{N}}$, all G^+-sets are open and all F^+-sets are closed. If $\mathscr{R} = [0,1]$ then all open(closed) sets are G^+- (F^+-) sets. Similar relations hold for the classes of G^+_δ-, G_δ-, F^+_σ-, and F_σ-sets.

Since $\mathscr{X}$ satisfies the countable chain condition, for every $E \subset \mathscr{R}$, we can easily find a G^+_δ-set B such that $E \subset B$ and $\Phi^*(E) = \Phi(B)$. The algebra $\mathscr{E}$ in both cases consists of such sets E that admit two-sided approximation:

$$A \subset E \subset B, \quad \Phi(B \setminus A) = \mathbf{0}, \tag{8}$$

where A is an F^+_σ-set and B is a G^+_δ-set. (The equality is another form of the relation $C\Phi^*(E) = \Phi^*(CE)$.)

The algebra $\mathscr{E}$ can be defined in another way. Using the σ-completeness of the algebra and the ideal I, it is easy to verify that $\mathscr{E} = \mathscr{B} +_2 I$ (i.e., is the totality of all sets of the form $b +_2 q$, where $b \in B$ and $q \in I$).

We now return to the question of the properties of I which define the properties of the Boolean algebra $\mathscr{X}$. It is clear that the ideal must

obey the condition (α) in 9.4.1.[16] We may also assume that (β) holds. The regularity of the algebra ensures the two additional properties of the ideal:

(γ) *each set belongs to* I *together with some* G_δ^+*-superset;*

(δ) *if* $\{M_i\}_{i=1}^\infty$ *is a sequence of downward-directed countable systems of* G^+*-sets and the containment*

$$\bigcap_{G\in M_i} G \in I$$

holds for all $i = 1, 2, \ldots$ *then there exists a countable family of "selections"*

$$\{G_i^{(k)}\}_{i=1}^\infty \quad (k = 1, 2, \ldots), \quad G_i^{(k)} \in M_i$$

such that

$$\bigcap_k \bigcup_i G_i^{(k)} \in I.$$

For brevity, an ideal I with the properties (α), (β), (γ), and (δ) will be referred to as *regular*. (By the property (δ), each regular ideal is always a σ-ideal.) The kernel of the homomorphism constructed in Theorems 9 and 10 is regular. The converse is also true. The following theorem refers to both cases in which $\mathscr{R} = X_{\mathbb{N}}$ and $\mathscr{R} = [0, 1]$. Moreover, it is still valid in the inseparable case when $\mathscr{R} = X_T$, where T is an uncountable set.

Theorem 11. *If* I *is a regular ideal then the system* $\mathscr{E}$ *of all sets that admit approximation in the sense of* (8) *is the* σ*-algebra coinciding with* $\mathscr{B} +_2 I$*; and the quotient algebra* $\mathscr{E}|_I$ *is complete and regular.*

(We regard I as an ideal in $\mathscr{E}$.)

First of all, it is clear that $\mathscr{E} \subset \mathscr{B} +_2 I$. Prove that $\mathscr{E}$ is a σ-algebra; this will imply the reverse inclusion.

Proposition. *Let* E *include an* F_σ^+*-set* A *and* $E \setminus A \in I$*. Then there exists a* G_δ^+*-set* $B \supset E$ *such that* $B \setminus E \in I$*. (And then* $B \setminus A \in I$*.)*

The set A can be presented as $A = \bigcup_{k=1}^\infty Q_k$, where each summand has the following form:

$$Q_k = \bigcap_{s=1}^\infty e_{ks}, \quad e_{ks} \in \mathscr{E}_0 \ (k, s = 1, 2, \ldots).$$

We can assume that $e_{k1} \supset e_{k2} \supset \ldots$. Then sets $e'_{ks} \equiv e_{ks} \setminus Q_k$ form a decreasing sequence (for all $k = 1, 2, \ldots$). These sets are σ-open. Given

[16]In the condition (α), the "Baire sets" are members of the σ-algebra $\mathscr{B}$.

k, we have $\bigcap_s e'_{ks} = \varnothing \in I$. Using ($\delta$), distinguish a countable system of selections

$$\{e'_{s_k^{(n)}}\}_{k=1}^{\infty} \quad (n = 1, 2, \ldots)$$

such that

$$\bigcap_n \bigcup_k e'_{s_k^{(n)}} \in I.$$

Denote for brevity: $P_n \equiv \bigcup_k e'_{s_k^{(n)}}$ and $Q \equiv \bigcap_n P_n$. The sets P_n are σ-open, Q belongs to I and is a G_δ^+-set. We have:

$$A \cup P_n = \Bigl(\bigcup_k Q_k\Bigr) \cup \Bigl(\bigcup_k e'_{s_k^{(n)}}\Bigr) = \bigcup_k \Bigl(Q_k \cup \Bigl(e_{s_k^{(n)}} \setminus Q_k\Bigr)\Bigr) = \bigcup_k e_{s_k^{(n)}}.$$

This set is σ-open. Therefore,

$$A' \equiv \bigcap_n (A \cup P_n) = A \cup \bigcap_n P_n = A \cup Q$$

is a G_δ^+-set; it includes A and $A' \setminus A \subset Q \in I$. Therefore, $A' \setminus A \in I$.

On the other hand, $E \setminus A \in I$; therefore, there exists a G_δ^+-set $A'' \supset E \setminus A$ such that $A'' \in I$. The union $A' \cup A''$ is also a G_δ^+-set and

$$E = A \cup (E \setminus A) \subset A' \cup A'',$$

$$(A' \cup A'') \setminus E \subset (A' \cup A'') \setminus A \subset (A' \setminus A) \cup A'' \in I.$$

Thus, we can take $A' \cup A''$ as B. The proposition is proved.

So, in order to verify the inclusion $E \subset \mathscr{E}$, it is sufficient to establish the possibility of ONE-SIDED approximation "from inside" by an F_σ^+-set $A \subset E$.

Now we can easily show that the system $\mathscr{E}$ is closed under countable intersections. Let $E = \bigcup_n E_n$, where $E_n \in \mathscr{E}$. Select F_σ^+-set $A_n \subset E_n$ so that the containment $E_n \setminus A_n \in I$ holds.

The union $A = \bigcup_n A_n$ is an F_σ^+-set. Since I is a σ-ideal, it is clear that $E \setminus A \subset I$. Hence $E \in \mathscr{E}$.

It is clear from the definition of $\mathscr{E}$ that this system is closed under complementation. The above makes it clear that $\mathscr{E}$ is a σ-algebra; therefore, $\mathscr{E} = \mathscr{B} +_2 I$. In view of the condition (α), the quotient algebra $\mathscr{E}|_I$ satisfies the countable chain condition and is complete. It remains only to verify the diagonal principle. Let $\{x_{nm}\}$ be a double sequence such that $x_{nm} \in \mathscr{E}|_I$ and $x_{nm} \downarrow \mathbf{0}$ as $n \longrightarrow \infty$ for all $m = 1, 2, \ldots$. Denote by X_{nm} a representative of the class x_{nm}. We can assume that $\bigcap_k X_{km} \in I$.

We write down the representatives as intersections

$$X_{nm} = \bigcap_{k=1}^{\infty} E_{nm}^{(k)}, \quad E_{nm}^{(1)} \supset E_{nm}^{(2)} \supset \dots ,$$

with $E_{nm}^{(k)}$ open. Define the sets M_m as follows: M_m consists of all finite intersections of the form

$$E_{1m}^{(k_1)} \cap E_{2m}^{(k_2)} \cap \dots \cap E_{sm}^{(k_s)}.$$

All M_m are σ-open. It is clear that M_m is directed downward and countable, while the intersection of each M_m coincides with the intersection $\bigcap_k X_{km}$ and, therefore, belongs to the ideal I.

Using (δ), arrange the selections

$$\{G_m^{(k)}\}_{m=1}^{\infty} \quad (k = 1, 2, \dots)$$

such that

1) $G_m^{(k)} \in M_m$;

2) $\bigcap_k \bigcup_m G_m^{(k)} \in I$.

Assign $\widetilde{G}_m \equiv G_m^{(1)} \cap G_m^{(2)} \cap \dots \cap G_m^{(m)}$ $(m = 1, 2, \dots)$. These sets constitute one more selection: $\widetilde{G}_m \in M_m$ $(m = 1, 2, \dots)$ (since the sets M_m possesses the finite intersection property).

The inclusion

$$\bigcup_m G_m^{(k)} \supset \bigcup_{m=k}^{\infty} \widetilde{G}_m$$

holds for all $k = 1, 2, \dots$; therefore, from 2) we obtain

$$\bigcap_k \bigcup_{m=k}^{\infty} \widetilde{G}_m \subset \bigcap_k \bigcup_m G_m^{(k)} \in I.$$

Choose the indices $\{n_m\}$ such that

$$X_{n_m m} \subset \widetilde{G}_m \quad (m = 1, 2, \dots).$$

This is possible since $\widetilde{G}_m \in M_m$ implying that

$$\widetilde{G}_m = E_{1m}^{(k_1)} \cap E_{2m}^{(k_2)} \cap \dots \cap E_{sm}^{(k_s)} \supset X_{1m} \cap X_{2m} \cap \dots \cap X_{sm} = X_{sm}.$$

We may assign $n_m = s$. Finally, we have

$$\bigcap_k \bigcup_{m=k}^{\infty} X_{n_m m} \in I \quad \text{and} \quad x_{n_m m} \xrightarrow{(o)} \mathbf{0}.$$

The "diagonal" sequence is constructed. So, $\mathscr{E}|_I$ is a complete regular Boolean algebra.

Instead of $\mathscr{E}|_I$, we can consider $\mathscr{B}|_{I''}$, where $I'' \equiv I \cap \mathscr{B}$. Both quotient algebras are obviously isomorphic.

REMARK. The condition (β) was not used in the proof of Theorem 11. Moreover, we may require that the condition (α) be valid only for disjoint systems of closed sets. In fact, a regular ideal is defined by its stock of closed sets.

We see that there is a correspondence between regular ideals in Cantor discontinua and regular Boolean algebras: each regular ideal generates a regular algebra and each regular algebra can be obtained in this manner. The situation is similar when the role of $\mathscr{R}$ is played by an interval or a cube of an arbitrary dimension.

Chapter 6

VECTOR LATTICES AND BOOLEAN ALGEBRAS

The theory of vector lattices (in other words, "the theory of ordered vector spaces") is closely connected with the subject of the present book. Many facts concerning Boolean algebras become more explicit when they are considered from the "vector" point of view. It is not a gross exaggeration to say that the theories of Boolean algebras and ordered vector spaces merge into a single huge area of functional analysis.

Many monographs are devoted to order vector spaces, in Russian, in particular.[1] This explains the summarizing style of exposition in this chapter.

1. *K*-spaces and the related Boolean algebras

1.1 Basic concepts

A *vector lattice* or *K-lineal* is a real vector space equipped with some partial order making this space into a lattice so that the usual axioms hold:

(a) if $x < y$ then $x + z < y + z$ for all z;
(b) if $x < y$ then $\lambda x < \lambda y$ for all real $\lambda > 0$.

Thus, the definition of a vector lattice contains the requirement for every finite set to have the least upper and greatest lower bounds.

We now require additionally that each bounded set has the least upper and greatest lower bounds, so arriving at the definition of a *Dedekind complete vector lattice* or *K-space* (= "Kantorovich space"). The *posi-*

[1] L. V. Kantorovich, B. Z. Vulikh, and A. G. Pinsker [1]; B. Z. Vulikh [1]; L. V. Kantorovich and G. P. Akilov [1]; G. P. Akilov and S. S. Kutateladze [1].

tive elements of a vector lattice R are elements satisfying the inequality $x \geq \mathbf{0}$, where $\mathbf{0}$ is the usual zero or origin of the vector space.

The set R_+ of all positive elements in R is a cone; i.e., a set closed under vector addition and multiplication by every positive scalar which contains no nonzero opposite elements. In this case, each element $x \in R$ is represented uniquely as $x = x_+ - x_-$, where $x_+ \wedge x_- = \mathbf{0}$. The elements x_+ and x_- are called the *positive part* and the *negative part* of x respectively; and their sum

$$|x| \equiv x_+ + x_-$$

is called by the *modulus* of x.

We say that x and y are *disjoint* if $|x| \wedge |y| = \mathbf{0}$. Likewise in the case of Boolean algebra, we speak about the *disjoint complement* E^d of a given set E and defined a *band*[2] as a set E satisfying the condition $E^{dd} = E$.

In order to give another equivalent definition for a band, we should introduce two extra concepts. A subspace $R_0 \subset R$ is called *regular* provided that the supremum and infimum of each bounded subset in R_0 belong to R_0. Next, a subspace R_0 is called *solid*[3] provided that the conditions $x \in R_0$, $y \in R$, and $|y| \leq x$ imply $y \in R_0$. Soon, we will meet many substantial examples of solid subspaces.

A band in a K-space R is characterized completely as a regular and solid subspace of R. In other words, a band is a linear subset $R_0 \subset R$ containing the suprema and infima of all subsets in R_0 bounded in R and, moreover, if x belongs to R_0 then so does each $y \in [-|x|, |x|]$. Each band can be considered as a K-space in its own right, if need be.

Henceforth, we often presume tacitly that every vector lattice under consideration is Dedekind complete, leaving it to the reader to find out when and how we use this assumption.

1.2 The base of a K-space

We consider the family of all bands in a given K-space R equipping this family with the natural order by inclusion. Denote this partially ordered set by H_R. We almost repeat the proof of Theorem 7 in Chapter 2 and obtain the main fact of our book. This seems to be first stated by G. Birkhoff.[4]

Theorem 1. *The set H_R is a complete Boolean algebra. In this Boolean algebra, the space R plays the role of unity, the Boolean com-*

[2] A band is a component in the literature of Russian provenance. (S. S. Kutateladze)
[3] Compare these concepts with the concepts of regular subalgebra and solid set (p. 85 and p. 48).
[4] A more general assumption is in the article [3] by A. G. Pinsker.

plement of each band $U \subset H_R$ is its disjoint complement U^d; and the greatest lower bound of an arbitrary family of bands coincides with their intersection.

Thus, the complete Boolean algebra H_R is connected with a vector lattice R that can be studied instead of R. The algebra H_R reflects the deepest properties of the ordered vector space R.

We call this algebra the *base*[5] of R. In many important cases, the base of a K-space may be realized as some set of elements in this space. For this purpose, it is necessary for R to have a *weak unit*; i.e., a positive element $\mathbf{1}$ such that there no nonzero element disjoint from $\mathbf{1}$. In general, R can possess no weak unit.

However, if a weak unit $\mathbf{1}$ exists then the family E_R of all elements e satisfying

$$e \wedge (\mathbf{1} - e) = \mathbf{0}$$

(those are *unit elements*) comprise a complete Boolean algebra with respect to the order induced from R, with the elements $\mathbf{0}$ and $\mathbf{1}$ serving as the "Boolean" zero and unity. The complement of $e \in E_R$ is the difference $\mathbf{1} - e$. *The Boolean algebra H_R is isomorphic to the algebra of unit elements*; such an isomorphism may be given, for instance, by the formula

$$\Psi(U) = \bigvee_{x \in U} (\mathbf{1} \wedge x) \tag{1}$$

assigning to an arbitrary band $U \in H_R$ the element $\Psi(U) \in E_R$. Moreover, for every set $H' \subset H_R$, the following hold:

$$\sup \Psi(H') = \Psi(\sup H'), \quad \inf \Psi(H') = \Psi(\inf H').$$

On the left-hand sides of these relations the suprema and infima are calculated in R. Moreover, reflecting the isomorphism property, the following holds:

$$\Psi(CU) = \mathbf{1} - \Psi(U) = C\Psi(U).$$

Thus it is possible to embed the Boolean algebra H_R (which is the base of R) in R preserving the order as well as suprema and infima. There are infinitely many ways for such an embedding. The reason behind this is that the choice of a weak unit (if it is possible) can be done in infinitely many fashions. If some unit $\mathbf{1}$ is distinguished then we assume always that the embedding is done with the help of the "canonical" isomorphism Ψ defined by (1).

---

[5]We somewhat deviate from the terminology of the literature on ordered vector spaces.

Note that under such an embedding the SUM of two disjoint elements in the ALGEBRA coincides with their SUM as the elements of the vector SPACE R. If the elements are not disjoint in the algebra then we may speak about their sum in R; in the last case, the addition drives us out of the algebra. The SYMMETRIC DIFFERENCE of elements in the algebra after the embedding into R may be interpreted as the modulus of the usual difference. This justifies our choice of notation in Chapter 0.

We illustrate the above by a very simple example. Let Q be an arbitrary nonempty set and let F_Q be the family of all finite real functions on Q with the pointwise order. The operations of addition and scalar multiplication are defined in F_Q by the usual formulas:

$$(f+g)(q) = f(q) + g(q), \quad (\alpha f)(q) = \alpha f(q),$$

where $f, g \in F_Q$ and α is a real number. Each bounded set $E \subset F_Q$ has a supremum and a infimum. They are found by the formulas:

$$(\sup E)(q) = \sup_{y \in E} y(q); \quad (\inf E)(q) = \inf_{y \in E} y(q).$$

In particular,

$$(f \vee g)(q) = \max\{f(q);\ g(q)\}, \quad (f \wedge g)(q) = \min\{f(q);\ g(q)\};$$

$$f_+(q) = \max\{f(q), 0\}, \ f_-(q) = -\min\{f(q), 0\}; \ |f|(q) = |f(q)|. \quad (2)$$

It is clear, that every finite set in F_Q is bounded. Obviously, the axioms a) and b) in the definition of vector lattice hold too. We see that F_Q is a K-space. The positive cone of it comprises all possible functions positive in the usual sense.

The formulas (2) shows that the concepts of modulus, positive part, and negative part of an element are interpreted as usual. Functions in F_Q are disjoint if they differ from zero on disjoint sets. At each point $q \in Q$, the value of one of the two disjoint functions must equal zero.

Now it is easy to describe the bands of our space: each of them is connected with some subset $Q_0 \subset Q$ and consists of all functions vanishing beyond Q_0.

Therefore, the Boolean algebra of bands in our example can be identified with the algebra 2^Q which is well known to us already. There is a weak unit in the space F_Q. Each positive function can play this role if it does not take zero values. Most often the identically one function is used as the unity. Then the unit elements are all characteristic functions of subsets in Q.

Instead of the space F_Q of real functions on Q, we may consider a narrower space M_Q consisting of all BOUNDED functions. The algebraic operations and order are introduced in M_Q in the same manner as in F_Q.

The supremum and infimum exist for each bounded set. Everything holds that was said above interpretation of the modulus, positive part and negative part of an element, disjointness and the concept of a band. As in the case of F_Q, the base of M_Q may be identified with the algebra 2^Q.

We may consider other spaces of functions on Q whose bases are isomorphic to 2^Q. For example, if Q is the sequence of natural numbers $\{1, 2, \dots\}$ then we can consider the l^p spaces popular in analysis which consist of the sequences $f = (f_1, f_2, \dots, f_n, \dots)$ satisfying the condition

$$\sum_{n=1}^{\infty} |f_n|^p < +\infty.$$

Here p can be assumed to be a strictly positive number.[6]

The proof is not difficult of the fact that l^p is a vector lattice. It is clear that the formulas (2) are valid. We need to prove only that the functions $f \vee g$, $f \wedge g$, f_+, f_-, and $|f|$, defined by them, belong to l^p. The l^p space does not contain the identically one function. The role of a weak unit is played in it by every vector $f = (f_1, f_2, \dots)$ whose coordinates are strictly positive. We leave it to the reader to find the structure of the base and to prove that this base is isomorphic to 2^Q in this case too. Also we may consider an analogous space with an uncountable Q. But there is no weak unit in this case.

The spaces M_Q and l^p are examples of solid subspaces of the corresponding space F_Q. The spaces F_Q and their solid subspaces are convenient models for the first acquaintance with vector lattices in much the same way as the discrete algebras 2^Q are good illustrations of the simplest facts of the theory of Boolean algebras. However, the most interesting lattices are connected with continuous Boolean algebras, for example, with the Lebesgue algebra E_0. We consider these spaces below.

We obtain a classical example of a vector lattice that is not a K-space when we consider the vector lattice $C_{[0,1]}$ of all continuous real functions on $[0, 1]$ (the order and vector structure are usual). The suprema and infima of finite sets are calculated in $C_{[0,1]}$ in the same way as in $F_{[0,1]}$ by (2). However, for example, the set of continuous functions whose graphs are in Fig. 1 is bounded below by zero but has no infimum in the class of continuous functions.

Thus, the lattice $C_{[0,1]}$ is not Dedekind complete. The reason behind this is the "very good" (or, if you prefer the contrary, "bad") topological properties of the interval $[0, 1]$. The continuous functions, defined not on

[6]The usual assumption $p \geq 1$ is unnecessary here.

an interval but rather on some totally disconnected Hausdorff compact space, comprise a Dedekind complete vector lattice, as it can be easily shown. Moreover, the theorem is valid claiming that every K-space is realized in such a manner. If the reader wants to acquaint himself or herself in detail with realization of ordered vector spaces, then he or she will consult the book of B. Z. Vulikh [1].

1.3 Band projections

In the theory of vector lattices, the operation of projecting onto a band plays an important role. Namely, each band h of a K-space R is connected with the *band projection* P_h defined by the equalities:

$$P_h(x) \equiv \bigvee_{y \in h, 0 \leq y \leq x} y$$

for $x \geq \mathbf{0}$ and

$$P_h(x) = P_h(x_+) - P_h(x_-)$$

for an arbitrary x.

This operator assigns to each $x \in R$ the element $P_h(x)$ which is called the *projection* of x to h. It can be shown that the operator P_h is additive, homogeneous, and positive (if $x \geq \mathbf{0}$ then always $P_h(x) \geq \mathbf{0}$). Note the main property of band projections: if the bands $\{h_\xi\}_{\xi \in \Xi}$ are mutually disjoint and have the supremum in H_R equal to unity,[7] then the elements $P_{h_\xi}(x)$ are pairwise disjoint, and

$$\bigvee_{\xi \in \Xi} P_{h_\xi}(x) = x.$$

[7]Recall that the unity of H_R is the band coincident with the whole space R.

In this event we say that the family of bands $\{h_\xi\}$ comprises a (*disjoint*) *decomposition* of R.

If a family of bands is finite then the last supremum coincides with the usual sum. It is easy to verify that the unit elements are projections of unity to various bands. We propose to the reader to interpret the operation of band projection in the spaces F_Q and M_Q.

Note an important fact: *for each K-space, there exists a decomposition into bands with weak units.*

1.4 Convergence and order topology in K-spaces

This subsection is devoted to some topological questions of the theory of ordered vector spaces. Almost all context of Chapter 2 can be translated here but we confine exposition to a few facts. All further considerations concern some K-space R with base $\mathscr{X}$.

We first define the concept of (o)-*convergence.* Considering the conditional completeness of R, for this purpose we may use upper and lower limits. Namely, the *upper limit* of a bounded generalized sequence $\{x_\alpha\}_{\alpha\in A}$ is defined as

$$\overline{\lim}_\alpha x_\alpha \equiv \bigwedge_\alpha \bigvee_{\beta\succ\alpha} x_\beta;$$

while the *lower limit,* as

$$\underline{\lim}_\alpha x_\alpha \equiv \bigvee_\alpha \bigwedge_{\beta\succ\alpha} x_\beta.$$

The (o)-*limit* of a bounded sequence $\{x_\alpha\}_{\alpha\in A}$ is the common value of its upper and lower limits provided that the latter coincide. In other words,

$$x = (o)\text{-}\lim x_\alpha = \bigwedge_\alpha \bigvee_{\beta\succ\alpha} x_\beta = \bigvee_\alpha \bigwedge_{\beta\succ\alpha} x_\beta.$$

In this event we also write $x_\alpha \xrightarrow{(o)} x$. Other definitions of (o)-limit are available; some of them do not presume boundedness.

In much the same way as in the case of a Boolean algebra we may prove the "(o)-continuity" of the main operations. Namely, from the relations

$$x_\alpha \xrightarrow{(o)} x, \quad y_\alpha \xrightarrow{(o)} y,$$

it follows that

$$x_\alpha + y_\alpha \xrightarrow{(o)} x + y,$$

$$x_\alpha \vee y_\alpha \xrightarrow{(o)} x \vee y,$$

$$x_\alpha \wedge y_\alpha \xrightarrow{(o)} x \wedge y,$$

$$\lambda x_\alpha \xrightarrow{(o)} \lambda x \quad (\lambda \in \mathbb{R}),$$

$$|x_\alpha| \xrightarrow{(o)} |x|;$$

etc. Also, "(o)-continuous" are all band projections: $x_\alpha \xrightarrow{(o)} x$ implies $P_h(x_\alpha) \xrightarrow{(o)} P_h(x)$ for every band $h \in H_R$.

As far as the upper and lower limits are concerned, they are SEMIADDITIVE: for all generalized sequences

$$\{x_\alpha^{(1)}\}_{\alpha\in A}, \{x_\alpha^{(2)}\}_{\alpha\in A}, \dots, \{x_\alpha^{(m)}\}_{\alpha\in A}$$

we have

$$\overline{\lim}(x_\alpha^{(1)} + x_\alpha^{(2)} + \dots + x_\alpha^{(m)}) \le \overline{\lim}\, x_\alpha^{(1)} + \overline{\lim}\, x_\alpha^{(2)} + \dots + \overline{\lim}\, x_\alpha^{(m)},$$

$$\underline{\lim}(x_\alpha^{(1)} + x_\alpha^{(2)} + \dots + x_\alpha^{(m)}) \ge \underline{\lim}\, x_\alpha^{(1)} + \underline{\lim}\, x_\alpha^{(2)} + \dots + \underline{\lim}\, x_\alpha^{(m)}.$$

The simplest example illustrating the concept of (o)-convergence is provided by the space F_Q: here (o)-convergence coincides with the ordinary "pointwise" convergence of functions (at every point $q \in Q$). For (o)-convergence in M_Q we additionally need the uniform boundedness of all functions (with a common constant).

Using the concept of (o)-convergence, in much the same way as in the case of a Boolean algebra we may define the concept of (o)-*topology* as the strongest of those topologies in which (o)-convergence implies topological convergence. Closed in this topology are the sets that contain the limits of all (o)-convergent generalized sequences of their elements.

Let us agree finally on how to use the symbol "$+\infty$." The formulas like

$$\sup A < +\infty, \quad \sup A = +\infty$$

speak respectively about upper boundedness or unboundedness of A.

In vector lattices there exists another type of convergence, *relatively uniform convergence* or, in other words, *convergence with regulator.* Assume that

$$|x_n - x| \le \varepsilon_n r, \quad x_n, x, r \in R,\ \varepsilon_n \in \mathbb{R},\ \varepsilon_n \downarrow 0$$

for all $n = 1, 2, \dots$. In this event the element r is called the *regulator of convergence*, and we say that the sequence $\{x_n\}$ *converges to x with regulator r* or *relatively uniformly.* This convergence always implies (o)-convergence. In the spaces of real functions with the identically one

function **1** as a weak unit, convergence with regulator **1** amounts to the usual uniform convergence.

We call a K-space R *regular* provided that

1) every bounded disjoint set is at most countable;

2) each (o)-convergent sequence converges relatively uniformly (with the regulator depending on the sequence);

3) to each sequence $\{x_n\}$ there is a numerical sequence $\{\lambda_n\}$ satisfying $\bigvee_n |\lambda_n x_n| < +\infty$.

The class of regular spaces is wide; it contains many of those K-spaces that are most important in functional analysis.

1.5 A K-space over a Boolean algebra

The above Theorem 1 has already justified the appearance of vector lattices in this book on Boolean algebras. However, the question still remains open as to whether or not each complete BA is the base of some K-space.

The next theorem answers this question in the affirmative.

Theorem 2. *To each complete BA $\mathscr{X}$ there is a K-space with base isomorphic to $\mathscr{X}$.*

Many proofs are known of this important theorem. The shortest path to the aim relates to using the extremally disconnected compact Stone space $\mathfrak{Q}$ of this algebra. The vector lattice $C_{\mathfrak{Q}}$ of all finite real continuous functions on $\mathfrak{Q}$ turns out to be a sought K-space.

The construction of an isomorphism reduces to providing a one-to-one correspondence between the bands of $C_{\mathfrak{Q}}$ and the clopen subsets of $\mathfrak{Q}$: To each of these sets e there corresponds the band of all functions vanishing beyond e.

Moreover, each band is of this provenance while distinct clopen sets generates distinct bands. A detailed proof is given in the monograph by B. Z. Vulikh [1].

A K-space with base isomorphic to a given BA is said to be *over* this algebra. The above sketch of the proof of Theorem 2 has a shortcoming: it does not reveal the important fact that a K-space over an algebra is not unique. Even in the case of the discrete algebra 2^Q we have established the existence of the two generally distinct vector lattices F_Q and M_Q each with base 2^Q (they coincide only in the trivial case when Q is finite).

In this event, the space F_Q is in a sense the widest among the K-spaces over the algebra 2^Q.

Namely, each of these spaces is isomorphic to some solid subspace of F_Q. A similar situation is observed in the general case.

Theorem 3. *To each complete BA $\mathscr{X}$ there corresponds a K-space $\mathfrak{S}_{\mathscr{X}}$ with weak unit over $\mathscr{X}$ such that each K-space R over $\mathscr{X}$ is isomorphic to some solid subspace of $\mathfrak{S}_{\mathscr{X}}$. Moreover, if R contains a weak unit then each isomorphism i_0 from BA E_R to $E_{\mathfrak{S}_{\mathscr{X}}}$ admits a unique extension to an isomorphism i from R to some solid subspace $i(R) \subset \mathfrak{S}_{\mathscr{X}}$ satisfying $(i(R))^d = \{\mathbf{0}\}$.*

This theorem belongs to A. G. Pinsker; the space $\mathfrak{S}_{\mathscr{X}}$ is called the *universally complete* or *extended* K-space over a complete BA $\mathscr{X}$.

A universally complete K-space is determined from a given algebra up to isomorphism; the symbol $\mathfrak{S}_{\mathscr{X}}$ will stand for EACH of these universally complete spaces. From the properties of $\mathfrak{S}_{\mathscr{X}}$ we may soundly judge the properties of $\mathscr{X}$ and vice versa. The class of universally complete K-spaces (i.e., the spaces representable as $\mathfrak{S}_{\mathscr{X}}$, with $\mathscr{X}$ a complete BA) possesses the following easy properties:

1°. *A space isomorphic to a universally complete space is universally complete itself.*

2°. *If R_1 and R_2 are universally complete and R_1 is a solid subspace of R_2 such that $\mathbf{1}_{R_2} \in R_1$ then $R_1 = R_2$.*

3°. *R is a universally complete space if and only if every disjoint subset of R is bounded.*

Note also that the regularity of a K-space $\mathfrak{S}_{\mathscr{X}}$ amounts to the regularity of the BA $\mathscr{X}$.

The description of the space $\mathfrak{S}_{\mathscr{X}}$, and so of an arbitrary K-space over $\mathscr{X}$, may be implemented in several ways. Thus, in the language of the Stone space we may characterize $\mathfrak{S}_{\mathscr{X}}$ as the space of all continuous real functions on the compact space $\mathfrak{Q}(\mathscr{X})$ which are finite everywhere but possibly a rare set depending on the function. This space is denoted by $C_\infty(\mathfrak{Q})$.

We will not overcome the difficulties of defining operations over these functions (namely, the definitions of addition, suprema and infima, etc.). We only show the manner in which we embed the BA $\mathscr{X}$ into the space $C_\infty(\mathfrak{Q})$ the former has generated. This is done by assigning to each member $x \in \mathscr{X}$ the characteristic function ("indicator") of the clopen set that is the image of x in the Stone space of $\mathscr{X}$. (The characteristic function of a clopen set is always continuous.)

We call this embedding *canonical*, following the routine practice of similar situations. We often identify x and its canonical image for simplicity.

There is another way of interpreting $\mathfrak{S}_{\mathscr{X}}$ as the system of all possible RESOLUTIONS OF THE IDENTITY. The last concept is the topic of the following section.

2. Spectral families and resolutions of the identity. Spectral measures

2.1 Spectral families

We start with the simplest model. Let a vector lattice R consist of real functions on some set Q. To each function f we assign the family of sets

$$E_\lambda(f) = \{q \mid q \in Q,\ f(q) < \lambda\}, \tag{3}$$

which depends on a real parameter $\lambda \in [-\infty, +\infty]$. This is the "spectral family" corresponding to f; clearly, we may uniquely recover f from the family (3). Therefore, we may replace the study of real functions with the study of the above-defined families. It is easy to give the conditions for an arbitrary family of sets to be generated by some real function. We first note that we may consider the elements of an arbitrary BA $\mathscr{X}$ in place of the point subsets. In this event the role of real functions will be played by the elements of some vector lattice over $\mathscr{X}$. However, there is no necessity to mention some vector lattice in the definition of spectral family; it is "intrinsic" to the BA $\mathscr{X}$.

A *spectral family* or *spectral function* is a family $\{e_\lambda\}_{\lambda \in [-\infty,+\infty]}$ of a given complete $\mathscr{X}$ which depends on a real parameter and possesses the properties:

1. $e_\lambda \leq e_\mu$ for $\lambda < \mu$;

2. $e_{+\infty} = \bigvee_{-\infty \leq \lambda \leq +\infty} e_\lambda = \mathbf{1}, \quad e_{-\infty} = \bigwedge_{-\infty \leq \lambda \leq +\infty} e_\lambda = \mathbf{0}.$

If the equalities hold

$$\bigvee_{\lambda < \lambda_0} e_\lambda = e_{\lambda_0} = \bigwedge_{\lambda > \lambda_0} e_\lambda$$

then we say that λ_0 is a *continuity point* of the spectral family $\{e_\lambda\}$. It is clear also what we will imply speaking about *left continuity* or *right continuity*.

Observe that the family in (3) is spectral in the sense of our definition; moreover, it possesses the additional property of left continuity. Sometimes the requirement of one-sided continuity is included in the definition of a spectral function. In the sequel $\pm\infty$ are always continuity points.

We say that two spectral functions $\{e'_\lambda\}$ and $\{e''_\lambda\}$ *almost coincide* provided that the inequalities hold

$$e'_\lambda \leq e''_\mu, \quad e''_\lambda \leq e'_\mu$$

for $\lambda < \mu$.

It is easy to check that the so-introduced relation of "almost coincidence" is reflexive, symmetric, and transitive; therefore, we may split the class of spectral families into disjoint subclasses so that the membership of two spectral families in the same class means that they almost coincide. We will call this subclasses the *resolutions of the identity* and denote them in general by the lower-case Gothic letters. Among all spectral functions $\{e_\lambda\}$ generating a given resolution of the identity $\mathfrak{f}$ we may distinguish two representatives that are in a sense "extreme": namely, there exist spectral families $\{e^-_\lambda(\mathfrak{f})\}$ and $\{e^+_\lambda(\mathfrak{f})\}$ such that the class $\mathfrak{f}$ coincides with the set of all spectral functions that satisfy the inequalities

$$e^-_\lambda(\mathfrak{f}) \leq e_\lambda \leq e^+_\lambda(\mathfrak{f}) \tag{4}$$

for all λ while $\{e^-_\lambda(\mathfrak{f})\}$ is left continuous and $\{e^+_\lambda(\mathfrak{f})\}$ is right continuous.

To construct these "boundary" families we take an arbitrary family $\{e_\lambda\}$ among those constituting the resolution of the identity $\mathfrak{f}$ under study and put

$$e^-_\lambda(\mathfrak{f}) \equiv \bigvee_{\mu<\lambda} e_\mu, \quad e^+_\lambda(\mathfrak{f}) \equiv \bigwedge_{\mu>\lambda} e_\mu.$$

It is easy to see that these two families possess the needed properties (in particular, each of them belongs to the class $\mathfrak{f}$). In general, if we are given a pair of almost coinciding spectral functions one of which is right continuous and the other is left continuous then the resolution of the identity is available that consists of all spectral families "jammed" between the two given. In this manner we may identify the resolutions of the identity with such pairs of spectral functions.

If one of the spectral functions generating a given resolution of the identity is continuous at some point λ_0 then each of the remaining functions of the class $\mathfrak{f}$ is continuous at λ_0; moreover,

$$e^-_{\lambda_0}(\mathfrak{f}) = e^+_{\lambda_0}(\mathfrak{f}).$$

Note also that all spectral functions constituting a given resolution of the identity have the same constancy intervals. The closed set of the real axis, complementary to the union of all constancy intervals, is called the *spectrum* of the resolution of the identity. The points of the spectrum are the same as the points of growth: each of the functions constituting this

resolution of the identity must increase essentially in each neighborhood about such a point. The spectrum may contain also the points $+\infty$ and $-\infty$. This will be exactly so if for all finite λ we have either $e_\lambda < \mathbf{1}$ or $e_\lambda > \mathbf{0}$.

We furnish the set of all resolutions of the identity with some order. Namely, we agree to write $\mathfrak{f}' \le \mathfrak{f}''$ provided that the inequality $\lambda < \mu$ implies $e_\lambda^+(\mathfrak{f}'') \le e_\mu^-(\mathfrak{f}')$. In other words, for $\lambda < \mu$ we must have $e_\lambda'' \le e_\mu'$ for arbitrary spectral families representing the classes $\mathfrak{f}'$ and $\mathfrak{f}''$ respectively. It is an easy matter to check the validity of the axioms of partial order (see Appendix A).

2.2 Spectral measures

We show that the concept of resolution of the identity is embraced in fact in a more general concept of homomorphism. In more detail, to each resolution of the identity there corresponds the unique homomorphism of the Boolean algebra $\mathscr{R}_{[-\infty,+\infty]}$ (cf. p. 68) to the algebra $\mathscr{X}$ under study and, conversely, such a homomorphism always generated a resolution of the identity. We now describe this correspondence.

Assume given some resolution of the identity $\mathfrak{f}$. Denote by E the set of all elements of the form $\{e_\lambda^-(\mathfrak{f})\}$ and $\{e_\lambda^+(\mathfrak{f})\}$. To define the homomorphism $\Phi_\mathfrak{f} \equiv \Phi$ of $\mathscr{R}_{[-\infty,+\infty]}$ to $\mathscr{X}$ (more exactly, onto $\mathscr{X}\langle E\rangle$), we first put

$$\Phi(\Delta_\lambda^+) \equiv e_\lambda^+(\mathfrak{f}), \quad \Phi(\Delta_\lambda^-) \equiv e_\lambda^-(\mathfrak{f}), \tag{5}$$

and then extend this mapping from the system of generators to the entire algebra $\mathscr{R}_{[-\infty,+\infty]}$. Such an extension is possible in view of the arguments on p. 66 since the system of generators of the form $\Delta_\lambda^-, \Delta_\lambda^+$ is linearly ordered and the formulas (5) determine an isotonic mapping. Thus, the initial resolution of the identity has produced the homomorphism Φ of $\mathscr{R}_{[-\infty,+\infty]}$ onto $\mathscr{X}\langle E\rangle$. It is easy to write out the formulas for the values of Φ at the intervals:

$$\begin{aligned} \Phi((\alpha,\beta)) &= e_\beta^- - e_\alpha^+, \\ \Phi([\alpha,\beta]) &= e_\beta^+ - e_\alpha^-, \\ \Phi([\alpha,\beta)) &= e_\beta^- - e_\alpha^-, \\ \Phi((\alpha,\beta]) &= e_\beta^+ - e_\alpha^+. \end{aligned} \tag{6}$$

In particular,

$$\begin{aligned} \Phi([-\infty,\beta)) &= e_\beta^-, \\ \Phi([-\infty,\beta]) &= e_\beta^+. \end{aligned} \tag{7}$$

Therefore, the resolution of the identity $\mathfrak{f}$ is, in turn, uniquely determined from the homomorphism $\Phi \equiv \Phi_\mathfrak{f}$. It is easy to see that in this event we

have

$$\Phi([-\infty,\beta)) = \bigvee_{\beta'<\beta} \Phi([-\infty,\beta')),$$

$$\Phi([-\infty,\beta]) = \bigwedge_{\beta'>\beta} \Phi([-\infty,\beta']),\ \ \Phi(\{\pm\infty\}) = \mathbf{0}.$$

Each homomorphism with these properties is generated by some resolution of the identity by the formulas (6) and (7). Agree to call these homomorphisms "spectral."

We adduce an important EXAMPLE. Let u be an arbitrary element of an algebra $\mathscr{X}$. Using this element, we define the spectral function $\{e_\lambda^u\}$ as

$$e_\lambda^u \equiv e_\lambda = \begin{cases} \mathbf{0}, & \text{if} \quad \lambda < 0, \\ Cu, & \text{if} \quad 0 \le \lambda < 1, \\ \mathbf{1}, & \text{if} \quad \lambda \ge 1. \end{cases} \tag{8}$$

The spectral family (8) is right continuous; the corresponding resolution of the identity is composed of all spectral functions whose values at each λ lie in the interval $[e_\lambda^-, e_\lambda^+]$, where $e_\lambda^+ = e_\lambda$ and e_λ^- is determined from the equality

$$e_\lambda^- = \begin{cases} \mathbf{0}, & \text{if} \quad \lambda \le 0, \\ Cu, & \text{if} \quad 0 < \lambda \le 1, \\ \mathbf{1}, & \text{if} \quad \lambda > 1. \end{cases}$$

At last, we describe the homomorphism Φ that corresponds to this resolution of the identity. The law of its generation is very simple: if the set $e \in \mathscr{R}_{[-\infty,+\infty]}$ does not contain 0 and 1 then $\Phi(e) = \mathbf{0}$; if $1 \in e$ and $0 \notin e$ then $\Phi(e) = u$; for $1 \notin e,\ 0 \in e$ we have $\Phi(e) = Cu$; and, finally, $\Phi(e) = \mathbf{1}$ for the sets containing 0 and 1.

We are in a similar situation in mathematical analysis when constructing the Lebesgue–Stieltjes measure given the "distribution function" with jumps at the points 0 and 1 and of constant width between the endpoints. The homomorphism Φ is an abstract analog of this measure. Once we have agreed to call the homomorphisms on algebras of sets "Boolean measures" (p. 68). Among them are listed the homomorphisms corresponding to the resolutions of the identity; intending to emphasize their provenance, we will call them *spectral measures.* As regards the spectral families comprising a given resolution of the identity, we will say that they generate the corresponding spectral measure. It is easy to see that *the spectral measure is always supported by the spectrum of the generating resolution of the identity.*

The theory of spectral measures originated with the spectral theory of operators where the role of the BA $\mathscr{X}$ is performed by the Boolean

algebra of invariant subspaces of some selfadjoint operator in a Hilbert space.[8] The general concept of spectral measure was introduced by V. I. Sobolev in [1], [2] (under the name of "semiordered measure").[9]

The algebra $\mathscr{R}_{[-\infty,+\infty]}$ consists of the finite unions of intervals and so it is rather poor; it fails to be even a σ-algebra of sets. Therefore, we naturally come to the problem of extending a spectral measure to a wider algebra, for instance, the Borel algebra. Of course, it is desirable to obtain at least a σ-continuous extension. The following theorem is rather easy.

Theorem 4.[10] *Each spectral homomorphism from the algebra*

$$\mathscr{R} \equiv \mathscr{R}_{[-\infty,+\infty]}$$

to a complete BA $\mathscr{X}$ satisfies the condition (E_σ).[11]

Note that the condition (E_σ^*), guaranteeing the existence of a σ-continuous extension, may fail in regard to the spectral measure Φ; everything depends on the properties of $\mathscr{X}$. As a sufficient condition we may for instance take the regularity of $\mathscr{X}$. However, as was first shown by J. D. M. Wright and A. G. Kusraev and S. A. Malyugin [1], the σ-continuous extension of a spectral measure is always possible provided only that $\mathscr{X}$ is a complete BA.

Theorem 5. *If $\mathfrak{f}$ is a resolution of the identity of a complete BA $\mathscr{X}$ then the corresponding spectral measure Φ admits a σ-continuous extension to some σ-algebra $\widetilde{\mathscr{R}} \supset \mathscr{R}$.*

The proof of this theorem rests on Lemma 4 of Chapter 5. In this event it is convenient to assume that $\mathscr{X} = \mathscr{X}^0 \equiv \mathscr{C}O(\mathfrak{Q}(\mathscr{X}))$. As usual, this involves no loss of generality. Alongside the homomorphism Φ from $\mathscr{R}$ to $\mathscr{X}$, consider another homomorphism $\Psi : \mathscr{R} \longrightarrow \mathscr{B}_0(\mathfrak{Q}(\mathscr{X}))$. Namely, if

$$\Phi((a,b)) = \bigcup_{\lambda<b} e_\lambda^- - \bigwedge_{\lambda>a} e_\lambda^+, \quad \Phi([a,a]) = \bigwedge_{\lambda>a} e_\lambda^- - \bigcup_{\lambda<a} e_\lambda^+,$$

then[12]

$$\Psi((a,b)) \equiv \bigcup_{\lambda<b} e_\lambda^- \setminus \bigcap_{\lambda>a} e_\lambda^+, \quad \Psi([a,a]) \equiv \bigcap_{\lambda>a} e_\lambda^- \setminus \bigcup_{\lambda<a} e_\lambda^+$$

[8] N. Dunford and J. T. Schwartz [2]; A. I. Plesner [1]; A. I. Plesner and V. A. Rokhlin [1]; B. Z. Vulikh [1,2]; E. Lorch [1]; F. Riesz and B. Szökefalvi-Nagy [1].

[9] Also see B. Z. Vulikh [2], A. G. Porishkin [1], and D. A. Vladimirov [2].

[10] B. Z. Vulikh [1, 2].

[11] The condition (E_σ) is formulated in Chapter 5, p. 249.

[12] We briefly write $e_\lambda^\pm$ instead of $e_\lambda^\pm(\mathfrak{f})$.

(on the remaining sets Ψ is defined naturally in much the same way as Φ). The resultant Ψ is a spectral measure with values in $\mathscr{B}_0(\mathfrak{Q}(\mathscr{X}))$. By Theorem 4 Ψ satisfies the condition (E_σ). Clearly, the sets $\Phi(E)$ and $\Psi(E)$, with $E \in \mathscr{R}$, lie in $\mathfrak{Q}(\mathscr{X})$ and differ by a rare set. So by the already-mentioned Lemma 4, σ-continuous extension is possible.

Thus every spectral measure is σ-continuous ("countably additive") on the σ-algebra $\widetilde{\mathscr{R}}$ and, consequently, on the Borel σ-algebra. However, the case is worth noting in which $\mathscr{X}$ is a weakly σ-distributive algebra; for instance, a regular algebra. In this case the measure Φ may be constructed on appealing to Theorem 5 of the previous chapter and it coincides with the "outer measure" Φ^* at Borel sets. In the general case this is not so. If $\mathscr{X}$ is not a weakly σ-distributive BA then there exist closed sets $F_1, F_2, \dots$ such that $\Phi^*(F_n) = \mathbf{0}$ and $\Phi^*(\bigcup F_n) = \mathbf{1}$.[13] Clearly, we always have $\Phi(E) \le \Phi^*(E)$ (by the σ-continuity of Φ), and so

$$\Phi(F_n) = \mathbf{0}, \quad \Phi\Big(\bigcup_n F_n\Big) = \mathbf{0} \neq \Phi^*\Big(\bigcup_n F_n\Big).$$

If $\mathscr{X}$ is a regular BA then the value $\Phi(E)$, with E a Borel set, coincides with $\Phi(E_1)$, where E_1 is some G_δ-set and $E_1 \supset E$. In the general case this is nor so either.

Theorem 5 may be abstracted. To this end, we need reveal the connection between the resolutions of the identity of a BA and the continuous functions on the Stone space of this BA. This will drive us to a better understanding of the proof of Theorem 5 as well as the nature of the homomorphism Ψ.

2.3 Continuous functions and resolutions of the identity

Identifying a complete Boolean algebra $\mathscr{X}$ and its representation $\mathscr{X}^0 \equiv \mathscr{C}O(\mathfrak{Q}(\mathscr{X}))$ again, we now establish a bijection between the continuous real functions on $\mathfrak{Q}$ and resolutions of the identity.

Assume given a continuous function f (possibly taking the values $\pm\infty$ on rare sets). Consider the family of its level sets

$$E_\lambda \equiv E_\lambda(f) \equiv \{q \in \mathfrak{Q} \mid f(q) < \lambda\},$$
$$\widetilde{E}_\lambda \equiv \widetilde{E}_\lambda(f) \equiv \{q \in \mathfrak{Q} \mid f(q) \le \lambda\}.$$

If $\mathscr{X}$ is a nondiscrete BA (we will study only this case) then none of the sets E_λ and $\widetilde{E}_\lambda$ must be clopen and belong to $\mathscr{X}^0$. Clopen are the sets

$$e_\lambda^- \equiv clos\, E_\lambda(f), \quad e_\lambda^+ \equiv Int\, \widetilde{E}_\lambda(f). \tag{9}$$

[13] D. A. Vladimirov [2].

Since $E_\lambda \subset Int\,\widetilde{E}_\lambda$; therefore, $e_\lambda^- \subset e_\lambda^+$. We have two obviously equivalent spectral families that determine some resolution of the identity $\mathfrak{f}$:

$$e_\lambda^- \equiv e_\lambda^-(\mathfrak{f}), \quad e_\lambda^+ \equiv e_\lambda^+(\mathfrak{f}).$$

It is easy to check the one-sided continuity of the families e_λ^- and e_λ^+ as well as to derive the following:

$$E_\lambda(f) = \bigcup_{\mu<\lambda} e_\mu^- = \bigcup_{\mu<\lambda} e_\mu^+, \quad \widetilde{E}_\lambda(f) = \bigcap_{\mu>\lambda} e_\mu^- = \bigcap_{\mu>\lambda} e_\mu^+.^{14} \tag{10}$$

On the other hand, the resolution of the identity $\mathfrak{f}$ allows us to uniquely restore the continuous functions f. Namely, the value of $f(q)$ at each point $q \in \mathfrak{Q}$ is calculated from the following condition: $f(q)$ is the unique number $a \in \overline{\mathbb{R}}$ such that

$$q \in \bigcap_{\lambda>a} e_\lambda^- \cap \bigcap_{\lambda<a} Ce_\lambda^+ \equiv \bigcap_{\lambda>a} e_\lambda^- \setminus \bigcup_{\lambda<a} e_\lambda^+.$$

(The proof of the unique existence of this number is readily available.) The set on the right-hand side of the last equality is a level set of f. It is easy to check that the level sets E_λ and $\widetilde{E}_\lambda$ are connected with the resolution of the identity $\mathfrak{f}$ exactly by the formulas (9) and (10); this implies, first, the continuity of f (for E_λ are open and $\widetilde{E}_\lambda$ are closed) and, second, f generates $\mathfrak{f}$ (the formula (9)). Clearly, the correspondence $f \longleftrightarrow \mathfrak{f}$ is one-to-one.

We emphasize once again that the clopen sets e_λ^- and e_λ^+ do not coincide in general with the level sets E_λ and $\widetilde{E}_\lambda$. The partition of a compact space into level sets is not Boolean as a rule. We have mentioned this fact in Chapter 3 and will return to it somewhat later.

2.4 Spectral homomorphisms of a general form

Let a resolution of the identity $\mathfrak{f}$ and a continuous function f be connected so as described in 2.3. Then, given a and b ($a < b$) we easily see that

$$f^{-1}((a,b)) = \bigcup_{\lambda<b} e_\lambda^- \setminus \bigcap_{\lambda>a} e_\lambda^+ = \Psi((a,b)),$$

where Ψ is the homomorphism constructed in the proof of Theorem 5. In much the same way, for the singletons we infer

$$f^{-1}([a,a]) = \widetilde{E}_a(f) \setminus E_a(f) = \bigcap_{\lambda>a} e_\lambda^- \setminus \bigcap_{\lambda<a} e_\lambda^+ = \Psi([a,a]).$$

[14] From (10) it follows that these are Baire sets.

Clearly, the equality $f^{-1}(e) = \Psi(e)$ holds for all $e \in \mathscr{R}_{[-\infty,+\infty]}$. This gives us a simple interpretation of Ψ. It is now clear without recourse to Theorem 4 that Ψ satisfies the condition (E_σ) which reduces in this case to the familiar property of inverse images:

$$f^{-1}\Big(\bigcup_n e_n\Big) = \bigcup_n f^{-1}(e_n). \tag{11}$$

The idea we outlined admits further elaboration. Consider the space $R_\Xi \equiv \overline{\mathbb{R}}^\Xi$, the cartesian power of the extended real axis $\overline{\mathbb{R}} \equiv [-\infty, +\infty]$. Here Ξ is an arbitrary set, and the elements of R_Ξ are all families $r = \{r_\xi\}_{\xi\in\Xi}$, $r_\xi \in \overline{\mathbb{R}}$. The *standard half-spaces* are the sets of the form

$$L_\xi^{c-} \equiv \{r \in R_\Xi \mid r_\xi < c\}, \quad L_\xi^{c+} \equiv \{r \in R_\Xi \mid r_\xi \le c\} \quad (c \in \overline{\mathbb{R}}).$$

The algebra $\mathscr{R}^{[\Xi]}$, generated by these half-spaces, consists of all unions of the form $E = E_1 \cup E_2 \cup \ldots \cup E_s$, where each E_i is the cartesian product of intervals

$$E_i = \prod_{\xi\in\Xi} \Delta_i^\xi$$

in which all but finitely many Δ_i^ξ coincide with $\overline{\mathbb{R}}$. We let $\mathbb{B}^{[\Xi]}$ stand for the σ-algebra generated by $\mathscr{R}^{[\Xi]}$.

To each $\xi \in \Xi$ there corresponds the subalgebra $\mathscr{R}_\xi^{[\Xi]} \subset \mathscr{R}^{[\Xi]}$ generated by the half-spaces of the form $L_\xi^{c\pm}$ $(c \in \overline{\mathbb{R}})$. Clearly, the mapping

$$L_\xi^{c-} \longrightarrow [-\infty, c), \quad L_\xi^{c+} \longrightarrow [-\infty, c]$$

extends uniquely to an isomorphism of $\mathscr{R}_\xi^{[\Xi]}$ onto $\mathscr{R}_{[-\infty,+\infty]}$. This ("canonical") isomorphism will be denoted by T_ξ (cf. 1.1.5).

Assume that to each $\xi \in \Xi$ there is associated some homomorphism $\theta_\xi : \mathscr{R}_{[-\infty,+\infty]} \longrightarrow \mathscr{Y}$, where $\mathscr{Y}$ is some BA. Denote by $\overline{\theta}_\xi$ the result of its "translation" to the algebra $\mathscr{R}_\xi^{[\Xi]}$:

$$\overline{\theta}_\xi \equiv \theta_\xi \circ T_\xi.$$

Using the Sikorski Theorem it is easy to check that all homomorphism $\overline{\theta}_\xi$ has a unique simultaneous extension: there is precisely one homomorphism $\overline{\theta} : \mathscr{R}^{[\Xi]} \longrightarrow \mathscr{Y}$ satisfying the condition $\overline{\theta}(L_\xi^{c\pm}) = \overline{\theta}_\xi(L_\xi^{c\pm})$ for all $\xi \in \Xi$ and $c \in \overline{\mathbb{R}}$.

Assume now that to each $\xi \in \Xi$ there is assigned a resolution of the identity $\mathfrak{f}_\xi$ of a BA $\mathscr{X}$, which as we have just seen, generates the numerical function $f_\xi : \mathfrak{Q} \longrightarrow \overline{\mathbb{R}}$ and the homomorphism $\Psi_\xi :\ \Psi_\xi(E) \equiv f_\xi^{-1}(E)$

from $\mathscr{R}_{[-\infty,+\infty]}$ to $\mathscr{B}_0(\mathfrak{Q})$. Consider the mapping $f : \mathfrak{Q} \longrightarrow R_\Xi$ defied by the formula $f(q) \equiv \{f_\xi(q)\}_{\xi\in\Xi}$ and the corresponding homomorphism Ψ; i.e.,

$$\Psi(E) \equiv f^{-1}(E), \quad E \in \mathscr{R}^{[\Xi]}.$$

This homomorphism Ψ acts from $\mathscr{R}^{[\Xi]}$ to $\mathscr{B}_0(\mathfrak{Q})$; the values of Ψ at the standard half-spaces $L_\xi^{c\pm}$ are calculated by the formulas:

$$\Psi(L_\xi^{c-}) = \Psi_\xi([-\infty, c)) = \bigcup_{\lambda<c} e_\lambda^+(\mathfrak{f}_\xi) = f_\xi^{-1}([-\infty, c)),$$

$$\Psi(L_\xi^{c+}) = \Psi_\xi([-\infty, c]) = \bigcap_{\lambda>c} e_\lambda^-(\mathfrak{f}_\xi) = f_\xi^{-1}([-\infty, c]).$$

We may simultaneously determine the spectral homomorphisms Φ_ξ that correspond to the resolutions of the identity $\mathfrak{f}_\xi$ and the "multidimensional" spectral homomorphism $\overline{\Phi}$ that is given at the standard half-spaces by the equalities

$$\overline{\Phi}(L_\xi^{c-}) = \bigvee_{\lambda<c} e_\lambda^+(\mathfrak{f}_\xi) = e_c^-(\mathfrak{f}_\xi) = \Phi_\xi([-\infty, c)),$$

$$\overline{\Phi}(L_\xi^{c+}) = \bigwedge_{\lambda>c} e_\lambda^-(\mathfrak{f}_\xi) = e_c^+(\mathfrak{f}_\xi) = \Phi_\xi([-\infty, c]).$$

In other words, $\overline{\Phi}$ is constructed by the general scheme: here

$$\theta_\xi \equiv \Phi_\xi, \ \overline{\theta}_\xi \equiv \overline{\Phi}_\xi, \ \overline{\theta} \equiv \overline{\Phi}.$$

Given ξ and c, we have the inclusions

$$\Psi(L_\xi^{c-}) \subset \overline{\Phi}(L_\xi^{c-}), \Psi(L_\xi^{c+}) \supset \overline{\Phi}(L_\xi^{c+});$$

moreover, the differences

$$\overline{\Phi}(L_\xi^{c-}) \setminus \Psi(L_\xi^{c-}) = \bigvee_{\lambda<c} e_\lambda^+(\mathfrak{f}_\xi) \setminus \bigcup_{\lambda<c} e_\lambda^+(\mathfrak{f}_\xi),$$

$$\Psi(L_\xi^{c+}) \setminus \overline{\Phi}(L_\xi^{c+}) = \bigcap_{\lambda>c} e_\lambda^-(\mathfrak{f}_\xi) \setminus \bigwedge_{\lambda>c} e_\lambda^-(\mathfrak{f}_\xi)$$

are rare sets, implying that the difference $|\overline{\phi}(E) - \Psi(E)|$ is rare for all $E \in \mathscr{R}^{[\Xi]}$. Clearly, the homomorphism Ψ now satisfies the condition (E_σ) (the equality (11)). Hence, we may apply Lemma 5.4 and, consequently the multidimensional spectral homomorphism $\overline{\Phi}$ extends to a σ-continuous homomorphism from $\mathbb{B}^{[\Xi]}$ to $\mathscr{X}$. We have thus arrived to the following theorem whose particular case is Theorem 5.

Theorem 5*. *Let $\mathscr{X}$ be a complete BA, and let $\{\mathfrak{f}_\xi\}_{\xi\in\Xi}$ be an arbitrary family of resolutions of the identity of $\mathscr{X}$. There exists a unique σ-continuous homomorphism $\widetilde{\Phi} : \mathbb{B}^{[\Xi]} \longrightarrow \mathscr{X}$ such that*

$$\widetilde{\Phi}(L_\xi^{c\pm}) = e_c^{\pm}(\mathfrak{f}_\xi)$$

for all $c \in \overline{\mathbb{R}}$ and $\xi \in \Xi$.

We now turn to a PROBABILITY BA $\{\mathscr{X}, \mu\}$. In this event, it stands to reason to call $\{\mathfrak{f}_\xi\}$ a "random process." The formula $m(e) \equiv \mu\widetilde{\Phi}(e)$ defines on $\mathbb{B}^{[\Xi]}$ a quasimeasure that satisfies the conditions

$$m(L_{\xi_1}^{c_1\pm} \cap \ldots \cap L_{\xi_s}^{c_s\pm}) = \mu(e_{c_1}^{\pm}(\mathfrak{f}_{\xi_1}) \wedge \ldots \wedge e_{c_s}^{\pm}(\mathfrak{f}_{\xi_s}))$$

for all $\xi_1, \ldots, \xi_s \in \Xi$ and $c_1, \ldots, c_s \in \overline{\mathbb{R}}$. The existence of this quasimeasure follows also from the celebrated Kolmogorov Theorem (A. N. Kolmogorov [1]).

3. Separable Boolean algebras and σ-algebras of sets. Measurable functions

At the end of Chapter 5, the representation theorems were stated for the Boolean algebras satisfying the countable chain condition, in particular, for regular Boolean algebras. For instance, Theorem 10 asserts that a separable Boolean algebra satisfying the countable chain condition is a σ-HOMOMORPHIC image of a certain σ-algebra of subsets of an interval. We sketch now a different approach to the proof of this theorem.

Let $\mathfrak{f}$ be an arbitrary resolution of the identity for a BA $\mathscr{X}$ and let $\Phi \equiv \Phi_{\mathfrak{f}}$ be the corresponding spectral measure; this measure can be assumed σ-continuous and defined on some σ-algebra $\mathscr{A}_{\mathfrak{f}}$ of subsets of the real axis $\mathbb{R}$. This σ-algebra (we often denote it simply by $\mathscr{A}$) contains all Borel subsets.

Put $I = \ker \Phi$. The quotient algebra $\mathscr{A}|_I$ is isomorphic to the regular subalgebra $\mathscr{X}_0 \equiv \Phi(\mathscr{A}) \subset \mathscr{X}$. If $\mathscr{X}$ is separable then it is possible to choose the initial resolution of the identity so that this image coincides with $\mathscr{X}$,[15] and we obtain another version of the representation theorem for a separable BA satisfying the countable chain condition. Observe that the "actual" representation space is the SPECTRUM of $\mathfrak{f}$; every set disjoint from this spectrum belongs to the ideal I; its "Boolean measure" equals $\mathbf{0}$. The spectrum can coincide with the interval $[0, 1]$; in this case, exactly the same Theorem 10 of Chapter 5 is obtained. The new proof of this theorem allows us to specify the structure of the σ-algebra $\mathscr{A}$,

[15]See Theorem 7 below.

taking into account the properties of the resolution of the identity which defines the representation.

In this construction, a narrower σ-algebra $\mathscr{B}$, the Borel algebra of the representation space, may be used instead of the algebra $\mathscr{A}$. If we denote by I_0 the σ-ideal in $\mathscr{B}$ consisting of all Borel sets of the ideal I, then the quotient algebras $u\mathscr{A}|_I$ and $\mathscr{B}|_{I_0}$ become naturally isomorphic, so that either of them can be regarded as a representation of the BA $\mathscr{X}_0$ which can be given as a homomorphic image:

$$\mathscr{X}_0 = \Phi_0(\mathscr{B}),$$

where $\Phi_0 = \Phi_{\mathscr{B}}$.

This construction originated in the classical scheme of Lebesgue–Stieltjes measure. The σ-algebra $\mathscr{A}$ itself resembles the ordinary algebra of Lebesgue–measurable sets, and the theory of algebras of this kind is a generalization of the traditional theory of functions in a real variable. We give examples of the stock of available theorems. The words "almost everywhere" mean "everywhere except some member of I." From now on we assume the algebra $\mathscr{X}$ REGULAR. Then the ideal I possesses the "regularity properties" (α)–(δ) of Chapter 5. These properties guarantee validity of the following theorems:

Egorov Theorem. *If f and f_n $(n = 1, 2, \dots)$ are $\mathscr{A}$-measurable almost everywhere finite functions and $f_n(x) \longrightarrow f(x)$ almost everywhere then there exists a sequence of $\mathscr{A}$-measurable sets $\{E_m\}$ such that $\Phi(E_m) \xrightarrow{(o)} \mathbf{1}_{\mathscr{X}}$ and f_n converges uniformly to f on each E_m.*

Luzin Theorem. *For each $\mathscr{A}$-measurable almost everywhere finite function f, there exists a sequence of continuous functions $\{f_n\}$ such that all sets $q_n \equiv \{x | f_n(x) \neq f(x)\}$ are $\mathscr{A}$-measurable and $\Phi(q_n) \xrightarrow{(o)} \mathbf{0}_{\mathscr{X}}$.*

(It is clear that all continuous functions are $\mathscr{A}$-measurable.)

Riesz Theorem. *If f and f_n $(n = 1, 2, \dots)$ are $\mathscr{A}$-measurable almost everywhere finite functions and, for each $\sigma > 0$,*

$$\Phi\{x \mid |f_n(x) - f(x)| \geq \sigma\} \longrightarrow \mathbf{0}_{\mathscr{X}} \quad \textit{in the } (o)\textit{-topology},$$

then there exists a sequence $\{f_{n_k}\}$ $(n_1 < n_2 < \cdots)$ convergent to f almost everywhere.

The idea of deriving these theorems is clear: take the "classical" formulation and make no mentions of the measure and its numerical values; leave only such expressions as "almost everywhere." In the proofs of the three theorems given above, only the regularity of $\mathscr{X}$ is used. But we do not assert that each theorem in the theory of functions turns automatically into a valid statement after the modification described. The

situation is the same as in the elementary geometry where some theorems are "absolutely" valid, but the validity of other theorems depends on the parallel axiom and they become false in a non-Euclidean geometry (despite their formulations still make sense).

D. Maharam proved in 1947 that the negation of the Suslin hypothesis[16] implies the existence of a regular nonnormable BA. Therefore, we may speak about the "Suslin" and "non-Suslin" versions of analysis. The second is rather rich; for example, it contains the three theorems cited above. As far as we know, a thorough search of other theorems was not performed. As for the above three theorems, their proofs are not difficult, although they demand some knowledge of regular K-spaces.

4. The integral with respect to a spectral measure and the Freudenthal Theorem. The space $\mathfrak{S}_{\mathscr{X}}$ as the family of resolutions of the identity. Functions of elements

4.1 The spectral integral

Let R be a K-space with weak unit $\mathbf{1}$ and let H_R be the Boolean algebra of bands, the *base* of R. We have already noted that H_R is isomorphic to the naturally ordered system E_R of all unit elements. Taking the liberty of inaccuracy, we call E_R the base of R as well. Given some resolution of the identity of the algebra E_R, let Φ be the corresponding homomorphism from $\mathscr{R}_{[-\infty,+\infty]}$ into E_R (in other words, a spectral measure). The values of this measure can be regarded as elements of R. Following the routine scheme of mathematical analysis, it is possible to introduce the concept of "spectral integral" with respect to the measure Φ for a real function on the real axis. It is desirable to suppose that the measure Φ is extended onto the widest possible algebra of sets. Theorem 5 of this chapter shows that the spectral measure Φ can be assumed countably additive and defined on the σ-algebra $\widetilde{\mathscr{R}} \supset \mathscr{R}_{[-\infty,+\infty]}$. Let f be a $\widetilde{\mathscr{R}}$-measurable and Φ-almost everywhere finite real function. This means that the algebra $\widetilde{\mathscr{R}}$ contains the inverse images $f^{-1}(\Delta)$ of all intervals and the set

$$e_\infty \equiv \{x \mid x \in (-\infty, +\infty),\ |f(x)| = +\infty\};$$

moreover, $\Phi(e_\infty) = \mathbf{0}$.

[16]See p. 396 below. The negation of the Suslin hypothesis is consistent with the axioms of set theory.

The *spectral integral*

$$I(f,\Phi) \equiv \int\limits_{(-\infty,+\infty)} f\,d\Phi \tag{12}$$

is, by the definition, the supremum in R of all sums of the form

$$S = \sum_{k=1}^{m} m_k \Phi(e_k),$$

where the sets e_k are disjoint and belong to $\widetilde{\mathscr{R}}$, and the sum of them is the whole real axis;[17] while the numbers m_k are the least upper bounds of f over the sets e_k. The sums of this kind (they may be called "Darboux sums") are elements of the K-space R; for the integral to exist, it is sufficient that the set of these sums is bounded above. It may be shown that in the case of a universally complete K-space R the integral exists for every $\widetilde{\mathscr{R}}$-measurable and Φ-almost everywhere finite function f. Thus we can interpret the spectral integral as an OPERATOR sending a certain set of real functions into R. This operator is additive while its domain is a linear subspace. As for other properties of the integral, we mention positivity (the integral of a nonnegative function is a positive element of R) and orthogonality: if the functions f and g are equal to zero on disjoint sets then their integrals are disjoint.

The integral (12) is often denoted by the pattern of the Stieltjes integral:

$$I(f,\Phi) \equiv \int\limits_{-\infty}^{+\infty} f(\lambda)\,de_\lambda.$$

This notation calls to mind the spectral family $\{e_\lambda\}$ generating the spectral measure Φ. Of course, the letter λ in this formula is "fictitious," it is a "bound variable."

An especially important case is the function defined by the equality $f(\lambda) = \lambda$. It turns that the formula

$$x = \int\limits_{-\infty}^{+\infty} \lambda\,de_\lambda \tag{13}$$

gives a general form of the elements of the K-space R.

---

[17] Here we imply the unextended real axis $\mathbb{R} \equiv (-\infty,+\infty)$. Recall that $e^-_{+\infty} = \mathbf{1}$ and $e^+_{-\infty} = \mathbf{0}$; therefore, $\Phi(\overline{\mathbb{R}} \setminus \mathbb{R}) = \mathbf{0}$.

Theorem 6 (H. Freudenthal). *To each element x of a K-space R there is a unique spectral measure Φ_x satisfying*

$$x = \int\limits_{(-\infty,+\infty)} f \, d\Phi_x, \tag{14}$$

with $f(\lambda) = \lambda$ for all λ.

The last formula, as was mentioned above, can be rewritten as

$$x = \int\limits_{-\infty}^{+\infty} \lambda \, de_\lambda^x, \tag{15}$$

keeping in mind that the spectral function $\{e_\lambda^x\}$ generates the measure Φ_x. The resolution of the identity, composed of these functions, is called the *characteristic* of x, and the spectrum of this resolution is called the spectrum of x. In particular, the resolution of the identity we gave as an example on p. 290 coincides with the characteristic of the element $u \in E_R \subset R$.

Let R be a K-space with weak unit and let, as usual, E_R be the system of all unit elements. The elements of the form

$$x = c_1 e_1 + c_2 e_2 + \cdots + c_m e_m, \tag{16}$$

where $e_i \in E_R$, are called *step functions.* The characteristic of the step function of the type (16) is very simple: if $c_1 < c_2 < \cdots < c_m$, $\sum_{k=1}^m e_k = \mathbf{1}$, then

$$e_\lambda^- = \begin{cases} \mathbf{0}, & \text{if} \quad \lambda \le c_1, \\ e_1, & \text{if} \quad c_1 < \lambda \le c_2, \\ e_1 + e_2, & \text{if} \quad c_2 < \lambda \le c_3, \\ \dots & \dots \quad \dots \\ e_1 + e_2 + \cdots + e_m = \mathbf{1}, & \text{if} \quad c_{m-1} < \lambda \le c_m \end{cases}$$

(it is a generalization of (8)).

The equality (16) itself demonstrates the Freudenthal Theorem in this simple situation. Denote the set of all step functions by R^0. It is a vector sublattice of the K-space R. The Freudenthal Theorem implies that each element $x \in R$ is represented as the supremum of a some set of step functions.

As a rule, the set R^0 is not conditionally complete, and it is not a K-space. But its closure in norm

$$||x||_b \equiv \inf\{\lambda \ge 0 \mid \ |x| \le \lambda \mathbf{1}\}$$

is already a K-space; namely, it is a solid subspace of R. We call it the "*space of bounded elements*," and denote it by R^b. Note that the constitution of this space depends on the choice of the weak unit $\mathbf{1}$. The spectrum of each element $x \in R^b$ is supported in the interval $[-\|x\|_b, \|x\|_b]$ and the integration in the Freudenthal formula is carried out over this interval.

If the role of a weak unit is played by the identically one function, the Stone transform sends R^b into the familiar space $C(\mathfrak{Q})$ of bounded continuous real functions. And the norm $\|\cdot\|_b$ is simply the Chebyshev ("uniform") norm making R^b into a Banach space.

As for R^0, this set consists of all finite continuous functions. It is clear that R^0 is dense in R^b with respect to the norm $\|\cdot\|_b$.

4.2 The space of resolutions of the identity

The Freudenthal Theorem determines a one-to-one correspondence between the elements of R and the certain resolutions of the identity of the algebra E_R. It is clear that the concept of resolution of the identity has an intrinsic nature with respect to the Boolean algebra; its definition goes with no mention of the elements of R. At the same time, Theorem 6 allows us to identify the elements of R and the corresponding resolutions of the identity or, equivalently, the spectral measures.

These arguments suggest a method for constructing a K-space over a given algebra $\mathscr{X}$ as the space consisting of the resolutions of the identity or, if you please, of the spectral measures. Namely, in addition to the partial order introduced in 6.2.1, it is necessary in the same intrinsic way to define the algebraic operations (addition and scalar multiplication) of resolutions of the identity so as to obtain a universally complete K-space R that includes $\mathscr{X}$ in the capacity of E_R. The elements of this space must coincide with theirs characteristics.

This program was first implemented by L. V. Kantorovich,[18] so giving seemingly the first proof of Theorem 2. The space of ALL resolutions of the identity of the BA $\mathscr{X}$ is universally complete[19] and it can be identified with $\mathfrak{S}_{\mathscr{X}}$; we may also consider its subspaces that contain not ALL resolutions of the identity. Thus *each K-space may be realized as a space of spectral measures.*

Another proof of Theorem 2 (it was sketched in 6.1.5) bases on the Stone representation of a BA. Thus we possess the two interpretations of $\mathfrak{S}_{\mathscr{X}}$; first, as the space $C_\infty(\mathfrak{Q})$ of continuous functions; second, as the space of resolutions of the identity.

---

[18] L. V. Kantorovich, B. Z. Vulikh, and A. G. Pinsker [1, Section 4, § 1].

[19] The weak unit is determined for example from the formula (8) with $u = \mathbf{1}_{\mathscr{X}}$.

The bridge between these two approaches was actually found in 6.2.3 where the correspondence

$$\mathfrak{f} \longleftrightarrow f$$

is defined identifying the continuous functions and the resolutions of the identity. This correspondence is in fact an isomorphism between the two models of universally complete K-spaces over a complete BA.

We now rivet the reader's attention on the "Stone" interpretation of the SPECTRUM. Namely, the spectrum of the resolution of the identity $\mathfrak{f}$ coincides with the set of all values (including the infinity) of a continuous function f. The points of the discrete spectrum corresponds to the clopen sets on which f is constant. The proof is not complicated; for the reader familiar with the theory of Banach algebras this situation seems well-known.

Let $R = \mathfrak{S}_{\mathscr{X}}$. Then, as we know, there exists an isomorphism between $\mathscr{X}$ and E_R. Suppose that R is the space of resolutions of the identity. The formula (8) assigns to an element $u \in \mathscr{X}$ the spectral family $\{e^u_\lambda\}$; in the sequel we will note the corresponding resolution of the identity by $u^\bullet$. It is easy to see that the mapping $u \longrightarrow u^\bullet$ is an isotonic isomorphism which embeds $\mathscr{X}$ into $\mathfrak{S}_{\mathscr{X}}$. The element $\mathbf{1}^\bullet$ is a weak unit in $\mathfrak{S}_{\mathscr{X}}$, and all unit elements $u^\bullet$ comprise a Boolean algebra isomorphic to $\mathscr{X}$. We will denote it by $\mathscr{X}^\bullet$ but often omit the dot, so identifying u and $u^\bullet$ as well as $\mathscr{X}$ and $\mathscr{X}^\bullet \equiv E_R$. This is the *canonical* embedding. Thus, the BA $\mathscr{X}$ can be "canonically" embedded into $\mathfrak{S}_{\mathscr{X}}$ in two ways: the first corresponds to the interpretation of the elements of $\mathfrak{S}_{\mathscr{X}}$ as continuous functions, and the second views $\mathfrak{S}_{\mathscr{X}}$ as consisting of the resolutions of the identity, or equivalently, of the spectral measures.

In 6.4.3, another realization of the space $\mathfrak{S}_{\mathscr{X}}$ will be presented which is appropriate for the algebras with the countable chain condition.

4.3 Representation of $\mathfrak{S}_{\mathscr{X}}$ as a space of measurable functions

In Section 3 of this chapter we will again see that a separable BA $\mathscr{X}$ with the countable chain condition can be represented as a quotient algebra of some σ-algebra $\mathscr{A}$ of subsets of an interval. In the non-separable case the interval is replaced by a Cantor set X_Ξ of uncountable weight. Alongside the algebra $\mathscr{X}$ appears the universally complete K-space $\mathfrak{S}_{\mathscr{X}}$. We will describe this representation explicitly. Just as above, the expressions "almost everywhere," "*mod* 0," "equivalent," and "negligible" correspond to the σ-ideal I and the kernel of the homomorphism Φ ("Boolean measure") that maps $\mathscr{A}$ onto $\mathscr{X}$. The words "measurable function" always mean "$\mathscr{A}$-measurable." Some facts about

these functions are given (in the "regular" case) in 6.3 (the Luzin and Egorov Theorems, and so on).

We consider the class of all measurable almost everywhere finite real functions on the given interval and carry out factorization by identifying equivalent (almost everywhere coincident) functions. After this identification we obtain the K-space whose elements are the cosets of equivalent functions (of "functions $mod\,0$"). We will denote it by $\mathbf{S}_{\mathscr{X}}$ or simply, by $\mathbf{S}$. With each individual function we associate the family of Lebesgue sets of the form (3). After factorization, these families turn into the spectral families of elements of $\mathscr{X}$. To the almost coinciding spectral families there corresponds the same element of $\mathbf{S}_{\mathscr{X}}$, the same "function $mod\,0$." Thus we establish a one-to-one correspondence between the elements of $\mathbf{S}_{\mathscr{X}}$ and all resolutions of the identity, the elements of $\mathfrak{S}_{\mathscr{X}}$. It is easy that this correspondence is a linear and order isomorphism. Therefore we can sometimes identify the spaces $\mathbf{S}_{\mathscr{X}}$ and $\mathfrak{S}_{\mathscr{X}}$, considering $\mathbf{S}_{\mathscr{X}}$ as a universally complete K-space over $\mathscr{X}$. Once again we see that the Boolean algebra $\mathscr{X}$ is "canonically" embedded in $\mathbf{S}_{\mathscr{X}}$. Precisely with its elements there are associated the cosets of equivalent characteristic functions, the "indicators." With each resolution of the identity $\mathfrak{f}, \mathfrak{g}, \dots$ we associate the coset of equivalent functions for which as a typical notation we again use the same Latin letters: $f, g, \dots$ and so on. Let us agree to call these functions the $\mathfrak{f}$-functions, $\mathfrak{g}$-functions and so on. (Now the correspondence $\mathfrak{f} \longleftrightarrow f$ is not one-to-one.)

We can obtain some knowledge about the properties of the resolution of the identity $\mathfrak{f}$ by studying the family (3) of the level sets E_λ of some $\mathfrak{f}$-function f. With the POINTS OF DISCONTINUITY of the spectral families comprising $\mathfrak{f}$ there are associated the LEVEL SETS of POSITIVE MEASURE of the function f; if some value λ_0 is taken by f on a set of Boolean measure zero then

$$e^{+}_{\lambda_0}(\mathfrak{f}) = e^{-}_{\lambda_0}(\mathfrak{f})$$

and at the point λ_0 all spectral families comprising $\mathfrak{f}_0$ are continuous. For each of these families the CONSTANCY INTERVALS are the intervals that almost everywhere contain no values of f; the inverse image $f^{-1}(\Delta)$ of the constancy interval Δ is a negligible set. At last, the SPECTRUM of the resolution of the identity $\mathfrak{f}$ consists of such points whose every neighborhood contains the values of EACH $\mathfrak{f}$-function f. It may be said that the points of the spectrum are the values that we cannot eliminate by changing the function on a negligible set. In particular, the spectrum contains all discontinuity points of the spectral function which comprise its discrete part.

Now, let $\mathscr{X}$ be a separable algebra furnished with a totally additive measure, for example, the Lebesgue algebra E_0 or another "metric struc-

ture." Such an algebra is always regular and all what was said above applies to it. The existence of a measure on $\mathscr{X}$ means that there exists a "measure" on $\mathscr{A}$ in a traditional sense (in the case of E_0, the usual Lebesgue measure on an interval).

In this case, alongside the space $\mathbf{S}$ we may consider some subspaces of $\mathbf{S}$: namely, L, the space of summable functions; L^p; and so on. They also are K-spaces over the same algebra of measurable $mod\,0$ sets, although they fail to be universally complete.

We now dwell on a "probabilistic" interpretation of the vector lattice over a Boolean algebra.

Interpreting the algebra as a system of events we simultaneously interpret the resolutions of the identity as a RANDOM VARIABLES. In this case we can say that $\mathfrak{S}_{\mathscr{X}}$ is the subspace of all random variables.

Each element $e^+_\lambda(\mathfrak{f})$ can be visually interpreted as the event: "the value of the random variable $\mathfrak{f}$ is at most λ."

If, moreover, the given algebra is equipped with a probability measure μ ("probability") then, associating to the given resolution of identity $\mathfrak{f}$ the real function

$$M(\lambda) \equiv \mu e^+_\lambda(\mathfrak{f}),$$

we obtain the *distribution function* of the random variable $\mathfrak{f}$, and the Lebesgue–Stieltjes integral

$$\int\limits_{-\infty}^{+\infty} \lambda \, dM(\lambda)$$

gives its mathematical expectation. We will return to these questions in Chapter 7.

At last, we take as $\mathscr{X}$ the Boolean algebra of invariant subspaces of a selfadjoint operator A acting in a Hilbert space H (cf. Example 4 of Chapter 1). The universally complete K-space $\mathfrak{A}$ over this BA is the space of all selfadjoint operators that are functions[20] of A, i.e., that commute with each bounded selfadjoint operator commuting with A (see F. Riesz and B. Szökefalvi-Nagy [1]). The order on $\mathfrak{A}$ is usual: $B \geq C$ means that the operator $B - C$ is positive.

The characteristic of $\mathfrak{A}$ is the usual spectral function of the selfadjoint operator A. The concept of spectrum acquires the well-known meaning. We note that the "operator-theoretic" interpretation does not essentially differ from the "function-theoretic." The matter is that the algebra of invariant subspaces is normable and permits a representation in the

[20] See Section 7 below.

form of the metric structure associated with some measure space. Then the space $\mathfrak{A}$ turns out isomorphic to the corresponding space **S**. It is easy to characterize the functions (up to equivalence) that constitutes the last space. Namely, these are all real functions f for which the selfadjoint operator $f(A)$ makes sense; the space $\mathfrak{A}$ consists precisely of these operators.

4.4 Separable subalgebras and resolutions of the identity

Recall again that we call an algebra "separable" if it is fully generated by some countable set. We now consider only complete algebras and their subalgebras satisfying the countable chain condition.

Let R be an arbitrary interval and let $\mathscr{B} \equiv \mathscr{B}(R)$ be the Borel σ-algebra of R. Assume that Φ is a σ-homomorphism of this σ-algebra into a complete BA $\mathscr{X}$ satisfying the countable chain condition. Then the image $\Phi(\mathscr{B})$ is a regular and, moreover, separable subalgebra of $\mathscr{X}$. We may compose a countable fully generated set in $\mathscr{X}$ of the elements $\Phi(\Delta)$, where Δ is an interval with rational endpoints. The presence of the σ-homomorphism Φ is equivalent to the fact that we are given some resolution of the identity with spectrum R (of course, the equality $R = \mathbb{R}$ is not excluded).

Thus (given $\mathscr{X}$ satisfying the countable chain condition), each resolution of the identity generates the unique separable regular subalgebra of $\mathscr{X}$. On the other hand, each of these subalgebras has the same origin as follows from

Theorem 7. I. *Let $\mathscr{Y}$ be a complete continuous separable BA and assume that $\{u_n\}_{n=1}^{\infty}$ is a sequence in $\mathscr{Y}$ generating $\mathscr{Y}$; let $\Delta \equiv [a, b]$ be an arbitrary nondegenerate interval (it is not excluded that $a = -\infty$ and $b = +\infty$). Then there exists a spectral family $\{e_\lambda\}$ with the following properties:*

1) $e_a = e_{a+0} = \mathbf{0}$, $e_b = e_{b-0} = \mathbf{1}$; $e_{t+0} = e_{t-0}$ *for all* $t \in [a, b]$*;*
2) *the inequalities* $a \le t_1 < t_2 \le b$ *and* $e_{t_1} < e_{t_2}$ *are equivalent;*
3) *every element* u_n $(n = 1, 2, \dots)$ *is representable as*

$$u_n = (e_{r_1''} - e_{r_1'}) + (e_{r_2''} - e_{r_2'}) + \dots + (e_{r_{k_n}''} - e_{r_{k_n}'}),$$

where $r_1' < r_1'' \le r_2' < r_2'' \le \dots \le r_{k_n}' < r_{k_n}''$ *are rational numbers in* $[a, b]$.

II. *Let $\mathscr{Y}$ be a complete discrete separable BA and let $\{u_n\}_{n=1}^{\infty}$ be a sequence fully generating $\mathscr{Y}$. Assume that $\{t_n\}_{n=1}^{\infty}$ is an arbitrary strictly increasing sequence of real numbers:*

$$-\infty < t_1 < t_2 < \dots < +\infty.$$

Then there is a spectral family $\{e_\lambda\}$ *possessing the properties* 2) *and* 3) *whose spectrum consists of the points* $t_1, t_2, \ldots$.

It is clear that this theorem applies to the algebras having both continuous and discrete bands. In this case there also exists a spectral family $\{e_\lambda\}$ possessing the property 3).

Theorem 7 in its essential ("continuous") part belongs to A. G. Pinsker.[21]

Let us apply this theorem to the case in which $\mathscr{Y} = \mathscr{X}_0$ is a regular separable subalgebra of a BA $\mathscr{X}$ satisfying the countable chain condition. The spectral family $\{e_\lambda\}$ (more precisely, the corresponding resolution of the identity) generates the spectral measure Φ_0, a σ-continuous epimorphism of the σ-algebra $\mathscr{B}$ onto $\mathscr{X}_0$:

$$\mathscr{X}_0 = \Phi_0(\mathscr{B}). \tag{17}$$

Here $\mathscr{B} \equiv \mathscr{B}(\mathbb{R})$ is the Borel σ-algebra of the real axis. As always, the kernel of Φ_0 contains all Borel sets disjoint from the spectrum of the resolution of the identity which generates this homomorphism.

We may thus summarize the above as follows:

Theorem 8. *The general form of a separable regular subalgebra* $\mathscr{X}_0$ *of a complete BA* $\mathscr{X}$ *satisfying the countable chain condition is given by the formula* (17), *where* Φ_0 *is a* σ*-continuous homomorphism of the* σ*-algebra* $\mathscr{B} \equiv \mathscr{B}(\mathbb{R})$ *into* $\mathscr{X}$.

The presence of a σ-homomorphism Φ_0 amounts to the fact that we are given some resolution of the identity of $\mathscr{X}$. To each of these resolutions of the identity there corresponds a regular separable subalgebra, and each of these subalgebras is generated by some resolution of the identity. Of course, this correspondence is not one-to-one. The homomorphism Φ_0 may be extended from $\mathscr{B}$ onto a wider σ-algebra $\mathscr{A}$ whose structure we have already considered in 6.3. Thus, alongside (17) the formula

$$\mathscr{X}_0 = \Phi(\mathscr{A}) \tag{18}$$

is valid, where Φ is a σ-continuous homomorphism of the σ-algebra $\mathscr{A}$ onto $\mathscr{X}_0$. But the algebra $\mathscr{A}$ is connected with the original resolution of the identity, and so the formula (17) has a more universal character.

We now consider the subalgebra $\mathscr{X}_0$ generated by some resolution of the identity $\mathfrak{f}$ from the "Stone" point of view. Let $\mathscr{X}_0$ be infinite. Each resolution of the identity is characterized by the two spectral functions: $\{e^+_\lambda(\mathfrak{f})\}$ and $\{e^-_\lambda(\mathfrak{f})\}$. We already know (see 6.3) that there is a unique

[21] L. V. Kantorovich, B. Z. Vulikh, and A. G. Pinsker [1, Section 13, § 4].

continuous function $f \in C_\infty(\mathfrak{Q})$ on the compact space $\mathfrak{Q} \equiv \mathfrak{Q}(\mathscr{X})$ corresponding to this resolution of the identity. On the other hand, to the subalgebra $\mathscr{X}_0$ there corresponds some continuous partition ξ_0 of the same compact space $\mathscr{X}_0 \equiv \mathscr{X}_{\xi_0}$. Together with this partition, it is natural to consider the two other partitions: the partition ξ_1 corresponding to the subalgebra $\Phi_0(\mathscr{R}_{[-\infty,+\infty]})$, and the partition η whose elements are precisely the level sets of f. (Just as above, Φ_0 is the σ-homomorphism generated by the resolution of the identity $\mathfrak{f}$.) The partition η is not Boolean (p. 146). The elements of ξ_1 are separated by the images $\Phi_0(\Delta)$, where Δ is an interval with rational endpoints. We know that only the whole subalgebra $\mathscr{X}_0$ can separate the elements of ξ_0, whereas no countable subset of $\mathscr{X}_0$ can accomplish this. Hence ξ_0 is essentially finer than ξ_1. More precisely, ξ_0 is the coarsest of all continuous partitions finer than ξ_1. We now recall (see 6.3) that

$$e^+_\lambda(\mathfrak{f}) = Int\,\widetilde{E}_\lambda(\mathfrak{f}), \quad e^-_\lambda(\mathfrak{f}) = clos\,E_\lambda(\mathfrak{f}).$$

Hence, it follows easily that the level sets (of the form $\widetilde{E}_\lambda(\mathfrak{f}) \setminus E_\lambda(\mathfrak{f})$) are separated by the elements $e^+_\lambda(\mathfrak{f})$ and $e^-_\lambda(\mathfrak{f})$; i.e., they are separated by the algebra $\Phi_0(\mathscr{R}_{[-\infty,+\infty]})$. We come to the following conclusion: ξ_0 *is essentially finer than* ξ_1 *while* ξ_1 *is essentially finer than* η.

Returning to what has been said on p. 293 we note that "non-Boolean" partition η is coarse; it is impossible to "compose" of its elements the sets of the form e^+_λ and e^-_λ other than η-sets. These elements (level sets) are of the form

$$\{q \mid f(q) = a\} = \bigcap_{\lambda > a, \mu < a} e^+_\lambda \cap Ce^+_\mu = \bigcap_{\lambda > a, \mu < a} e^-_\lambda \cap Ce^-_\mu,$$

and they are separated by the corresponding sets e^+_a and e^-_a, as a rule.

4.5 Regular subspaces and subalgebras

Let us consider a K-space R with weak unit $\mathbf{1}$, and put $E_R = \mathscr{X}$. We will establish a one-to-one correspondence between the regular subalgebras of $\mathscr{X}$ and the regular subspaces of R. If $\mathscr{X}_0 \subset \mathscr{X}$ is a regular subalgebra, f is an element of R, and $\mathfrak{f}$ is the corresponding resolution of the identity (the characteristic) of f then in the case $e^\pm_\lambda(\mathfrak{f}) \in \mathscr{X}_0$ we agree to say that f is *measurable with respect to* $\mathscr{X}_0$ for all $\lambda \in [-\infty, +\infty]$. We denote by $R_{(\mathscr{X}_0)}$ the set of all elements f measurable with respect to $\mathscr{X}_0$. It is easy to check that $R_{(\mathscr{X}_0)}$ is a regular subspace of R.

Conversely, let a regular subspace R_0, with $\mathbf{1} \in R_0$, be given. It is uniquely represented as $R_0 = R_{(\mathscr{X}_0)}$ on taking $\mathscr{X}_0$ to be the smallest regular subalgebra such that all $f \in R_0$ are measurable with respect to $\mathscr{X}_0$, i.e., $\mathscr{X}_0 = R_0 \cap E_R$.

Thus, there exists a one-to-one correspondence

$$\mathscr{X}_0 \longleftrightarrow R_{(\mathscr{X}_0)}$$

between the regular subalgebras and the regular subspaces with unity. Of the utmost interest is the case in which the subalgebra $\mathscr{X}_0$ is separable. Soon we will give a complete characterization of these subalgebras and corresponding subspaces. For this purpose we need the concept of "function of an element."

4.6 Functions of elements

We begin with a simple example. If g and f are two real functions defined on $\mathbb{R}$ then we can consider another function

$$F = g \circ f. \tag{19}$$

It is natural to say that F is the "function of f." Also, we can easily find a condition under which a given function F is a "function of f," i.e., F is representable as (19). This (necessary and sufficient) condition is very simple: F must be constant on the level sets of f. In other words, if η_F and η_f are the partitions of $\mathbb{R}$ into the level sets of F and f respectively, then η_F must be coarser than η_f.

This simple example concerns the space $F_{\mathbb{R}}$. Also, there are well known examples of the "functional calculus" in the spaces of operators. (We have already dealt with this topic in 6.4.5.) We now consider a similar general construction related to ordered vector spaces.

Let $\mathscr{X}$ be a complete BA satisfying the countable chain condition, and let $\mathfrak{S}$ be a universally complete K-space. Assume that $E_{\mathfrak{S}} = \mathscr{X}$ (thus $\mathfrak{S}$ is one of the realizations of $\mathfrak{S}_{\mathscr{X}}$). The Freudenthal Theorem asserts that each element $\mathfrak{S}$ is representable by the formula (15); its characteristic is composed of the elements of $\mathscr{X}$.

Given an arbitrary resolution of the identity $\mathfrak{f}$, we distinguish in $\mathscr{X}$ the separable regular subalgebra generated by $\mathfrak{f}$. In other words, this subalgebra is defined by the equality (17). We will denote it by $\mathscr{X}_{\langle\mathfrak{f}\rangle}$. We select in $\mathfrak{S}$ the set $\mathfrak{S}^f$ of all elements whose characteristics belong to $\mathscr{X}_{\langle\mathfrak{f}\rangle}$. This set is a regular subspace in $\mathfrak{S}$, and it is itself a universally complete K-space over the BA $\mathscr{X}_{\langle\mathfrak{f}\rangle}$ (it is a "variant" of the space $\mathfrak{S}_{\mathscr{X}_{\langle\mathfrak{f}\rangle}}$). Clearly, $\mathscr{X}_{\langle\mathfrak{f}\rangle} = E_{\mathfrak{S}^f}$.

We know that a universally complete K-space over $\mathscr{X}_{\langle\mathfrak{f}\rangle}$ can be represented in another way; namely, as the space of measurable functions. The resolution of the identity $\mathfrak{f}$ generates, as it was said in 6.3, the σ-algebra $\mathscr{A} \equiv \mathscr{A}_{\mathfrak{f}}$ of subsets of the real axis $\mathbb{R}$ and the σ-continuous epimorphism $\Phi \equiv \Phi_{\mathfrak{f}}$ of $\mathscr{A}$ onto $\mathscr{X}_{\langle\mathfrak{f}\rangle}$. Put $I \equiv I_{\mathfrak{f}} \equiv \ker \Phi$. In much the

same way as in 6.4 we define the space of all $\mathscr{A}$-measurable I-almost everywhere finite real functions and let $\mathbf{S} \equiv \mathbf{S}_{\mathfrak{f}}$ stand for the quotient space of this space by the ideal I. Note that the elements of $\mathbf{S}$ are the cosets of I-equivalent functions. The coset containing the identically one function is a weak unit of $\mathbf{S}$. Similarly, the unit elements of $\mathbf{S}$ are cosets containing the characteristic functions of sets. As above, we denote the set of these functions by $E_{\mathbf{S}}$. It is a complete BA isomorphic to $\mathscr{X}_{\langle\mathfrak{f}\rangle}$.

We already know that $\mathbf{S}$ is just another model for the space $\mathfrak{S}_{\mathscr{X}_{\langle\mathfrak{f}\rangle}}$; there exists an isomorphism between $\mathbf{S}$ and $\mathfrak{S}^{f}$ which we will describe in more detail. In fact, this isomorphism is a "lifting" of the epimorphism Φ. Let us explain this more explicitly. Consider instead of Φ the mapping $\overline{\Phi}$, that is in some sense "equivalent" to Φ and defined on $\mathscr{A}$ rather than on $E_{\mathbf{S}}$. It is determined from the condition $\overline{\Phi}(e) \equiv \Phi(\varepsilon)$, where ε is an arbitrary set in $\mathscr{A}$ whose characteristic function belongs to the coset e (the value $\Phi(\varepsilon)$ does not depend on the choice of ε). It is easily seen that $\overline{\Phi}$ is an isomorphism of $E_{\mathbf{S}}$ onto $\mathscr{X}_{\langle\mathfrak{f}\rangle}$. We must translate this isomorphism from the Boolean algebras onto the K-spaces. This is done by spectral integration. Given $\widehat{h} \in \mathbf{S}$, put

$$T_{\mathfrak{f}}(\widehat{h}) \equiv \int\limits_{-\infty}^{+\infty} h(\lambda)\, de_{\lambda} = \int\limits_{\mathbb{R}} h\, d\Phi.$$

Here $\{e_{\lambda}\}$ is some spectral family representing the resolution of the identity $\mathfrak{f}$, and h is some function from the coset (the spectral integral is independent of how we choose it).

Using the main properties of the spectral integral, we easily check that the operator $T_{\mathfrak{f}}$ defined by the last equality is a linear and order isomorphism of $\mathbf{S}$ onto some space $\mathfrak{S}' \subset \mathfrak{S}$. Show that $\mathfrak{S}' = \mathfrak{S}^{\mathfrak{f}}$. Indeed, let $e \in E_{\mathbf{S}}$. Then we obviously have

$$T_{\mathfrak{f}}(e)\overline{\Phi}(e).$$

Thus, $T_{\mathfrak{f}}$ coincides with $\overline{\Phi}$ at the unit elements. It is well known (Theorem 3) that the isomorphism $\overline{\Phi}$ extends uniquely to the isomorphism U sending $\mathbf{S}$ onto some solid subspace $\mathfrak{S}''$ of $\mathfrak{S}^{\mathfrak{f}}$ (because the last space is universally complete). For the step functions $x \in \mathfrak{S}^{0}$, as well as for the unit elements, we have

$$T_{\mathfrak{f}}(x) = U(x).$$

Hence, by the order continuity of $T_{\mathfrak{f}}$ and U, we easily derive that $T_{\mathfrak{f}}$ and U agree on the whole $\mathbf{S}$. So

$$\mathfrak{S}'' = \mathfrak{S}'$$

and $\mathfrak{S}'$ is a solid subspace of $\mathfrak{S}^{\mathfrak{f}}$. At the same time, $\mathfrak{S}'$ is a universally complete space (as an isomorphic image of a universally complete space). We have two universally complete spaces and one of them is a solid subspace of the other. This is possible only in the case of their coincidence.[22] Thus, $\mathfrak{S}' = \mathfrak{S}^{f}$. Keeping the preceding notation, we finally state the following

Theorem 9. *Let $\mathscr{X}$ be a complete BA satisfying the countable chain condition and assume that f is an arbitrary element of $\mathfrak{S}$. Let $\mathfrak{f}$ be the characteristic of f. Then for each element $h \in \mathfrak{S}$ the following are equivalent:*

(I) *There exists an element $\widehat{g} \in \mathbf{S}_{\mathfrak{f}}$ such that*

$$h = \int\limits_{-\infty}^{+\infty} g(\lambda)\, de_\lambda(\mathfrak{f})$$

for every choice of the representing function $g \in \widehat{g}$;

(II) $h \in \mathfrak{S}^{f}$.

The element h satisfying the properties (I) and (II) is called the "function of f." We will denote it by $g(f)$. This notation expresses the essence of the matter: in the most important cases the function of an element is constituted by the usual composition. For example, let $\mathfrak{S} = S([0,1])$ (i.e., the space of Lebesgue measurable almost everywhere finite functions with the usual identification). If $\widehat{f} \equiv \mathfrak{f}$ is an element of this space and $\widehat{g} \in \mathbf{S}_{\mathfrak{f}}$ then for all $f \in \widehat{f}$ and $g \in \widehat{g}$ (g is a function measurable with respect to $\mathscr{A}_{\mathfrak{f}}$ and f is Lebesgue measurable) the composition $F = g \circ f$ is Lebesgue measurable and generates the element $\widehat{F}$ of $S([0,1])$ which we denote by $\widehat{g}(\widehat{f})$. Note that the measurability of g with respect to $\mathscr{A}_{\mathfrak{f}}$ is equivalent to the Lebesgue measurability of $g \circ f$. As is well known, the Lebesgue measurability of g is not sufficient for this function. On the other hand, every Borel measurable ("Baire") and, all the more, every continuous function is measurable with respect to each σ-algebra $\mathscr{A}_{\mathfrak{f}}$.

In this subsection we have always assumed that the algebra under study satisfies the countable chain condition. The function of elements of the form (1) may be considered without this assumption, but Theorem 8 fails since in general a regular (even separable) subalgebra may possess

[22]See 6.1.5 (the property 2°). In our case $\mathfrak{S}'$ contains the unity of $\mathfrak{S}^{\mathfrak{f}}$; i.e., $\mathbf{1}_{\mathfrak{S}^f} \equiv \mathbf{1}_{\mathscr{X}_{\aleph\mathfrak{f}\rangle}} \equiv \mathbf{1}_{\mathscr{X}} = T_{\mathfrak{f}}(\mathbf{1}_{\mathbf{S}})$.

arbitrarily large cardinality, but the number of elements of the form (1) is, in any case, at most $2^{\aleph_0}$.

As far as Theorem 9 is concerned, it remains valid for the algebras not satisfying the countable chain condition; however, defining the subalgebra $\mathscr{X}_{\langle \mathfrak{f} \rangle}$ and hence the subspace $\mathfrak{S}^f$, we must use the formula (17). In this event $\mathscr{X}_{\langle \mathfrak{f} \rangle}$ may fail to be the "regular subalgebra generated by the resolution of the identity $\mathfrak{f}$," and $\mathfrak{S}^f$ may fail to be a regular subspace.

We can approach the concept of "function of element" from the "Stone space" standpoint. To the resolution of the identity $\mathfrak{f}$ there corresponds some continuous function $f \in C_\infty(\mathfrak{Q})$. The function $g \circ f$ is also continuous, where g is a continuous real function on the spectrum of $\mathfrak{f}$ (i.e., on the set $f(\mathfrak{Q})$). It is easy to check that this function, belonging to $C_\infty(\mathfrak{Q})$, is exactly the element $g(f)$ in the sense of the last definition. This approach was abstracted to many discontinuous functions (G. Ya. Lozanovskiĭ [1]). The "Stone" approach is especially convenient for defining the functions of several elements of a K-space. We dwell only on the case of $g(x, y) = xy$; that is, we describe the operation of MULTIPLICATION OF ELEMENTS introduced firstly by B. Z. Vulikh.[23]

We will assume that the space $\mathfrak{S}_{\mathscr{X}}$ consists of resolutions of the identity. Take $\mathfrak{f}_1, \mathfrak{f}_2 \in \mathfrak{S}_{\mathscr{X}}$, and let f_1 and f_2 be the corresponding functions[24] of $C_\infty(\mathfrak{Q})$. For simplicity we consider the case in which $\mathfrak{f}_1$ and $\mathfrak{f}_2$ are BOUNDED ELEMENTS, that is $f_1, f_2 \in C(\mathfrak{Q})$. Then the usual product $f \equiv f_1 f_2$ also belongs to $C(\mathfrak{Q})$; the corresponding resolution of the identity $\mathfrak{f}$ is the product of $\mathfrak{f}_1$ and $\mathfrak{f}_2$; in symbols, $\mathfrak{f} = \mathfrak{f}_1 \mathfrak{f}_2$. Clearly, this multiplication is commutative and distributive with respect to addition; the role of the multiplicative unity is played by the unity of $\mathscr{X}$ (more precisely, by its canonical image $\mathbf{1}^\bullet$). This multiplication is extended to unbounded elements as well; thus $\mathfrak{S}_{\mathscr{X}}$ becomes a commutative ring (moreover, an algebra).

The space of bounded elements $(\mathfrak{S}_{\mathscr{X}})^b$ (or, equivalently, $C(\mathfrak{Q})$) becomes a subring of this ring (a subalgebra). The BA $\mathscr{X}$ itself, more exactly its canonical image $\mathscr{X}^\bullet$, is closed under multiplication: if $x_1, x_2 \in \mathscr{X}$ then $x_1^\bullet, x_2^\bullet \in \mathscr{X}^\bullet$ and $x_1^\bullet x_2^\bullet = (x_1 \wedge x_2)^\bullet$. We will usually write, if need be, the Boolean meet as multiplication: $x_1 x_2 \equiv x_1 \wedge x_2$. As far as the products of the form $x^\bullet \mathfrak{f}$ are concerned, they are understood in a simple sense: they are the projections of $\mathfrak{f}$ to the band generated by the element $x^\bullet$; i.e., $x^\bullet \mathfrak{f} = P_U(\mathfrak{f})$, where U is the least band in $\mathfrak{S}_{\mathscr{X}}$ containing $x^\bullet$. This is clear: the product of a function $f \in C_\infty(\mathfrak{Q})$ by

[23] B. Z. Vulikh [1].

[24] Here we bear in mind the canonical isomorphism assigning to the unity $\mathbf{1}^\bullet$ of the BA $\mathscr{X}^\bullet$ the identically one function.

the characteristic function of a clopen set x is a function that equals f on this set and equals zero beyond it; and this is exactly the result of projecting to the band generated by x (Fig. 2).

We may naturally define the double integral for a resolution the identity. In this event, the following formula holds:

$$\mathfrak{f}_1\mathfrak{f}_2 = \int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} \lambda\mu \, de_\lambda(\mathfrak{f}_1)\, de_\mu(\mathfrak{f}_2).$$

Multiplication of elements of an ARBITRARY universally complete space is interpreted evidently by passage to resolutions of the identity (to the characteristics of the factors) or directly to continuous functions.

4.7 The space $\mathfrak{S}^f$

Comparing Theorems 7 and 9 of this chapter, we obtain the final result:

Theorem 10. *Let $\mathscr{X}$ be a complete BA satisfying the countable chain condition, let $\mathfrak{S}$ be the universally complete K-space over $\mathscr{X}$, and let $\mathscr{X}_0$ be a regular subalgebra of $\mathscr{X}$. For the space $\mathfrak{S}_{(\mathscr{X}_0)}$ to consist of the functions of a sole element, it is necessary and sufficient that the subalgebra $\mathscr{X}_0$ be separable.*

The same holds for an arbitrary $R_{(\mathscr{X}_0)}$, with R a solid subspace of $\mathfrak{S}$.

Thus the spaces $\mathfrak{S}_{(\mathscr{X}_0)}$ over separable subalgebras and their solid subspaces consist (under our assumptions on $\mathscr{X}_0$) of the functions of a sole element, and so they are solidly embedded in $\mathfrak{S}^f$. These spaces occur, for example, in the theory of operators in connection with the celebrated Riesz–von Neumann Theorem.

Let $\mathfrak{f}$ be an element of $\mathfrak{S}$ and let $\mathfrak{S}^f$ be the corresponding subspace constituted by all functions of $\mathfrak{f}$; $\mathfrak{S}^f \subset \mathfrak{S}$ and $\mathscr{X} = E_{\mathfrak{S}^f}$.

Among the functions of $\mathfrak{f}$ there are all possible powers of this element and also polynomials of the form

$$p(\mathfrak{f}) = p_0 + p_1\mathfrak{f} + \cdots + p_n\mathfrak{f}^n$$

(with real coefficients). These polynomials belong to the space $\mathfrak{S}^f$ and they play the same role as the usual polynomials in the spaces of real functions. Therefore, each element $h \in \mathfrak{S}^f$ may be expressed as the limit of some sequence of polynomials:

$$h = (o)\text{-}\lim p_n(\mathfrak{f}). \tag{20}$$

If h has the form $g(\mathfrak{f})$, where g is a continuous function; then we may achieve the uniform convergence in (20) (with regulator $\mathbf{1}$).

In particular, the space $\mathfrak{S}^f$ contains the whole spectral family $\{e_\lambda^{\pm}(\mathfrak{f})\}$ because, for example,

$$e_\lambda^-(\mathfrak{f}) = g(\mathfrak{f}), \quad \text{where} \quad g(t) = \begin{cases} 1, & t < \lambda, \\ 0, & t \geq \lambda \end{cases}$$

(since g is a Baire function; therefore, $g \in \mathbf{S}$.)

To approximate a spectral family according to (20), we may use some standard polynomials of approximation theory. For example, consider the simple case in which $\mathbf{0} \leq \mathfrak{f} \leq \mathbf{1}$. Then it is convenient to take the Bernstein polynomials and we obtain the formula

$$e_\lambda^-(\mathfrak{f}) = (o)\text{-}\lim_n \sum_{0 \leq k < \lambda n} \binom{n}{k} \mathfrak{f}^k (\mathbf{1} - \mathfrak{f})^{n-k}. \tag{21}$$

If there is a probability measure μ on the BA $\mathscr{X}$ then (21) implies the formula that reconstructs the distribution function of a random variable from its moments:

$$\mu(e_\lambda^-(\mathfrak{f})) = \lim_n \sum_{0 \leq k < \lambda n} \binom{n}{k} \int \mathfrak{f}^k (\mathbf{1} - \mathfrak{f})^{n-k}\, d\mu.$$

We may easily write down a similar formula for an arbitrary bounded random variable. It is well known that for unbounded variables no similar formula is possible; in this case the distribution function is not reconstructed from its moments.

s

II

METRIC THEORY OF BOOLEAN ALGEBRAS

Chapter 7

NORMED BOOLEAN ALGEBRAS

In this section we begin a systematic study of the algebras furnished with a measure or measure algebras. Recall that a measure is an essentially positive totally additive function. The most important class is constituted by probability measures (see 0.3.5). Each algebra with probability measure may be interpreted as a system of events, with the measure itself the probability on this system. The most part of this and subsequent chapters admits translation into the language of probability theory.

1. Normed algebras

1.1 The definition and topological properties of a normed Boolean algebra

Definition. *A normed Boolean algebra* (briefly, an NBA) is a pair $\{\mathscr{X}, \mu\}$, where $\mathscr{X}$ is complete BA and μ is a measure on $\mathscr{X}$.[1] Routinely abusing the language, we will often speak of an "NBA $\mathscr{X}$." If μ is a probability measure then we sometimes use the term *probability algebra.*

The measure is always an outer measure (Chapter 4). Therefore, all assertions proven for complete Boolean algebra with outer measure apply to an NBA as well. In particular, each NBA $\{\mathscr{X}, \mu\}$ turns into a metric space on introducing the metric

$$\rho_\mu \equiv \rho: \quad \rho(x, y) = \mu(|x - y|).$$

[1]Observe that D. A. Vladimirov spoke about the unordered pair in contrast to the conventional practice preferring ordered tuples on this and similar occasions. (S. S. Kutateladze)

Since the measure is totally additive, the condition $x_n \downarrow \mathbf{0}$ implies that $\rho(x_n, \mathbf{0}) = \mu x_n \longrightarrow 0$. Indeed,

$$x_n = \bigvee_{k=n}^{\infty} (x_k \wedge Cx_{k+1}), \quad \mu x_n = \sum_{k=n}^{\infty} \mu(x_k \wedge Cx_{k+1}).$$

Theorem 1. *The topology of the metric ρ_μ on $\mathscr{X}$ coincides with the (o)-topology.*

This theorem is a particular case of Theorem 4.8. It suffices to recall that the metric topology satisfies the condition (os) as follows from the preceding remarks.

Corollary 1. *Let μ and ν be two measures on a complete BA. Then to each real $\varepsilon > 0$, there is some $\delta > 0$ such that the inequality $\mu x < \delta$ implies $\nu x < \varepsilon$.*

To prove, it suffices to observe that the relations $\mu x_n \longrightarrow 0$ and $\nu x_n \longrightarrow 0$, by virtue of Theorem 1, express the same: the convergence to zero in the (o)-topology. This convergence may be called "convergence in measure." It is clear that *each measure is a continuous function.*

Corollary 1 shows that the concept of absolute continuity, while important for set functions, becomes senseless in our case. For each complete BA $\mathscr{X}$ admitting measure, all metric spaces $\{\mathscr{X}, \mu\}$ are homeomorphic to one another.

Corollary 2. *For the relation $\mu x_n \longrightarrow 0$ to hold, it is necessary and sufficient that for each strictly increasing sequence of indices $\{n_k\}$ there is a subsequence $\{n_{k_i}\}$ $(k_1 < k_2 < \dots)$ such that $x_{n_{k_i}} \xrightarrow{(o)} \mathbf{0}$.*

In fact, keeping in mind the coincidence of the (o)- and (os)-topologies, it is enough to refer to Theorem 4.7.

Corollary 2 shows that each sequence $\{x_n\}$ vanishing in measure contains an (o)-convergent subsequence $\{x_{n_k}\}$. This assertion can be made more precise on pointing out the method of choosing such a subsequence: for (o)-convergence it is sufficient that the series $\sum_{k=1}^{\infty} \mu x_{n_k}$ converges. We will prove this.

Lemma 1. *If $\sum_{i=1}^{\infty} \mu y_i < +\infty$ then $y_i \xrightarrow{(o)} \mathbf{0}$.*

Proof. Given m and k, we have

$$\mu\Big(\bigvee_{i=m}^{m+k} y_i\Big) \le \sum_{i=m}^{m+k} \mu y_i \le \sum_{i=m}^{\infty} \mu y_i.$$

Since the measure is (o)-continuous; therefore,

$$\mu\Big(\bigvee_{i=m}^{\infty} y_i\Big) \leq \sum_{i=m}^{\infty} \mu y_i.$$

The right-hand side of this inequality vanishes, and so

$$0 \leq \mu(\overline{\lim}_i\, y_i) = \mu\Big(\bigwedge_{m=1}^{\infty} \bigvee_{i=m}^{\infty} y_i\Big) \leq \inf_m \sum_{i=m}^{\infty} \mu y_i = 0.$$

Since the measure is essentially positive, we see that

$$\overline{\lim}_i\, y_i = \mathbf{0} \quad \text{or} \quad (o)\text{-}\lim y_i = \mathbf{0}.$$

The lemma is proved.[2]

Using this lemma, we will consider the problem of metric completeness of a Boolean algebra.

Theorem 2. *A complete BA $\mathscr{X}$ endowed with the metric ρ_μ is a complete metric space.*

Proof. Let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence with respect to ρ_μ. We can choose a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ $(n_1 < n_2 < \ldots)$ such that the series $\sum_{k=1}^{\infty} \rho(x_{n_k}, x_{n_{k+1}})$ converges. Since $\rho(x_{n_k}, x_{n_{k+1}}) = \rho(|x_{n_k} - x_{n_{k+1}}|, \mathbf{0})$; therefore, by virtue of Lemma 1, $|x_{n_k} - x_{n_{k+1}}| \xrightarrow{(o)} \mathbf{0}$. So, by Lemma 4.3, we have

$$\mathbf{0} \leq \overline{\lim}\, x_{n_k} - \underline{\lim}\, x_{n_k} = \overline{\lim}\, |x_{n_k} - x_{n_{k+1}}| = \mathbf{0}.$$

Hence, there exists an element x such that $x_{n_k} \xrightarrow{(o)} x$. Now, from the inequality

$$\rho(x_n, x) \leq \rho(x_{n_k}, x) + \rho(x_n, x_{n_k})$$

we easily find that

$$\rho(x_n, x) \longrightarrow 0,$$

since $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence. The completeness of $\{\mathscr{X}, \mu\}$ is proved.

Remark. *Theorem* 2 *remains valid if μ is an outer measure;* to demonstrate, it suffices to slightly change the proof of Theorem 2. We have already mentioned this in Chapter 4 (see p. 216).

[2] This proposition is known as the "Borel–Cantelli Lemma" in probability theory.

1.2 Properties of (o)-convergence

Leaning on Lemma 1, it is easy to prove another important theorem originated with M. Fréchet.

Theorem 3. *Every NBA is regular.*

In fact, in Chapter 4 we have established an even more general fact: the regularity of an algebra with outer measure (p. 221).

Thus, each NBA can be considered as a metric space. What else can we say about the properties of this space? For example, will it be connected? The answer is negative in general, since among normed algebras there are listed all finite BAs which are clearly disconnected.

Theorem 4. *A continuous NBA is an arcwise connected metric space.*

This theorem implies in particular that the set of values of a measure μ coincides (under the conditions of the theorem) with the interval $[0, \mu\mathbf{1}]$. But this fact has been proved previously (Proposition 2° of Section 2.2): it is a base for the proof of Theorem 4 to which we proceed now. So, let two elements $x_0, x_1 \in \mathscr{X}$ be given. We will construct a path between this elements. This is done in a few steps.

1. Show that for all $u, v \in \mathscr{X}$, $u \neq v$, there exists $w \in \mathscr{X}$ such that $\rho(u, w) = \rho(v, w) = \frac{1}{2}\rho(u, v)$. Assume that $u' \equiv u \wedge Cv$ and $v' \equiv v \wedge Cu$. These elements are disjoint but not equal to zero simultaneously and

$$\rho(u', v') = \mu|u' - v'| = \mu|u - v| = \rho(u, v) = \mu u' + \mu v'.$$

Assume for definiteness that $\mu u' > 0$ and $\mu u' \geq \mu v'$. Using Proposition 3° of Section 2.2, we construct some elements u'' and v'' with the following properties

$$u'' \leq u', \;\; v'' \leq v', \;\; \mu u'' = \frac{1}{2}\mu u', \;\; \mu v'' = \frac{1}{2}\mu v'.$$

Put $w = u'' + v'' + u \wedge v$. Then

$$\rho(w, u) = \mu(u' - u'') + \mu v'' = \frac{1}{2}(\mu u' + \mu v') = \frac{1}{2}\rho(u, v).$$

Similarly, $\rho(w, v) = \frac{1}{2}\rho(u, v)$.

2. Now, putting at first that $u \equiv x_0$ and $v \equiv x_1$, we as above construct the element $w \equiv x_{01}$ whose distance from x_0 and x_1 is $\frac{1}{2}\rho(x_0, x_1)$; furthermore, we define x_{001} and x_{011} likewise; continuing this process, we will map into our BA the set of all numbers $r\rho(x_0, x_1)$, where $r \in [0, 1]$ is dyadic-rational. Moreover, the so-constructed mapping γ_0 is uniformly continuous: $\rho(\gamma_0(t'), \gamma_0(t'')) \leq |t' - t''|$.

3. It remains to extend this mapping by continuity onto the whole interval $\Delta = [0, \rho(x_0, x_1)]$. We obtain some continuous function γ that maps Δ into the BA $\mathscr{X}$ so that $\gamma(0) = x_0$ and $\gamma(\rho(x_0, x_1)) = x_1$. It is a desired path between the points x_0 and x_1. The proof of the theorem is complete. We append to it the very important

REMARK. *In a continuous NBA every set of the form*

$$M_\alpha \equiv \{x \mid \mu x = \alpha\}$$

is arcwise connected, with $\alpha \in [0, \mu\mathbf{1}]$.

It is clear from the proof of the theorem: if $\mu u = \mu v = \alpha$ then $\mu w = \alpha$. The set M_α is closed. Therefore, each path γ connecting $x_0, x_1 \in M_\alpha$ lies entirely in M_α.

This remark will allow us later to prove the celebrated Lyapunov Theorem on vector measures.

Definition. A complete BA is called *normable* if it admits some measure.

Each normable BA possesses many measures in general; but they are equivalent in some sense as Corollary 1 of Theorem 1 shows: they define the same metric topology, the order topology. Theorems 3 and 4 are obviously valid for normable algebras.

In essence, a normable BA is a complete BA $\mathscr{X}$ considered together with the system of measures on $\mathscr{X}$. Moreover, it is assumed that this system is not empty. Henceforth we denote it by $\mathbb{M}(\mathscr{X}) \equiv \mathbb{M}$. We also consider the wider class $\mathbb{K}$ of all totally additive quasimeasures on $\mathscr{X}$. Each of those quasimeasures is an order continuous function. If $\nu \in \mathbb{K}$ then the set $\Omega \equiv \{x \mid \nu x = 0\}$ is a principal ideal of the form $\mathscr{X}_u$. The restriction of the quasimeasure ν to the complementary band $\mathscr{X}_{Cu}$ is a measure. Usually, we will consider the PROBABILITY measures satisfying the condition $\mu\mathbf{1} = 1$. We denote the set of probability measures by $\mathbb{M}_0(\mathscr{X})$. Similarly, we put

$$\mathbb{K}_0(\mathscr{X}) \equiv \{\nu \in \mathbb{K} \mid \nu\mathbf{1} = 1\}.$$

Furthermore, we may consider the class Π_0 of all quasimeasures π normalized by the condition $\pi(\mathbf{1}) = 1$. It is clear that $\mathbb{K}_0 \subset \Pi_0$. These classes (and their subclasses) are encountered in mathematical statistics under the name "statistical structures."[3]

1.3 Isomorphisms between NBAs

Let $\{\mathscr{X}, \mu\}$ and $\{\mathscr{Y}, \nu\}$ be the two NBA. A mapping f of some subset $E \subset \mathscr{X}$ (for example, a subalgebra or band) into $\mathscr{Y}$ is called *measure*

[3] J. Soler [1].

preserving if $\nu(f(x)) = \mu x$ for all $x \in E$. First of all, we are interesting in homomorphisms and isomorphisms. Speaking of a *homomorphism* or *homomorphism* of $\{\mathscr{X}, \mu\}$ into $\{\mathscr{Y}, \nu\}$, we will always mean a measure preserving homomorphism (isomorphism) of $\mathscr{X}$ into $\mathscr{Y}$. The method of 5.3.3 is often useful for constructing these isomorphisms. The following theorem rests on this method.

Theorem 5. *If $\mathscr{X}_0 \subset \mathscr{X}$ and $\mathscr{Y}_0 \subset \mathscr{Y}$ are two everywhere (o)-dense subalgebras of $\mathscr{X}$ and $\mathscr{Y}$ then each measure preserving monomorphism Φ_0 of $\mathscr{X}_0$ onto $\mathscr{Y}_0$ extends to a measure preserving isomorphism Φ of $\mathscr{X}$ onto $\mathscr{Y}$. Such an extension is unique.*

Proof. The algebras $\mathscr{X}$ and $\mathscr{Y}$ are regular, and so we will apply Theorem 5.5. Measure preservation means that the homomorphisms Φ_0 and Φ_0^{-1} satisfy the condition (E_σ). Indeed, let

$$x = \bigvee_{n=1}^{\infty} x_n, \quad x, x_n \in \mathscr{X}_0.$$

Put

$$x'_1 = x_1, \quad x'_2 = x_2 \wedge Cx'_1, \quad x'_3 = x_3 \wedge C(x'_1 \vee x'_2), \dots .$$

Then

$$\nu\Big(\bigvee_n \Phi_0(x_n)\Big) \geq \nu\Big(\bigvee_n \Phi_0(x'_n)\Big)$$
$$= \sum_n \nu\Phi_0(x'_n) = \sum_n \mu x'_n = \mu x = \nu\Phi_0(x).$$

Hence,

$$\Phi_0(x) = \bigvee_{n=1}^{\infty} \Phi_0(x_n).$$

The last equality means that the condition (E_σ) is fulfilled. In our case it implies (E_σ^*).

Thus the homomorphism Φ_0 satisfies the condition (E_σ^*). By analogy, we check that this condition is satisfied for Φ_0^{-1}. The BA $\mathscr{X}$ and $\mathscr{Y}$ satisfy the countable chain condition; as is easily seen, for such algebras the conditions (E_σ) and (E_σ^*) are equivalent to (E) and (E^*). In this case (p. 250), the extension Φ of the monomorphism Φ_0 is an isomorphism of $\mathscr{X}$ onto $\mathscr{Y}$. Show that it is measure preserving. To each $x \in \mathscr{X}$ there exists a sequence $\{x_n\}$ in $\mathscr{X}_0$ convergent to x. Passing to the limit in the relation

$$\nu\Phi(x_n) = \nu\Phi_0(x_n) = \mu x_n,$$

we come to the desired equality $\nu\Phi(x) = \mu x$. The uniqueness of the so-obtained extension is evident. The proof of the theorem is complete.

We adduce two important examples in which Theorem 5 is applied. In this examples $\mathscr{X}$ and $\mathscr{Y}$ are two normed algebras, and μ and ν are two probability measures on these algebras.

I. Let $\mathscr{X}$ and $\mathscr{Y}$ contain everywhere dense free subalgebras $\mathscr{X}_0$ and $\mathscr{Y}_0$ respectively. Assume that these subalgebras include not only independent but also METRICALLY independent systems of generators (with respect to some measures μ and ν) of the same cardinality. Let $E_{\mathscr{X}_0}$ be a μ-independent system of generators for $\mathscr{X}_0$, and let $E_{\mathscr{Y}_0}$ be a ν-independent system of generators for $\mathscr{Y}_0$. At last, we assume for simplicity that

$$\mu x = \nu y = \frac{1}{2}$$

for all $x \in E_{\mathscr{X}_0}$ and $y \in E_{\mathscr{Y}_0}$. In this event, each one-to-one mapping φ from $E_{\mathscr{X}_0}$ onto $E_{\mathscr{Y}_0}$ is extendible to an isomorphism Φ_0 of $\mathscr{X}_0$ onto $\mathscr{Y}_0$. It is easily verified that this isomorphism is measure preserving. In fact, if

$$x = x_1 \wedge x_2 \wedge \cdots \wedge x_k, \tag{1}$$

where $x_i \in E_{\mathscr{X}_0} \cup CE_{\mathscr{X}_0}$ $(i = 1, 2, \ldots, k)$ and $x_i \neq x_j$ for $i \neq j$ then $\mu x = \frac{1}{2^k}$. At the same time,

$$\Phi_0(x) = \varphi(x_1) \wedge \varphi(x_2) \wedge \cdots \wedge \varphi(x_k),$$

and also $\varphi(x_i) \in E_{\mathscr{Y}_0} \cup CE_{\mathscr{Y}_0}$ $(i = 1, 2, \ldots, k)$, $\varphi(x_i) \neq \varphi(x_j)$ for $i \neq j$. Hence, $\nu\Phi_0(x) = \frac{1}{2^k} = \mu x$. It remains to note that each element of $\mathscr{X}_0$ is a finite sum of disjoint elements of the form (1). Thus, by Theorem 5, there exists a measure preserving isomorphism of $\mathscr{X}$ onto $\mathscr{Y}$.

II. Now, let us consider a typical situation when there exist two pairs of regular subalgebras $\widetilde{\mathscr{X}}, \mathscr{X}' \subset \mathscr{X}$ and $\widetilde{\mathscr{Y}}, \mathscr{Y}' \subset \mathscr{Y}$ with the following properties:

1) $\mu(\widetilde{x} \wedge x') = \mu\widetilde{x}\mu x'$ for all $\widetilde{x} \in \widetilde{\mathscr{X}}$ and $x' \in \mathscr{X}'$; similarly, $\nu(\widetilde{y} \wedge y') = \nu\widetilde{y}\nu y'$ for all $\widetilde{y} \in \widetilde{\mathscr{Y}}$ and $y' \in \mathscr{Y}'$;

2) $\overline{\mathscr{X}\langle\widetilde{\mathscr{X}}, \mathscr{X}'\rangle} = \mathscr{X}$ and $\overline{\mathscr{Y}\langle\widetilde{\mathscr{Y}}, \mathscr{Y}'\rangle} = \mathscr{Y}$.

Assume that there exist measure preserving isomorphisms $\widetilde{\Phi}$ and Φ' of $\widetilde{\mathscr{X}}$ onto $\widetilde{\mathscr{Y}}$ and of $\mathscr{X}'$ onto $\mathscr{Y}'$ respectively. We will show that in this event there exists a measure preserving isomorphism Φ of $\mathscr{X}$ onto $\mathscr{Y}$ which is a common extension of $\widetilde{\Phi}$ and Φ'.

To this end, we introduce the subalgebras $\mathscr{X}_0 \equiv \mathscr{X}\langle\widetilde{\mathscr{X}}, \mathscr{X}'\rangle$ and $\mathscr{Y}_0 \equiv \mathscr{Y}\langle\widetilde{\mathscr{Y}}, \mathscr{Y}'\rangle$. Since $\overline{\mathscr{X}_0} = \mathscr{X}$ and $\overline{\mathscr{Y}_0} = \mathscr{Y}$, it suffices, using Theorem 5, to construct a measure preserving isomorphism of $\mathscr{X}_0$ onto $\mathscr{Y}_0$. Put

$$\varphi(x) = \begin{cases} \widetilde{\Phi}(x), & x \in \widetilde{\mathscr{X}}, \\ \Phi'(x), & x \in \mathscr{X}'. \end{cases}$$

We have just defined a one-to-one mapping of $\widetilde{\mathscr{X}} \cup \mathscr{X}'$ onto $\widetilde{\mathscr{Y}} \cup \mathscr{Y}'$. Show by using Theorem 5 that it extends to an isomorphism of $\mathscr{X}$ onto $\mathscr{Y}$. As is easily seen, all elementary polynomials in the generators of the subalgebras $\mathscr{X}_0$ and $\mathscr{Y}_0$ are represented as

$$\begin{aligned} x &= \widetilde{x} \wedge x', \quad \widetilde{x} \in \widetilde{\mathscr{X}},\ x' \in \mathscr{X}', \\ y &= \widetilde{y} \wedge y', \quad \widetilde{y} \in \widetilde{\mathscr{Y}},\ y' \in \mathscr{Y}'. \end{aligned} \tag{2}$$

Therefore, the equality $x = \widetilde{x} \wedge x' = \mathbf{0}$ means that, by the assumption 1), either $\widetilde{x} = \mathbf{0}$ or $x' = \mathbf{0}$ and so

$$\varphi(\widetilde{x}) \wedge \varphi(x') = \mathbf{0}. \tag{3}$$

Similarly, (3) implies $\widetilde{x} \wedge x' = \mathbf{0}$. We see that for each of the mappings φ and φ^{-1} the conditions of Theorem 2.18 are satisfied. Thus there exists an isomorphism Φ_0 that extends φ from $\mathscr{X}_0$ onto $\mathscr{Y}_0$. It is obvious that Φ_0 is measure preserving, since each element of $\mathscr{X}_0$ is the sum of finitely many disjoint elementary polynomials of the form (2) for which measure preservation is guaranteed by the assumption 1). By Theorem 5, the isomorphism Φ_0 is uniquely extendible to a measure preserving isomorphism Φ of $\mathscr{X}$ onto $\mathscr{Y}$.

Now, we again meet the situation that has already appeared in 5.3.7. The isomorphism Φ, constructed above, is exactly the DIRECT PRODUCT of $\widetilde{\Phi}$ and Φ'.

We mention the theorem of I. Ya. Dorfman which asserts that the isomorphism of $\{\mathscr{X}, \mu\}$ and $\{\mathscr{Y}, \nu\}$ is equivalent to the ISOMETRY of the metric spaces $\{\mathscr{X}, \rho_\mu\}$ and $\{\mathscr{Y}, \rho_\nu\}$. This theorem is valid for all BAs with essentially positive quasimeasure.

This "metric" part of the book is devoted to both NBAs (with distinguished measures) and NORMABLE algebras. In the latter case we speak of the properties connected only with the POSSIBILITY of introducing a measure but not with such a measure itself. In fact, we consider here the two categories. The first is the category of normable algebras. It is a subcategory of the category $\mathbb{B}\mathrm{oole}$. Its morphisms are order continuous (or equivalently, metrically continuous) homomorphisms. The second is the category of normed probability algebras. Its morphisms, as was said above, are measure preserving homomorphisms. The same applies to isomorphisms. A distinction between this two categories must be taken into account in particular when we deal with an isomorphism of subalgebras and quotient algebras ("subobjects" and "quotient objects"). In what follows, the isomorphisms of NBAs will be called *metric*, and the isomorphisms of normable BAs, *algebraic* or *Boolean.*

2. Extension of a countably additive function. The Lebesgue–Carathéodory Theorem

All constructions of measure theory rest on the fundamental extension theorem originated with A. Lebesgue whose contemporary formulation is due to C. Carathéodory. We have already referred to this theorem not once, and now, for the sake of completeness, we give its proof.

Let $\mathscr{X}$ be a σ-complete BA. Consider some subalgebra $\mathscr{X}_0$ with a quasimeasure φ. We now undertake the following task: construct a σ-regular subalgebra $\widetilde{\mathscr{X}}$ that includes $\mathscr{X}_0$ and a countably additive quasimeasure $\widetilde{\varphi}$ on $\widetilde{\mathscr{X}}$ such that $\widetilde{\varphi}(x) \leq \varphi(x)$, if it is possible, $\widetilde{\varphi}(x) = \varphi(x)$ for all $x \in \mathscr{X}_0$. To fulfill the task, we define an outer quasimeasure φ^* by associating with each $x \in \mathscr{X}$ the set S_x that consists of all at most countable families ("covers") $\sigma \subset \mathscr{X}_0$ satisfying the condition $\sup \sigma \geq x$, and by putting

$$\varphi^*(x) \equiv \inf_{\sigma \in S_x} \sum_{y \in \sigma} \varphi(y).$$

We note the main properties of this outer quasimeasure.

1°. *If $x \leq x_0 \in \mathscr{X}_0$ then $\varphi^*(x) \leq \varphi(x_0)$.*

This property is evident. It implies that φ^* is finite. The next property is evident too.

2°. *The outer measure φ^* is monotone: $x \leq y$ implies $\varphi^*(x) \leq \varphi^*(y)$.*

3°. *If $x = \bigvee_k x_k$ then $\varphi^*(x) \leq \sum_k \varphi^*(x_k)$.*

Indeed, given an arbitrary $\varepsilon > 0$, associate with each x_k the family $\sigma_k \in S_{x_k}$ so that the inequality

$$\sum_{y \in \sigma_k} \varphi(y) \leq \varphi^*(x_k) + \frac{\varepsilon}{2^k}$$

holds. Clearly, the cover $\sigma = \bigcup_k \sigma_k$ belongs to S_x; therefore,

$$\varphi^*(x) \leq \sum_{y \in \sigma} \varphi(y) = \sum_k \sum_{y \in \sigma_k} \varphi(y) \leq \sum_k \varphi^*(x_k) + \varepsilon.$$

Since ε is arbitrary, we obtain the required inequality.

We now arrange the set $\widetilde{\mathscr{X}}$ of all elements $x \in \mathscr{X}$ such that the equality

$$\varphi^*(u \wedge x) + \varphi^*(u \wedge Cx) = \varphi^*(u) \tag{4}$$

holds for all $u \in \mathscr{X}$. It is clear that always

$$\varphi^*(u \wedge x) + \varphi^*(u \wedge Cx) \geq \varphi^*(u),$$

so that we are left only with validating the reverse inequality.

Lemma 2. *Let $z_n \in \widetilde{\mathscr{X}}$ $(n = 1, 2, \dots)$ and let the sequence $\{z_n\}$ tend to z monotonically. Then*

$$\varphi^*(u \wedge z) \leq \lim_{n\to\infty} \varphi^*(u \wedge z_n)$$

for all $u \in \mathscr{X}$.

PROOF. All is evident if $\{z_n\}$ decreases. Assume that $z_n \uparrow z$ and $z_1 = \mathbf{0}$. Then

$$u \wedge z = \sum_{n=1}^{\infty} u \wedge (z_{n+1} - z_n),$$

$$\varphi^*(u \wedge z) \leq \sum_{n=1}^{\infty} \varphi^*[u \wedge (z_{n+1} - z_n)] = \sum_{n=1}^{\infty} \varphi^*[u \wedge z_{n+1} \wedge Cz_n]$$

$$= \sum_{n=1}^{\infty} [\varphi^*(u \wedge z_{n+1}) - \varphi^*(u \wedge z_n)] = \lim_{n\to\infty} \varphi^*(u \wedge z_n).$$

The lemma is proved.

Lemma 3. *The set $\widetilde{\mathscr{X}}$ is a σ-regular subalgebra.*

PROOF. In the main condition (4) the elements x and Cx have equal status; therefore, $\widetilde{\mathscr{X}}$ contains each element together with its complement.

Take $x, y \in \widetilde{\mathscr{X}}$ and put $z = x \wedge y$. Given $u \in \mathscr{X}$, we have

$$\varphi^*(u) = \varphi^*(u \wedge x) + \varphi^*(u \wedge Cx)$$

$$= \varphi^*(u \wedge x \wedge y) + \varphi^*(u \wedge x \wedge Cy) + \varphi^*(u \wedge Cx \wedge y) + \varphi^*(u \wedge Cx \wedge Cy)$$

$$\geq \varphi^*(u \wedge x \wedge y) + \varphi^*[(u \wedge x \wedge Cy) \vee (u \wedge Cx \wedge y) \vee (u \wedge Cx \wedge Cy)]$$

$$= \varphi^*(u \wedge z) + \varphi^*(u \wedge Cz),$$

since

$$(u \wedge x \wedge Cy) \vee (u \wedge Cx \wedge y) \vee (u \wedge Cx \wedge Cy)$$

$$= u \wedge [(x \wedge Cy) \vee (Cx \wedge y) \vee (Cx \wedge Cy)] = u \wedge C(x \wedge y) = u \wedge Cz.$$

Thus, $z \in \widetilde{\mathscr{X}}$. We see that $\widetilde{\mathscr{X}}$ is subalgebra. It remains to show that $\widetilde{\mathscr{X}}$ is σ-regular. Let $\{x_1, x_2, \dots, x_n, \dots\}$ be a countable subset of $\widetilde{\mathscr{X}}$ and $x = \bigvee_{n=1}^{\infty} x_n$. Put $y_n \equiv \bigvee_{k=1}^{n} x_k$. It is clear that $y_n \in \widetilde{\mathscr{X}}$ and $y_n \uparrow x$, $Cy_n \downarrow Cx$. Given $u \in \mathscr{X}$, we see that

$$\varphi^*(u) = \varphi^*(u \wedge y_n) + \varphi^*(u \wedge Cy_n).$$

By the preceding lemma,

$$\varphi^*(u \wedge x) \le \lim_n \varphi^*(u \wedge y_n),$$

$$\varphi^*(u \wedge Cx) \le \lim_n \varphi^*(u \wedge Cy_n).$$

Whence,

$$\varphi^*(u \wedge x) + \varphi^*(u \wedge Cx) \le \varphi^*(u).$$

Thus, $x \in \widetilde{\mathscr{X}}$. It is proved that $\widetilde{\mathscr{X}}$ is a σ-regular subalgebra.

Lemma 4. $\mathscr{X}_0 \subset \widetilde{\mathscr{X}}$.

PROOF. Take $x \in \mathscr{X}_0$. Choose an element u and a cover $\sigma \in S_u$ arbitrarily. The families of elements of the form $x \wedge z$ $(z \in \sigma)$ and $Cx \wedge z$ $(z \in \sigma)$ are covers of the elements $x \wedge u$ and $Cx \wedge u$ respectively. Thus,

$$\sum_{z \in \sigma} \varphi(z) = \sum_{z \in \sigma} (\varphi(z \wedge x) + \varphi(z \wedge Cx))$$

$$= \sum_{z \in \sigma} \varphi(z \wedge x) + \sum_{z \in \sigma} \varphi(z \wedge Cx) \ge \varphi^*(x \wedge u) + \varphi^*(Cx \wedge u).$$

Since σ is arbitrary, we conclude that

$$\varphi^*(u) \ge \varphi^*(x \wedge u) + \varphi^*(Cx \wedge u).$$

Thus, $x \in \widetilde{\mathscr{X}}$. The lemma is proved.

Lemma 5. *If* $\varphi^*(x) = 0$ *then* $x \in \widetilde{\mathscr{X}}$.

PROOF. Given $u \in \mathscr{X}$, infer that

$$\varphi^*(u) \ge \varphi^*(x \wedge u) + \varphi^*(Cx \wedge u)$$

(the second summand is equal to zero because of the monotonicity of φ^*).

Lemma 6. *The function* $\widetilde{\varphi}$, *defined on a subalgebra* $\widetilde{\mathscr{X}}$ *by the equality* $\widetilde{\varphi}(x) \equiv \varphi^*(x)$, *is countably additive.*

PROOF. Assume that $x, y \in \widetilde{\mathscr{X}}$ and $x\, d\, y$. Then

$$\widetilde{\varphi}(x \vee y) = \varphi^*(x \vee y) = \varphi^*[(x \vee y) \wedge x] + \varphi^*[(x \vee y) \wedge Cx]$$

$$= \varphi^*(x) + \varphi^*(y) = \widetilde{\varphi}(x) + \widetilde{\varphi}(y).$$

So, we observe the instance of finite additivity. Hence, for every disjoint sequence $\{x_n\}$ in $\widetilde{\mathscr{X}}$, we have

$$\widetilde{\varphi}\Big(\bigvee_{n=1}^{\infty} x_n\Big) \ge \widetilde{\varphi}\Big(\bigvee_{n=1}^{m} x_n\Big) = \sum_{n=1}^{m} \widetilde{\varphi}(x_n),$$

and so

$$\widetilde{\varphi}\Big(\bigvee_{n=1}^{\infty} x_n\Big) \ge \sum_{n=1}^{\infty} \widetilde{\varphi}(x_n).$$

Considering the inequality

$$\widetilde{\varphi}\Big(\bigvee_{n=1}^{\infty} x_n\Big) = \varphi^*\Big(\bigvee_{n=1}^{\infty} x_n\Big) \le \sum_{n=1}^{\infty} \varphi^*(x_n) = \sum_{n=1}^{\infty} \widetilde{\varphi}(x_n),$$

which holds always, we arrive at the equality

$$\widetilde{\varphi}\Big(\bigvee_{n=1}^{\infty} x_n\Big) = \sum_{n=1}^{\infty} \widetilde{\varphi}(x_n)$$

expressing the countable additivity of $\widetilde{\varphi}$.

We study the structure of the subalgebra $\widetilde{\mathscr{X}}$ on which our quasimeasure $\widetilde{\varphi}$ is defined.

Given an arbitrary $x \in \widetilde{\mathscr{X}}$, choose the covers $\sigma_n \in S_x$ $(n = 1, 2, \dots)$ so that the inequalities

$$\widetilde{\varphi}(x) = \varphi^*(x) \le \sum_{y \in \sigma_n} \varphi(y) < \widetilde{\varphi}(x) + \frac{1}{n} \quad (n = 1, 2, \dots)$$

hold. We now put $\overline{x} \equiv \bigwedge_n \sup \sigma_n$. It is clear that $x \le \overline{x}$ and $\overline{x}$ belongs to the subalgebra $\overset{\cdots}{\mathscr{X}\langle\mathscr{X}_0\rangle}$ (the least σ-regular subalgebra that includes $\mathscr{X}_0$), and also

$$\widetilde{\varphi}(x) \le \widetilde{\varphi}(\overline{x}) \le \widetilde{\varphi}(\sup \sigma_n) \le \sum_{y \in \sigma_n} \varphi(y) < \widetilde{\varphi}(x) + \frac{1}{n}.$$

Hence,

$$\widetilde{\varphi}(x) = \widetilde{\varphi}(\overline{x}), \quad \widetilde{\varphi}(\overline{x} - x) = \varphi^*(\overline{x} - x) = 0.$$

Thus, each element $x \in \widetilde{\mathscr{X}}$ is representable as

$$x = \overline{x} - u, \tag{5}$$

where $\overline{x} \in \overset{\cdots}{\mathscr{X}\langle\mathscr{X}_0\rangle}$, $\varphi^*(u) = 0$, and $\widetilde{\varphi}(x) = \widetilde{\varphi}(\overline{x})$.

On the other hand, by Lemma 5, the elements of $\mathscr{X}$ admitting such a representation belong to $\widetilde{\mathscr{X}}$.

We have thus proved the following

Lemma 7. *The formula* (5) *gives a general form of an arbitrary element* $x \in \widetilde{\mathscr{X}}$.

In the just-considered situation it is evident that $u \in \widetilde{\mathscr{X}}$ and $\varphi^*(u) = \widetilde{\varphi}(u) = 0$. As for the element $\overline{x}$, it belongs to $\overset{\cdots\cdots}{\mathscr{X}}\langle \mathscr{X}_0 \rangle$ and also has the special form

$$\overline{x} = \bigwedge_{i=1}^{\infty} \bigvee_{k=1}^{\infty} y_{ik},$$

where $y_{ik} \in \mathscr{X}_0$ (so $\overline{x}$ is a "$\sigma\delta$-element").

This lemma has a dual version whose statement is left to the reader.

Approximation of the elements $\widetilde{\mathscr{X}}$ by the elements of the ORIGINAL subalgebra $\mathscr{X}_0$ is also possible, but precision will be less than in Lemma 7. Namely, we have the following

Lemma 8. *For all $x \in \widetilde{\mathscr{X}}$ and $\varepsilon > 0$ there exists $x_0 \in \mathscr{X}_0$ satisfying*

$$\varphi^*(|x - x_0|) < \varepsilon.$$

As in the preceding proof, we start with choosing $\sigma \in S_x$ so that the inequality

$$\widetilde{\varphi}(\overline{x}) \leq \sum_{y \in \sigma} \varphi(y) < \widetilde{\varphi}(x) + \frac{\varepsilon}{2}$$

holds. Put $\overline{x} \equiv \sup \sigma$ and find

$$|\overline{x} - x| = \overline{x} - x, \quad \widetilde{\varphi}(|\overline{x} - x|) = \widetilde{\varphi}(\overline{x}) - \widetilde{\varphi}(x) \leq \sum_{y \in \sigma} \widetilde{\varphi}(y) - \widetilde{\varphi}(x) \leq \frac{\varepsilon}{2}.$$

Furthermore, since σ is countable, there exists a countable subset $\sigma_0 \subset \sigma$ such that

$$\widetilde{\varphi}(\overline{x}) - \frac{\varepsilon}{2} < \widetilde{\varphi}(\sup \sigma_0) < \widetilde{\varphi}(\overline{x}),$$

whence

$$\widetilde{\varphi}(|\overline{x} - \sup \sigma_0|) = \widetilde{\varphi}(\overline{x}) - \widetilde{\varphi}(\sup \sigma_0) < \frac{\varepsilon}{2}.$$

We now may take $\sup \sigma_0$ as a sought x_0. Thus,

$$\widetilde{\varphi}(|x - x_0|) \leq \widetilde{\varphi}(|x - \overline{x}|) + \widetilde{\varphi}(|\overline{x} - x_0|) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

The lemma is proved.

Definition. A quasimeasure ν on a subalgebra $\mathscr{Y} \subset \mathscr{X}$ is called *complete* if the conditions $x \leq y$, $y \in \mathscr{Y}$, $x \in \mathscr{X}$, and $\nu y = 0$ imply $x \in \mathscr{Y}$ (and, obviously, $\nu x = 0$).

It follows from Lemma 6 that *the quasimeasure $\widetilde{\varphi}$ that we have constructed is complete.*

Lemma 9. *Let ψ be a complete countably additive quasimeasure on some σ-regular subalgebra $\mathscr{Y}$ including $\mathscr{X}_0$. Assume that $\psi(x) \leq \varphi(x)$ for all $x \in \mathscr{X}_0$. Then*

1) $\mathscr{Y} \supset \widetilde{\mathscr{X}}$;

2) $\psi(x) \le \widetilde{\varphi}(x)$ *for all* $x \in \widetilde{\mathscr{X}}$.

PROOF. Since ψ is countably additive, for all $x \in \mathscr{Y}$ and $\sigma \in S_x$ we have the inequality

$$\psi(x) \le \psi(\sup \sigma) \le \sum_{y\in\sigma} \psi(y) \le \sum_{y\in\sigma} \varphi(y).$$

Hence, $\psi(x) \le \varphi^*(x)$. Applying this to an element $x \in \widetilde{\mathscr{X}}$, note that $\psi(x) \le \widetilde{\varphi}(x)$. It remains to prove the first assertion of the lemma.

We first show that if $\varphi^*(v) = 0$ then $v \in \mathscr{Y}$ and $\psi(v) = 0$. Indeed, by Lemma 8, there is a $\sigma\delta$-element $\overline{v} \ge v$ satisfying $\varphi^*(\overline{v}) = \widetilde{\varphi}(\overline{v}) = 0$. All $\sigma\delta$-elements belong to $\mathscr{X}\langle\mathscr{X}_0\rangle$ and, hence, to $\mathscr{Y}$. Thus $\overline{v} \in \mathscr{Y}$ and

$$0 \le \psi(\overline{v}) \le \varphi^*(\overline{v}) = 0.$$

The quasimeasure ψ is complete; hence, $v \in \mathscr{Y}$ and $\psi(v) = 0$. Now it is clear that each $x \in \widetilde{\mathscr{X}}$ belongs to $\mathscr{Y}$, since x may be presented as $x = \overline{x} - u$, where $\overline{x}$ is a $\sigma\delta$-element and $\varphi^*(u) = 0$. The inclusion $\widetilde{\mathscr{X}} \subset \mathscr{Y}$ is proved, implying the lemma.

REMARK. We need the completeness of ψ only in the proof of 1).

Lemma 10. *If*

$$\varphi(\sup \sigma) \le \sum_{y\in\sigma} \varphi(y)$$

for every countable subset σ *of* $\mathscr{X}_0$ *such that* $\sup \sigma \in \mathscr{X}_0$ *then* $\widetilde{\varphi}(x) = \varphi(x)$ *for all* $x \in \mathscr{X}_0$, *i.e., the quasimeasure* $\widetilde{\varphi}$ *is an extension of* φ.

(It is easy to see that in the hypotheses of this lemma we simply require the *countable additivity* of φ.)

Indeed, in this case $\widetilde{\varphi}(x) = \varphi^*(x) \ge \varphi(x)$ for all $x \in \mathscr{X}_0$, whence, the property 1° of an outer measure yields the desired equality.

The above Lemmas 3–10 are summarized in the following

Theorem 6. *Let* $\mathscr{X}$ *be a* σ*-complete BA, let* $\mathscr{X}_0$ *be an arbitrary subalgebra of* $\mathscr{X}$, *and let* φ *be a quasimeasure on* $\mathscr{X}_0$. *Then there exist* a) *a* σ*-regular subalgebra* $\widetilde{\mathscr{X}}$ *including* $\mathscr{X}_0$, *and* b) *a countably additive quasimeasure* $\widetilde{\varphi}$ *on* $\widetilde{\mathscr{X}}$ *possessing the following properties:*

1) $\widetilde{\varphi}(x) \le \varphi(x)$ *for all* $x \in \mathscr{X}_0$;

2) *for every countably additive quasimeasure* ψ *on* $\widetilde{\mathscr{X}}$ *and for all* $x \in \mathscr{X}_0$ *satisfying* $\psi(x) \le \varphi(x)$, *the inequality* $\psi(x) \le \widetilde{\varphi}(x)$ *holds for all* $x \in \widetilde{\mathscr{X}}$.

If, moreover, φ is countably additive then the equality

$$\varphi(x) = \widetilde{\varphi}(x)$$

holds for all $x \in \mathscr{X}_0$; in other words, $\widetilde{\varphi}$ is an extension of φ.

We call the quasimeasure $\widetilde{\varphi}$ that is constructed in the proof of Theorem 6 the *standard countably additive quasimeasure generated by a quasimeasure φ*. In the case when $\widetilde{\varphi}$ is an extension of φ, it is called the *standard extension of a quasimeasure φ.* The term "Lebesgue extension" is also used. The above "constructive" definition may be replaced with the equivalent "descriptive" definition:

The standard extension is the narrowest among all complete countably additive extensions. (See Lemma 9.)

We have already mentioned that this main theorem of measure theory in the present form is due to C. Carathéodory who was also one of the pioneers in the "algebraization" of measure theory.[4] The above proof is similar to that given in the book of G. Birkhoff [1] but differs from it in some details.

Of course, the Lebesgue–Carathéodory Theorem is primarily applied to algebras of sets.

Lemma 8 implies that the subalgebra $\widetilde{\mathscr{X}}$ can be considered as the closure of $\mathscr{X}_0$ with respect to the quasimetric

$$\rho(x, y) = \varphi^*(|x - y|).$$

This opens up another way of proving the main Theorem 5; it is possible to extend a quasimeasure φ on the closure of its domain by considering φ as a function uniformly continuous in ρ. At present, this approach is used rather often in the presentation of the basics of measure theory.[5]

Theorem 6 has an important corollary. Let us agree to call a quasimeasure φ on $\mathscr{X}$ *purely finitely additive* if the unique countably additive quasimeasure ψ, satisfying the inequality

$$\psi(x) \leq \varphi(x)$$

for all x, is identically zero. We now apply Theorem 6 to an arbitrary quasimeasure φ on the whole Boolean algebra $\mathscr{X}$. In this case the standard quasimeasure $\widetilde{\varphi}$ is also defined on $\widetilde{\mathscr{X}} = \mathscr{X}$ and the difference $\varphi - \widetilde{\varphi} \equiv \psi$ is a purely finitely additive quasimeasure (by the maximality of $\widetilde{\varphi}$). Thus, we have proved the following

---

[4] C. Carathéodory [1, 2].

[5] L. Ya. Savel′ev [2] and A. A. Borovkov [1].

Theorem 7. *Each quasimeasure φ on a σ-complete Boolean algebra is uniquely represented as the sum of two quasimeasures $\varphi = \varphi_1 + \varphi_2$, where φ_1 is countably additive and φ_2 is purely finitely additive.*[6]

We mention one important example of application of the Lebesgue–Carathéodory Theorem.

In Chapter 1 we have already met the "Bernoulli measures" β_{pq} (see p. 80). These measures are defined only on the clopen algebra $\mathscr{D}_\xi$ of the Cantor discontinuum X_ξ. It was noted that β_{pq} is an essentially positive totally additive function. By Theorem 6, it has a countably additive Lebesgue extension which we will still denote by β_{pq} and call the "Bernoulli measure."

In fact, it is a countably additive quasimeasure that "transforms" by factorization into a genuine measure on the metric structure of the probability space $\{X, \mathscr{E}_{pq}, \beta_{pq}\}$. Here $\mathscr{E}_{pq}$ is the σ-algebra to which β_{pq} is extended by the Lebesgue–Carathéodory process. This measure depends essentially on p and q, i.e., on the coin whose tossing is characterized by the measure β_{pq}. For example, it is well known that to the distinct coins there correspond the singular "measures."[7] The structure of the algebra $\mathscr{E}_{pq}$ is clear: $\mathscr{E}_{pq}$ is obtained by adjoining all subsets of β_{pq}-negligible Baire sets to the Baire algebra $\mathscr{B}_0(X_\xi)$.

We will call $\mathscr{E}_{pq}$ the *Bernoulli σ-algebra* and call the measure space $\{X, \mathscr{E}_{pq}, \beta_{pq}\}$ the *Bernoulli space.* An especially important role is played by the "binary" case: $p = q = \frac{1}{2}$ when the Bernoulli "measure" is identifiable in some sense with Lebesgue "measure" on the interval $[0, 1]$.

We will return to the question of extension of an additive function in Section 5 of this chapter.

3. NBAs and the metric structures of measure spaces

3.1 The metric structure of a measure space

Assume given a measure space $\{\Omega, \mathscr{E}, m\}$. To this space there corresponds the "metric structure,"[8] i.e., the Boolean algebra $\widehat{\mathscr{E}}$ resulting from factorization of the σ-algebra $\mathscr{E}$ by the ideal of negligible sets. This metric structure is a complete BA endowed with the measure $\widehat{m}$. Taking some liberty, we will apply the term "metric structure" to the NBA $\{\widehat{\mathscr{E}}, \widehat{m}\}$ as well. Thus, each metric structure, associated with some metric space ("the algebra of *mod* 0 measurable sets"), provides an example

[6] See K. Yosida and E. Hewitt [1].
[7] W. Feller [1, p. 177].
[8] See 1.3.3.

of an NBA. It turns out that each NBA can be represented in this form. Before formulating the corresponding theorem, we dwell on the concept of isomorphism.

In this book we pay attention mostly to the Boolean isomorphisms; i.e., the isomorphisms of metric structures. However, measure theory uses another important concept of isomorphism between SPACES. It occurs in several variants, but in all cases we deal with a "pointwise" mapping of one space onto the other. From numerous definitions we choose only those that are actually used in this part of the book.

Consider the two categories $\mathfrak{M}$ and $\overline{\mathfrak{M}}$ whose objects are all measure spaces of the form $\{\Omega, \mathscr{E}, m\}$. The morphisms of the first category are all possible mappings $\mathfrak{F} : \Omega_1 \longrightarrow \Omega_2$ possessing the following properties:

a) $\mathfrak{F}^{-1}(e) \in \mathscr{E}_1$ for all $e \in \mathscr{E}_2$;

b) if $m_2 e = 0$ then $m_1 \mathfrak{F}^{-1}(e) = 0$.

If, moreover,

c) $m_1 \mathfrak{F}^{-1}(e) = m_2 e$

for all $e \in \mathscr{E}_2$ ("measure preservation") then $\mathfrak{F}$ is called a *metric homomorphism.* These homomorphisms are taken as the morphisms of the category $\overline{\mathfrak{M}}$.

In fact, each object in the category $\mathfrak{M}$ is defined by the space Ω, the algebra $\mathscr{E}$, and the ideal of negligible sets; the quasimeasure m itself plays a secondary role.

To these two categories there correspond also two forms of isomorphisms. Later we will call an isomorphism between two measure spaces $\{\Omega_1, \mathscr{E}_1, m_1\}$ and $\{\Omega_2, \mathscr{E}_2, m_2\}$ AS OBJECTS of $\mathfrak{M}$ simply an isomorphism of these spaces. Such an isomorphism is a *bijective mapping* $\mathfrak{F} : \Omega_2 \longrightarrow \Omega_1$ possessing the properties a) and b) together with its inverse mapping $\mathfrak{F}^{-1}$. Moreover, if $\mathfrak{F}$ (hence, $\mathfrak{F}^{-1}$) possesses the property c) then we call $\mathfrak{F}$ a *metric isomorphism* (an isomorphism in the category $\overline{\mathfrak{M}}$).

If an isomorphism (metric isomorphism) is available between two measure spaces then we say that these spaces are isomorphic (metrically isomorphic).

Measure spaces are called *isomorphic mod* 0 (*metrically isomorphic mod* 0) if, after removing some negligible sets, we obtain isomorphic (metrically isomorphic) spaces.

It is easily seen that an isomorphism of spaces[9] implies a Boolean isomorphism of the corresponding metric structures, and the metric iso-

[9]As usual, the word "isomorphism" means both a mapping and the property of the spaces to be isomorphic to one another.

morphism implies an isomorphism of normed Boolean algebras. The converse fails in general but it may be valid if we can successfully restrict the class of measure spaces under consideration.

Theorem 8. *For each NBA* $\{\mathscr{X}, \mu\}$ *there exists a measure space* $\{\Omega, \mathscr{E}, m\}$ *such that the NBAs* $\{\mathscr{X}, \mu\}$ *and* $\{\widehat{\mathscr{E}}, \widehat{m}\}$ *are isomorphic.*

In other words, each NBA is isomorphic to some metric structure. We mean here a METRIC (measure preserving) isomorphism.

PROOF. We may use each of the three representation theorems: the Loomis–Sikorski Theorem with its proof, Theorem 5.9 or Theorem 5.10. The last two theorems are applicable, since each normed algebra is regular. According to one of these theorems, we can find for our NBA $\mathscr{X}$ a space Ω, a σ-algebra of subsets $\mathscr{E}$ of Ω, and a σ-ideal I of this algebra so that there exists an isomorphism Φ of the quotient algebra $\mathscr{E}|_I$ onto $\mathscr{X}$. In our case, there is a measure μ on $\mathscr{X}$; this makes it possible to define a countably additive quasimeasure m on $\mathscr{E}$ by the following condition:

$$m(e) \equiv \mu\Phi(\theta(e)).$$

Here θ is a canonical σ-homomorphism of $\mathscr{E}$ onto $\mathscr{E}|_I$. Since μ is essentially positive on $\mathscr{X}$, we easily establish that the ideal I coincides with the ideal of m-negligible sets, and Φ is a required isomorphism of the NBA $\{\mathscr{E}|_I, \widehat{m}\}$ onto the NBA $\{\mathscr{X}, \mu\}$. Measure preservation is evident.

Thus, we have obtained the three variants of the proof of Theorem 8, the three measure spaces of the form $\{\Omega, \mathscr{E}, m\}$ whose metric structure is isomorphic to the NBA $\{\mathscr{X}, \mu\}$. In the first case the role of the main space Ω plays the Stone space $\mathfrak{Q}(\mathscr{X})$; in the second and third cases the main space is "standard," connected only with the weight of $\mathscr{X}$ but not with the algebra itself. Namely, using Theorem 5.9 we obtain as R the Cantor space of weight $\tau(\mathscr{X})$; Theorem 5.10 leads to the product of $\tau(\mathscr{X})$ copies of the interval $[0, 1]$ (or another nontrivial interval). Of the utmost interest is the question: What measure space is better as regards the representation of an NBA?

3.2 The Stone space of an NBA

Consider the Stone space $\mathfrak{Q} \equiv \mathfrak{Q}(\mathscr{X})$ of a complete BA $\mathscr{X}$. We will identify $\mathscr{X}$ and $\mathscr{X}^0 \equiv \mathscr{C}O(\mathfrak{Q})$. Every quasimeasure μ_0 on $\mathscr{X}^0$ satisfies the assumption of the Lebesgue–Carathéodory Theorem and admits a countably additive extension to a quasimeasure μ on some σ-algebra $\mathscr{E} \equiv \mathscr{E}_\mu$. If the quasimeasure μ_0 is actually an essentially positive and countably additive measure then this algebra can be described in more detail. We now launch into this description.

1°. Note first that *every rare set belongs to $\mathscr{E}$.* Let a set e be rare. Its closure $\overline{e}$ is also rare. Consider the complementary open set $G \equiv \mathfrak{Q} \setminus \overline{e}$; it is represented as the union of clopen sets

$$G = \bigcup_{t \in T} x_t, \quad x_t \in \mathscr{X} \quad (t \in T).$$

Since $\overline{e}$ is rare; therefore, $\bigvee_{t \in T} x_t = \mathbf{1}$. Our algebra $\mathscr{X}^0$ is normable (since the measure μ_0 is available); hence, it satisfies the countable chain condition and the unity can be represented as the supremum of an ordinary sequence

$$\mathbf{1} = \bigvee_{n=1}^{\infty} x_{t_n} \quad (t_n \in T, \; n = 1, 2, \ldots).$$

We may assume that $x_{t_1} \leq x_{t_2} \leq \ldots$. Applying the countable additivity of μ_0, we infer that $\mu_0 x_{t_n} \longrightarrow \mu_0 \mathbf{1}$. Then

$$\mu^* e \leq \mu^* \overline{e} \leq \mu_0(C x_{t_n}) = \mu_0 \mathbf{1} - \mu_0 x_{t_n} \longrightarrow 0, \quad \mu^* e = 0$$

and $e \in \mathscr{E}_\mu$.

2°. The algebra $\mathscr{X}^0$, as each normable algebra, is regular, and so the rare sets comprise a σ-ideal. Denote it by I and consider the σ-algebra of sets (which we have already encountered in Chapter 5)

$$\Sigma \equiv \Sigma(\mathfrak{Q}) = \Sigma'(\mathfrak{Q}).$$

This σ-algebra consists of all sets of the form

$$e = x +_2 q, \quad x \in \mathscr{X}^0, \; q \in I.$$

It is clear that by 1°, we have $\Sigma \subset \mathscr{E}$.

3°. Clearly, some countably additive extension of μ_0 does exist on Σ. Namely, such is the quasimeasure $\mu' \equiv \mu_0 \circ U_{\mathfrak{Q}}$, where $U_{\mathfrak{Q}}$ is the Birkhoff–Ulam σ-homomorphism (see p. 252). By the essential positivity of μ_0 the equality $\mu' e = 0$ is equivalent to $e \in \ker U_{\mathfrak{Q}} = I$. The quasimeasure μ' is complete, since the conditions $e' \subset e$ and $\mu' e = 0$ imply $e' \in I$ and $e' \in \Sigma$.

4°. Both quasimeasures μ and μ' are countably additive and complete while extending the measure μ_0. Since μ is the Lebesgue extension; therefore, $\mathscr{E} \subset \Sigma$ and $\mu'|_{\mathscr{E}} = \mu$. At the same time, we see that $\Sigma \subset \mathscr{E}$.

5°. So we draw the following conclusions. *If $\mathscr{X}^0$ is an NBA, μ_0 is a measure on $\mathscr{X}^0$, and μ is its Lebesgue extension then*

1) *the domain of μ is the σ-algebra $\Sigma(\mathfrak{Q})$, i.e., it consists of all sets of the form $x +_2 q$, where $x \in \mathscr{X}^0$ and q is rare;*

2) *the μ-negligible sets are precisely the rare subsets of the compact space $\mathfrak{Q}$.*

Thus, all measures on $\mathscr{X}^0$ extend onto the same σ-algebra with the same stock of negligible sets.

The mapping $x +_2 I \longrightarrow x$ $(x \in \mathscr{X})$ is in this case the isomorphism Φ that is discussed in Theorem 8, the isomorphism of the quotient algebra $\Sigma|_I$ onto $\mathscr{X} \equiv \mathscr{X}^0$. As was said above, this is an isomorphism between normed Boolean algebras, since Φ is measure preserving:

$$\widehat{\mu}(x +_2 I) = \mu(x +_2 q) = \mu x = \mu_0 x.$$

(Here q is an arbitrary rare set.)

But if $\mathscr{X}$ does not coincide with $\mathscr{X}^0$ then Φ induces an isomorphism between $\mathscr{X}$ and $\mathscr{X}^0$.

We see that every NBA $\{\mathscr{X}, \mu\}$ is isomorphic to the metric structure of the space $\{\mathfrak{Q}(\mathscr{X}), \Sigma(\mathfrak{Q}(\mathscr{X})), \mu\}$. Each isomorphism between NBAs $\{\mathscr{X}, \mu\}$ and $\{\mathscr{X}', \mu'\}$ induces a homeomorphism between the Stone spaces and an isomorphism between measure spaces.

Thus, *for the class of "Stone representations," each "Boolean" isomorphism between metric structures implies an isomorphism between the corresponding spaces.*

However, the "Stone representation" of a normed algebra has many shortcomings. It never occurs as an original object in real situations; the representation space, the Stone space, is bizarre: it has as a rule a huge cardinality containing many "superfluous" points. This compels us to prefer other representations that are more adapted to solving concrete problems of probability, functional analysis, and ergodic theory. As regards separable algebras, the most adequate approach to representation was suggested by V. A. Rokhlin.

3.3 Lebesgue–Rokhlin spaces

As was said in 5.4.2, the Cantor discontinuum $X \equiv X_T$ can be used as the representation space of an NBA $\mathscr{X}$, where $card\, T = \tau(\mathscr{X})$. Indeed, there exists a σ-ideal I of the Baire algebra of the extremally disconnected compact space X and an isomorphism Φ of the quotient algebra $\mathscr{B}_0|_I$ onto $\mathscr{X}$. Recall that the Baire algebra $\mathscr{B}_0$ is the least σ-algebra including the clopen algebra $\mathfrak{D}_T$.

If $\mathscr{X}$ is furnished with a measure μ then this measure "translates" into the quasimeasure m on the algebra $\mathscr{B}_0$ by the rule $m\, e = \mu\Phi(e +_2 I)$. In this event the ideal $I \equiv I_m$ becomes exactly the σ-ideal of m-negligible Baire sets. The metric structure of the measure space $\{X, \mathscr{B}_0, m\}$ is practically the NBA $\{\mathscr{X}, \mu\}$ (up to isomorphism).

We obtain the same metric structure if we replace $\mathscr{B}_0$ with the "completed" σ-algebra

$$\mathscr{B}_m \equiv \mathscr{B}_0 + \widetilde{I}_m,$$

where $\widetilde{I}_m$ consists of all sets bounded by the elements of the ideal I_m (see the remark at the end of 5.4.2). The "measure" m extends naturally onto $\mathscr{B}_m$ exactly as the isomorphism Φ.

This representation has important peculiarities connected with the exceptional role of the algebra $\mathfrak{D}_T$. This algebra possesses the following properties:

1) Each centered system of nonempty sets in $\mathfrak{D}_T$ has nonempty intersection. This is clear because $\mathfrak{D}_T$ consists of compact sets.

2) Each set in $\mathscr{B}_m$ coincides *mod* 0 (i.e., modulo the ideal $\widetilde{I}_m$) with some set in the σ-algebra generated by $\mathfrak{D}_T$ (i.e., in $\mathscr{B}_0$). This means precisely that there exists a standard extension of the quasimeasure $m|_{\mathscr{D}_T}$.

3) $\mathfrak{D}_T$ separates the points of the original space. (The proof is not difficult.)

The property 1) is called the *compactness* property[10] of the algebra $\mathfrak{D}_T$; the properties 2) and 3) mean by definition that $\mathfrak{D}_T$ is a *base* of $\{X_T, \mathscr{B}_m, m\}$.

In the case of an arbitrary "measure space" $\{\Omega, \mathscr{E}, m\}$ the subalgebra $\mathscr{E}_0 \subset \mathscr{E}$ is also called a *compact base* if it possesses the properties 1), 2), and 3) (of course, on replacing X_T and $\mathfrak{D}_T$ with Ω and $\mathscr{E}_0$ respectively.)

A measure space $\{\Omega, \mathscr{E}, m\}$ with a complete "measure" m is called a *Lebesgue–Rokhlin space* if it has a countable compact base. The theory of such spaces is completely developed by V. A. Rokhlin in 1940; he called them "Lebesgue spaces."[11] We will usually denote a Lebesgue–Rokhlin space by a single letter Ω.

From what was said at the beginning of this subsection, it is clear that every separable[12] NBA is (up to metric isomorphism) the metric structure of some Lebesgue–Rokhlin space. Because of the countability of the base, the metric structure of a Lebesgue–Rokhlin space is always separable. We will not present the theory of these spaces in detail, referring the reader to the literature and stating several main results without proofs. In the sequel m is assumed to be a probability "measure," i.e., $m\Omega = 1$.

[10] The "compactness property" of an algebra (or, general, of a system of sets) should not be confused with the "compactness" of a topological space, although these concepts are related to one another. The closed sets of a compact space comprise a system with the "compactness property."

[11] V. A. Rokhlin [1] (the first publication). See also V. G. Vinokurov, B. A. Rubshteĭn, and A. L. Fëdorov [1]; A. A. Samorodnitskiĭ [1].

[12] Recall that this means $\tau(\mathscr{X}) = \alpha_0$.

Let $\{\Omega_i, \mathscr{E}^i, m_i\}$ $(i = 1, 2)$ be two Lebesgue–Rokhlin spaces. Each of them has at most countably many atoms (loaded points). Allowing the zero loads for the sake of generality, we enumerate the loads in decreasing order. Let $\{m_1^1, m_2^1, \ldots\}$ and $\{m_1^2, m_2^2, \ldots\}$ be the loads in Ω_1 and Ω_2 respectively, where $m_1^i \geq m_2^i \geq \ldots$ $(i = 1, 2)$.[13] It turns out that this collection of loads is the sole invariant enabling us to classify the Lebesgue–Rokhlin spaces metrically. Namely,

1) *If $m_k^i = 0$ for all i and k then the spaces are metrically isomorphic (a "pure" isomorphism). In this case, the metric structures are metrically isomorphic too, which is obvious.*

2) *If $m_k^1 = m_k^2$ for all $k = 1, 2, \ldots$ then the spaces $\{\Omega_1, \mathscr{E}^1, m_1\}$ and $\{\Omega_2, \mathscr{E}^2, m_2\}$ are metrically isomorphic $\bmod 0$, and their metric structures are metrically isomorphic.*

3) *If the metric structures are metrically isomorphic then $m_k^1 = m_k^2$ for all $k = 1, 2, \ldots,$ and the spaces are metrically isomorphic $\bmod 0$.*

The proof of this "classification theorem" is available; for example, see the monograph of V. G. Vinokurov, B. A. Rubshteĭn, and A. L. Fëdorov [1]. It follows from this theorem that *the Lebesgue–Rokhlin spaces with isomorphic metric structures are isomorphic $\bmod 0$, and if these structures are continuous ("nonatomic"),*[14] *then there is a "pure" isomorphism between these Lebesgue–Rokhlin spaces.* (Speaking of isomorphism of metric structures, we now mean an isomorphism between NBAs.)

The isomorphic classification of Lebesgue–Rokhlin spaces is equivalent to classification of the corresponding NBAs.

A *subspace* of a measure space $\{\Omega, \mathscr{E}, m\}$ is a measure space $\{E, \mathscr{E}_E, m_E\}$, where $E \in \mathscr{E}$, $mE > 0$, and $m_E = m|_{\mathscr{E}_E}$. If $mE = m\Omega$ then both spaces have the same metric structure. More precisely, there is a canonical isomorphism $\{\widehat{\mathscr{E}}, \widehat{m}\}$ onto $\{\widehat{\mathscr{E}}_E, \widehat{m}_E\}$ naturally generated by the mapping $e \longrightarrow e \cap E$ $(e \in \mathscr{E})$. The details of this trivial construction are left to the reader. We now note only the fundamental fact: *a subspace of a Lebesgue–Rokhlin space is again a Lebesgue–Rokhlin space.* Sometimes, the concept of subspace is interpreted more loosely, not assuming E to be measurable. Thus, if $m^*E = m\Omega$ but $E \notin \mathscr{E}$, then we can still make E into a measurable space naturally and, moreover, with the same metric structure, but no Lebesgue–Rokhlin space will be produced.

[13] We mean here the measures that are supported at isolated points. The equality $m_k^i = m_{k+1}^i = \cdots = m_{k+s}^i > 0$ means that there exist $s+1$ points in Ω_i with a given (nonzero) load. The formulas $m_k^i > 0$, $m_{k+1}^i = m_{k+2}^i = \cdots = 0$ say that the total number of loaded points in Ω_i is equal to k.

[14] Or even have continuous bands.

Clearly, each space $\{\Omega, \mathscr{E}, m\}$ isomorphic to some Lebesgue–Rokhlin space is a Lebesgue–Rokhlin space too. If we restrict the cardinality of spaces to at most that of the continuum then the same will be true for the isomorphisms $mod\,0$. Thus, the class of Lebesgue–Rokhlin spaces is "well-designed" as a whole.

The Lebesgue–Rokhlin spaces comprise a "well-designed class," but also they are remarkable in regard to their intrinsic structure. Postponing a detailed discussion of this topic to the next subsection, we now list the main models. We begin with the continuous case in which all Lebesgue–Rokhlin spaces are isomorphic to one another.

In fact, one model has already been exhibited; it is the Bernoulli space $\{X_{\mathbb{N}}, \mathscr{E}_{pq}, \beta_{pq}\}$. We will prefer the case $p = q = \frac{1}{2}$ (a "symmetric coin") and briefly write $\mathscr{E}_{\frac{1}{2}\frac{1}{2}} \equiv \mathscr{E}$ and $\beta_{\frac{1}{2}\frac{1}{2}} \equiv \beta$. As a countable compact base here we may take the clopen algebra $\mathfrak{D}_{\mathbb{N}}$. Existence of such a base means precisely that the Bernoulli space is a Lebesgue–Rokhlin space. We have already noted that in this case the point loads are absent; therefore, our model is a model of a continuous Lebesgue–Rokhlin space, and each of these spaces must be isomorphic to the Bernoulli space. In particular, all Bernoulli spaces constructed on the discontinuum $X_{\mathbb{N}}$ (of countable weight) are metrically isomorphic. (Although, the distinct measures β_{pq} are pairwise singular.) A second model of a continuous Lebesgue–Rokhlin space is the Lebesgue space $\{I^n, \mathscr{E}_l^{(n)}, l\}$, where I^n is the n-dimensional cube (see p. 61). In the previous chapters we have described the mapping $\psi : X_{\mathbb{N}} \longrightarrow [0, 1]$ acting by the formula

$$\psi(\chi) \sum_{n=1}^{\infty} \frac{\chi_n}{2^n}.$$

Similar mappings can be constructed onto the square $[0, 1] \times [0, 1]$, the cube, etc. All they are "almost" bijective and continuous, while translating the Bernoulli "measure" $\beta = \beta_{\frac{1}{2}\frac{1}{2}}$ into Lebesgue "measure": $l(\psi(e)) = \beta(e)$.

The mapping ψ is not bijective at the elements $X_{\mathbb{N}}$ of the form $(\chi_1, \chi_2, \dots, \chi_n, 0, 0, \dots)$ and $(\chi_1, \chi_2, \dots, \chi_n, 1, 1, \dots)$, which correspond to the binary rational points of the interval. They form a countable set S, and its image $\psi(S)$ consists of all binary rational points of the interval. Let ω be some bijection of S onto $\psi(S)$ (both sets are countable). The mapping $\widetilde{\psi}$ given by

$$\widetilde{\psi}(\chi) = \begin{cases} \psi(\chi), & \chi \notin S, \\ \omega(\chi), & \chi \in S, \end{cases}$$

is clearly an isomorphism of the Bernoulli space onto the Lebesgue interval. This confirms that the Lebesgue interval is a Lebesgue–Rokhlin space and every continuous Lebesgue–Rokhlin space is isomorphic to it. The same is true for the Lebesgue square, cube, and etc. We need to say that the most appropriate model for a continuous Lebesgue–Rokhlin space is simply the Lebesgue square.

A purely discrete Lebesgue–Rokhlin space is represented more simply as a countable or finite family of points with loads $m_1, m_2, \dots$. A model of another type is the interval $[a, b]$ with a distinguished family of points $x_1, x_2, \dots$ at which the measure is entirely supported. Coincidence of the corresponding loads implies, as was mentioned, an isomorphism $mod\,0$; but no genuine isomorphism is possible, since these spaces have different cardinalities and so do not admit any bijection.

At last, the most general model of Lebesgue–Rokhlin space is the Lebesgue interval, or square (or the Bernoulli space) with an at most countable family of loaded points adjoined to it. Each Lebesgue–Rokhlin space is metrically isomorphic $mod\,0$ to one of these "model" spaces. To simplify the exposition, this fact is often taken as a basis for the initial definition of Lebesgue–Rokhlin space.

The above implies in particular that *a separable continuous probability BA is always represented as the metric structure of the Lebesgue interval, square, or cube of an arbitrary dimension; i.e., it is isomorphic to each of the NBAs*

$$E_0^1, E_0^2, \dots .$$

This fact has been stated independently of the theory of Lebesgue–Rokhlin spaces by many authors.[15] We have still provided no proof, deferring it to the end of this chapter. Now, we note again that the theorem of isomorphism between continuous Lebesgue–Rokhlin spaces is stronger than the theorem of isomorphism between metric structures, since not every "Boolean" isomorphism is produced in general by a point mapping. By now, we spoke only about representation of a separable NBA. For the algebras of uncountable weight, there are also available[16] the most convenient classes of representation of measure spaces; many facts of the theory of Lebesgue–Rokhlin spaces are carried over to these classes. However, the Rokhlin theory is not abstracted for the nonseparable case to a full extent. We will return to these questions in the subsequent chapters.

[15] C. Carathéodory [2], P. R. Halmos and J. von Neumann [1], L. V. Kantorovich, B. Z. Vulikh, and A. G. Pinsker [1].
[16] V. G. Vinokurov [1] and A. A. Samorodnitskiĭ [1].

3.4 Boolean isomorphisms and point mappings

As was mentioned in 7.3.1, the presence of an isomorphism between metric structures $\{\widehat{\mathscr{E}}_1, \widehat{m}_1\}$ and $\{\widehat{\mathscr{E}}_2, \widehat{m}_2\}$ follows from isomorphisms of the corresponding measure spaces but the converse fails in general. Here we meet the general problem of "pointwise" representation of a Boolean isomorphism.

Each measure preserving isomorphism $\mathfrak{F}$ of a measure space $\{\Omega_1, \mathscr{E}_1, m_1\}$ onto another space $\{\Omega_2, \mathscr{E}_2, m_2\}$ induces the Boolean isomorphism

$$\widehat{\mathfrak{F}} : \{\widehat{\mathscr{E}}_2, \widehat{m}_2\} \longrightarrow \{\widehat{\mathscr{E}}_1, \widehat{m}_1\}$$

by the formula $\widehat{\mathfrak{F}}(\widehat{e}) = \widehat{\mathfrak{F}^{-1}(e)}$, where e is an arbitrary representation of $\widehat{e}$. The same is true in the case when $\mathfrak{F}$ is an isomorphism *mod* 0. The question posed first by J. von Neumann is as follows: Does every metric isomorphism between metric structures have such provenance? The crux of the matter lies here in the properties of the representation spaces Ω_1 and Ω_2. J. von Neumann introduced a class of "normal" spaces for which this question has a positive answer. D. Maharam extended this result to uncountable products of Lebesgue intervals (the metric structures of the form E^{Γ}). It is clear that the answer is also positive for Stone spaces. This problem has a voluminous literature,[17] but we confine exposition to the formulation of the main result: *if* $\{\Omega_1, \mathscr{E}_1, m_1\}$ *and* $\{\Omega_2, \mathscr{E}_2, m_2\}$ *are Lebesgue–Rokhlin spaces and* Φ *is a Boolean isomorphism of* $\{\widehat{\mathscr{E}}_2, \widehat{m}_2\}$ *onto* $\{\widehat{\mathscr{E}}_1, \widehat{m}_1\}$ *then there exists* $\mathfrak{F}$, *an isomorphism mod* 0 *of* Ω_1 *onto* Ω_2 *such that* $\Phi = \widehat{\mathfrak{F}}$. We do not presume measure preservation, but if $\widehat{m}_2 = \widehat{m}_1 \circ \Phi$ then $m_1 = m_2 \circ \mathfrak{F}$.

3.5 An example of a nonseparable NBA

There is an abundance of such examples, and we will describe one of the most interesting. There exists an NBA of weight $2^{2^{\aleph_0}}$ including the Lebesgue algebra E_0 as a regular subalgebra and such that the same interval $[0,1]$ serves as its representation space. Moreover, the "translations" defined by the formula

$$\mathfrak{F}_\alpha : \quad \mathfrak{F}_\alpha(x) \equiv x + \alpha \pmod 1$$

induce the metric automorphisms of this NBA. The existence of such an algebra follows from the theorem first proved by S. Kakutani and J. Oxtoby [1] which asserts the existence of nonseparable translation-invariant extensions of Lebesgue measure. The theory of such extensions

---

[17] P. Halmos and J. von Neumann [1]; D. Maharam [3]; and A. A. Samorodnitskiĭ [1].

was independently developed, in particular, by some mathematicians (of the Georgian school: Mkhakadze, Kharazishvili, et al.). Thus, there exists a measure space $\{I, \widetilde{\mathscr{E}}, \widetilde{m}\}$, where $I = [0, 1]$, while $\widetilde{\mathscr{E}}$ contains all Lebesgue measurable sets and $\widetilde{m}$ is a translation-invariant extension of Lebesgue measure of the form $\mathfrak{F}_\alpha$. As was mentioned, this space can be constructed so that the equality $\tau(\widehat{\mathscr{E}}) = 2^{2^{\aleph_0}}$ holds.

It can be shown that the metric structure $\{\widehat{\widetilde{\mathscr{E}}}, \widehat{m}\}$ admits $2^{2^{2^{\aleph_0}}}$ distinct metric isomorphisms. The (cardinal) number of these isomorphisms is essentially larger than that of the automorphisms of an interval, and so the overwhelming majority of them has no pointwise representation.

4. Totally additive functions and resolutions of the identity of a normed algebra

In this section we first derive some integral representation of an arbitrary totally additive function on a normed Boolean algebra. The reader familiar with measure theory will recognize the celebrated Radon–Nikodým Theorem.

4.1 A general form of a totally additive function

Let $\mathscr{X}$ be a complete NBA, and let μ be a measure on $\mathscr{X}$. To each spectral family $\{e_\lambda\}$, we may put into correspondence the family $\{M_x\}_{x \in \mathscr{X}}$ of increasing real functions on the real axis $(-\infty, +\infty)$, by putting

$$M_x(\lambda) \equiv \mu(e_\lambda \wedge x), \quad -\infty < \lambda < +\infty.$$

We refer to a spectral family $\{e_\lambda\}$ as *μ-summable* provided that all Lebesgue–Stieltjes integrals

$$\int_{(-\infty,+\infty)} |\lambda| \, dM_x(\lambda)$$

are finite. For a such family, the equality

$$\varphi(x) \equiv \int_{(-\infty,+\infty)} \lambda \, dM_x(\lambda) \tag{6}$$

defines some real function φ on $\mathscr{X}$. It is not difficult to verify that this function is countably additive and, hence, totally additive. Moreover, the value of the integral (6) is clearly the same for every spectral family coinciding almost everywhere with $\{e_\lambda\}$. We show now that (6) gives a general form of a totally additive function on a normed Boolean algebra.

Theorem 9. *To each totally additive real function φ on a complete NBA $\mathscr{X}$ with measure μ, there exists a μ-summable spectral family $\{e_\lambda\}$ such that* (6) *holds for all $x \in \mathscr{X}$. This spectral family is almost uniquely determined from φ and μ.*[18]

Proof. Given a real number λ, we put

$$P_\lambda = \{x \mid \varphi(x) \le \lambda\mu x\}, \quad Q_\lambda = \{x \mid \varphi(x) > \lambda\mu x\}.$$

The sets P_λ and Q_λ are d-regular. Therefore, by Theorem 2.4 their solid cores are disjoint bands. Hence, putting

$$e_\lambda \equiv \sup(P_\lambda)^0,$$

we see that

$$e_\lambda \in (P_\lambda)^0, \quad Ce_\lambda \in (Q_\lambda)^0, \tag{7}$$

and also $Ce_\lambda = \sup(Q_\lambda)^0$.

Verify that the family $\{e_\lambda\}$ is a spectral function. First of all, we have monotonicity: $e_\lambda \le e_\mu$ for all $\lambda < \mu$. Further, observe that the intersections

$$\bigcap_\lambda P_\lambda, \quad \bigcap_\lambda Q_\lambda$$

cannot contain nonzero elements.[19] At the same time,

$$\bigwedge_\lambda e_\lambda \in \bigcap_\lambda P_\lambda, \quad C\Big(\bigvee_\lambda e_\lambda\Big) \in \bigcap_\lambda Q_\lambda,$$

whence

$$\bigwedge_\lambda e_\lambda = \mathbf{0}, \quad \bigvee_\lambda e_\lambda = \mathbf{1}.$$

Thus, we have really constructed a spectral family. Show that it is a sought family. Given an arbitrary element $x \in \mathscr{X}$, we define the distribution function M_x as shown above. We denote the corresponding Lebesgue–Stieltjes measure by m_x. If α and β are some continuity points of the distribution function then

$$m_x(\alpha, \beta) = m_x[\alpha, \beta] = M_x(\beta) - M_x(\alpha) = \mu[(e_\beta - e_\alpha) \wedge x].$$

Whence we infer the inequality

$$\alpha\mu[(e_\beta - e_\alpha) \wedge x] \le \int\limits_{(\alpha,\beta)} \lambda \, dM_x(\lambda)$$

[18] I.e., up to "almost coincidence" (see p. 288).

[19] Recall that all real functions in this book are finite.

$$= \int\limits_{[\alpha,\beta]} \lambda\, dM_x(\lambda) \leq \beta\mu[(e_\beta - e_\alpha) \wedge x].$$

Let α and β be real numbers of the same sign. Then, from (7) we obtain the estimate

$$\frac{\alpha}{\beta}\varphi[(e_\beta - e_\alpha) \wedge x] \leq \int\limits_{(\alpha,\beta)} \lambda\, dM_x(\lambda)$$

$$= \int\limits_{[\alpha,\beta]} \lambda\, dM_x(\lambda) \leq \frac{\beta}{\alpha}\varphi[(e_\beta - e_\alpha) \wedge x]. \tag{8}$$

Partition the intervals $(0, +\infty)$ and $(-\infty, 0)$ into disjoint intervals by some points λ_k and μ_k ($k = 0, \pm 1, \pm 2, \dots$). Assume that

$$0 < \dots < \lambda_{-1} < \lambda_0 < \lambda_1 < \dots < +\infty,$$

$$-\infty < \dots < \mu_{-1} < \mu_0 < \mu_1 < \dots < 0,$$

and the distribution function M_x is continuous at these points. For brevity, we put

$$e' \equiv \sum_{k=-\infty}^{+\infty} (e_{\lambda_k} - e_{\lambda_{k-1}}), \quad e'' \equiv \sum_{k=-\infty}^{+\infty} (e_{\mu_k} - e_{\mu_{k-1}}),$$

$$x' \equiv x \wedge e', \quad x'' \equiv x \wedge e''.$$

From the trivial equalities

$$e' = \mathbf{1} - \bigwedge_{\lambda>0} e_\lambda, \quad e'' = \bigvee_{\mu<0} e_\mu - \mathbf{0} = \bigvee_{\mu<0} e_\mu$$

it follows that the element $\widetilde{e} \equiv C(e' + e'') \equiv Ce' \wedge Ce''$ belongs to the intersection $\bigcap_{\lambda>0,\mu<0} P_\lambda \cap Q_\mu$; therefore, $\varphi(\widetilde{e} \wedge x) = \mathbf{0}$ and

$$\varphi(x' + x'') = \varphi(x). \tag{9}$$

Then from (8) we infer

$$\begin{aligned}
&\sum_{k=-\infty}^{+\infty} \frac{\lambda_k}{\lambda_{k+1}}\varphi[(e_{\lambda_{k+1}} - e_{\lambda_k}) \wedge x] \\
&\leq \int\limits_{(0,+\infty)} \lambda\, dM_x(\lambda) \lambda \sum_{k=-\infty}^{+\infty} \frac{\lambda_{k+1}}{\lambda_k}\varphi[(e_{\lambda_{k+1}} - e_{\lambda_k}) \wedge x], \\
&\sum_{k=-\infty}^{+\infty} \frac{\mu_k}{\mu_{k+1}}\varphi[(e_{\lambda_{k+1}} - e_{\lambda_k}) \wedge x] \\
&\leq \int\limits_{(-\infty,0)} \lambda\, dM_x(\lambda) \lambda \sum_{k=-\infty}^{+\infty} \frac{\mu_{k+1}}{\mu_k}\varphi[(e_{\lambda_{k+1}} - e_{\lambda_k}) \wedge x].
\end{aligned} \tag{10}$$

It is clear that the ratios $\frac{\lambda_k}{\lambda_{k+1}}, \dots, \frac{\mu_{k+1}}{\mu_k}$ in (10) can be made arbitrarily close to 1 simultaneously.[20] Thus, in fact

$$\int\limits_{(0,+\infty)} \lambda\, dM_x(\lambda) = \sum_{k=-\infty}^{+\infty} \varphi[(e_{\lambda_{k+1}} - e_{\lambda_k}) \wedge x] = \varphi(x'),$$

$$\int\limits_{(-\infty,0)} \lambda\, dM_x(\lambda) = \sum_{k=-\infty}^{+\infty} \varphi[(e_{\lambda_{k+1}} - e_{\lambda_k}) \wedge x] = \varphi(x''),$$

whence, using (9) and taking it into account that

$$\int\limits_{[0,0]} \lambda\, dM_x(\lambda) = 0,$$

we obtain the equality

$$\int\limits_{(-\infty,+\infty)} \lambda\, dM_x(\lambda) = \varphi(x).$$

Furthermore, we see that

$$\int\limits_{(-\infty,+\infty)} |\lambda|\, dM_x(\lambda) = - \int\limits_{(-\infty,0)} \lambda\, dM_x(\lambda)$$
$$+ \int\limits_{(0,+\infty)} \lambda\, dM_x(\lambda) = \varphi(x') - \varphi(x'');$$

hence, the spectral function $\{e_\lambda\}$ is μ-summable.

It remains to prove that the spectral measure $\{e_\lambda\}$ is "almost unique." Assume that, besides the given above spectral function $\{e_\lambda\}$, there exists another family $\{e'_\lambda\}$ that also satisfies the equality

$$\varphi(x) = \int\limits_{(-\infty,+\infty)} \lambda\, dM'_x(\lambda)$$

for all $x \in \mathscr{X}$, where $M'_x(\lambda) = \mu(e'_\lambda \wedge x)$. Assume that $\lambda_1 < \lambda_2$. Take some element $x \leq e'_{\lambda_1}$. Given $\lambda \geq \lambda_1$, we have

$$M'_x(\lambda) = \mu(x \wedge e'_\lambda) = \mu x.$$

[20]Recall that the continuity points of a monotone function are everywhere dense, since the set of discontinuity points is at most countable.

Since the function M'_x is constant for $\lambda > \lambda_1$, we obtain

$$\varphi(x) = \int\limits_{(-\infty,\lambda_1]} \lambda\, dM'_x(\lambda) \leq \lambda_1 M'_x(\lambda_1) = \lambda_1 \mu x.$$

Recalling the definition of $\{e_\lambda\}$, we may conclude that $e'_{\lambda_1} \leq e_\lambda$ for all $\lambda \geq \lambda_1$. In particular,

$$e'_{\lambda_1} \leq e_{\lambda_2}. \tag{11}$$

Now, let $x \leq Ce'_{\lambda_2}$. Given $\lambda \leq \lambda_2$, we have

$$M'_x(\lambda) = \mu(x \wedge e'_\lambda) = 0.$$

As in the preceding case, we then obtain

$$\varphi(x) = \int\limits_{[\lambda_2,+\infty)} \lambda\, dM'_x(\lambda) \geq \lambda_2[M'_x(+\infty) - M'_x(\lambda_2 - 0)] = \lambda_2 \mu x.$$

We see that for all $\lambda < \lambda_2$ and $x \leq Ce'_{\lambda_2}$, $x > \mathbf{0}$, the inequality

$$\varphi(x) > \lambda \mu x$$

holds. In other words, $Ce'_{\lambda_2} \in (Q_\lambda)^0$. Hence, $Ce'_{\lambda_2} \leq Ce_\lambda$ since $Ce_\lambda = \sup(Q_\lambda)^0$. In particular, $Ce_{\lambda_1} \geq Ce'_{\lambda_2}$ whence $e_{\lambda_1} \leq e'_{\lambda_2}$, which together with the inequality (11) means "almost coincidence" of the families $\{e_\lambda\}$ and $\{e'_\lambda\}$. The proof of the theorem is complete.

It is worth observing that the spectral family constructed in the proof of this theorem is left continuous.

Note that to the functions φ_+ and φ_- there correspond the spectral families

$$e_\lambda^{(+)} \equiv \begin{cases} e_\lambda, & \lambda > 0, \\ \mathbf{0}, & \lambda \leq 0, \end{cases}$$

and

$$e_\lambda^{(-)} \equiv \begin{cases} Ce_{-\lambda}, & \lambda > 0, \\ \mathbf{0}, & \lambda \leq 0. \end{cases}$$

Consider a simple example. Let a function φ be defined as $\varphi(x) \equiv \mu(x \wedge u)$, where u is some nonzero element. Then for $\lambda \geq 1$ we have $\varphi(x) = \mu(x \wedge u) \leq \lambda \mu x$ for every x; so in this case $P_\lambda = (P_\lambda)^0 = \mathscr{X}$ and $e_\lambda = \mathbf{1}$. If $0 \leq \lambda < 1$ then $(P_\lambda)^0$ contains only elements disjoint from u; consequently, $e_\lambda = Cu$. At last, for $\lambda < 0$ we have $e_\lambda = \mathbf{0}$. Thus,

$$e_\lambda = \begin{cases} \mathbf{0}, & \lambda < 0, \\ Cu, & 0 \leq \lambda < 1, \\ \mathbf{1}, & \lambda \geq 1. \end{cases}$$

We have obtained a spectral family that was already considered in Chapter 6 (p. 290).

Theorem 9 associates with each totally additive real function φ on $\mathscr{X}$ not a single spectral family but the whole class named the RESOLUTION OF THE IDENTITY; we can use each spectral family of this class in (6). We will consider below these resolutions of the identity in more detail.

4.2 The space of summable resolutions of the identity

Let $\mathfrak{f}$ be some resolution of the identity of a normed algebra $\{\mathscr{X}, \mu\}$. By definition, $\mathfrak{f}$ is the class of almost coincident spectral families. If one of the families is μ-summable then so are the remaining; this gives us a reason to call $\mathfrak{f}$ itself *μ-summable.*

By Theorem 9, there is a one-to-one correspondence between the totally additive functions on a normed algebra and the summable resolutions of the identity. We have noted in Chapter 6 that the resolutions of the identity may be treated as elements of the universally complete K-space $\mathfrak{S}_{\mathscr{X}}$ over algebra $\mathscr{X}$. As for the μ-summable resolutions, they constitute a solid subspace of $\mathfrak{S}_{\mathscr{X}}$; we will denote this subspace by $\mathfrak{L}_{\mathscr{X}}(\mu)$ (or simply by $\mathfrak{L}_{\mathscr{X}}$ or $\mathfrak{L}$).

It is easy to understand that the order on $\mathfrak{S}_{\mathscr{X}}$ we have introduced in Chapter 6 is compatible for μ-summable resolutions with the "natural" order of the set of additive functions: the inequality $\mathfrak{f}_1 \leq \mathfrak{f}_2$ (in the sense of Chapter 6) means that for the corresponding additive functions φ_1 and φ_2 the inequality $\varphi_1(x) \leq \varphi_2(x)$ holds for all $x \in \mathscr{X}$. To the algebraic operations on $\mathfrak{S}_{\mathscr{X}}$ (we did not describe them in detail) there correspond the naturally defined algebraic operations on the class of additive functions as well.[21]

Thus, the correspondence established by Theorem 8 between the naturally ordered linearized system of all additive functions and the K-space $\mathfrak{L}$ is a linear and order isomorphism. It is appropriate to call the resolution of the identity $\mathfrak{f}$ associated to the given totally additive function φ the *density* or *Radon–Nikodým derivative* of φ. We often denote it by $\frac{d\varphi}{d\mu}$.

The value itself of the function

$$\varphi(u) = \int\limits_{(-\infty,+\infty)} \lambda \, dM_u(\lambda)$$

[21] This remark suggests a way of defining the algebraic operations on resolutions of the identity in the case when the main algebra is normed.

is called *the integral of the resolution of the identity* $\mathfrak{f}$ *over the band* $\mathscr{X}_u$. It is convenient even to write

$$\varphi(u) = \int\limits_{\mathscr{X}_u} \mathfrak{f}\, d\mu$$

or

$$\varphi(u) = \int\limits_{u} \mathfrak{f}\, d\mu. \tag{12}$$

For $u = \mathbf{1}$ we agree to denote this integral simply by $\int \mathfrak{f}\, d\mu$ ("the integral over the whole space"). In this case the analogy with the usual integral is more expressive. This analogy is not accidental: Representing $\mathscr{X}$ as a metric structure connected with some measure space, we can indeed convert the expressions of the form (12) into the usual integrals, the bands into measurable sets, and the μ-summable resolutions of the identity into summable functions. The space $\mathfrak{S}$ turns into the well-known space S, and the space $\mathfrak{L}$ turns into the space L of summable functions.

If $\mathscr{X}$ is a probability algebra then we will often call the elements of $\mathfrak{L}_{\mathscr{X}}$ *random variables* which agrees completely with the terminology of probability theory.

Wishing to compute $\varphi(u)$ by the formula (12), we must integrate some of the $\mathfrak{f}$-functions[22] over the measurable set that corresponds to the band $\mathscr{X}_u$. We have meant precisely this when we said that the expressions of the type (12) "turn into" the usual integrals. We have already dealt with these integrals in Chapter 6 (Section 3). The integral of a random variable is also called *expectation*.

We cite some important properties of the integral which are immediate from the definition. The analogous facts are well known in the traditional integration theory. Assume the following notations:

$$M_u(\mathfrak{f}) \equiv \inf\{\lambda \mid \mathfrak{f}\cdot u^\bullet \le \lambda u^\bullet\}, \quad m_u(\mathfrak{f}) \equiv \sup\{\lambda \mid \mathfrak{f}\cdot u^\bullet \ge \lambda u^\bullet\}$$

(recall that $u^\bullet$ is the canonical image of u under the embedding in $\mathfrak{S}_{\mathscr{X}}$). Here $u \in \mathscr{X}$, $\mathfrak{f} \in \mathfrak{S}_{\mathscr{X}}$, and a multiplication has the same meaning as in Section 4 of Chapter 6.

1°. *The functional, defined on* $\mathfrak{L}$ *by the equality* $l(\mathfrak{f}) \equiv \int \mathfrak{f}\, d\mu$, *is linear and essentially positive, i.e., for* $\mathfrak{f} > \mathbf{0}$ *we have* $l(\mathfrak{f}) > 0$.

2°. *The function* φ, *defined by*

$$\varphi(x) \equiv \int\limits_{x} \mathfrak{f}\, d\mu \quad (x \in \mathscr{X},\ \mathfrak{f} \in \mathfrak{L}), \tag{$*$}$$

[22]See p. 303.

is totally additive; for every $u \in \mathscr{X}$ the following inequalities hold:

$$m_u(\mathfrak{f})\mu u \leq \int\limits_u \mathfrak{f}\, d\mu \equiv \varphi(u) \leq M_u(\mathfrak{f})\mu u. \qquad (**)$$

Conversely, every additive function satisfying the inequalities $(**)$ for all $u \in \mathscr{X}$ is representable in the form $(*)$.

3°. *Let $\mathfrak{f} \in \mathfrak{L}_{\mathscr{X}}$, and let g be a real function on $(-\infty, +\infty)$ measurable with respect to the Lebesgue–Stieltjes measure and generated by the distribution*

$$M(\lambda) \equiv \mu e_\lambda^+(\mathfrak{f}).$$

Here $e_\lambda^+(\mathfrak{f}) = \bigwedge_{\mu>\lambda} e_\mu(\mathfrak{f})$ in contrast to the definition of e_λ^+ in this chapter. Let $F \equiv g(\mathfrak{f})$ (the function of $\mathfrak{f}$ in the sense of Chapter 6). If the integral

$$I = \int\limits_{(-\infty,+\infty)} g(\lambda)\, dM(\lambda),$$

is finite then $I = \int F\, d\mu$. In particular, if $\mathfrak{f} \geq \mathbf{0}$ then

$$\int \mathfrak{f}^k\, d\mu = \int\limits_0^{+\infty} \lambda^k\, dM(\lambda) \qquad (k > 0)$$

under the condition that the rightmost integral exists.

4°. *If $\mathfrak{g}, \mathfrak{f} \geq \mathbf{0}$ then*

$$\int \mathfrak{f}\mathfrak{g}\, d\mu = \int\limits_0^{+\infty} dy \int\limits_{Ce_y(\mathfrak{f})} \mathfrak{g}\, d\mu$$

(the D. K. Faddeev formula).

We now spend some time on the properties of $\mathfrak{L}_{\mathscr{X}}$. The latter is not only a K-space but also Banach space under the norm

$$||\mathfrak{f}|| = \int |\mathfrak{f}|\, d\mu.$$

The main properties of this norm are as follows:

1°. *Monotonicity: if $|\mathfrak{f}_1| \leq |\mathfrak{f}_2|$ then $||\mathfrak{f}_1|| \leq ||\mathfrak{f}_2||$.*

2°. *If $\mathfrak{f}_n \downarrow \mathbf{0}$ then $||\mathfrak{f}_n|| \downarrow 0$; if $\mathfrak{f}_n \uparrow +\infty$ then $||\mathfrak{f}_n|| \uparrow +\infty$.*

The K-spaces whose norms satisfy 1° and 2° are called KB-spaces (“Banach–Kantorovich spaces”). Each of them is norm complete and its (o)-topology coincides with the norm topology. Furthermore, each

norm convergent sequence contains a relatively uniformly convergent subsequence. As for the $\mathfrak{L}_{\mathscr{X}}$, its norm has some property specific for this space.

3°. *If* $\mathfrak{f}, \mathfrak{g} \geq \mathbf{0}$ *then* $||\mathfrak{f} + \mathfrak{g}|| = ||\mathfrak{f}|| + ||\mathfrak{g}||$ ("additivity of the norm").

The spaces with such a norm are known as "L-spaces." We have already mentioned that the representation of $\mathscr{X}$ as a metric structure transforms the space $\mathfrak{L}_{\mathscr{X}}$ into the classical space L of summable functions. The above-listed properties of the norm say precisely the same. Alongside the space $\mathfrak{L}_{\mathscr{X}}$ of the utmost interest are its solid subspaces.

First of all, this is the $\mathfrak{L}^p_{\mathscr{X}}(\mu)$ spaces ($p > 1$) consisting of all $\mathfrak{f} \in \mathfrak{L}_{\mathscr{X}}(\mu)$ for which $|\mathfrak{f}|^p \in \mathfrak{L}_{\mathscr{X}}(\mu)$. This is also a KB-space under the norm

$$||\mathfrak{f}||_{\mathfrak{L}^p} \equiv \Big(\int |\mathfrak{f}|^p \, d\mu \Big)^{\frac{1}{p}}.$$

For $p = 2$ we obtain a Hilbert space.

Note the celebrated *Hölder inequalities*

$$\left| \int \mathfrak{f}\mathfrak{g} \, d\mu \right| \leq ||\mathfrak{f}||_{\mathfrak{L}^p} ||\mathfrak{g}||_{\mathfrak{L}^{p'}} \quad \Big(p' = \frac{p}{p-1} \Big).$$

For $p = 2$ we have the *Bunyakovskiĭ inequality*:[23]

$$\left| \int \mathfrak{f}\mathfrak{g} \, d\mu \right| \leq ||\mathfrak{f}||_{\mathfrak{L}^2} ||\mathfrak{g}||_{\mathfrak{L}^2} = \sqrt{\int \mathfrak{f}^2 \, d\mu} \sqrt{\int \mathfrak{g}^2 \, d\mu}.$$

An especial role is played by the "ultimate" space $\mathfrak{L}^\infty_{\mathscr{X}}$. In fact, we are already familiar with it; its norm does not depend on μ and is given by the formula

$$||\mathfrak{f}||_{\mathfrak{L}^\infty} \equiv ||\mathfrak{f}||_b \equiv \inf\{\lambda \mid \; |\mathfrak{f}| \leq \lambda \mathbf{1}\}.$$

This spaces is usually identified with the dual space $(\mathfrak{L}_{\mathscr{X}})^*$. Namely, for each μ, the general form of a continuous linear functional on $\mathfrak{L}_{\mathscr{X}}(\mu)$ is given by the formula

$$l(\mathfrak{f}) = \int \mathfrak{f} \cdot \mathfrak{g}_l \, d\mu,$$

where $\mathfrak{g}_l$ is an element of $\mathfrak{L}^\infty_{\mathscr{X}}$ uniquely determined by l. Hence $l \longleftrightarrow \mathfrak{g}_l$ is a linear and isometric isomorphism.

[23]This inequality is known in the Russian literature as the "Cauchy–Bunyakovskiĭ inequality" while the Western literature mostly uses the term the "Cauchy–Schwarz inequality." (S. S. Kutateladze)

We mention also the Orlicz classes. To each convex positive function on $[0, +\infty)$ with the properties

$$\Phi(0) = 0, \quad \frac{\Phi(u)}{u} \uparrow +\infty \text{ as } u \longrightarrow \infty,$$

there corresponds the class $\mathfrak{L}^{(\Phi)}_{\mathscr{X}}(\mu)$ of the elements of $\mathfrak{L}_{\mathscr{X}}(\mu)$ such that

$$\Phi(|\mathfrak{f}|) \in \mathfrak{L}_{\mathscr{X}}(\mu).$$

In particular, among such classes (called *Orlicz classes*) we encounter all $\mathfrak{L}^p_{\mathscr{X}}(\mu)$ spaces. The role of these classes is explained by the following proposition:

Each element $\mathfrak{f} \in \mathfrak{L}_{\mathscr{X}}(\mu)$ *belongs to some Orlicz class.*

An algebraic approach to the main concepts of integral calculus must be attributed primarily to C. Carathéodory.[24]

The spaces $\mathfrak{L}_{\mathscr{X}}$ and $\mathfrak{L}^p_{\mathscr{X}}$ are listed among K-spaces over $\mathscr{X}$ (see Chapter 5). They are all solid subspaces of the universally complete space $\mathfrak{S}_{\mathscr{X}}$. In our case, when the algebra $\mathscr{X}$ has a measure μ, all these spaces, after making $\mathscr{X}$ the metric structure of the measure space, "turn into" the well-known function spaces. Namely, if we are given a measure space $\{\Omega, \mathscr{E}, m\}$ and the BA $\mathscr{X}$ is identified with $\widehat{\mathscr{E}}$ while the measure μ is identified with $\widehat{m}$ then each K-space $\mathfrak{F}$ over $\mathscr{X}$ is naturally interpreted as the quotient space of some solid subspace F of real $\mathscr{E}$-measurable functions by the ideal of m-negligible functions.[25] For example, taking as F the set of all measurable functions, by factorization we obtain the classical space S or, which is the same, the K-space $\mathfrak{S}_{\mathscr{X}}$. In much the same way, $\mathfrak{L}^p$ can be identified with the usual L^p space, $\mathfrak{L}^\infty$ with L^∞, etc. The choice of the letter $\mathfrak{L}$ reflects only our desire to expose the matter intrinsically in the Boolean terms avoiding representations. The elements $\mathfrak{f} \in \mathfrak{S}_{\mathscr{X}}$ are by nature the resolutions of the identity of the Boolean algebra. If μ is a probability measure then, as was mentioned, it is better to call them random variables.

In conclusion, observe that if Ω is the Stone space ($\Omega = \mathfrak{Q}(\mathscr{X})$) then $\mathfrak{L}^\infty$ is identified with $C(\mathfrak{Q})$. In much the same way, we can treat in this case all resolutions of the identity as continuous functions (possibly, equal to $\pm\infty$ on a rare set depending on a function). These questions have already been discussed in Chapter 6.

[24] C. Carathéodory [1].

[25] Abusing the language, we call the elements of these spaces ("functions *mod* 0") "functions."

4.3 The essential positivity band of a totally additive quasimeasure

Each nonzero totally additive quasimeasure φ on a complete BA $\mathscr{X}$ determines some band $\mathscr{X}_{[\varphi]}$ with the following property: $\varphi(x) > 0$ for all $x \in \mathscr{X}^{+}_{[\varphi]}$. This band is defined as follows: take $u \equiv C \sup\{v \,|\, \varphi(v) = 0\}$. The total additivity implies that $\varphi(u) > 0$. If $0 < x < u$ then $\varphi(x) > 0$ (otherwise the quasimeasure is identically zero on $[0, x]$ and $C \sup\{v \mid \varphi(v) = 0\} \leq u - x < u$). Thus, $\mathscr{X}_u$ is the sought band $\mathscr{X}_{[\varphi]}$. It is called the *essential positivity band* or *carrier* of φ. Even if $\mathscr{X}$ is not normed, the band $\mathscr{X}_{[\varphi]}$ is an NBA: as a measure we may take the restriction of φ to this band.

We now consider the case in which the BA $\mathscr{X}$ is normed and take the form of the metric structure $\{\widehat{\mathscr{E}}, \widehat{m}\}$ of the measure space $\{\Omega, \mathscr{E}, m\}$. Let φ be a countably additive (hence totally additive) quasimeasure on $\mathscr{X}$. Represent it in the integral form, and let $\mathfrak{f}$ be its density. To distinguish the resolution of the identity $\mathfrak{f}$ amounts to that we are given a summable real function f satisfying

$$\varphi(\widehat{e}) = \int\limits_{e} f \, d\mu$$

for all $e \in \widehat{e}$. If we take the set $E = \{t \in \Omega \mid f(t) > 0\}$ then the band $\widehat{\mathscr{E}}_{\widetilde{E}}$ is exactly the essential positive band of φ. We may say that φ is supported in E. If, instead of a quasimeasure φ on $\widehat{\mathscr{E}}$, we consider the function φ' such that

$$\varphi'(e) = \int\limits_{e} f \, dm$$

then for all $e_1 \in \widehat{e}$ (i., e., for all e_1 coinciding m-almost everywhere with e) we have $\varphi'(e_1) = \varphi'(e) = \varphi(\widehat{e})$. The function φ' is a countably additive quasimeasure on $\mathscr{E}$ absolutely continuous with respect to m. The mapping $\varphi \longrightarrow \varphi'$ gives a one-to-one correspondence between countably additive quasimeasures on $\widehat{\mathscr{E}}$ and the countably additive quasimeasures on $\mathscr{E}$ ABSOLUTELY CONTINUOUS with respect to m. We may even identify φ and φ'; in this connection countable additivity on $\widehat{\mathscr{E}}$ is equivalent to absolute continuity with respect to m. The absolute continuity of φ'_1 and φ'_2 with respect to one another amounts exactly to the coincidence of the bands $\widehat{\mathscr{E}}_{[\varphi_1]}$ and $\widehat{\mathscr{E}}_{[\varphi_2]}$.

Closing this section, we return to the question of arcwise connectedness of the sets of the form

$$M_\alpha \equiv \{x \mid \mu x = \alpha\}.$$

The remark on Theorem 4 (p. 321) assumes that μ is a measure. Nevertheless, it remains valid if μ is a totally additive probability quasimeasure. The BA $\mathscr{X}$ is assumed to be normable as before. Indeed, take $x_0, x_1 \in M_\alpha$. These elements may be represented as

$$x_0 \equiv x_0' + x_0'',\ \ x_1 \equiv x_1' + x_1'',\ \ x_0', x_1' \in \mathscr{X}_{[\mu]},\ \ \mu x_0'' = \mu x_1'' = 0.$$

In the algebras $\mathscr{X}_{[\mu]}$ and $\mathscr{X}^d_{[\mu]}$ there exist paths γ' and γ'' joining x_0' and x_1' and x_0'' and x_1'' respectively, where the path γ' lies in M_α. Let γ be the path defined as

$$\gamma(t) \equiv \gamma'(t) + \gamma''(t)$$

and joining x_0 and x_1; all values of $\gamma(t)$ also lie in M_α. This yields the arcwise connectedness of M_α.

5. Subalgebras of a normed Boolean algebra

Each regular subalgebra $\mathscr{X}_0$ of an NBA $\mathscr{X}$ is evidently a normed algebra endowed, for example, with the measure induced from $\mathscr{X}$. A subalgebra $\mathscr{X}_0$, considered by itself, is either continuous, or discrete, or contains both discrete and continuous bands.

5.1 Examples of subalgebras

A good opportunity to demonstrate examples of subalgebras is given by the "metric structures" associated with the simplest measure spaces. We consider now the most important model: take as $\mathscr{X}$ the BA E_0^2 mentioned in Chapter 1 (p. 61) (the algebra of Lebesgue measurable *mod* 0 subsets of the unit square I^2). We usually take as μ Lebesgue measure (more precisely, the measure on E_0^2 that is obtained from Lebesgue measure by factorization).

Before constructing the examples of subalgebras, we make one general remark. Let $\mathscr{Y}$ and $\mathscr{X}$ be two BAs and let Φ be an automorphism from $\mathscr{Y}$ into $\mathscr{X}$. It follows from the definitions of homomorphism and subalgebra that the homomorphic image $\Phi(\mathscr{Y}_0) \equiv \mathscr{X}_0$ of an arbitrary algebra $\mathscr{Y}_0 \subset \mathscr{Y}$ is a subalgebra of $\mathscr{X}$; slightly abusing the language, we may view Φ as an epimorphism from $\mathscr{Y}_0$ onto $\mathscr{X}_0$.

It is easy to understand that each subalgebra of $\mathscr{X}$ is a homomorphic image of some subalgebra $\mathscr{Y}_0 \subset \mathscr{Y}$, which gives a general description of all subalgebras of $\mathscr{X}$. These considerations are applicable in particular in any case when the role of $\mathscr{X}$ is played by a quotient algebra of some BA $\mathscr{Y}$; in this case Φ is the canonical homomorphism associating with each element $y \in \mathscr{Y}$ the coset containing it.

The algebra E_0^2, as each "metric structure," is an instance of a quotient algebra; here the role of $\mathscr{Y}$ is played by the algebra of all Lebesgue

measurable subsets of the unit square I^2. Let us denote this algebra by $\mathscr{E}^{(2)}$ and the canonical homomorphism of $\mathscr{E}^{(2)}$ into E_0^2, by Ψ.

Having described some subalgebra of

$$\mathscr{Y}_0 \subset \mathscr{E}^{(2)}$$

(that is technically simpler), we simultaneously describe some subalgebra

$$\mathscr{X}_0 = \Psi(\mathscr{Y}_0) \subset E_0^2.$$

We give now a few examples.

I. Assume that $\mathscr{Y}_0$ consists of the Lebesgue measurable subsets of the square I^2 which are unions of vertical segments. Clearly, it is a subalgebra. The corresponding subalgebra $\mathscr{X}_0$ is interpreted as the algebra of cosets of equivalent sets; each coset must contain an element of $\mathscr{Y}_0$.

We will call this algebra the "algebra of vertical $mod\,0$ cylinders"; the "algebra of horizontal $mod\,0$ cylinders" is defined similarly. It is easily seen that both algebras are isomorphic to the BA E_0, the algebra of measurable $mod\,0$ subsets of the interval. The reader may easily find this isomorphism.

Fig. 3 is an attempt to show approximately to what sets there correspond the elements of these algebras. (The "bases" of cylinders are arbitrary measurable sets in I^1.)

II. Now, define an algebra $\mathscr{Y}_0$ as follows: first, we enlist as a member of $\mathscr{Y}_0$ the whole triangle lying over the secondary diagonal; next, add to it the sets lying under this diagonal, i.e., the sets representable as the union of vertical segments between the secondary diagonal and the base of the square.

Moreover, the subalgebra must, as usual, contain all possible sums of sets of these two types. As above, we obtain some subalgebra of the

algebra E_0^2 by factorization. This subalgebra is depicted in Fig. 4. It has both continuous and discrete bands. The discrete band consists of a single atom.

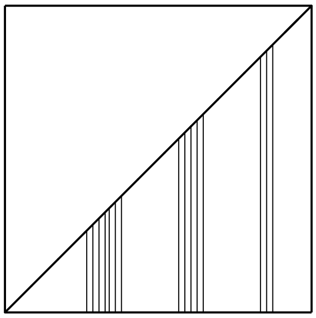

Fig. 4

III. Split now the triangle over the secondary diagonal into countably many disjoint sets of positive measure and enlist each of them in $\mathscr{Y}_0$; the sets lying under the diagonal are the same as in the preceding example. This is depicted in Fig. 5.

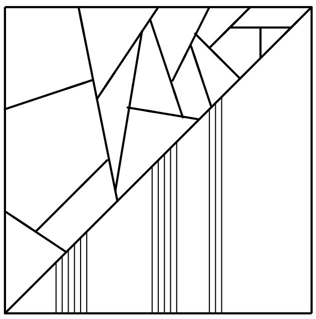

Fig. 5

We again have both continuous and discrete bands. The discrete band is infinite.

These examples are rather typical, and our figures can be associated visually with the word "subalgebra" giving a base for intuitive inferences.

Note a characteristic feature of the subalgebras of I–III: each example rests on some partition of the unit square I^2 into disjoint sets. In the first example there was a partition into vertical (horizontal) segments; in the second, into the triangle and vertical segments; and in the third example this triangle splits additionally into pieces. Every time the generated

subalgebra $\mathscr{Y}_0$ consists of all measurable sets composed of the elements of the partition. Such a situation is typical of separable algebras, and the reason behind this will become clear soon.

5.2 The $\mathfrak{L}_{\mathscr{X}_0}$ space

In Chapter 6 we have already considered vector lattices over a regular subalgebra $\mathscr{X}_0$. It was noted that such a lattice can be naturally treated as the band of the space over the whole algebra which consists of all elements $\mathfrak{f}$ of this space, resolutions of the identity, satisfying the following: $e_\lambda^+(\mathfrak{f}),\ e_\lambda^-(\mathfrak{f}) \in \mathscr{X}_0$. This specifically concerns the $\mathfrak{L}_{\mathscr{X}_0}$ space of summable functions, which in this case is a regular (possibly not solid) subspace of $\mathfrak{L}_{\mathscr{X}}$. Namely, $\mathfrak{L}_{\mathscr{X}_0}$ consists of all resolutions of the identity $\mathfrak{f}$ for which the following conditions

$$e_\lambda^+(\mathfrak{f}),\ e_\lambda^-(\mathfrak{f}) \in \mathscr{X}_0 \quad (-\infty < \lambda < +\infty),$$

$$\int\limits_{(-\infty,+\infty)} |\lambda|\, dM_{\mathfrak{f}}(\lambda) < +\infty$$

are satisfied. So, in the example I, $\mathfrak{L}_{\mathscr{X}_0}$ is the space of summable functions on the square, depending in fact only on the first argument. In just the same way the spaces $\mathfrak{L}^p_{\mathscr{X}_0}$, $\mathfrak{L}^{(\Phi)}_{\mathscr{X}_0}$, etc. are introduced.

5.3 Conditional measure

The Radon–Nikodým Theorem gives the possibility of introducing the concept of conditional measure which is important in probability theory.

We again let $\mathscr{X} \equiv \{\mathscr{X}, \mu\}$ be an NBA, and let $\widetilde{\mathscr{X}}$ be a regular subalgebra of $\mathscr{X}$. Taking an element $u \in \mathscr{X}$ arbitrarily, consider the function φ_u defined on $\widetilde{\mathscr{X}}$ by the following equality

$$\varphi_u(x) \equiv \mu(u \wedge x).$$

By Theorem 9, there exists a spectral family $\{e_\lambda^{(u)}\}$ of the subalgebra $\widetilde{\mathscr{X}}$ such that for every $x \in \widetilde{\mathscr{X}}$ the following equality holds

$$\varphi_u(x) = \int\limits_{(-\infty,+\infty)} \lambda\, dM_x^{(u)}(\lambda) \tag{13}$$

where the distribution function $M_x^{(u)}$ is defined as above by the equality

$$M_x^{(u)}(\lambda) \equiv \mu(x \wedge e_\lambda^{(u)}).$$

We call the family $\{e_\lambda^{(u)}\}$ and the resolution of the identity it generates the *conditional measure* of u with respect to $\widetilde{\mathscr{X}}$. The concept of *conditional measure* implies the OPERATOR that associates to each element $u \in \mathscr{X}$ the corresponding spectral family $\{e_\lambda^{(u)}\}$ or the resolution of identity. Of course, the measure μ is assumed to be distinguished.

We note the most important properties of conditional measure.

1°. *For every $u \in \mathscr{X}$ the obvious equality*

$$\mu u = \int\limits_{(-\infty,+\infty)} \lambda \, dM_{\mathbf{1}}^{(u)}(\lambda) \tag{14}$$

is true. Thus, each $\{e_\lambda^{(u)}\}$ is an element of $\mathfrak{L}_{\widetilde{\mathscr{X}}}$.

2°. *Take $y \in \widetilde{\mathscr{X}}$, and let a real number $\lambda(y)$ satisfy the condition*

$$e_{\lambda(y)+0}^{(u)} - e_{\lambda(y)-0}^{(u)} \geq y. \tag{15}$$

Then, calculating $\varphi_u(y)$ by the formula (13), *we obtain*

$$\mu(u \wedge y) = \lambda(y)\mu y. \tag{15$'$}$$

Of course, only the case when $y > \mathbf{0}$ is of interest. In this case the point $\lambda(y)$ must be a jump discontinuity of the spectral function $\{e_\lambda^{(u)}\}$.

3°. We now consider the case in which the subalgebra $\widetilde{\mathscr{X}}$ is discrete. Let $\{y_1, y_2, \ldots\}$ be the collection of all its atoms. It is clear that to each element y_k there corresponds some $\lambda_k \equiv \lambda(y_k)$ satisfying the condition (15). In this case, the integral equality (14) assumes the form

$$\mu u = \sum_k \lambda_k \mu y_k. \tag{16}$$

4°. Let $u \in \widetilde{\mathscr{X}},\ u > \mathbf{0}$. Then, as we have already seen,

$$e_\lambda^{(u)} = \begin{cases} \mathbf{0}, & \lambda < 0, \\ Cu, & 0 \leq \lambda < 1, \\ \mathbf{1}, & \lambda \geq 1. \end{cases} \tag{17}$$

We emphasize that the conditional measure of an element is not a measure in the exact meaning of the word but rather some spectral function that makes it possible to calculate the measures of all elements of the form $u \wedge x$ with $x \in \widetilde{\mathscr{X}}$. In the simplest case (the properties 2° and 3°), this calculation reduces to multiplication of the measure of an element x by a factor that is a discontinuity point of the spectral function $e_\lambda^{(u)}$

constant on some band. In this (and only this) case we may interpret the number λ as the "conditional probability of one event u occurring given that another event x has happened." The equality (15) expressed the familiar multiplication theorem of the elementary probability theory; and (16) is the celebrated total probability formula. In general case it is replaced by (14). Recalling what was said about the $\mathfrak{L}_{\mathscr{X}}$ space, we can give to this formula a "functional" interpretation using the following simple model.

Let $\mathscr{X}$ be the algebra E_0^2 of Lebesgue measurable subsets of the square $I^2 \equiv [0,1] \times [0,1]$ with usual identifications; let $\widetilde{\mathscr{X}}$ be the subalgebra of $\mathscr{X}$ consisting of all "vertical cylinders" which can be naturally identified with the algebra of measurable subsets of the interval $[0,1]$; and let u be an arbitrary element of $\mathscr{X} \setminus \widetilde{\mathscr{X}}$. Denote by u' one of sets comprising the coset u. To the spectral function $\{e_\lambda^{(u)}\}$ there corresponds the measurable function g_u on $[0,1]$ whose almost all values are equal to the linear Lebesgue measure of the intersection of u' with the corresponding vertical lines. The formula (14) itself is equivalent to the relation

$$\mu u' = \int\limits_{[0,1]} g_u(x)\,dx$$

well-known in integration theory.

The properties of conditional measure depend on the manner in which $\widetilde{\mathscr{X}}$ is embedded into the ambient algebra $\mathscr{X}$. For example, it is easy to prove the following

Theorem 10. *For a regular subalgebra $\widetilde{\mathscr{X}}$ of an NBA $\mathscr{X}$ to saturate the band $\mathscr{X}_{u_0}$ ($u_0 \in \widetilde{\mathscr{X}}$), it is necessary and sufficient that for all $u \in \mathscr{X}_{u_0}$ the conditional measure be defined by the formula* (17).

We pass from the conditional measures (probabilities) to *conditional integrals (conditional expectations)* which, in essence, are extension of the operator of conditional measure from $\mathscr{X}$ onto $\mathfrak{L}_{\mathscr{X}}$. More precisely, the conditional integral is defined as follows. Let $\mathfrak{f} \in \mathfrak{L}_{\mathscr{X}}$ be a resolution of the identity. Consider the totally additive function φ defined on $\mathscr{X}$ by the following equality:

$$\varphi(x) \equiv \int\limits_x \mathfrak{f}\,d\mu \quad (x \in \widetilde{\mathscr{X}}).$$

This function is representable as

$$\varphi(x) \equiv \int\limits_x \mathfrak{f}'\,d\mu,$$

where $\mathfrak{f}'$ is the resolution of the identity in $\widetilde{\mathscr{X}}$ which is uniquely determined from $\mathfrak{f}$ (i.e., $\mathfrak{f}' \in \mathfrak{L}_{\widetilde{\mathscr{X}}}$). It is taken as the value of the operator of conditional integration at the point $\mathfrak{f}$. We denote this operator by $E^{\widetilde{\mathscr{X}}}$. Thus,

$$E^{\widetilde{\mathscr{X}}}(\mathfrak{f}) = \mathfrak{f}'.$$

In other words, the main (defining) property of the operator $E^{\widetilde{\mathscr{X}}}$ from $\mathfrak{L}_{\mathscr{X}}$ into $\mathfrak{L}_{\widetilde{\mathscr{X}}}$ is the following equality

$$\int\limits_x E^{\widetilde{\mathscr{X}}}(\mathfrak{f})\, d\mu = \int\limits_x \mathfrak{f}\, d\mu$$

for all $x \in \widetilde{\mathscr{X}}$. We call the value $E^{\widetilde{\mathscr{X}}}(\mathfrak{f})$ the "conditional integral" ("conditional expectation") of $\mathfrak{f}$. It is again some resolution of the identity. In particular, if the resolution of the identity is defined by the formula (17) then the conditional integral coincides with the conditional measure. We note some other properties of this operator.

1°. *$E^{\widetilde{\mathscr{X}}}$ is a linear operator from $\mathfrak{L}_{\mathscr{X}}(\mu)$ into $\mathfrak{L}_{\mathscr{X}_0}(\mu)$.*

2°. *The operator $E^{\widetilde{\mathscr{X}}}$ is essentially positive: if $\mathfrak{f} > \mathbf{0}$ then $E^{\widetilde{\mathscr{X}}}(\mathfrak{f}) > \mathbf{0}$.*

3°. *It is idempotent: $(E^{\widetilde{\mathscr{X}}})^2 = E^{\widetilde{\mathscr{X}}}$.*

In other words, this operator is a *projection*; the set of its values is $\mathfrak{L}_{\widetilde{\mathscr{X}}}$ which is simultaneously the set of its fixed points. In particular, $E^{\widetilde{\mathscr{X}}}(\mathbf{1}) = \mathbf{1}$.

4°. *For each $p \geq 1$ the operator $E^{\widetilde{\mathscr{X}}}$ acts in $\mathfrak{L}^p_{\mathscr{X}}(\mu)$, and has norm 1. It projects the $\mathfrak{L}^p_{\mathscr{X}}(\mu)$ space onto the subspace $\mathfrak{L}^p_{\widetilde{\mathscr{X}}}(\mu)$. For the Orlicz classes $\mathfrak{L}^{(\Phi)}_{\mathscr{X}}(\mu)$ the situation is analogous. For $p = 2$ (in the Hilbert space $\mathfrak{L}^2_{\mathscr{X}}(\mu)$), $E^{\widetilde{\mathscr{X}}}$ is an orthoprojection.*

5°. *If $\widetilde{\mathscr{X}_1} \subset \widetilde{\mathscr{X}_2}$ then $E^{\widetilde{\mathscr{X}_1}} E^{\widetilde{\mathscr{X}_2}} = E^{\widetilde{\mathscr{X}_2}} E^{\widetilde{\mathscr{X}_1}} = E^{\widetilde{\mathscr{X}_1}}$.*

6°. *If $\mathfrak{g} \in \mathfrak{L}^{\infty}_{\widetilde{\mathscr{X}}}$ then for every $\mathfrak{f} \in \mathfrak{L}_{\widetilde{\mathscr{X}}}$ the equality*

$$E^{\widetilde{\mathscr{X}}}(\mathfrak{g}\mathfrak{f}) = \mathfrak{g} E^{\widetilde{\mathscr{X}}}(\mathfrak{f})$$

holds. We mean here the product of elements of a K-space in the sense of Section 6.4.

The listed properties 1°–3° are characteristic of the operator of conditional integration. Many authors gave various collections of the properties also characterizing conditional integration.[26] The proofs of the above

[26] V. G. Kulakova [1].

facts can be found in many popular books (for example, see J. Neveu [1] and K. Parthasarathy [1]).

5.4 Again on extension of a countably additive function

We discuss once again a question of extension of a countably additive[27] real function from a subalgebra onto the whole algebra. In general case such extension is clearly impossible, but for a normed BA the following easy theorem is true:

Theorem 11. *Let $\{\mathscr{X}, \mu\}$ be an NBA, let $\widetilde{\mathscr{X}}$ be a regular subalgebra in $\mathscr{X}$, and let $\widetilde{\varphi}$ be a countably additive real function defined on $\widetilde{\mathscr{X}}$. Then there exists a countably additive real function φ on $\mathscr{X}$ extending $\widetilde{\varphi}$:*

$$\widetilde{\varphi} = \varphi|_{\widetilde{\mathscr{X}}}.$$

The proof starts with applying the Radon–Nikodým Theorem to the NBA $\{\widetilde{\mathscr{X}}, \widetilde{\mu}\}$ ($\widetilde{\mu} \equiv \mu|_{\widetilde{\mathscr{X}}}$). There exists a resolution of the identity $\mathfrak{f}$ of the BA $\widetilde{\mathscr{X}}$ such that

$$\widetilde{\varphi}(x) = \int_x \mathfrak{f}\, d\widetilde{\mu} \quad (x \in \widetilde{\mathscr{X}}).$$

It is clear that $\mathfrak{f}$ can be considered as a resolution of the identity of the algebra $\mathscr{X}$; therefore, the last integral makes sense for all $x \in \mathscr{X}$, and the equality

$$\varphi(x) = \int_x \mathfrak{f}\, d\mu$$

defines a required function φ.

Remark. If $\widetilde{\varphi}$ is a measure then φ is a measure too. Indeed, each of these conditions means that the density $\mathfrak{f}$ is essentially positive; i.e., $u \cdot \mathfrak{f} > \mathbf{0}$ for all $u \in \mathscr{X}^+$.

Assume now that an additive real function φ_0 is defined on a subalgebra $\mathscr{X}_0$ which is not assumed regular now. By the Lebesgue–Carathéodory Theorem, the countable additivity of φ_0 implies extendibility of φ_0, first onto $\overline{\mathscr{X}_0}$, and then by Theorem 11, onto the whole BA $\mathscr{X}$ (we assume that $\mathscr{X}$ is an NBA). But countable additivity is not always easy

[27]Recall that for the algebras satisfying the countable chain condition (in particular, for normed algebras), the "total" and "countable" additivity properties coincide.

to establish; therefore, we need some simpler conditions for extendibility. We present now a theorem that bases on the ideas originated with Ch. Vallée–Poussin[28] and F. Riesz. Introduce the preliminary notations:

1) $\mathscr{K}_0 = \{Q\}$ is the class of all continuous real functions increasing indefinitely on $[0, +\infty)$ and satisfying $Q(0) = 0$.

2) $\mathscr{K} = \{\Phi\}$ is the class of all functions of the form

$$\Phi : \quad \Phi(u) = uQ(u), \quad Q \in \mathscr{K}_0.$$

Let $\{\mathscr{X}, \mu\}$ be the NBA; let $\mathscr{X}_0$ be a subalgebra in $\mathscr{X}$; let $A \subset \mathscr{X}_0$ be a set such that each $x \in \mathscr{X}_0$ is a finite disjoint sum of some elements of A; and let T be the class of all finite disjoint partitions of unity composed of the elements of A. The following theorem is true:

Theorem 12. *For an additive real function α_0 on $\mathscr{X}_0$ to admit a countably additive extension α onto the whole BA $\mathscr{X}$, it is necessary and sufficient that there exist a function $\Phi \in \mathscr{K}$ having the property*

$$\mathbf{C} \equiv \sup_{\tau \in T} \sum_{e \in \tau} \Phi\Big(\frac{|\alpha_0(e)|}{\mu e}\Big)\, \mu e < +\infty. \tag{18}$$

PROOF. Let a countably additive extension α exist. Then, by the Radon–Nikodým Theorem, there is a summable density $\mathfrak{f}$:

$$\alpha(x) = \int\limits_x \mathfrak{f}\, d\mu.$$

Choose a function $\Phi \in \mathscr{K}$ such that

$$\int \Phi(|\mathfrak{f}|)\, d\mu < +\infty.$$

We may assume that Φ is a convex function.

Now, for $\tau \in T$ we have

$$\sum_{e \in \tau} \Phi\Big(\frac{|\alpha_0(e)|}{\mu e}\Big)\, \mu e \le \sum_{e \in \tau} \Phi\Big(\frac{1}{\mu e} \int\limits_e |\mathfrak{f}|\, d\mu\Big)\, \mu e \le \int \Phi(|\mathfrak{f}|)\, d\mu.$$

(Here we use the classical Jensen inequality.)

Thus,

$$\sup_{\tau \in T} \sum_{e \in \tau} \Phi\Big(\frac{|\alpha_0(e)|}{\mu e}\Big)\, \mu e \le \int \Phi(|\mathfrak{f}|)\, d\mu < +\infty.$$

[28] Ch. Vallée–Poussin [1].

This completes the proof of necessity. We now prove SUFFICIENCY. Let Φ be a function discussed in the formulation of the theorem. We begin with ESTIMATION of the values of α_0. Given $\varepsilon > 0$, we put

$$K_\varepsilon = \inf\left\{t \;\middle|\; \frac{\Phi(t)}{t} \geq \frac{1}{\varepsilon}\right\}.$$

Taking an arbitrary element

$$x = u_1 + u_2 + \dots + u_n \in \mathscr{X}_0 \quad (u_i \in A,\ u_i > \mathbf{0}),$$

represent it as $x = x' + x''$, where

$$x' = \sum_{i:\, \frac{|\alpha_0(u_i)|}{\mu u_i} \geq K_\varepsilon} u_i, \qquad x'' = \sum_{i:\, \frac{|\alpha_0(u_i)|}{\mu u_i} < K_\varepsilon} u_i.$$

For brevity, put

$$Q(u) \equiv \frac{\Phi(u)}{u}, \quad Q(0) \equiv 0.$$

We have

$$|\alpha_0(x')| \leq \sum_{i:\, \frac{|\alpha_0(u_i)|}{\mu u_i} \geq K_\varepsilon} |\alpha_0(u_i)|$$

$$= \sum_{i:\, \frac{|\alpha_0(u_i)|}{\mu u_i} \geq K_\varepsilon} \frac{|\alpha_0(u_i)|}{\mu u_i} Q\left(\frac{|\alpha_0(u_i)|}{\mu u_i}\right) \frac{\mu u_i}{Q\left(\frac{|\alpha_0(u_i)|}{\mu u_i}\right)}$$

$$= \sum_{i:\, \frac{|\alpha_0(u_i)|}{\mu u_i} \geq K_\varepsilon} \Phi\left(\frac{|\alpha_0(u_i)|}{\mu u_i}\right) \frac{\mu u_i}{Q\left(\frac{|\alpha_0(u_i)|}{\mu u_i}\right)} \leq \frac{\mathbf{C}}{Q(K_\varepsilon)} = \mathbf{C}\varepsilon;$$

$$|\alpha_0(x'')| \leq K_\varepsilon \mu x'' \leq K_\varepsilon \mu x.$$

Thus

$$|\alpha_0(x)| \leq K_\varepsilon \mu x + \mathbf{C}\varepsilon \tag{19}$$

for all $\varepsilon > 0$. This inequality implies the two corollaries:

1) All values of α_0 are bounded both from above and from below. Whence, this function has the finite variations α_0^+ and α_0^- satisfying the same inequalities:

$$|\alpha_0^{\pm}(x)| \leq K_\varepsilon \mu x + \mathbf{C}\varepsilon.$$

These variations are quasimeasures whose extensions give an extension of α_0. Hence, we can assume the function α_0 to be positive from the very beginning, and we apply the Lebesgue–Carathéodory Theorem.

2) It is clear from (19) that for every sequence $\{x_k\}_1^\infty$ ($x_k \in \mathscr{X}_0$) the following implications hold:

$$(x_k \downarrow \mathbf{0} \text{ in } \mathscr{X}) \;\rightarrow\; (\mu x_k \longrightarrow 0) \;\rightarrow\; (\alpha_0(x_k) \longrightarrow 0).$$

This yields the countable additivity of α_0 and the possibility of extending this function with preservation of countable additivity, first onto $\overline{\mathscr{X}_0}$ and then, by Theorem 11, onto the whole BA $\mathscr{X}$. The proof of the theorem is complete.[29]

Thus, (18) guarantees extendibility of α_0 onto the whole BA $\mathscr{X}$. Assume, for simplicity, that $\mathscr{X} = \overline{\mathscr{X}_0}$. Denote the countably additive extension whose existence has just been proved by α. By the Radon–Nikodým Theorem, this function is represented as the integral

$$\alpha(x) = \int\limits_x \mathfrak{f}\, d\mu = \int\limits_{(-\infty,+\infty)} \lambda\, de_\lambda(\mathfrak{f}), \quad \mathfrak{f} \in \mathfrak{L}.$$

What can we say under these conditions about the density $\mathfrak{f}$? We will assume that $\alpha = \alpha^+$ implying that $\mathfrak{f} \geq \mathbf{0}$.

Given an arbitrary $\varepsilon > 0$, choose the numbers λ_n ($0 < \lambda_0 < \lambda_1 < \dots,\ \lambda_n \uparrow +\infty$) such that for all $n = 1, 2, \dots$ we have

$$\Phi(\lambda_n) - \Phi(\lambda_{n-1}) \leq \varepsilon.$$

Put

$$u_n \equiv e_{\lambda_n}(\mathfrak{f}) - e_{\lambda_{n-1}}(\mathfrak{f}) \quad (n = 1, 2, \dots),$$

$$s_\varepsilon \equiv \sum_1^\infty \Phi(\lambda_{n-1})\mu u_n, \quad S_\varepsilon \equiv \sum_1^\infty \Phi(\lambda_n)\mu u_n,$$

$$\sigma_\varepsilon \equiv \sum_1^\infty \Phi\left(\frac{\alpha(u_n)}{\mu u_n}\right)\mu u_n = \sum_1^\infty \Phi\left(\frac{\int\limits_{u_n} \mathfrak{f}\, d\mu}{\mu u_n}\right)\mu u_n.$$

Clearly,

$$s_\varepsilon \leq \sigma_\varepsilon \leq S_\varepsilon \leq s_\varepsilon + \varepsilon \leq \sigma_\varepsilon + \varepsilon \leq \mathbf{C} + \varepsilon.$$

It follows from these equalities that

$$\int\limits_{(-\infty,+\infty)} \Phi(\lambda)\, de_\lambda(\mathfrak{f}) = \int\limits_{(0,+\infty)} \Phi(\lambda)\, de_\lambda(\mathfrak{f}) = \int \Phi(\mathfrak{f})\, d\mu \leq \mathbf{C}.$$

[29] It is clear from the proof that the sufficiency part of this theorem is true for μ a totally additive function essentially positive on $\mathscr{X}_0$. In this event, naturally, there will be $\alpha(x) = 0$ for $\mu x = 0$.

This means that $\Phi(\mathfrak{f}) \in \mathfrak{L}$; in other words, $\mathfrak{f}$ belongs to the ORLICZ CLASS generated by Φ (as above, we can assume Φ convex). In particular, the relation

$$\sup_{\tau} \sum_{e \in \tau} \frac{[\alpha_0(e)]^p}{\mu e^{p-1}} \equiv \mathbf{C} < +\infty \tag{20}$$

means that $\mathfrak{f} \in \mathfrak{L}^p \quad (p > 1)$.

It is easy to verify that, conversely, the membership of $\mathfrak{f}$ in an Orlicz class (the $\mathfrak{L}^p$ class) implies (18) (respectively, (20)). Recall that each density $\mathfrak{f} \in \mathfrak{L}$ belongs to some Orlicz class (but, not necessarily to $\mathfrak{L}^p$).

5.5 Martingales

Let, as usual, $\{\mathscr{X}, \mu\}$ be an NBA with μ a probability measure.

Consider the following situation: let $\{\mathfrak{f}_n\}_1^\infty$ and $\{\mathscr{X}_n\}_1^\infty$ be two sequences, where $\mathfrak{f}_n \in \mathfrak{L}_{\mathscr{X}}(\mu) \equiv \mathfrak{L}_{\mathscr{X}}$ and $\mathscr{X}_n$ are regular subalgebras in $\mathscr{X}$ such that $\mathscr{X}_1 \subset \mathscr{X}_2 \subset \dots,\ \mathfrak{f}_n = E^{\mathscr{X}_n}(\mathfrak{f}_{n+1})\ (n = 1, 2, \dots)$. In such cases we say that $\mathfrak{f}_1, \mathfrak{f}_2, \dots$ form a *martingale*. Assume that there exists a function $\Phi \in \mathscr{K}$ satisfying

$$\mathbf{C} \equiv \sup \int \Phi(|\mathfrak{f}_n|)\, d\mu < +\infty. \tag{21}$$

In this case, as we will see, the sequence $\{\mathfrak{f}_n\}$ has a limit. We assume that the union $\bigcup \mathscr{X}_n$ is dense in $\mathscr{X}$. Otherwise, we will consider the corresponding regular subalgebra $\overline{\mathscr{X}\langle \bigcup \mathscr{X}_n \rangle}$ instead of $\mathscr{X}$.

1°. Under our conditions, the equalities

$$\int\limits_x \mathfrak{f}_n\, d\mu = \int\limits_x \mathfrak{f}_{n+1}\, d\mu = \int\limits_x \mathfrak{f}_{n+2}\, d\mu = \dots$$

hold for all $x \in \mathscr{X}_n$. Assuming

$$\varphi_0(x) \equiv \lim \int\limits_x \mathfrak{f}_n\, d\mu,$$

we obtain an additive function on the subalgebra $\mathscr{X}' \equiv \bigcup_{n=1}^{+\infty} \mathscr{X}_n$. Consider a resolution of the identity $\tau = \{e_k\}_1^{\mathscr{N}}, e_k \in \mathscr{X}'\ (k = 1, 2, \dots, \mathscr{N})$ and estimate the sum

$$S \equiv \sum_1^{\mathscr{N}} \Phi\Big(\frac{\varphi_0(e_k)}{\mu e_k}\Big)\, \mu e_k.$$

Let m be a number such that all $e_1, e_2, \ldots, e_{\mathscr{N}}$ lie in $\mathscr{X}_m$. Given $n_0 > m$, we can estimate the sum S as follows:

$$S = \sum_{k=1}^{\mathscr{N}} \Phi\left(\frac{\int\limits_{e_k} |\mathfrak{f}_{n_0}|\, d\mu}{\mu e_k}\right) \mu e_k \leq \int \Phi(|\mathfrak{f}_{n_0}|\, d\mu \leq \mathbf{C}$$

(like in the proof of Theorem 12). This means that the function φ_0 is extendible to a totally additive function φ. Let $\mathfrak{f}$ be the density of this function. We have already noted that $\mathfrak{f} \in \mathfrak{L}^{(\Phi)}$.

2°. Now, we show that $\mathfrak{f}_n = E^{\mathscr{X}_n}(\mathfrak{f})$ for all $n = 1, 2, \ldots$. This follows from the definition of $\mathfrak{f}$:

$$\int\limits_x \mathfrak{f}\, d\mu = \int\limits_x \mathfrak{f}_n\, d\mu$$

for all $x \in \mathscr{X}_n$. These equalities characterize, as it is well known, the conditional integrals $E^{\mathscr{X}_n}(\mathfrak{f})$.

3°. At last, establish the convergence of $\mathfrak{f}_n$ to $\mathfrak{f}$ with respect to the metric of $\mathfrak{L}_{\mathscr{X}}(\mu)$. First of all, note that for $\mathfrak{f} \in \mathfrak{L}^2_{\mathscr{X}}(\mu)$ the element $\mathfrak{f}_n = E^{\mathscr{X}_n}(\mathfrak{f})$ (as the orthogonal projection) is the point of $\mathfrak{L}^2_{\mathscr{X}_n}(\mu)$ nearest to $\mathfrak{f}$. From the Cauchy–Bunyakovskiĭ inequality we obtain

$$\|\mathfrak{f} - \mathfrak{f}_n\| = \|\mathfrak{f} - E^{\mathscr{X}_n}(\mathfrak{f})\| \leq \|\mathfrak{f} - E^{\mathscr{X}_n}(\mathfrak{f})\|_{\mathfrak{L}^2_{\mathscr{X}}}.$$

The union of the subalgebras $\bigcup \mathscr{X}_n$ is dense in $\mathscr{X}$; hence, the union of the corresponding subspaces $\mathfrak{L}^2_{\mathscr{X}_n}$ is dense in $\mathfrak{L}^2_{\mathscr{X}}$ and

$$\|\mathfrak{f} - E^{\mathscr{X}_n}(\mathfrak{f})\|_{\mathfrak{L}^2_{\mathscr{X}}} \downarrow 0.$$

This implies that $\mathfrak{f}_n$ converges to $\mathfrak{f}$ in $\mathfrak{L}_{\mathscr{X}}$ (even in $\mathfrak{L}^2_{\mathscr{X}}$).

Now, take $\mathfrak{f} \in \mathfrak{L}_{\mathscr{X}} \setminus \mathfrak{L}^2_{\mathscr{X}}$, $\mathfrak{f} \geq \mathbf{0}$. Put

$$\mathfrak{f}_n^{(\mathscr{N})} \equiv \mathfrak{f}_n \cdot e_{\mathscr{N}}(\mathfrak{f}_n),\ \ \mathfrak{f}^{(\mathscr{N})} \equiv \mathfrak{f} \cdot e_{\mathscr{N}}(\mathfrak{f}),\ \ \mathfrak{g}_n^{(\mathscr{N})} \equiv \mathfrak{f}_n - \mathfrak{f}_n^{(\mathscr{N})},\ \ \mathfrak{g}^{(\mathscr{N})} \equiv \mathfrak{f} - \mathfrak{f}^{(\mathscr{N})}.$$

Since $\mathfrak{f} \geq \mathbf{0}$, we have $\mathfrak{f}_n \geq \mathbf{0}$ and $\mathfrak{f}_n^{(\mathscr{N})} \leq \mathscr{N} e_{\mathscr{N}}(\mathfrak{f}_n)$.

Given arbitrary $\varepsilon > 0$, choose $\mathscr{N}$ so as

$$Q(\mathscr{N}) > \frac{1}{\varepsilon}.$$

(As in Theorem 12, $Q(u) \equiv \frac{\Phi(u)}{u} \uparrow +\infty$.) Next, we have

$$\int \mathfrak{g}_n^{(\mathscr{N})}\, d\mu = \int\limits_{Ce_{\mathscr{N}}(\mathfrak{f})} \mathfrak{g}_n^{(\mathscr{N})}\, d\mu$$

$$= \int\limits_{Ce_{\mathscr{N}}(\mathfrak{f})} \frac{\mathfrak{g}^{(\mathscr{N})} \cdot Q(\mathfrak{g}_n^{(\mathscr{N})})}{Q(\mathfrak{g}_n^{(\mathscr{N})})} \, d\mu \leq \varepsilon \int \Phi(\mathfrak{f}_n) \, d\mu \leq \mathbf{C}\varepsilon$$

for all $n = 1, 2, \ldots$; similarly,

$$\int \mathfrak{g}^{(\mathscr{N})} \, d\mu \leq \mathbf{C}\varepsilon.$$

By what was proved above, $\mathfrak{f}_n^{(\mathscr{N})} \longrightarrow \mathfrak{f}^{(\mathscr{N})}$ (they are elements of $\mathfrak{L}^2_{\mathscr{X}}$). Furthermore,

$$\|\mathfrak{f} - \mathfrak{f}_n\| \leq \|\mathfrak{f}^{(\mathscr{N})} - \mathfrak{f}_n^{(\mathscr{N})}\| + \|\mathfrak{f}_n^{(\mathscr{N})} - \mathfrak{f}^{(\mathscr{N})}\| + \|\mathfrak{f}^{(\mathscr{N})} - \mathfrak{f}\|$$

$$= \int \mathfrak{g}_n^{(\mathscr{N})} \, d\mu + \|\mathfrak{f}_n^{(\mathscr{N})} - \mathfrak{f}^{(\mathscr{N})}\| + \int \mathfrak{g} \, d\mu = 2\mathbf{C}\varepsilon + \|\mathfrak{f}_n^{(\mathscr{N})} - \mathfrak{f}^{(\mathscr{N})}\|.$$

For n sufficiently large, the second term is less than ε and

$$\|\mathfrak{f} - \mathfrak{f}_n\| \leq (2\mathbf{C} + 1)\varepsilon.$$

The assumption $\mathfrak{f} \geq \mathbf{0}$ is inessential, since we can always represent $\mathfrak{f}$ as $\mathfrak{f}_+ - \mathfrak{f}_-$. Summarizing this reasoning, we formulate "J. Doob's Martingale Theorem":

Theorem 13. *Let $\{\mathscr{X}, \mu\}$ be an NBA with probability measure, let $\mathscr{X}_1 \subset \mathscr{X}_2 \subset \ldots$ be an increasing sequence of regular subalgebras, and let $\widetilde{\mathscr{X}} \equiv \overline{\mathscr{X}\langle \bigcup \mathscr{X}_n \rangle}$. If $\mathfrak{f}_n \in \mathfrak{L}_{\mathscr{X}_n}$ and $\mathfrak{f}_n$ constitute a martingale and if for some function $\Phi \in \mathscr{K}$ the condition* (21) *holds, then there exists an element $\mathfrak{f} \in \mathfrak{L}_{\widetilde{\mathscr{X}}}$ such that*

1) *$\mathfrak{f}_n \longrightarrow \mathfrak{f}$ in the metric of $\mathfrak{L}_{\widetilde{\mathscr{X}}}$* (or, *which is the same, in the metric of $\mathfrak{L}_{\mathscr{X}}$*);

2) *$\mathfrak{f}_n = E^{\mathscr{X}_n}(\mathfrak{f})$ for all $n = 1, 2, \ldots$;*

3) *$\mathfrak{f} \in \mathfrak{L}^{(\Phi)}_{\widetilde{\mathscr{X}}}$.*

Remark 1. *The existence of a function Φ with the property mentioned in the theorem is not only sufficient but also necessary for the validity of the conclusion of the theorem.*

Remark 2. *If $\mathscr{X}$ is a KB-space, in particular, $\mathfrak{L}_{\mathscr{X}}$; then each norm convergent sequence contains an* (o)*-convergent subsequence. This concerns also our case.* (It is sufficient to choose $n_k \uparrow +\infty$ so that $\sum \|\mathfrak{f} - \mathfrak{f}_{n_k}\| < +\infty$.) Moreover, it is well known that only under the sole condition of boundedness of the norms $\mathfrak{f}_n$ in $\mathfrak{L}$ (and, all the more,

under the conditions of Theorem 13) the (o)-convergence of the martingale $\{\mathfrak{f}_n\}$ takes place in the space $\mathfrak{S}_{\mathscr{X}}$ wider than $\mathfrak{L}_{\mathscr{X}}$. In this case the limit random variable $\mathfrak{f}$ belongs to $\mathfrak{L}_{\mathscr{X}}$, but convergence in $\mathfrak{L}_{\mathscr{X}}$ may fail. Such convergence is exactly equivalent to the conditions of Theorem 13, or equivalently to the condition $\|\mathfrak{f}_n\| \longrightarrow \|\mathfrak{f}\|$.

Martingale theory, which we touch slightly, is an established section of probability theory. The reader can find a more detailed presentation in the special literature.[30] We rivet the reader's attention on a very general theorem published in 1937 by L. V. Kantorovich. This theorem deals with the question of convergence of linear operators in a partially ordered space but applies in particular to what was later named "martingale." It follows from this theorem that for $\mathfrak{f} \in \mathfrak{L}^p$ $(p > 1)$ the martingale (o)-converges in $\mathfrak{L}^p$. L. V. Kantorovich also constructed an example which show that, under conditions of our last theorem, the (o)-convergence of the whole sequence $\mathfrak{f}_n$ in $\mathfrak{L}$ may fail. All these results are gathered in the book of L. V. Kantorovich, B. Z. Vulikh, and A. G. Pinsker [1, Chapter 11, § 1 and § 3].

5.6 Measurable Lebesgue–Rokhlin partitions

We once again consider a Lebesgue–Rokhlin space $\{\Omega, \mathscr{E}, m\}$. Let $\mathscr{E}_0$ be a countable compact base for it. Let ξ be a partition of Ω. This means that ξ is a set whose elements are measurable disjoint sets of Ω and, moreover,

$$\Omega = \bigcup_{z \in \xi} z.$$

A partition ξ is called *measurable* if $\mathscr{E}$ includes a countable subalgebra $\mathscr{A}_0$ that generates ξ in the sense of 1.1.3. The sets $a \in \mathscr{A}_0$ are ξ-saturated; i.e., they consist of the whole elements of the partition ξ.

Denote by $\mathscr{A}$ the σ-algebra generated by $\mathscr{A}_0$; it consists of ξ-saturated sets too. Finally, let $\mathscr{E}_\xi$ be the family of all measurable ξ-saturated sets. Clearly, $\mathscr{E}_\xi$ is a σ-algebra. It is uniquely defined by ξ (which is not so for $\mathscr{A}_0$ and $\mathscr{A}$).

Dealing with partitions, we as usual introduce the "natural projection" $\pi \equiv \pi_\xi$ that associates with each point $\omega \in \Omega$ the element ξ containing this point. The family of all subsets

$$\widehat{\mathscr{E}} \equiv \widehat{\mathscr{E}}_\xi \equiv \{E \subset \xi \mid \pi^{-1}(E) \in \mathscr{E}_\xi\} \equiv \{E \subset \xi \mid \pi^{-1}(E) \in \mathscr{E}\}$$

[30] See, for example, J. Doob [1], J. Neveu [1], P. A. Meyer [1], R. De Marr [1], and R. Sh. Liptser and A. N. Shiryaev [1].

is a σ-algebra of sets in ξ. If we define the "measure" $\widehat{m} \equiv \widehat{m}_\xi$ by the equality

$$\widehat{m}(E) \equiv \widehat{m}_\xi(E) \equiv m\pi^{-1}(E)$$

then we come to the measure space $\{\xi, \widehat{\mathscr{E}}_\xi, \widehat{m}_\xi\}$ which, as we can show, is a Lebesgue–Rokhlin space. It is called the *quotient space* of the initial space. (But we do not call $\widehat{\mathscr{E}}$ the "quotient algebra"; the mapping $E \longrightarrow \pi^{-1}(E)$ embeds $\widehat{\mathscr{E}}$ in $\mathscr{E}$ as a subalgebra but not as a quotient algebra.)

So we let ξ be a measurable partition, and let $\mathscr{A}_0$ be a subalgebra that was mentioned in the definition, the base of the partition. This algebra is countable; hence, it is easy to construct a countable system $\{\tau_n\}$ of finite partitions of the space Ω so that $\tau_1 \prec \tau_2 \prec \dots$ and each τ_n has the form $\tau_n = \{e_n^k\}_{k=1}^{k_n}$, where $e_n^k \in \mathscr{A}_0$, $me_n^k > 0$; moreover, each point $z \in \xi$ is uniquely representable in the form

$$z = \bigcap_{n=1}^{\infty} e_n^{k_n}.$$

Taking $e \in \mathscr{E}$ arbitrarily, we associate with this set the sequence of summable functions $\{f_n^e\}$ defined by the following formulas:

$$f_n^e(\omega) \equiv \sum_{k=1}^{k_n} \frac{m(e \cap e_n^k)}{me_n^k} \chi_{e_n^k}(\omega). \tag{22}$$

(If $me_n^k = 0$ then the corresponding term is equal to zero.) Denoting by $\mathfrak{f}_n^e$ the corresponding elements of the $\mathfrak{L}_{\widehat{\mathscr{E}}}$ space ("the functions *mod* 0"),[31] we easily verify that these elements constitute a martingale; moreover, $\mathbf{0} \le \mathfrak{f}_n^e \le \mathbf{1}$. We are obviously in the conditions of Theorem 13, and hence there exists an element $\mathfrak{f} \in \mathfrak{L}_{\widehat{\mathscr{E}}}$ to which $\mathfrak{f}_n^e$ converges in the norm of $\mathfrak{L}_{\widehat{\mathscr{E}}}$. Moreover, as was observed above, passing to a subsequence, we can even obtain the (o)-convergence in $\mathfrak{L}_{\widehat{\mathscr{E}}}$. In the language of the Lebesgue–Rokhlin space $\{\Omega, \mathscr{E}, m\}$, this means that some subsequence $\{f_{n_k}^e\}$ converges to a summable function f m-almost everywhere on Ω (and it has even a summable majorant). We will assume that $n_k = k$. Remove from Ω the m-negligible set on which $f_n^e(\omega) \not\longrightarrow f(\omega)$. We observe convergence at all of the REMAINING points of Ω_e. Take $\omega \in \Omega_e$. Then we have

$$f(\omega) = \lim f_n^e(\omega) = \lim \frac{m(e \cap e_n^{k_n})}{me^{k_n}},$$

[31]More exactly, the resolutions of the identity of the BA $\widehat{\mathscr{E}}$.

where $e_n^{k_n} \equiv e_n^{k_n}(\omega)$ is the unique element of the partition τ_n which contains the point ω, or equivalently, that contains $\pi(\omega)$.

In constructing the sets e_n^k, we can ensure that the algebra generated by these sets coincides with $\mathscr{A}_0$ and hence the generated σ-algebra coincides with $\mathscr{A}$; moreover, the same Theorem 13 implies the equalities

$$\int\limits_{e_n^k} f\,dm = \int\limits_{e_n^k} f_n^e\,dm = \frac{m(e \cap e_n^k)}{me_n^k} \cdot me_n^k = m(e \cap e_n^k)$$

valid for all $n = 1, 2, \dots$ and $k = 1, 2, \dots, k_n$. The same remains true after replacing e_n^k with an arbitrary set in the algebra $\mathscr{A}_0$, and hence with $a \in \mathscr{A}$:

$$\int\limits_a f\,dm = m(e \cap a).$$

In particular,

$$\int\limits_\Omega f\,dm = me.$$

It is clear that the set Ω_e belongs to $\mathscr{A}$ and the function f is $\mathscr{A}$-measurable. In fact, f is the conditional measure of e with respect to $\mathscr{A}$. We further denote it by f_e. The base $\mathscr{E}_0$ of our space is countable; therefore, the set

$$\Omega_0 = \bigcap_{e \in \mathscr{E}_0} \Omega_e$$

belongs to $\mathscr{A}$ and $m\Omega_0 = 1$. The above construction associates with each set $e \in \mathscr{E}_0$ the function f that is the pointwise limit on the set Ω_0 (which is common for all sets) of the corresponding functions of the form (22). We put $f_e(\omega) = 0$ outside Ω_0. For each $\omega \in \Omega_0$, the set function

$$p_\omega: \quad p_\omega(e) \equiv f_e(\omega)$$

is positive and additive, i.e., it is a quasimeasure on the algebra $\mathscr{E}_0$. The compactness of $\mathscr{E}_0$ provides extendibility of all quasimeasures of this family $\{p_\omega\}$ to the σ-algebra $\mathscr{E}^\sigma$ generated by $\mathscr{E}_0$. (Each quasimeasure p_ω extends further to a "complete measure" but the corresponding σ-algebras wider than $\mathscr{E}^\sigma$ depend on ω.)

As a preliminary, we summarize this in the following

Proposition 1. *Let ξ be a measurable partition of the Lebesgue–Rokhlin space $\{\Omega, \mathscr{E}, m\}$, let $\mathscr{E}_0$ be a countable compact base of this space, let $\mathscr{A}_0$ be a countable base of the partition ξ, and let $\mathscr{E}^\sigma$ and $\mathscr{A}$ be the σ-algebras generated by $\mathscr{E}_0$ and $\mathscr{A}_0$ respectively. Under these conditions there exist*

1) *a set* $\Omega_0 \in \mathscr{A}$, *with* $m\Omega_0 = 1$;

2) *a family of countably additive quasimeasures* $\{p_\omega\}_{\omega\in\Omega_0}$ *on* $\mathscr{E}^\sigma$ *such that,*

> *for each* $e \in \mathscr{E}^\sigma$, *the function* f_e *defined on* Ω_0 *by the equality* $f_e(\omega) \equiv p_\omega(e)$ *and equal to zero outside* Ω_0 *is* $\mathscr{A}$*-measurable and has the property*
> $$\int\limits_a f_e \, dm = m(e \cap a)$$
> *for all* $a \in \mathscr{A}$.

The $\mathscr{A}$-measurability of the function f_e implies in particular that f_e is constant on the elements of the partition ξ: if $\pi(\omega_1) = \pi(\omega_2)$ then $f_e(\omega_1) = f_e(\omega_2)$.

It is desirable to proceed further and connect each quasimeasure p_ω not with the general algebra $\mathscr{E}^\sigma$ but with the algebra of SUBSETS of the corresponding ELEMENT of the partition $z = \pi(\omega)$. To this end we (temporarily) introduce an extra assumption and prove the following

Proposition 2. *Let the conditions of Proposition* 1 *be satisfied, and assume that a countable compact base* $\mathscr{E}_0$ *can be chosen so that it includes* $\mathscr{A}_0$. *Then each quasimeasure* p_ω *is supported in the subsets of the element* $\pi(\omega)$. *More precisely, in this case every set* $\pi(\omega)$ *belongs to* $\mathscr{E}^\sigma$ *and* $p_\omega(\pi(\omega)) = 1$ *for all* $\omega \in \Omega_0$. *The quasimeasure* p_ω *may be other than zero only at the sets* $e \subset \pi(\omega)$, *with* $e \in \mathscr{E}^\sigma$, *comprising a* σ*-algebra of subsets of* $\pi(\omega)$.

Indeed, arbitrarily taking $e \in \mathscr{E}^\sigma$ disjoint from some $z \equiv \pi(\omega)$ ($\omega \in \Omega_0$), choose a sequence $a_n \equiv e_n^k(\omega) \in \mathscr{A}_0$ whose intersection is exactly z. Then

$$Ca_n \cap e \uparrow e, \quad Ca_n, Ca_n \cap e \in \mathscr{E}^\sigma, \quad p_\omega(Ca_n) = p_\omega(Ca_n \cap e) = 0,$$

and $p_\omega(e) = \lim p_\omega(Ca_n \cap e) = 0$. Moreover, $z \in \mathscr{E}^\sigma$ as the intersection of the countable sequence $\{a_n\}$, where $a_n \in \mathscr{A}_0 \subset \mathscr{E}^\sigma$.

This proof implies that, under the conditions of Proposition 2, for each $\omega \in \Omega_0$ the equality

$$p_\omega(e) = p_\omega(e \cap \pi(\omega))$$

holds for all $e \in \mathscr{E}^\sigma$, and also $p_\omega(\Omega) = p_\omega(\pi(\omega)) = 1$.

The family of quasimeasures $\{p_\omega\}$ plays a significant role in the theory of Lebesgue–Rokhlin spaces; however, this rests on the assumption $\mathscr{A}_0 \subset \mathscr{E}_0$. This assumption is very stringent and we now eliminate it. Denote

by $\mathscr{E}_1$ the algebra generated by the union $\mathscr{E}_0 \cup \mathscr{A}_0$. It is countable but not necessarily compact, although it includes the countable algebra $\mathscr{E}_0$.

Construct the canonical $\mathscr{E}_1$-extension $\overline{\Omega}$ of the space Ω. Recall that $\overline{\Omega}$ is the Stone space of $\mathscr{E}_1$ whose points are interpreted as ultrafilters of this algebra. For simplicity, we identify the trivial ultrafilter generated by a point $\omega \in \Omega$, i.e., the set

$$\{e \in \mathscr{E}_1 \mid \omega \in e\},$$

with the same point. Thus Ω is considered as a part of $\overline{\Omega}$.[32] Associating with each $e \in \mathscr{E}_1$ its closure $\overline{e}$, we obtain the algebra of all clopen subsets of $\overline{\Omega}$ which we denote by $\overline{\mathscr{E}_1}$. The correspondence $e \leftrightarrow \overline{e}$ between the sets of $\mathscr{E}_1$ and their closures is, as was noted in Chapter 3, an isomorphism between $\mathscr{E}_1$ and $\overline{\mathscr{E}_1}$. Denote by $\overline{m}$ the image of the quasimeasure $m|_{\mathscr{E}_1}$ under this isomorphism: $\overline{m}(\overline{e}) \equiv m(e)$. The algebra $\overline{\mathscr{E}_1}$ is always compact; hence, $\overline{m}$ is extendible in a standard way to a complete countably additive "measure" which we denote as above by $\overline{m}$. It is defined on some σ-algebra $\overline{\mathscr{E}}$; while $\{\overline{\Omega}, \overline{\mathscr{E}}, \overline{m}\}$ is obviously a Lebesgue–Rokhlin space, and the algebra $\overline{\mathscr{E}_1}$ is a countable compact base of this space.

Existence in $\mathscr{E}_1$ of a compact subalgebra $\mathscr{E}_0$ makes it possible, as we have seen, to construct a retraction $r : \overline{\Omega} \longrightarrow \Omega$. Now, the following is important: the preimage $r^{-1}(e)$ of a set $e \in \mathscr{E}_0$ coincides with the closure $\overline{e}$. Furthermore, it is easy to prove that

$$\overline{\Omega} \setminus \Omega = \bigcup_{e \in \mathscr{E}_1} (\overline{e} \setminus r^{-1}(e)).$$

If we prove that all terms on the right-hand side are $\overline{\mathscr{E}}$-measurable then this will imply the following: first, they are all $\overline{m}$-negligible (because $\overline{m}\overline{\Omega} = m\Omega = 1$ and the inner measure of $\overline{\Omega} \setminus \Omega$ is zero); second, $\overline{\Omega} \setminus \Omega \in \overline{\mathscr{E}}$ and $\overline{m}(\overline{\Omega} \setminus \Omega) = 0$ (the sum is countable); and, third, Ω itself is measurable.

As for the measurability of differences of the type $\overline{e} \setminus r^{-1}(e)$ $(e \in \mathscr{E}_1)$, it is a consequence of the following arguments. For $e \in \mathscr{E}_0$, as was said, we have $r^{-1}(e) = \overline{e}$ and $me = \overline{m}\overline{e}$. Hence, the mapping $e \longrightarrow r^{-1}(e)$ isomorphically maps $\mathscr{E}_0$ onto $\overline{\mathscr{E}_0}$ and also transforms the quasimeasure m into $\overline{m}$. The algebra $\mathscr{E}_0$ is a base of $\mathscr{E}$; hence, each m-negligible set q of $\mathscr{E}$ is included in some $\widetilde{q}$ representable as the disjoint sum: $\widetilde{q} = \bigcup e_n$, $e_n \in \mathscr{E}_0$, where the value of $\sum me_n$ can be chosen arbitrarily small. Since

$$r^{-1}(q) \subset r^{-1}(\widetilde{q}), \quad \overline{m}r^{-1}(\widetilde{q}) = \sum \overline{m}r^{-1}(e_n) = \sum me_n,$$

[32]For the points of $\overline{\Omega}$ we further use the standard notation $\overline{\omega}$. The absence of the bar over ω reminds us that the points in question belong to Ω.

The concept of canonical extension was introduced in 3.1.5.

it is clear that the preimage $r^{-1}(q)$ is $\overline{m}$-negligible. Write now $e \in \mathscr{E}$ as $e = a \setminus q$, where

$$a = \bigcap_n \bigcup_k e_{nk}, \quad e_{nk} \in \mathscr{E}_0, \quad mq = 0.$$

We see that

$$r^{-1}(e) = \bigcap_n \bigcup_k r^{-1}(e_{nk}) \setminus r^{-1}(q),$$

whence $r^{-1}(e) \in \overline{\mathscr{E}}$. We have proved the $\overline{\mathscr{E}}$-measurability of all preimages $r^{-1}(e)$ $(e \in \mathscr{E})$; and, hence, of all differences $\overline{e} \setminus r^{-1}(e)$, $e \in \mathscr{E}_1$.

We have thus established that under the canonical $\mathscr{E}_1$-extension of the space $\{\Omega, \mathscr{E}, m\}$, the set Ω turns into a measurable set of the Lebesgue–Rokhlin space

$$\{\overline{\Omega}, \overline{\mathscr{E}}, \overline{m}\}, \quad \Omega \subset \overline{\Omega}, \quad \overline{m}(\overline{\Omega} \setminus \Omega) = 0.$$

The algebra $\overline{\mathscr{E}_1}$ is a base of $\overline{\Omega}$; it is countable and compact.

Now we recall our partition ξ. Under the canonical embedding, its base $\mathscr{A}_0$ turns into the countable algebra $\overline{\mathscr{A}_0} \subset \overline{\mathscr{E}_1}$ that generates in $\overline{\Omega}$ a measurable partition $\overline{\xi}$. At this time, the conditions of Propositions 1 and 2 are fulfilled; hence, there exist probability quasimeasures $\overline{p}_{\overline{\omega}}$ supported in the elements $\pi_{\overline{\xi}}(\overline{\omega})$ which have all properties listed in these propositions (where Ω, $\mathscr{E}$, $\mathscr{E}_0$, m, $\mathscr{A}_0$, and ξ are replaced with $\overline{\Omega}$, $\overline{\mathscr{E}}$, $\overline{\mathscr{E}_1}$, $\overline{m}$, $\overline{\mathscr{A}_0}$, and $\overline{\xi}$ respectively). In place of $\mathscr{E}^\sigma$ and $\mathscr{A}$ we now consider the σ-algebras $(\overline{\mathscr{E}_1})^\sigma$ and $\mathscr{A}_1$ generated by the algebras $\overline{\mathscr{E}_1}$ and $\overline{\mathscr{A}_0}$. We will assume that quasimeasures $\overline{p}_{\overline{\omega}}$ are defined for $\overline{\omega} \in \widetilde{\Omega}^0$ where $\widetilde{\Omega}^0 \in \mathscr{A}_1$ and $\overline{m}\widetilde{\Omega}^0 = 1$. It is easy to see that

$$\mathscr{E}_1^\sigma = \{e \mid e = \widetilde{e} \cap \Omega,\ \widetilde{e} \in (\overline{\mathscr{E}_1})^\sigma\},$$

$$\mathscr{A} = \{e \mid e = \widetilde{e} \cap \Omega,\ \widetilde{e} \in \mathscr{A}_1\}.$$

Moreover, in both cases $me = \overline{m}\widetilde{e}$. In fact, this equality is valid for $e \in \mathscr{E}_1$ when $\widetilde{e}$ necessarily coincides with $\overline{e}$; it extends onto the other $e \in \mathscr{E}_1^\sigma$ by the countable additivity of m and $\overline{m}$.

Note several simple facts. Denote by S one of the algebras $\mathscr{E}_1^\sigma$ and $\mathscr{A}$, and by $\widetilde{S}$, respectively, $(\overline{\mathscr{E}_1})^\sigma$ or $\mathscr{A}_1$.

1) Let $\widetilde{f}$ be a $\widetilde{S}$-measurable function on a set $\widetilde{E} \in \widetilde{S}$. Then its restriction $f \equiv \widetilde{f}|_E$, where $E = \widetilde{E} \cap \Omega$, is an S-measurable function. This is clear: for each interval Δ, we have $\widetilde{f}^{-1}(\Delta) \in \widetilde{S}$; hence, $f^{-1}(\Delta) = \Omega \cap \widetilde{f}^{-1}(\Delta) \in S$.

2) Under the same conditions

$$\int\limits_E f\,dm = \int\limits_{\widetilde{E}} \widetilde{f}\,d\overline{m}.$$

Indeed, for every y the following equality holds:

$$m\{\omega \mid f(\omega) < y\} = \overline{m}\{\overline{\omega} \mid \widetilde{f}(\overline{\omega}) < y\}.$$

From here we infer the equality between the integrals.

3) To each $z \in \xi$ there corresponds the unique element $\overline{z} \in \overline{\xi}$ satisfying $z = \overline{z} \cap \Omega$. This element is constructed as follows: Represent z as the intersection $z = \bigcap_{a \in \mathscr{A}_0} a'$ where a' is one of the elements a and Ca; then the formula $\overline{z} = \bigcap_{a \in \mathscr{A}_0} \overline{a'}$ defines a sought element $\overline{z}$.

We already know that $\Omega \in \overline{\mathscr{E}}$ and $\overline{m}\Omega = 1$. The algebra $\overline{\mathscr{E}_1}$ is a base of $\overline{\Omega}$; hence, there exists a set $\widetilde{\Omega} \in (\overline{\mathscr{E}_1})^\sigma$ for which the following conditions hold:

$$\overline{m}\widetilde{\Omega} = 1, \quad \widetilde{\Omega} \subset \Omega.$$

Since $\widetilde{\Omega} = \widetilde{\Omega} \cap \Omega$; therefore, $\widetilde{\Omega} \in \mathscr{E}_1^\sigma$. Put

$$f_{\widetilde{\Omega}}(\overline{\omega}) \equiv \overline{p}_{\overline{\omega}}(\widetilde{\Omega}) \quad (\overline{\omega} \in \widetilde{\Omega}^0).$$

It is clear that

$$0 \le f_{\widetilde{\Omega}}(\overline{\omega}) \le 1 \quad (\overline{\omega} \in \widetilde{\Omega}^0), \qquad \int\limits_{\widetilde{\Omega}} f_{\widetilde{\Omega}}(\overline{\omega})\,d\overline{m} = 1,$$

where we have an $\mathscr{A}_1$-measurable function under the integral sign in the last integral. Therefore, there is $\widetilde{\Omega}_0 \in \mathscr{A}_1$ satisfying

$$\overline{m}\widetilde{\Omega}_0 = 1, \quad \widetilde{\Omega}_0 \subset \widetilde{\Omega}^0, \quad f_{\widetilde{\Omega}}(\overline{\omega}) = 1$$

for all $\overline{\omega} \in \widetilde{\Omega}_0$. Each set $e \in \mathscr{E}_1^\sigma$ lying in $\widetilde{\Omega}$ has the form

$$e = \widetilde{\Omega} \cap e = \widetilde{\Omega} \cap \Omega \cap \widetilde{e} = \widetilde{\Omega} \cap \widetilde{e},$$

where $\widetilde{e} \in (\overline{\mathscr{E}_1})^\sigma$; and, hence, $e \in (\overline{\mathscr{E}_1})^\sigma$. So we may put

$$p_{\overline{\omega}}(e) \equiv \overline{p}_{\overline{\omega}}(e \cap \widetilde{\Omega})$$

for all

$$e \in \mathscr{E}_1^\sigma, \ \overline{\omega} \in \widetilde{\Omega}_0.$$

In particular, this makes sense for $e \in \mathscr{E}^\sigma$. Put $\Omega_0 \equiv \Omega \cap \widetilde{\Omega}_0$. Since $\widetilde{\Omega}_0 \in \mathscr{A}_1$; therefore, $\Omega_0 \in \mathscr{A}$. Moreover, $m\Omega_0 = \overline{m}\widetilde{\Omega}_0 = 1$.

We will consider the quasimeasures p_ω only for $\overline{\omega} \equiv \omega \in \Omega_0$. What properties has this family?

a) The set Ω_0 on which the family $\{p_\omega\}$ is defined belongs to the σ-algebra $\mathscr{A}$; and hence it is ξ-saturated. Moreover, $m\Omega_0 = 1$.

b) The quasimeasures p_ω itself are defined on the σ-algebra $\mathscr{E}_1^\sigma$ and supported in the corresponding elements $z = \pi(\omega)$ of the partition ξ.

c) For every $e \in \mathscr{E}_1^\sigma$, the function $f_e(\omega) \equiv p_\omega(e) \equiv \overline{p}_\omega(e \cap \Omega)$ is the restriction of an $\mathscr{A}_1$-measurable function

$$\widetilde{f}_{e\cap\widetilde{\Omega}}(\overline{\omega}) \equiv \overline{p}_{\overline{\omega}}(e \cap \widetilde{\Omega})$$

on $\widetilde{\Omega}_0$. Namely, $f_e = \widetilde{f}_{e\cap\widetilde{\Omega}}|_{\Omega_0}$. In view of 1) and 2) the function f_e is $\mathscr{A}$-measurable for all $e \in (\overline{\mathscr{E}_1})^\sigma$ and

$$\int\limits_{\Omega_0} f_e \, dm = \int\limits_{\widetilde{\Omega}_0} \widetilde{f}_{e\cap\widetilde{\Omega}} \, d\overline{m}.$$

d) Let $\omega \in \Omega_0$. Then $f_{\widetilde{\Omega}}(\omega) = 1$,

$$p_\omega(\pi_\xi(\omega)) = \overline{p}_\omega(\pi_{\overline{\xi}}(\omega) \cap \widetilde{\Omega}).$$

Since $\omega \in \Omega_0 \subset \widetilde{\Omega}^0$, the quasimeasure $\overline{p}_\omega$ is supported in the set $\pi_{\overline{\xi}}(\omega)$; moreover,

$$\overline{p}_\omega(\widetilde{\Omega} \setminus \pi_{\overline{\xi}}(\omega)) = 0, \quad \overline{p}_\omega(\widetilde{\Omega} \cap \pi_{\overline{\xi}}(\omega)) = \overline{p}_\omega(\widetilde{\Omega}) = \widetilde{f}_{\widetilde{\Omega}}(\omega) = 1$$

and $p_\omega(\pi_\xi(\omega)) = 1$. Thus, for $\omega \in \Omega_0$, all p_ω are probability quasimeasures and they are supported in the corresponding elements of the partition.

Let z be an element of the partition ξ. Assume that $z = \pi(\omega)$, $\omega \in \Omega_0$. This means merely that $\omega \in z$. The system $\mathscr{E}^\sigma(z)$ of all possible intersections $e \cap z$ $(e \in \mathscr{E}^\sigma)$ is a σ-algebra where z plays the role of unity. The quasimeasure p_ω, introduced above, is supported precisely in the sets of $\mathscr{E}^\sigma(z)$, its values are independent of the choice of $\omega \in z$. We will denote by m_z the countably additive quasimeasure that is defined on $\mathscr{E}^\sigma(z)$ by the equality

$$m_z(e) \equiv p_\omega(e),$$

z where ω is an element of z. We complete it by adding to the domain all possible subsets of m_z-negligible sets of $\mathscr{E}^\sigma(z)$. We thus come to the new σ-algebra of subsets z which is denoted by $\mathscr{E}(z)$. We retain the former notation m_z for the so-completed measure. We thus have the family

$\{m_z\}_{z\in\pi(\Omega_0)}$ of complete countably additive quasimeasures defined on the corresponding σ-algebras $\mathscr{E}(z)$. We mention the main properties of these quasimeasures.

1°. *For each $e \in \mathscr{E}$ there exists a set $a_e \in \mathscr{A},\ a_e \subset \Omega_0,\ ma_e = 1$ such that $e \cap \pi(\omega) \in \mathscr{E}(\pi(\omega))$ for all $\omega \in a_e$ and the function*

$$f_e: \quad f_e(\omega) = \begin{cases} m_{\pi(\omega)}(e \cap \pi(\omega)) \equiv p_\omega(e), & \omega \in a_e, \\ 0, & \omega \notin a_e, \end{cases}$$

is $\mathscr{A}$-measurable. (Thus, the function f_e is now defined everywhere.)

2°. *The following is valid:*

$$m(a \cap e) = \int\limits_a f_e\, dm \quad (a \in \mathscr{A}).$$

In particular,

$$me = \int\limits_\Omega f_e\, dm.$$

Here f_e stands for the function in 1° or another function m-equivalent to it.

These two assertions are actually stated for $e \in \mathscr{E}^\sigma$.[33] Since each $e \in \mathscr{E}$ is representable as $e' \setminus q$, where $e' \in \mathscr{E}^\sigma$, $mq = 0$; therefore, it is now enough to consider the case $me = 0$. There exists $e' \in \mathscr{E}^\sigma$ with the properties $e' \supset e$ and $me' = 0$. By what was proved above, for this set we have

$$\int f_{e'}\, dm = 0,$$

where the function $f_{e'}$ is $\mathscr{A}$-measurable. Then $f_e(\omega) = 0$ for almost all ω. More precisely, the set $a' \equiv \{\omega \mid f_{e'}(\omega) = 0\}$ lies in $\mathscr{A}$ and $ma' = 1$. Then by completeness of the measures m_z, the inclusions $e \cap z \subset e' \cap z$ imply the $\mathscr{E}(z)$-measurability of the sets $e \cap z$ for all $z \in \pi(a')$. Hence, the function f_e, defined by the formula of 1°, is identically zero. In our case, this is certainly sufficient for the validity of 1° and 2°.

We summarize the above as the following

Theorem 14. *Let $\{\Omega, \mathscr{E}, m\}$ be a Lebesgue–Rokhlin space, and let ξ be a measurable partition generated by a countable algebra $\mathscr{A}_0$. Let $\mathscr{A}$ be the σ-algebra generated by $\mathscr{A}_0$. Then there exist a set $\Omega_0 \in \mathscr{A}$ with $m\Omega_0 = 1$ and a family $\{m_z\}_{z\in\pi(\Omega_0)}$ of complete countably additive probability quasimeasures each of which is defined on the corresponding σ-algebra $\mathscr{E}(z)$; moreover, $\mathscr{E}(z) \subset 2^z \cap \mathscr{E}$.*

[33]When $a_e = \Omega_0$ and it does not depend of e.

This family possesses the following properties:

A. *For each $e \in \mathscr{E}$ there exists some set $a_e \in \mathscr{A}, a_e \subset \Omega_0$, with $ma_e = 1$ such that $e \cap z \in \mathscr{E}(z)$ for all $z \in \pi(\Omega_0)$ and the function f_e defined by*

$$f_e(\omega) = \begin{cases} m_{\pi(\omega)}(e \cap \pi(\omega)), & \omega \in a_e, \\ 0, & \omega \notin a_e, \end{cases}$$

is measurable with respect to the σ-algebra $\mathscr{A}$;

B. *Given $a \in \mathscr{A}$, we have*

$$\int\limits_a f_e \, dm = m(a \cap e).$$

In particular,

$$\int\limits_\Omega f_e \, dm = me.$$

This theorem follows immediately from the preceding considerations. We explain only the inclusion $\mathscr{E}(z) \subset 2^z \cap \mathscr{E}$. Clearly, $\mathscr{E}(z) \subset 2^z$; if $mz = 0$ then, in view of the completeness of the quasimeasure m, we have $2^z \subset \mathscr{E}$. Otherwise, if $mz > 0$ then $\mathscr{E}(z)$ consists simply of the intersections $e \cap z$ and it is a part of $\mathscr{E}$.

Theorem 14 admits another formulation in terms of "quotient space" and "quotient measure." This formulation will be given later. The system $\{m_z\}$ we have constructed is connected with the algebras $\mathscr{A}_0$ and $\mathscr{A}$. We will consider the question that naturally arises after the first acquaintance with measurable partitions. Alongside the σ-algebra $\mathscr{A}$, we also have the σ-algebra $\mathscr{E}_\xi$ which is connected with the partition ξ but not connected with any base. What relationship is there between these algebras? Clearly, $\mathscr{A} \subset \mathscr{E}_\xi$; the reverse inclusion is false as a rule since $\mathscr{E}_\xi$ is not usually a countably generated σ-algebra. Let $\mathscr{A}^*$ be the family of all sets of the form $e +_2 q$, where $e \in \mathscr{A}$ and q is a saturated set of measure zero. It is clear that $\mathscr{A}^*$ is a σ-algebra and $\mathscr{A}^* \subset \mathscr{E}_\xi$. Formally, it is also connected with $\mathscr{A}_0$. In fact, we have

Theorem 15. *For every choice of a countable base $\mathscr{A}_0$ of a measurable partition ξ, the algebras $\mathscr{A}^*$ and $\mathscr{E}_\xi$ coincide.*

Proof. It suffices to verify that each $e \in \mathscr{E}_\xi$ belongs to $\mathscr{A}^*$. Let the sets Ω_0, a_e and the family $\{m_z\}$ be chosen in accordance with Theorem 14, and let

$$me = \int\limits_{\Omega_0} f_e \, dm = \int\limits_{a_e} f_e \, dm.$$

Given $\omega \in a_e$, we have $f_e(\omega) = 0, 1$. Namely, $f_e(\omega) = 1$ for $\pi(\omega) \subset e$, and $f_e(\omega) = 0$ for $\pi(\omega) \cap e = \varnothing$. The third case is impossible, since e is a ξ-saturated set. The function f_e is $\mathscr{A}$-measurable for $a_e \in \mathscr{A}$; hence, the set

$$e_0 \equiv \{\omega \mid \omega \in a_e,\ \pi(\omega) \subset e\} \equiv \{\omega \mid f_e(\omega) = 1\}$$

belongs to $\mathscr{A}$. Clearly, $e_0 \subset e$. The difference $e \setminus e_0$ is a ξ-saturated set beyond a_e; therefore, $m(e \setminus e_0) = 0$. We see that e differs from e_0 by a saturated negligible set; hence, $e \in \mathscr{A}^*$. The proof of the theorem is complete.

Theorem 16. *Let ξ be a measurable partition of a Lebesgue–Rokhlin space $\{\Omega, \mathscr{E}, m\}$. Then there exist a set $\Omega_0 \in \mathscr{E}_\xi$ with $m\Omega_0 = 1$ and a family of complete countably additive probability quasimeasures $\{m_z\}_{z\in\pi(\Omega_0)}$ on the corresponding σ-algebras $\mathscr{E}(z) \subset \mathscr{E} \cap 2^z$ having the following properties:*

A. *For each $e \in \mathscr{E}$, there is a set $\Omega^e \in \mathscr{E}_\xi$ with $m\Omega^e = 1$ such that $e \cap z \in \mathscr{E}(z)$ for all $z \in \pi(\Omega^e)$ and the function f_e defined by*

$$f_e(\omega) \equiv \begin{cases} m_{\pi(\omega)}(e \cap \pi(\omega)), & \text{if } \ \omega \in \Omega^e, \\ 0, & \text{if } \ \omega \notin \Omega^e, \end{cases}$$

is measurable with respect to the σ-algebra $\mathscr{E}_\xi$.

B. *For all $b \in \mathscr{E}_\xi$ we have*

$$\int_b f_e\, dm = m(e \cap b).$$

In particular,

$$\int_\Omega f_e\, dm = me.$$

This theorem is almost equivalent to Theorem 14 from which the latter is immediate. It does not refer to the algebra $\mathscr{A}_0$ and used often in another formulation. We have already noted that the projection $\pi \equiv \pi_\xi$ establishes an isomorphism between $\mathscr{E}_\xi$ and $\widehat{\mathscr{E}_\xi}$. For $\mathscr{E}_\xi$-measurable functions, integration with respect to the "measure" m is equivalent, in fact, to integration with respect to the "quotient measure" $\widehat{m}$. More precisely,

$$\int_b g\, dm = \int_{\widehat{b}} \widehat{g}\, d\widehat{m};$$

$$\widehat{b} \equiv \pi(b), \quad \widehat{g} \equiv f \circ \pi^{-1}$$

for all $b \in \mathscr{E}_\xi$; in this case $\pi(b) \in \widehat{\mathscr{E}}_\xi$. Theorem 16 can be formulated in this new language as follows:

Theorem 17. *In the context of Theorem 16, there exist a set $\widehat{\Omega}_0 \in \widehat{\mathscr{E}}_\xi$, with $\widehat{m}\widehat{\Omega}_0 = 1$, and a family of countably additive probability quasimeasures $\{m_z\}$ on the corresponding σ-algebras $\mathscr{E}(z) \subset \mathscr{E} \cap 2^z$ such that*

A. *For each $e \in \mathscr{E}$, there exists a set $\widehat{\Omega}^e \in \widehat{\mathscr{E}}_\xi$, with $\widehat{m}\widehat{\Omega}^e = 1$, such that $e \cap z \in \mathscr{E}(z)$ for all $z \in \widehat{\Omega}^e$ and the function $\widehat{f}_e$, defined by*

$$\widehat{f}_e(z) \equiv \begin{cases} m_z(e \cap z), & \textit{if} \quad z \in \widehat{\Omega}^e, \\ 0, & \textit{if} \quad z \notin \widehat{\Omega}^e, \end{cases}$$

is measurable with respect to $\widehat{\mathscr{E}}_\xi$.

B. *For every set $\widehat{b} \in \widehat{\mathscr{E}}_\xi$, the following equality holds:*

$$\int\limits_{\widehat{b}} \widehat{f}_e \, d\widehat{m} = m(b \cap e).$$

In particular,

$$\int\limits_{\widehat{\Omega}^e} \widehat{f}_e \, d\widehat{m} = me.$$

Each family $\{m_z\}_{z \in \xi}$ of this theorem is called *a canonical system of conditional quasimeasures associated with a given partition.* Clearly, we can assume that the quasimeasures are defined (arbitrarily) for $z \notin \widehat{\Omega}_0$. We can easily prove the following:

1) *Each space $\{z, \mathscr{E}_z, m_z\}$ is a Lebesgue–Rokhlin space;*

2) *A canonical system with the properties listed in the last theorem exists only for a measurable partition;*

3) *A canonical system is unique up to choice of the set $\widehat{\Omega}_0$;*

4) *The space $\{\xi, \widehat{\mathscr{E}}_\xi, \widehat{m}\}$ is a Lebesgue–Rokhlin space.*

It follows from 3) that if $\{m_z\}$ and $\{m'_z\}$ are two canonical systems then the corresponding spaces $\{z, \mathscr{E}(z), m_z\}$ and $\{z, \mathscr{E}'(z), m'_z\}$ coincide $\widehat{m}$-almost everywhere.

Look again at Theorem 14. It is applicable to the case when ξ is a partition into singletons. We can take as $\mathscr{A}_0$ the base $\mathscr{E}_0$ of the Lebesgue–Rokhlin space under consideration. We see that in such space the partition into singletons is always measurable. Assume that our

Lebesgue–Rokhlin space $\{\Omega, \mathscr{E}, m\}$ has an extension $\{\Omega, \mathscr{E}', m'\}$, where $\mathscr{E} \subset \mathscr{E}'$, $m = m'|_{\mathscr{E}}$, and this extension is a Lebesgue–Rokhlin space too. In our case, the role of the algebra $\mathscr{A}_0$ is played by $\mathscr{E}_0$, and the algebra $\mathscr{A}^*$ is $\mathscr{E}$. In our case, we should take as $\mathscr{E}_\xi$ exactly $\mathscr{E}'$, the algebra of all subsets $e \in \mathscr{E}'$ composed of the whole elements of our partition (i.e., of singletons). By Theorem 15, we obtain $\mathscr{E} = \mathscr{E}'$. We come to the important conclusion: *no Lebesgue–Rokhlin space* $\{\Omega, \mathscr{E}, m\}$ *admits an essential extension* $\{\Omega, \mathscr{E}', m'\}$ *that is a Lebesgue–Rokhlin space.* (The "maximality" property of Lebesgue–Rokhlin spaces.) It means certainly that all partitions have the same basic set Ω.

5.7 Subalgebras and measurable partitions

We have begun this section with examples of regular subalgebras of the NBA E_0^2. The construction of these examples proceeds similarly by the following general scheme: we first distinguish a partition of the square I^2 and then we take the algebra of measurable sets saturated with respect to this partition; in conclusion, we apply to this algebra the canonical homomorphism Ψ. The Lebesgue square is a Lebesgue–Rokhlin space; the partitions that we have considered in the examples are measurable. The scheme we use for construction has in fact a universal character.

Let $\{\Omega, \mathscr{E}, m\}$ be a Lebesgue–Rokhlin space. Assume also that Ψ stands as above for the canonical epimorphism from $\mathscr{E}$ onto the metric structure $\widehat{\mathscr{E}}$. Taking an arbitrary measurable partition ξ, we apply this homomorphism to the σ-algebra $\mathscr{E}_\xi$. As a result, we obtain a regular subalgebra of the complete BA $\widehat{\mathscr{E}}$ as the image $\Psi(\mathscr{E}_\xi)$. Denote this subalgebra by $\widehat{\mathscr{E}}_\xi$. We have the mapping $\xi \longrightarrow \widehat{\mathscr{E}}_\xi$ that associates with each measurable partition a regular subalgebra of the metric structure.

We note several properties of this mapping.

1°. *If* $\xi_1 \prec \xi_2$ *then* $\widehat{\mathscr{E}}_{\xi_1} \subset \widehat{\mathscr{E}}_{\xi_2}$. "The finer is a partition, the larger is the subalgebra."

All this is clear; it is also clear that $\widehat{\mathscr{E}}_{\xi_1}$ is a *regular subalgebra of* $\widehat{\mathscr{E}}_{\xi_2}$. If $E \in \mathscr{E}$ and ξ is a partition of Ω then the sets

$$z \cap E \ \ (z \in \xi)$$

constitute a partition of E, the "trace" of the partition ξ. We denote it by ξ_E. Let $E \in \mathscr{E}$. Then E can be considered as a Lebesgue–Rokhlin space, a subspace of the original space Ω. It is easy to see that ξ_E is a measurable partition of this space. Say that two measurable partitions are *equivalent* (in other words, "coincident $mod\,0$") if for some E, with $mE = 1$, the traces ξ_{1E} and ξ_{2E} coincide. We denote this equivalence traditionally by $\sim$.

2°. *The relations* (1) $\xi_1 \sim \xi_2$ *and* (2) $\widehat{\mathscr{E}}_{\xi_1} = \widehat{\mathscr{E}}_{\xi_2}$ *are equivalent.*

PROOF. The implication (1) $\longrightarrow$ (2) is almost evident; it rests on the canonical isomorphism between $\widehat{\mathscr{E}}$ and $\widehat{\mathscr{E}}_E$.

Assume now that the same subalgebra is determined by the two partitions: $\widehat{\mathscr{E}}_{\xi_1} = \widehat{\mathscr{E}}_{\xi_2}$. Consider countable bases of these partitions $\mathscr{A}_0^1$ and $\mathscr{A}_0^2$ respectively. Let $\mathscr{A}^1$ and $\mathscr{A}^2$ be the σ-algebras generated by them. We may assume that $\mathscr{A}_0^2 \supset \mathscr{A}_0^1$ (i.e., $\xi_2 \succ \xi_1$). Otherwise we pass to the subalgebra generated by the union $\mathscr{A}_0^1 \cup \mathscr{A}_0^2$; this would not be reflected in the subalgebra $\widehat{\mathscr{E}}_{\xi_1} = \widehat{\mathscr{E}}_{\xi_2}$ itself.

For each set $e \in \mathscr{A}_0^2$ there exists $\widetilde{e} \in \mathscr{A}^1$ such that

$$m[(e \setminus \widetilde{e}) \cup (\widetilde{e} \setminus e)] = 0.$$

We will assume that such $\widetilde{e}$ is distinguished for each e (the axiom of choice.) Consider the countable algebra $\mathscr{B}^0$ generated by $\{\widetilde{e} \mid e \in \mathscr{A}_0^2\}$. This algebra is a part of $\mathscr{A}^1$; therefore, the partition ξ generated by it is coarser than ξ_1 (evidently, $\mathscr{A}_0^1$ and $\mathscr{A}^1$ generate the same partition ξ_1).

Using the countability of $\mathscr{A}_0^2$, we can unite all sets $(e \setminus \widetilde{e}) \cup (\widetilde{e} \setminus e)$ $(e \in \mathscr{A}_0^2)$ into a sole m-negligible set Q. On the complementary set $E = \Omega \setminus Q$, all sets e of $\mathscr{A}_0^2$ coincide with the corresponding $\widetilde{e}$. Hence, the partitions ξ_2 and ξ coincide on E. Recalling that $\xi_2 \succ \xi_1 \succ \xi$ and $mE = m\Omega$, we infer the equivalence of ξ_1 and ξ_2.

3°. *Each regular subalgebra of* $\widehat{\mathscr{E}}$ *has the form* $\widehat{\mathscr{E}}_\xi$ *for some measurable partition* ξ.

PROOF. Let $\widehat{\mathscr{E}}'$ be a regular subalgebra of $\widehat{\mathscr{E}}$. It is separable (as a subset of the separable metric space $\widehat{\mathscr{E}}$). Consequently, there is a countable dense (metrically) subalgebra $\widehat{\mathscr{A}_0}$ in $\widehat{\mathscr{E}}'$. Choose some representative a in each $\widehat{a} \in \widehat{\mathscr{A}_0}$. We may assume that these representatives also constitute some subalgebra $\mathscr{A}_0$ of $\mathscr{E}$ and this subalgebra is countable.[34] Introduce the partition ξ_1 generated by $\mathscr{A}_0$; it is measurable. We are left with the easy equality $\widehat{\mathscr{E}}' = \widehat{\mathscr{E}}_\xi$.

If we arrange the quotient of the set of all measurable partitions by the equivalence $\sim$ then our mapping $\xi \longrightarrow \widehat{\mathscr{E}}_\xi$ turns into a bijection between the resultant quotient set and the set of all regular subalgebras. Thus, we can say that for the metric structures of Lebesgue–Rokhlin spaces the concept of regular subalgebra coincides "in essence" with the concept of coset in the set of measurable partitions. We see that the method for defining subalgebras "by partitions" which was applied in 7.5.1 is in fact universal for the separable normed algebras which are always the metric structures of Lebesgue–Rokhlin spaces.

---

[34]We can also take as $\mathscr{A}_0$ the subalgebra generated by these representatives.

In conclusion, we return to conditional measures. The knowledge of a canonical system $\{m_z\}$ for some ξ permits us to construct the conditional measure operator for the corresponding subalgebra $\widehat{\mathscr{E}_\xi}$. Namely, given an arbitrary $\widehat{e} \in \widehat{\mathscr{E}}$, we choose a "representative" e, a concrete set of the σ-algebra $\mathscr{E}$. Let a_e and f_e be defined in accordance with Theorem 14. Now, if we take the element $\widehat{f_e}$ (the coset containing the function f_e) then it is just the conditional measure of the element $\widehat{e}$: for each $\widehat{e}'$ of $\widehat{\mathscr{E}_\xi}$ we have

$$\int\limits_{\widehat{e}'} \widehat{f}_e \, d\widehat{m} = \int\limits_{e'} f_e \, dm = m(e' \cap e) = \widehat{m}(\widehat{e}' \wedge \widehat{e})$$

(here $e' \in \mathscr{E}_\xi$ is an arbitrary representative of the coset $\widehat{e}'$).

Now, we can translate in the language of canonical systems the condition under which the regular subalgebra $\widehat{\mathscr{E}_\xi}$ saturates some band of the algebra $\widehat{\mathscr{E}}$. It suffices to study the case in which this band coincides with $\widehat{\mathscr{E}}$. (If $\widehat{\mathscr{E}}_{\widehat{u}}$ is an arbitrary nonzero band then we consider the trace ξ_u instead of ξ.)

Theorem 18. *The following conditions are equivalent:*

(1) $\widehat{\mathscr{E}} = \widehat{\mathscr{E}_\xi}$;

(2) *$\widehat{m}$-almost all elements of the partition ξ are singletons;*

(3) *$\widehat{m}$-almost all quasimeasures m_z are two-valued* (*i.e., each assumes only the values* 0 *and* 1).

Proof. The implication (1) $\longrightarrow$ (2) follows from what was proved above: since the partition into singletons ε necessarily generates the whole algebra $\widehat{\mathscr{E}}$ (i.e., $\widehat{\mathscr{E}} = \widehat{\mathscr{E}_\varepsilon}$), the partitions ε and ξ are equivalent; i.e., they coincide on some set whose complement is negligible.

The implication (2) $\longrightarrow$ (3) is evident, since each nontrivial measure is two-valued on a singleton space.

We will prove that (3) $\longrightarrow$ (1). Take $e \in \mathscr{E}$ arbitrarily. We have

$$me = \int\limits_{\xi} \varphi \, d\check{m} = \int\limits_{\check{\Omega}'} \varphi \, d\check{m} = \check{m}\check{\Omega}',$$

where $\varphi(z) \equiv m_z(e \cap z)$ and $\check{\Omega}' \equiv \{z \in \xi \mid \varphi(z) = 1\}$. The last set is measurable, since the function φ is measurable. It consists of those $z \in \mathscr{E}$ that m_z-almost entirely lie in e. The set $\Omega' \equiv \bigcup_{m \in \check{\Omega}'} m$ belongs to $\mathscr{E}_\xi$, lies in e m-almost entirely, and has the same measure: $me = \check{m}e =$

$\check{m}\check{\Omega}' = m\Omega'$. Hence, each $e \in \mathscr{E}$ is equivalent to some ξ-set, whence $\widehat{\mathscr{E}} = \widehat{\mathscr{E}}_\xi$.

Theorem 19. *For the subalgebra $\widehat{\mathscr{E}}_\xi$ to saturate no nonzero band, it is necessary and sufficient that for $\check{m}$-almost all $z \in \xi$ the quasimeasure m_z be purely continuous.*

We will not give a detailed proof of this theorem. Such a proof rests on the following fact.[35] If the set of such $z \in \xi$ that m_z has an atom is not negligible then there exists a "one-layer" measurable set of positive measure which has at most one common point with each element of the partition. Then it is clear that the subalgebra $\widehat{\mathscr{E}}_\xi$ saturates the corresponding band.

V. A. Rokhlin gave a complete classification of measurable partitions.

6. Fundamental systems of partitions

Assume that $\mathscr{X}$ is a complete BA and let T be a family of finite partitions of unity of this algebra; let $\mathscr{X}(T)$ be the subalgebra generated by all partitions of this family. We assume that T is directed with respect to the relation $\succ$. Each element $x_0 \in \mathscr{X}^+(T)$ can be written as $x_0 = u_1 + u_2 + \dots + u_m$, where $u_1, u_2, \dots, u_m$ belongs to the same partition $\tau \in T$ depending, in general, on x_0. We will always assume that $u_i > \mathbf{0}$. If the subalgebra $\mathscr{X}(T)$ is dense in $\mathscr{X}$ (in the sense of the (o)-topology) then we say that T is a *fundamental system of partitions in $\mathscr{X}$*. Clearly, we can also speak about a fundamental system in some regular subalgebra $\widetilde{\mathscr{X}} \subset \mathscr{X}$. If $\mathscr{X}$ is a separable BA then $\mathscr{X}$ has a countable fundamental system of partitions $T = \{\tau_1, \tau_2, \dots\}$; in this case we may assume that $\tau_1 \prec \tau_2 \prec \dots$ and adjoin to T the trivial partition $\{\mathbf{1}\}$.

With each partition $\tau = \{u_1, u_2, \dots, u_m\}$ we associate the operator U_τ acting in the $\mathfrak{L}_{\mathscr{X}}(\mu)$ space by the formula

$$U_\tau(\mathfrak{f}) \equiv \sum_{i=1}^{m} \Big(\frac{1}{\mu u_i} \int\limits_{u_i} \mathfrak{f}\, d\mu \Big) u_i.$$

It is the already-known conditional expectation $E^{\mathscr{X}_\tau}$, the projection onto the finite-dimensional subspace that corresponds to the partition τ. Let $T = \{\tau_1, \tau_2, \dots\}$ be a fundamental sequence of partitions such that each entry refines the previous entry. (Thus, the algebra $\mathscr{X}$ is assumed separable.) In this case, for every $\mathfrak{f}$, the elements $\{U_{\tau_n}(\mathfrak{f})\}$ constitute a martingale (see p. 364 above). Now, we restrict exposition to the case in

[35] V. G. Vinokurov, B. A. Rubshteĭn, and A. L. Fëdorov [1, p. 51].

which $\mathfrak{f} \in \mathscr{X}$. (The unit element of the $\mathfrak{L}_{\mathscr{X}}$ space.[36]) In view of the general martingale convergence theorems (for example, we may apply the Kantorovich Theorem which was mentioned in 6.3.5) for every $\mathfrak{f} \in \mathscr{X}$, we have the (o)-convergence in each $\mathfrak{L}^p_{\mathscr{X}}$:

$$U_{\tau_n}(\mathfrak{f}) \xrightarrow{(o)} \mathfrak{f}$$

(here $\mathfrak{f} \in \mathscr{X} \subset \mathfrak{L}^p_{\mathscr{X}}$ for all $p \geq 1$). In the $\mathfrak{L}^p$ spaces (as well as in all "regular" K-spaces), the (o)-convergence is equivalent to the "convergence with regulator."[37] In our case this means the following: there exist an element $r \in \mathfrak{L}^p$ ("regulator") and a sequence of numbers $\varepsilon_n \downarrow 0$ such that

$$|U_{\tau_n}(\mathfrak{f}) - \mathfrak{f}| \leq \varepsilon_n r.$$

Whence for each k we infer the inequality

$$|U_{\tau_n}(\mathfrak{f}) - \mathfrak{f}| \cdot e_k^+(r) \leq k\varepsilon_n e_k^+(r), \quad n = 1, 2, \ldots. \tag{23}$$

Recalling that $e_k^+(r) \uparrow \mathbf{1}$, we may derive the following: *for each $\delta > 0$, there exists an element $e \in \mathscr{X}^+$ for which*

1) $\mu e > 1 - \delta$,

2) *the elements $|U_{\tau_n}(\mathfrak{f}) - \mathfrak{f}| \cdot e$ vanish uniformly (i.e., with the "regulator" $\mathbf{1}$).*

We can take as e the element $e_k^+(r)$ for k sufficiently large.

The above conclusion (with the exception of $r \in \mathfrak{L}^p$) can be obtained by using the celebrated Egorov Theorem.

In our case $\mathfrak{f} \in \mathscr{X}$, $\int_{u_i} \mathfrak{f}\, d\mu = \mu(\mathfrak{f} \wedge u)$, and

$$U_{\tau_n}(\mathfrak{f}) = \sum_{i=1}^{m_n} \frac{\mu(\mathfrak{f} \wedge u_i^{(n)})}{\mu u_i^{(n)}} u_i^{(n)} \leq 1 \quad (\tau_n = \{u_1^{(n)}, \ldots, u_{m_n}^{(n)}\}).$$

The formula (23) opens the opportunity to estimate $U_{\tau_n}(\mathfrak{f})$:

$$\mathfrak{f} \wedge e \geq \mathfrak{f} \cdot U_{\tau_n}(\mathfrak{f}) \cdot e \geq (1 - k\varepsilon_n) e \wedge \mathfrak{f}.$$

Here k is some sufficiently large number and $\mu e > 1 - \delta$. The last inequality means that for every $n = 1, 2, \ldots$ and for every i with the property $u_i \wedge \mathfrak{f} \wedge e > \mathbf{0}$, the following inequality holds:

$$(1 - k\varepsilon_n)\mu u_i^{(n)} \leq \mu(\mathfrak{f} \wedge u_i^{(n)}) \leq \mu u_i^{(n)}.$$

[36]Here, in fact, $\mathfrak{f} \in \mathscr{X}^\bullet$ but we omit the dot, as was agreed upon before (cf. 6.4.3).
[37]The so-called "Egorov property."

In other words, those $u_i^{(n)}$ that meet $\mathfrak{f} \wedge e$ are "almost filled" by the element $\mathfrak{f}$; moreover, $\mathfrak{f} \wedge e$ "almost coincides" with $\mathfrak{f}$. The final result can be formulated precisely as follows:

Theorem 20. *Let $\{\tau_n\}$ be a fundamental sequence of measurable partitions and let $\mathfrak{f} \in \mathscr{X}^+$ be some element. For all $\varepsilon > 0$ and $\delta > 0$, there exist a number $n_0 \in \mathbb{N}$ and an element $e \in \mathscr{X}$ such that $\mu e > 1 - \delta$, and for each $u \in \tau_{n_0}$ satisfying the condition $u \wedge \mathfrak{f} \wedge e > \mathbf{0}$ the following inequality holds:*

$$1 - \varepsilon \leq \frac{\mu(\mathfrak{f} \wedge u)}{\mu u} \leq 1. \tag{24}$$

For brevity, we put

$$\overline{u} \equiv \sum_{u \in \tau_{n_0}, u \wedge \mathfrak{f} \wedge e > \mathbf{0}} u$$

to obtain

$$1 - \varepsilon \leq \frac{\mu(\mathfrak{f} \wedge \overline{u})}{\mu \overline{u}} \leq 1.$$

The element $\overline{u} \in \mathscr{X}_{\tau_n}$ approximates $\mathfrak{f}$ up to ε. But the theorem asserts much more: along with the "global" approximation of the "whole" element $\mathfrak{f}$ we simultaneously observe "local" approximation on the majority of the elements of the partition and, moreover, with the same precision. Perhaps, the reader has already recalled the celebrated theorem about density points in the theory of functions of a real variable and the related results.[38]

The above-proven theorem concerns only separable normed algebras. If an algebra is not separable then the theorem applies to the elements of the least regular subalgebra including the set $\bigcup_{n=1}^{\infty} \tau_n$.

Each directed system of partitions T generates two subalgebras. The first is the subalgebra GENERATED by the set $\bigcup_{\tau \in T} \mathscr{X}_\tau$. Since T is directed, this subalgebra is simply the union $\bigcup_{\tau \in T} \mathscr{X}_\tau$. The second is the REGULAR subalgebra $\overline{\mathscr{X}\langle \bigcup_{\tau \in T} \mathscr{X}_\tau \rangle}$ fully generated by the preceding; the latter is the (o)-closure of the former. We will denote the second subalgebra by $\mathscr{X}[T]$. If T is a fundamental system then this subalgebra coincides with $\mathscr{X}$:

$$\mathscr{X}[T] = \overline{\mathscr{X}\langle \bigcup_{\tau \in T} \mathscr{X}_\tau \rangle} = \mathscr{X}.$$

In this chapter we study measure algebras and their subalgebras. In the case when the algebra or subalgebra under consideration is CONTINUOUS, a fundamental system is often constructed from the partitions of

[38] I. P. Natanson [1].

unity into the ELEMENTS HAVING THE SAME MEASURE. This follows from the following proposition:

Lemma 11. *Let $\mathscr{X}$ be a continuous complete BA, and let μ be a probability measure on $\mathscr{X}$. Assume given*

1) *an element $x \in \mathscr{X}^+$,*

2) *natural numbers n and m,*

3) *a partition $\tau = \{u_1, u_2, \dots, u_m\}$ such that $\mu u_i = \frac{1}{m}$ for all $i = 1, 2, \dots, m$.*

Then there exists a partition τ' with the following properties:

a) *$\tau' \succ \tau$;*

b) *each element of τ' has the measure $\frac{1}{mn}$;*

c) *$\rho(x, \mathscr{X}_{\tau'}) \le \frac{1}{n}$.*

(In item c) we mean the distance from x to the subalgebra generated by the partition τ'.)

PROOF. Consider an arbitrary element of the form $\overline{x}_i \equiv x \wedge u_i$ ($i = 1, 2, \dots, m$) and apply Theorem 4 of this chapter (or Proposition 2° of Section 2 of Chapter 2) to the NBA $\mathscr{X}_{x \wedge u_i}$. If $\mu(x \wedge u_i) \ge \frac{1}{mn}$ then there exists $u^{(i)} < u$ such that $\mu u^{(i)} = \frac{1}{nm}$. Repeating this procedure finitely many times, we obtain disjoint elements $u_1^{(i)}, u_2^{(i)}, \dots, u_s^{(i)}, \dots, u_n^{(i)}$ not exceeding u_i; namely,

$$\mu u_k^{(i)} = \frac{1}{nm}, \quad u_1^{(i)}, \dots, u_s^{(i)} \le \overline{x}_i, \quad \mu(\overline{x}_i - \sum_{k=1}^{s} u_k^{(i)}) < \frac{1}{nm}.$$

Collecting all $u_k^{(i)}$ together, we obtain a sought partition τ'.

REMARK. It is clear that the inequality $\rho(x, \mathscr{X}_{\tau''}) < \varepsilon$ also remains true for all $\tau'' \succ \tau'$.

Denote by T_0 the system of all partitions into elements of the same measure.

Theorem 21. *For a continuous separable NBA the system T_0 includes a fundamental sequence*

$$\tau_0 \prec \tau_1 \prec \tau_2 \prec \cdots \prec \tau_n \prec \dots .$$

PROOF. Using the separability of the BA, take some sequence $\{x_n\}$ dense in $\mathscr{X}$. We inductively construct $\tau_0 = \{\mathbf{1}\}, \tau_1, \tau_2, \dots$ applying

Lemma 11 at each step; here we take $x \equiv x_k$, $n = k$, $\tau = \tau_k$ and $\varepsilon = \frac{1}{k}$. Then τ' will represent τ_{k+1}. It turns out that the sequence $\{\tau_k\}$ is fundamental; moreover, $\tau_0 \prec \tau_1 \prec \dots$.

In the above proof, we put $n \equiv n_k = k$ at each step. Clearly, it is essential only that $n_k \longrightarrow \infty$. The most often and convenient choice is $n_0 = 1$ and $n_k = 2^s$, where s is some natural number. The simplest solution is $s = 1$. We will call this sequence of partitions "binary." We conveniently write down such partitions on enumerating their elements by binary symbols:

$$\delta \equiv (0, \delta_1, \delta_2, \dots, \delta_p), \;\; \delta_i = 0, 1;$$

$$\tau_0 = \{u_0 \equiv \mathbf{1}\}, \;\; \tau_1 = \{u_{00}, u_{01}\}, \;\; \tau_2 = \{u_{000}, u_{001}, u_{010}, u_{011}\}, \dots, \tag{25}$$

$$u_0 = u_{00} + u_{01}, \;\; u_{00} = u_{000} + u_{001}, \;\; u_{01} = u_{010} + u_{011}, \dots,$$

$$\mu u_0 = 1, \;\; \mu u_{00} = \mu u_{01} = \frac{1}{2}, \;\; \mu u_{000} = \mu u_{001} = \mu u_{010} = \mu u_{011} = \frac{1}{2^2}, \dots.$$

As was just proved above, such a fundamental system is available for each separable continuous NBA or a separable continuous subalgebra of a continuous NBA.

Assign to each binary symbol some "binary rational" interval by the following scheme:

$$\Delta_0 \equiv [0, 1]; \;\; \Delta_{00} \equiv \Big[0, \frac{1}{2}\Big], \;\; \Delta_{01} \equiv \Big[\frac{1}{2}, 1\Big];$$

$$\Delta_{000} \equiv \Big[0, \frac{1}{2^2}\Big], \;\; \Delta_{001} \equiv \Big[\frac{1}{2^2}, \frac{1}{2}\Big], \dots.$$

In the NBA E_0, assign to them $\widehat{\Delta}_0, \widehat{\Delta}_{00}, \dots$.

If we are given an arbitrary continuous separable NBA $\{\mathscr{X}, \mu\}$ with μ a probability measure; then, choosing a "binary" system of the form (25) arbitrarily, we easily establish a bijection between this system and the system of all $\widehat{\Delta}_\delta$. Namely, to each u_δ we put into correspondence $\widehat{\Delta}_\delta$ with the same index. This correspondence is measure preserving. (We imply Lebesgue measure λ on E_0; so, $\mu u_\delta = \lambda \widehat{\Delta}_\delta$ for all δ.) Hence (by Theorem 5 of this chapter), there exists an isomorphism between $\{\mathscr{X}, \mu\}$ and $\{E_0, \lambda\}$. We have thus proved the following important theorem which was mentioned more than once before.

Theorem 22. *Each separable continuous NBA $\{\mathscr{X}, \mu\}$ with probability measure is isomorphic to the NBA $\{E_0, \lambda\}$, the metric structure of the Lebesgue interval.*

This theorem seems to belong to C. Carathéodory [2]. It is exposed also in the work by P. Halmos and J. von Neumann [1]. A more general theorem belonging to D. Maharam will be given in Chapter 9.

Corollary. *Every NBA $\mathscr{X}$ including a system of the form* (25) *is continuous.*

Indeed, the regular subalgebra $\mathscr{X}_0$ fully generated by such a system is isomorphic to E_0; and, consequently, it is continuous. Furthermore, a complete BA including a regular continuous subalgebra cannot have atoms: if we suppose the existence of an atom $u \in \mathscr{X}$ then the element $\inf\{x \in \mathscr{X}_0 \mid x \geq u\}$ would be an atom in $\mathscr{X}_0$.

7. Systems of measures and the Lyapunov Theorem

Assume given a finite family of measures on a BA $\mathscr{X}$. This determines some VECTOR MEASURE with values in a finite dimensional vector space. We will show in this section that the set of the values that this vector measure takes at some sufficiently large subalgebra is one-dimensional. Moreover, the set of ALL values is always convex. The last fact is the celebrated Lyapunov Theorem whose proof is given here.

We first consider the following question of interest in its own right.

Let $\alpha_1, \alpha_2, \dots, \alpha_n$ be totally additive quasimeasures on the same NBA $\mathscr{X}$. It is clear that they coincide at the trivial subalgebra $\{\mathbf{0}, \mathbf{1}\}$. Can we exhibit a "richer" subalgebra with the same property? The answer is given by the following

Theorem 23. *If an algebra $\mathscr{X}$ is continuous then there exists a continuous regular subalgebra $\widetilde{\mathscr{X}}$ on which $\alpha_1, \alpha_2, \dots, \alpha_n$ coincide.*

PROOF. We first consider the case $n = 2$. Take the subset $E \equiv \{x \in \mathscr{X} \mid \alpha_1(x) = \frac{1}{2}\}$ of $\mathscr{X}$. As we have seen above, it is arcwise connected. Show that there exists an element u in E such that $\alpha_2(u) = \frac{1}{2}$. Indeed, take an arbitrary element $y \in E$ and suppose that $\alpha_2(y) \neq \frac{1}{2}$; for example, $\alpha_2(y) > \frac{1}{2}$. Then $Cy \in E$ and $\alpha_2(Cy) < \frac{1}{2}$. By the arcwise connectedness of E and the continuity of α_2, there exists $u \in E$ satisfying $\alpha_2(u) = \alpha_1(u) = \frac{1}{2}$. Repeat this argument for the elements u and Cu and the measures $\alpha_1' \equiv 2\alpha_1$ and $\alpha_2' \equiv 2\alpha_2$. Then each of the elements u and Cu splits into two:

$$u = v_1 + v_2,\ \ Cu = w_1 + w_2,\ \ \alpha_1(v_1) = \alpha_1(v_2)$$

$$= \alpha_2(v_1) = \alpha_2(v_2) = \alpha_1(w_1) = \alpha_1(w_2) = \alpha_2(w_1) = \alpha_2(w_2) = \frac{1}{2^2}.$$

Continuing this process we come to the system of elements T having the form (25) and, by the corollary to Theorem 22, generating the continuous subalgebra $\mathscr{X}'$. Moreover, the quasimeasures α_1 and α_2 obviously coincide on T; hence, they also coincide on $\mathscr{X}'$.

The case in which n is arbitrary needs induction. Assume the theorem valid for $n = k$. Consider the quasimeasures $\alpha_1, \alpha_2, \dots, \alpha_{k+1}$. By what was proved, the measures α_k and α_{k+1} coincide on some continuous regular subalgebra $\mathscr{X}'$. We apply the inductive hypothesis to this subalgebra and the quasimeasures $\alpha_1, \alpha_2, \dots, \alpha_k$. There exists a regular continuous subalgebra $\widetilde{\mathscr{X}} \subset \mathscr{X}'$ which is a subalgebra of $\mathscr{X}$ such that $\alpha_1, \alpha_2, \dots, \alpha_k$ coincide on $\widetilde{\mathscr{X}}$ with α_{k+1}. The proof of the theorem is complete.

Corollary. *For each finite system $\{\varphi_1, \varphi_2, \dots, \varphi_m\}$ of totally additive real functions on a complete continuous BA $\mathscr{X}$ and for each probability measure μ on $\mathscr{X}$, there exists a continuous regular subalgebra $\widetilde{\mathscr{X}} \subset \mathscr{X}$ on which each of the functions φ_i is proportional to μ; i.e.,*

$$\varphi_i(x) = \lambda_i \mu(x) \quad (x \in \widetilde{\mathscr{X}}),$$

where $\lambda_1, \dots, \lambda_m$ are constants.

To prove this, recall that by the Radon–Nikodým Theorem, each of the functions φ_i is the difference of two quasimeasures $\varphi_i = \mu_i' - \mu_i''$. It remains to normalize μ_i' and μ_i'' and apply the last theorem to the resulting system.

The family of functions $\{\varphi_1, \varphi_2, \dots, \varphi_m\}$ defines a totally additive vector measure[39] φ on $\mathscr{X}$ with values in $\mathbb{R}^m$:

$$\varphi(x) = (\varphi_1(x), \varphi_2(x), \dots, \varphi_m(x)).$$

We see that in these conditions the image $\varphi(\widetilde{\mathscr{X}})$ of some continuous regular subalgebra $\widetilde{\mathscr{X}}$ consists of the vectors of the form

$$\mu x \cdot \lambda, \ x \in \widetilde{\mathscr{X}}, \ \lambda = (\lambda_1, \lambda_2, \dots, \lambda_m) \in \mathbb{R}^m, \ \lambda = \varphi(\mathbf{1});$$

i.e., it represents a one-dimensional set. Now, this remark allows us to easily obtain the Lyapunov Theorem on vector measures.

Theorem 24. *Let $\mathscr{X}$ be a continuous NBA, and let φ be a totally additive vector measure with values in $\mathbb{R}^m$. Then the image $\varphi(\mathscr{X})$ is a convex set.*

PROOF. Take $x, y \in \mathscr{X}$ and $\lambda \in (0, 1)$ arbitrarily. We need to show that $\lambda\varphi(x) + (1 - \lambda)\varphi(y) \in \varphi(\mathscr{X})$. Put $u \equiv x - x \wedge y$ and $v \equiv y - x \wedge y$. Apply the last corollary to the disjoint bands $\mathscr{X}_u$ and $\mathscr{X}_v$. Then there

[39] We use here the traditional term "vector measure" although the members $\varphi_1, \varphi_2, \dots, \varphi_m$ need not be measures in our sense.

exist regular continuous subalgebras $\widetilde{\mathscr{X}_u}$ and $\widetilde{\mathscr{X}_v}$ in these bands such that

$$\varphi(t) = p(t) \cdot \varphi(u), \quad \varphi(t') = q(t') \cdot \varphi(v),$$

where $t \in \widetilde{\mathscr{X}_u}$ and $t' \in \widetilde{x}_v$ while p and q are probability measures on $\widetilde{\mathscr{X}_u}$ and $\widetilde{\mathscr{X}_v}$ respectively. By Proposition 3 (Chapter 2, Section 2), there exist elements $\widetilde{t} \in \widetilde{\mathscr{X}_u}$ and $\widetilde{t'} \in \widetilde{\mathscr{X}_v}$ with the property $p(\widetilde{t}) = \lambda, q(\widetilde{t'}) = 1 - \lambda$. Put $w = \widetilde{t} + \widetilde{t'} + x \wedge y$. Then

$$\varphi(\mathscr{X}) \ni \varphi(w) = \varphi(\widetilde{t}) + \varphi(\widetilde{t'}) + \varphi(x \wedge y) = p(\widetilde{t})\varphi(u) + q(\widetilde{t'})\varphi(v) + \varphi(x \wedge y)$$

$$= \lambda\varphi(u) + (1-\lambda)\varphi(v) + \varphi(x \wedge y) = \lambda\varphi(u) + \lambda\varphi(x \wedge y) + (1-\lambda)\varphi(v) + \varphi(x \wedge y)$$

$$- \lambda\varphi(x \wedge y) = \lambda\varphi(u + x \wedge y) + (1-\lambda)\varphi(v + x \wedge y) = \lambda\varphi(x) + (1-\lambda)\varphi(y).$$

The proof of the theorem is complete.

This theorem belongs to A. A. Lyapunov[40] and has numerous applications in mathematical statistics, control theory and measure theory. A. A. Lyapunov showed also that the image $\varphi(\mathscr{X})$ is closed in $\mathbb{R}^m$ and studied the nature of extreme points of this set. He took as $\mathscr{X}$ not an NBA but rather a σ-algebra of sets. This distinction is immaterial.

Clearly, the proof of the theorem essentially uses the FINITENESS of the family $\{\varphi_1, \varphi_2, \dots, \varphi_m\}$. As was shown by A. A. Lyapunov himself, his theorem fails in general for measures with values in infinite-dimensional spaces. However, there exist situations when in an infinite-dimensional space the image $\varphi(\mathscr{X})$ turns out convex. This problem was investigated by many authors (I. Kluvanek and G. Knowles [1]; G. Knowles [1–2]; J. F. C. Kingman and A. R. Robertson [1]; and I. I. Bazhenov [1–5]). In particular, I. I. Bazhenov investigated the possibility of extending a given vector measure to a "Lyapunov measure" with the convex range. To this end, the algebra on which the extended measure is defined must be sufficiently large. One of the results of I. I. Bazhenov is as follows: A nonatomic vector measure on the algebra of Lebesgue measurable subsets of $[0, 1]$ with values in a Banach space of the cardinality of the continuum always has a "Lyapunov extension." This extension is also defined on some σ-algebra of subsets of the same interval which contains all Lebesgue measurable sets but which is far larger than the Lebesgue algebra. The algebras of this type were known earlier in measure theory;[41] we have already referred to them in this chapter (cf. 7.3.5).

---

[40] A. A. Lyapunov [1]; P. Halmos [1].

[41] S. Kakutani and J. Oxtoby [1]; D. Kappos [1]; Sh. S. Pkhakadze [1]; and A. B. Kharazishvili [1].

Chapter 8

EXISTENCE OF A MEASURE

This section is devoted to the PROBLEM OF NORMABILITY: *What conditions guarantee the existence of a measure on a complete BA $\mathscr{X}$?* If we consider the elements of the algebra as events then our problem admits the following formulation: *When does a countably additive probability measure exist on the set of events $\mathscr{X}$ which is equal to zero only for the impossible event?*

We rank this problem as central in the whole theory of Boolean algebras. At present, many conditions for normability belonging to several authors are available, but this problem is still far from a complete solution. Moreover, we believe that a satisfactory solution in "classical" terms is impossible in principle; the problem of normability relates in fact to the foundations of mathematics and demands the methods of contemporary model theory, mathematical logic, and set theory. Leaving aside the purely logical aspects, we will try to throw light on the state of the art of research into this inexhaustible problem.

1. Conditions for existence of a measure

1.1 Necessary conditions

Throughout this section, $\mathscr{X}$ is a complete BA. Speaking about a measure, we always imply a probability measure.

First of all, note an obvious SUFFICIENT condition of normability: *Every NBA is always regular in the sense of L. V. Kantorovich; and, in particular, it satisfies the countable chain condition.* Thus, each algebra without the countable chain condition and each non-weakly-σ-distributive algebra is not normable. The algebra of regular open subsets of the interval $(0, 1)$ (see p. 103) satisfies the countable chain condition

but is not regular. Consequently, it has no measure. Nevertheless, this algebra has an essentially positive quasimeasure (which coincides with the usual Lebesgue "measure"). This quasimeasure is not countably additive in this event. We now elaborate details.

Consider the intervals of the form

$$e_n = \Big(r_n - \frac{\varepsilon}{2^n},\, r_n + \frac{\varepsilon}{2^n}\Big),$$

where $0 < \varepsilon < 1$ and $r_1, r_2, \dots$ are rational numbers of $(0,1)$. All e_n and the interval $(0,1)$ are regular open sets. The same is true for their suprema

$$E_n = e_1 \vee e_2 \vee \dots \vee e_n = Int\,(\overline{e_1 \cup e_2 \cup \dots \cup e_n}).$$

These sets differ from the unions only in finitely many points; hence,

$$l(E_n) = l(e_1 \cup e_2 \cup \dots \cup e_n) \le \varepsilon\Big(\frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^n}\Big) < \varepsilon$$

(l is Lebesgue "measure"). However, $\bigvee_n E_n = (0,1)$ since r_n is everywhere dense. Thus,

$$1 = l(0,1) > \varepsilon \le \lim\, l(E_n).$$

Hence, the quasimeasure l is not countably additive.

1.2 The Pinsker, Kelley, and Maharam Theorems

We give one sufficient condition for existence of a measure on a Boolean algebra. The theorem we formulate here is due to A. G. Pinsker;[1] somewhat later practically the same result was obtained by J. Kelley.[2]

Theorem 1. *If a regular algebra $\mathscr{X}$ has an essentially positive quasimeasure then $\mathscr{X}$ is a normable BA.*

Proof. Let φ be an essentially positive quasimeasure on $\mathscr{X}$. By Theorem 7.6, φ can be represented as the sum $\varphi_1 + \varphi_2$, where φ_1 is a standard countably additive quasimeasure generated by φ. Our theorem will be proved if we establish the essential positivity of φ_1. Assume that $\varphi_1(x) = 0$ for some $x \in \mathscr{X}^+$. Then there exists a sequence of covers $\sigma_k \in S_x$ $(k = 1, 2, \dots)$ satisfying the following condition:

$$\sum_{y \in \sigma_k} \varphi(y) < \frac{1}{k} \quad (k = 1, 2, \dots).$$

[1] L. V. Kantorovich, B. Z. Vulikh, and A. G. Pinsker [1, pp. 428–430].
[2] J. Kelley [1].

Assume that the elements σ_k are enumerated: $\sigma_k = \{y_1^k, y_2^k, \dots\}$. Put

$$u_s^k \equiv \Big(\bigvee_{i \le s} y_i^k\Big) \wedge x \quad (k, s = 1, 2, \dots).$$

Clearly, $u_s^k \uparrow x$ for all $k = 1, 2, \dots$. Using the diagonal principle, choose a sequence of indices $s_1 < s_2 < \dots < s_k < \dots$ so that

$$u_{s_k}^k \xrightarrow{(o)} x.$$

Then the sequence $\{z_k\}_{k=1}^{\infty}$, defined by

$$z_k = \bigwedge_{i \ge k} u_{s_i}^i \quad (k = 1, 2, \dots),$$

INCREASES and tends to x; moreover, for k sufficiently large we have

$$0 < \varphi(z_k) \le \varphi(u_{s_k}^k) \le \sum_{m=1}^{\infty} \varphi(y_m^k) < \frac{1}{k}.$$

There exist two indices k_1 and k_2 such that

$$\varphi(z_{k_1}) > \varphi(z_{k_2});$$

at the same time, $z_{k_2} \ge z_{k_1}$. The evident inconsistency of the last two inequalities proves that the equality $\varphi_1(x) = 0$ is false. The proof of the theorem is complete.

The Pinsker Theorem shows that for a regular BA the problem of existence of a measure reduces to an easier problem of existence of an essentially positive quasimeasure. This last problem was considered by several authors. Here we formulate the result of J. Kelley. With each nonempty subset E of the BA $\mathscr{X}$ we associate some real number $K(E)$ called the "Kelley number." This number is defined by the following equality

$$K(E) \equiv \inf_{S \in \Sigma(E)} \frac{i(S)}{n(S)},$$

where $\Sigma(E)$ is the class of all finite families of elements of E, $n(S)$ is the number of elements of the family $S \in \Sigma(E)$, and $i(S)$ is the maximal number of elements of this family with nonzero infimum. Further, if we embed the algebra $\mathscr{X}$ in a vector lattice over $\mathscr{X}$ (namely, in the Banach space of all bounded continuous functions on the Stone space) then we can see that the inequality $K(E) > 0$ implies impossibility of approximating zero by the elements of the convex hull of E. Therefore, choosing

a hyperplane that separates E from zero, we can define a positive functional or, which is the same, a QUASIMEASURE on $\mathscr{X}$, so that the values of this quasimeasure on E be essentially positive. It is rather clear how to formulate the solution of our problem in these terms: *for existence of an essentially positive quasimeasure, it is necessary and sufficient that the family of all nonzero elements of the algebra can be decomposed into countably many sets with strictly positive Kelley numbers.*

An algebra with an essentially positive quasimeasure may fail to be normable. However, it can always be embedded in a measure algebra.

Theorem 2. *Let $\mathscr{X}$ be a BA, and let φ be an essentially positive quasimeasure on $\mathscr{X}$. There exists a complete normed algebra including an everywhere dense subalgebra isomorphic to $\mathscr{X}$.*

PROOF. Again denote by $\mathscr{X}^0$ the clopen algebra of the Stone space of $\mathscr{X}$. Let Φ be an isomorphism of $\mathscr{X}$ onto $\mathscr{X}^0$; the equality $\mu e = \varphi(\Phi^{-1}(e))$ defines a quasimeasure on $\mathscr{X}^0$ that satisfies all conditions of the Lebesgue–Carathéodory Theorem. It is clear that μ is countably additive on $\mathscr{X}^0$, because in each equality of the form

$$e = \bigcup_n e_n, \quad e, e_n \in \mathscr{Y}_0,$$

we may consider only finitely many summands on the right-hand side. Take as a sought algebra $\mathscr{Y}$ the algebra of measurable $mod\, 0$ sets obtained by the standard extension of μ from the subalgebra $\mathscr{X}^0$ (the role of the underlying algebra is played by 2^{Ω}).

REMARK. Clearly, the embedding of $\mathscr{X}$ into $\mathscr{Y}$ is an isometry (a measure preserving mapping).

Theorem 2 can be proved in another way by using the idea of "uniform completion" on considering $\mathscr{X}$ as a metric space.

Apparently, the first work in which the question of normability of a BA was posed belongs to D. Maharam.[3] In 1947, she published the following test for normability:

A complete BA $\mathscr{X}$ is normable if and only if $\mathscr{X}$ is regular and has a countable base $\{V_n\}_1^\infty$ of neighborhoods of a zero such that:

a) *if $x \in V_{n+1},\ y \notin V_n$ then $|x-y| \notin V_{n+1}$;*

b) *if $x, y \notin V_{n+1}$ and $x\, d\, y$ then $x \vee y \notin V_n$.*

We imply the neighborhoods of the (o)-topology: hence, it is impossible that the relations $(o)\text{-}\lim x_n = \mathbf{0}$ and $x_n \notin V_n$ $(n = 1, 2, \dots)$ are valid simultaneously.

[3] D. Maharam [2].

Clearly, the above condition is NECESSARY for existence of a measure: if μ is a measure on $\mathscr{X}$ then we can take as V_n the "balls"

$$V_n \equiv \Big\{x \mid \mu x \leq \frac{1}{2^n}\Big\}. \tag{$*$}$$

On the other hand, as was shown in the article cited above, if the sequence $\{V_n\}$ possesses the above properties then there always exists a measure μ for which V_n are "balls" of the form $(*)$. It is shown in the same article that the existence of a countable base of (o)-neighborhoods of zero (without the conditions a) and b)) is already equivalent to the existence of a continuous outer quasimeasure (see Chapter 4 above).

1.3 An example by H. Gaifman

Assume that an algebra $\mathscr{X}$ admits an essentially positive quasimeasure μ. Then the set of nonzero elements of $\mathscr{X}$ can be represented as

$$\mathscr{X}^+ = \bigcup_{n=1}^{\infty} B_n,$$

where each B_n has the following property: every system of disjoint elements in B_n contains at most n elements. Indeed, we may take

$$B_n \equiv \Big\{x \mid \mu x \geq \frac{1}{n}\Big\}.$$

We call this property the "strong countable chain condition" (clearly, it implies the usual countable chain condition).

H. Gaifman [1] constructed an example of a complete BA possessing the strong countable chain condition and having no essentially positive quasimeasure. This algebra is obtained as a Dedekind completion of the quotient algebra $\mathscr{D}|_I$ where $\mathscr{D}$ is the clopen algebra of the Cantor discontinuum of weight $\mathfrak{c}$ and I is a suitable ideal of this algebra. Omitting minute details, we restrict exposition to describing this ideal.

Take as T the interval $(0,1)$ and consider the Cantor space $X_T \equiv X_{(0,1)}$. Its elements are the functions $\chi = \{\chi_t\}_{t\in T}$ where $\chi_t = 0, 1$. Assume that all rational intervals in $(0,1)$ are enumerated into some sequence $\{T_i\}$. Take a sequence of natural numbers $\{n_i\}$ such that $\frac{n_i}{i} \longrightarrow +\infty$. Now, in each T_i we distinguish the disjoint intervals

$$T_{i1}, T_{i2}, \dots, T_{in_i}.$$

The independent generators of the clopen algebra of X_T are the elements of the form

$$b^x \equiv \{\chi \mid \chi_x = 1\}, \quad x \in T = (0,1).$$

Consider the elementary polynomial

$$b = b^{x_1} \cap \cdots \cap b^{x_j} \cap Cb^{x_{j+1}} \cap \cdots \cap Cb^{x_{j+s}}, \tag{$**$}$$

where $x_1, x_2, \dots$ are distinct numbers of $(0, 1)$. Each clopen set in X_T is a finite sum of disjoint sets of the form $(**)$. We select those among them that have the following property:

$(***)$ for some $i = 1, 2, \dots,$ we may choose i members among $x_1, x_2, \dots, x_s$ that belong to the pairwise distinct intervals

$$T_{ik_1}, T_{ik_2}, \dots, T_{ik_i}.$$

We denote by I the ideal generated by the sets of such type. This is a sought ideal. H. Gaifman showed that the algebra $\mathscr{X}$ presenting a Dedekind completion of the quotient algebra $\mathscr{D}|_I$ possesses the strong countable chain condition but admits no essentially positive quasimeasure. This algebra is not regular.

1.4 The Suslin hypothesis and normability

The connection between the Suslin hypothesis and the problem of existence of a measure seems to be first revealed in the same article by D. Maharam.[4] Among numerous equivalent formulations of the Suslin hypothesis, we choose those that were used in this article.

A *Suslin system* is defined to be a set A of sets possessing the following properties:

(1) Every two elements $a, b \in A$ are either disjoint or comparable (by inclusion);

(2) Every disjoint subset $A' \subset A$ is at most countable;

(3) Every chain $A'' \subset A$ is at most countable.

The Suslin hypothesis reads:

($\mathbb{SH}$) *Every Suslin system is at most countable.*

D. Maharam has proved that $\mathbb{SH}$ is a consequence of the following assertion:

(M) *Each continuous regular complete BA $\mathscr{X}$ admits a real function ν with the following properties:*

[4] D. Maharam [2].

(a) $0 \le \nu(x) < +\infty$ *for all* $x \in \mathscr{X}$; $\nu(x) = 0$ *if and only if* $x = \mathbf{0}$;

(b) *if* $x \le y$ *then* $\nu(x) \le \nu(y)$;

(c) $\nu(x \vee y) \le \nu(x) + \nu(y)$;

(d) *for each* $x > \mathbf{0}$ *there exists* $y < x$ *such that* $0 < \nu(y) < \nu(x)$

(so ν is a "nontrivial outer quasimeasure").

We describe the main steps of the proof of this theorem.

Assume that the Suslin hypothesis is false; hence, there exists an uncountable Suslin system. This system can be modified in another uncountable Suslin system $\mathbf{C}$ possessing some additional properties:

1) $\mathbf{C}$ has the form $\mathbf{C} = \bigcup_{\alpha<\omega_1} \mathbf{C}_\alpha$ where ω_1 is the first uncountable ordinal number and $\mathbf{C}_\alpha$ is a system of disjoint nonempty sets composed of the elements of some distinguished space S.

2) For $\alpha < \beta < \omega_1$, each $c_\beta \in \mathbf{C}_\beta$ is contained in a unique $c_\alpha \in \mathbf{C}_\alpha$, and each $c_\alpha \in \mathbf{C}_\alpha$ includes at least two sets $c_\beta \in \mathbf{C}_\beta$.

3) Put $S_\alpha \equiv \bigcup_{c_\alpha \in \mathbf{C}_\alpha} c_\alpha$ and $N_\alpha \equiv S \setminus S_\alpha$. For $\alpha < \beta$ the inclusion $N_\beta \supset N_\alpha$ holds.

4) If $\alpha < \beta < \omega_1$ then for every $c_\alpha \in \mathbf{C}_\alpha$ the following holds:

$$c_\alpha \subset \bigcup_{c_\beta \subset c_\alpha} c_\beta \cup N_\beta.$$

Moreover, $c_\alpha \not\subset N_\beta$.

Let $\mathscr{D}_\alpha$ be the set of all possible unions of distinct $c_\alpha \in \mathbf{C}_\alpha$. Put $\mathscr{D} = \bigcup_\alpha \mathscr{D}_\alpha$. Thus, $\mathscr{D}$ contains the sets of the form

$$d = \bigcup_{c_\alpha \in K_d} c_\alpha, \quad K_d \subset \mathbf{C}_\alpha$$

with some $\alpha \equiv \alpha(d)$.

We simultaneously constitute the class N of all sets $n \subset S$ lying in N_α; namely, $n \subset N_\alpha$ for some $\alpha \equiv \alpha(n)$. Basing on the above properties 1)–4), we can easily verify that the sets

$$(d \setminus n) \cup (n \setminus d) \quad (n \in N,\ d \in \mathscr{D})$$

together with the sets of N comprise a σ-algebra $\mathscr{X}$ of subsets of S and N is a σ-ideal of this algebra. Now, define the quotient algebra

$$\widehat{\mathscr{X}} \equiv \mathscr{X}|_N.$$

The properties of a Suslin system easily imply completeness as well as the countable chain condition and weak σ-distributivity of this quotient

algebra. Thus, $\widehat{\mathscr{X}}$ is regular. Nevertheless, no nontrivial outer measure exists on $\widehat{\mathscr{X}}$. This fact can be proved by the following argument.[5]

Assume to the contrary that ν is a nontrivial outer quasimeasure on $\widehat{\mathscr{X}}$. Take $c \equiv c_\alpha \in \mathbf{C}$ arbitrarily, and consider the coset $\widehat{c}$ containing c. This coset is clearly a nonzero element of $\widehat{\mathscr{X}}$ since (in view of 4)) none of the c_α's is contained in N_α, and hence does not belong to N. Therefore, $\nu(\widehat{c}) > 0$. Henceforth, we denote it simply by $\nu(c)$. Let r be a rational number such that $r > \nu(c)$. Consequently, taking the "larger" sets $c_\beta \supset c$ and moving "backwards" along the transfinite numbers, we arrive in finitely many steps at a maximal $g \in \mathbf{C}$ satisfying the inequality $r > \nu(g)$ (i.e., for $c \supset g,\ c \in \mathbf{C}$, this inequality is false). Then assign to each rational r the set G_r of all these maximal g's. This set does not contain two comparable elements; hence, it consists of the pairwise disjoint sets and, so it is at most countable.

Now, take an arbitrary c. Since ν is "nontrivial," there exists $d \in \mathscr{D}$ such that $\nu(\widehat{d}) < \nu(c)$ and $d \setminus c \in N$ (i.e., $\widehat{d} \le \widehat{c}$). For all $c' \subset d$ we have $c' \subset c$, since c and c' cannot be disjoint. It is clear that $\nu(c') < \nu(c)$, and for some rational r we have $\nu(c') < r < \nu(c)$. Then we can replace c' with a MAXIMAL set $g \in G_r$, where $c' \subset g \subset c$. Thus, the elements of $\mathbf{C}$ are decomposed into the classes $C(g)$ consisting of all $c \supset g$ for $g \in \bigcup_r G_r$. Each class $C(g)$ consists of pairwise comparable sets (it contains no disjoint sets); hence, it is countable or finite. The union $\bigcup_r G_r$ is countable. Thus, our Suslin system cannot be uncountable.

We have seen that the negation of $\mathbb{SH}$ implies the existence of a regular BA $\mathscr{X}$ having no nontrivial outer measure. In consequence, this algebra is not normable.

The BA $\mathscr{X}$ constructed by D. Maharam is interesting in other respects. We can associate with each $\alpha < \Omega$ a PARTITION of unity into disjoint nonzero elements $\widehat{c}$ ($c \in \mathbf{C}_\alpha$). These partitions are at most countable and refine one another as α increases. The cardinality of them is equal to $\aleph_1$. The set of elements of these partitions is a minorant and, so each countable set of elements of $\widehat{\mathscr{X}}$ is contained in a discrete regular subalgebra generated by one of these partitions. This implies, for example, that every separable[6] regular subalgebra is discrete, although the whole algebra $\mathscr{X}$ is atomless. Each resolution of the identity has a purely discrete spectrum (since such a resolution of the identity generates a separable subalgebra). Each continuous function on $\mathfrak{Q}(\widehat{\mathscr{X}})$ is

[5]D. Maharam furnished an evidence that this beautiful argument was proposed by K. Gödel.
[6]Recall again that in our book separability means the fact that the topological weight is countable.

a step function with countably many values. Of course, all this ensues from the negation of the Suslin hypothesis.

As for $\mathbb{SH}$ itself, it is independent of the other axioms of set theory[7] (i.e., the axioms of $\mathbb{ZFC}$). If it seems to us that at present it is not the Suslin hypothesis stated in 1920 but rather its negation that agrees better with the every-day mathematical intuition.

Outer quasimeasures play in the theory of Boolean algebras the role resembling that of the seminorms in the theory of vector spaces. In connection with this analogy, a hypothesis arises naturally that some theorem similar to the Hahn–Banach theorem must be true. The question is as follows: suppose that θ is an outer quasimeasure on a BA $\mathscr{X}$, does there always exist a nontrivial "dominated" quasimeasure φ satisfying the condition

$$\varphi(x) \leq \theta(x) \quad (x \in \mathscr{X})?$$

Certainly, of the utmost interest here is the case in which θ is continuous; then this quasimeasure φ will also be continuous and hence totally additive. It is not known in general whether the existence of an outer quasimeasure continuous in some sense implies the existence of a countably additive quasimeasure.

Some examples of outer quasimeasures not admitting "dominating" quasimeasures were given by many authors (V. A. Popov [1]; M. Talagrand [1]; and J. Christensen and W. Herer [1]). A necessary and sufficient condition for existence of a quasimeasure φ continuous with respect to a given outer quasimeasure θ was suggested by N. Kalton and J. Roberts in the article [1]. Namely, θ must be *uniformly exhaustive*: for each $\varepsilon > 0$ there exists a natural $n = n_\varepsilon$ such that for every disjoint family $x_1, x_2, \ldots, x_n$ we have

$$\min_{1 \leq i \leq n} \theta(x_i) \leq \varepsilon.$$

2. Existence of a measure invariant under the automorphism group

2.1 Statement of the problem

In the historical development of probability theory, it is usual to distinguish two stages: the classical stage in which mostly the finite algebras of events were studied successfully; and the modern stage which is connected with the axiomatization of probability theory. Investigations of

[7] R. M. Solovay and S. Tennenbaum [1].

the mathematicians of the first stage were based on the classical definition of probability which reduces the determination of probability to calculation of the number of equally possible events. This approach was mostly justified when confined to the finite systems of events; however, since practice arose interest in infinite trials, the use of the classical definition was hampered. Even the simple problems of a geometric nature turned to be untractable by a sufficiently sharp analysis; as an example we can refer to the "paradox" of J. Bertrand which is included in textbooks. Everything demanded a renewal of probability theory; as the frontier between the classical and modern stages, it is generally accepted to acclaim the appearance of the famous monograph of A. N. Kolmogorov.[8]

In axiomatization of probability theory, a measure on an algebra of events is considered to be given a priori; the question of its origin is carried out of mathematics into the sphere of experiment. Nevertheless, a reasonable mathematical statement of this question is possible which is suggested by the classical definition of probability. First, we analyze the latter.

Let a finite system of events S be given. The "classic" who wants to define the probability of all events seeks in S a complete subsystem S' of mutually exclusive and equally possible "elementary" events from which the others would be constructed as from bricks.

In this connection, the main question for the "classic" is to justify the equiprobability of "elementary" events. This is usually done by referring to some symmetry available in the problem to be solved. For instance, a well reshuffled pack of cards might be in the 52! states. The result of reshuffling depends only on the physical properties of cards; since they are the same for all cards, the possible states of the pack are physically indistinguishable from each other. The "equiprobability" here is just the word reflecting this "indiscernibility."

Thus, in our example, we have the set S' of 52! equiprobable states of the pack. Now, let S be the algebra of events that may happen during a "purely gambling" card game.[9] In such game the result of each event is completely predetermined by the state of a pack. In other words, the system S' minorizes S and each $e \in S$ has the form

$$\sum_{x \in S', x \leq e} x.$$

[8]A. N. Kolmogorov [1].

[9]One of those games is implied in which the role of gamblers reduces to passive observation of the combinations of cards.

After distinguishing the subsystem S', the "classic" ascribes to the "elementary" events the same probability equal to $\frac{1}{m}$, where m is the number of elements of S'; the other probabilities are determined automatically. In other words, the original system S must be a discrete algebra of the type $2^{S'}$; and the sought probability coincides with the well-known "basic" measure on $2^{S'}$. We have already noted that the basic measure is uniquely determined by the invariance property under all automorphisms of the BA $2^{S'}$. Moreover, it suffices to require invariance under the automorphisms constituting some ergodic group $\mathfrak{A}$. The ergodicity condition means the $\mathfrak{A}$-congruence of each two "elementary" events of S'. The automorphism group itself reflects a physical symmetry inherent in the statement of the problem. The "classical probability" may be characterized as the unique probability measure on S invariant under all automorphisms of some ergodic group.

The above discussion shows that in the classical definition of probability some automorphism group is always present implicitly. This circumstance permits us to extend the classical method for defining a probability measure on an infinite algebra when there is no complete disjoint set S' of equiprobable generators but there is an automorphism group reflecting the actual symmetry of a physical (or other) system. We demonstrate that, under some conditions, THE AUTOMORPHISM GROUP OF A BA $\mathscr{X}$ GENERATES A MEASURE on $\mathscr{X}$ which is invariant under all automorphisms belonging to this automorphism group and EACH measure of this type is obtainable so.

In this section (among the others) we consider the two related problems:

I. *Let $\mathscr{X}$ be a complete BA, and let $\mathfrak{A}$ be some automorphism group on $\mathscr{X}$. What are the necessary and sufficient conditions for $\mathscr{X}$ to admit an $\mathfrak{A}$-invariant measure?*

II. *The same problem with an a priori assumption that the algebra $\mathscr{X}$ is normable.*

We will see that the second problem is essentially easier than the first.

2.2 Necessary conditions for existence of an invariant measure

Let $\mathfrak{A}$ be an automorphism group of a complete BA $\mathscr{X}$. The elements x and y are called *congruent* (if need be, $\mathfrak{A}$-congruent) if for some $A \in \mathfrak{A}$ we have $y = Ax$. We call x and y *equipartite* (respectively $\mathfrak{A}$-equipartite) if they can be represented as

$$x = \sum_{\alpha} x_{\alpha}, \quad y = \sum_{\alpha} y_{\alpha}, \tag{1}$$

where x_α and y_α are congruent for all α of the index set (which may be of arbitrary cardinality). If both sums in (1) are finite then we say that x and y are *finitely equipartite*. We denote the congruence by the symbol $\approx$ or $\approx_{\mathfrak{A}}$, the equipartite relation by $\sim$ or $\sim_{\mathfrak{A}}$, and the finite equipartite relation by $\simeq$ or $\simeq_{\mathfrak{A}}$. *These relations are symmetric and transitive.* We only prove the transitivity of $\sim$. Let $x \sim y$ and $y \sim z$. This means that $x = \sum_\alpha x_\alpha$, $y = \sum_\alpha y_\alpha$, and also $y_\alpha = A_\alpha x_\alpha$. At the same time, $y = \sum_\beta y'_\beta$, $z = \sum_\beta z_\beta$, and $y'_\beta = A'_\beta z_\beta$. Put

$$y_{\alpha\beta} \equiv y_\alpha \wedge y'_\beta,\ \ z_{\alpha\beta} \equiv {A'}_\beta^{-1} y_{\alpha\beta},\ \ x_{\alpha\beta} \equiv A_\alpha^{-1} y_{\alpha\beta}.$$

It is clear that

$$x = \sum_{\alpha,\beta} x_{\alpha\beta},\ \ z = \sum_{\alpha,\beta} z_{\alpha\beta},\ \ z_{\alpha\beta} = {A'}_\beta^{-1} A_\alpha x_{\alpha\beta},$$

whence $x \sim z$.

We now formulate some conditions that characterize the set $\mathfrak{A}$ and are undoubtedly *necessary* for existence of an $\mathfrak{A}$-invariant measure:

(C_1) *From $x_n \longrightarrow \mathbf{0}$, it follows that $A_n x_n \longrightarrow \mathbf{0}$ for all $A_n \in \mathfrak{A}$.*

Here and in the sequel, the arrow $\longrightarrow$ denotes convergence in the (o)-topology or, equivalently, the metric convergence ("in measure").

(C_2) *From $x_n \approx x_{n+1}$ $(n = 1, 2, \dots)$ and $x_n\, d\, x_m$ $(n \neq m)$, it follows that $x_1 = x_2 = \dots = \mathbf{0}$.*

(C_3) *If $x_n = \sum_\alpha x_{n\alpha}$ and $x_n \longrightarrow \mathbf{0}$ then $\bigvee_\alpha A_{n\alpha} x_{n\alpha} \longrightarrow \mathbf{0}$ for all $A_{n\alpha} \in \mathfrak{A}$ (here the index α ranges over an arbitrary set depending on α).*

(C_4) *The relations $x < y$ and $x \sim y$ are incompatible.*

(C_5) *For each $x > \mathbf{0}$ there is a quasimeasure φ invariant under $\mathfrak{A}$ such that $\varphi(x) > \mathbf{0}$.*

We can rephrase the condition (C_5) saying that the group $\mathfrak{A}$ *admits sufficiently many invariant quasimeasures.*

These conditions (call them the conditions of the type (C)) speak about some equicontinuity of automorphisms in the group (what is especially concerned with the condition (C_1)). They say that in $\mathfrak{A}$, there are "not so many" automorphisms.

It is readily seen that (C_3) is stronger than (C_1); in turn, (C_1) is stronger than (C_2). If we replace the sequence $\{x_n\}$ in (C_3) with an arbitrary directed set $\{x_\gamma\}$ then yet a stronger condition is obtained which we denote by (C_3^+). It is easily seen that if the algebra $\mathscr{X}$ is regular then (C_3^+) is equivalent to (C_3).

Lemma 1. (C_3) *implies* (C_4).

PROOF. Assume that (C_4) is not fulfilled. Then there exist x and y such that $x < y$ and

$$x = \sum_\alpha x_\alpha, \ \ y = \sum_\alpha y_\alpha, \ \ y_\alpha = A_\alpha x_\alpha, A_\alpha \in \mathfrak{A}.$$

Put

$$y'_\alpha \equiv y_\alpha \wedge x, \ \ x'_\alpha \equiv A_\alpha^{-1} y'_\alpha, \ \ y''_\alpha \equiv y_\alpha \wedge (y - x),$$
$$x''_\alpha \equiv A_\alpha^{-1} y''_\alpha, \ \ x' \equiv \sum_\alpha x'_\alpha, \ \ x'' \equiv \sum_\alpha x''_\alpha = x - x'.$$

(It is clear that x_α and x''_α are disjoint and the use of the symbol $\sum$ is legitimate.) Note that $x = \sum_\alpha y'_\alpha = \sum_\alpha A_\alpha x'_\alpha \sim x'$. By transitivity of the relation $\sim$, we have $y \sim x'$. Further,

$$y - x = \sum_\alpha y''_\alpha = \sum_\alpha A_\alpha x''_\alpha \sim x''.$$

Now, put $y - x \equiv x_1$ and $x'' \equiv x_2$ and repeat the above arguments on taking x' as x. Continuing this process, we come to the sequence of elements $\{x_n\}$ having the following properties:

a) $x_n \, d \, x_m \quad (n \neq m)$,

b) $x_{n+1} \sim \sum_{k=1}^n x_k \quad (n = 1, 2, \ldots)$.

Clearly, $x_n \longrightarrow \mathbf{0}$. It follows from (C_3) that $\sum_{k=1}^n x_k \longrightarrow \mathbf{0}$. Then $x_1 = \mathbf{0}$ and $x = y$. The lemma is proved.

REMARK 1. We see from the proof of the lemma that there is a sequence with the properties a) and b) whenever (C_4) holds. For each $n = 1, 2, \ldots$, it is not difficult to construct an element $x'_n \leq x_n$ equipartite with x_1. We can conclude that (C_4) follows from a weaker assumption than (C_3): there is no infinite sequence of disjoint and equipartite nonzero elements. It is easy to see that this assumption is in fact equivalent to the condition (C_4). We observe that if

$$z_n \sim z_m, \ \ z_n \, d \, z_m \ \ (n \neq m)$$

and $z_n > \mathbf{0}$ $(n = 1, 2, \ldots)$ then the elements $u = \sum_{n=1}^\infty z_n$ and $v = \sum_{n=2}^\infty z_n$ are equipartite, although $v < u$.

REMARK 2. *Assume that the algebra $\mathscr{X}$ is regular. Then the condition (C_4) follows from the following weaker condition:*

(C_3^-) *If $x_n \simeq x_{n+1}$ $(n = 1, 2, \ldots)$ and $x_n \, d \, x_m$ $(n \neq m)$ then $x_1 = x_2 = \cdots = \mathbf{0}$.*

In fact, if (C_4) is not satisfied then by Remark 1, there exists a sequence $\{x_n\}_{n=1}^{\infty}$ with the following properties:

$$x_n > \mathbf{0} \ (n = 1, 2, \ldots), \quad x_n = \sum_{k=1}^{\infty} x_{nk}, \ x_1 = \sum_{k=1}^{\infty} y_{nk},$$

$$y_{nk} = A_{nk} x_{nk}, \ A_{nk} \in \mathfrak{A} \ (n = 2, 3, \ldots; \ k = 1, 2, \ldots).$$

Since $\sum_{k=1}^{s} y_{nk} \uparrow x_1$ for all $n = 1, 2, \ldots$, by regularity of the algebra there is a sequence $\{s_n\}$ satisfying

$$x_1 = (o)\text{-}\lim \sum_{k=1}^{s_n} y_{nk}.$$

We may find an element $y > \mathbf{0}$ such that $\sum_{k=1}^{s_n} y_{nk} \geq y$ for $n \geq n_0$. Put

$$\widetilde{x}_n \equiv \sum_{k=1}^{s_n} A_{nk}^{-1}(y \wedge y_{nk}) \quad (n = 1, 2, \ldots).$$

We see that the disjoint elements $\widetilde{x}_n$ are finitely equipartite with y, and so with one another. But this is incompatible with (C_3^-). Thus, in the regular case, (C_3^-) implies (C_4).

Lemma 2. (C_5) *implies* (C_3^-) (*and* (C_4) *in a regular algebra*).

Proof. If (C_3^-) is not satisfied then there exists a sequence $\{x_n\}$ such that

$$x_1 \simeq x_2 \simeq \ldots; \ x_n \, d \, x_m \ (n \neq m); \ x_n > \mathbf{0}.$$

Denoting one of such elements by x, we use the quasimeasure φ whose existence is guaranteed by the condition (C_5). Clearly, $\varphi(x_1) = \varphi(x_2) = \cdots = \varphi(x) > 0$. At the same time, the series $\sum_n \varphi(x_n)$ must converge, since $\sum_{n=1}^{m} \varphi(x_n) \leq \varphi(\mathbf{1})$ for all m. This contradiction proves the lemma.

Lemma 3. *Let $\mathscr{X}$ be a normable BA. Suppose that the group $\mathfrak{A}$ does not satisfy the property (C_1). Then there exist:* a) *a disjoint sequence $\{x_i\}$ of elements of $\mathscr{X}$;* b) *a sequence $\{A_i\}$ of automorphisms of $\mathfrak{A}$ such that*

$$\overline{\lim}\, abs\, A_i x_i > \mathbf{0}$$

(*the definition of $\overline{\lim}\, abs$ was given in Chapter* 4).

Proof. If the condition (C_1) is not satisfied then for some sequence $\{v_n\}$ of elements of the algebra $\mathscr{X}$ we will have $v_n \longrightarrow \mathbf{0}$ and $\mu B_n v_n > \eta_0$ simultaneously, where B_n are some automorphisms belonging to $\mathfrak{A}$, while μ is a measure in $\mathscr{X}$ and η_0 is a positive constant. Theorem 7.1 permits

us to suppose that $v_n \xrightarrow{(o)} \mathbf{0}$ (otherwise, this can be obtained by dropping to a subsequence). Let

$$y_n \equiv B_n v_n, \quad v_{nm} \equiv v_n \wedge C \bigvee_{i \geq m} v_i,$$

$$u_{nm} \equiv B_n v_{nm} \equiv y_n \wedge C \bigvee_{i \geq m} B_n v_i \quad (n, m = 1, 2, \ldots).$$

For each $n = 1, 2, \ldots$ the automorphism B_n, as every automorphism, preserves (o)-convergence. Therefore,

$$u_{nm} \xrightarrow{(o)} y_n \quad \text{as} \quad m \longrightarrow \infty \quad (n = 1, 2, \ldots)$$

(since $v_i \xrightarrow{(o)} \mathbf{0}$, i.e., $\bigvee_{i \geq m} v_i \xrightarrow{(o)} \mathbf{0}$). Thus,

$$\mu u_{nm} \longrightarrow \mu u_n > \eta_0,$$

and we can choose the indices $1 < m_1 < m_2 < \cdots < m_n < \ldots$ so that

$$\mu u_{nm_n} > \eta_0 \quad (n = 1, 2, \ldots).$$

Since obviously $u_{nm_n} \not\longrightarrow \mathbf{0}$, it can be assumed that

$$\overline{\lim}\, abs\, u_{nm_n} \equiv y > \mathbf{0}$$

(as above, we can easily achieve this by rarefying the sequence). Define the sequence of indices $\{n_i\}$ by the rule

$$n_1 \equiv 1, \;\; n_2 \equiv m_{n_1}, \ldots, n_{i+1} \equiv m_{n_i}, \ldots .$$

We now put

$$A_i \equiv B_{n_i}, \quad x_i \equiv v_{n_i m_{n_i}} \quad (i = 1, 2, \ldots).$$

Let $i_1 < i_2$. Then $n_{i_2} \geq m_{n_{i_1}}$. Therefore,

$$x_{i_2} = v_{n_{i_2} m_{n_{i_2}}} \leq v_{n_{i_2}} \leq \bigvee_{k \geq m_{n_{i_1}}} v_k$$

and

$$x_{i_2} \wedge x_{i_1} = x_{i_2} \wedge \Big(v_{n_{i_1}} \wedge C \bigvee_{k \geq m_{n_{i_1}}} v_k \Big) = \mathbf{0}.$$

We see that $\{x_i\}$ is a disjoint sequence. At the same time, using the main property of $\overline{\lim}\, abs$ (see p. 199), we infer that

$$\overline{\lim}\, abs\, A_i(x_i) = \overline{\lim}\, abs\, B_{n_i}(v_{nm_{n_i}}) = \overline{\lim}\, u_{n_i m_{n_i}} = \overline{\lim}\, u_{nm_n} > \mathbf{0}.$$

The proof of the lemma is complete.

From Lemma 3 we derive an important

Corollary. *In the case of a normed algebra,* (C_4) *implies* (C_1).

PROOF. If (C_1) is not satisfied then there exist sequences $\{x_i\}$ and $\{A_i\}$ with the properties listed in the statement of Lemma 3. Divide the positive integers into countably many disjoint indefinitely increasing sequences $\{i_{sj}\}_{s=1}^{\infty}$ $(j = 1, 2, \ldots)$. By the main property of $\overline{\lim}\, abs$, for all $j = 1, 2, \ldots$, we have

$$\bigvee_{s=1}^{\infty} A_{i_{sj}}(x_{i_{sj}}) \geq \overline{\lim}\, A_{i_{sj}}(x_{i_{sj}}) = \overline{\lim}\, A_i(x_i) \equiv y.$$

Given i, put

$$y_i \equiv (A_i(x_i)) \wedge y, \quad x_i' \equiv A_i^{-1}(y_i).$$

All elements x_i' are disjoint (since $x_i' \leq x_i$). Moreover,

$$\bigvee_{s=1}^{\infty} y_{i_{sj}} = y \wedge \bigvee_{s=1}^{\infty} A_{i_{sj}}(x_{i_{sj}}) = y \quad (j = 1, 2, \ldots).$$

Further, put

$$\widetilde{y}_{1j} \equiv y_{i_{1j}},\ \widetilde{y}_{2j} \equiv y_{i_{2j}} \wedge C y_{i_{1j}}, \ldots,\ \widetilde{y}_{sj} \equiv y_{i_{sj}} \wedge C \bigvee_{k=1}^{s-1} y_{i_{kj}}, \ldots;$$

$$\widetilde{x}_{sj} \equiv A_{i_{sj}}^{-1}(\widetilde{y}_{sj}) \quad (s, j = 1, 2, \ldots).$$

The inequalities

$$\widetilde{y}_{sj} \leq y_{sj},\ \widetilde{x}_{sj} \leq A_{i_{sj}}^{-1}(y_{sj}) = x_{i_{sj}}', \ldots$$

show that $\widetilde{x}_{sj}$ are disjoint. Moreover, the elements $\widetilde{y}_{sj}$ are disjoint too, and also for all $j = 1, 2, \ldots$, we have

$$\sum_s \widetilde{y}_{sj} = \sum_s A_{i_{sj}}(\widetilde{x}_{sj}) = y.$$

Putting

$$\overline{x}_j \equiv \sum_s \widetilde{x}_{sj} \quad (j = 1, 2, \ldots),$$

we see that $\overline{x}_j \sim y$ for all $j = 1, 2, \ldots$. This implies that the elements $\overline{x}_j$ constitute an infinite sequence of disjoint equipartite elements. The existence of such sequence contradicts (C_4) (in view of the Remark 1).

The assumption of normability of an algebra $\mathscr{X}$, which was stipulated in Lemma 3 and its corollary, was introduced only to simplify proofs. In fact, it suffices to assume the regularity of $\mathscr{X}$. Moreover, elaborating the proof, we can derive from (C_4) not only (C_1) but also (C_3). We mentioned that all conditions of the type (C) are necessary for existence of an invariant measure. This can be easily shown by Theorem 7.1. Also, it is not difficult to see that (C_2) and (C_5) are necessary for existence of an essentially positive invariant quasimeasure.

Henceforth, we agree to call an automorphism group *Liouville*[10] if it admits an invariant measure. Each automorphism generating a Liouville group of powers is also called *Liouville*.

Thus, a Liouville group always possesses the properties of the type (C). However, the conditions of the type (C) are insufficient by themselves: there might be no measure on an algebra whereas the group consisting of the sole identity automorphism satisfies all conditions of this type.

2.3 Existence of an invariant measure on a fully homogeneous algebra. Conditions of normability

In this subsection we establish that the "strong" equicontinuity of automorphisms of an ergodic group, expressed by the conditions (C_3) and (C_4), is not only necessary but also sufficient for existence of an invariant measure. A characterization of "homogeneous" normed algebras in terms not connected with the concept of measure will be given simultaneously. In conclusion, some necessary and sufficient conditions for normability of a complete Boolean algebra are given.

Lemma 4. *Let $\mathscr{X}$ be a complete BA, and let $\mathfrak{A}$ be an ergodic group of automorphisms $\mathscr{X}$ satisfying (C_4). There exists an ergodic group $\mathfrak{A}^*$ of automorphisms of $\mathscr{X}$ containing $\mathfrak{A}$ such that*

a) *the relations*

$$x \sim_{\mathfrak{A}} y, \quad x \sim_{\mathfrak{A}^*} y, \quad x \approx_{\mathfrak{A}^*} y$$

are equivalent;

b) *the group $\mathfrak{A}^*$ has the property (C_4) as before.*

Proof. Let $x \sim_{\mathfrak{A}} y$ and $x, y \neq \mathbf{1}$ (in view of the condition (C_4), x and y may equal to unity only simultaneously). The equipartite relation

[10]The first theorem on an invariant measure is connected historically with the name of J. Liouville.

means that

$$x = \sum_\eta x_\eta,\ \ y = \sum_\eta y_\eta,\ \ y_\eta = A_\eta(x_\eta),\ \ A_\eta \in \mathfrak{A}.$$

Put $\overline{x} \equiv Cx$ and $\overline{y} \equiv Cy$. It follows from ergodicity that there exists $B_1 \in \mathfrak{A}$ for which

$$\overline{y} \geq B_1(\overline{x}) \wedge \overline{y} \equiv \overline{y}_1 > \mathbf{0}.$$

Put $\overline{x}_1 \equiv B_1^{-1}(\overline{y}_1)$. Clearly, $\overline{x}_1 \leq \overline{x}$. Assume that for the ordinal number α_0 we have already constructed

$$\overline{x}_1, \overline{x}_2, \dots, \overline{x}_\alpha;\ \overline{y}_1, \overline{y}_2, \dots, \overline{y}_\alpha \quad (\alpha < \alpha_0),$$

where $\overline{x}_\alpha \, d \, \overline{x}_{\alpha'}$, $\overline{y}_\alpha \, d \, \overline{y}_{\alpha'}$ for $\alpha \neq \alpha'$, and $\overline{y}_\alpha = B_\alpha(\overline{x}_\alpha)$.

We could treat the elements

$$C\Big(\sum_\eta x_\eta + \sum_{\alpha<\alpha_0} \overline{x}_\alpha\Big), \quad C\Big(\sum_\eta y_\eta + \sum_{\alpha<\alpha_0} \overline{y}_\alpha\Big)$$

in much the same manner as $\overline{x}$ and $\overline{y}$. In spite of this, we will construct the elements $\overline{x}_{\alpha_0}, \overline{y}_{\alpha_0}$ and the automorphism $B_{\alpha_0} \in \mathfrak{A}$ such that

$$B_{\alpha_0}(\overline{x}_{\alpha_0}) = \overline{y}_{\alpha_0},$$

$$\overline{x}_{\alpha_0} \leq C\Big(\sum_\eta x_\eta + \sum_{\alpha<\alpha_0} \overline{x}_\alpha\Big), \quad \overline{y}_{\alpha_0} \leq C\Big(\sum_\eta y_\eta + \sum_{\alpha<\alpha_0} \overline{y}_\alpha\Big).$$

This process terminates provided that $\sum_{\alpha<\alpha_0} \overline{x}_\alpha = \overline{x}$ or $\sum_{\alpha<\alpha_0} \overline{y}_\alpha = \overline{y}$. Show that, in fact, these equalities can be only fulfilled simultaneously. Indeed, for example if $\sum_{\alpha<\alpha_0} \overline{x}_\alpha = \overline{x}$ then, arranging all $x_1, x_2, \dots$; $\overline{x}_1, \overline{x}_2, \dots$ into a transfinite sequence $\{z_\gamma\}$ and all operators $A_1, A_2, \dots$; $B_1, B_2, \dots$ into a sequence $\{C_\gamma\}$, we see that $\sum_\gamma z_\gamma = \mathbf{1}$. Then by (C_4), we have $\sum_\gamma C_\gamma z_\gamma = \mathbf{1}$ and, obviously, $\sum_{\alpha<\alpha_0} \overline{y}_\alpha = \overline{y}$. Analogously, the second equality implies the first.

Define an automorphism U of the algebra $\mathscr{X}$ by the following formula:

$$U(z) \equiv \sum_\gamma C_\gamma(z \wedge z_\gamma). \tag{2}$$

Soundness of this definition is almost evident. It is easy to check that the terms of this sum are mutually disjoint, and that the formula (2) really defines an automorphism. Namely, it is obvious that $y = U(x)$. We denote by $\mathfrak{A}^*$ the set of all automorphisms having a similar structure. More exactly, an operator T is included in $\mathfrak{A}^*$ if and only if there exist

a) a family of disjoint elements $\{u_\xi\}$ with sum the unity of the algebra;

b) a family of automorphisms $A_\xi \in \mathfrak{A}$ having the properties $A_\xi(u_\xi) \wedge A_{\xi'}(u_{\xi'}) = \mathbf{0}$ $(\xi \neq \xi')$, and such that

$$T(u) = \sum_\xi A_\xi(u \wedge u_\xi)$$

for all $u \in \mathscr{X}$.

Each of these operators is, obviously, an automorphism. The inverse automorphism is defined by the formula $u = T^{-1}(v) = \sum_\xi A_\xi^{-1}(v \wedge v_\xi)$, where $v_\xi = A_\xi(u_\xi)$. It is included in $\mathfrak{A}^*$ too. Clearly, $\mathfrak{A} \subset \mathfrak{A}^*$.

Assume that $A, B \in \mathfrak{A}^*$, $T \equiv AB$, $A(y) = \sum_\xi A_\xi(y \wedge y_\xi)$, and $B(x) = \sum_\xi B_\xi(x \wedge x_\xi)$. We show that $T \in \mathfrak{A}^*$. It is easy to check that

$$T(x) = \sum_{\xi,\eta} A_\xi B_\eta(z_{\xi\eta} \wedge x),$$

where $z_{\xi\eta} \equiv x_\eta \wedge B_\eta^{-1}(y_\xi)$. Moreover, $\sum_{\xi,\eta} z_{\xi\eta} = \mathbf{1}$, and $A_\xi B_\eta(z_{\xi\eta})$ are disjoint.

Thus, $T \in \mathfrak{A}^*$. We see that $\mathfrak{A}^*$ is an automorphism group. The equivalence of the relations $x \sim_{\mathfrak{A}} y$, $x \sim_{\mathfrak{A}^*} y$, and $x \approx_{\mathfrak{A}^*} y$ is immediate from the definition of $\mathfrak{A}^*$.

In much the same way, it is obvious that $\mathfrak{A}^*$ is ergodic and has the property (C_4). The proof of the lemma is complete.

We will write $x \succ_{\mathfrak{A}^*} y$ (sometimes simply $x \succ y$) if there exists $u \leq x$ such that $u \approx_{\mathfrak{A}^*} y$. It is clear from the proof of Lemma 4 that *we always have either $x \approx_{\mathfrak{A}^*} y$ or $x \succ_{\mathfrak{A}^*} y$, or $x \prec_{\mathfrak{A}^*} y$.* We will express this property of the group by saying that the group (in our case $\mathfrak{A}^*$) *equates* each two elements. Note some facts connected with this property of a group. We write further simply, omitting the letter $\mathfrak{A}^*$.

1) *If $u + v \approx u' + v'$ and $u \approx u'$ then $v \approx v'$.*

Indeed, if for instance $v \approx v'' < v$ then $u'+v' \approx u+v \approx u'+v'' < u'+v'$ which is impossible by the condition (C_4). In much the same way, the relation $v \approx v'' > v'$ is impossible. Then $v \approx v'$, since the group equates the elements.

2) *If $x \succ y$ then $Cx \prec Cy$.*

Indeed, otherwise we would have $Cx \succ Cy$ and $Cy \approx v < Cx$, and simultaneously $y \approx u \leq x$ and $\mathbf{1} = y + Cy \approx u + v < \mathbf{1}$, which again contradicts (C_4).

3) *If $x \succ y$ then $v \approx x$ for some $v \geq y$.*

In fact, we have $Cx \prec Cy$, $Cx \approx u \leq Cy$. According to 1), $x \approx Cu \geq y$. We can take $v = Cu$.

4) *If $x \prec y \prec z$, $x \approx z$, then $x \approx y \approx z$.*

By what was proved above, $z \approx x \approx u \leq y \leq v \approx z$. If $u < y$ or $y < v$ then we obtain $u < v$, $u \approx v$, which is impossible. Thus, $u = y = v$, $x \approx y \approx z$.

5) *Let $x \approx y$, $x = x_1 + x_2 + \cdots + x_m$, and $y = y_1 + y_2 + \cdots + y_m$. If $x_1 \approx x_2 \approx \cdots \approx x_m$ and $y_1 \approx y_2 \approx \cdots \approx y_m$ then*

$$x_i \approx y_i \ \ (i, j = 1, 2, \dots, m).$$

Assume the contrary. Then either $x_1 \succ y_1$ $(x_1 \not\approx y_1)$ or $x_1 \prec y_1$ $(x_1 \not\approx y_1)$. For example, consider the first case. We have $x_1 \approx x_2 \approx \cdots \approx x_m$ and $y_1 \approx y_2 \approx \cdots \approx y_m$ whence $x_i \succ y_i$ $(i = 1, 2, \dots, m)$ and $x_i \not\approx y_i$. There exist $u_i < x_i$ such that $u_i \approx y_i$ $(i = 1, 2, \dots, m)$. Hence,

$$y \approx x = x_1 + \cdots + x_m \approx u_1 + \cdots + u_m < y.$$

We again come to a contradiction with (C_4).

Definition. A complete BA is called *fully homogeneous* if it has an ergodic automorphism group satisfying (C_3).

We will see further that every fully homogeneous algebra is regular. If we a priori assume regularity then, as it may be shown, (C_3) in the definition given above can be replaced with another condition of the type (C). The full homogeneity of $\mathscr{X}$ means the existence in $\mathscr{X}$ of an automorphism group $\mathfrak{A}$ which, on the one hand, is sufficiently rich (ergodic) but also, on the other hand, has the property of STRONG EQUICONTINUITY. Namely, there must exist a base of neighborhoods of zero in the (o)-topology, each of which contains all elements of the form $\bigvee_\alpha A_\alpha(x_\alpha)$, $A_\alpha \in \mathfrak{A}$ together with the element $x = \sum_\alpha x_\alpha$. (This is equivalent to the condition (C_3^+); in the regular case we use only the finite sums.) It is easy to see that a fully homogeneous algebra may be either discrete or continuous. It is also easy to verify that a discrete algebra is fully homogeneous if and only if this algebra is finite.

Theorem 3. *A fully homogeneous algebra is always normable. In this case, if $\mathfrak{A}$ is an ergodic automorphism group of an algebra satisfying (C_3) then there exists a unique probability measure invariant under $\mathfrak{A}$.*

PROOF. We will rely upon the condition (C_4) which is fulfilled by Lemma 1. The main Lemma 4 permits us to suppose that the relations $x \approx y$ and $x \sim y$ are equivalent and the group $\mathfrak{A}$ equates the elements. The last assertion means, as we have agreed, that for all $x, y \in \mathscr{X}$ one of the three relations $x \succ y$, $x \prec y$, $x \approx y$ is always valid. We finally assume the algebra $\mathscr{X}$ to be continuous. (Otherwise, as was mentioned, the algebra would be finite and the theorem would have a trivial proof.)

Now, we prove some auxiliary propositions. We will say that an element $x \in \mathscr{X}$ is *divisible* if x can be represented as the sum of two congruent elements: $x = x_1 + x_2,\ x_1 \approx x_2$. The following holds:

1°. *If $x = \sum_\alpha x_\alpha$, with all x_α divisible, then x is divisible too.*

Proof. We have $x_\alpha = x'_\alpha + x''_\alpha$ and $x'_\alpha \approx x''_\alpha$. Put $x_1 \equiv \sum_\alpha x'_\alpha$ and $x_2 \equiv \sum_\alpha x''_\alpha$. It is clear that $x = x_1 + x_2$ and $x_1 \approx x_2$.

2°. *Every element $x \in \mathscr{X}$ is divisible.*

Proof. By 1°, it suffices to demonstrate that the divisible elements comprise the set that minorizes $\mathscr{X}$. Take an arbitrary $x > \mathbf{0}$. By continuity of the algebra, x can be represented as the sum of two nonzero elements: $x = y_1 + y_2$. We may assume that $y_1 \succ y_2$. This means that $y_1 = y + y''$ and $y'' \approx y_2$. The element $y = y'' + y_2$ is divisible and $y \le x$. It remains to refer to the exhaustion principle (the corollary to Theorem 2.2).

Item 2° immediately yields *the existence of a decreasing sequence $\{x_n\}_{n=1}^\infty$ with the following properties:*

1) $x_0 = \mathbf{1}$,

2) $x_n - x_{n+1} \approx x_{n+1}$.

3°. *There exists a sequence $\{\overline{x}_n\}_{n=1}^\infty$ with disjoint entries such that $\overline{x}_n \approx x_n$ for all n.*

In fact, we can put $\overline{x}_{n+1} \equiv x_n - x_{n+1}$, $n = 0, 1, 2, \ldots$.

4°. *Let*

$$x \equiv \sum_{k=1}^p y_k,\ \ y_k \approx x_{n_k}\ \ (k = 1, 2, \ldots, p),\ \ \sigma_0 \equiv \sum_{k=1}^p \frac{1}{2^{n_k}},\ \ m \ge \max_{1 \le k \le p} n_k.$$

Then there exists a representation

$$x = x^{(1)} + x^{(2)} + \cdots + x^{(s)},\ \ x^{(i)} \approx x_m\ \ (i = 1, 2, \ldots, s),\ \ s = 2^m \sigma_0.$$

To prove this proposition we observe above all that for $m > n$, each element x_n can be represented as a disjoint sum of 2^{m-n} terms congruent with x_m. This can be established by induction on $k = m - n$. Indeed, for $k = 1$ we have

$$x_n = x_{n+1} + \overline{x}_{n+1},\ \ x_{n+1} \approx \overline{x}_{n+1}.$$

Suppose that the required representation exists for $k = s$. Then $x_n = u_1 + u_2 + \cdots + u_{2^s}$, where $u_i \approx x_{n+s}$ $(i = 1, 2, \ldots, 2^s)$. Split each of the elements $u_1, u_2, \ldots, u_{2^s}$ into two congruent parts: $u_1 = u'_1 + u''_1, \ldots, u_{2^s} = u'_{2^s} + u''_{2^s}$. By 5) (see p. 410), all u'_i and u''_i are congruent,

and since, moreover, $x_{n+s} = x_{n+s+1} + \overline{x}_{n+s+1}$ and $x_{n+s+1} \approx \overline{x}_{n+s+1}$, they are congruent with x_{n+s+1}. Collecting all terms, we obtain a sought representation for x_n when $m - n = s + 1$.

Now, taking $x = \sum_1^p y_k$, with $y_k \approx x_{n_k}$, we divide each x_{n_k} into 2^{m-n_k} disjoint terms that are congruent with x_m, and then collect them. We thus come to the sum indicated in the formulation with the following number of terms:

$$\sum_1^p 2^{m-n_k} = 2^m \sum_1^p \frac{1}{2^{n_k}} = 2^m \sigma_0.$$

5°. *Let*

$$x = \sum_{k=1}^r u_k, \;\; y = \sum_{k=1}^s v_k, \;\; u_k \approx x_{n'_k} \;\; (k = 1, 2, \dots, r), \;\; v_k \approx x_{n''_k}$$

$$(k = 1, 2, \dots, s); \quad \sigma' = \sum_{k=1}^r \frac{1}{2^{n'_k}}, \;\; \sigma'' = \sum_{k=1}^s \frac{1}{2^{n''_k}}, \;\; \sigma' \le \sigma''.$$

Then $x \prec y$. In this case, if $\sigma' < \sigma''$ then x and y cannot be congruent.

To prove this proposition, let m be a maximal number of $n'_1, \dots, n''_s$. Using 4°, we can write

$$x = x^{(1)} + \dots + x^{(\mu)}, \;\; y = y^{(1)} + \dots + y^{(\nu)},$$

$$x^{(1)} \approx x^{(2)} \approx \dots \approx y^{(1)} \approx \dots \approx y^{(\nu)}.$$

Since $\mu = 2^m \sigma'$ and $\nu = 2^m \sigma''$; therefore, $\mu \le \nu$. It is clear that $x \prec y$, and for $\sigma' < \sigma''$, the elements x and y are not congruent.

Corollary. *If $x \approx y$, $x = \sum_{k=1}^{\mu} u_k$, and $y = \sum_{k=1}^{\nu} v_k$, where $u_k \approx x_{n'_k}$ and $v_k \approx x_{n''_k}$, then*

$$\sum_{k=1}^{\mu} \frac{1}{2^{n''_k}} = \sum_{k=1}^{\nu} \frac{1}{2^{n''_k}}.$$

6°. *Let*

$$x = \sum_{k=1}^p u_k, \;\; y = \sum_{k=1}^q v_k, \;\; u_k \approx x_{n'_k}, \;\; v_k \approx x_{n''_k},$$

$$x > y, \quad \sigma' = \sum_{k=1}^p \frac{1}{2^{n'_k}}, \quad \sigma'' = \sum_{k=1}^q \frac{1}{2^{n''_k}}.$$

Then there exists a representation of the element $w \equiv x - y$ as the finite sum $w = \sum_{k=1}^{r} w_k$, with $w_k \approx x_{n_k}$. Moreover, $\sum_{k=1}^{r} \frac{1}{2^{n_k}} = \sigma' - \sigma''$.

PROOF. By 4°, we have

$$x = \sum_{k=1}^{\mu} z_k, \;\; y = \sum_{k=1}^{\nu} z'_k, \;\; x_m \approx z_1 \approx z_2 \approx \cdots \approx z'_1 \approx z'_2 \approx \cdots,$$

where $m \equiv \max(n'_1, n'_2, \dots, n'_p, n''_1, n''_2, \dots, n''_q)$. It follows from (C_4) that $\mu > \nu$. Put $\overline{x} \equiv z_1 + z_2 + \cdots + z_\nu$ and $\overline{\overline{x}} \equiv z_{\nu+1} + z_{\nu+2} + \cdots + z_\mu$. We see that $\overline{x} \approx y$. Then $\overline{\overline{x}} \approx w$.[11] The second assertion follows from the above corollary.

7°. *Suppose that the element x is represented in two different ways as a finite or infinite sum:*

$$x = \sum_k u_k, \;\; x = \sum_k v_k, \;\; u_k \approx x_{n_k}, \;\; v_k \approx x_{n'_k}.$$

Then[12]

$$\sum_k \frac{1}{2^{n'_k}} = \sum_k \frac{1}{2^{n''_k}}.$$

PROOF. Suppose the contrary. In this event we may assume that $\sigma' < \sigma''$. Choose m so that the inequality $2^{-m} < \sigma'' - \sigma'$ be valid. If the sum $\sum_k v_k$ contains finitely many terms then we will assume that its last term v_{k_0} is such that $n''_{k_0} \geq m$. (Otherwise, we can represent it as $v_{k_0} = \sum_i \overline{v}_i$, where $\overline{v}_1 \approx \overline{v}_2 \approx \cdots \approx \overline{v}_{2^{m-n''_{k_0}}} \approx x_m$.) Now, we may assume that for some p_0 the following inequalities hold:

$$\sigma' < \sum_{k=1}^{p_0} \frac{1}{2^{n''_k}}, \quad \overline{x}_1 \equiv \sum_{k=1}^{p_0} v_k < x.$$

By 5°, there exists $\overline{u}_1 \approx u_1$ such that $\overline{u}_1 < \overline{x}_1$. Put $\overline{x}_2 \equiv \overline{x}_1 - \overline{u}_1$. We infer from 6° that

$$\overline{x}_2 = \sum_{k=1}^{s} w_k, \;\; w_k \approx x_{\overline{n}_k},$$

and also

$$\sum_{k=1}^{s} \frac{1}{2^{\overline{n}_k}} = \sum_{k=1}^{p_0} \frac{1}{2^{n''_k}} - \frac{1}{2^{n'_1}} > 0.$$

---

[11] By 1) (see p. 409).

[12] In the previous notation, $\sigma' = \sigma''$.

There exists an element $\overline{u}_2 \approx u_2$ such that $\overline{u}_2 < \overline{x}_2$. Now, put $\overline{x}_3 \equiv \overline{x}_2 - \overline{u}_2$. Repeating this argument, we obtain a sequence $\overline{u}_1, \overline{u}_2, \ldots$ (finite or infinite) such that

$$\sum_k \overline{u}_k \leq \overline{x}_1, \ \ \overline{u}_k \approx u_k, \ \ k = 1, 2, \ldots .$$

Then $\sum_k \overline{u}_k \approx \sum_k u_k = x$ and $\sum_k \overline{u}_k \leq \overline{x}_1 < x$. We come to a contradiction with (C_4). Proposition 7° is proved.

8°. *$\mathscr{X}$ minorizes the set of all Ax_n $(n = 1, 2, \ldots)$ such that $A \in \mathfrak{A}$.*

Otherwise, since the group equates the elements, there exists a sequence $\{A_n\}_{n=1}^{\infty}$ such that $x \equiv \bigwedge_n A_n \overline{x}_n > \mathbf{0}$. Putting $\overline{\overline{x}}_n \equiv A_n^{-1} x$, we see that all $\overline{\overline{x}}_n$ are disjoint, and $\overline{\overline{x}}_1 \approx \overline{\overline{x}}_2 \approx \cdots \approx x > \mathbf{0}$. Then

$$\sum_{n=1}^{\infty} \overline{\overline{x}}_n \approx \sum_{n=1}^{\infty} \overline{\overline{x}}_{2n},$$

which contradicts the condition (C_4). Now, Theorem 3 can be proved with no efforts. Each element $x > \mathbf{0}$ is represented by 8° as the sum $x = \sum_{\alpha} u_{\alpha}$, where each u_{α} is congruent with one of the elements x_n. This sum contains at most countably many terms; otherwise there would exist an infinite sequence of disjoint and congruent terms, which is impossible (for example, see the proof of 8°). Thus, we always have

$$x = \sum_k u_k, \quad u_k \approx x_{n_k}. \tag{3}$$

The sum $\sum_k \frac{1}{2^{n_k}}$ does not depend (by 7°) on the choice of the representation of x in the form (3). Denote the value of this sum by $\psi(x)$. It is clear that

a) $\psi(x) > 0$,

b) $\psi\Big(\sum_k y_k\Big) = \sum_k \psi(y_k)$ (this follows from 7°),

c) $\psi(\mathbf{1}) = 1$ (also by 7°),

d) $\psi(x) \leq \psi(\mathbf{1}) < +\infty$.

Moreover, putting $\psi(\mathbf{0}) = 0$, we have constructed a measure. Its invariance is obvious; the uniqueness property follows immediately from the ergodicity of the group (cf. 2.2.2). The proof of the theorem is complete.

Corollary. *The ergodic group satisfying (C_4) also obeys (C_3).*

Indeed, the proof of the theorem relies on (C_4). We now see that if $x_n \longrightarrow \mathbf{0}$, $x_n = \sum_\alpha x_{n\alpha}$, and $y_n = \bigvee_\alpha A_{n\alpha}(x_{n\alpha})$; then $\psi(y_n) \longrightarrow 0$ since

$$\psi(y_n) \leq \sum_\alpha \psi(A_{n\alpha}(x_{n\alpha})) = \psi(x_n),$$

and so $y_n \longrightarrow \mathbf{0}$. Thus, the group $\mathfrak{A}$ also satisfies the condition (C_3) that is the strongest among the conditions of the type (C).

Thus, a fully homogeneous algebra is an algebra with an ergodic automorphism group such that the inequality $x < y$ cannot be valid for any equipartite elements x and y.

Clearly, a fully homogeneous BA is always homogeneous by weight; see Chapter 2. Even the existence of an ergodic automorphism group is sufficient for a "simple" homogeneity. In the next chapter we will give a complete list of fully homogeneous BAs, and show that each NBA splits into fully homogeneous bands. After that the discussion of normability will be over.

In the proof of Theorem 3, the automorphisms constituting the group $\mathfrak{A}$ did not explicitly appear anywhere. Only the main properties of the relation $\approx$ were used. We list these properties once more.

I) The relation $\approx$ is an equivalence; in other words, it is reflexive, symmetric, and transitive.

II) The equipartite property with respect to the relation $\approx$ coincides with this the latter. More precisely, if $x = \sum_{\alpha \in A} x_\alpha$, $y = \sum_{\alpha \in A} y_\alpha$ and $y_\alpha \approx x_\alpha$ for all $\alpha \in A$ then $x \approx y$. (Here A is a set of an arbitrary cardinality; it follows from the theorem that, in fact, A cannot be uncountable.)

III) The relations $x < y$ and $x \approx y$ are incompatible.

IV) For all x and y, we have either $x \prec y$ or $x \succ y$.

All other properties, used in the proof, follows from these four.

Thus, the following theorem is in fact available.[13]

Theorem 4. *If there is a relation $\approx$ with the properties* I)–IV) *in a complete $\mathscr{X}$ then there exists a unique probability measure μ on $\mathscr{X}$ such that $x \approx y$ implies $\mu x = \mu y$.*

In what follows, it will become clear that an algebra homogeneous by weight with such equivalence is fully homogeneous, and so it has an ergodic automorphism group with the properties of the type (C).

[13] The above proof of uniqueness is not literally preserved but in fact implies another proof. The matter is that the measures of the elements x_n are uniquely determined implying the same for the remaining elements x.

Each function φ on a BA $\mathscr{X}$ generates the equivalence $\widetilde{\varphi}$ corresponding to the partition of $\mathscr{X}$ into the level sets of φ; i.e.,

$$x \sim_\varphi y \longleftrightarrow \varphi(x) = \varphi(y).$$

Theorem 5. *Let φ be a mapping from a complete continuous BA $\mathscr{X}$ into a Hausdorff topological space U with the following properties:*

1) *$\varphi(x) \neq \varphi(y)$ for $x < y$;*

2) *for every chain $\mathbf{C} \subset \mathscr{X}$ we have*

$$\varphi(\sup \mathbf{C}) = \lim_{x \in \mathbf{C}} \varphi(x); \quad \varphi(\inf \mathbf{C}) = \lim_{x \in \mathbf{C}} \varphi(x)$$

(in the first case, $\mathbf{C}$ is considered as an increasing directed set; and in the second case, as decreasing);

3) *there exists a function $f : U \times U \longrightarrow U$ such that*

$$\varphi(x + y) = f(\varphi(x), \varphi(y))$$

for all disjoint $x, y \in \mathscr{X}$;

4) *the mapping φ takes all its values on each maximal chain $\mathbf{C}$: $\varphi(\mathbf{C}) = \varphi(\mathscr{X})$; it follows* (from 1)) *that each value is taken on $\mathbf{C}$ exactly once*).

Then $\mathscr{X}$ is a normable BA. There exists a unique probability measure μ connected with φ by the equality

$$\varphi(x) = \Phi(\mu(x)),$$

where Φ is a continuous one-to-one function from $[0, 1]$ to U.

The proof of this theorem reduces to checking that the relation $\sim_\varphi$ satisfies the conditions of Theorem 4.

The conditions I) and III) are obviously fulfilled. For finite partitions, the condition II) follows from 3), and for arbitrary partitions, from 2) by transfinite induction on α (we may assume always that the sums in II) are enumerated by ordinal numbers). It remains to verify I). Each element $x \in \mathscr{X}$ can be included in a maximal chain by the Kuratowski–Zorn Lemma. Hence, for each $y \in \mathscr{X}$, there is an element z in this chain equivalent to y, which is exactly the condition IV).

To complete the proof, we construct the function Φ. For $\alpha \in [0, 1]$, the set $A_\alpha \equiv \{x \mid \mu x = \alpha\}$ is simultaneously the level set of φ; hence, we can denote by $\Phi(\alpha)$ the common value taken by φ at the points A_α.

Find the element $x \in \mathscr{X}$ of measure α and let $\alpha_n \uparrow \alpha$. By continuity of the algebra, there exists a sequence $x_n \uparrow x$ satisfying $\mu x_n = \alpha_n$. Then

$$\Phi(\alpha_n) = \varphi(x_n), \quad \Phi(\alpha) = \varphi(x).$$

Thus, Φ is left continuous; the right continuity is proved similarly. Clearly, Φ is a one-to-one mapping.

Of course, the case in which $U = \mathbb{R}$ is the field of real numbers plays the most important role. In this case the conditions 2) and 4) may be combined if we demand that the function φ is continuous on every chain in $\mathscr{X}$. In this case the function Φ turns out to be strictly monotone.

Theorem 5 was first proved, precisely in this version, in the article [1] by L. I. Potepun. This article also contains a more general theorem that is similar to Theorem 4.

Returning to Theorem 5, consider an important case in which U is a vector space furnished with a separated topology (in particular, $\mathbb{R}$) and $f(u, v) = u + v$, i.e., φ is an additive mapping. The theorem shows that such a mapping with the properties 2) and 4) may exist only on a normable algebra; moreover, the theorem gives a general form of this mapping: $\varphi(x) = \Phi(\mu x)$, where μ is a "φ-invariant" measure. Obviously, the additivity of φ implies the additivity of Φ:

$$\Phi(\mu x + \mu y) = \Phi(\mu(x + y)) = \varphi(x + y) = \varphi(x) + \varphi(y) = \Phi(\mu x) + \Phi(\mu y)$$

(arbitrary nonnegative numbers whose sum is not greater than 1 can be considered as μx and μy, so that the function Φ is additive on the interval $[0, 1]$).

3. The Potepun Theorem

In this section we give one more theorem of normability of a complete BA. The proof of this theorem belonging to A. V. Potepun is published here for the first time.

3.1 Maximal chains

We will often use the Kuratowski–Zorn Lemma. The concept of maximal chain is our main tool.

The greatest and least elements of a chain are called its *endpoints*. If a chain has the endpoints a and b, then we say that it *connects* a and b. Among these there exist maximal chains; we are interesting mainly in the latter. The term a *maximal chain from* a *to* b $(a \leq b)$ *or between* a and b means that this chain is maximal among chains with endpoints a and b (clearly, there are many such chains). For $a = \mathbf{0}$ and $b = \mathbf{1}$, we

usually say simply: a *maximal chain.* Note some simple properties of maximal chains. The symbol E_a^b denotes a maximal chain from a to b. The algebra $\mathscr{X}$ is, as usual, complete and continuous.

1) *The chain E_a^b contains the supremum and infimum of its every subset and is closed in the* (o)*-topology (the order topology).*

2) *The order topology of E_a^b is induced by the* (o)*-topology of the algebra* $\mathscr{X}$.

3) *The chain E_a^b is connected in the* (o)*-topology.*

4) *Let f be a monotone mapping from a maximal chain E_a^b into the Boolean algebra $\mathscr{Y}$ which is continuous with respect to the order topologies of E_a^b and $\mathscr{Y}$. Then the image $f(E_a^b)$ is a maximal chain from $f(a)$ to $f(b)$.*

5) *The set $C(E_a^b) \equiv \{Cx \mid x \in E_a^b\}$ is a maximal chain between Ca and Cb.*

6) *The set $x_0 \vee E_a^b \equiv \{x_0 \vee y \mid y \in E_a^b\}$ is a maximal chain between $x_0 \vee a$ and $x_0 \vee b$.*

7) *The set $x_0 \wedge E_a^b \equiv \{x_0 \wedge y \mid y \in E_a^b\}$ is a maximal chain between $x_0 \wedge a$ and $x_0 \wedge b$.*

3.2 The main theorem

Assume that a complete BA $\mathscr{X}$ admits a measure μ. Thus, to each $\alpha \in (0, \mu\mathbf{1})$ there corresponds the level set $A \equiv A_\alpha \equiv \{x \mid \mu x = \alpha\}$ (A_0 and $A_{\mu\mathbf{1}}$ are singletons and so they are uninteresting). What can we say about A_α? We list its main properties. Assume that $\mathscr{X}$ is *continuous.*

I. *Every maximal chain E in $\mathscr{X}$ meets the set A_α; i.e., $A_\alpha \cap E \neq \varnothing$.* Indeed, $\mathbf{0}, \mathbf{1} \in E$, and so all sets

$$E^+ \equiv \{x \in E \mid \mu x > \alpha\}, \quad E^- \equiv \{x \in E \mid \mu x < \alpha\}$$

are nonempty. Put $\underline{x} \equiv \sup E^-$ and $\overline{x} \equiv \inf E^+$. Clearly, $\underline{x} \leq \overline{x}$, $\mu \leq \underline{x} \leq \alpha\mu\overline{x}$; $\overline{x}, \underline{x} \in E$ (we apply the continuity of μ and the maximality of E). If we assume that $A_\alpha \cap E = \varnothing$ then $E = E^+ \cup E^-$. After that we must have $\underline{x} = \overline{x}$, since otherwise, by continuity of the algebra $\mathscr{X}$, we find an element x_1 lying strictly between $\underline{x}$ and $\overline{x}$ which may complete our chain which would contradict its maximality. Thus, $\underline{x} = \overline{x}$, $\mu\underline{x} \leq \alpha \leq \mu\overline{x} = \mu\underline{x}$, whence $\mu\overline{x} = \mu\underline{x} = \alpha$ and $A_\alpha \cap E \neq \varnothing$.

We agree to call the set with the property I (i.e., touching each maximal chain) *wide.* Thus, A_α is always a wide set.

II. *The intersection $A_\alpha \cap E$ contains at most one element.* It is evident, since A_α cannot contain two comparable distinct elements. We will call a set with this property *thin.* The properties I and II together mean that for $\alpha \in (0, \mu\mathbf{1})$ the set A_α is wide and thin. Now, let F be some nonempty set in $\mathscr{X}$. We say that this set is *planar* if for all

$x, y, z \in F$ the condition

$$x \wedge y \leq z \leq x \vee y \tag{+}$$

implies

$$u \equiv (x \wedge y \wedge z) \vee (x \wedge Cy \wedge Cz) \vee (Cx \wedge y \wedge Cz) \in F. \tag{++}$$

This condition becomes clearer if we translate it into the terms of vector lattices. Let R be some K-space over $\mathscr{X}$. Considering x, y, and z as elements of R, we can sum and subtract them. As the reader has remembered since Chapter 6, the sum turns out to be an element of $\mathscr{X}$ if and only if the summands are disjoint. The condition (+) says that $x + y - z \in \mathscr{X}$; i.e., the linear combination of elements of $\mathscr{X}$ calculated in R is again the element of $\mathscr{X}$. This element is exactly the element u of the condition (++). So the property of the set to be "planar" means: if $x, y, z \in F$ and $x + y - z \in \mathscr{X}$ then $x + y - z \in F$.

III. *The set A_α is always planar.* It is evident, since the measure of an element of a BA may always be considered as the integral of the corresponding element of the vector lattice over the BA: if x, y, and z have the same integrals then $x + y - z$ has the same integral; so if $x, y, z \in \mathscr{X} \cap F$ then $\mu(x + y - z) = \mu x + \mu y - \mu z = \alpha$. Thus, in the presence of a measure μ, all sets A_α $(0 < \alpha < \mu\mathbf{1})$ are wide, thin, and planar. It turns out that these families completely characterize the sets of the form A_α, even without the a priori assumption that such a measure is available. Further, we consider only continuous algebras. (In the case of a discrete algebra, there is no normability problem at all.)

Theorem 6. *Let $\mathscr{X}$ be a complete continuous BA, and let F be a set in $\mathscr{X}$. For $\mathscr{X}$ to admit a probability measure μ such that*

$$F = A_\alpha \equiv \{x \mid \mu x = \alpha\}$$

for some $\alpha \in (0, 1)$, it is necessary and sufficient that the set F be wide, thin, and planar. If these conditions are satisfied then such a probability measure μ is unique.

Corollary. *A complete continuous BA is normable if and only if it includes a wide, thin, and planar set.*

Necessity has already been proved; we prove sufficiency. Assume that F is a wide, thin, and planar set.

Let us agree upon terminology. An element $x \in \mathscr{X}$ is called "upper" ("lower") if for some $y \in F$ we have $x \geq y$ $(x \leq y)$. We denote the set of all upper (lower) elements by F^+ (F^- respectively). It is easy to see

that $F^+ \cap F^- = F$ and that each maximal chain connecting lower and upper elements intersects F necessarily at a single point. If there exists z with the properties $z\, d\, (x \vee y)$; $x + z, y + z \in F$ then we write $x \sim y$. Finally, call an element $x \in F^-$ "small" if for some y we will have $y \sim x$, $y\, d\, x$, and $C(x \vee y) \in F^+$. We note several facts. Denote by L the set of all small elements.

1°. *The binary relation $\sim$ is reflexive, symmetric, and defined for all pairs of ("lower") elements of F^-.* (This is clear.)

2°. *If $x \sim y \in F^-$ and $z \le x$ then $t \sim z$ for some $t \le y$.*

PROOF. Since $x \sim y$; therefore, there exists $u\, d\ (x \vee y)$ satisfying $x+u \in F$ and $y+u \in F$. Put $u_1 \equiv u+(x-z) \in \mathscr{X}$. Since $u_1 \le u+x \in F$, we have $u_1 \in F^-$, and from $u_1 \vee y \ge u \vee y = u + y \in F$ we see that $u_1 \vee y \in F^+$. The elements u_1 and $u_1 \vee y$ may be included in a maximal chain that contains an element $v \in F$ satisfying the inequality $u_1 \le v \le u_1 \vee y$ (it is clear that the inequalities $v < u$ and $v > u_1 \vee y$ are contradictory). Put $t \equiv v - u_1$. This is a required element. Indeed, $t = v - u_1 \le (u_1 \vee y) - u_1 \le y$. Further, the element $u_1 = (u + x) - z$ is disjoint from z and $z + u_1 = x + u \in F$. Further, $t\, d\, u_1,\ t + u_1 = v \in F$. Whence $z \sim t$.

3°. *If $x_1 \ge x_2,\ x_1 \sim y_1,\ x_2 \sim y_2$, and y_1 is comparable with y_2 then $y_1 \ge y_2$.*

PROOF. Let $y_1 < y_2$. Since $x_1 \ge x_2,\ x_1 \sim y_1$, by 2° for some y we have $y \le y_1,\ y \sim x_2$. Thus, $y \le y_1 < y_2,\ y \sim x_2,\ y_2 \sim x_2$. There exists t_1 disjoint from x_2 and y and such that $x_2 + t_1, y + t_1 \in F$. For the same reason, there exists t_2 disjoint from x_2 and y_2, $x_2 + t_2, y_2 + t_2 \in F$. Since $y < y_2$, $y_2\, d\, t_2$, we have $y\, d\, t_2$. The element $y + t_2 \in \mathscr{X}$ can be written as

$$y + t_2 = (x_2 + t_2) + (y + t_1) - (x_2 + t_1).$$

This is a linear combination of three elements of F in the vector lattice over $\mathscr{X}$; since it belongs to $\mathscr{X}$, it also belongs to $F:\ y + t_2 \in F$. Thus, the two elements $y + t_2$ and $y_2 + t_2$ belong to F and the first is strictly less than the second. This is impossible, and so the inequality $y_1 < y_2$ is false.

4°. *If $x \in F^-$ then each chain contains at most one element x' such that $x' \sim x$.* This immediately follows from 3°; the same relates to the following property 5°.

5°. *If $x_1 > x_2,\ x_1 \sim y_1,\ x_2 \sim y_2$, and y_1 and y_2 are comparable then $y_1 > y_2$.*

6°. *Let $x \in F^-$ and $Cx \in F^+$. Then each maximal chain between* $\mathbf{0}$ *and* $\mathbf{1}$ *contains exactly one element y with the property $y \sim x$. Moreover, $y \in F^-$ and $Cy \in F^+$.*

Proof. Let E be a maximal chain between $\mathbf{0}$ and $\mathbf{1}$. Then the set $x \vee E$ is a maximal chain between x and $\mathbf{1}$, while $C(x \vee E)$ is a maximal chain between $\mathbf{0}$ and Cx, and finally, $x \vee C(x \vee E)$ is a maximal chain between x and $\mathbf{1}$. Since $x \in F^-$ and $\mathbf{1} \in F^+$; therefore, $x \vee C(x \vee y_0) \in F$ for some $y_0 \in E$. Put $t \equiv C(x \vee y_0)$ and consider the "interval" $E_0 \equiv E \cap [\mathbf{0}, y_0]$. The set $t \vee E_0$ is a maximal chain between t and $t \vee y_0$. Since $x + t = x \vee t \in F$, we have $t \in F^-$. Further:

$$t \vee y_0 = C(x \vee y_0) \vee y_0 = (Cx \wedge Cy_0) \vee y_0$$
$$= (Cx \vee y_0) \wedge (Cy_0 \vee y_0) = Cx \vee y_0 \geq Cx.$$

By assumption, $Cx \in F^+$; hence, $t \vee y_0 \in F^+$. In the chain $t \vee E_0$, there exists an element of F. In other words, $t \vee y \in F$ for some $y \in E_0$. It is clear by the definition of t that $t\, d\, y_0$; hence (since $y \leq y_0$) $t\, d\, y$; we see that $x + t \in F,\ \ y + t \in F,\ \ t\, d\, (x \vee y)$; consequently, $x \sim y$. We have found an element $y \sim x$ in $E_0 \subset E$; clearly, it is unique ($4°$). It remains to prove that $Cy \in F^+$. We have the following: $t \vee (y \wedge Cx) \leq t \vee y = t + y \in F$. Therefore, $t \vee (y \wedge Cx) \in F^-$. The inequalities $t \leq Cx$ and $y \wedge Cx \leq Cx$ imply that $t \vee (y \wedge Cx) \leq Cx \in F^+$. Each maximal chain connecting $t \vee (y \wedge Cx)$ and Cx contains some $u \in F$. For this element, we will have

$$u \wedge (x + t) \leq Cx \wedge (x \vee t) = (Cx \wedge x) \vee (Cx \wedge t) = Cx \wedge t \leq t \leq y \vee t$$
$$\leq x \vee y \vee t = x \vee (y \wedge Cx) \vee t \leq x \vee u \vee t = u \vee (x + t),$$

i.e.,

$$u \wedge (x + t) \leq y \vee t = y + t \leq u \vee (x + t).$$

But, in this case (by the main property of F), we must have

$$u_1 \equiv u + (x + t) - (y + t) \in F \ \ (u, x + t, y + t \in F).$$

By the choice of u, we obtain $Cx \geq u$; hence,

$$Cy = \mathbf{1} - y = Cx + (x + t) - (y + t) \geq u + (x + t) - (y + t) = u_1 \in F.$$

We have thus proved that $Cy \in F^+$.

$7°$. *If $x_1 \sim x_2,\ x_2 \sim x_3$, and $C(x_2 \vee x_3) \in F^+$ then there exists an element t satisfying*

$$t\, d\, (x_1 \vee x_2 \vee x_3), x_1 + t, x_2 + t, x_3 + t \in F.$$

Proof. Let E be a maximal chain between $\mathbf{0}$ and $C(x_1 \vee x_2 \vee x_3)$. Then $x_1 \vee E$ is a maximal chain between x_1 and $x_1 \vee C(x_1 \vee x_2 \vee x_3)$. We have

$$x_1 \vee C(x_1 \vee x_2 \vee x_3) = x_1 \vee (Cx_1 \wedge C(x_2 \vee x_3))$$

$$= (x_1 \vee Cx_1) \wedge (x_1 \vee C(x_2 \vee x_3)) = x_1 \vee C(x_2 \vee x_3) \geq C(x_2 \vee x_3).$$

By hypotheses, $x_1 \vee C(x_1 \vee x_2 \vee x_3) \in F^+$. At the same time, $x_1 \in F^-$; therefore, there exists an element of F in the chain $x_1 \vee E$, i.e., there exists $t \in E$ such that $x_1 \vee t \in F$. Since $t \leq C(x_1 \vee x_2 \vee x_3)$, we have $t\,d\,(x_1 \vee x_2 \vee x_3)$. Show that the elements $x_1 + t$, $x_2 + t$, and $x_3 + t$ belong to F. Indeed, this is clear for x_1, since $x_1 + t = x_1 \vee t$. Since $x_1 \sim x_2$ and $x_1 + z \in F$ for some z, we have $x_2 + z \in F$. Then $x_2 + t = (x_1 + t) + (x_2 + z) - (x_1 + z)$, and by the main property of F (all terms on the right-hand side belong to F) we have $x_2 + t \in F$. Further, using the property $x_2 \sim x_3$ and the inclusion just proved, we infer for some u that

$$x_2 + u \in F,\ \ x_3 + u \in F,\ \ x_3 + t = (x_2 + t) + (x_3 + u) - (x_2 + u) \in F.$$

(In both cases, it is important that $x_2, x_3\,d\,t$ and $x_2 + t, x_3 + t \in \mathscr{X}$.) Thus, $x_1 + t, x_2 + t, x_3 + t \in F$. It is clear that under the conditions of 7° we have $x_1 \sim x_3$.

8°. *If $x_0 \in L$ and $x \sim x_0$ then $C(x_0 \vee x) \in F^+$.*

Proof. Suppose first that $x\,d\,x_0$. Since x_0 is a small element, for some y the conditions $x_0\,d\,y$, $x_0 \sim y$, and $C(x_0 \vee y) \in F^+$ hold. Further, $x\,d\,x_0$; hence, $x \leq Cx_0$, $x \wedge Cy \leq Cx_0 \wedge Cy = C(x_0 \vee y)$. Since $x \sim x_0$, we have $x \in F^-$ and $x \wedge Cy \in F^-$. There exists an element $u \in F$ between $x \wedge Cy$ and $C(x_0 \vee y)$. The inequality $u \leq C(x_0 \vee y)$ implies that $u\,d\,y$, $u \wedge y = \mathbf{0} \leq x$. On the other hand, $u \vee y \geq (x \wedge Cy) \vee y = x \vee y \geq x$. We have obtained $u \wedge y \leq x \leq u \vee y$; whence, the element $u + y - x$ belongs to $\mathscr{X}$. Apply 7° to the elements x, x_0, and y (it is possible since $C(x_0 \vee y) \in F^+$). For some t we will have $(x_0 \vee x \vee y)\,d\,t$ and $x + t, y + t \in F$. We also have $u + y - x = u + (y + t) - (x + t)$. The left-hand side, as was mentioned, belongs to $\mathscr{X}$, and all terms on the right-hand side belong to F too; hence, by the main property of F, $u + y - x \in F$. Further, $C(x_0 \vee y) = \mathbf{1} - x_0 - y \geq u$; therefore, $C(x_0 \vee x) = \mathbf{1} - x_0 - x = (\mathbf{1} - x_0 - y) + y - x \geq u + y - x \in F$. We see that $C(x_0 \vee x) \in F^+$.

We consider now the general case in which we simply have $x \sim x_0$. Moreover, $x_0 \sim y$ and $C(x_0 \vee y) \in F^+$. Hence (by 7°) there exists t such that $x_0 + t, x + t, y + t \in F$.

Consider a maximal chain E between $x \wedge Cx_0$ and $(x \wedge Cx_0) \vee y$. Then the chain $t \vee E$ is a maximal chain connecting $(x \wedge Cx_0) \vee t$ and $(x \wedge Cx_0) \vee y \vee t$. The relation $(x \wedge Cx_0) \vee t \leq x \vee t = x + t \in F$ implies $(x \wedge Cx_0) \vee t \in F^-$. Simultaneously, $(x \wedge Cx_0) \vee y \vee t \geq y \vee t = y + t \in F$; hence, $(x \wedge Cx_0) \vee y \vee t \in F^+$. There exists $x_1 \in E$ satisfying $x_1 \vee t \in F$. Further, $x_1 \leq (x \wedge Cx_0) \vee y$ and $x \wedge Cx_0 \leq x$. The elements $x \wedge Cx_0$ and y are disjoint from t, and hence $x_1\,d\,t$, $x_1 \vee t = x_1 + t \in F$. The

relations $x \wedge Cx_0 \leq Cx_0$ and $y \leq Cx_0$ imply $(x \wedge Cx_0) \vee y \leq Cx_0$, $x_1 \leq (x \wedge Cx_0) \vee y$, and $x_1 \, d \, x_0$. Therefore, $x_1 + t \in F$, $x_0 + t \in F$, hence $x_1 \sim x_0$. Since $x_1 \, d \, x_0$, by what was proved at the beginning, we have $C(x_0 \vee x_1) \in F^+$. But $x \vee x_0 = (x \wedge Cx_0) \vee x_0 \leq x_1 \vee x_0$ (since $x_1 \in E$ and $x \wedge Cx_0 = \min E$). Thus, $C(x_0 \vee x) \geq C(x_0 \vee x_1)$. Hence, $C(x_0 \vee x) \in F^+$.

Corollary. *If $x_0 \in L$ then $Cx_0 \in F^+$.*

(For $x_0 \sim x_0$).

We now consider the restriction of $\sim$ to the set of small elements L. Denote this restriction by the same symbol $\sim$.

9°. *The relation $\sim$ (restricted onto the set L) is an equivalence.*

Proof. We have already noted in 1° that the relation $\sim$ is reflexive and symmetric. It remains to prove transitivity. Assume that $x_1, x_2, x_3 \in L$, $x_1 \sim x_2$, and $x_2 \sim x_3$. In this case, by 8° we have $C(x_2 \vee x_3) \in F^+$; and by 7°, $x_1 \sim x_3$.

Remark. Actually, we use the fact that $x_2 \in L, x_1 \sim x_2$, and $x_3 \sim x_2$.

10°. If $x \in L$ and $y \sim x$ then $y \in L$.

Proof. First, let $x \, d \, y$. In this case 8° is applicable and $C(y \vee x) \in F^+$. But then by the definition of small element, $y \in L$. We now take some element $y \sim x$. There exists a maximal chain E containing $C(x \vee y)$. By the corollary to 8°, we have $Cx \in F^+$. Using 6°, we find that $z \sim x$, $z \in E$. By 8° from $x \in L$ and $y \sim x$ it follows that $C(x \vee y) \in F^+$. Thus, $z \in F^-$ and $C(x \vee y) \in F^+$; these two elements are comparable (both lie in E), and hence $z \leq C(x \vee y)$ and $z \, d \, x$, $z \sim x$. By what was proved above, $z \in L$. Further, $x \in L$, $x \sim y$, and $x \sim z$; whence $y \sim z$ (see the remark on 9°). Moreover, $z \, d \, y$ (since $z \leq C(x \vee y)$). We have $z \in L$, $y \, d \, z$, and $y \sim z$; hence, $y \in L$ (we lean here on what was proved above).

11°. *If $x_1 \leq x \in L$ then $x_1 \in L$.*

Proof. By the definition of "small" element, for some y we have $y \, d \, x$, $y \sim x$, and $C(x \vee y) \in F^+$. Further, by 2°, there exists $y_1 \leq y$ such that $y_1 \sim x_1$. These conditions imply that $y_1 \, d \, x_1$. It follows from the inequality $x_1 \vee y_1 \leq x \vee y$ that $C(x_1 \vee y_1) \geq C(x \vee y)$, i.e., $C(x_1 \vee y_1) \in F^+$. Hence, $x_1 \in L$.

12°. *If $x, y \in L$, $x \sim y$, $z \leq x$, $t \leq y$, and $z \sim t$ then $x - z \sim y - t$.*

Proof. The relation $x \sim y$ implies the existence of an element u with the property $x + u, y + u \in F$. Furthermore, u is disjoint from x and y; hence, $u \leq C(x \vee y)$. Extend the chain $\{\mathbf{0}, u, C(x \vee y)\}$ to a maximal chain E_0 between zero and $C(x \vee y)$. Since $x \in L$, $y \sim x$; therefore, $C(x \vee y) \in F^+$ (8°). The set $z \vee E_0$ is a maximal chain between z and $z \vee C(x \vee y)$. The inequality $z \leq x$ implies $z \in F^-$. Thus, the chain $z \vee E_0$ has a nontrivial intersection with F, and for some $v \in E_0$ we have

$z \vee v \in F$. It follows from the conditions $z \leq x$ and $v \leq C(x \vee y)$ that $v\,d\,z$ and $v+z \in F$. Moreover, $t \leq y$ implies $v\,d\,t$ and $t+v \in \mathscr{X}$. Finally, it follows from the condition $z \sim t$ that there is an element v_1 satisfying $z+v_1, t+v_1 \in F$. We have $t \vee v = t+v = (t+v_1)+(z+v)-(z+v_1)$. We note again that all terms on the left-hand side belong to F; consequently, $t+v \in F$. The elements u and v are comparable (they lie in E_0). We will show that $u \leq v$.

Indeed, if $u > v$ and $x \geq z$ then $x+u > z+v$. But $x+u, z+v \in F$, which is impossible, since the elements of F are pairwise incomparable. Thus, $u \leq v$; hence, $v-u \in \mathscr{X}$ (it is an element of the Boolean algebra). Now, we prove that $x-z \sim v-u$. It can be seen from the conditions $z \leq x$ and $x\,d\,u$ that $z\,d\,u$ and $z \vee u = z+u \in \mathscr{X}$.

Further, $(x-z)+(z+u) = x+u \in F$ and $(v-u)+(z+u) = z+v \in F$. Therefore, $x-z \sim v-u$. In the same way, $(y-t)+(u+t) = y+u \in F$, $(v-u)+(t+u) = t+v \in F$, and $u+t \in \mathscr{X}$; hence, $y-t \sim v-u$. The inequality $x-z \leq x$ implies $x-z \in L$. By 10°, this implies $v-u, y-t \in L$, and by 9°, we have $x-z \sim y-t$ (we use the transitivity of the relation $\sim$ on L which was stated in 9°).

13°. *If $x, y \in L$ then either for some $z \leq x$ we have $y \sim z$ or for some $t \leq y$, $x \sim t$ holds.*

Proof. By the corollary to 8°, we obtain $Cx \in F^+$. Extend the chain $\{\mathbf{0}, y, \mathbf{1}\}$ to a maximal chain E. By 6°, there exists exactly one element $t \in E$ such that $t \sim x$; this element is comparable with y. If $t \leq y$ then the proof is complete. Otherwise, if $t \geq y$ then we use 2°: the relations $t \sim x$ and $t \geq y$ imply the existence of $z \leq x$ such that $z \sim y$.

14°. *In each maximal chain E, there exists $x_0 \in L$, $x_0 > \mathbf{0}$. Moreover, $E \cap L = [\mathbf{0}, x_0] \cap E$.*

Proof. Consider the chain $C(E)$ (constituted by the complements). It is maximal too. The intersections of F with E and $C(E)$ are singletons:

$$E \cap F \equiv \{y_1\}, \quad C(E) \cap F \equiv \{z_1\}.$$

The zero and unity elements do not belong to F; hence,

$$\mathbf{0} < y_1,\ \ z_1 < \mathbf{1}.$$

Since $z_1 \in C(E)$; therefore, $Cz_1 \in E$ and $Cz_1 > \mathbf{0}$. Put $y_0 \equiv y_1 \wedge Cz_1 = \min(y_1, Cz_1) > \mathbf{0}$ (both elements belong to the chain and are not equal to zero).

Similarly, $z_0 \equiv Cy_1 \wedge z_1 = \min(Cy_1, z_1) > \mathbf{0}$.

Now, put $E_1 = E \cap [\mathbf{0}, y_0]$ and $E_2 = C(E) \cap [\mathbf{0}, z_0]$. We will prove that E_1 and E_2 are order isomorphic.

First of all it is clear that $y_0 \in F^-$ and $Cy_0 \in F^+$ (since $y_0 \leq y_1 \in F$ and $Cy_0 \geq z_1 \in F$).

Given $x \in E_1$, we see that $x \leq y_0$, $x \in F^-$, and $Cx \in F^+$. Then by 6°, there exists a unique $y \in C(E)$ such that $y \sim x$, $y \in F^-$, and $Cy \in F^+$. Since $z_1 \in C(E) \cap F$, we have $y \leq z_1$.

Further, $Cy \in E$, $Cy \in F^+$, and $y_1 \in E \cap F$. Hence, $y_1 \leq Cy$ and $y \leq Cy_1$. Thus, $y \leq z_1 \wedge Cy_1 = z_0$; consequently, $y \in E_2$. Thus, to each $x \in E_1$ there corresponds a unique $y \in E_2$ such that $y \sim x$; we denote it by $f(x)$. Similarly, we can construct the mapping $g : E_2 \longrightarrow E_1$ such that $g(y) \sim y$. Each chain has only one element equivalent to a given element; so $g(f(x)) = x$ and $f(g(y)) = y$ for all x and y. By 5°, the mappings f and g preserve the order, and hence they are mutually inverse order isomorphisms.

Now, define a new mapping $\Phi : E_1 \longrightarrow \mathscr{X}$. Namely, put

$$\Phi(x) = C(x \vee f(x)).$$

Since the (o)-convergence in a maximal chain is the same as in $\mathscr{X}$ and f is an isomorphism of E_1 onto E_2; therefore, for every sequence of elements of E_1 from $x_\alpha \xrightarrow{(o)} x$ it follows that $\Phi(x_\alpha) \xrightarrow{(o)} \Phi(x))$. Whence we infer the continuity of Φ with respect to the (o)-topologies on E_1 and on $\mathscr{X}$. Then $\Phi(E_1)$ is a maximal chain in $\mathscr{X}$ between $\Phi(\mathbf{0})$ and $\Phi(y_0)$.

The isomorphism always translates the least element into the least and the greatest element into the greatest. Hence,

$$f(\mathbf{0}) = \mathbf{0},\ \ f(y_0) = z_0,\ \ \Phi(\mathbf{0}) = C(\mathbf{0} \vee \mathbf{0}) = \mathbf{1},\ \ \Phi(y_0) = C(y_0 \vee z_0).$$

We consider the following two cases:

1) $\Phi(y_0) = C(y_0 \vee z_0) \in F^+$. We have

$$y_0 \wedge z_0 = (y_1 \wedge Cz_1) \wedge (Cy_1 \wedge z_1) = y_1 \wedge Cz_1 \wedge Cy_1 \wedge z_1 = \mathbf{0}.$$

Moreover, $z_0 = f(y_0)$ whence $y_0 \sim z_0$. The above means that $y_0 \in L$. To this end, show that $y_0 = \max L \cap E$. It suffices to verify that $y_0 = y_1 \in F$. There is no element of L greater than all elements of F. We know that $y_0 = \min(y_1, Cz_1)$. If $Cz_1 < y_1$ then

$$y_0 = Cz_1, \quad z_0 = Cy_1 \wedge z_1 = Cy_1,$$

$$C(y_0 \vee z_0) = C(Cz_1 \vee Cy_1) = z_1 \wedge y_1 < y_1$$

(since $y_1, z_1 \in F$ and these elements cannot be comparable). On the other hand, $C(y_0 \vee z_0) \in F^+$; whence $C(y_0 \vee z_0) \geq u \in F$. We have obtained $u \leq C(y_0 \vee z_0) < y_1$. At the same time, $u, y_1 \in F$ and these elements are not comparable.

This contradiction shows that $y_0 = y_1 \in F$.

2) $\Phi(y_0) = C(y_0 \vee z_0) \notin F^+$. Then there is no element of F less than or equal to $\Phi(y_0)$. We complete the chain $\Phi(E_1)$ to a maximal chain E_3 (from $\mathbf{0}$ to $\mathbf{1}$). In E_3 there exists an element $u \in F$ comparable with $\Phi(y_0)$. It is clear that $u > \Phi(y_0)$. Since $\Phi(E_1)$ is a maximal chain between $\Phi(y_0)$ and $\mathbf{1}$; therefore, $u \in \Phi(E_1)$. Then there exists $x_0 \in E_1$ with the properties

$$\Phi(x_0) = C(x_0 \vee f(x_0)) = u \in F, \ \ x_0 \leq y_0, \ \ f(x_0) \leq z_0, \ \ y_0 \, d \, z_0$$

(the last has already been proved in 1)). We see that

$$x_0 \, d \, f(x_0), \ \ x_0 \sim f(x_0), \ \ C(x_0 \vee f(x_0)) \in F^+.$$

Hence, $x_0 \in L$ and also $x_0 > \mathbf{0}$ (since $C(\mathbf{0} \vee f(\mathbf{0})) = \mathbf{1} \notin F$). Suppose that there exists $x_1 > x_0$ such that $x_1 \in E \cap L$.

In this case, by the corollary to 8°, $Cx_1 \in F^+$. Further, $x_1 \in F^-$ and $x_1 \in E$; i.e., x_1 is comparable with y_1. This means that $x_1 \leq y_1$. Similarly, $Cx_1 \in F^+$, $Cx_1 \in C(E)$, and $Cx_1 \geq z_1$, or else $x_1 \leq Cz_1$, and then $x_1 \leq y_1 \wedge Cz_1 = y_0$; i.e., $x_1 \in E_1$.

By assumption, $x_1 \in L$ and $f(x_1) \sim x_1$; hence, $C(x_1 \vee f(x_1)) \in F^+$. But $f(x_1) \in E_2$, whence $f(x_1) \, d \, x_1$ and $x_1 \vee f(x_1) = x_1 + f(x_1)$. Moreover, $x_1 > x_0$ implies $f(x_1) > f(x_0)$ (see 5°). Then

$$x_1 + f(x_1) > x_0 + f(x_0), \ \ C(x_1 \vee f(x_1)) < C(x_0 + f(x_0)) \in F.$$

We arrive at a contradiction with $C(x_1 \vee f(x_1)) \in F^+$. Thus, $x_0 = \max L \cap E$. It follows from 11° that the whole segment of the chain E from $\mathbf{0}$ to y_1 (in the case 1)) or from $\mathbf{0}$ to x_0 (in the case 2)) lies in L.

Corollary. *The set L minorizes $\mathscr{X}$.*

15°. *If $x_n \downarrow \mathbf{0}$ then for n sufficiently large we have $x_n \in L$.*

Proof. Extend the chain $\{x_1, x_2, \ldots\}$ to a maximal chain E. By 14°, there exists $x_0 \in L \cap E$ such that $x_0 > \mathbf{0}$. This element is comparable with all $x_1, x_2, \ldots$. If we assume that $x_n \geq x_0$ for all $n = 1, 2, \ldots$, then also $\bigwedge_1^\infty x_n \geq x_0 > \mathbf{0}$, although $x_n \downarrow \mathbf{0}$. Thus, $x_n < x_0$ for n sufficiently large, and $x_n \in L$ (the solidity of L is used).

16°. *If $x_n, x \in L$, $x_n \xrightarrow{(o)} \mathbf{0}$, and $x_n \sim x$; then $x = x_n = \mathbf{0}$ ($n = 1, 2, \ldots$).*

Proof. Put $z_n = \bigvee_{k=n}^\infty x_k$. We have $z_n \downarrow \mathbf{0}$. Extend the chain $\{z_1, z_2, \ldots\}$ to a maximal chain E. Since $x \in L$; therefore, $Cx \in F^+$ (the corollary to 8°). By 6°, there exists $z \in E$, $z \sim x$, $z \in L$. According to 5°, for $x > \mathbf{0}$ we must have $z > \mathbf{0}$. All z_n are comparable with z, and if $z > \mathbf{0}$ then there exists $z_n < z$ (otherwise $z_n \not\longrightarrow \mathbf{0}$). On the other hand, $z_n \geq x_n \sim x \sim z$, i.e., $z > x_n \sim z$ which is inconsistent with 5°. Thus, $x = \mathbf{0}$, and so $x_n = \mathbf{0}$.

Corollary. *There are no "wandering" elements: if x_n are disjoint and $x_1 \sim x_2 \sim \dots$ then $x_1 = x_2 = \dots = \mathbf{0}$.*

Indeed, each disjoint sequence always (o)-converges to zero.

17°. *There exists a finite disjoint set*

$$\{x_1, x_2, \dots, x_n\} \subset L,$$

such that $x_1 + x_2 + \dots + x_n = \mathbf{1}$.

Proof. Given a maximal chain E, we see that $x_1 \in E \cap L,\ x_1 > \mathbf{0}$. The proof proceeds by recursion. Assume that we have already constructed disjoint and equivalent elements $x_1, x_2, \dots, x_k \in L$. Complete the chain $\{\mathbf{0}, C(x_1 \vee \dots \vee x_k), \mathbf{1}\}$ to a maximal chain E_1. Since $x_1 \in L$, we have $Cx_1 \in F^+$ (the corollary to 8°). By 6°, we infer that there exists $y \in E_1$, $y \sim x$. By 10°, $y \in L$. This element is comparable with $C(x_1 \vee \dots \vee x_k)$; and the two cases are possible:

1) $y < C(x_1 \vee \dots \vee x_k)$. Then we put $x_{k+1} \equiv y$ and continue recursion.

2) $y \geq C(x_1 \vee \dots \vee x_k)$. In this case, from 11° we obtain $C(x_1 \vee \dots \vee x_k) \in L$ and

$$\mathbf{1} = x_1 + \dots + x_k + C(x_1 \vee \dots \vee x_k).$$

All terms on the right-hand side belong to L, and the construction is over.

Thus, if at each step we observe the situation of 1) then we can obtain a sequence of disjoint and equivalent elements. Since $x_1 > \mathbf{0}$, they are all nonzero, which is impossible (see 10°). Hence, in finitely many steps we arrive at the situation of 2) and the "exhaustion" process terminates.

18°. *If $x_1\, d\, y_1,\ x_2\, d\, y_2,\ x_1 \sim x_2,\ y_1 \sim y_2$, and $x_1 + y_1 \in L$ then $x_1 + y_1 \sim x_2 + y_2$ and $x_2 + y_2 \in L$.*

Proof. Complete the chain $\{\mathbf{0}, x_2, x_2 + y_2, \mathbf{1}\}$ to a maximal chain E. Since $x_1 + y_1 \in L$; therefore, $C(x_1 + y_1) \in F^+$ and there exists $z \in E$ such that $x_1 + y_1 \sim z$. We have $x_1 + y_1 \sim z,\ x_1 \sim x_2$, where z is comparable with x_2. Therefore, $z \geq x_2$ (see 3°).

By 12°, we have $(x_1 + y_1) - x_1 \sim z - x_2$, i.e., $y_1 \sim z - x_2$. At the same time, the element z is comparable with $x_2 + y_2$. Whence, $z - x_2$ is comparable with y_2. We have $y_1 \sim y_2$, $y_1 \sim z - x_2$. By 4°, this implies that $y_2 = z - x_2$, i.e., $z = x_2 + y_2$ and $x_1 + y_1 \sim x_2 + y_2$, $x_2 + y_2 \in L$ (we have used 10°).

Remark. The above is easily generalized: *if $x_1, \dots, x_n \in L$ are disjoint as well as $y_1, \dots, y_n$ and $x_i \sim y_i$ for all i, and if $x_1 + \dots + x_n \in L$ then $y_1 + \dots + y_n \in L$ and $x_1 + \dots + x_n \sim y_1 + \dots + y_n$.*

19°. *If $x, y \in L$, $x \sim y$, and $x = \sum_1^n x_i$ (x_i are disjoint) then there exist disjoint $y_1, \dots, y_n$ such that $x_i \sim y_i$ ($i = 1, 2, \dots, n$) and $y = \sum_1^n y_i$.*

We prove this by induction on n. For $n = 1$, it is clear. Assume that this assertion is proved for $n = k$, and let

$$x = \sum_{i=1}^{k+1} x_i = \sum_{i=1}^{k} x_i + x_{k+1} \equiv x_0 + x_{k+1}.$$

Since $x_{k+1} \leq x$, there exists an element $y_{k+1} \leq y$, $y_{k+1} \sim x_{k+1}$ (see 2°). We have by 12° that

$$x_0 = x - x_{k+1} \sim y - y_{k+1}, \quad x_0 = \sum_{i=1}^{k} x_i.$$

By the inductive assumption, $y - y_{k+1} = \sum_{i=1}^{k} y_i$, $y_i \sim x_i$ $(i = 1, \dots, k)$. Whence, $y = \sum_1^{k+1} y_i$, $y_i \sim x_i$ $(i = 1, 2, \dots, k+1)$.

Now, we introduce a new relation $\approx$. By definition, $x \approx y$ *if there exist finite disjoint partitions* $x = \sum_1^k x_i$ *and* $y = \sum_1^k y_i$ *such that* $x_i, y_i \in L$ *and* $x_i \sim y_i$ $(i = 1, 2, \dots, k)$. In this definition x and y are elements of $\mathscr{X}$ (not necessarily small). The "additivity" of this relation is clear: if $a = a_1 + a_2$, $b = b_1 + b_2$, $a_1 \approx b_1$, and $a_2 \approx b_2$ then $a \approx b$.

20°. *The relation* $\approx$ *is an equivalence on* $\mathscr{X}$.

It suffices only to prove transitivity. Let $x \approx y$ and $y \approx z$ hold. Then $x = \sum_1^m x_i$, $y = \sum_1^m y_i$, $x_i \sim y_i$, $y = \sum_1^n y'_j$, $z = \sum_1^n z'_j$, $y'_j \sim z'_j$, and $x_i, y_i, y'_j, z'_j \in L$. Putting $t_{ij} \equiv y_i \wedge y'_j$, we have

$$y_i = \sum\nolimits_j t_{ij} \ (i = 1, 2, \dots, m); \quad y'_j = \sum\nolimits_i t_{ij} \ (j = 1, 2, \dots, n).$$

According to 19°, the elements x_i can be represented as

$$x_i = \sum\nolimits_{j=1}^n u_{ij}, \quad \text{where} \quad u_{ij} \sim t_{ij}.$$

Then $x = \sum_i \sum_j u_{ij}$. Similarly,

$$z'_j = \sum_i v_{ij}, \quad \text{where} \quad v_{ij} \sim t_{ij}, \quad \text{and} \quad z = \sum_i \sum_j v_{ij}.$$

Using 11° and 9°, we come to the desired conclusion: $x \approx z$.

21°. *For all* $x, y \in \mathscr{X}$, *either* $z \approx y$ *holds for some* $z \leq x$ *or* $t \approx x$ *holds for some* $t \leq y$.

Proof. We use 17° and represent the unity of $\mathscr{X}$ as the sum

$$\mathbf{1} = \sum_{i=1}^{n} t_i,$$

where t_i are disjoint and $t_i \in L$. Given $x \in \mathscr{X}$, we have $x = \sum_1^n (x \wedge t_i)$. Here the summands are also disjoint and belong to L. Thus, *each element of the BA $\mathscr{X}$ is a finite disjoint sum of small elements.*

Let $x = \sum_1^m x_i$ and $y = \sum_{j=1}^n y_j$, with $x_i, y_j \in L$. We proceed by induction on $m + n$. If $m = n = 1$ then $x, y \in L$ and it suffices to refer to 13°. Assume that our assertion is true for all pairs (m, n) such that $m + n \leq k$. Let $m + n = k + 1$. According to 13°, either $y_1 \sim z \leq x_1$ or $x_1 \sim t \leq y_1$. In the first case we consider the elements $x - z = \sum_2^m x_i + (x_1 - z)$, $y - y_1 = \sum_2^n y_j$. By solidity, $x_1 - z \in L$. The number of the elements of the partitions $x - z$ and $y - y_1$ is equal to $m + n - 1$ if $x_1 - z > \mathbf{0}$, and to $m + n - 2$ if $x_1 - z = \mathbf{0}$. By the inductive assumption, we have either (A): $y - y_1 \approx u \leq x - z$ or (B): $x - z \approx v \leq y - y_1$. In the case (A), $u \approx y - y_1$, $z \approx y_1$, $u \leq x - z$, hence $u\, d\, z$. Then $u + z \approx (y - y_1) + y_1 = y$ and (since $u \leq x - z$) $u + z \leq x$. In the case (B) $v \approx x - z$, $y_1 \approx z$, $v \leq y - y_1$, hence $v\, d\, y_1$. We have $v + y_1 \approx (x - z) + z = x$, $v \leq y - y_1$, $v + y_1 \leq y$. The second case is considered similarly.

22°. *If $x \approx y$, $z \leq x$, and $z \in L$ then there exists $t \leq y$ with the properties $t \sim z$ and $t \in L$.*

PROOF. We have $x = \sum_1^n x_i$ and $y = \sum_1^n y_i$, with $x_i \sim y_i$ and $x_i, y_i \in L$. Then $z = \sum_1^n z_i$. Here $z_i = z \wedge x_i$ and $z_i \in L$. Since $z_i \leq x_i$, $x_i \sim y_i$; therefore, there exist $t_i \leq y_i$ and $z_i \sim t_i$. These elements t_i are disjoint, since so are y_i. Further, $z = \sum_1^n z_i$, $z \in L$, and $t_i \sim z_i$; whence $t = \sum_1^n t_i \sim z$ and $t \in L$ (see 18° and the remark).

Since $t_i \leq y_i$ $(i = 1, 2, \dots, n)$; therefore, $t \leq y$.

23°. *If $x \approx x_n$ $(n = 1, 2, \dots)$, $x_n \xrightarrow{(o)} \mathbf{0}$ then $x = \mathbf{0}$.*

Let $x > \mathbf{0}$. Since L is a minorant (the corollary to 14°), for some $y \in L$ we have $\mathbf{0} < y \leq x$. Then, by 22°, for all n there exist $y_n \in L$ such that $y_n \leq x_n$, $y_n \sim y$. It is clear that $y_n \xrightarrow{(o)} \mathbf{0}$, and we arrive at a contradiction with 16°. In this connection we can derive the following

Corollary. *There are no "wandering" elements for the relation $\approx$. This means that $x_n\, d\, x_m$ $(n \neq m)$, $x_n \approx x_m$, $x_n \neq \mathbf{0}$ $(n, m = 1, 2, \dots)$ are impossible.*

24°. *If $x \geq y$ and $x \approx y$ then $x = y$.*

PROOF. By definition, $x = \sum_i x_i$ and $y = \sum_i y_i$, with $x_i, y_i \in L$ and $x_i \sim y_i$ $(i = 1, 2, \dots, n)$. Put $x_i \wedge y \equiv x_i'$ and $x_i \wedge (x - y) \equiv x_i''$. Then $x_i = x_i' + x_i''$, $\sum_i x_i' = y$, and $\sum_i x_i'' = x - y$.

By 19°, the relations $x_i \sim y_i$ imply that for some y_i and y_i'' we have $y_i = y_i' + y_i''$, $x_i' \sim y_i'$, and $x_i'' \sim y_i''$. Then $y = \sum_i x_i' \approx \sum_i y_i' \equiv \overline{y}$ and $x - y = \sum_i x_i'' \approx \sum_i y_i'' = y - \overline{y}$. Thus, there exists $\overline{y} \leq y$ such that $\overline{y} \approx y$ and $x - y \approx y - \overline{y}$. Continuing this process, we obtain

the sequence $y = y_1 \geq y_2 \geq \dots$ such that $x \approx y_1 \approx y_2 \approx \dots$ and $x - y \approx y_1 - y_2 \approx y_2 - y_3 \approx \dots$. The elements $y_k - y_{k+1}$ are disjoint. It follows from 23° that $x - y = \mathbf{0}$, $x = y$.

Corollary. *If $a + b \approx a' + b'$ and $a \approx a'$ then $b \approx b'$. In particular, $x \approx y$ implies $Cx \approx Cy$.*

Assume that $b \neq b'$. Then either $b \approx b^* < b'$ or $b' \approx b^{**} < b$. Consider the first case. We have $a+b \approx a'+b^*$ and $a'+b' \approx a+b \approx a'+b^* < a'+b'$, which is impossible.

In the case when $x \approx u < y$, we as before use the symbol $\prec$ and write $x \prec y$. From 24° we infer that if $x \approx u \leq y$ then the relation $x \approx y$ is impossible. Otherwise, by the transitivity of $\approx$ we would have $y \approx u < y$.

Now, 21° can be expressed as follows: *for all x and y, either* (a) $x \prec y$ *or* (b) $y \prec x$. *Moreover, if both relations* (a) *and* (b) *are fulfilled then* $x \approx y$. (The last follows from 24°.)

Thus, we have defined the equivalence $\approx$ on $\mathscr{X}$. If we show that $\approx$ satisfies the conditions of Theorem 4 then the normability of $\mathscr{X}$ will be established. First, we prove that the algebra $\mathscr{X}$ satisfies the countable chain condition.

25°. *The algebra $\mathscr{X}$ satisfies the countable chain condition.*

Assume that $\{x_t\}_{t \in T}$ is a family of distinct nonzero elements. Let $\{x'_n\}$ be an arbitrary sequence monotonically converging to zero: $x'_n \downarrow \mathbf{0}$, $x'_n > \mathbf{0}$. By 23°, to each $t \in T$, we can assign the least n_t with the following property: $x'_{n_t} \prec x_t$ (otherwise, if $x_t \prec x'_n$ $(n = 1, 2, \dots)$ then $x_t = \mathbf{0}$). If T is uncountable then for at least some n the set $T_n \equiv \{t \mid n = n_t\}$ is infinite, and then the element x'_n is wandering, which is impossible (see 23°). Thus, T is always countable and $\mathscr{X}$ satisfies the countable chain condition.

Now, we consider our relation $\approx$. We see that it possesses the properties I), III), and IV) (see Theorem 4). It remains to verify the property II).

We agree now to denote the equipartite relation relative to $\approx$ by the sign $\overset{*}{\sim}$, and we prove that $x \overset{*}{\sim} y$ is equivalent to $x \approx y$. It suffices to establish that $x \overset{*}{\sim} y$ implies $x \approx y$. Thus, let $x \overset{*}{\sim} y$. Either $x \succ y$ or $x \prec y$ is true. Assume for definiteness that the first $x \succ y$ is true. Since $x \overset{*}{\sim} y$, there exist partitions $x = \sum_n x_n$ and $y = \sum_n y_n$, with $x_n \approx y_n$. These partitions are countable by 25°. Let $x \not\approx y$. Then the "inequality" $x \succ y$ means that $x > x' \approx y$. In this case we have $Cx' \approx Cy$. The following two situations may occur:

$$\text{a) } x - x' \approx u \leq Cy;$$

$$\text{b) } Cy \approx v \leq x - x'.$$

In case (a), we have $u + y \approx x$ (since $x' \approx y$ and $x - x' \approx u$). For every $n = 1, 2, \ldots$ we have $\sum_1^n x_k \approx \sum_1^n y_k$ and (the corollary to 24°) $x - \sum_1^n x_k \approx u + \sum_{n+1}^\infty y_k$. The left-hand side (o)-converges to zero; the right-hand side, to $u > \mathbf{0}$. We can select some elements $x'_n < x - \sum_1^n x_k$ that are equivalent to u; then $x'_n \xrightarrow{(o)} \mathbf{0}$, as before, and we arrive at a contradiction with 23°. This proves that in fact $x \approx y$.

Consider now the case b). In this case $\mathbf{1} = y + Cy \approx x' + v$. Then $x' + v = \mathbf{1}$ and $v = Cx' \le x - x'$; i.e., $x = \mathbf{1}$. We have

$$\mathbf{1} = \sum_1^\infty x_n, \;\; y = \sum_1^\infty y_n, \;\; x_n \approx y_n, \;\; y \approx x' < \mathbf{1}.$$

Hence, $y < \mathbf{1}$. By the remark on 18°, we have $\sum_1^n x_k \approx \sum_1^n y_k$. Then by the corollary to 24°, the relation $\mathbf{1} - \sum_1^n x_k \approx \mathbf{1} - \sum_1^n y_k$ holds; hence, $\sum_{n+1}^\infty x_k \approx Cy + \sum_{n+1}^\infty y_k$. The left-hand side (o)-converges to $\mathbf{0}$; the right-hand side, to $Cy > \mathbf{0}$. After that, as in (a), we arrive at the relation $x \approx y$.

We see that all conditions of Theorem 4 are satisfied; hence, there exists a unique probability measure (essentially positive and countably additive quasimeasure) μ on $\mathscr{X}$ invariant under $\approx$ (on L it is invariant under $\sim$ since these two relations coincide on L).

It remains to show that $F = \{x \mid \mu x = \alpha\}$ for some $\alpha \in (0, 1)$.

In 14°, we establish that for each maximal chain E the intersection $E \cap L$ of E with L is an interval (in E) having the greatest element $x_0 \in L$. We agree, for a short time, to call such an element x_0 "maximal." Denote the set of all these elements by M.

26°. *If $x, y \in M$ then $x \sim y$; if $x \in M$, $y \sim x$ then $y \in M$ ($y \in L$ by 10°).*

Proof. Take $x, y \in M$. Draw maximal chains E_1 and E_2 between x and y such that $E_1 \cap L \equiv [\mathbf{0}, x]$ and $E_2 \cap L \equiv [\mathbf{0}, y]$. By the corollary to 8°, for $x, y \in L$ we must have $Cx, Cy \in F^+$, and so (see 6°) there exist $x_1 \in E_1$ and $y_1 \in E_2$ such that $x_1 \sim y$ and $y_1 \sim x$. By 10°, $x_1, y_1 \in L$; hence, by the maximality of x and y, in the intervals of the chains E_1 and E_2 we have $x_1 \le x$ and $y_1 \le y$. Thus, $x \sim y_1$ and $x_1 \le x$. By 6°, there exists $y_2 \le y_1$, $y_2 \sim x_1$. We now have $x_1 \sim y$, $x_1 \sim y_2$, and $y_2 \le y_1 \le y$. By 4°, only one element equivalent to x_1 may exist in the chain; whence $y_2 = y_1 = y$. Thus, $x \sim y_1 = y$.

Now, let $x \in M$ and $y \sim x$. Take a maximal chain E containing y so that it also contains a maximal element y_0 (first, we must take a maximal chain in L containing y, and then extend it to a maximal chain in $\mathscr{X}$). By what was proved above, $y_0 \sim x$; and by 4°, $y_0 = y$; hence, $y \in M$.

Corollary. *The measure μ is constant on M.*

(For $\approx$ and $\sim$ are equivalent on $M \subset L$.)

Put $A_\alpha \equiv \{x \mid \mu x = \alpha\}$, $\alpha \in (0,1)$.

27°. *The set F coincides with one of the sets A_α.*

In the proof of 14°, it was actually established that for $x_0 \in M$, at least one of the following two conditions is satisfied:

1) $x_0 \in F$;

2) there exists $y\, d\, x$, $y \sim x_0$ such that $C(x_0 \vee y) \in F$.

In the case 1) we have the following situation: if $x_0 \in M \cap F$ then all $x \in F$ will be maximal by 10° (it is clear that every two elements of F are connected by $\sim$). Hence, by the corollary to 26°, the measure μ is constant on F. Consider the case 2). In this case also we may observe the already considered situation 1); but we will assume that this is not the case and $F \cap M = \varnothing$.

Let $x \in F$, let E be a maximal chain containing Cx, and let x_0 be a maximal element in this maximal chain: $x_0 \equiv \max E \cap L$. Since Cx and x_0 are comparable, Cx_0 and x are comparable too. Then $Cx_0 \in F^+$ (the corollary to 8°), $Cx_0 \geq x$, and $x_0 \leq Cx$. This means that we can consider in $\mathscr{X}$ the difference $Cx - x_0$. We assume that $M \cap F = \varnothing$; hence, for some y, we have $y\, d\, x_0$, $y \sim x_0$, and $C(x_0 \vee y) \in F$ (the case 2)). Consider the element $w \equiv C(y \vee (Cx - x_0)) = Cy \wedge (x \vee x_0)$ and take a maximal chain E_1 from $\mathbf{0}$ to w. We have $y \vee w = y \vee x \vee x_0 \geq x \in F$, i.e., $y \vee w \in F^+$. On the other hand, $y \in F^-$; hence, there exists an element of F in the chain $y \vee E_1$. Since all elements of E_1 are disjoint from y, there exists $t \in E_1$ such that $y + t \in F$.

We have $t \leq w$; hence, $t\, d\, (Cx - x_0)$ and we may consider the element $Cx - x_0 + t \in \mathscr{X}$. Note that

$$Cx - x_0 + t = \mathbf{1} - x - x_0 + t = (\mathbf{1} - x_0 - y) + (y + t) - x.$$

Since $x_0\, d\, y$; therefore, $\mathbf{1} - x_0 - y = C(x_0 \vee y) \in F$. We see that all elements on the right-hand side of the preceding equality belong to the planar set F, whence the left-hand side also belongs to F: $Cx - x_0 + t \in F$. Since $y + t \in F$, it follows that $Cx - x_0 \sim y \sim x_0$. But then by 25° the element $Cx - x_0$ is maximal together with x_0 : $Cx - x_0 \in M$. Let β be the common value of μ at the elements of M. Then $\mu Cx = 2\beta$ independently of the choice of $x \in F$. Let $\alpha \equiv 1 - 2\beta$. We have seen that the set $A = \{x \mid \mu x = \alpha\}$ is thin, wide, and planar. At the same time, $F \subset A$. The coincidence of A and F will be established if we show that each planar, thin, and wide set does not admit any essential extension to a set with same properties. This is almost evident. Assume for example that $F \subset A$ but $F \neq A$. Then there exist an element $x \in A \setminus F$ and a maximal chain E containing this element and not touching F, since it intersects the thin and wide set A at a single point. Consequently, the

set F is not wide, which contradicts the main condition of our theorem. The proof is complete.

As regards necessity, the theorem just proved above is almost evident. But it is far from trivial in part of sufficiency. We spare ample room for the proof because the normability test of the theorem seems to us to be the best of those available. To explain our opinion, we consider the superstructure (in the sense of A. Robinson) over a BA $\mathscr{X}$. The "floors" of this superstructure are the sets

$$S_0 = \mathscr{X}, \quad S_1 = S_0 \cup 2^{S_0}, \quad S_2 = S_1 \cup 2^{S_1}, \dots .$$

Thus, the subsets of $\mathscr{X}$ are on the ground floor (of S_1), the subsets of the subsets are on the first floor, the mappings from $\mathscr{X}$ to $\mathscr{X}$ are on the third (see M. Davis [1]), etc.

All (necessary and sufficient) normability tests of this chapter have the following logical structure: $(\exists\, A \in S_i)\ \mathscr{P}(A)$, where $\mathscr{P}$ is some predicate. The index i characterizes, in some sense, the "logical level" of the problem. In the Maharam and Kelley Theorems, $i = 2$; in our Theorem 3, $i = 5$. But for the test of A. V. Potepun, $i = 1$. Here, the existence is postulated of a SET OF ELEMENTS of a BA which possesses specific properties. The predicative formula that describes these properties is also bounded by the class S_1 in the sense that its quantifiers are restricted to S_1. Thus, this test has the simplest logical structure.

It follows from Theorem 6 that the normability property, as well as the completeness property of a BA, is expressible in the second order language. However, there is an essential distinction between these properties: completeness is described by a "universal" formula ("for each set $E \subset \mathscr{X}$, there exists $\sup E$"), whereas the test of A. V. Potepun is written as an "existential" formula ("there exists a wide, thin, and planar set").

We add the following remark: appending the condition of A. V. Potepun to the axioms of a complete continuous BA, we obtain the theory that, in particular, includes the order theory of the interval $[0, 1]$, since in a continuous algebra with a probability measure there always exists a subset isomorphic to the interval. At the same time, the axioms themselves never mention real numbers.

4. Automorphisms of normable algebras and invariant measures

4.1 Automorphisms

Here we return to studying automorphisms, on assuming that the main algebra $\mathscr{X}$ is normable. Such an algebra can always be realized

as the metric structure of some measure space $\{\Omega, \mathscr{E}, m\}$. We simply assume that $\mathscr{X} \equiv \widehat{\mathscr{E}} \equiv \mathscr{E}|_I$, where I is the σ-ideal of m-negligible sets and $\mu \equiv \widehat{m}$. Let $\mathscr{A}$ be an automorphism of $\{\Omega, \mathscr{E}, m\}$ (an isomorphism onto itself). This automorphism generates the automorphism of the Boolean algebra

$$A(\widehat{e}) \equiv \widehat{\mathscr{A}^{-1}(e)}, \tag{4}$$

where $e \in \mathscr{E}$ is an arbitrarily chosen representative of the coset $\widehat{e} \in \widehat{\mathscr{E}}$ and $\mathscr{A}^{-1}(e)$ is its inverse image under the mapping $\mathscr{A}$; the symbol $\widehat{\ }$ denotes, as usual, the passage from a measurable set to the coset (the "set $mod\,0$"); i.e.,

$$\widehat{E} = E +_2 I$$

for all $E \in \mathscr{E}$.

The question whether each automorphism A of the algebra $\widehat{\mathscr{E}}$ is generated in such a way (i.e., it corresponds to some point mapping) has different answers, depending on the choice of a representation space.[14]

It is obvious that a positive answer can be given in the case when the Stone space $\mathfrak{Q}(\widehat{\mathscr{E}})$ is taken as a representation space. In this case all Borel automorphisms are generated by homeomorphisms of $\mathfrak{Q}(\widehat{\mathscr{E}})$ (see p. 138). We are in a similar situation if the representation measure space is a Lebesgue–Rokhlin space. However, there may occur cases when no point mapping corresponds to a Borel automorphism of the quotient algebra $\widehat{\mathscr{E}}$. This happens in the case of an "unfortunate" choice of a representation space.[15]

The traditional ergodic theory deals precisely with point mappings of measurable spaces. However, the properties under consideration are related to automorphisms and endomorphisms of the corresponding metric structure. Therefore, the further content of this section is actually related to ergodic theory in particular to the section of this theory that is devoted to the problem of an invariant measure.

4.2 Automorphisms and operators

We now assume $\mathscr{X}$ to be a normed BA. Take one of the measures on $\mathscr{X}$ as the main measure; let it be μ. Consider an arbitrary automorphism A to which there correspond the two measures $\mu_A \equiv \mu \circ A$ and $\mu_{A^{-1}} \equiv \mu \circ A^{-1}$. (For example, $\mu_A x = \mu(Ax)$.) We will consider the elements $x \in \mathscr{X}$ simultaneously as the (unit) elements of the universally complete K-space $\mathfrak{S}_{\mathscr{X}}$ over $\mathscr{X}$.

[14]We have addressed this problem in 7.3.4.
[15]An example was given in 7.3.5.

To each automorphism A of the BA $\mathscr{X}$ there uniquely corresponds its *canonical extension* $\overline{A}$. Namely: $\overline{A}$ is the unique automorphism of the K-space $\mathfrak{S}_{\mathscr{X}}$ which has the unity $\mathbf{1}$ as a fixed point and coincides with A on $\mathscr{X}$. In case $\mathfrak{S}_{\mathscr{X}}$ is the space S of "measurable *mod* 0 functions" and the automorphism A is given by the formula (4), the canonical extension coincides with the so-called "substitution operator":

$$\overline{A}(\widehat{f}) = \widehat{f \circ \mathscr{A}}$$

(f is some measurable function representing the coset $\widehat{f}$).

We interpret $\overline{A}$ similarly when $\mathfrak{S}_{\mathscr{X}} = C_\infty(\mathfrak{Q}(\mathscr{X}))$. However, it is convenient again to interpret the elements $\mathfrak{S}_{\mathscr{X}}$ as RESOLUTIONS OF THE IDENTITY of the BA $\mathscr{X}$. In this case the operator $\overline{A}$ is defined by the condition

$$e^{\pm}_{\lambda}(\overline{A}\mathfrak{f}) = Ae^{\pm}_{\lambda}(\mathfrak{f}), \qquad \lambda \in [-\infty, +\infty].$$

As before we will speak about $\overline{A}$ as about an "extension" of A. Although in our case $\mathscr{X} \not\subset \mathfrak{S}_{\mathscr{X}}$, we have already grown accustomed to identify each element $x \in \mathscr{X}$ with its canonical image $x^\bullet \in \mathscr{X}^\bullet$ and we may assume that $\overline{A}$ is an automorphism of the BA $\mathscr{X}$ which, in this case, coincides with $E_{\mathfrak{S}_{\mathscr{X}}}$. The operator $\overline{A}$ acts in $\mathfrak{L}^\infty_{\mathscr{X}}$. Moreover, $\overline{A}$ is an automorphism of this space and it is also a metric isomorphism, a *rotation*:

$$\|\overline{A}\mathfrak{f}\|_{\mathfrak{L}^\infty} = \|\mathfrak{f}\|_{\mathfrak{L}^\infty} \quad \text{for all} \quad \mathfrak{f} \in \mathfrak{L}^\infty_{\mathscr{X}}(\mu).$$

The spaces $\mathfrak{L}_{\mathscr{X}}(\mu)$ and $\mathfrak{L}^\infty_{\mathscr{X}}$ are set in duality by the bilinear form

$$(\mathfrak{f}, \mathfrak{g}) \equiv \int \mathfrak{f}\mathfrak{g}\, d\mu \quad (\mathfrak{f} \in \mathfrak{L}_{\mathscr{X}}(\mu),\ \ \mathfrak{g} \in \mathfrak{L}^\infty_{\mathscr{X}}),$$

enabling us to associate with each linear operator $B : \mathfrak{L}^\infty \longrightarrow \mathfrak{L}^\infty$ the unique transpose B^* in accord with the usual condition

$$(B^*\mathfrak{f}, \mathfrak{g}) = (\mathfrak{f}, B\mathfrak{g}).$$

The transpose B^* is a linear operator acting in $\mathfrak{L}_{\mathscr{X}}$. Of course, we are interesting in the case when $B \equiv \overline{A}$ is the operator defined by an automorphism A of the initial algebra. We note some formulas that may be useful in the sequel.

1°. *Given $x \in \mathscr{X}$, we have*

$$\mu_A x \equiv \mu(Ax) = (\overline{A}x, \mathbf{1}) = (x, \overline{A}^*\mathbf{1}) \equiv \int_x \overline{A}^*(\mathbf{1})\, d\mu.$$

This means that $\overline{A}^*(\mathbf{1}) = \frac{d\mu_A}{d\mu}$. Similarly, $(\overline{A}^{-1})^*(\mathbf{1}) = \frac{d\mu_{A^{-1}}}{d\mu}$.

2°. *For each* $\mathfrak{f} \in \mathfrak{L}_{\mathscr{X}}^{\infty}$, *the following hold:*

$$\int \overline{A}\mathfrak{f}\, d\mu_{A^{-1}} = \int \overline{A}\mathfrak{f} \frac{d\mu_{A^{-1}}}{d\mu}\, d\mu = \int \overline{A}\mathfrak{f} \cdot (\overline{A}^{-1})^*(\mathbf{1})\, d\mu$$

$$= (\overline{A}\mathfrak{f}, (\overline{A}^{-1})^*(\mathbf{1})) = (\mathfrak{f}, \mathbf{1}) = \int \mathfrak{f}\, d\mu.$$

Hence, using the MULTIPLICATIVITY of the operator A, we find that

$$\int_{x} \mathfrak{f}\, d\mu_{A^{-1}} = \int \mathfrak{f} \cdot x\, d\mu_{A^{-1}} = \int \overline{A}\,\overline{A}^{-1}(\mathfrak{f} \cdot x)\, d\mu_{A^{-1}}$$

$$= \int \overline{A}^{-1}(\mathfrak{f} \cdot x)\, d\mu = \int \overline{A}^{-1}\mathfrak{f} \cdot \overline{A}^{-1}x\, d\mu = \int_{A^{-1}x} \overline{A}^{-1}\mathfrak{f}\, d\mu$$

for all $\mathfrak{f} \in \mathfrak{L}_{\mathscr{X}}$ and $x \in \mathscr{X}$.

These two formulas actually express the well-known change-of-variable rules under the integral sign.

The $\mathfrak{L}_{\mathscr{X}}^{\infty}$ space as well as $\mathfrak{S}_{\mathscr{X}}$ is invariant under the operators of this type. It seems impossible to find in general another invariant subspace without additional assumptions about the automorphism A. If A preserves the main measure (i.e., $\mu_A = \mu$) then, in particular, all $\mathfrak{L}_{\mathscr{X}}^{p}(\mu)$ $(p \geq 1)$ spaces and all Orlicz spaces are invariant. This is not so in general. However, "correcting" the operator A by means of additional factors, we can obtain the invariance of each of the $\mathfrak{L}_{\mathscr{X}}^{p}$ spaces. We consider the case $p = 2$. Define the new operator $U_A^{(2)}$ by the formula

$$U_A^{(2)}(\mathfrak{f}) \equiv \overline{A}\mathfrak{f} \cdot [(\overline{A}^{-1})^*(\mathbf{1})]^{\frac{1}{2}} = \overline{A}\mathfrak{f} \cdot \left[\frac{d\mu_{A^{-1}}}{d\mu}\right]^{\frac{1}{2}}.$$

(Here A is, as above, an automorphism of the BA $\mathscr{X}$, which is not measure preserving in general.) It is clear that $U_A^{(2)}$ is a linear operator. The equalities

$$\|U_A^{(2)}(\mathfrak{f})\|^2_{\mathfrak{L}^2_{\mathscr{X}}(\mu)} = \int (U_A^{(2)}(\mathfrak{f}))^2\, d\mu = \int (\overline{A}\mathfrak{f})^2 (\overline{A}^{-1})^*(\mathbf{1})\, d\mu$$

$$= \int \mathfrak{f}^2\, d\mu = \|\mathfrak{f}\|^2_{\mathfrak{L}^2_{\mathscr{X}}(\mu)}$$

show (we have used the formula 2°) that $U_A^{(2)}$ acts from $\mathfrak{L}^2_{\mathscr{X}}(\mu)$ into $\mathfrak{L}^2_{\mathscr{X}}(\mu)$, and it is a ROTATION of this space. Analogously, similar rotations are built for each $\mathfrak{L}_{\mathscr{X}}^{p}$ $(p \geq 1)$:

$$U_A^{(p)}(\mathfrak{f}) = \overline{A}\mathfrak{f} \cdot [(\overline{A}^{-1})^*(\mathbf{1})]^{\frac{1}{p}}.$$

The Banach Theorem is available claiming that for $p \neq 2$ there are no other rotations of $\mathfrak{L}^p$. For $p \neq 2$, the rotation group is considerably smaller than one in the case of the Hilbert space in which $p = 2$. In particular, it is remarkable that for $p \neq 2$, every rotation preserves disjointness of the elements, the property which seems to be purely "order" rather than "linearly metric." The Banach Theorem was also generalized to other spaces over the NBA; particularly, to complex symmetric Hilbert spaces,[16] real Lorentz spaces[17] and others. It seems strange that, up to now, most difficulties are connected with the real case.

4.3 The theorem of an invariant measure

The problem of an invariant measure has already been considered in Section 8.2. We have seen that it simplifies essentially if the main BA $\mathscr{X}$ is assumed to be normed or at least regular. (Lemma 3 and its corollaries.) In this case all conditions of the type (C) are equivalent; we will call an automorphism group satisfying these conditions *equicontinuous.* (This term exactly reflects the sense of the conditions of the type (C).)

The main theorem, which we now give, belongs to A. B. Hajian and K. Ito. It crowns a long series of results obtained by many mathematicians from 1932 on.

Theorem 7. *For existence of a measure on an NBA $\mathscr{X}$ which is invariant under an automorphism group $\mathfrak{A}$ of this algebra, it is necessary and sufficient that $\mathfrak{A}$ be equicontinuous. In other words, a Liouville group on an NBA is equicontinuous.*

As regards NECESSITY, this theorem has already been proved (we noted the necessity of the conditions of the type (C) in Section 8.2). The proof of SUFFICIENCY leans on a general idea of functional analysis: the problem of an invariant measure reduces to the problem of existence of a common fixed point for some family of operators.

Let $\mathfrak{A}$ be an equicontinuous group of automorphisms of a complete NBA $\mathscr{X}$. Distinguish a probability measure μ and consider the set Γ of all operators of the form U_A^2, $A \in \mathfrak{A}$ (see 8.3.2). It is easy to see that Γ is a group of operators.

Consider the "orbit of unity," i.e., the set of all elements $\mathfrak{L}^2_{\mathscr{X}}(\mu)$ of the form $U_A^{(2)}(\mathbf{1})$ $(A \in \mathfrak{A})$. Let S be the closed convex hull of this orbit, $\lambda_0 \equiv \inf_{\mathfrak{f} \in S} \|\mathfrak{f}\|$. The sets $K_\varepsilon \equiv S \cap \{\mathfrak{f} \mid \|\mathfrak{f}\| \leq \lambda_0 + \varepsilon\}$ $(\varepsilon > 0)$ are nonempty, convex, bounded, and closed; whence they are weakly compact. These sets have the finite intersection property; hence, their

[16] M. G. Zaĭdenberg [1].
[17] V. A. Biktasheva [1].

intersection

$$K_0 \equiv \bigcap_{\varepsilon > 0} K_\varepsilon$$

is nonempty and closed. The $\mathfrak{L}^2_{\mathscr{X}}$ space is strictly convex; hence, the ball $\{\mathfrak{f} \mid \|\mathfrak{f}\| \leq \lambda_0 + \varepsilon\}$ intersects the set K_0 only at a sole point $\mathfrak{f}_0$. Now, let

$$\nu(x) \equiv \int\limits_x \mathfrak{f}_0^2 \, d\mu.$$

We have obtained a countably additive quasimeasure. Prove its strict positivity. Take an arbitrary $x \in \mathscr{X}^+$ and show that the integrals $I_A \equiv \int_x U_A(\mathbf{1}) \, d\mu$ are separated from zero. Otherwise we would have $I_{A_n} \longrightarrow 0$ for some sequence $\{A_n\}$. But then, as was noted in 7.4.2, some subsequence $\{U_{A_{n_k}}(\mathbf{1}) \cdot x\}$ would converge to zero relatively uniformly, i.e., there will be an element $\mathfrak{r} \in \mathfrak{L}_{\mathscr{X}}(\mu)$ such that $\mathbf{0} \leq U_{A_{n_k}}(\mathbf{1}) \cdot x \leq \varepsilon_k \cdot \mathfrak{r}$ where $\varepsilon_k \downarrow 0$. Select an element $u \in \mathscr{X}_x^+$ so that $\mathfrak{r} \cdot u \leq Mu$ holds, where M is a positive constant (we may take $u \equiv e_\lambda^+(\mathfrak{r}) \wedge x$ for a sufficiently large λ and $M \equiv \lambda$). Recalling that $u(\mathbf{1}) = \left[\frac{d\mu_{A_{n_k}^{-1}}}{d\mu}\right]^{\frac{1}{2}}$, we see for all $k = 1, 2, \ldots$ that

$$\frac{d\mu_{A_{n_k}^{-1}}}{d\mu} \cdot u \leq \varepsilon_k^2 \mathfrak{r}^2 \cdot u \leq \varepsilon_k^2 M^2 \cdot u$$

and

$$\mu A_{n_k}^{-1}(u) = \mu_{A_{n_k}^{-1}}(u) = \int\limits_u \frac{d\mu_{A_{n_k}^{-1}}}{d\mu} \, d\mu \leq \varepsilon_k^2 M^2 \mu u \longrightarrow 0.$$

But $u > \mathbf{0}$; hence, by equicontinuity (the property (C_1)), we must have $\inf_k \mu A_{n_k}^{-1}(u) > \mathbf{0}$. This contradiction proves that the integrals $I_A = \int U_A(\mathbf{1}) \, d\mu$ are greater than some strictly positive constant. The same is true for the integrals $\int \mathfrak{g} \, d\mu$, where $\mathfrak{g} \in S$, and among them for the integral $\int_x \mathfrak{f}_0 \, d\mu$. So, $\mathfrak{f}_0 \cdot x > \mathbf{0}$ and $\nu(x) \equiv \int_x \mathfrak{f}_0^2 \, d\mu > 0$. These arguments prove the essential positivity of ν. Thus, ν is really a measure.

It remains to establish the invariance of ν. First, we demonstrate that $\mathfrak{f}_0$ is a *common fixed point* of all operators of the group Γ. Indeed, the set S is evidently invariant under this group: $U_A^{(2)}(S) \subset S$ for all $A \in \mathfrak{A}$. The point $\mathfrak{f}_0$ is nearest to zero; moreover, such point is unique. All $U_A^{(2)}$ are isometric operators. Therefore, $\|U_A^{(2)} \mathfrak{f}_0\| = \|\mathfrak{f}_0\| = \lambda_0$ and $U_A^{(2)}(\mathfrak{f}_0) = \mathfrak{f}_0$ for all $A \in \mathfrak{A}$.

Now, for all $x \in \mathscr{X}$ and $A \in \mathfrak{A}$ we have:

$$\nu(Ax) = \int\limits_{Ax} \mathfrak{f}_0^2 \, d\mu = \int\limits_{Ax} \overline{A}^{-1}\overline{A}\mathfrak{f}_0^2 \, d\mu = \int\limits_{x} \overline{A}^{-1}\mathfrak{f}_0^2 \, d\mu$$

$$= \int\limits_{x} (\overline{A}^{-1}\mathfrak{f}_0)^2 \frac{d\mu_A}{d\mu} \, d\mu = \int\limits_{x} \Big(U_{A^{-1}}^{(2)}(\mathfrak{f}_0)\Big)^2 \, d\mu = \int\limits_{x} \mathfrak{f}_0^2 \, d\mu = \nu(x).$$

Here we used the multiplicativity of $\overline{A}$ and the second formula of 2° (replacing A by A^{-1}). The invariance of the measure ν is proved.

Of course, this theorem applies to the case in which the group consists of the powers of a single automorphism A; we then obtain tests for existence of a measure invariant under this automorphism. We further agree to call an automorphism with the equicontinuous group of powers $\mathfrak{A} = \{A^n \mid n = 0, \pm 1, \pm 2, \dots\}$ *strongly continuous.* It follows immediately from Theorem 7 that an *automorphism of an NBA is Liouville if and only if it is strongly continuous.*

The next theorem of invariant measure is easily obtained by the synthesis of the Hajian–Ito Theorem with the ideas going back to Ch. Vallée –Poussin which we have repeatedly used in this book. Again, let $\{\mathscr{X}, \mu\}$ be a probability BA, and let $\mathfrak{A}$ be an automorphism group on $\mathscr{X}$.

Theorem 8. *For existence of an $\mathfrak{A}$-invariant measure, it is necessary and sufficient that there exist a function $\Phi \in \mathscr{K}$ and a constant K such that, for each finite partition of unity τ and each $A \in \mathfrak{A}$, the following hold:*

$$\sum_{e \in \tau} \Phi\Big(\frac{\mu(Ae)}{\mu e}\Big) \mu e \le K < +\infty.$$

Indeed, by the hypothesis of the theorem means the equicontinuity of the measures $\mu \circ A$, which is equivalent to each of the condition of the type (C).

For example, taking $\Phi(u) = u^p$ $(p > 1)$, we see that the inequality

$$\sup_{\tau} \sum_{e \in \tau} \frac{[\mu(Ae)]^p}{(\mu e)^{p-1}} < +\infty$$

implies the existence of an invariant measure with density in $\mathfrak{L}^p_{\mathscr{X}}(\mu)$.

5. Construction of a normed Boolean algebra given a transformation group

In this monograph the stance prevails in which the initial object under study is a complete Boolean algebra (a system of events), and the main

problem consists in constructing a (probability) measure on this algebra, possibly invariant under some automorphism group. An another approach is possible (and constantly practiced) in which the Boolean algebra is created simultaneously with a measure from a transformation group of some set. The group itself may be used as this set; the classical example is the Haar theory whose various versions are exposed in many famous textbooks.[18]

5.1 Amenable groups

A group G is called *amenable* if there is a nontrivial finite translation-invariant quasimeasure α on the boolean of G. As ever, this means that for all $g \in G$ and $e \subset G$ we have

$$\alpha(ge) = \alpha e = \alpha(eg)$$

(recall that the set ge consists of all elements of the form gg', with $g' \in e$; the set eg is defined similarly).

We always assume that $\alpha G = 1$. Most often, the quasimeasure α is not countably additive; however, it is always defined on ALL subsets.

We will denote the integral over an invariant quasimeasure α by

$$\int_E f\,d\alpha \equiv \int_E f(g)\,d\alpha_g \quad (E \subset G) \tag{5}$$

(it is convenient to introduce the bound variable g in notation of the integral). This integral exists for every bounded function.

The characteristic property of this integral is its invariance. For each bounded function f and all $g_0 \in G$ the following equalities hold:

$$\int_G f\,d\alpha = \int_G {}_{g_0}f\,d\alpha = \int_G f_{g_0}\,d\alpha,$$

where ${}_{g_0}f(g) \equiv f(g_0 g)$ and $f_{g_0}(g) \equiv f(g g_0)$.

The source of amenable groups lies probably in the articles of S. Banach [1, 2]. The class of amenable groups itself was first introduced by J. von Neumann [1] in 1929; he called these groups "measurable." In 1936 A. A. Markov (junior) proved[19] the existence of an invariant quasimeasure for every commutative family of mappings from a set into itself. In particular, this immediately implies the amenability of

[18] See, for example, the books of S. Sacks [1] (with an appendix written by S. Banach), P. Halmos [3], and K. Parthasarathy [1].

[19] A. A. Markov [1].

all commutative groups. All solvable and finite groups are amenable too. A further development of this interesting theory is concerned with the contributions of J. Dixmier, E. Følner, G. M. Adel′son-Vel′skiĭ and Yu. A. Shreĭder, R. I. Grigorchuk, and other authors.[20] The famous theorems of F. Hausdorff and S. Banach and A. Tarski on the "paradoxical" partitions are closely connected with nonamenability of the rotation group of the three-dimensional space.

We give some simplest examples.

Example 1. Let $G = \mathbb{R}$ be the additive group of reals. It is abelian and, hence, amenable. To each invariant quasimeasure α there corresponds two numbers

$$a_\alpha = \alpha(-\infty, x), \quad b_\alpha = \alpha(x, +\infty).$$

These numbers do not depend on x. In this case we always have $a_\alpha, b_\alpha \geq 0$, $a_\alpha + b_\alpha = 1$. For every bounded set E, the equality $\alpha E = 0$ holds. If f is a bounded function having the limit values $f(+\infty)$, $f(-\infty)$ then

$$\int\limits_{\mathbb{R}} f(x)\, d\alpha_x = a_\alpha f(-\infty) + b_\alpha f(+\infty).$$

One of the numbers a_α, b_α can be zero.

Example 2. We now take as G the interval $[0, 1]$, and define the group operation as addition modulo 1. (We may interpret G as the rotation group of the circle with radius $\frac{1}{2\pi}$.) This group is abelian too; the invariant quasimeasures on it are "Banach measures," the integrals over them are "Banach integrals." There are many such "measures" and integrals, but each integral of a continuous function coincides with its Riemann integral.

Example 3. $G = \mathbb{Z}$ is the additive group of integers. Let

$$\mathbb{Z}^{(n)} \equiv \{k \in \mathbb{Z} \mid |k| \leq n\} \quad (n = 1, 2, \dots).$$

Denote by the symbol *Lim* one of the "Banach limits." These limits, as is well known, exist for each bounded numerical sequence. By the formula

$$\alpha(E) \equiv Lim \frac{1}{2n} card\, (E \cap \mathbb{Z}^{(n)})$$

we define an invariant quasimeasure on $2^{\mathbb{Z}}$.

Amenable groups appear constantly in ergodic theory as the groups of powers of automorphisms, flows, and etc.

[20] F. Greenleaf [1]; G. M. Adel′son-Vel′skiĭ and Yu. A. Shreĭder; R. I. Grigorchuk [1]; A. Yu. Ol′shanskiĭ [1]; V. A. Kaĭmanovich and A. M. Vershik [1].

5.2 Construction of an NBA given a transformation group

Consider the following situation. Let R be a nonempty set, let Γ be an amenable group of its bijective transformations, and let M_R be the space of all real functions bounded on R and endowed with the Chebyshev norm $\|f\| = \sup_{x\in R} |f(x)|$. Select in M_R some Γ-invariant subspace M' containing all constants, together with the identically one function $\mathbf{1}$. Finally, take an arbitrary positive functional $\varphi \in M^*$ satisfying the condition $\varphi(\mathbf{1}) = 1$.

The existence of an invariant quasimeasure α on Γ allows us to define on M' the functional[21]

$$L_\varphi : \qquad L_\varphi(f) \equiv \int\limits_\Gamma \varphi(f \circ \gamma)\, d\alpha_\gamma. \tag{6}$$

It is clear that this functional is positive and satisfies the condition $L_\varphi(\mathbf{1}) = 1$. Moreover, it is invariant; i.e., $L_\varphi(f \circ \eta) = L_\varphi(f)$ for all $\eta \in \Gamma$. Moreover, $\|L_\varphi\| \le \|\varphi\|$.

It may happen that in R there exists a sufficiently rich Γ-invariant algebra $\mathscr{E}$ such that $\chi_e \in M'$ for all $e \in \mathscr{E}$. Then we can define on this algebra the following Γ-invariant quasimeasure:

$$p : \qquad p(e) \equiv \int\limits_\Gamma \varphi(\chi_{\gamma^{-1}(e)})\, d\alpha_\gamma \equiv L_\varphi(\chi_e).$$

The countable additivity of this quasimeasure would allow us to use the Lebesgue–Carathéodory Theorem and construct an invariant countably additive quasimeasure on a σ-algebra wider than $\mathscr{E}$. We can give various conditions that guarantee such a possibility. However, we will not do this but rather study in detail the case in which R is furnished with a topological structure.

Let R be a compact space, and let Γ be an amenable group of its homeomorphisms. Take as M' the space $C(R)$ of continuous real functions. It may contain a few characteristic functions; therefore, the approach described above is not suitable for this case: the algebra $\mathscr{E}$ is very poor. Nevertheless, the functional L_φ, in accord with the celebrated Riesz Theorem, generates a regular quasimeasure l_φ on the Borel σ-algebra and is

[21]We may assume that the functional φ (and hence, L_φ) is defined on the whole M_R; nevertheless, its particular properties are often important that are connected with M'.

represented via this quasimeasure by the formula

$$L_\varphi(f) = \int\limits_R f\,dl_\varphi.$$

Such measure is unique, and invariant together with the functional L_φ. Moreover, $l_\varphi(\mathbb{R}) = 1$. Thus, under the above assumptions on the Borel algebra (which is obviously invariant under homomorphisms), there exists a Γ-invariant countably additive regular quasimeasure l_φ.[22] The action of the group Γ is naturally translated to the metric structure that corresponds to this quasimeasure (cf. 1.3.3). We thus obtain a complete Boolean algebra with an amenable automorphism group and probability measure invariant under this group. In the construction described now an essential role is played by the functional φ. It can be arbitrary. For example, as φ we can take the *delta-function* supported at some point r_0; i.e., $\delta_{r_0}(f) = f(r_0)$.

The group of all powers of one homeomorphism is evidently amenable. In this case our construction transforms into the proof of the celebrated theorem of N. N. Bogolyubov and N. M. Krylov on an autoinvariant measure.[23]

Amenability of a group allows us to avoid in the preceding discussion the problem of existence of the integral in the formula (6). But the same can be achieved by using other assumptions. In many important cases the integral (6) can be treated as the integral with respect to the Haar measure. To this end, it suffices that the group Γ be furnished with a compact topology such that the integrand $F(\gamma) = \varphi(f \circ \gamma)$ is a continuous or Baire function with respect to this topology. We indicate the most important case: let R be a metrizable compact space, and let Γ be the group of all isometries. It is well known[24] that Γ is a compact transformation group (with respect to the topology of uniform convergence).

Given $\varphi \in C^*(R)$ and $f \in C(R)$, the function $F(\gamma) = \varphi(f \circ \gamma)$ is continuous on Γ, and if we now denote by α the Haar measure on Γ then we will have the integral (6). Hence, in this case there exists a Borel measure on R invariant under the group Γ.

By way of summarizing, we formulate the theorem that concerns the most important cases.

Theorem 9.

Assume that one of the following two conditions holds:

---

[22]If $R = \Gamma$ is a compact group then l_φ is the Haar measure. This construction uses amenability.

[23]N. N. Bogolyubov and N. M. Krylov [1].

[24]For example, see L. S. Pontryagin [1, § 24].

a) *R is an arbitrary compact space, and Γ is an amenable group of homomorphisms of R.*

b) *R is a metrizable compact space, and Γ is a group of all isometries of this compactum.*

Then there exists a Borel quasimeasure on R invariant under all transformations in Γ. The automorphism group $\widehat{\Gamma}$ preserving this quasimeasure acts in the metric structure generated by this quasimeasure; moreover, the group $\widehat{\Gamma}$ is the homomorphic image (representation) of Γ.

This theorem follows from the preceding consideration.

The case b) in the last theorem, for example, comprises the group SO_n of all rotations of the n-dimensional Euclidean space which acts on the unit sphere of this space. For $n > 2$ this group is not amenable. The Borel quasimeasure, mentioned in the theorem, is a suitably normalized n-dimensional Lebesgue "measure" on this sphere. The corresponding metric structure is isomorphic (as a normed Boolean algebra) to the Lebesgue algebra $E^{(n-1)}$. This example is important in view of the following: It is easy to see that translating the action of the group SO_n to the metric structure we come to a FAITHFUL representation of this group; the automorphism group of the NBA arising in this way (it is obviously ergodic) is isomorphic to the original rotation group. We can see in this case that the rotation group concisely (but adequately) reflects the probability situation that arises in the "absolutely target-free" shooting when all directions are equiprobable. The group defines everything: the complete Boolean algebra of events and the unique invariant probability measure on this algebra. Such cases occur not so often. For example, we apply the case a) of Theorem 8 to the group of Example 1 (p. 441), taking as R the compact extended axis $[-\infty, +\infty]$ (the points $\pm\infty$ are assumed to remain immovable under translations). It is easy to see that the Borel measure, mentioned in the theorem, is concentrated at the points $\pm\infty$; the metric structure consists of (at most) four elements but the automorphism group is trivial and in no way resembles the original translation group.

Formulating the last theorem, we emphasize only the most important cases a) and b); a series of intermediate variants was not considered and awaits further research.

Not only groups but also semigroups may be "amenable." For example, we can consider the semigroups of homomorphisms (not necessary one-to-one). The amenability of such a semigroup facilitates the construction of an invariant measure.

Chapter 9

STRUCTURE OF A NORMED BOOLEAN ALGEBRA

1. Structure of a normed algebra

1.1 Decomposition of a homogeneous normed algebra into a product of metrically independent simple subalgebras

Note that a BA is called homogeneous in this book whenever all bands have the same weight. (In other words, such a BA is τ-homogeneous in the sense of Chapter 2.) Each complete BA decomposes into homogeneous bands. We have seen in Chapter 2 that each homogeneous BA decomposes always into the product of simple (four-element) subalgebras. Now, on assuming that our BA is $\mathscr{X}$ normed, we will essentially strengthen this result.

Namely, we decompose a homogeneous BA into a product of μ-independent simple subalgebras.

As was mentioned, we say that some subalgebras $\mathscr{X}_1, \mathscr{X}_2, \dots, \mathscr{X}_m$ of an NBA $\{\mathscr{X}, \mu\}$ are μ-independent whenever given $x_1, x_2, \dots, x_m$ such that $x_i \in \mathscr{X}_i$ $(i = 1, 2, \dots, m)$ we have

$$\mu(x_1 \wedge x_2 \wedge \cdots \wedge x_m) = \mu x_1 \mu x_2 \dots \mu x_m.$$

We assume that μ is a probability measure, i.e., $\mu\mathbf{1} = 1$.

Granted a class $\mathfrak{P}$ of subalgebras, we call it *μ-independent* if so is its every finite subset. The meaning of the term "μ-independent system" is also evident.

A simple subalgebra $\{u, Cu, \mathbf{0}, \mathbf{1}\}$ of an NBA $\mathscr{X}$ is called *μ-simple* provided that $\mu u = \mu Cu = \frac{1}{2}$.

We now exhibit an example of importance for the sequel. Assume that a normed complete algebra $\mathscr{X}$ decomposes into 2^n pairwise disjoint

bands

$$\mathscr{X}_1 \equiv \mathscr{X}_{u_1},\ \ \mathscr{X}_2 \equiv \mathscr{X}_{u_2}, \dots, \mathscr{X}_{2^n} \equiv \mathscr{X}_{u_{2^n}},$$

where $\mu u_1 = \mu u_2 = \dots = \mu u_{2^n} = \frac{1}{2^n}$. (Here we suppose that μ is a probability measure on $\mathscr{X}$.)

Consider the subalgebra $\mathscr{X}_0 \equiv \mathscr{X}\langle u_1, u_2, \dots, u_{2^n}\rangle$. It is discrete and regular while containing 2^{2^n} elements. By Theorem 1.7 this is a free subalgebra on some independent generators $z_1, z_2, \dots, z_n$. Since μ induces the BASIC measure on $\mathscr{X}_0$; therefore (cf. p. 73) the system $z_1, z_2, \dots, z_n$ is also metrically independent. In other words, it is a μ-independent system of elements of $\mathscr{X}$; moreover, $\mu z_1 = \mu z_2 = \dots = \mu z_n = \frac{1}{2}$.

Thus, the subalgebra $\mathscr{X}_0$ in our example is the product of some μ-independent n-point system of μ-simple subalgebras.

Theorem 1. *Let $\{\mathscr{X}, \mu\}$ be an NBA with μ a probability measure; and let $\widetilde{\mathscr{X}}$ stand for a regular subalgebra of $\{\mathscr{X}, \mu\}$ other than $\mathscr{X}$. If $\mathscr{X}$ is a $\widetilde{\mathscr{X}}$-homogeneous BA then there is a μ-independent class $\mathfrak{P}$ of μ-simple subalgebras of cardinality $\sigma(\widetilde{\mathscr{X}}, \mathbf{1})$ such that the regular subalgebra*

$$\prod_{Z \in \mathfrak{P}} Z = \overline{\mathscr{X}\Big\langle \bigcup_{Z \in \mathfrak{P}} Z \Big\rangle}$$

it generates is a μ-independent complement of $\widetilde{\mathscr{X}}$.

This theorem is a "metric analog" of Theorem 2.17. However, we now observe a few principal features: the subalgebras of $\mathfrak{P}$ are μ-simple rather than simple; the class $\mathfrak{P}$ itself is μ-independent and the regular subalgebra it generates is a μ-independent complement of $\widetilde{\mathscr{X}}$.

PROOF. In much the same way as in the proof of Theorem 2.17, the first steps are Lemmas 14 and 15 of Chapter 2. However, we have to replace Lemma 16 with a less trivial proposition to be formulated soon. The following lemma "collecting" most difficulties is in order now.

Lemma 1. *Assume that $\mathscr{X}$ is an NBA, μ is a probability measure on $\mathscr{X}$, and $\widetilde{\mathscr{X}}$ is a regular subalgebra saturating no nonzero band. To each $u \in \mathscr{X}$ and each natural number n, there is a disjoint system $Z \equiv Z_{\widetilde{\mathscr{X}},n} = \{z_1, z_2, \dots, z_n\}$ of n members such that*

1) *given $x \in \widetilde{\mathscr{X}}$, we have*

$$\mu(x \wedge z_k) = \mu x \mu z_k \quad (k = 1, 2, \dots, n);$$

2) *there is some u' in $\mathscr{X}\langle\widetilde{\mathscr{X}}, Z\rangle$ satisfying*

$$\mu|u - u'| \le \frac{2}{n}.$$

The proof rests on a series of auxiliary propositions.

Given $x \in \widetilde{\mathscr{X}}^+$ and $y \in \mathscr{X}$, put

$$\overline{h}(x,y) \equiv \sup_{x' \in \widetilde{\mathscr{X}}_x^+} \frac{\mu(x' \wedge y)}{\mu x'}, \quad \underline{h}(x,y) \equiv \inf_{x' \in \widetilde{\mathscr{X}}_x^+} \frac{\mu(x' \wedge y)}{\mu x'}.$$

We may call these numbers the "outer width" and "inner width" of y with respect to $\widetilde{\mathscr{X}_x}$. Clearly, $0 \leq \underline{h}(x,y) \leq \overline{h}(x,y)$.

The equality $\underline{h}(x,y) = \overline{h}(x,y)$ means that $\mu(x' \wedge y) = \eta\mu x'$ holds for all $x' \in \widetilde{\mathscr{X}_x}$.

1°. *To all $u \in \mathscr{X}$, $\overline{x} \in \widetilde{\mathscr{X}}^+$ and each real $\varepsilon > 0$ there is some $\overline{\overline{x}}$ in $\widetilde{\mathscr{X}}$ such that $\mathbf{0} < \overline{\overline{x}} \leq \overline{x}$ and*

$$\overline{h}(\overline{\overline{x}},u) - \underline{h}(\overline{\overline{x}},u) \leq \varepsilon.$$

To prove, put $a \equiv \underline{h}(\overline{x},u) + \varepsilon$ and consider the set A comprising $x \in \widetilde{\mathscr{X}_{\overline{x}}}$ such that $\mu(x \wedge u) < a\mu x$.

By the definition of $\underline{h}(\overline{x},u)$ there is a nonzero $x' \in \widetilde{\mathscr{X}_{\overline{x}}}$ satisfying

$$\mu(x' \wedge u) < a\mu x'.$$

Clearly, A^+ is nonempty since $x' \in A^+$. Noting that $A' \equiv \widetilde{\mathscr{X}_{\overline{x}}} \setminus A$ is obviously a d-regular set, apply Theorem 2.5 to $\widetilde{\mathscr{X}_{\overline{x}}}$. By this theorem the solid core A^0 contains a nonzero element $\overline{\overline{x}}$. Since the whole band $\widetilde{\mathscr{X}_{\overline{\overline{x}}}}$ belongs to A; therefore,

$$\overline{h}(\overline{\overline{x}},u) \equiv \sup_{x \in \widetilde{\mathscr{X}}_{\overline{\overline{x}}}^+} \frac{\mu(x \wedge u)}{\mu x} \leq a.$$

Also, $\underline{h}(\overline{\overline{x}},u) \geq \underline{h}(\overline{x},u)$, implying

$$\overline{h}(\overline{\overline{x}},u) - \underline{h}(\overline{\overline{x}},u) \leq \overline{h}(\overline{\overline{x}},u) - \underline{h}(\overline{x},u) \leq \varepsilon,$$

which completes the proof.

2°. *To each $v \in \mathscr{X}_{\overline{\overline{x}}}$ and each real $\varepsilon > 0$ there is some $w \in \mathscr{X}_v^+$ satisfying*

$$\overline{h}(\mathbf{1},w) \leq \varepsilon.$$

To prove, use Lemma 2.15. By hypothesis the subalgebra $\widetilde{\mathscr{X}}$ saturates no nonzero band. Hence, to each $y > \mathbf{0}$ we may assign some $z_y < y$ so as to satisfy the inequalities

$$z_y \wedge x > \mathbf{0}, \quad (y - z_y) \wedge x > \mathbf{0}$$

for all $x \in \widetilde{\mathscr{X}}^+$, $x \leq \overline{y} \equiv \inf\{x \mid x \in \widetilde{\mathscr{X}}, x \geq y\}$. Applying Lemma 2.15 successively to the nonzero elements

$$y_0 \equiv v,\ \ y_1 \equiv y_0 - z_{y_0},\ \ y_2 \equiv y_1 - z_{y_1}, \dots,$$

find the disjoint sequence $\{z_{y_k}\}_{k=1}^{\infty}$ whose entries satisfy the inequalities

$$x \wedge z_{y_k} > \mathbf{0}, \quad x \wedge (y_k - z_{y_k}) > \mathbf{0} \tag{1}$$

for all $x \in \widetilde{\mathscr{X}}^+$, $x \leq \overline{y}_k = \overline{y}_0$ (from the second inequality it follows by induction that $y_k > \mathbf{0}$). Clearly, $\mu z_{y_k} \longrightarrow 0$.

Distinguish an arbitrary $x_0 \in \widetilde{\mathscr{X}_{\overline{v}}} \equiv \widetilde{\mathscr{X}_{\overline{x}_0}}$ and choose an index $\overline{k}$ so that

$$\mu(x_0 \wedge z_{y_{\overline{k}}}) < \varepsilon \mu x_0.$$

Let A be the set comprising all $x \in \widetilde{\mathscr{X}_{\overline{v}}}$ that satisfy the inequality

$$\mu(x \wedge z_{y_{\overline{k}}}) < \varepsilon \mu x.$$

Note that A^+ is nonempty while $A' \equiv \widetilde{\mathscr{X}} \setminus A$ is d-regular. Applying Theorem 2.7 once again, conclude that the solid core A^0 contains some nonzero element x^*. Put $w \equiv z_{y_{\overline{k}}} \wedge x^*$. Then

$$\frac{\mu(x \wedge w)}{\mu x} = \frac{\mu(x \wedge x^* \wedge z_{y_{\overline{k}}})}{\mu x} \leq \frac{\varepsilon \mu(x \wedge x^*)}{\mu x} \leq \varepsilon$$

for all $x \in \widetilde{\mathscr{X}}^+$. In other words, $\overline{h}(\mathbf{1}, w) \leq \varepsilon$. At the same time $x^* \in \widetilde{\mathscr{X}_{\overline{v}}^+}$ and so from the first inequality of (1) it follows that

$$v \equiv y_0 \geq w \equiv x^* \wedge z_{y_{\overline{v}}} > \mathbf{0},$$

i.e., $w \in \mathscr{X}_v^+$. This yields 2°.

Take $y \in \mathscr{X}^+$, and let η be an arbitrary positive real. Arrange the set D_η by comprising those elements $w \in \mathscr{X}_y$ that satisfy $\overline{h}(\mathbf{1}, w) \leq \eta$.

3°. *The set D_η has maximal elements.*

We are to check that we may apply the Kuratowski–Zorn Lemma to D_η. To this end consider an arbitrary chain $\mathbf{C} \subset D_\eta$. Given $w \in \mathbf{C}$, we have $\overline{h}(\mathbf{1}, w) \leq \eta$. This yields the inequality $\mu(w \wedge x) \leq \eta \mu x$ that holds for all $x \in \widetilde{\mathscr{X}}$. By continuity,

$$\mu(\sup \mathbf{C} \wedge x) \leq \eta \mu x.$$

Since this holds for all $x \in \widetilde{\mathscr{X}}$; therefore,

$$\sup_{x \in \widetilde{\mathscr{X}}^+} \frac{\mu(\sup \mathbf{C} \wedge x)}{\mu x} \equiv \overline{h}(\mathbf{1}, \sup \mathbf{C}) \leq \eta.$$

This means that $\sup \mathbf{C} \in D_\eta$. So, every chain $\mathbf{C} \subset D_\eta$ is bounded above in D_η, which implies that D_η contains maximal elements by the Kuratowski–Zorn Lemma.

4°. *If* $\underline{h}(\overline{x}, u) \geq \eta \geq 0$ *then there is some* $z \leq \overline{x} \wedge u$ *satisfying*

$$\underline{h}(\overline{x}, u) = \overline{h}(\overline{x}, u) = \eta.$$

To prove, arrange the set D_η as above on taking $\overline{x} \wedge u$ as y. Let z be a maximal element of D_η which exists by 3°.

Show that

$$\mu(x \wedge z) = \eta \mu x$$

for all $x \in \widetilde{\mathscr{X}_{\overline{x}}}$. Clearly, $\mu(x \wedge z) \leq \eta \mu x$. Assume that for some $x^* \in \widetilde{\mathscr{X}_{\overline{x}}}$ we have the strict inequality

$$\eta \mu x^* - \mu(x^* \wedge z) > 0.$$

Choose $\varepsilon > 0$ so that

$$(\eta - \varepsilon)\mu x^* - \mu(x^* \wedge z) > 0$$

and consider the set A comprising $x \in \widetilde{\mathscr{X}_{\overline{x}}}$ such that $(\eta - \varepsilon)\mu x - \mu(x \wedge z) > 0$. This set contains a nonzero element x^*; moreover, the complement $A' \equiv \widetilde{\mathscr{X}_{\overline{x}}} \setminus A$ is d-regular. Therefore, there is a nonzero element x^+ belonging to the solid core A^0.

Given $x \in \widetilde{\mathscr{X}_{x^+}}$, note that $\mu(x \wedge z) \leq (\eta - \varepsilon)\mu x$. We now put $v \equiv u \wedge x^+ \wedge Cz$. This element differs from zero: Otherwise the inequality would hold $u \wedge x^+ \leq z$, implying the estimate

$$\underline{h}(\overline{x}, u) \leq \frac{\mu(u \wedge x^+)}{\mu x^+} \leq \frac{\mu(z \wedge x^+)}{\mu x^+} \leq \eta - \varepsilon < \eta,$$

which is impossible since $\eta \leq \underline{h}(\overline{x}, u)$.

Using 2°, find a nonzero $w \in \mathscr{X}_v$ so that

$$\overline{h}(\mathbf{1}, w) \leq \varepsilon.$$

Put $z^* \equiv z + w$. Clearly, $\overline{x} \wedge u \geq z^* > z$. At the same time, given $x \in \widetilde{\mathscr{X}^+}$ we have

$$\mu(x \wedge z^*) = \mu\{(x \wedge Cx^+ \wedge z^*) + (x \wedge x^+ \wedge z^*)\} = \mu(x \wedge Cx^+ \wedge z)$$
$$+ \mu(x \wedge x^+ \wedge z) + \mu(x \wedge x^+ \wedge w) \leq \eta\mu(x \wedge Cx^+) + (\eta - \varepsilon)\mu(x \wedge x^+)$$
$$+ \varepsilon\mu(x \wedge x^+) = \eta[\mu(x \wedge Cx^+) + \mu(x \wedge x^+)] = \eta\mu x,$$

yielding $\overline{h}(\mathbf{1}, z^*) \le \eta$.

In other words, $z^* \in D_\eta$. This is impossible since $z^* > z$ and z is a maximal element of D_η. Thus, given $x \in \widetilde{\mathscr{X}}_{\overline{x}^+}$, we have

$$\frac{\mu(x \wedge z)}{\mu x} = \eta,$$

i.e., $\overline{h}(\overline{x}, z) = \underline{h}(\overline{x}, z) = \eta$. This completes the proof of 4°.

Assume now that $x \in \widetilde{\mathscr{X}}$ and $z \in \mathscr{X}$ satisfy

$$\overline{h}(x, z) = \underline{h}(x, z) \equiv \eta_0 > 0.$$

Given $n = 2, 3, \dots$, put

$$\eta = \frac{1}{n}\eta_0$$

and apply 4°. We obtain $z_1 \le z$ such that $\overline{h}(x, z_1) = \underline{h}(x, z_1) = \eta$. It is easy to see that we now have

$$\overline{h}(x, z - z_1) = \underline{h}(x, z - z_1) = \eta_0 - \eta = \frac{n-1}{n}\eta_0 \ge \eta.$$

We may thus apply 4° to x and $z - z_1$ again and proceed likewise exactly n times. This yields:

5°. *If $\overline{h}(x, z) = \underline{h}(x, z) \equiv \eta > 0$ then to each natural number n there is a decomposition of z into n disjoint summands $z_1, z_2, \dots, z_n$ satisfying*

$$\overline{h}(x, z_k) = \underline{h}(x, z_k) = \frac{1}{n}\eta \quad (k = 1, 2, \dots, n).$$

We now turn to the concluding step of the proof of the lemma. Put $\varepsilon = \frac{2}{n}$ and arrange the set

$$E \equiv \left\{x \mid x \in \widetilde{\mathscr{X}},\ \overline{h}(x, u) - \underline{h}(x, u) < \frac{\varepsilon}{2}\right\}.$$

By 1° this is a minorant for $\widetilde{\mathscr{X}}$. Hence, there is a disjoint system S in E whose supremum equals $\mathbf{1}$. To each $x \in S$ we assign a nonnegative integer m_x so that

$$\underline{h}(x, u) - \frac{\varepsilon}{2} \le \frac{m_x}{n} < \underline{h}(x, u).$$

Using 4° and given $x \in S$, find $z \equiv z(x) \le x \wedge u$ satisfying

$$\underline{h}(x, z) = \overline{h}(x, z) = \frac{m_x}{n}.$$

Then it is clear that

$$\underline{h}(x, x - z) = \overline{h}(x, x - z) = \frac{n - m_x}{n}.$$

Applying 5° to z and $x - z$, find a disjoint system $z_1(x), z_2(x), \dots, z_n(x)$ such that for all $k = 1, 2, \dots, n$ we will have

$$\underline{h}(x, z_k(x)) = \overline{h}(x, z_k(x)) = \frac{1}{n}$$

and

$$\bigvee_{k=1}^{m_x} z_k(x) = z(x) \leq x \wedge u.$$

We now put $z_k \equiv \sum_{x \in S} z_k(x)$. Since $z_k(x)$, $x \in S$, are pairwise disjoint for all $k = 1, 2, \dots, n$; given $x_0 \in \widetilde{\mathscr{X}}$, we see that

$$\mu(x_0 \wedge z_k) = \sum_{x \in S} \mu(x_0 \wedge z_k(x) \wedge x) = \frac{1}{n} \sum_{x \in S} \mu(x_0 \wedge x) = \frac{1}{n} \mu x_0.$$

Putting $x_0 = \mathbf{1}$, find the measure z_k: it turns out equal to $\frac{1}{n}$. Hence, $\mu(x_0 \wedge z_k) = \mu x_o \mu z_k$ for all $x_0 \in \widetilde{\mathscr{X}}$, $k = 1, 2, \dots$. The elements $z_1, z_2, \dots, z_n$ comprise the sought system Z. We are left with constructing the approximant u'. Put

$$u' \equiv \sum_{x \in S} \bigvee_{k=1}^{m_x} z_k(x) = \sum_{x \in S} z(x).$$

Clearly, $u' \in \overline{\mathscr{X}\langle \widetilde{\mathscr{X}}, Z \rangle}$. Moreover, $u' \leq u$ and

$$\mu|u - u'| = \mu(u - u') = \sum_{x \in S} \mu[(u - u') \wedge x] \leq \sum_{x \in S} \overline{h}(x, u - u') \mu x.$$

Given $x \in S$, obtain

$$\overline{h}(x, u - u') = \sup_{x'} \frac{\mu(x' \wedge (u - u'))}{\mu x'} \leq \sup_{x'} \Big[\frac{\mu(x' \wedge u)}{\mu x'} - \underline{h}(x, u') \Big]$$

$$\leq \overline{h}(x, u) - \underline{h}(x, u') = \overline{h}(x, u) - \frac{m_x}{n} \leq \underline{h}(x, u) + \frac{\varepsilon}{2} - \underline{h}(x, u) + \frac{\varepsilon}{2} = \varepsilon.$$

We thus conclude that

$$\mu(|u - u'|) \leq \varepsilon \sum_{x \in S} \mu x = \frac{2}{n}.$$

We now prove a proposition that serves as an analog of Lemma 2.16 in the "metric" situation.

Lemma 2. *In the context of Lemma 1, to each $u \in \mathscr{X}$ there is a countable family Z of μ-independent elements which possesses the following properties:*

1) $u \in \overline{\mathscr{X}\langle \widetilde{\mathscr{X}}, Z\rangle}$,
2) *the subalgebras* $\widetilde{\mathscr{X}}$ *and* $\overline{\mathscr{X}\langle Z\rangle}$ *are μ-independent,*
3) $\mu z = \frac{1}{2}$ *for all* $z \in Z$.

Condition 1) means that u belongs to the product of $\widetilde{\mathscr{X}}$ and $\mathscr{X}\langle Z\rangle$.

Proof. Using the preceding lemma, find a chain of disjoint families $\{Z^n\}$ and subalgebras $\{\mathscr{Y}_n\}$ on successively putting

$$\begin{gathered}
\mathscr{Y}_0 \equiv \widetilde{\mathscr{X}}; \quad Z^1 \equiv Z_{\mathscr{Y}_0,2}, \quad \mathscr{Y}_1 \equiv \overline{\mathscr{X}\langle \mathscr{Y}_0, Z^1\rangle}; \\
Z^2 \equiv Z_{\mathscr{Y}_1,2^2}, \quad \mathscr{Y}_2 \equiv \overline{\mathscr{X}\langle \mathscr{Y}_1, Z^2\rangle}; \dots \\
Z^n \equiv Z_{\mathscr{Y}_{n-1},2^n}, \quad \mathscr{Y}_n \equiv \overline{\mathscr{X}\langle \mathscr{Y}_{n-1}, Z^n\rangle}; \dots
\end{gathered} \tag{2}$$

The soundness of this definition is easy on noting that each of the subalgebras $\mathscr{Y}_n$ fails to saturate any band as generated by the subalgebra $\widetilde{\mathscr{X}}$ and the finite set $\bigcup_{k=1}^{n} Z^k$.

Consequently, we may use Lemma 1 at every step of construction, and the families are indeed definable by the formulas of (2) from each other. Moreover, it is evident that each of the subalgebras $\mathscr{Y}_n$ contains some element whose distance to u is at most $\frac{1}{2^{n-1}}$. Hence, by closing the union of these subalgebras, we come to a set containing u.

The construction of Z^k shows that these families comprise a μ-independent class of sets (which is easy by induction on recalling the inclusions $Z^k \subset \mathscr{Y}_k$). Let us consider the finite subalgebras

$$\widetilde{\mathscr{X}_k} \equiv \mathscr{X}\langle Z^k\rangle.$$

Each of them is generated by a disjoint partition of unity; consequently, each member of $x \in \widetilde{\mathscr{X}_k}$ is a finite sum of elements of Z^k. Hence, as follows by simple calculation, not only the families Z^k but also the subalgebras $\widetilde{\mathscr{X}_k}$ generated by them comprise a μ-independent class of sets.

Further, each of the subalgebras $\widetilde{\mathscr{X}_k}$, generated by a partition of unity in 2^k addends of the same measure, includes the μ-independent system of k generators $\{z_1^k, z_2^k, \dots, z_k^k\}$; moreover, $\mu z_1^k = \mu z_2^k = \dots = \mu z_k^k = \frac{1}{2}$ (cf. the example on p. 446).

We now arrange a countable family Z by collecting all elements z_i^k. This is a μ-independent family, since so are the subalgebras $\widetilde{\mathscr{X}_k}$ and

the generators z_i^k are μ-independent inside each of these subalgebras. Further, the family Z satisfies the condition 3), since u belongs obviously to the subalgebra $\overline{\mathscr{X}\langle\widetilde{\mathscr{X}}, Z\rangle}$ (which contains all $\mathscr{Y}_n$). Finally, it is easy to check that the subalgebras $\widetilde{\mathscr{X}}$ and $\mathscr{X}\langle Z\rangle$ are μ-independent implying that so are $\widetilde{\mathscr{X}}$ and $\overline{\mathscr{X}\langle Z\rangle}$ by continuity of the measure. The proof of the lemma is complete.

The subalgebra $\overline{\mathscr{X}\langle Z\rangle}$ we have constructed in the proof of Lemma 2 is a product of countably many μ-simple subalgebras.

We are now ready to prove Theorem 1. The proof proceeds along the same lines as the proof of the "algebraic" theorem of Chapter 2. The difference consists in using Lemma 2 in place of a simpler Lemma 16 of Chapter 2.

Let E be again a set of cardinality $\sigma(\widetilde{\mathscr{X}}, \mathbf{1})$ complementing the subalgebra $\widetilde{\mathscr{X}}$ to $\mathscr{X}$. Assume first that E is UNCOUNTABLE. Arrange its elements in a transfinite sequence $\{x_\alpha\}_{\alpha<\widetilde{\omega}}$, where $\widetilde{\omega}$ is the first transfinite of cardinality $\sigma(\widetilde{\mathscr{X}}, \mathbf{1})$. As before, we construct a net of regular subalgebras $\{\widetilde{\mathscr{X}^{(\alpha)}}\}_{0\le\alpha<\widetilde{\omega}}$ with the following properties:

1) $\{\widetilde{\mathscr{X}^{(\alpha)}}\}$ is a μ-independent family;

2) $\widetilde{\mathscr{X}^{(0)}} = \widetilde{\mathscr{X}}$; for $\alpha \ge 1$ each subalgebra $\widetilde{\mathscr{X}^{(\alpha)}}$ has the form

$$\widetilde{\mathscr{X}^{(\alpha)}} = \prod_{Z\in\mathfrak{P}_\alpha} Z,$$

where $\mathfrak{P}_\alpha$ is a μ-independent countable class of μ-simple subalgebras;

3) given $\alpha \ge 1$ we have

$$x_\alpha \in \prod_{0\le\beta\le\alpha} \widetilde{\mathscr{X}^{(\beta)}}.$$

The construction of these subalgebras proceeds by induction, step by step. Namely, having constructed the subalgebras $\{\widetilde{\mathscr{X}^{(\beta)}}\}_{\beta<\alpha_0}$, we put

$$\mathscr{Y} \equiv \prod_{0\le\beta<\alpha_0} \widetilde{x}^{(\beta)}.$$

Each of the subalgebras $\widetilde{\mathscr{X}^{(\beta)}}$, $\beta \ge 1$, may be presented as a product

$$\widetilde{\mathscr{X}^{(\beta)}} = \prod_{Z\in\mathfrak{P}_\beta} Z,$$

with factors μ-independent μ-simple subalgebras. By μ-independence of $\widetilde{\mathscr{X}^{(\beta)}}$, putting

$$\mathfrak{P}_0 \equiv \{\widetilde{\mathscr{X}^{(0)}}\},$$

we may represent $\mathscr{Y}$ as

$$\mathscr{Y} = \prod_{Z \in \bigcup\limits_{0 \le \beta < \alpha_0} \mathfrak{P}_\beta} Z.$$

The cardinality of the system of factors is strictly less than $\sigma(\widetilde{\mathscr{X}}, \mathbf{1})$ (recall that every $\mathfrak{P}_\beta$ ($\beta \ge 1$) is countable, and $\widetilde{\omega}$ is an uncountable transfinite). Likewise in the proof of Theorem 17 of Chapter 2 this implies that the subalgebra $\mathscr{Y}$ saturates no nonzero band.

Apply Lemma 2 to $\mathscr{Y}$ with x_{α_0} in place of u. The resultant subalgebra is the sought $\widetilde{\mathscr{X}}^{(\alpha_0)}$. We proceed likewise until $\alpha < \widetilde{\omega}$.

Consider the case in which E is COUNTABLE. Let $E = \{x_1, x_2, \dots\}$. Put $\widetilde{\omega} \equiv \omega$, $\widetilde{\mathscr{X}}^{(1)} \equiv \widetilde{\mathscr{X}}_1$, and $\widetilde{\mathscr{X}}^{(2)} \equiv \widetilde{\mathscr{X}}_2, \dots$ (cf. the proof of Lemma 2). The condition 3) is replaced with the inequality

$$\rho\Big(x_n, \prod_{0 \le k \le n} \widetilde{\mathscr{X}_k}\Big) < \frac{2}{n} \quad (n = 1, 2, \dots)$$

ensuing from Lemma 1. All classes $\mathfrak{P}_1, \mathfrak{P}_2, \dots$ are finite in this case and so the process of their construction can never terminate.

We now put

$$\mathfrak{P}^* = \bigcup_{0 \le \alpha < \widetilde{\omega}} \mathfrak{P}_\alpha.$$

This is a μ-independent class of subalgebras. Since

$$E \cup \widetilde{\mathscr{X}} \subset \overline{\mathscr{X}\langle \bigcup_{Z \in \mathfrak{P}^*} Z \rangle},$$

it follows that

$$\overline{\mathscr{X}\langle \bigcup_{Z \in \mathfrak{P}^*} Z \rangle} = \mathscr{X}. \tag{3}$$

Finally, consider the class

$$\mathfrak{P} \equiv \bigcup_{0 < \alpha < \widetilde{\omega}} \mathfrak{P}_\alpha,$$

consisting only of μ-simple subalgebras. As a part of $\mathfrak{P}^*$, it possesses the same property of μ-independence; its cardinality is $\sigma(\widetilde{\mathscr{X}}, \mathbf{1})$. By μ-independence of the class $\mathfrak{P}^*$, the subalgebras

$$\widetilde{\mathscr{X}} \equiv \widetilde{\mathscr{X}}^{(0)} \quad \text{and} \quad Z' \equiv \mathscr{X}\langle \bigcup_{Z \in \mathfrak{P}} Z \rangle$$

are μ-independent. The equality (3) shows that

$$\mathscr{X} = \widetilde{\mathscr{X}} \times Z'.$$

We are left with checking that the subalgebra

$$\widetilde{Z} \equiv \prod_{Z \in \mathfrak{P}} Z,$$

which is simply the closure of Z', is a μ-independent complement of $\widetilde{\mathscr{X}}$. It suffice to validate the μ-independence of $\widetilde{\mathscr{X}}$ and $\widetilde{Z}$. However, it follows easily from the μ-independence of $\widetilde{\mathscr{X}}$ and Z' and the continuity of the measure.

In much the same way as in Chapter 2, we now take as $\widetilde{\mathscr{X}}$ the degenerate subalgebra $\{\mathbf{0}, \mathbf{1}\}$. In this event "$\widetilde{\mathscr{X}}$-homogeneity" is what we have called the "homogeneity of the BA $\mathscr{X}$" (or "homogeneity by weight"; cf. p. 119). We thus come to the following fundamental theorem:

Theorem 2. *Each homogeneous probability algebra $\{\mathscr{X}, \mu\}$ may be presented as the product*

$$\mathscr{X} = \prod_{Z \in \mathfrak{P}} Z, \tag{$*$}$$

where $\mathfrak{P}$ is a μ-independent class of μ-independent subalgebras which has cardinality $\tau(\mathscr{X})$.

(We will call such a BA *μ-decomposable.*)

This theorem is included in the previous one: If $\widetilde{\mathscr{X}}$ is the degenerate subalgebra then

$$\mathscr{X} = \widetilde{Z}' = \widetilde{Z} = \prod_{Z \in \mathfrak{P}} Z.$$

Theorem 2 belongs in actuality to D. Maharam who proved an equivalent proposition in 1942.[1] The transfinite induction we use in the proof of Theorem 1 stems from her article. Similar problems were addressed by A. N. Kolmogorov; he suggested a formulation of the theorem[2] of the structure of a homogeneous normed BA which is similar to that above.

We now sketch an interpretation of this theorem by treating $\mathscr{X}$ as an algebra of events. Theorem 2 shows that a homogeneous algebra of events is similar to the game of "heads or tails" or the coin tossing (infinite in general). We have an independent family of "basic" events

[1] D. Maharam [1].
[2] A. N. Kolmogorov [2, 3].

(the occurrence or not occurrence of the obverse of a coin in a consecutive toss); the outcome of an arbitrary event is uniquely determined from the outcome of basic events. We may say also that the outcomes of all events of the algebra under study are determined from implementing a series of independent trials with two outcomes of equal probability (a particular case of the so-called "Bernoulli trials").

Theorem 2 and Theorem 17 of Chapter 2 may be partially combined as follows: *Each homogeneous complete BA $\mathscr{X}$ is the closure of some free subalgebra of cardinality equal to the weight of $\mathscr{X}$.* However, Theorem 2 is much richer in fact: it asserts (in the case of an NBA) that such a free subalgebra is generated by some METRICALLY independent family of elements.

We now show that Theorem 2 admits abstraction: The structure it describes (μ-decomposability) is characteristic precisely of homogeneous normed BAs. Namely, we have

Theorem 3. *If an NBA $\{\mathscr{X}, \mu\}$ is representable as the product*

$$\mathscr{X} = \prod_{Z \in \mathfrak{P}} Z \qquad (*)$$

whose factors are μ-simple subalgebras of some μ-independent infinite class $\mathfrak{P}$ then $\{\mathscr{X}, \mu\}$ is homogeneous and the weight $\tau(\mathscr{X})$ equals the cardinality of $\mathfrak{P}$.

This theorem ensues from the following lemma which is of the utmost importance in its own right:

Lemma 3. *The group of measure preserving automorphisms of a normed BA $\{\mathscr{X}, \mu\}$ satisfying the conditions of Theorem* 3 (*i.e., of the form* $(*)$) *is always ergodic.*

We show first how Theorem 3 follows from Lemma 3. Recall (cf. Chapter 2) that $\mathscr{X}$, as an arbitrary algebra, decomposes into homogeneous bands; we will show that it is impossible that the weights of these bands do not coincide. Assume that $u_1 u_2 > \mathbf{0}$ and $\tau(\mathscr{X}_{u_1}) \neq \tau(\mathscr{X}_{u_2})$, with $\mathscr{X}_{u_1}$ and $\mathscr{X}_{u_2}$ homogeneous bands. The automorphism group $\mathfrak{A}$ of $\mathscr{X}$ is ergodic. Hence, there is an $A \in \mathfrak{A}$ satisfying

$$A(u_1) \wedge u_2 \equiv z > \mathbf{0}.$$

The bands $\mathscr{X}_z$ and $\mathscr{X}_{A^{-1}(z)}$ are clearly isomorphic to one another. But then they must have the same weight, whereas by assumption

$$\tau(\mathscr{X}_z) = \tau(\mathscr{X}_{u_2}) \neq \tau(\mathscr{X}_{u_1}) = \tau(\mathscr{X}_{A^{-1}(z)}).$$

This contradiction shows that the weights are the same of all homogeneous bands comprising the decomposition of our algebra. Two cases

are open. Either the general value τ of these weights is equal to 1, in which case $\mathscr{X}$ is consequently a discrete algebra,[3] or τ is an infinite cardinal. But the first case is excluded since it is easy to see that an infinite discrete algebra can never have an ergodic group of measure preserving automorphisms; we are so left with the second case. The algebra $\mathscr{X}$ satisfies the countable chain condition and so the cardinality of the set of homogeneous bands of each decomposition is at most countable. Hence, the weight of every nonzero band, which is at least τ on the one hand, is at the same time at most $\aleph_0 \cdot \tau = \tau$. This means in fact that $\mathscr{X}$ is a homogeneous algebra. We now show the equality

$$card\,\mathfrak{P} = \tau(\mathscr{X}).$$

Clearly, the subalgebra $\mathscr{X}\langle\bigcup_{Z\in\mathfrak{P}} Z\rangle$ has the same cardinality as $\mathfrak{P}$; since it is dense in $\mathscr{X}$, it follows that

$$\tau(\mathscr{X}) \le card\,\mathfrak{P}.$$

Suppose that

$$\tau(\mathscr{X}) < card\,\mathfrak{P}. \tag{4}$$

Then there would exist a dense set E in $\mathscr{X}$ of cardinality strictly less that of $\mathfrak{P}$. The elements z' and z'', belonging to different subalgebras $Z', Z'' \in \mathfrak{P}$, are at the distance of $\frac{1}{2}$, which is shown by simple calculation. Choosing an arbitrary element z from each subalgebra $Z \in \mathfrak{P}$, combine them in the set R. Clearly,

$$card\,\mathfrak{P} = card\,R.$$

Assigning to each $z \in R$ some element $e(z) \in E$ at the distance from z at most $\frac{1}{4}$, we would come to a one-to-one mapping from R to E, which

---

[3]The following note is now in order. Recalling the definition given in Chapter 2 of the weight of a complete BA $\mathscr{X}$ as the least cardinality of the sets $E \subset \mathscr{X}$ that fully generate $\mathscr{X}$, i.e., $\mathscr{X} = \overline{\mathscr{X}\langle E\rangle}$, we are impelled to consider the number 0 to be the weight of the trivial BA $\{\mathbf{0}, \mathbf{1}\}$ (likewise the weight of each trivial homogeneous band $\mathscr{X}_u = \{\mathbf{0}, u\}$ generated by an atom u) since the empty subset of the trivial BA (and the trivial band) fully "generates" the latter as the least improper regular subalgebra including the empty set of its members. Since in the previous monograph D. A. Vladimirov considered the weight of the trivial BA $\{\mathbf{0}, \mathbf{1}\}$ equal to 2 (cf. D. A. Vladimirov [7, p. 274]) and it never came to my head to discuss this "trifle" with him, no corrections are introduced in the main text. Both views of D. A. Vladimirov have a simple explanation: first, in many cases a complete BA appears as a completion of some of its subalgebras and the least cardinality of such a subalgebra is 2; second, using the definition of the weight of a BA in this book, we may presume that at least one element is needed for generating a ("however small") regular subalgebra since this definition involves closure with respect to a particular topology. It seemed to me that the zero weight of the trivial BA will not contradict the terminology of this book: prior to considering topological closure, it is necessary to construct the least subalgebra we will then try to close. On the other hand, in my opinion, the weight 1 belongs to each simple algebra $\{\mathbf{0}, u, Cu, \mathbf{1}\}$, with $\mathbf{0} < u < \mathbf{1}$, which is not homogeneous. (A. A. Samorodnitskiĭ)

is impossible by the inequality (4). The so-obtained contradiction shows that

$$card\, \mathfrak{P} = \tau(\mathscr{X}).$$

The last theorem has the following two corollaries.

Corollary 1. *If an NBA $\{\mathscr{X}, \mu\}$ is the product of two homogeneous μ-independent regular subalgebras $\mathscr{X}_1$ and $\mathscr{X}_2$ then $\{\mathscr{X}, \mu\}$ is homogeneous too and the weight of $\{\mathscr{X}, \mu\}$ may be calculated by the formula*

$$\tau(\mathscr{X}) = \max\{\tau(\mathscr{X}_1), \tau(\mathscr{X}_2)\}.$$

Indeed, the subalgebras $\mathscr{X}_1$ and $\mathscr{X}_2$ are the products:

$$\mathscr{X}_1 = \prod_{Z \in \mathfrak{P}_1} Z, \quad \mathscr{X}_2 = \prod_{Z \in \mathfrak{P}_2} Z,$$

where $\mathfrak{P}_1$ and $\mathfrak{P}_2$ are infinite μ-independent classes of μ-simple subalgebras of cardinality $\tau(\mathscr{X}_1)$ and $\tau(\mathscr{X}_2)$ respectively. By μ-independence of $\mathscr{X}_1$ and $\mathscr{X}_2$ we have

$$\mathscr{X} = \prod_{Z \in \mathfrak{P}_1 \cup \mathfrak{P}_2} Z.$$

Consequently, $\mathscr{X}$ is a homogeneous BA by Theorem 3 and the weight of $\mathscr{X}$ equals the cardinality of $\mathfrak{P}_1 \cup \mathfrak{P}_2$, i.e., the greatest of the cardinalities $\tau(\mathscr{X}_1)$ and $\tau(\mathscr{X}_2)$.

Corollary 2. *Each normed algebra representable as the product of an arbitrary μ-independent system of homogeneous subalgebras is homogeneous itself and its weight equals the sum of the weights of the factors.*

The proof is completely analogous to the previous one.

Corollary 3. *The μ-independent complement, as constructed in Theorem 1, is homogeneous.*

We are now ready to prove the lemma.

Note that

$$\mathscr{X} = \prod_{Z \in \mathfrak{P}} Z,$$

where the subalgebras Z have the form $Z = \{z, Cz, \mathbf{0}, \mathbf{1}\}$ and are μ-simplest. Introduce the subalgebra $\mathscr{X}_0 \equiv \mathscr{X}\langle \bigcup_{Z \in \mathfrak{P}} Z \rangle$ whose (o)-closure coincides with $\mathscr{X}$. Each member of this subalgebra is representable as a finite sum of pairwise disjoint elementary polynomials of the form

$$u = u_1 \wedge u_2 \wedge \cdots \wedge u_k, \tag{5}$$

where $u_i \in Z_i \in \mathfrak{P}$, $u_i \neq \mathbf{1}$ and all Z_i are pairwise distinct. Call k the *length* of an elementary polynomial u. Choosing a representative (other than zero and unity) of each subalgebra of the class $\mathfrak{P}$, we come to a μ-independent family of generators of the subalgebra $\mathscr{X}_0$; as mentioned in Section 1 of Chapter 7 (example 1 of Section 3), each bijective mapping from such a family to itself extends to a measure preserving automorphism of the whole algebra $\mathscr{X}$. It is therefore easy to see that every two nonzero polynomials of the form (5), having the same length, may be transformed into each other by measure preserving automorphisms of $\mathscr{X}$. Let $\mathfrak{A}$ stand for the group of these automorphisms.

We show now that μ is a unique probability measure invariant under all automorphisms in $\mathfrak{A}$. Indeed, choosing an arbitrary collection $Z_1, Z_2, \dots, Z_s$ of pairwise distinct subalgebras in $\mathfrak{P}$, consider all possible elementary polynomials of the form (5) on assuming that $u_i \neq \mathbf{0}, \mathbf{1}$. The number of these polynomials is 2^k; they are all pairwise disjoint and comprise a partition of unity. Since such polynomials transform into each other by automorphisms in $\mathfrak{A}$; therefore, to each probability measure ν invariant under this group we must have

$$\nu(u) = \frac{1}{2^k} = \mu u.$$

It is now clear that the values of the measures ν and μ must coincide on the whole subalgebra $\mathscr{X}_0$ and so on $\mathscr{X}$ (because $\overline{\mathscr{X}_0} = \mathscr{X}$).

We pass finally to the concluding step of the proof of the lemma: we will demonstrate that $\mathfrak{A}$ is an ergodic group. Assume to the contrary that $\mathfrak{A}$ is not an ergodic group. Then

$$\mathbf{0} < \overline{x} \equiv \bigvee_{A \in \mathfrak{A}} Ax < \mathbf{1}$$

for some $x \in \mathscr{X}$. The equality

$$A_0\overline{x} = \bigvee_{A \in \mathfrak{A}} A_0 Ax = \bigvee_{A \in \mathfrak{A}} Ax = \overline{x}$$

shows that the band $\mathscr{X}_{\overline{x}}$, together with the complementary band $\mathscr{X}_{C\overline{x}}$, is invariant under every automorphism $A_0 \in \mathfrak{A}$. Take arbitrary numbers p and q such that

$$0 < p, q < 1, \quad p + q = 1, \quad p \neq \mu\overline{x},$$

and define the measure ν by the rule

$$\nu x = \frac{p}{\mu\overline{x}}\mu(x \wedge \overline{x}) + \frac{q}{\mu C\overline{x}}\mu(x \wedge C\overline{x}).$$

Clearly, ν is an invariant probability measure other than μ, which is impossible as we have already checked. This contradiction yields the lemma.

REMARK. In the concluding part of the proof of the lemma we have established the following important fact: A unique invariant measure is available only in the case of an ergodic group. Considering what was told in Chapter 2 (p. 92), we come to the following ERGODICITY TEST: *A group of automorphisms preserving some probability measure is ergodic if and only if this group admits no other invariant probability measure.*

Closing this section, we return to the question of whether there exists an independent complement of a subalgebra.

The conditions of Theorem 1 ensure the existence of an independent complement but are far from being necessary. Even in the case in which we have nonzero bands saturated by the subalgebra $\widetilde{\mathscr{X}}$, it may happen that any independent complement of $\widetilde{\mathscr{X}}$ is absent. Some information on the picture we see in this case is collected in the following proposition.

Lemma 4. *Let U and V be a μ-independent pair of subalgebras of a BA $\mathscr{X}$, and let w be a nonzero element such that the band $\mathscr{X}_w$ is saturated by U. Then w is disjoint from the continuous band of V.*

PROOF. Assume the lemma false. Then there is some v belonging to the continuous band of V and such that

$$\overline{w} \equiv w \wedge v > \mathbf{0}.$$

Using the continuity of the band V_v, choose $v_0 \in V_v$ so that $\mu v_0 = \frac{1}{2}\mu v$. Put

$$v_1 \equiv v - v_0, \;\; w_0 \equiv \overline{w} \wedge v_0, \;\; w_1 \equiv \overline{w} \wedge v_1.$$

Take an arbitrary member $u \geq \overline{w}$ of U. By the main hypothesis of the lemma there is some $u_0 \in U$, $u_0 \leq u$ satisfying $u_0 \wedge \overline{w} \equiv w_0$. Let $u_1 \equiv u - u_0$. Since $w_0 \leq u_0, v_0$; therefore, $w_0 \, d \, u_1$ and $w_0 \, d \, v_1$, implying that

$$u_0 \wedge v_1 \wedge \overline{w} = w_0 \wedge v_1 = \mathbf{0}, \quad v_0 \wedge u_1 \wedge \overline{w} = w_0 \wedge u_1 = \mathbf{0},$$

and

$$\overline{w} \leq u_0 \wedge v_0 + u_1 \wedge v_1.$$

We now use the μ-independence:

$$\mu(u_0 \wedge v_0) + \mu(u_1 \wedge v_1) = \frac{\mu v}{2}(\mu u_0 + \mu u_1) = \frac{\mu u \mu v}{2},$$

which yields

$$\mu\overline{w} \leq \mu(u_0 \wedge v_0) + \mu(u_1 \wedge v_1) = \frac{\mu u \mu v}{2}.$$

Note that on letting

$$w' \equiv \overline{w} \wedge u_0 \wedge v_0, \quad w'' \equiv \overline{w} \wedge u_1 \wedge v_1,$$

we may repeat the previous argument, replacing $\overline{w}, u, v$ first with w', u_0, v_0 and second, with w'', u_1, v_1. We so obtain

$$\mu\overline{w} = \mu w' + \mu w'' \leq \frac{1}{2}[\mu(u_0 \wedge v_0) + \mu(u_1 \wedge v_1)] = \frac{1}{2^2}\mu u \mu v.$$

Consecutively repeating this argument, for each n we find $\mu\overline{w} \leq \frac{1}{2^n}\mu u \mu v$, implying that $\overline{w} = \mathbf{0}$, which is a contradiction.

We now elaborate the most important case for applications, i.e., the case of a SEPARABLE normed algebra $\mathscr{X}$ furnished with a probability measure μ. All subalgebras of $\mathscr{X}$ are also separable. We have already mentioned that in this event the $\widetilde{\mathscr{X}}$-homogeneity of an algebra amounts to the absence of nonzero bands saturated by $\widetilde{\mathscr{X}}$.

It is easy to check that the homogeneity of a separable algebra means that $\mathscr{X}$ has no bands of finite weight or, equivalently, no discrete bands. Thus, *the concepts of "homogeneity" and "continuity" coincide for a separable BA.*

By Lemma 4 a subalgebra $\widetilde{\mathscr{X}}$ with a continuous μ-independent complement cannot saturate nonzero bands. On the other hand, in the separable case the absence of saturated bands guarantees the $\widetilde{\mathscr{X}}$-homogeneity of the algebra under study, implying the existence of a μ-independent complement by Theorem 1. We thus arrive at the following conclusion:

For a subalgebra $\widetilde{\mathscr{X}}$ of a separable probability BA $\{\mathscr{X}, \mu\}$ to have a homogeneous μ-independent complement, it is necessary and sufficient that $\widetilde{\mathscr{X}}$ saturate no nonzero band.

A. A. Samorodnitskiĭ generalized this result to the nonseparable case by showing that the existence of a homogeneous μ-independent complement of a subalgebra $\widetilde{\mathscr{X}}$ always implies the $\widetilde{\mathscr{X}}$-homogeneity of $\mathscr{X}$ (A. A. Samorodnitskiĭ [1, p. 104]).

Therefore, the fact that a given regular subalgebra has a homogeneous μ-independent complement does not depend on what measure μ is implied: either such a complement exists for whatever probability measure μ or there is no such a complement at all. The existence of a nonhomogeneous μ-independent complement may be determined from some special properties of the measure itself rather than the subalgebra in question.

The problem of existence of an independent complement is a particular case of a more general problem that we will address in Section 3 of this chapter.

By now we spoke about an independent complement of a single subalgebra. Of interest is the following more general problem which seems to

be first raised by V. N. Sudakov: Given subalgebras $\widetilde{\mathscr{X}_1}, \widetilde{\mathscr{X}_2}, \dots, \widetilde{\mathscr{X}_n} \subset \mathscr{X}$, find another subalgebra $\mathscr{Y} \subset \mathscr{X}$ serving as a μ-independent complement to each of the subalgebras $\widetilde{\mathscr{X}_1}, \dots, \widetilde{\mathscr{X}_n}$. We confine exposition to one relevant theorem.

Theorem 4. *Assume that $\mathscr{X}$ is a separable NBA and μ is a probability measure on $\mathscr{X}$; assume further that $\widetilde{\mathscr{X}_1}, \widetilde{\mathscr{X}_2}, \dots, \widetilde{\mathscr{X}_n}$ are regular continuous subalgebras comprising a μ-independent family and $\mathscr{X} = \widetilde{\mathscr{X}_1} \times \widetilde{\mathscr{X}_2} \times \dots \times \widetilde{\mathscr{X}_n}$. Then there is a regular subalgebra $\widetilde{\mathscr{Y}} \subset \mathscr{X}$ that is μ-independent with respect to each of the subalgebras $\widetilde{\mathscr{X}_1}, \dots, \widetilde{\mathscr{X}_n}$ and, moreover, $\mathscr{X} = \widetilde{\mathscr{X}_i} \times \widetilde{\mathscr{Y}}$ for all $i = 1, 2, \dots, n$.*

Proof. To each subalgebra $\widetilde{\mathscr{X}_i}$ we assign a fundamental sequence of partitions $\{\tau_j^{(i)}\}_{j=1}^{\infty}$ $(i = 1, 2, \dots, n)$ so that the following holds: Each $\tau_j^{(i)}$ consists of the elements $u_{j1}^{(i)}, u_{j2}^{(i)}, \dots, u_{j2^j}^{(i)} \in \widetilde{\mathscr{X}_i}$, with $\mu u_{j1}^{(i)} = \dots = \mu u_{j2^j}^{(i)} = \frac{1}{2^j}$ (cf. p. 385). Given $s = 1, 2, \dots$, define the partition

$$\sigma_s = \{\sigma_{k_1 k_2 \dots k_n}^{(s)} \equiv u_{sk_1}^{(1)} \wedge u_{sk_2}^{(2)} \wedge \dots \wedge u_{sk_n}^{(n)}\}_{1 \le k_1, k_2, \dots, k_n \le 2^s}.$$

We call the elements of the form $\sigma_{k_1 k_2 \dots k_n}^{(s)}$ *elements of rank s*. The set Σ of all these elements fully generates $\mathscr{X}$. We now construct one more sequence of partitions $\{\tau_s^*\}_{s=0}^{\infty}$ so that

1) $\tau_0^* = \{\mathbf{1}\}$, $\tau_0^* \prec \tau_1^* \prec \dots \prec \tau_s^* \prec \dots$;

2) each partition τ_s^* has $(2^{(n-1)})^s$ elements;

3) each element of τ_s^* is the disjoint sum of 2^s elements of rank s;

4) each element of τ_s^* is at the same time the disjoint sum of 2^{n-1} elements of τ_{s+1}^*;

5) all elements of the partitions τ_s^* are μ-independent with respect to each of the subalgebras $\widetilde{\mathscr{X}_1}, \dots, \widetilde{\mathscr{X}_n}$;

6) for all $s = 1, 2, \dots$ and $k = 1, 2, \dots, n$, each element of rank s is representable as $u \wedge v$, with $u \in \widetilde{\mathscr{X}_k}$, $u \in \tau_s^{(k)}$, $v \in \tau_s^*$.

The sequence $\{\tau_s^*\}$ is constructed by recursion. The initial entry is determined from the condition 1). Assume that $\tau_0^*, \tau_1^*, \dots, \tau_s^*$ are constructed. Consider some $a \in \tau_s^*$. It is representable (uniquely) as a sum of 2^s addends of the form $\sigma \equiv \sigma_{k_1 k_2 \dots k_n}^{(s)}$; look at one of these addends. It has the form

$$\sigma = u_{sk_1}^{(1)} \wedge u_{sk_2}^{(2)} \wedge \dots \wedge u_{sk_n}^{(n)}.$$

Here $u_{sk}^{(i)}$ are the members of the partition $\tau_s^{(i)}$. Each of these elements is uniquely written as

$$u_{sk}^{(i)} = v_{k0}^{(i)} + v_{k1}^{(i)},$$

where $v_{k0}^{(i)}$ and $v_{k1}^{(i)}$ are some elements of

$$\tau_{s+1}^{(i)}: \quad v_{k0}^{(i)} \equiv u_{s+1j_0}^{(i)}, \quad v_{k1}^{(i)} \equiv u_{s+1j_1}^{(i)}.$$

Note that

$$\begin{aligned} \sigma &= (v_{k_1 0}^{(1)} + v_{k_1 1}^{(1)}) \wedge (v_{k_2 0}^{(2)} + v_{k_2 1}^{(2)}) \wedge \dots \\ &= \sum_{i_1, i_2, \dots, i_n = 0,1} v_{k_1 i_1}^{(1)} \wedge v_{k_1 i_2}^{(2)} \wedge \dots \wedge v_{k_n i_n}^{(n)} \\ &= \sum_{i_1, i_2, \dots, i_{n-1} = 0,1} \Big(v_{k_1 i_1}^{(1)} \wedge \dots \wedge v_{k_{n-1} i_{n-1}}^{(n-1)} \wedge v_{k_n 0}^{(n)} \\ &\quad + v_{k_1 (1-i_1)}^{(1)} \wedge \dots \wedge v_{k_{n-1}(1-i_{n-1})}^{(n-1)} \wedge v_{k_n 1}^{(n)} \Big). \end{aligned}$$

Denote the elements in the parentheses by $\rho_{i_1 \dots i_{n-1}}^{k_1 \dots k_n}$. The number of these elements is 2^{n-1}; they are pairwise disjoint and each of them is constructed from two elements of rank $s+1$. Take $a \in \tau_s^*$ and arrange the sum

$$\rho_{i_1 \dots i_{n-1}}^{[a]} \equiv \sum \rho_{i_1 \dots i_{n-1}}^{k_1 \dots k_n},$$

where summation ranges over the tuples $(k_1 \dots k_n)$ satisfying $\sigma_{k_1, \dots, k_n}^{(s)} \le a$. (The number of these tuples is 2^s, as mentioned above.) The elements of the form $\rho_{i_1 \dots i_{n-1}}^{[a]}$ $(a \in \tau_s^*,\ i_1, \dots, i_{n-1} = 0, 1)$ comprise the partition τ_{s+1}^*. It is easy to check that this partition has the desired properties.

The subalgebras $\mathscr{X}_{\tau_s^*}$ make an increasing sequence and are μ-independent with respect to each of the subalgebras $\widetilde{\mathscr{X}_1}, \dots, \widetilde{\mathscr{X}_n}$. The same holds for the subalgebra

$$\widetilde{\mathscr{Y}} \equiv \overline{\mathscr{X} \Big\langle \bigcup_{s=1}^{\infty} \mathscr{X}_{\tau_{s+1}^*} \Big\rangle}$$

they fully generate.

By the property 6) each elements of the fully generating set Σ belongs to $\mathscr{X} \langle \widetilde{\mathscr{X}_k}, \widetilde{\mathscr{Y}} \rangle$ for all $k = 1, 2, \dots, n$, and so $\mathscr{X} = \widetilde{\mathscr{X}_k} \times \widetilde{\mathscr{Y}}$. The sought subalgebra is constructed. We may see in Fig. 6 a few elements of the partitions τ_1^* and τ_2^* for $n = 3$.

In case $n = 2$ V. N. Sudakov has obtained a stronger result as far back as in 1976: existence was established of a mutually μ-independent complements to the subalgebras $\widetilde{\mathscr{X}_1}$ and $\widetilde{\mathscr{X}_2}$ in the situation when these subalgebras, failing to be μ-independent, are still independent with respect to some measure ν.

For $n > 2$ this theorem is not yet established as far as we know this. From 1961 on, the series of articles (I. V. Romanovskiĭ and V. N. Sudakov [1], V. N. Sudakov [1], and H. G. Kellerer [1]) address the problem of existence of some subalgebra that is μ-independent with respect to each member of a given family and rather rich (whereas possibly failing to complement). Study was launched in the language of measure spaces, primarily, the Lebesgue–Rokhlin spaces. Instead of subalgebras this study dealt usually with the corresponding measurable partitions, while independence was achieved with respect to a few measures simultaneously. These problems are rooted in mathematical statistics.

2. Classification for normed algebras

Recall that some normed BAs $\{\mathscr{X}, \mu\}$ and $\{\mathscr{Y}, \nu\}$ are *isomorphic* provided that there is a measure preserving isomorphism T from $\mathscr{X}$ to BA $\mathscr{Y}$ (or vice versa). "Measure preservation" means that $\nu = \mu \circ T^{-1}$; i.e., $\mu x = \nu T(x)$ $(x \in \mathscr{X})$.

If this condition is not presumed then we use the attribute *algebraic* or *Boolean* isomorphism (in contract to "metric").

In the sequel we will give a test for isomorphism between NBAs. We start with the simplest case in which $\mathscr{X}$ and $\mathscr{Y}$ are homogeneous algebras. As before, we denote the weights of these algebras by $\tau(\mathscr{X})$ and $\tau(\mathscr{Y})$. Assume that μ and ν are probability measures.

Theorem 5. *The equality $\tau(\mathscr{X}) = \tau(\mathscr{Y})$ is necessary and sufficient for normed BAs $\{\mathscr{X}, \mu\}$ and $\{\mathscr{Y}, \nu\}$ to be isomorphic.*

Only the sufficiency part needs proving. By Theorem 2 we may represent $\mathscr{X}$ and $\mathscr{Y}$ as the products

$$\mathscr{X} = \prod_{Z \in \mathfrak{P}} Z, \quad \mathscr{Y} = \prod_{W \in \mathfrak{R}} W,$$

where $\mathfrak{P}$ and $\mathfrak{R}$ are equipollent metrically independent classes of μ-simple and ν-simple subalgebras. Choosing representatives $z, w \neq \mathbf{0}, \mathbf{1}$ in the respective subalgebras Z and W, we come to two equipollent metrically independent families similar to those that appear in the example of Section 1 of Chapter 7 (p. 323). We have already found out that each bijective correspondence between the members of these families extends to a measure preserving isomorphism between these algebras. In other words, the normed BAs $\{\mathscr{X}, \mu\}$ and $\{\mathscr{Y}, \nu\}$ are isomorphic (metrically, in the capacity of NBAs).

Theorem 5 shows that homogeneous normed algebras may be classified to within a measure preserving isomorphism by a sole property, weight. This sole invariant has an algebraic rather than metric character. Theorem 5 thus has the following important corollary:

Corollary. *If two homogeneous probability BAs are algebraically isomorphic then there is a measure preserving isomorphism between them.*

Remark. To each cardinal τ there is a complete homogeneous normed algebra of weight τ.

Indeed, we know that there is a free BA $\mathscr{X}_0$ on an independent set of generators E of cardinality τ. Assign to each elementary polynomial

$$e = e_1 \wedge e_2 \cdots \wedge e_p \wedge Ce_{p+1} \wedge \cdots \wedge Ce_m,$$

$$e_i \in E \quad (i = 1, 2, \ldots, m), \quad e_i \neq e_j \ (i \neq j)$$

the number

$$\mu_0(e) = \frac{1}{2^m}.$$

Each $x \in \mathscr{X}_0^+$ may be presented as a polynomial in the generators and so it belongs to some subalgebra of the form $\mathscr{X}_0\langle E'\rangle$, where E' is a finite subset of E. We may assume that E' is the least of these sets. Taking x as the canonical polynomial

$$x = e^1 + e^2 + \cdots + e^n,$$

with $e^i,\ i = 1, 2, \ldots, n$, elementary polynomials in the members of E' (such a representation is well known to be unique), put

$$\mu_0 x = \sum_{i=1}^{n} \mu_0(e^i).$$

We further put $\mu_0(\mathbf{0}) = 0$. We have so defined an essentially positive function on $\mathscr{X}_0$; the reader will check that it is additive and serves as a quasimeasure. By Theorem 7.2 we may now assume that $\mathscr{X}_0$ is an

everywhere dense subalgebra of some complete BA $\mathscr{X}$. Moreover, $\mathscr{X}$ is furnished with some probability measure μ extending μ_0. Clearly, $\mathscr{X}$ is a μ-decomposable algebra. By Theorem 3 it is homogeneous and has weight τ. Soon we will encounter some particular instances of homogeneous normed algebras.

We now refute the supposition that we consider homogeneous algebras. Let $\mathscr{X}$ be an arbitrary complete normed algebra with a probability measure μ. Apply Lemma 2.17 to it, on taking as $\widetilde{\mathscr{X}}$ the degenerate subalgebra $\{\mathbf{0}, \mathbf{1}\}$. By this lemma we obtain a disjoint decomposition of $\mathscr{X}$ into at most countably many nonzero bands $\mathscr{X}_{v_1}, \mathscr{X}_{v_2}, \dots$ each of which is a homogeneous Boolean algebra by itself. Put $\sigma_n \equiv \tau(\mathscr{X}_{v_n})$. It might happen that for some (finite or infinite) subfamily of bands $\mathscr{X}_{v_{n_1}}, \mathscr{X}_{v_{n_2}}, \dots$ we have $\sigma_{n_1} = \sigma_{n_2} = \dots = \sigma$. Let $v \equiv \bigvee_k v_{n_k}$. Check that the band $\mathscr{X}_v$ is homogeneous and has the same weight σ. Given an arbitrary $v' \in \mathscr{X}_v^+$, put $v'_k \equiv v' \wedge v_{n_k},\ k = 1, 2, \dots$. In each of the bands $\mathscr{X}_{v'_k}$ there is an everywhere dense subset D_k of cardinality at most σ. Arrange the set D by collecting the suprema of finite subsets of $\bigcup_k D_k$. Clearly, D is dense in $\mathscr{X}_{v'}$ and $card\, D \le \sigma$. (By the continuity of the algebra, σ is an infinite cardinal.) So, $\tau(\mathscr{X}_{v'}) \le \sigma$. On the other hand, if $v'_k > \mathbf{0}$ then from $\mathscr{X}_{v'} \supset \mathscr{X}_{v'_k}$ it follows that

$$\tau(\mathscr{X}_{v'}) \ge \tau(\mathscr{X}_{v'_k}) = \sigma.$$

Consequently, $\tau(\mathscr{X}_{v'}) = \sigma$; in particular, $\tau(\mathscr{X}_v) = \sigma$.

This remark enables us, by "enlarging" the bands if need be, to obtain a disjoint decomposition of $\mathscr{X}$ into nonzero bands $\mathscr{X}_{u_1}, \mathscr{X}_{u_2}, \dots$ of pairwise distinct weights. Put $\tau_n \equiv \tau(\mathscr{X}_{u_n})$ and assume that the bands are enumerating so that the weights increase: $\tau_1 < \tau_2 < \dots$ (enumeration proceeds up to some finite or countable ordinal). In this event we call the family $\{\mathscr{X}_{u_n}\}$, as well as its every band, *canonical*. It is easy to see that each homogeneous band must be included in one of the canonical bands so that the canonical decomposition is the "hugest" of all decompositions into homogeneous bands.

The *passport* of a normed BA $\{\mathscr{X}, \mu\}$ we call the matrix

$$\begin{pmatrix} \tau_1 & \tau_2 & \dots \\ \mu u_1 & \mu u_2 & \dots \end{pmatrix}.$$

Here the rows are enumerated with ordinals of cardinality at most $\tau(\mathscr{X})$.

Theorem 6. *For some NBAs $\{\mathscr{X}, \mu\}$ and $\{\mathscr{Y}, \nu\}$ to be isomorphic it is necessary and sufficient that their passports coincide.*

This theorem is practically immediate from the previous. The coincidence of passports means that the canonical families $\{\mathscr{X}_{u_k}\}$ and $\{\mathscr{Y}_{v_k}\}$

possess the following properties:

$$\tau(\mathscr{X}_{u_k}) = \tau(\mathscr{Y}_{v_k}), \quad \mu u_k = \nu v_k, \;\; k = 1, 2, \ldots .$$

Furnishing $\mathscr{X}_{u_k}$ and $\mathscr{Y}_{v_k}$ with the probability measures

$$\mu_k x \equiv \frac{\mu x}{\mu u_k}, \;\; x \in \mathscr{X}_{u_k}, \quad \nu_k y \equiv \frac{\nu y}{\nu v_k}, \;\; y \in \mathscr{Y}_{v_k},$$

note that by Theorem 5 to each k there is a measure preserving isomorphism Φ_k from $\mathscr{X}_{u_k}$ to $\mathscr{Y}_{v_k}$. Clearly, the mapping Φ, acting by the rule

$$\Phi(x) \equiv \sum_k \Phi_k(x \wedge u_k), \quad x \in \mathscr{X},$$

is a measure preserving isomorphism from $\mathscr{X}$ onto $\mathscr{Y}$. The proof is complete. In fact, this theorem was established by D. Maraham [1].

Remark. Inspecting the proof of Theorem 5, it is easy to see that *for two normed algebras to be algebraically isomorphic, it is necessary and sufficient that topmost rows of their passports coincide.*

We will call Theorem 6 and the remark on it the "main classification theorems" for normed (respectively, normable) algebras.

Assume now that $\mathscr{X}$ is a separable normed algebra. In the most general case, the weight of a homogeneous band $\mathscr{X}_u$ may equal either 1 (which means that u is an atom[4]) or $\aleph_0$. Therefore, *a separable NBA is homogeneous if and only if it is continuous.* Theorem 5 straightforwardly implies

Theorem 7. *All complete continuous separable probability algebras are isomorphic to one another.*

(Another version of Theorem 7.22.)

The BA E_0, the metric structure of all Lebesgue measurable subsets of the interval $[0, 1]$, may serve as a common model for the class of all these algebras. We may naturally define on this algebra the measure λ that corresponds to the usual Lebesgue "measure" (cf. p. 62). Theorem 2 allows us to conclude that E_0 is representable as the product of some countable λ-independent family of λ-simple subalgebras. (This well-known fact is easy to derive straightforwardly by using the binary notation of numbers in the interval $[0, 1]$. The matter is practically the same with the algebras E_0^n $(n = 1, 2, \ldots)$.

Another principal model of a continuous separable probability we obtain by taking the metric structure of the Bernoulli space, the Cantor

[4] Cf. the footnote on p. 457.

discontinuum $X_{\mathbb{N}}$ with the Bernoulli measure $\beta_{\frac{1}{2}\frac{1}{2}}$. From the "Boolean" viewpoint, there is no difference between the "Lebesgue interval" and the "Bernoulli space" since in both cases we have the algebra of events occurring in countably many coin tosses. A one-time selection of a "random number" amounts to an infinite game of coin tossing.

Consider a complete normed algebra $\mathscr{X}$ with a measure μ presented as the product of a metrically independent family of subalgebras of the form E_0 (more exactly, of subalgebras isomorphic to E_0). Each of the "factors" is in turn representable as the product of some μ-independent class of μ-simple subalgebras. This shows that $\mathscr{X}$ itself is μ-decomposable and, hence, homogeneous. The weight of $\mathscr{X}$ depends only on the cardinality of the family of factors; if this cardinality equals τ then the weight of $\mathscr{X}$ is the product

$$\aleph_0 \cdot \tau = \max\{\tau, \aleph_0\}.$$

In particular, in the case of $\tau \leq \aleph_0$ we obtain a homogeneous algebra of countable weight, i.e., practically the same algebra E_0. We have met similar products in the form of the algebras E^Γ (cf. p. 62). Obviously, each BA $\mathscr{X}$ of this type presents the product of some family $\{\mathscr{X}_\gamma\}_{\gamma\in\Gamma}$ of metrically independent subalgebras which we may view as "extra copies" of E_0. We see that the key role belongs to the cardinality of Γ coinciding with the weight τ. Therefore, we may find an example of a homogeneous normed algebra of an arbitrary weight among the algebras E^Γ. By Theorem 5, we may conclude that each complete homogeneous probability algebra admits an isomorphism to one of the algebras E^Γ. This fact was first revealed in the article by D. Maharam [1]. In the "Kolmogorov" version of this theorem the metric structures of the Bernoulli spaces of weight τ take the place of E^Γ (A. N. Kolmogorov [2, 3]).

The above exposition relates to the case of a continuous normed algebra. This constraint is justified since each algebra decomposes into the continuous and discrete bands, while there is no isomorphism problem for discrete algebras: Every infinite discrete BA with the countable chain condition is normable and all these algebras are isomorphic to each other. The presence of a measure preserving isomorphism for discrete algebras is, if you please, a lucky chance. For this purpose we need that the atoms of the two algebras can be put into some measure preserving one-to-one correspondence, which is an entirely nonalgebraic requirement related only to the properties of the measure itself.

The following theorem of this chapter demonstrates "universality" of homogeneous normed algebras.

Theorem 8. *Let $\mathscr{X}$ be an arbitrary complete BA furnished with a probability measure μ, and let $\mathscr{Y}$ be a homogeneous complete algebra*

furnished with a probability measure ν. If $\tau(\mathscr{X}) \leq \tau(\mathscr{Y})$ then there is a measure preserving monomorphism from $\mathscr{X}$ to $\mathscr{Y}$.

We will just sketch the basic steps of the proof of this almost obvious theorem. Start with considering the case in which $\mathscr{X}$ is homogeneous. In this event, decomposing $\mathscr{Y}$ into the product of some metrically independent μ-simple subalgebras by Theorem 1, distinguish some portion of "factors" of cardinality $\tau(\mathscr{X})$ in this product. The corresponding "partial" product will be a regular subalgebra of $\mathscr{Y}$ to which there is a measure preserving isomorphism by Theorems 3 and 5.

We now consider an arbitrary normed algebra $\mathscr{X}$. It is representable as the disjoint sum of at most countable family of pairwise disjoint bands $\mathscr{X}_{u_1}, \mathscr{X}_{u_2}, \dots$ each of which is either homogeneous or degenerate. The weight of every band $\mathscr{X}_{u_k}$ is at most $\tau(\mathscr{X})$. It is an easy matter to construct a disjoint collection of elements $v_1, v_2, \dots$ of $\mathscr{Y}$ so that for all $k = 1, 2, \dots$ we have $\mu u_k = \nu v_k$. Each of the bands $\mathscr{Y}_{v_k}$ is homogeneous and

$$\tau(\mathscr{Y}_{v_k}) = \tau(\mathscr{Y}) \geq \tau(\mathscr{X}) \geq \tau(\mathscr{X}_{u_k}).$$

Hence, there is a v_k-subalgebra serving as the image of $\mathscr{X}_{u_k}$ under some measure preserving mapping Φ_k (if $\tau(\mathscr{X}_k) = 1$[5] then $\Phi_k(\mathbf{0}) = \mathbf{0}$ and $\Phi_k(v_k) = v_k$). "Pasting" the mappings Φ_k as this was done in the proof of Theorem 6, we obtain a desired monomorphism.

Theorem 8 enables us to consider each NBA as a regular subalgebra of one of the algebras of the form E^{Γ}. To use this circumstance is in fact a routine practice. For instance, the algebra $\mathscr{X}_0$ of events is nonhomogeneous which relates to tossing an asymmetric coin or dice (and a real coin or dice is always asymmetric). Recall however what we do when trying to calculate the probability of a face of the dice or a side of the coin. We usually assume that this probability is proportional to the solid angle at which this face or side is seen from the barycenter O of the dice or coin. This means that we replace the finite algebra $\mathscr{X}_0$ with a wider algebra of Lebesgue measurable *mod* 0 sets on the circumference of the sphere of unit area with center O. We make the usual assumption of the problems of "geometric probability" the probability of the event that the vertical line from O downwards intersects the sphere within some set is the area of this set. In our case, we speak about the set resulting from the face of the dice or the side of the coin we are interested in by central projection. The algebra $\mathscr{X}$ is now fully homogeneous and isomorphic to E_0. The rotation group may be taken as a group generating the measure. In our opinion, this tiny example refutes the

[5]Cf. the footnote on p. 457.

prevailing view that the symmetry arguments lose their force and become of no avail for determining probability in the problems similar to the one we have just considered.

Discussing the problem of normability of a BA we have introduced the concept of fully homogeneous BA and proved that such an algebra is always normable and homogeneous. From Lemma 3 of this chapter it follows that a homogeneous algebra with a measure is in turn fully homogeneous in all cases. We so acquire an opportunity to complete the study of the problems of normability we have started in Chapter 8 and to provide an abstract (nonmetric) characterization of the classical probability algebras. Namely, the results of this chapter imply that *the following are equivalent:*

A. $\mathscr{X}$ *is a normable BA;*

B. $\mathscr{X}$ *is a disjoint union of at most countably many homogeneous bands;*

C. $\mathscr{X}$ *is a regular subalgebra of a fully homogeneous BA.*

Moreover, a complete BA is isomorphic to one of the algebras E^{Γ} *if and only if it is fully homogeneous; it is isomorphic to* E_0 *if and only if it is fully homogeneous and separable* (in both cases we imply METRIC isomorphism).

3. Interlocation of subalgebras of a normed Boolean algebra

3.1 Statement of the problem. The case of two subalgebras

Let a normable BA $\mathscr{X}$ have a family $\{\mathscr{X}_\xi\}$ of regular subalgebras. What interlocation of these subalgebras is possible? We are first of all interested in the question of whether they are independent in some sense. It might happen for instance that these subalgebras are generated by random variables (resolutions of the identity) $\mathfrak{f}_\xi$.

The independence of $\{\mathscr{X}_\xi\}$ with respect to some probability measure means that the variables $\mathfrak{f}_\xi$ are independent, which is well known to be important in probability theory.

Assume that for some probability measure μ_0 the family $\{\mathscr{X}_\xi\}$ is μ_0-independent. In this event we say that this family is *metrically independent*. Clearly this implies algebraic independence.

We elaborate the case of a finite family. Everything is essentially seen in the case of two subalgebras already. We thus consider some pair

$$\{U, V\}$$

of continuous regular subalgebras.

Assume that they are *algebraically independent* and $\mathscr{X} = U \times V$; i.e., $\mathscr{X} = \overline{\mathscr{X}\langle U, V\rangle}$. It turns out that the metric independence problem is connected with the extension problem for measures and automorphisms. If μ and ν are probability measures on U and V respectively then their *direct product* $\mu \times \nu$ is by definition a probability measure σ on $\mathscr{X}$ such that its restrictions (the so-called "marginal measures") $\sigma|_U$ and $\sigma|_V$ coincide with μ and ν; moreover,

$$\sigma(u \wedge v) = \sigma(u)\sigma(v) \equiv \mu u \nu v$$

for all $u \in U,\ v \in V$. It is easy to check that such a measure is unique (since $\mathscr{X} = U \times V$). Existence of at least one of these products means that U and V are metrically independent.

Recall that the product $A \times B$ of automorphisms $A : U \longrightarrow U$ and $B : V \longrightarrow V$ is their natural joint extension to an automorphism of $\mathscr{X}$. Clearly, this extension may fail to exist at all: everything depends as we will show on the interlocation of U and V.

Theorem 9. *The following are equivalent:*

I) *U and V are metrically independent subalgebras;*

II) *all probability measures μ and ν on U and V respectively have the product $\mu \times \nu$;*

III) *there is an ergodic strongly continuous*[6] *automorphism A_0 of the subalgebra U such that the product exists*

$$A = A_0 \times I_V,$$

presenting a strongly continuous automorphism of $\mathscr{X}$ (with I_V standing for the identity transformation of V).

IV) *to all strongly continuous automorphisms A and B of the subalgebras U and V respectively, the product $A \times B$ is available also presenting a strongly continuous automorphism.*

Proof of the Theorem.

1) I) $\longrightarrow$ II). Let σ be the probability measure on $\mathscr{X}$ with respect to which U and V are independent. Denote its restrictions as follows: $\sigma_U \equiv \sigma|_U, \quad \sigma_V \equiv \sigma|_V$. For whatever probability measures μ and ν on the subalgebras U and V, respectively, there exist integral representations with densities $\mathfrak{f}_\mu$ and $\mathfrak{f}_\nu$ with respect to σ_U and σ_V. These densities are resolutions of the identity of the subalgebras; however, we may treat them as resolutions of the identity of the whole algebra $\mathscr{X}$. Let $\mathfrak{f} \equiv \mathfrak{f}_\mu \cdot \mathfrak{f}_\nu$

[6] In other words, a Liouville automorphism.

(the product in the sense of Chapter 6). Put

$$\alpha(x) \equiv \int\limits_x \mathfrak{f}\, d\sigma \quad (x \in \mathscr{X}).$$

Then α is a totally additive function. It is essentially positive since if $\alpha(x) = 0$ for some $x > \mathbf{0}$ then x would be dominated by an element of the form $u_0 \vee v_0$, where $u_0 \in U,\ v_0 \in V$ and $\mu u_0 = \nu v_0 = 0$ ($\mathfrak{f}_\mu$ and $\mathfrak{f}_\nu$ annihilate on u_0 and v_0). This would contradict the essential positivity of μ and ν. Thus, α is a measure on $\mathscr{X}$. Now, taking $u \in U$ and $v \in V$, note that

$$\alpha(u \wedge v) = \int\limits_u \mathfrak{f}_\mu\, d\sigma_U \cdot \int\limits_v \mathfrak{f}_\nu\, d\sigma_V = \mu u \cdot \nu v.$$

In particular, $\alpha(\mathbf{1}) = 1$, $\alpha(u) = \alpha(u \wedge \mathbf{1}) = \mu u$, and $\alpha(v) = \alpha(\mathbf{1} \wedge v) = \nu v$. The product of μ and ν is so constructed: $\alpha = \mu \times \nu$.

2) II) $\longrightarrow$ IV). Let A and B be strongly continuous automorphisms of our subalgebras. To these there are invariant measures μ and ν on the respective subalgebras. By II) there is a product $\mu \times \nu \equiv \sigma$. We are so in the context of Example 2 of Chapter 7 (Section 1). It was shown that in these conditions the product $A \times B$ exists and is strongly continuous (since it preserves the measure σ).

3) IV) $\longrightarrow$ III). It suffices to prove that there is a strongly continuous automorphisms on U. However, such an automorphism exists on every continuous NBA.

4) III) $\longrightarrow$ I). Let A_0 be the automorphism whose existence is claimed in III).

This A_0 generates the automorphism group $\mathfrak{A}^*_{A_0} \equiv \mathfrak{A}^*$ of U whose construction is given in Chapter 8 (p. 407). This construction shows that the products $B_0 \times I_V$ exist for all $B_0 \in \mathfrak{A}^*$, and the collection of all these products is an equicontinuous group. To the latter there is an invariant measure μ. Taking an arbitrary $u \in U$ of measure $\frac{1}{2}$,[7] choose in the group $\mathfrak{A}^*$ (since it is ergodic) an automorphism B_0 with the property $B_0(u) = Cu$. Putting $B = B_0 \times I_V$, we see that

$$\mu(u \wedge v) = \mu(B(u \wedge v)) = \mu(B_0 u \wedge v) = \mu(Cu \wedge v) = \frac{1}{2}\mu v$$

for all $v \in V$. In exactly the same manner we establish the equality

$$\mu(u \wedge v) = \mu u \cdot \mu v$$

[7] Recall that we consider only continuous groups here.

for all $u \in U$ with measure $\frac{1}{2^n}$ and proceed to the proof for an arbitrary u with binary rational measure. These elements are dense in U, and so the last equality holds for all $u \in U,\ v \in V$, i.e., U and V are μ-independent.

REMARK. We have not used the continuity of V in this proof.

The above theorem assumes that the subalgebras U and V are regular and algebraically independent. However, there is a fragment of this theorem that does not concern the regularity of V. Namely, we have the following

SUPPLEMENT TO THEOREM 9. *If there is an ergodic strongly continuous automorphism A_0 of a regular subalgebra U such that the product $A = A_0 \times I_V$ exists and serves as a strongly continuous automorphism of $\mathscr{X}$ then U and V are metrically independent.*

3.2 The Kakutani Theorem

In the proof of Theorem 9 (the implication I) $\longrightarrow$ II)) there was no doubt that multiplication of the densities $\mathfrak{f}_\mu = \frac{d\mu}{d\sigma_U}$ and $\mathfrak{f}_\nu = \frac{d\nu}{d\sigma_V}$ yields the product $\mathfrak{f} = \mathfrak{f}_1 \cdot \mathfrak{f}_2$, i.e., the density of the sought measure α. Moreover, we have demonstrated that $\mathfrak{f}$ is a strictly positive density, and so α is a measure indeed. The same holds also in the case of an arbitrary finite family of subalgebras. However, the situation changes in passing to an infinite family, for instance, to the sequence $\{U_n\}_{n=1}^{\infty}$, and the product of (even probability) measures $\mu_1, \mu_2, \dots$ on the respective subalgebras $U_1, U_2, \dots$ may fail to exist sometimes. The next theorem which we left unproved provides an exhaustive solution to this problem. We start with introducing a numerical characteristic for a pair of independent measures which describes the deviation of one measure from the other:

$$P(\mu, \nu) = \int \sqrt{\frac{d\mu}{d\nu}}\, d\nu \equiv \int \sqrt{\frac{d\nu}{d\mu}}\, d\mu.$$

This is often written as the HELLINGER INTEGRAL:

$$P(\mu, \nu) = \int \sqrt{d\mu d\nu}.$$

The last form is more symmetric. It should be recalled that the coincidence of measures is characterized by the equality $P(\mu, \nu) = 1$, that is why $-\log P$ appears of in place of P as a measure of deviation.

Kakutani Theorem.[8] *Let $U_1, U_2, \dots$ be independent subalgebras with respect to a probability measure σ. Let $\sigma_1, \sigma_2, \dots$ stand for the restrictions of this measure to the subalgebras $U_1, U_2, \dots$ respectively,*

[8]S. Kakutani [1].

and let $\mu_1, \mu_2, \ldots$ be probability measures on these subalgebras. Assume finally that $\mathfrak{f}_i \equiv \frac{d\mu_i}{d\sigma_i}$ are the appropriate densities. Then the following are equivalent:

I) *There is a product of the measures $\mu_1, \mu_2, \ldots$; i.e., such a probability measure μ that $\mu_i = \mu|_{U_i}$ $(i = 1, 2, \ldots)$ and the subalgebras $U_1, U_2, \ldots$ are μ-independent.*

II) *There is an infinite product*

$$\mathfrak{f} = \prod_1^{\infty} \mathfrak{f}_i = (o)\text{-}\lim \prod_1^{n} \mathfrak{f}_i$$

in the sense of (o)-convergence in the space $\mathfrak{S}_{\mathscr{X}}$ which is itself a strictly positive random variable.

III) *The numerical infinite product*

$$\prod_1^{\infty} P(\mu_n, \sigma_n)$$

converges. Moreover, violation of either of these conditions amounts to $\prod_1^n \mathfrak{f}_n \xrightarrow{(o)} \mathbf{0}$ in $\mathfrak{S}_{\mathscr{X}}$.

The integrals of the form $P(\mu, \nu)$, introduced by S. Kakutani, are of interest in their own right.

Consider by way of example the finite algebra $\mathscr{X}_n$ of all subsets of the space of binary tuples

$$X = \{0,1\}^n \equiv \{\chi \mid \chi = (\chi_1, \chi_2, \ldots, \chi_n),\ \ \chi_i = 0, 1\}$$

and the family comprising n its subalgebras $\widetilde{\mathscr{X}_k} \equiv \{\varnothing, X, Q_k, CQ_k\}$, where $Q_k = \{\chi \in X \mid \chi_k = 1\}$. We have already dealt with these algebras which arise while considering n tosses of a coin.

The subalgebras $\widetilde{\mathscr{X}_k}$ are algebraically independent. Introduce the set M_n that consists of all probability measures μ such that $\mu Q_k = \frac{1}{2}$ $(k = 1, 2, \ldots)$. Moreover, on $\mathscr{X}_n$ there are given some Bernoulli measures β_{pq} $(p+q = 1,\ \ p, q > 0)$ which are well known to be defined by the condition

$$\beta_{pq}(\{\chi\}) = p^{s(\chi)} q^{n-s(\chi)},$$

where $s(\chi) = \chi_1 + \cdots + \chi_n$ is the number of 1's among $\chi_1, \chi_2, \ldots, \chi_n$. Distinguish one of these measures: $\beta \equiv \beta_{pq}$ $(q < p)$. All measures are equivalent on a finite space X; however, the "degree of equivalence," i.e., the deviation P characterizing the proximity of measures can vary. In particular, we may posed a question about the quantity

$$\delta_n \equiv \max_{\mu \in M_n} P(\mu, \beta).$$

"How equivalent" might be a measure of the class M_n with respect to β? Using the conventional analytical technique, we may determine the behavior of δ_n at large n. Namely, $\delta_n \longrightarrow \frac{1}{\sqrt{2p}}$. If $p = q = \frac{1}{2}$ (a symmetric coin) then the measure β itself belongs to M_n, and as n increases this set M_n "concentrates" stronger and stronger near the measure β. However, we must remember that all β_{pq} are pairwise singular in the infinite-dimensional space $X_{\mathbb{N}} \equiv \{\chi \mid \chi = (\chi_1, \chi_2, \ldots),\ \chi_i = 0, 1\}$: "how many coins, so many types."

4. Isomorphism between subalgebras

4.1 Statement of the problem

When we deal with some family of subalgebras, it is important to know not only their interlocation but also the manner in which they are embedded in the main algebra. For instance, assume given two subalgebras $\widetilde{\mathscr{X}}'$ and $\widetilde{\mathscr{X}}''$ of some complete BAs $\mathscr{X}'$ and $\mathscr{X}''$ respectively.

We will say that they are *isomorphic as subalgebras* provided that there is an isomorphism $\Phi : \mathscr{X}' \longrightarrow \mathscr{X}''$ sending $\widetilde{\mathscr{X}}'$ to $\widetilde{\mathscr{X}}''$. We must draw a distinction between an isomorphism of subalgebras and an *individual isomorphism* (when these subalgebras are viewed as BAs by themselves.)

In other words, the subalgebras are treated in fact as the subobjects of the category $\mathbb{B}\mathbb{o}\mathbb{o}\mathbb{l}\mathbb{e}$, and the isomorphism we discuss in this section is the isomorphism between subobjects.[9] All properties of subalgebras split into the two groups:

1) The properties that they possess exactly as subalgebras; these properties are connected in particular with the manner of embedding of the subalgebras in question into the main BA.

2) The "individual properties" belonging to them since they are Boolean algebras themselves.

The properties of the first group are invariant under isomorphisms between subalgebras; the properties of the second group are preserved already by individual isomorphisms.

HOMOGENEITY may serve as an example of a property of the second group. When we say that $\widetilde{\mathscr{X}}$ is a homogeneous subalgebra, we imply that $\widetilde{\mathscr{X}}$ as such is a homogeneous BA (by weight).

By analogy we understand the expressions a "continuous subalgebra," a "discrete subalgebra," etc. REGULARITY is a typical example of a property of the first group. (That is why we never call a regular subalgebra

[9]Herein we will still accept the traditional understanding of a subalgebra as a SUBSET that is "verbatim" embedded into the main BA (cf. p. 161).

"complete" in contrast to the common parlance; "completeness" is an individual property, a property of the first group.)

We illustrate this by a simple example. Consider the following "tiny" model: assume given two discrete BAs, $\mathscr{X}' = 2^{Q'}$ and $\mathscr{X}'' = 2^{Q''}$. Let $\widetilde{\mathscr{X}'} \subset \mathscr{X}'$ and $\widetilde{\mathscr{X}''} \subset \mathscr{X}''$ be regular subalgebras. We know that these subalgebras are generated by the partitions $\widetilde{\mathscr{X}'} \equiv \mathscr{X}'_{\tau'}$ and $\widetilde{\mathscr{X}''} \equiv \mathscr{X}''_{\tau''}$. For existence of an individual isomorphism of $\widetilde{\mathscr{X}'}$ and $\widetilde{\mathscr{X}''}$ it is necessary and sufficient that these partitions be equipollent: each bijection $\varphi : \tau' \longrightarrow \tau''$ generates such an isomorphism.

However, if we want to extend this isomorphism to an isomorphism from $\mathscr{X}'$ onto $\mathscr{X}''$ then we need that our bijection φ sends the elements of the partition τ' to EQUIPOLLENT elements of τ''. Existence of such a bijection amounts to the fact that for every cardinal n we have the equality

$$card\,\{x' \in \tau' \mid card\,x' = n\} = card\,\{x'' \in \tau'' \mid card\,x'' = n\}.$$

This is a necessary and sufficient condition for our subalgebras to be isomorphic "in actuality" (as subobjects).

The isomorphism between subalgebras we have defined before may be called *algebraic*. Another type is a metric isomorphism. This concept applies only to the subalgebras of normed BAs.

Namely, we will say that some subalgebras $\widetilde{\mathscr{X}'}$ and $\widetilde{\mathscr{X}''}$ of NBAs $\{\mathscr{X}', \mu'\}$ and $\{\mathscr{X}'', \mu''\}$ are *metrically isomorphic* provided that there is a MEASURE PRESERVING isomorphism $\Phi : \mathscr{X}' \longrightarrow \mathscr{X}''$ satisfying $\Phi(\widetilde{\mathscr{X}'}) = \widetilde{\mathscr{X}''}$ (preservation of measure means as before that $\mu' = \mu'' \circ \Phi$). This concept of isomorphism relates to the subobjects of the category of normed BAs.

Another problem is close to the problem of isomorphism between subalgebras, i.e., the classification problem for functions. We explain the essence of this problem again by a simple example. We once more consider the discrete BAs $\mathscr{X}' = 2^{Q'}$ and $\mathscr{X}'' = 2^{Q''}$. Assume that f' and f'' are numerical functions on Q' and Q'' respectively. The question is: When are they "isomorphic," i.e., when $f'' = f' \circ \varphi$, where φ is some bijection from Q'' onto Q'? It is an easy matter to give a necessary and sufficient condition for such an isomorphism: for every y the level sets

$$E^y_{f'} \equiv \{q' \in Q' \mid f'(q') = y\}$$

and

$$E^y_{f''} \equiv \{q'' \in Q'' \mid f''(q'') = y\}$$

must be of the same cardinality. In this event the subalgebras $\widetilde{\mathscr{X}'} \equiv \mathscr{X}'_{\tau'}$ and $\widetilde{\mathscr{X}''} \equiv \mathscr{X}''_{\tau''}$, generated by the functions (τ' and τ'' consist of

level sets), are isomorphic as subalgebras and, all the more, they are isomorphic individually.

We now give a more interesting example. Considering a normed BA $\{\mathscr{X}, \mu\}$, assume given two resolutions of the identity $\mathfrak{f}_1$ and $\mathfrak{f}_2$. In other words, we speak of two measurable functions (random variables) on the realization spaces of the algebras. The question is: When are they "metrically isomorphic," i.e., when is there a metric automorphism Φ sending $\mathfrak{f}_1$ to $\mathfrak{f}_2$:

$$e_\lambda^{\pm}(\mathfrak{f}_2) = \Phi(e_\lambda^{\pm}(\mathfrak{f}_1))$$

for all λ?

Introduce the subalgebras $\widetilde{\mathscr{X}_1}$ and $\widetilde{\mathscr{X}_2}$ that are generated by the corresponding spectral families. These subalgebras are separable. We are to prove that for the existence of an individual metric isomorphism between them it is necessary and sufficient that the distribution functions coincide:

$$\mu(e_\lambda^{\pm}(\mathfrak{f}_2)) = \mu(e_\lambda^{\pm}(\mathfrak{f}_1)).$$

If this condition is fulfilled then there is an "individual" isomorphism

$$\Phi_0 : \widetilde{\mathscr{X}_1} \longrightarrow \widetilde{\mathscr{X}_2}$$

sending $\mathfrak{f}_1$ to $\mathfrak{f}_2$, and Φ_0 is a measure preserving isomorphism.

However, for an isomorphism between $\mathfrak{f}_1$ and $\mathfrak{f}_2$ we need exactly their isomorphism AS SUBALGEBRAS in the sense of the preceding definition. Namely, we need that the isomorphism Φ_0 extends to an automorphism Φ of the NBA $\mathscr{X}$. In this and only this event, the subalgebras $\widetilde{\mathscr{X}_1}$ and $\widetilde{\mathscr{X}_2}$ are the "same" not only as such but also as embedded in the "same" manner in $\mathscr{X}$; then we may view $\mathfrak{f}_1$ and $\mathfrak{f}_2$ as the "same" too.

In the separable case, a complete solution to the problem we have just posed was given by V. A. Rokhlin in the article [2]. It turns out that in addition to the distribution functions some other functions must coincide of a similar type which jointly comprise a complete system of invariants. We will consider this problem in more detail later. We just note now that it may be addressed in two versions: "metric" (we have discussed above) and "algebraic" in dependence on what type of isomorphism is implied.

Interplay between subalgebras and the manner of embedding of a subalgebra into an algebra may be visualized in the case when $\mathscr{X}$ is a separable continuous normable algebra. We will sometimes use the drafts that relate to the following model.

Consider the algebra E_0^2 (the metric structure of the Lebesgue square). As $\widetilde{\mathscr{X}}$ we take the subalgebra that corresponds to the partition of the square into vertical segments (cf. p. 354).

Assume also given a partition of the square into finitely or countably many sets of positive measure (more exactly, a partition of unity of E_0^2). As $\mathscr{X}$ we take not E_0^2 but rather some "intermediate" algebra, namely, the regular subalgebra of E_0^2 fully generated by $\widetilde{\mathscr{X}}$ and the above-mentioned partition. The relevant drafts follow; the last of them shows some elements of the BA $\mathscr{X}$ (Fig. 7).

For simplicity, we take a partition of two elements u_1 and u_2; the BA $\mathscr{X}$ consists of the elements of the form $u_1 \wedge \widetilde{x}_1 + u_2 \wedge \widetilde{x}_2$, $\widetilde{x}_1, \widetilde{x}_2 \in \widetilde{\mathscr{X}}$. The remaining elements of E_0^2 do not participate in this construction and we pretend to forget them.

By the theorem of isomorphism (Theorem 7) our algebra $\mathscr{X}$ is isomorphic with E_0, and so a similar interlocation may be implemented inside each separable continuous normed BA, in particular, inside E_0. Roughly speaking, we may always find in E_0 a regular subalgebra that is embedded in E_0 in the "same" manner as our subalgebra $\widetilde{\mathscr{X}}$ is embedded in $\mathscr{X}$. And this embedding is in no way the same as, say, the embedding of $\widetilde{\mathscr{X}}$ to E_0^2.

The algebra $\widetilde{\mathscr{X}}$ may be considered as a subalgebra of $\mathscr{X}$ and, if desire be, as a subalgebra of E_0^2; these subalgebras are not isomorphic despite their individual isomorphism raises no doubts.

We will now give some tests that allow us to judge presence or absence of an isomorphism between subalgebras as such. More exactly, we speak about finding invariants. The "isomorphism problem" is always the "invariant problem."

4.2 Preliminaries

The sign $\sim$ on the nearby pages over a letter signifying some element will remind us that this element belongs to the corresponding subalgebra: $\widetilde{x} \in \widetilde{\mathscr{X}}$, $\widetilde{y} \in \widetilde{\mathscr{Y}}$, and so on. This sign however is sometimes omitted.

The TRACE of a subalgebra $\widetilde{\mathscr{X}}$ on the band $\mathscr{X}_u$, as said in Chapter 2, consists of all possible elements of the form $\widetilde{x} \wedge u$, with $\widetilde{x} \in \widetilde{\mathscr{X}}$; it is

denoted by $[\widetilde{\mathscr{X}}]_u$. This trace is a subalgebra of $\mathscr{X}_u$ or, in other words, a "u-subalgebra."

Among the elements of $\widetilde{\mathscr{X}}$ that dominate a given $x \in \mathscr{X}$, there is always a least element (see Fig. 8). We denote this element by $\overline{x}_{\widetilde{\mathscr{X}}}$ or simply $\overline{x}$ if we need not mentioning the subalgebra:

$$\overline{x}_{\widetilde{\mathscr{X}}} \equiv \overline{x} \equiv \inf\{\widetilde{x} \in \widetilde{\mathscr{X}} \mid x \leq \widetilde{x}\}.$$

The subalgebra $\widetilde{\mathscr{X}}$ is regular and so $\overline{x} \in \widetilde{\mathscr{X}}$. (The sign $\sim$ is omitted in this case.) Clearly, the inequality $x_1 \leq x_2$ implies always that $\overline{x_1} \leq \overline{x_2}$. Each element $x \in [\widetilde{\mathscr{X}}]_u$ may be represented as $x = \widetilde{x} \wedge u$ in many ways; the most convenient choice of $\widetilde{x}$ is exactly $\overline{x}$: $x = \overline{x} \wedge u$. This representation is always possible since if $x = \widetilde{x} \wedge u$ $(\widetilde{x} \in \widetilde{\mathscr{X}})$ then $x \leq u$, $x \leq \overline{x}$, implying that $x \leq \overline{x} \wedge u$. The reverse inequality follows from the fact that $\overline{x} \leq \widetilde{x}$, $\overline{x} \wedge u \leq \widetilde{x} \wedge u = x$.

The next obvious fact is constantly used in what follows: *if* $p \in \widetilde{\mathscr{X}}$ *and* $p\, d\, u$ *then* $p\, d\, \overline{u}$.

Distinguish the band $\widetilde{\mathscr{X}}_{\overline{u}}$ in $\widetilde{\mathscr{X}}$ (with u an arbitrary member of $\mathscr{X}^+$). Note the *canonical mapping* F_u of this band to the trace $[\widetilde{\mathscr{X}}]_u$:

$$F_u(x) \equiv x \wedge u$$

(i.e., F_u is the restriction of the band projection P_u to $\widetilde{\mathscr{X}}_{\overline{u}}$.) It is easy to check that F_u is an *isomorphism whose inverse sends each* $x \in [\widetilde{\mathscr{X}}]_u$ *to* $\overline{x}$, *the only element of* $\widetilde{\mathscr{X}}_{\overline{u}}$ *whose projection is* x. *In particular, if* $\widetilde{x} \in \widetilde{\mathscr{X}}$ *then*

$$F_u^{-1}(\widetilde{x} \wedge u) = \widetilde{x} \wedge \overline{u}.$$

This remains valid for another regular subalgebra $\widetilde{\mathscr{Y}}$ of an arbitrary complete BA $\mathscr{Y}$. Intending to reflect the role of a subalgebra, we often write $F_u \equiv F_{u,\widetilde{\mathscr{X}}}$ or $F_v \equiv F_{v,\widetilde{\mathscr{Y}}}$ and so on.

If $[\widetilde{\mathscr{X}}]_u = \mathscr{X}_u$ then we have agreed to say that the algebra $\widetilde{\mathscr{X}}$ SATURATES the band $\mathscr{X}_u$; in this event we also call the element u *saturated.* The set of all elements of a BA $\mathscr{X}$ saturated by a subalgebra $\widetilde{\mathscr{X}}$ is denoted by $U(\widetilde{\mathscr{X}})$ or simply U if there is no danger of confusion. Note a few simple facts.

1°. *$U([\widetilde{\mathscr{X}}]_b) = U(\widetilde{\mathscr{X}}) \cap \mathscr{X}_b$ for all $b \in \mathscr{X}$.* Let $x \in U(\widetilde{\mathscr{X}}) \cap \mathscr{X}_b$, $x' \leq x$. The element x' is representable as $x' = \widetilde{x} \wedge x$, with $\widetilde{x} \in \widetilde{\mathscr{X}}$. Therefore, $x' = x \wedge (\widetilde{x} \wedge b)$. Since $\widetilde{x} \wedge b \in [\widetilde{\mathscr{X}}]_b$, we conclude that x is saturated by the b-subalgebra $[\widetilde{\mathscr{X}}]_b$. Conversely, if $x \in U([\widetilde{\mathscr{X}_b}])$ then each $x' \leq x$ has the form $x' = x \wedge b \wedge \widetilde{x} = x \wedge \widetilde{x}$, $\widetilde{x} \in \widetilde{\mathscr{X}}$. Thus, $x \in U(\widetilde{\mathscr{X}}) \cap \mathscr{X}_b$.

We see that we may speak about "saturation" in regard to the elements $x \leq b$ drawing no distinction between the subalgebra $\widetilde{\mathscr{X}}$ and the b-subalgebra $[\widetilde{\mathscr{X}}]_b$.

2°. *U is solid: if $u_1 \leq u_2 \in U$ then $u_1 \in U$.* Indeed, we may write an arbitrary $x \in \mathscr{X}_{u_1}$ as $x = \overline{x} \wedge u_2$. Since $x \leq u_1$, we have $x = x \wedge u_1 = \overline{x} \wedge u_2 \wedge u_1 = \overline{x} \wedge u_1$, $\overline{x} \in \widetilde{\mathscr{X}}$.

3°. *Let $u_1, u_2 \in U$ and $u_1 \, d \, u_2$. For the validity of the containment $u_1 + u_2 \in U$ it is necessary and sufficient that $\overline{u_1} \, d \, \overline{u_2}$.* (In other words, u_1 and u_2 may be separated by elements of the subalgebra $\widetilde{\mathscr{X}}$.) *In this event $\overline{u_1 + u_2} = \overline{u_1} + \overline{u_2}$.*

We first prove SUFFICIENCY. Given $x \leq u_1 + u_2$, note that

$$x = x \wedge u_1 + x \wedge u_2 = u_1 \wedge \overline{(x \wedge u_1)} + u_2 \wedge \overline{(x \wedge u_2)}$$

$$= \{u_1 + u_2\} \wedge (\overline{x \wedge u_1} + \overline{x \wedge u_2}).$$

The parentheses on the right-hand side contain an element of the subalgebra; hence, $x \in [\widetilde{\mathscr{X}}]_{u_1+u_2}$. This proves that $u_1 + u_2 \in U$. Further, it is clear that $\overline{u_1 + u_2} \leq \overline{u_1} + \overline{u_2}$, and the obvious inequalities $\overline{u_1} \leq \overline{u_1 + u_2}$, $\overline{u_2} \leq \overline{u_1 + u_2}$ imply that $\overline{u_1} \vee \overline{u_2} = \overline{u_1} + \overline{u_2} \leq \overline{u_1 + u_2}$.

We now establish the NECESSITY of our condition. Let $u_1 + u_2 \in U$. Then $u_1, u_2 \in [\widetilde{\mathscr{X}}]_{u_1+u_2}$, $\overline{u_1} = F^{-1}_{u_1+u_2}(u_1)$, $\overline{u_2} = F^{-1}_{u_1+u_2}(u_2)$, and from the disjointness of u_1 and u_2 it follows that $\overline{u_1} \, d \, \overline{u_2}$. (We have used the properties of an isomorphism.)

Corollary. *If u' and u'' are saturated then the inequality $u' \leq u''$ holds only if u' and $u'' - u'$ are separated by elements of the subalgebra $\widetilde{\mathscr{X}}$.*

4°. *If $u_\alpha \uparrow u$ then $\overline{u_\alpha} \uparrow \overline{u}$.* Put $\widetilde{u} \equiv (o)\text{-}\lim \overline{u_\alpha}$. Clearly, $\widetilde{u} \le \overline{u}$, $\widetilde{u} \in \mathscr{X}$. Since for all α we have $u_\alpha \le \overline{u_\alpha}$; therefore, $u = (o)\text{-}\lim u_\alpha \le \widetilde{u}$. Hence, $\overline{u} \le \widetilde{u}$ and $\overline{u} = \widetilde{u}$.

5°. *If $u_\alpha \uparrow u$ and $u_\alpha \in U$ then $u \in U$.* Indeed, for each $x \le u$ we have

$$x \wedge u_\alpha = u_\alpha \wedge \overline{(u_\alpha \wedge x)}$$

for all α. Passing to the limit (and using 4°), we will obtain

$$x = x \wedge u = u \wedge \overline{(u \wedge x)} = u \wedge \overline{x},$$

i.e., $x \in [\widetilde{\mathscr{X}}]_u$ and $[\widetilde{\mathscr{X}}]_u = \mathscr{X}_u$ or $u \in U$.

6°. *If $\{u_t\}_{t\in T}$ is a family such that $u_t \in U$ $(t \in T)$ and all $\overline{u_t}$ are pairwise disjoint then*

$$\sum_{t\in T} u_t \in U, \quad \overline{\sum_{t\in T} u_t} = \sum_{t\in T} \overline{u_t}.$$

This is immediate from 3°–5°.

7°. *If $\overline{y} = \widetilde{a}$, $\widetilde{x} \in \widetilde{\mathscr{X}_{\widetilde{a}}}$ then $\overline{\widetilde{x} \wedge y} = \widetilde{x}$.* Note first of all that $\overline{\widetilde{x} \wedge y} \le \widetilde{x}$. Assume that $\widetilde{t} \equiv \widetilde{x} - \overline{\widetilde{x} \wedge y} > \mathbf{0}$. Then $\widetilde{t} \in \widetilde{\mathscr{X}}^+$, $\widetilde{t} \le \widetilde{x}$, $\widetilde{t}\, d\, \overline{\widetilde{x} \wedge y}$. This may happen only if $\widetilde{t}\, d\, y$. Bur then $\overline{y} \le \widetilde{a} - \widetilde{t} < \widetilde{a}$, which contradicts the hypothesis.

In the forthcoming formulations concerning various subalgebras, the upper line still relates to the "main" subalgebra $\widetilde{\mathscr{X}}$: $\overline{x} \equiv \overline{(x)}_{\widetilde{\mathscr{X}}}$.

8°. *If $x \le b$ then $(\overline{x})_{[\widetilde{\mathscr{X}}]_b} = \overline{x} \wedge b \equiv F_b(\overline{x})$.* Indeed, if $x \le b$ then $\overline{x} \le \overline{b}$; the inequalities $\widetilde{x} \ge x$ and $\widetilde{x} \wedge b \ge x$ are equivalent, and so

$$(\overline{x})_{[\widetilde{\mathscr{X}}]_b} = \bigwedge_{\widetilde{x} \in \widetilde{\mathscr{X}},\ \widetilde{x} \wedge b \ge x} (\widetilde{x} \wedge b) = \bigwedge_{\widetilde{x} \in \widetilde{\mathscr{X}},\ \widetilde{x} \ge x} (\widetilde{x} \wedge b) = \overline{x} \wedge b \equiv F_b(\overline{x}).$$

Corollary. *If $\overline{b} \equiv \overline{b}_{\widetilde{\mathscr{X}}} = \widetilde{c}$, $x \le b$, $(\overline{x})_{[\widetilde{\mathscr{X}}]_b} = b$ then $\overline{x} \equiv \overline{x}_{\widetilde{\mathscr{X}}} = \widetilde{c}$.*

This is clear since in our case $b = \overline{x} \wedge b$, i.e., $b \le \overline{x}$, $\overline{b} = \widetilde{c} \le \overline{x}$. The reverse inequality is obvious.

9°. *If $\mathbf{0} < x', x'' \le b$, $(\overline{x'})_{[\widetilde{\mathscr{X}}]_b}\, d\, (\overline{x''})_{[\widetilde{\mathscr{X}}]_b}$ then $\overline{x'}\, d\, \overline{x''}$.* Using 8°, find that $\overline{x'}$ and $\overline{x''}$ are disjoint as the inverse images of disjoint objects under the isomorphism F_b.

We now address the problem of interlocation of saturated elements. Assume first that we are given some saturated elements u_1 and u_2. Their infimum $u = u_1 \wedge u_2$ is also saturated, similarly as $u_1 - u$ and $u_2 - u$. Put $u \equiv u_1 \wedge u_2$. Then

$$\overline{u}, C\overline{u} \in \widetilde{\mathscr{X}},\ u_1 \wedge \overline{u} = u_2\overline{u} = u,\ (u_1 \wedge C\overline{u})\, d\, (u_2 \wedge C\overline{u}).$$

We so arrive at the following important conclusion:

10°. *To every two saturated elements* u_1 *and* u_2 *there is a decomposition*

$$\mathbf{1} = \widetilde{x}_1 + \widetilde{x}_2, \quad \widetilde{x}_1 \widetilde{x}_2 \in \widetilde{\mathscr{X}}$$

such that these elements coincide in the band $\mathscr{X}_{\widetilde{x}_1}$ *whereas they are disjoint in* $\mathscr{X}_{\widetilde{x}_2}$. *More precisely,* $u_1 \wedge \widetilde{x}_1 = u_2 \wedge \widetilde{x}_1$ *and* $u_1 \wedge \widetilde{x}_2 \, d \, u_2 \wedge \widetilde{x}_2$.

We give an obvious method for finding saturated elements.

11°. *If* $u_1, u_2 \in U$ *then every element of the form* $u_1 \wedge \widetilde{x} + u_2 \wedge C\widetilde{x}$, $\widetilde{x} \in \widetilde{\mathscr{X}}$ *is also saturated.*

4.3 Algebraic isomorphism between subalgebras

We consider the two problems.

(A) Given are two NBAs $\mathscr{X}'$ and $\mathscr{X}''$ and their regular subalgebras $\widetilde{\mathscr{X}}' \subset \mathscr{X}'$ and $\widetilde{\mathscr{X}}'' \subset \mathscr{X}''$. The question is: Under which conditions can we extend a given "individual" isomorphism Φ_0 sending $\widetilde{\mathscr{X}}'$ to $\widetilde{\mathscr{X}}''$ to some isomorphism $\Phi : \mathscr{X}' \longrightarrow \mathscr{X}''$?

(B) Under which general conditions does there exist an isomorphism of $\mathscr{X}'$ to $\mathscr{X}''$ sending $\widetilde{\mathscr{X}}'$ to $\widetilde{\mathscr{X}}''$? In other words, when are $\widetilde{\mathscr{X}}'$ and $\widetilde{\mathscr{X}}''$ isomorphic as subalgebras?

We emphasize that we are now interested in an ALGEBRAIC isomorphism, presuming no measure on $\mathscr{X}'$ nor on $\mathscr{X}''$. However, the EXISTENCE of a measure will be often important since it allows us to use classification theorems.

In the sequel we distinguish between the objects of the same type which are related to the algebras $\mathscr{X}'$ and $\mathscr{X}''$ (subalgebras, families of elements, measures, etc.) by using one or two primes in denotations. However, we will replace the symbol $\overline{x}$ by $(\overline{x})_{\widetilde{\mathscr{X}}'}$ if $x \in \mathscr{X}'$ and by $(\overline{x})_{\widetilde{\mathscr{X}}''}$ if $x \in \mathscr{X}''$. The same relates to zeros and unities whose denotations never vary with algebras.

The general approach we will pursue rests on PARTITIONS. Assume that in each of the algebras $\mathscr{X}'$ and $\mathscr{X}''$ we have distinguished some partitions of unity:

$$\mathbf{1}_{\mathscr{X}'} = \sum_{\xi \in \Xi} c'_\xi, \quad \mathbf{1}_{\mathscr{X}''} = \sum_{\xi \in \Xi} c''_\xi. \tag{6}$$

Digressing a little from the terminology we have agreed upon, here we treat a partition not as a set but rather as a FAMILY of pairwise disjoint elements: it is not excluded that for some ξ we have $c'_\xi = \mathbf{0}$ ($c''_\xi = \mathbf{0}$). Observe that in our case both families have the same system of indices ξ. To these partitions of unity there correspond the decompositions of

algebras into disjoint bands

$$\mathscr{X}' = \bigoplus \mathscr{X}_{c'_\xi}, \quad \mathscr{X} = \bigoplus \mathscr{X}_{c''_\xi}.$$

If Φ is an isomorphism and $\Phi(c'_\xi) = c''_\xi$ then we say that *the partition* $\{c'_\xi\}$ *goes to* $\{c''_\xi\}$ *under this isomorphism.* In this event it stands to reason to speak of *isomorphic* partitions. Observe that in this case these partitions must have the same system of indices. We introduce some simplifying notations:

$$\widetilde{\mathscr{X}}'_\xi \equiv [\widetilde{\mathscr{X}}']_{c'_\xi}, \ \ \widetilde{\mathscr{X}}''_\xi \equiv [\widetilde{\mathscr{X}}'']_{c''_\xi}, \ \ \mathscr{X}'_\xi \equiv \mathscr{X}'_{c'_\xi}, \ \ \mathscr{X}''_\xi \equiv \mathscr{X}''_{c''_\xi}.$$

If c'_ξ (c''_ξ) is an element of an appropriate subalgebra then $\widetilde{\mathscr{X}}'_\xi$ $(\widetilde{\mathscr{X}}''_\xi)$ is a band of it. In this event we call this element of the partition *internal*; otherwise, we call it *external* (with respect to the subalgebra).

We are primarily interested in partitions that possess the property of *invariance*. Now the rigorous definition is in order. Let $\mathscr{X}$ be a complete BA, and let $\widetilde{\mathscr{X}}$ be a subalgebra of $\mathscr{X}$. We call a partition of unity

$$\mathbf{1} = \sum_{\xi \in \Xi} c_\xi$$

invariant with respect to $\widetilde{\mathscr{X}}$ (or $\widetilde{\mathscr{X}}$*-invariant*) provided that for each automorphism Φ of $\mathscr{X}$, sending $\widetilde{\mathscr{X}}$ to $\widetilde{\mathscr{X}}$, we have $\Phi(c_\xi) = c_\xi$ for all ξ. The elements of an invariant partition are fixed points of every isomorphism keeping the subalgebra $\widetilde{\mathscr{X}}$ immovable. The following property of such a partition is obvious: If $\mathscr{X}'$ and $\mathscr{X}''$ are Boolean algebras, $\widetilde{\mathscr{X}}'$ and $\widetilde{\mathscr{X}}''$ are subalgebras, Φ is an isomorphism from $\mathscr{X}'$ to $\mathscr{X}''$ sending $\widetilde{\mathscr{X}}'$ to $\widetilde{\mathscr{X}}''$, then each $\widetilde{\mathscr{X}}'$-invariant partition of $\mathscr{X}'$ goes to an $\widetilde{\mathscr{X}}''$-invariant partition of $\mathscr{X}''$ under Φ.

A special role among invariant partitions is played by those partitions whose relation with the subalgebra in question has a CANONICAL character. In general, the attribute "canonical" in a mathematical context indicates the presence of some FUNCTOR. Specification of the term would require an accurate description of the relevant categories which is out of the agenda right away. We thus confine exposition to a few clarifications and examples.

The "functor" we have just mentioned is simply the mapping $\mathscr{F}$ that assigns to each subalgebra $\widetilde{\mathscr{X}} \subset \mathscr{X}$ (more exactly, to the pair $\langle \widetilde{\mathscr{X}}, \mathscr{X} \rangle$) some class of a uniquely determined partition $\xi \equiv \mathscr{F}[\widetilde{\mathscr{X}}, \mathscr{X}]$. All these partitions have the same system of indices ξ (related to the functor), and under each isomorphism $\Phi : \mathscr{X} \longrightarrow \mathscr{Y}$ the partition $\mathscr{F}[\widetilde{\mathscr{X}}, \mathscr{X}]$ goes to

$\mathscr{F}[\Phi(\widetilde{\mathscr{X}}), \Phi(\mathscr{X})]$. If such a functor is available, we say that the partition ξ is canonically related to the subalgebra or is "canonical." Clearly, the partition $\mathscr{F}[\widetilde{\mathscr{X}}, \mathscr{X}]$ is $\widetilde{\mathscr{X}}$-invariant. For instance, assume that to the subalgebra $\widetilde{\mathscr{X}} \subset \mathscr{X}$ there corresponds the partition $\xi = \mathscr{F}[\widetilde{\mathscr{X}}, \mathscr{X}] = \{u_1, u_2\}$ such that $\widetilde{\mathscr{X}_{u_1}}$ is the continuous band and $\widetilde{\mathscr{X}_{u_2}}$ is the discrete band of $\widetilde{\mathscr{X}}$. This partition ξ is canonically related to $\widetilde{\mathscr{X}}$.

The idea of canonical correspondence is naturally abstracted to arbitrary families of elements and other objects connected with algebras and subalgebras. Intending to view the system of indices ξ as a set, we always consider the classes of subalgebras of bounded cardinalities.

In the examples we will meet the "canonicity" of a partition, i.e., the existence of some functor $\mathscr{F}$ is always established straightforwardly on using the construction of the partition. *A partition is "canonical" whenever the properties characterizing it in a unique fashion are preserved under every isomorphism of the subalgebra.*

We exhibit another example. Let $\mathfrak{a}$ be an infinite cardinal, and let ξ stand for the set of all cardinals at most $\mathfrak{a}$. We will consider a complete BA with the countable chain condition of weight at most $\mathfrak{a}$. Let $\mathscr{X}$ be one of these algebras, and let $\widetilde{\mathscr{X}}$ be a regular subalgebra saturating no nonzero bands. To this subalgebra there corresponds the decomposition of $\mathscr{X}$ into $\widetilde{\mathscr{X}}$-homogeneous bands (p. 117). Using the countable chain condition, we may combine the band of the same degree of nonsaturation into a single one and find the partition $\mathscr{F}[\widetilde{\mathscr{X}}, \mathscr{X}] \equiv \{u_\xi\}_{\xi \in \xi}$ in which the nonzero elements u_ξ correspond to $\widetilde{\mathscr{X}}$-homogeneous bands; their degrees of nonsaturation $\sigma(\widetilde{\mathscr{X}}, u_\xi)$ are pairwise distinct. The number of nonzero u_ξ is at most countable. The countable chain condition has enabled us to take as u_ξ the MAXIMAL $\widetilde{\mathscr{X}}$-homogeneous bands. Clearly, this partition is canonically related with the subalgebra, since it is a result of a universal construction: the partition $\mathscr{F}[\widetilde{\mathscr{X}}, \mathscr{X}]$ is uniquely determined for every pair $\langle \widetilde{\mathscr{X}}, \mathscr{X} \rangle$ of the class we have described above; under every isomorphism Φ each $\widetilde{\mathscr{X}}$-homogeneous band goes to a $\Phi(\widetilde{\mathscr{X}})$-homogeneous band; the maximality property is also preserved.

In this example it was presumed that the saturated nonzero bands are absent. Therefore, all degrees of nonsaturation $\sigma(\widetilde{\mathscr{X}}, u_\xi)$ are infinite: a band of a finite nonzero degree of nonsaturation can never be homogeneous.[10] In the most general case a partition may contain saturated bands of zero degree of nonsaturation. However, such a partition

[10]This again corroborates the remark in the footnote on p. 457. (A. A. Samorodnitskiĭ.)

will fail to be canonically related with the subalgebra: as a rule, it is neither unique nor invariant.

We now return to the partitions (6); assume that they are canonically related to the subalgebras $\widetilde{\mathscr{X}'}$ and $\widetilde{\mathscr{X}''}$; moreover, they are related in the same manner, i.e. are obtainable by the same construction or the same functor. If $\widetilde{\mathscr{X}'}$ and $\widetilde{\mathscr{X}''}$ are isomorphic then so are these partitions. Each isomorphism $\Phi : \mathscr{X}' \longrightarrow \mathscr{X}''$, satisfying the condition

$$\Phi(\widetilde{\mathscr{X}'}) = \widetilde{\mathscr{X}''},$$

must also possess the following property:

$$\Phi(c'_\xi) = c''_\xi \quad (\xi \in \Xi). \tag{7}$$

The same must hold for internal and external elements of the partitions. Clearly, for external elements we must also have

$$\Phi(\overline{c}'_\xi) = \overline{c}''_\xi \quad (\xi \in \Xi). \tag{8}$$

If the partitions (6) are canonical then the last system of equalities expresses a NECESSARY condition for the given isomorphism $\Phi : \widetilde{\mathscr{X}'} \longrightarrow \widetilde{\mathscr{X}''}$ to admit an extension to an isomorphism of $\mathscr{X}'$ onto $\mathscr{X}''$, i.e., Problem (A) has a positive solution.[11] This condition is also sufficient if the partitions (6) are chosen properly. Prior to choosing partitions, we will discuss the plan of constructing an isomorphism on assuming the partitions (6) given (but not necessarily canonical). Thus, we suppose that there is an "individual" isomorphism Φ_0 satisfying $\Phi_0(\overline{c}'_\xi) = \overline{c}''_\xi$ for all $\xi \in \Xi$. We fist construct to each $\xi \in \Xi$ a PARTIAL isomorphism $\Phi_\xi : \mathscr{X}'_\xi \longrightarrow \mathscr{X}''_\xi$. We will try to do this so as to make the following hold simultaneously:

$$\Phi_\xi(\overline{\mathscr{X}}'_\xi) = \overline{\mathscr{X}}''_\xi \quad (\xi \in \Xi). \tag{9}$$

After that we will try to "paste together" these into a single isomorphism Φ by the formulas

$$\Phi(x) = \sum_\xi \Phi_\xi(x \wedge c'_\xi). \tag{10}$$

We indeed come to an isomorphism of $\mathscr{X}'$ onto $\mathscr{X}''$ (it is an easy matter to check this fact). But does this isomorphism extend Φ_0? Will it send

[11] In general, the condition (8) is necessary for Φ to admit an extension to an isomorphism sending the one of the partitions (6) to the other (irrespectively of whether or not these partitions are canonical).

$\widetilde{\mathscr{X}'}$ to $\widetilde{\mathscr{X}''}$ at least? The answers to these questions depend on the choice of partitions and our skill in defining Φ_ξ.

We first note the following: If the isomorphism Φ given by (10) actually coincides on $\widetilde{\mathscr{X}'}$ with a given "individual" isomorphism Φ_0 then for every $x \in \widetilde{\mathscr{X}'}_\xi$ we must have

$$\Phi_\xi(x) = \Phi(x) = \Phi(\overline{x} \wedge c'_\xi) = \Phi(\overline{x}) \wedge \Phi(c'_\xi)$$
$$= \Phi_0(\overline{x}) \wedge c''_\xi = (F_{c''_\xi} \circ \Phi_0 \circ F^{-1}_{c'_\xi})(x). \tag{11}$$

So, the restrictions of $\Phi^0_\xi \equiv \Phi_\xi|_{\widetilde{\mathscr{X}'}_\xi}$ *to* $\widetilde{\mathscr{X}'}_\xi$ *might be determined only from the formula*

$$\Phi^0_\xi = F_{c''_\xi} \circ \Phi_0|_{\widetilde{\mathscr{X}}_{\overline{c}''_\xi}} \circ F^{-1}_{c'_\xi} \tag{12}$$

(we have assumed that $\Phi_0(\overline{c}'_\xi) = \overline{c}''_\xi$ for all ξ, and so the formula (12) indeed determines an isomorphism sending $\widetilde{\mathscr{X}'}_\xi$ to $\widetilde{\mathscr{X}''}_\xi$. To extend Φ_0, we have to solve the two successive problems:

a) we are to extend each of the isomorphisms Φ^0_ξ given by the formula (12) to some isomorphism $\Phi_\xi : \mathscr{X}'_\xi \longrightarrow \mathscr{X}''_\xi$;

b) "pasting together" all Φ_ξ by the formula (10), we must demonstrate that the resultant Φ indeed extends Φ_0 and, in particular, sends $\widetilde{\mathscr{X}'}$ to $\widetilde{\mathscr{X}''}$.

To solve Problem a) successfully we need a special collection of partitions: $\widetilde{\mathscr{X}'}_\xi$ and $\widetilde{\mathscr{X}''}_\xi$ must be isomorphic as subalgebras of the corresponding bands $\mathscr{X}'_\xi$ and $\mathscr{X}''_\xi$. We thus reduce Problem (A) to an analogous problem for the bands $\mathscr{X}'_\xi$ and $\mathscr{X}''_\xi$, hoping to use some special properties of these bands.

As regards Problem b), it entails no new difficulties. Namely, we have

Lemma 5. *Assume given some isomorphisms $\Phi : \mathscr{X}' \longrightarrow \mathscr{X}''$ and $\Phi_0 : \widetilde{\mathscr{X}'} \longrightarrow \widetilde{\mathscr{X}''}$ so that for all $\xi \in \Xi$ and all $x \in \widetilde{\mathscr{X}'}_\xi$ the inequalities hold*

$$\Phi(x) \leq \Phi^0_\xi(x) \equiv c''_\xi \wedge \Phi_0(\overline{x}). \tag{13}$$

Then $\Phi(x) = \Phi_0(x)$ for all $x \in \widetilde{\mathscr{X}'}$.

Proof. Given $x \in \widetilde{\mathscr{X}'}$ and $\xi \in \Xi$, note that $x \wedge c'_\xi \in \widetilde{\mathscr{X}'}_\xi$. Hence,

$$\Phi(x) = \sum_\xi \Phi(x \wedge c'_\xi) \leq \sum_\xi (c''_\xi \wedge \Phi_0(\overline{x \wedge c'_\xi}))$$
$$\leq \sum_\xi c''_\xi \wedge \Phi_0(\overline{x}) = \Phi_0(\overline{x}) = \Phi_0(x).$$

(Since $\overline{x} = x$.) Considering that Φ and Φ_0 are isomorphisms, we whence infer that

$$C\Phi(x) = \Phi(Cx) \le \Phi_0(Cx) = C\Phi_0(x),$$
$$\Phi(x) \ge \Phi_0(x)$$

for all $x \in \widetilde{\mathscr{X}'}$. The proof of the lemma is complete. Normability of the algebras was not involved. In this lemma we do not assume the partitions canonical, it is the inequalities (13) that were material.

The last lemma entails the following result important in future. Let us suppose that the individual isomorphism Φ_0 satisfies the condition $\Phi_0(\overline{c}'_\xi) = \overline{c}''_\xi$ $(\xi \in \Xi)$.

Lemma 6. *Assume that for each $\xi \in \Xi$ the isomorphism Φ^0_ξ, given by* (12)*, extends to an isomorphism $\Phi_\xi : \mathscr{X}'_\xi \longrightarrow \mathscr{X}''_\xi$. Then there is an isomorphism $\Phi : \mathscr{X}' \longrightarrow \mathscr{X}''$ extending Φ_0.*

Indeed, such an isomorphism results by pasting together the mappings Φ_ξ by the formula (10).

We are thus left with Problem a). We have to find sufficient conditions under which this problem is solvable for the given bands $\mathscr{X}'_\xi$ and $\mathscr{X}''_\xi$. We start with the following two conditions:

(H) *the subalgebras $\widetilde{\mathscr{X}'}$ and $\widetilde{\mathscr{X}''}$ saturate the bands $\mathscr{X}_\xi$ and $\mathscr{X}''_\xi$*, i.e.,

$$\mathscr{X}'_\xi = \widetilde{\mathscr{X}'}_\xi, \quad \mathscr{X}''_\xi = \widetilde{\mathscr{X}''_\xi};$$

(O) *the bands $\mathscr{X}'_\xi$ and $\mathscr{X}''_\xi$ are $\widetilde{\mathscr{X}'}_\xi$- and $\widetilde{\mathscr{X}''_\xi}$-homogeneous, with*

$$\sigma(\widetilde{\mathscr{X}'}, c'_\xi) = \sigma(\widetilde{\mathscr{X}''}, c''_\xi).$$

Lemma 7. *Each of the conditions (H) and (O) is sufficient for the isomorphism Φ^0_ξ, given by the formula* (12)*, to be extendible to an isomorphism of $\mathscr{X}'_\xi$ onto $\mathscr{X}''_\xi$.*

Proof. If the condition (H) is satisfied that no extension is required at all: the mapping Φ^0_ξ itself is a sought isomorphism.

Consider the case in which (O) holds. Distinguish some probability measures μ' and μ'' on the bands $\mathscr{X}'_\xi$ and $\mathscr{X}''_\xi$ respectively. (We do not use the normability of our algebras at this stage.) By Theorems 1 and 3 the subalgebras $\widetilde{\mathscr{X}'}_\xi$ and $\widetilde{\mathscr{X}''_\xi}$ have the μ'-independent complement and the μ''-independent complement of the same weight, both homogeneous. Denote them by $\widehat{\mathscr{X}'}_\xi$ and $\widehat{\mathscr{X}''_\xi}$. Let

$$\mu'_1 \equiv \mu'|_{\widetilde{\mathscr{X}'}_\xi},\ \mu'_2 \equiv \mu'|_{\widehat{\mathscr{X}'}_\xi},\ \mu''_1 \equiv \mu''|_{\widetilde{\mathscr{X}''_\xi}},\ \mu''_2 \equiv \mu''|_{\widehat{\mathscr{X}''_\xi}}.$$

Since $\widehat{\mathscr{X}}'_\xi$ and $\widehat{\mathscr{X}}''_\xi$ are homogeneous and of the same weight, there is an "individual" isomorphism $\Psi : \widehat{\mathscr{X}}'_\xi \longrightarrow \widehat{\mathscr{X}}''_\xi$ satisfying $\mu''_2 = \mu'_2 \circ \Psi^{-1}$. Replace the measure μ''_1 with another one: $\nu_1 = \mu'_1 \circ (\Phi^0_\xi)^{-1}$. By Theorem 9 the product $\nu = \nu_1 \times \mu''_2$ exists presenting a probability measure on $\mathscr{X}''_\xi$ such that $\widetilde{\mathscr{X}}''_\xi$ and $\widehat{\mathscr{X}}''_\xi$ are ν-independent. Theorem 9 also enables us to construct the direct product $\Phi_\xi \equiv \Phi^0_\xi \times \Psi$. This is a sought isomorphism. The proof of the lemma is complete.

REMARK 1. Normability of the algebras was first involved in the second part of the proof of the lemma (in regard to the condition (O)).

REMARK 2. Instead of Φ^0_ξ in Lemma 7 we may take an arbitrary individual isomorphism sending $\widetilde{\mathscr{X}}'_\xi$ to $\widetilde{\mathscr{X}}''_\xi$. If desire be, we may abstain from relating the bands $\mathscr{X}'_\xi \equiv \mathscr{X}'_{c'_\xi}$ and $\mathscr{X}''_\xi \equiv \mathscr{X}''_{c''_\xi}$ with whatever partitions, treating c'_ξ and c''_ξ simply as some elements satisfying $\Phi^0_\xi(c'_\xi) = c''_\xi$.

REMARK 3. If $\mu''_1 = \mu'_1 \circ (\Phi^0_\xi)^{-1}$ then $\nu = \mu''$ and $\Phi_\xi = \Phi^0_\xi \times \Psi$ is a measure preserving isomorphism, i.e., it satisfies the condition $\mu'' = \mu' \circ \Phi_\xi^{-1}$ (and so it is a "metric isomorphism").

In the sequel we will give another condition guaranteeing solvability of Problem a), the so-called "decomposability."

Let $\mathscr{X}$ be a complete BA with the countable chain condition, and let $\widetilde{\mathscr{X}}$ be a regular subalgebra of $\mathscr{X}$. We will construct two principal "canonical" partitions canonically related to this subalgebra. We start with the partition which is referred to as the "standard external partition" in the sequel.

Lemma 8. *To each regular subalgebra $\widetilde{\mathscr{Y}}$ of a complete BA $\mathscr{Y}$ there is a unique partition of unity*

$$\mathbf{1} = u + w, \quad u, w \in \mathscr{Y},$$

such that the band $\mathscr{Y}_w$ contains no nonzero element saturated by $\widetilde{\mathscr{Y}}$, and these elements comprise a minorant for the band $\mathscr{Y}_u$.

PROOF. The set of nonsaturated elements is antisolid and so it is d-regular. Hence, by Theorem 2.4 its solid core is a band. Let this band be $\mathscr{Y}_w$, $u \equiv Cw$. By the same Theorem 2.4 the saturated elements comprise a minorant for $\mathscr{Y}_u$. The elements u and w make a sought partition whose uniqueness is obvious. Clearly, it is not excluded that $u = \mathbf{0}$ or $w = \mathbf{0}$. Since the properties characterizing u and w are preserved under an isomorphism between subalgebras, the partition $\{u, w\}$ is related to $\widetilde{\mathscr{Y}}$ canonically. To emphasize this relation we will sometimes write $u = u(\widetilde{\mathscr{Y}})$ and $w = w(\widetilde{\mathscr{Y}})$. The band $\mathscr{Y}_w$, if it is nonzero, we may decompose it further into $\widetilde{\mathscr{Y}}$-homogeneous bands.

We now apply the lemma to the basic situation for us in which $\mathscr{Y} = \mathscr{X}$ is a complete BA with the countable chain condition and $\widetilde{\mathscr{X}}$ is a regular subalgebra of $\mathscr{X}$. Let the elements $u \equiv u(\widetilde{\mathscr{X}})$ and $w \equiv w(\widetilde{\mathscr{X}})$ comprise the partition that is described by the lemma. In this event, decomposing $\mathscr{X}_w$ into $\widetilde{\mathscr{X}}$-homogeneous bands, we obtain an at most countable partition

$$w = w^1 + w^2 + \cdots, \tag{14}$$

where the degrees of nonsaturation $\sigma(\widetilde{\mathscr{X}}, w^i)$ are infinite and pairwise distinct. In other words, $\mathscr{X}_{w^i}$ are MAXIMAL $\widetilde{\mathscr{X}}$-homogeneous bands. The partition (14) is also canonically related to $\widetilde{\mathscr{X}}$.

As regards the band $\mathscr{X}_u$, its $\widetilde{\mathscr{X}}$-homogeneous bands are saturated. There are many decompositions of $\mathscr{X}_u$ into saturated bands and it is impossible to select some "canonical" representative. However, combining the addends in a special manner, we may construct such a decomposition. For brevity, put

$$Z \equiv \mathscr{X}_u, \quad \widetilde{Z} \equiv [\widetilde{\mathscr{X}}]_u.$$

We agree to call an element $a \in \widetilde{Z}^+$ *decomposable* provided that it may be represented as the sum

$$a = \sum_{p \in P} u_p, \tag{15}$$

where u_p are pairwise disjoint elements saturated by $\widetilde{\mathscr{X}}$ and possessing the property

$$\overline{u}_p = a \quad (p \in P).$$

(From here up to the end of 8° the upper bar refers to the subalgebra $\widetilde{Z}$, i.e., we always have $\overline{z} \equiv (\overline{x})_{\widetilde{Z}}$.)

We define the cardinal function $\mathscr{N}$ on $\widetilde{Z}^+$, letting $\mathscr{N}(a)$ equal to the least possible cardinality of the set P in (15) in case a is decomposable; if $a \in \widetilde{Z}^+$ is not decomposable, we put $\mathscr{N}(a) = \aleph_1$. Note a few facts important for the sequel.

1°. *If a is decomposable and* (15) *holds then* $\operatorname{card} P = \mathscr{N}(a)$. In other words it is impossible not only to diminish the number of addends in the formula (15) but also to increase this number.

Proof. It suffices to establish the following: If $a = u_1 + u_2 + \cdots + u_m$, u_i are saturated and $\overline{u}_i = a$ $(i = 1, 2, \dots)$ then there is no disjoint collection $u'_1, u'_2, \dots, u'_{m+1}$ also composed of saturated elements such that $\overline{u}'_j = a$ $(j = 1, 2, \dots, m + 1)$. If such a collection were available then, using Proposition 10° of 9.4.2 a few times, we would find an element

$b \in \widetilde{Z}$, $\mathbf{0} < b \leq a$, such that each element of the form $u_i \wedge b$ either coincides with is disjoint from each element of the form $u'_j \wedge b$ (they are all other than zero). Assign to each $j = 1, 2, \dots, m+1$ a number $i_j \leq m$ so that $u'_j \wedge b = u_{i_j} \wedge b$. This index is available because

$$\sum u_i \wedge b = b \geq \sum u'_j \wedge b.$$

Since the elements of u'_j are pairwise disjoint, the mapping $j \longrightarrow i_j$ would be a bijection. This is clearly impossible. The proof of 1° is thus complete.

2°. *If $\mathbf{0} < a' \leq a$, $a' \in Z$ then $\mathscr{N}(a') \leq \mathscr{N}(a)$, so that the function $\mathscr{N}$ is isotonic. Moreover, if $\mathscr{N}(a) \leq \aleph_0$ then $\mathscr{N}(a) = \mathscr{N}(a')$.*

Proof. Assume that $a = u_1 + \ldots + u_n$, the elements u_k are saturated, and $\overline{u}_k = a$. Then the elements $u'_k \equiv u_k \wedge a'$ are also saturated. Moreover, $a' = u'_1 + \cdots + u'_n$, $\overline{u}'_k = a'$. (This follows from 7° of 9.4.2.) By 1° of 9.4.3, we have $\mathscr{N}(a') = n = \mathscr{N}(a)$. The same holds also for $\mathscr{N}(a) = \aleph_0$. If $\mathscr{N}(a) = \aleph_1$ then in any case we note the inequality $\mathscr{N}(a') \leq \aleph_1 = \mathscr{N}(a)$.

We now introduce the set $S_n \equiv \{z \in \widetilde{Z}^+ \mid \mathscr{N}(z) = n\}$ $(1 \leq n \leq \aleph_1)$.

From 2° it is immediate that

3°. *The sets $S_n \cup \{\mathbf{0}\}$ are solid for all $n \leq \aleph_0$.*

In fact we have the following stronger assertion:

4°. *If $n \leq \aleph_0$ then $S_n \cup \{\mathbf{0}\}$ is a band.*

It suffices to show that S_n is d-regular. Consider the disjoint family $\{z_q\}_{q \in Q}$, where $z_q \in S_n$, $z_q = \sum_{1 \leq k \leq n} u_{qk}$ (or $z_q = \sum_{k=1}^{\infty} u_{qk}$), $\overline{u}_{qk} = z_q$, u_{qk} are saturated elements. Put

$$u^i = \sum_{q \in Q} u_{qk}.$$

The elements u^i are pairwise disjoint (6° of 9.4.2). Moreover,

$$\sum_i u^i = \sum_i \sum_q u_{qi} = \sum_q \sum_i u_{qi} = \sum_q z_q.$$

So, the element $\sum_q z_q$ is decomposed into n disjoint saturated addends $u^1, u^2, \dots,$ with $\overline{u}^i = \sum_q \overline{u}_{qi} = \sum_q z_q$. Clearly, $\sum_q z_q \in S_n$. We have thus checked the d-regularity of S_n. Hence, recalling 3° once again, we conclude that $S_n \cup \{\mathbf{0}\}$ is a band coinciding with the solid core S_n : $S_n \cup \{\mathbf{0}\} = S_n^0$.

5°. $\widetilde{Z}^+ = \bigcup_{1 \leq n \leq \aleph_1} S_n$. This equality is evident.

6°. *Each band S_n^0 with $n < \aleph_1$ coincides with the disjoint complement of the solid core of the set $\widetilde{Z} \setminus S_n \equiv \bigcup_{m \neq n,\ m \leq \aleph_1} S_m \cup \{\mathbf{0}\}$.* This follows from Theorem 2.4.

7°. *The set $S_{\aleph_1}$ is d-regular. Its solid core $S^0_{\aleph_1}$ is the complementary band to the disjoint sum of all bands S^0_n, $n < \aleph_1$. Therefore, $S^0_1, S^0_2, \dots, S^0_{\aleph_0}, S^0_{\aleph_1}$ comprise a disjoint decomposition of Z.*

The proof is not difficult but we omit it since the band $S^0_{\aleph_1}$ contributes nothing to the decomposition. We will demonstrate this.

8°. *The set $S^0_{\aleph_1}$ consists of the sole zero.*

PROOF. Assume by way of contradiction that there is some $z_0 \in S^0_{\aleph_1}$ such that $z_0 > \mathbf{0}$. The elements, saturated by $\widetilde{Z}$, comprise a minorant for Z. (Recall that $\widetilde{Z} \equiv [\widetilde{\mathscr{X}}]_u$.) Therefore, the elements of the form $\overline{z}$, with z saturated, comprise a minorant for $\widetilde{Z}$. This implies that z_0 (as well as each element of $\widetilde{Z}$) may be represented as the disjoint sum

$$z_0 = \sum_k \overline{z}_k,$$

where z_k are saturated elements. The latter are pairwise disjoint and separated by pairwise disjoin elements $\overline{z}_k \in \widetilde{Z}$; therefore, $z^1 \equiv \sum_k z_k$ is also saturated and $\overline{z^1} = z_0$ (6° of 9.4.2).

Assume now that to all ordinals $\alpha < \alpha_0$ there are defined pairwise disjoint saturated elements $z^1, z^2, \dots, z^\alpha, \dots$ satisfying the conditions:

$$\overline{z}^\alpha = z_0, \quad z_\alpha \, d \sum_{\gamma<\alpha} z^\gamma \quad (\alpha < \alpha_0).$$

The two cases are possible:

1) $\overline{z_0 - \sum_{\alpha<\alpha_0} z^\alpha} = z_0$,

2) $\overline{z_0 - \sum_{\alpha<\alpha_0} z^\alpha} < z_0$.

In case 1) we may continue recursion and define z^{α_0} in much the same way as z^1 (on replacing z_0 with the difference $z_0 - \sum_{\alpha<\alpha_0} z^\alpha$).

We explicate this construction in more detail. Let $z_0 - \sum_{\alpha<\alpha_0} z^\alpha \equiv b$ and $Z' \equiv [\widetilde{Z}]_b$. The elements of the form $(\overline{u})_{Z'}$, with u saturated, comprise a minorant for Z'. The element b plays the role of unity and to b there is a disjoint decomposition

$$b = \sum_k (\overline{v}_k)_{Z'},$$

with v_k saturated. (Here we imply the saturation by the b-subalgebra Z', but this amounts to the saturation by the subalgebra $\widetilde{Z}$ or $\widetilde{\mathscr{X}}$; cf. 1° of 9.4.2.) We now put $z^{\alpha_0} \equiv \sum_k v_k$. Since $(\overline{v}_k)_{Z'}$ are pairwise disjoint, we have $(\overline{z^{\alpha_0}})_{Z'} = \sum_k (\overline{v}_k)_{Z'} = b$. Apply the corollary in 8° of 9.4.2, on taking $c \equiv z_0$. Then $(\overline{z^{\alpha_0}})_{\widetilde{Z}} = \overline{b} = z_0$. Using 9° and 6° (in the same

place), we conclude that $(\overline{v}_k)_{\widetilde{Z}}$ are disjoint from $(\overline{v}_k)_{Z'}$ and the element z^{α_0} is saturated.

Case 2) means that there is an element $y \in \widetilde{Z}$ satisfying the inequality

$$\mathbf{0} < y = z_0 - \overline{z_0 - \sum_{\alpha<\alpha_0} z^\alpha} \leq \sum_{\alpha<\alpha_0} z^\alpha.$$

Put $y^\alpha \equiv y \wedge z^\alpha$. Clearly, y^α are saturated and $\sum_{\alpha<\alpha_0} y^\alpha = y$. Moreover, $\overline{y}^\alpha = y$ for all $\alpha < \alpha_0$ (7° of 9.4.2). We see that $y \in S_n$, where n is the cardinality of the ordinal α. Since our algebra satisfies the countable chain condition, we infer that $n \leq \aleph_0$. But this contradicts the containment $z_0 \in S^0_{\aleph_1}$. Consequently, case 2) is impossible and we may continue recursion indefinitely. On the other hand, unbounded recursion is impossible either, since the process must terminate after countably many steps. Thus, the band $S^0_{\aleph_0}$ may contain only zero.

Put $s_n \equiv \sup S^0_n$ (cf. Fig. 9). Some s_n may equal zero, but we always have

$$s_n \in S^0_n, \quad u = \sum_{1\leq n\leq\aleph_0} s_n.$$

Thus, we have the two disjoint partitions:

$$u = \sum_n s_n, \quad w = \sum_n w^n.$$

Together they comprise the partition of unity we are going to construct. We now describe it in the final form.

Denote by $K \equiv K(\mathscr{X})$ the set of all nonzero cardinals at most $max(\tau(\mathscr{X}), t(\mathscr{X}))$, and let $T \equiv T(\mathscr{X})$ stand for the set of all pairs of the form $t = (k, i)$, where $i = 0, 1,\ k \in K$ while $k \leq \aleph_0$ for $i = 0$ and

$k \geq \aleph_0$ for $i = 1$. If $\tau(\mathscr{X}) < \alpha_0$ then T contains only the pairs of the form $(k, 0)$. In the sequel we will view the symbol $\overline{x}$ as referring to the subalgebra $\widetilde{\mathscr{X}}$, i.e., $\overline{x} \equiv (\overline{x})_{\widetilde{\mathscr{X}}}$. We will often write down the pair t as (k_t, i_t).

Theorem 10. *To each regular subalgebra $\widetilde{\mathscr{X}} \subset \mathscr{X}$, where $\mathscr{X}$ is a complete BA with the countable chain condition, there corresponds a unique partition of unity $\{b_t\}_{t \in T}$ with the following properties:*

1) For $i_t = 1$ the band $\mathscr{X}_{b_t}$ is either zero or $\widetilde{\mathscr{X}}$-homogeneous with degree of nonsaturation $\sigma(\widetilde{\mathscr{X}}, b_t) = k_t$;

2) If $i_t = 0$ then either $b_t = \mathbf{0}$ or b_t may be written as

$$b_t = \sum_{q \in Q_t} u_q,$$

where the addends u_q are saturated by $\widetilde{\mathscr{X}}$,

$$\overline{u}_q = \overline{b}_t \ (q \in Q_t), \quad card\, Q_t = k_t.$$

In this event

$$b_t \in [\widetilde{\mathscr{X}}]_u \equiv [\widetilde{\mathscr{X}}]_{\sum\limits_{t:i_t=0} b_t}.$$

The condition $\overline{u}_q = \overline{b}_t$ may be written as

$$(\overline{u}_q)_{\widetilde{Z}} = F_u(\overline{u}_q) = F_u(\overline{b}_t) = (\overline{b}_t)_{\widetilde{Z}}.$$

(Cf. 8° of 9.4.2.)

As regards existence, the theorem is in fact proved: the elements of a sought partition are exactly the elements w^n and s_n we have already defined. We need only enumerate them with the index set T. Put

a) if $i_t = 0$ then $b_t \equiv b_{(k_t, 0)} \equiv s_{k_t}$;

b) if $i_t = 1$ then we consider b_t equal to the only element among w^n for which $\sigma(\widetilde{\mathscr{X}}, W^n) = k_t$; if there is no such an element w^n then $b_t = \mathbf{0}$.

It is clear that the conditions 1) and 2) are fulfilled.

We now prove that the partition with the properties 1) and 2) is unique. To this end, assume that, in addition to the above-constructed partition $\{b_t\}_{t \in T}$ there is another partition $\{b_t'\}$ also possessing the properties 1) and 2). Note first of all that for $i_{t_1} = 0,\ i_{t_2} = 1$ we clearly have $b_{t_1}'\, d\, b_{t_2}$ (since it is obvious that $b_{t_1} \leq u,\ b_{t_2} \leq w$). It is also clear that $b_t = b_t'$ for $i_t = 1$. We are left with establishing the same for the case of $i_t = 0$. The element b_t' is decomposable and so $b_t' \leq s_{k_t} = b_t$ (it is important here that $b_t' \in [\widetilde{\mathscr{X}}]_u$ yielding $b_t' \in S_{k_t}^0$). Thus, $b_t' \leq b_t$ for all t, and at the same time

$$\sum_t b_t = \sum_t b_t' = \mathbf{1}.$$

This is possible only if $b_t = b'_t$ $(t \in T)$. The proof of the theorem is complete.

REMARK. The elements b_t fall into two types: "decomposable" for which $i_t = 0$ and "homogeneous" for which $i_t = 1$. Note one important property of "decomposable" elements: *If $t_1 \neq t_2$ and $i_{t_1} = i_{t_2} = 0$ then $\overline{b}_{t_1}$ and $\overline{b}_{t_2}$ are disjoint.* Indeed, the above-mentioned construction makes it clear that b_{t_1} and b_{t_2} are disjoint elements of $\widetilde{Z}$, where

$$\widetilde{Z} \equiv [\widetilde{\mathscr{X}}]_u; \quad b_{t_1} = s_{k_{t_1}}, \quad b_{t_2} = s_{k_{t_2}}.$$

Therefore, they may be represented as

$$b_{t_1} = \overline{b}_{t_1} \wedge u = F_u(\overline{b}_{t_1}),\ \ b_{t_2} = \overline{b}_{t_2} \wedge u = F_u(\overline{b}_{t_2}).$$

The mapping F_u is an isomorphism, and the elements

$$\overline{F}_u^{-1}(b_{t_1}) = \overline{b}_{t_1},\ \ \overline{F}_u^{-1}(b_{t_2}) = \overline{b}_{t_2}$$

are disjoint whenever so are b_{t_1} and b_{t_2}.

It is this partition $b = \{b_t\}_{t\in T}$ we have just constructed that we will call the *external standard partition corresponding to the subalgebra $\widetilde{\mathscr{X}}$.* Sometimes we will denote it with more details:

$$b[\widetilde{\mathscr{X}}, \mathscr{X}] \equiv \{b_t[\widetilde{\mathscr{X}}, \mathscr{X}]\}_{t\in T(\mathscr{X})}.$$

Observe that it is related exactly with the subalgebra $\widetilde{\mathscr{X}} \subset \mathscr{X}$ and this relation is CANONICAL. This follows, first, from uniqueness and, second, from the fact that the properties 1) and 2), characterizing the partition uniquely, are preserves under isomorphisms between subalgebras as such. The elements b_t may be external for $w > \mathbf{0}$, i.e., they may lie beyond $\widetilde{\mathscr{X}}$. This explains the choice of the term. The family $\{\overline{b}_t\}$ is always related to $\widetilde{\mathscr{X}}$ canonically. Clearly, the set $\{t \in T \mid b_t > \mathbf{0}\}$ is at most countable. Hence, the nonzero elements of the form b_t or $\overline{b}_t$ comprise at most countable set.

The role of the external standard partition is clear from the following

Theorem 11. *Assume that $\mathscr{X}'$ and $\mathscr{X}''$ are NBAs of the same weight, $\widetilde{\mathscr{X}'} \subset \mathscr{X}'$ and $\widetilde{\mathscr{X}''} \subset \mathscr{X}''$ are regular subalgebras, and $\{b'_t\}_{t\in T}$ and $\{b''_t\}_{t\in T}$ are the corresponding external standard partitions.*

For an individual isomorphism $\Phi_0 : \widetilde{\mathscr{X}'} \longrightarrow \widetilde{\mathscr{X}''}$ to admit extension to an isomorphism $\Phi : \mathscr{X}' \longrightarrow \mathscr{X}''$ it is necessary and sufficient that

$$\Phi_0(\overline{b}'_t) = \overline{b}''_t \tag{16}$$

for all $t \in T$.

(Here $T \equiv T(\mathscr{X}') \equiv T(\mathscr{X}'')$.)

Proof. Let Φ be an isomorphism extending Φ_0. By the "canonicity" of the standard partition in question we must have $\Phi(b'_t) = b''_t$ for all $t \in T$. By the general properties of isomorphisms this implies that $\Phi(\bar{b}'_t) = \bar{b}''_t$ which is precisely the equality (16). The theorem is proved as regards necessity.

Assume now that (16) holds for all $t \in T$. By Lemma 6, to construct a desired isomorphism Φ it suffices, given $t \in T$, to implement extension of the "partial" mapping Φ_t^0 sending $\widetilde{\mathscr{X}}'_t \equiv [\widetilde{\mathscr{X}}']_{b'_t}$ to $\widetilde{\mathscr{X}}''_t \equiv [\widetilde{\mathscr{X}}'']_{b''_t}$ to an isomorphism $\Phi_t : \mathscr{X}'_{b'_t} \longrightarrow \mathscr{X}''_{b''_t}$. The existence of this extension is beyond a doubt if $i_t = 1$ (Lemma 7, case (O)). Assume now that $i_t = 0$. Then the elements b'_t and b''_t may be represented as

$$b'_t = \sum_{q \in Q_t} u'_q, \quad b''_t = \sum_{q \in Q_t} u''_q,$$

where u'_q and u''_q are saturated by the relevant subalgebras $\widetilde{\mathscr{X}}'$ and $\widetilde{\mathscr{X}}''$, $\overline{u}'_q = \bar{b}'_t$, $\overline{u}''_q = \bar{b}''_t$ for all $q \in Q_t$ and $card\, Q_t = k_t$. Put $\Phi_{tq} \equiv F_{u''_q} \circ \Phi_t^0|_{\widetilde{\mathscr{X}}_{\overline{u}'_q}} \circ F_{u'_q}^{-1}$. We are again in the context of Lemma 7, but now we see that the condition (H) is now satisfied for all $q \in Q_t$. Thus, by Lemma 6 there is an isomorphism $\Phi_t : \mathscr{X}'_{b'_t} \longrightarrow \mathscr{X}''_{b''_t}$ extending all Φ_{tq}, $q \in Q_t$, simultaneously. We are now left with applying Lemma 6 once again, so completing the proof.

We now describe another partition also canonically related to a subalgebra. Let $\mathscr{X}_0$ be the subalgebra generated in $\widetilde{\mathscr{X}}$ by the set of all elements of the form $\bar{b}_t$ $(t \in T)$, and let $\overline{\mathscr{X}_0}$ stand as usual for the closure of $\mathscr{X}_0$, the least regular subalgebra of the subalgebra $\widetilde{\mathscr{X}}$ (or, which is equivalent, of the algebra $\mathscr{X}$) which includes $\mathscr{X}_0$. This subalgebra is separable; it might happen even to be finite. It is related with $\widetilde{\mathscr{X}}$ canonically.

This follows from the "canonicity" of the family $\{\bar{b}_t\}$ we have already mentioned. The complete notation is as follows: $\overline{\mathscr{X}_0} \equiv \overline{\mathscr{X}_0}[\widetilde{\mathscr{X}}, \mathscr{X}]$.

Construct the external standard partition corresponding to the subalgebra $\overline{\mathscr{X}_0} \subset \widetilde{\mathscr{X}}$ ($\overline{\mathscr{X}_0}$ is treated now as a subalgebra of $\widetilde{\mathscr{X}}$). Denote it by $a[\widetilde{\mathscr{X}}, \mathscr{X}]$:

$$a \equiv a[\widetilde{\mathscr{X}}, \mathscr{X}] \equiv b[\overline{\mathscr{X}_0}, \widetilde{\mathscr{X}}], \tag{17}$$
$$a_t \equiv a_t[\widetilde{\mathscr{X}}, \mathscr{X}] \equiv b_t[\overline{\mathscr{X}_0}, \widetilde{\mathscr{X}}] \quad (t \in T(\widetilde{\mathscr{X}})).$$

We call this partition the *internal standard partition of $\mathscr{X}$ corresponding to the subalgebra $\widetilde{\mathscr{X}}$*. Its elements are internal; i.e., they belong to $\widetilde{\mathscr{X}}$. Note that $T(\widetilde{\mathscr{X}}) \subset T(\mathscr{X})$.

The pair of partitions a and b may be considered as a single family of elements. Namely, we define the new index set $H = H(\mathscr{X})$ by comprising all pairs of the form $h \equiv (t_h, j_h)$, with $t_h \in T(\mathscr{X})$, $j_h = 0, 1$. Let

$$H^0 \equiv \{h \mid j_h = 0\}, \ H^1 \equiv \{h \mid j_h = 1\},$$

$$e_h \equiv e_h[\widetilde{\mathscr{X}}, \mathscr{X}] = \begin{cases} b_{t_h}, & h \in H^0, \\ a_{t_h}, & h \in H^1, \ \ t_h \in T(\widetilde{\mathscr{X}}), \\ \mathbf{0}, & h \in H^1, \ \ T_h \in T(\mathscr{X}) \setminus T(\widetilde{\mathscr{X}}). \end{cases}$$

(In essence, H^0 and H^1 are two "copies" of the set $T(\mathscr{X})$ whose disjoint union makes H; in this event we factually use only that part of H^1 which "consists" of the members of $T(\widetilde{\mathscr{X}})$.)

The family

$$e \equiv e[\widetilde{\mathscr{X}}, \mathscr{X}] \equiv \{e_h[\widetilde{\mathscr{X}}, \mathscr{X}]\}$$

is referred to in the sequel as the standard family corresponding to $\widetilde{\mathscr{X}}$. It consists of the elements of the BA $\mathscr{X}$ which are the members of the external or internal partitions.

It is sometimes reasonable to extend the subalgebra $\widetilde{\mathscr{X}}$ by adjoining to it elements of the form $b_{(n,i)} \equiv e_{((n,i),0)}$. We will denote by $\widetilde{\widetilde{\mathscr{X}}}$ the regular subalgebra which is generated by these elements in $\mathscr{X}$ (as well as the elements of the original subalgebra $\widetilde{\mathscr{X}}$). It is easy to check that $\widetilde{\mathscr{X}} \subset \widetilde{\widetilde{\mathscr{X}}}$ and $b[\widetilde{\mathscr{X}}, \mathscr{X}] = b[\widetilde{\widetilde{\mathscr{X}}}, \mathscr{X}]$; moreover, for $\widetilde{\widetilde{\mathscr{X}}}$ the elements $b_{(n,i)}$ become internal. The external standard partition of $\widetilde{\widetilde{\mathscr{X}}}$ consists of internal elements and has the same members as the analogous partition of $\widetilde{\mathscr{X}}$. However, we have in general $a[\widetilde{\mathscr{X}}, \mathscr{X}] \neq a[\widetilde{\widetilde{\mathscr{X}}}, \mathscr{X}]$. The role of the subalgebra $\widetilde{\widetilde{\mathscr{X}}}$ may be seen from the following almost evident theorem:

Theorem 12. *In the context of Theorem* 11 *for an individual isomorphism* Φ_0 *be extendible to an isomorphism* $\Phi : \mathscr{X}' \longrightarrow \mathscr{X}''$ *it is sufficient that* Φ_0 *be extendible to an isomorphism* $\Phi_1 : \widetilde{\widetilde{\mathscr{X}}}' \longrightarrow \widetilde{\widetilde{\mathscr{X}}}''$ *such that* $\Phi_1(b'_t) = b''_t$ *for all* t.

Indeed, further extension from $\widetilde{\widetilde{\mathscr{X}}}'$ to $\mathscr{X}'$ is guaranteed by Theorem 11.

A standard family is related to the relevant subalgebra canonically. We must first know those h for which e_h differs from zero. We call the corresponding set

$$A \equiv A[\widetilde{\mathscr{X}}, \mathscr{X}] \equiv \{h \mid e_h > \mathbf{0}\}$$

the *certificate* of $\widetilde{\mathscr{X}}$. Define another family:

$$\widetilde{e}_h \equiv \begin{cases} \overline{e}_h \equiv (\overline{e}_h)_{\widetilde{\mathscr{X}}}, & \text{if} \quad h \in H^0, \\ (\overline{e}_h)_{\overline{\mathscr{X}_0}}, & \text{if} \quad h \in H^1. \end{cases}$$

Denote by $\mathscr{X}_1$ the subalgebra that is generated in $\widetilde{\mathscr{X}}$ by the elements $\widetilde{e}_h$. Clearly, $\overline{\mathscr{X}_0} \supset \mathscr{X}_1 \supset \mathscr{X}_0$ and this subalgebra, as well as its closure $\overline{\mathscr{X}_1} = \overline{\mathscr{X}_0}$, relates to $\widetilde{\mathscr{X}}$ canonically: if $\Phi : \mathscr{X}' \longrightarrow \mathscr{X}''$ and $\Phi(\widetilde{\mathscr{X}'}) = \widetilde{\mathscr{X}''}$ then $\Phi(\mathscr{X}_1') = \mathscr{X}_1''$ and $\Phi(\overline{\mathscr{X}_1}') = \overline{\mathscr{X}_1}''$.

The elements of A may be identified in a sense with the INDEPENDENT GENERATORS OF THE FREE BOOLEAN ALGEBRA $\mathscr{D}_A$ (cf. 1.3.7). Namely, consider the Cantor space X_A consisting of all "binary families" $\chi = \{\chi_h\}_{h\in A}$ $(\chi = 0, 1)$, and assign to each $h \in A$ the set $[h] \equiv \{\chi \in X_A \mid \chi_h = 1\}$ (Q_h in the previous notation). Those are the independent generators of the free BA $\mathscr{D}_A$ comprising all clopen subsets of the compact space X_A. The correspondence $h \longleftrightarrow [h]$ is one-to-one and so each mapping $\omega : A \longrightarrow \mathscr{X}$ generates a unique homomorphism $\Omega : \mathscr{D}_A \longrightarrow \mathscr{X}$ such that $\Omega([h]) = \omega(h)$ for all $h \in A$. We will apply this scheme to our situation, taking as ω the mapping $\omega(h) = \widetilde{e}_h$. Thus, the values $\omega(h)$ are precisely all NONZERO $\widetilde{e}_h$, the subalgebra $\Omega(\mathscr{D}_A)$, generated by them in $\mathscr{X}$, coincides with $\mathscr{X}_1$. Instead of $\Omega(d)$ $(d \in \mathscr{D}_A)$ we will mostly write x_d and call the family $\{x_d\}$ "parametrizing." (This term reminds us that the element x_d makes $\mathscr{X}_1$: this subalgebra is "parametrized" by the family $\{x_d\}$.) If we distinguish in $\mathscr{D}_A$ the subalgebra generated by only those $[h]$ for which $h \in H^0$ then its image under Ω will coincide with $\mathscr{X}_0$ and the corresponding restriction of the family $\{x_d\}$ will be parametrizing for this narrower subalgebra.

We define a quasimeasure M on $\mathscr{D}_A$ by the formula $M(d) \equiv \mu x_d$, where μ is some probability measure on $\mathscr{X}$. The function M may vanish also at nonzero elements of $\mathscr{D}_A$, but in any case it is countably additive[12] and extends to the Baire σ-algebra. We are however interested not in this quasimeasure but rather in its ("Hellinger") type: the whole class $\widehat{M}$ of equivalent quisimeasures (in the sense of mutual absolute continuity) on $\mathscr{D}_A$ which contains M. The type $\widehat{M}$ is independent of the initial choice of the measure μ.[13] It is eventually determined from the mapping ω,

[12]Recall that this quasimeasure is defined on the clopen subsets of the compact space X_A.
[13]The membership of the functions $M_1(d) \equiv \mu_1 x_d$ and $M_2(d) \equiv \mu_2 x_d$ in the same type (i.e., their equivalence) means precisely that the relations $\mu_1(x_{d_n}) \longrightarrow 0$ and $\mu_2(x_{d_n}) \longrightarrow 0$ are equivalent. However, either is simply convergence to zero in the order topology.

i.e., the subalgebra $\widetilde{\mathscr{X}}$. Therefore, we will denote it by $\widehat{M} \equiv \widehat{M}[\widetilde{\mathscr{X}}, \mathscr{X}]$ and call the *algebraic type* (or simply *type*) of the subalgebra.[14]

Consider the two subalgebras: $\widetilde{\mathscr{X}'} \subset \mathscr{X}'$ and $\widetilde{\mathscr{X}''} \subset \mathscr{X}''$. Assume that μ', μ'' and $\{x'_d\}, \{x''_d\}$ are probability measures and parametrizing families that are determined for $\mathscr{X}', \mathscr{X}''$ and $\widetilde{\mathscr{X}'}, \widetilde{\mathscr{X}''}$ respectively. Assume further that these subalgebras are isomorphic and $\Phi : \mathscr{X}' \longrightarrow \mathscr{X}''$ is an isomorphism that sends $\widetilde{\mathscr{X}'}$ to $\widetilde{\mathscr{X}''}$. It is easy to see that in these conditions we must have

$$A' \equiv A[\widetilde{\mathscr{X}'}, \mathscr{X}'] = A[\widetilde{\mathscr{X}''}, \mathscr{X}''] \equiv A'', \quad x''_d = \Phi(x'_d) \ (d \in \mathscr{D}_{A'} \equiv \mathscr{D}_{A''}),$$

and the quasimeasures

$$M' : \ M'(d) \equiv \mu'(x_d)$$

and

$$M'' : \ M''(d) \equiv \mu''(x_d)$$

determine the same type: $\widehat{M}[\widetilde{\mathscr{X}'}, \mathscr{X}'] = \widehat{M}[\widetilde{\mathscr{X}''}, \mathscr{X}'']$. A type is invariant under isomorphisms. (We imply here as usual the isomorphisms of subalgebras as such rather than individual isomorphisms.) We will soon see that the type $\widehat{M}$ carries in an encoded form full information about "internal" and "external" properties of the subalgebra. We start with a few simplest examples.

1. *Assume that* $\widetilde{\mathscr{X}} = \mathscr{X} = \{\mathbf{0}, \mathbf{1}\}$*; i.e., the algebra and subalgebra are both degenerate.* Here

$$K(\mathscr{X}) = \{1\}, \ T(\mathscr{X}) = \{(1,0)\},$$

$$H(\mathscr{X}) = \{((1,0),0), ((1,0),1)\}.$$

The external standard partition has a sole nonzero term $b_{(1,0)} = \mathbf{1}$. Clearly, $\mathscr{X}_0 = \overline{\mathscr{X}_0} = \widetilde{\mathscr{X}}$, and so $a_{(1,0)} = \mathbf{1}$. Consequently,

$$e_{((1,0),0)} = e_{((1,0),1)} = \mathbf{1}.$$

Obviously, $\widetilde{e}_h = e_h$ for all h and $\overline{\mathscr{X}_1} = \mathscr{X}_1 = \widetilde{\mathscr{X}} = \mathscr{X}$. The certificate A consists of two elements $h_1 = ((1,0),0), \ h_2 = ((1,0),1)$. The Cantor space X_A comprises the four points: 1) $\chi_{h_1} = \chi_{h_2} = 0$; 2) $\chi_{h_1} = \chi_{h_2} = 1$; 3) $\chi_{h_1} = 0, \ \chi_{h_2} = 1$; 4) $\chi_{h_1} = 1, \ \chi_{h_2} = 0$. The algebraic type $\widehat{M}$ consists of the sole "delta-function" supported at one of these points.

[14] A type is a class of quasimeasures on the CONCRETE algebra $\mathscr{D}_A$ (rather than on an abstract free BA). Therefore, alongside the type we are also given the certificate A.

2. *$\mathscr{X}$ is a continuous NBA, and $\widetilde{\mathscr{X}} = \mathscr{X}$.* The decomposition into homogeneous bands (by weight) consists of the principal ideals $\mathscr{X}_{u_1}, \mathscr{X}_{u_2}, \dots$, where

$$\sum u_k = \mathbf{1}, \quad u_k > \mathbf{0}, \quad \tau(\mathscr{X}_{u_k}) \equiv \tau_k.$$

The number of addends is finite or countable. We may assume that $\aleph_0 \leq \tau_1 < \tau_2 < \dots$, and so $\{u_k\}$ is a canonical partition.

The external standard partition consists of a single nonzero term $b_{(1,0)} = \mathbf{1}$; the subalgebra $\mathscr{X}_0 = \overline{\mathscr{X}_0}$ is trivial. The internal standard partition has nonzero terms $a_{(\tau_k,1)} = u_k$, the remaining terms are all equal to zero. Finally, we have

$$e_h = \widetilde{e}_h = \mathbf{1} \quad \text{for} \quad h = h_0 \equiv ((1,0),0),$$

$$e_h = u_k, \ \widetilde{e}_h = \mathbf{1} \quad \text{for} \quad h = h_k \equiv ((\tau_k,1),1) \ \ (k = 1,2,\dots),$$

$$\widetilde{e}_h = e_h = \mathbf{0} \quad \text{otherwise.}$$

The certificate A consists of the points $h_0, h_1, \dots$ whose number is finite or countable. The algebraic type of the subalgebra is determined by the "delta-function" supported at the point $\chi \in X_A$ whose coordinates are all equal to 1: $\chi_h = 1, h \in A$. This type is uniquely determined by the collection of weights $\tau_1, \tau_2, \dots$.

3. *$\mathscr{X}$ is a homogeneous algebra of weight $\tau_0 \equiv \tau(\mathscr{X}) \geq \aleph_0$; $\widetilde{\mathscr{X}}$ is the degenerate subalgebra: $\widetilde{\mathscr{X}} = \{\mathbf{0}, \mathbf{1}\}$.*

The external standard partition is as follows:

$$b_t = \begin{cases} \mathbf{1}, & t = t_0, \\ \mathbf{0}, & t \neq t_0, \end{cases}$$

where $t_0 \equiv (\tau_0, 1)$. For all t we have $\overline{b}_t = b_t$, and so the subalgebra $\mathscr{X}_0 = \overline{\mathscr{X}_0}$ is degenerate.

The internal standard partition is as follows:

$$a_t = \begin{cases} \mathbf{1}, & t = t_1, \\ \mathbf{0}, & t \neq t_1, \end{cases}$$

with $t_1 \equiv (1,0)$.

The standard family is

$$e_h = \widetilde{e}_h = \begin{cases} \mathbf{1}, & h \in A, \\ \mathbf{0}, & h \notin A, \end{cases}$$

where the certificate A consists of the two elements:

$$h_0 \equiv (t_{h_0}, 0) \equiv (t_0, 0) \equiv ((\tau_0, 1), 0); \quad h_1 \equiv (t_{h_1}, 1) \equiv (t_1, 1) \equiv ((1,0),1).$$

The certificate comprises two elements once again. The algebraic type reduces again to the sole "delta-function" that is now supported at the point of the Cantor space which is defined by the conditions: $\chi_{h_0} = \chi_{h_1} = 1$.

4. *$\mathscr{X}$ is again a homogeneous algebra of weight τ_0, while $\widetilde{\mathscr{X}}$ is a discrete subalgebra with n atoms* $(1 < n \leq \aleph_0)$.

The external standard partition is as follows:

$$b_t = \begin{cases} \mathbf{1}, & t = t_0, \\ \mathbf{0}, & t \neq t_0, \end{cases}$$

$$t_0 = (\tau_0, 1), \ \ \mathscr{X}_0 = \overline{\mathscr{X}_0} = \{\mathbf{0}, \mathbf{1}\}, \ \ \overline{b}_t = b_t.$$

The internal standard partition is as follows:

$$a_t = \begin{cases} \mathbf{1}, & t = t_1, \\ \mathbf{0}, & t \neq t_1, \end{cases}$$

where $t_1 \equiv (n, 0)$. Other features are the same as in the previous examples. The certificate A consists of the two points: $h_0 \equiv ((\tau_0, 1), 0)$ and $h_1 \equiv ((n, 0), 1)$; the only quasimeasure, making the algebraic type $\widehat{M}$, is the "delta-function" supported at the point of the Cantor space with the coordinates $\chi_{h_0} = \chi_{h_1} = 1$.

In various examples we have met the same four-point Cantor space X_A together with the same type as it seems at the first glance. However, we should abstain from identifying the spaces X_A in these events, since the information about a subalgebra is kept in particular in its certificate A and so we must consider X_A as a concrete set rather than an abstract Cantor space. In much the same way, the algebraic type exists not in its own right but rather as related to a given Cantor space X_A. (For instance, the "delta-functions" supported at different points present different types.)

Consider one example more.

5. We now assume that $\mathscr{X}$ is a *discrete algebra* with $n < \aleph_0$ atoms and $\widetilde{\mathscr{X}} \subset \mathscr{X}$ is an arbitrary subalgebra of $\mathscr{X}$. The BA $\mathscr{X}$ has the form 2^Q, while $\widetilde{\mathscr{X}}$ is generated by the partition $\tau = \{e_1, e_2, \dots, e_m\}$. Denote by m_k the number of those e_i that contain k points $(k = 1, 2, \dots, n)$. We further put

$$K^0 \equiv \{k \mid m_k > 0\}, \quad K^1 \equiv \{km_k \mid k \in K^0\}.$$

It is easy to see that in this example the external standard partition consists of the sets of the form

$$b_{(k,0)} = \bigcup_{i:\ card\, e_i = k} e_i, \quad k \in K^0,$$

while the internal standard partition consists of the sets

$$a_{(k',0)} = \bigcup_{s:\ sm_s = k'} b_{(s,0)}, \quad k' \in K^1.$$

The remaining b_t and a_t are equal to zero (i.e., they are empty).

The certificate splits into two parts: $A = A^0 \cup A^1$, where

$$A^0 \equiv \{h \mid h = ((k,0),0),\ k \in K^0\},$$

$$A^1 \equiv \{h \mid h = ((k',0),0), k' \in K^1\}.$$

The algebraic type of the subalgebra is presented by measures with nonzero loads at those and only at those points $\chi \in X_A$ that are characterized by the following property: among the coordinates χ_h there are precisely two of them equal to 1: χ_{h_0} and χ_{h_1}, where $h_0 \equiv ((k,0),0)$ and $h_1 \equiv ((k',0),1)$, with $k' = km_k$. The remaining χ_h are all equal to zero. Clearly, the choice of the main measure μ on $\mathscr{X}$ plays no role at all in these circumstances.

6. In conclusion we give a simple example of constructing a subalgebra with a given certificate. Let τ_0 be an arbitrary infinite cardinal; $h' = ((2,0),0)$ and $h'' = ((\tau_0,1),1)$. The set $A = \{h', h''\}$ serves as the certificate of the subalgebra $\widetilde{\mathscr{X}} \subset \mathscr{X}$ with the standard family

$$e_{h'} = b_{(2,0)} = \mathbf{1}, \quad e_{h''} = a_{(\tau_o,1)} = \mathbf{1},$$

while the remaining b_h are all equal to zero. There are two saturated disjoint elements b' and b'' such that $\overline{b}' = \overline{b}'' = \mathbf{1}$, $b' + b'' = \mathbf{1}$; the weight of $\mathscr{X}$ equals τ_0 and this algebra is homogeneous. The algebraic type consists of the "delta-function" supported at the point χ^0 with coordinates $\chi^0_{h'} = \chi^0_{h''} = 1$.

It is easy to see that the subalgebra in this example may be recovered from the certificate up to isomorphism. The algebraic type here is also determined by the certificate. Note that the measures of b' and b'' play no role whatsoever.

The problem of the last example may be called "inverse": it is required to find a subalgebra given a certificate. This problem is not solvable in general. For instance, the set

$$\{((1,0),0),((n,0),1),((m,0),1)\}$$

with $m \neq n$ is not the certificate of any subalgebra.

The certificate and algebraic type are related to a subalgebra canonically. If there are two isomorphic regular subalgebras $\widetilde{\mathscr{X}}' \subset \mathscr{X}'$ and

$\widetilde{\mathscr{X}}'' \subset \mathscr{X}''$ of normable BAs $\mathscr{X}'$ and $sCle\mathscr{X}''$ then we see with necessity that

$$\begin{aligned} A[\widetilde{\mathscr{X}}', \mathscr{X}'] = A[\widetilde{\mathscr{X}}'', \mathscr{X}''], \\ \widehat{M}[\widetilde{\mathscr{X}}', \mathscr{X}'] = \widehat{M}[\widetilde{\mathscr{X}}'', \mathscr{X}'']. \end{aligned} \tag{18}$$

(In fact, the first equality of (18) is included in the second since, as we have just mentioned, the type $\widehat{M}$ is related to a particular Cantor space X_A and so we are given the certificate A.)

Theorem 13. *The condition* (18) *is necessary and sufficient for* $\widetilde{\mathscr{X}}'$ *and* $\widetilde{\mathscr{X}}''$ *to be isomorphic.*

We need to prove this theorem only in regard to necessity. Assume that $\mathscr{X}'$ and $\mathscr{X}''$ are normable BAs, while $\widetilde{\mathscr{X}}'$ and $\widetilde{\mathscr{X}}''$ are regular subalgebras satisfying the condition (18). Denote by A the common certificate and by $\widehat{M}$, the common type of these subalgebras:

$$A \equiv A[\widetilde{\mathscr{X}}', \mathscr{X}'] \equiv A[\widetilde{\mathscr{X}}'', \mathscr{X}''],$$

$$\widehat{M} \equiv \widehat{M}[\widetilde{\mathscr{X}}', \mathscr{X}'] \equiv \widehat{M}[\widetilde{\mathscr{X}}'', \mathscr{X}''].$$

Choosing the measures μ' and μ'' on $\mathscr{X}'$ and $\mathscr{X}''$ arbitrarily, we as before put

$$M'(d) \equiv \mu' x'_d, \quad M''(d) \equiv \mu'' x''_d \quad (d \in \mathscr{D}_A).$$

Clearly $M', M'' \in \widehat{M}$, and so the relations

$$M'(d) \longrightarrow 0 \tag{19}$$

and

$$M''(d) \longrightarrow 0 \tag{20}$$

are equivalent. In particular,

$$M'(d) = 0 \quad \text{and} \quad M''(d) = 0$$

or, which is the same,

$$x'_d = \mathbf{0} \quad \text{and} \quad x''_d = \mathbf{0}.$$

Theorem 1.18* opens an opportunity to construct a homomorphism $\Theta : \mathscr{X}'_1 \longrightarrow \mathscr{X}''_1$ such that

$$\Theta(\widetilde{e}'_h) = \Theta(\omega(h)) = \Theta(x'_{[h]}) = x''_{[h]} = \widetilde{e}''_h. \tag{21}$$

This will be so also for all $d \in \mathscr{D}_A :\ \Theta(x'_d) = x''_d$. Simultaneously, we construct the inverse homomorphism with similar properties. Therefore,

Θ is in fact an isomorphism from $\mathscr{X}_1'$ onto $\mathscr{X}_1''$. The main condition (the equivalence between (20) and (21)) guarantees the continuity of Θ and Θ^{-1}: if $x_n' \downarrow \mathbf{0}$ then $\Theta(x_n') \downarrow \mathbf{0}$, and if $x_n'' \downarrow \mathbf{0}$ then $\Theta^{-1}(x_n'') \downarrow \mathbf{0}$. Using Theorem 5.4, conclude that there is an isomorphism $\Phi_0 : \overline{\mathscr{X}'_0} \longrightarrow \overline{\mathscr{X}''_0}$ extending Θ (recall that we always have $\overline{\mathscr{X}_0} = \overline{\mathscr{X}}$).

Clearly, for all t the following holds:

$$\Phi_0((\overline{a}_t')_{\overline{\mathscr{X}'_0}}) = (\overline{a}_t'')_{\overline{\mathscr{X}''_0}}, \tag{21$'$}$$

$$\Phi_0(\overline{b}_t') = \overline{b}_t''. \tag{21$''$}$$

The elements a_t' and a_t'' constitute, as we remember, the internal standard partitions for the subalgebras $\widetilde{\mathscr{X}'}$ and $\widetilde{\mathscr{X}''}$ or, which is the same, EXTERNAL partitions for $\overline{\mathscr{X}'_0}$ and $\overline{\mathscr{X}''_0}$[15] with respect to $\widetilde{\mathscr{X}'}$ and $\widetilde{\mathscr{X}''}$. (Cf. the formula (17).) The equalities (21$'$) tell us that the subalgebras $\overline{\mathscr{X}'_0} \subset \widetilde{\mathscr{X}'}$ and $\overline{\mathscr{X}''_0} \subset \widetilde{\mathscr{X}''}$ meet the main condition that guarantees the possibility of extending Φ_0 to some isomorphism $\Phi_1 : \widetilde{\mathscr{X}'} \longrightarrow \widetilde{\mathscr{X}''}$ (Theorem 11). In this event we have (21$''$) and we may once again apply Theorem 11 and extend Φ_1 to a sought isomorphism $\Phi : \mathscr{X}' \longrightarrow \mathscr{X}''$. The proof is complete.

The main classification theorem for normable algebras (the remark on Theorem 6) is a particular case of Theorem 13. Namely, we should take $\widetilde{\mathscr{X}} = \mathscr{X}$ (cf. Example 2).

4.4 Subalgebras of discrete type. Primitive and semiprimitive subalgebras

The algebraic type of each subalgebra in the examples of 9.4.3 was supported in a finite or countable subset of a Cantor space. Surely, these examples are rather simple and not very interesting. However, the type of a subalgebra turns out to be "discrete" in many practically important situations. We give a precise definition.

We say that $\widetilde{\mathscr{X}} \subset \mathscr{X}$ is a *subalgebra of discrete algebraic type* whenever there is at most countable set $S_0 \equiv S_0[\widetilde{\mathscr{X}}, \mathscr{X}] \subset X_A$ such that for whatever quasimeasure $M \in \widehat{M}[\widetilde{\mathscr{X}}, \mathscr{X}]$ each point $\chi \in S_0$ is an atom of M (i.e., $M\{\chi\} > 0$) and

$$\sum_{\chi \in S_0} M(\{\chi\}) = 1.$$

[15] Recall that $\overline{\mathscr{X}'_0} = \overline{\mathscr{X}'_1}$ and $\overline{\mathscr{X}''_0} = \overline{\mathscr{X}''_1}$.

The last equality means that $M(e) = 0$ for $e \subset X_A \setminus S_0,\ M \in \widehat{M}$. The set S_0 may be called the *support* of this algebraic type. For a subalgebra of discrete algebraic type all questions are solved more easily than in the general case, since in this event the type is simply the set S_0. The following theorem is obvious.

Theorem 14. *For some subalgebras of discrete algebraic type $\widetilde{\mathscr{X}}' \subset \mathscr{X}'$ and $\widetilde{\mathscr{X}}'' \subset \mathscr{X}''$ to be isomorphic it is necessary and sufficient that*

$$A[\widetilde{\mathscr{X}}', \mathscr{X}'] = A[\widetilde{\mathscr{X}}'', \mathscr{X}''],$$

$$S_0[\widetilde{\mathscr{X}}', \mathscr{X}'] = S_0[\widetilde{\mathscr{X}}'', \mathscr{X}''].$$

The subalgebras of Examples 1)–4) not only belong to some discrete algebraic type but also possess another property: the type of such a subalgebra is uniquely determined by its certificate. There is an important class of subalgebras (including these examples) within which we may "classify by certificate" without using the algebraic type.

A regular subalgebra $\widetilde{\mathscr{X}} \subset \mathscr{X}$ is called *primitive* provided that it satisfies the two conditions:

[I] *$\widetilde{\mathscr{X}}$ is a homogeneous or degenerate subalgebra;*

[II] *$\mathscr{X}_0$ is degenerate: $\mathscr{X}_0 = \overline{\widetilde{\mathscr{X}}}_0 = \{\mathbf{0}, \mathbf{1}\}$.*

The condition [II] amounts to the following:

[II$'$] *Each element of $\widetilde{e}_h$ equals zero or unity.*

Let A be the certificate of a primitive subalgebra, and let X_A stand for the corresponding Cantor space. It is easy to see that the algebraic type of a primitive subalgebra is always discrete: it is supported at the singleton $S_0 = \{\chi^0\}$, where χ^0 is the point of X_A whose all coordinates are equal to one: $\chi^0_h = 1,\ h \in A$.[16] This and Theorem 14 imply

Theorem 15. *Two primitive subalgebras are isomorphic if and only if their certificates coincide.*

It is easy to exhibit an example of two subalgebras with the same certificate one of which is primitive and the other is not. Certainly, they are not isomorphic.

The primitive subalgebras fall into two types: *trivially-primitive* (of the form $\{\mathbf{0}, \mathbf{1}\}$) and *homogeneously-primitive*, i.e., homogeneous (by weight, as always). The certificate of each subalgebra consists of the three parts:

$$A = A_0 \cup A'_0 \cup A_1,$$

[16] This property is another form of the conditions [II] and [II$'$].

where A_0 comprises the indices of the form $((n,0),0)$, while A'_0 contains only the indices of the form $((k,1),0)$ and the set A_1 is composed of the elements of the form $((m,i),1)$. What might these sets be if the subalgebra is primitive?

1) Note first that *the set* A_0 *is at most singleton for each primitive subalgebra*. This follows from the fact that $\overline{b}_{(n,0)} = \mathbf{0}, \mathbf{1}$, and all $\overline{b}_{(n,0)}$ are pairwise disjoint by the remark on Theorem 10.

2) For every primitive subalgebra, the set A_1 is always a singleton. Moreover, the two cases are possible:

a) the only element of $h^* \in A_1$ has the form $h^* = ((1,0),1)$;

b) $h^* = ((\tau(\widetilde{\mathscr{X}}),1),1)$.

In case a) the subalgebra $\widetilde{\mathscr{X}}$ is degenerate: $\widetilde{\mathscr{X}} = \overline{\widetilde{\mathscr{X}_0}} = \{\mathbf{0},\mathbf{1}\}$. In case b) $\widetilde{\mathscr{X}}$ is a homogeneous subalgebra of weight $\tau(\widetilde{\mathscr{X}})$.

3) An arbitrary primitive subalgebra may have as A'_0 whatever finite or countable (possibly, empty) set of indices of the form $((k_i,1),0)$, where $k_1, k_2, \dots$ are infinite and pairwise distinct while not exceeding $\tau\mathscr{X}$.

Note that if A_1 is the singleton containing $((1,0),1)$ and we know that $\widetilde{\mathscr{X}}$ is primitive then $\widetilde{\mathscr{X}}$ must be trivially-primitive. However, there are not primitive algebras with the same A_1.

The conditions 1)–3) completely characterize the certificates of primitive algebras. Given a set A satisfying these conditions, we may easily find a normable BA $\mathscr{X}$ and a subalgebra $\widetilde{\mathscr{X}}$ of $\mathscr{X}$ satisfying

$$A[\widetilde{\mathscr{X}}, \mathscr{X}] = A.$$

In the class of primitive subalgebras, $\widetilde{\mathscr{X}}$ is uniquely determined to within isomorphism of subalgebras.

The subalgebras we meet in "concrete" analysis are as rule such that if they fail to be primitive then they decompose into primitive bands. We agree to call an element $u \in \widetilde{\mathscr{X}}^+$ *primitive* provided that the band $\widetilde{\mathscr{X}_u}$ embeds in $\mathscr{X}_u$ as a primitive subalgebra. The meaning is also clear of the terms "trivially-primitive," "homogeneously-primitive," and "certificate" in regard to an element. If $u \in \widetilde{\mathscr{X}}^+$ is representable as a sum of trivially primitive with the same certificate we will call it *primitively-discrete*. Such an element itself is not primitive and its decompositions into trivially-primitive elements are not canonical in view of "indiscernibility" of addends. For instance, if $\mathscr{X}$ is a homogeneous algebra then its unity is primitively-discrete with respect to every regular subalgebra, while all decompositions into atoms are not canonical.

Let a primitively-discrete element u be a sum of m trivially-primitive elements each with the certificate A_0. The certificate of u appears from A_0 by replacing the index $((1,0),1) \in (A^0)_1$ with $((m,0),1)$.

The next propositions about primitive elements are immediate by definition or provable by standard arguments.

1°. *No primitive element may be split by any element* $\widetilde{e}_h$*: either the former is dominated by the latter or it the former disjoint from the latter.*

2°. *Assume that* A *is a set in* H*, and* U_A *is the set of all homogeneously-primitive elements with the certificate* A*. Then* $U_A \cup \{\mathbf{0}\}$ *is solid (in* $\widetilde{\mathscr{X}}$*) and d-regular.*

We leave the simple proof of this fact to the reader (as a pattern take the proof of Propositions 2° and 4° of 9.4.3).

3°. *If* $U_A \neq \varnothing$ *then there is* $u \in \widetilde{\mathscr{X}}^+$ *such that* $U_A \cup \{\mathbf{0}\}\widetilde{\mathscr{X}_u}$. This follows from 2° and the theorem on solid cores (Chapter 2).

4°. *Primitive elements with different certificates are disjoint.* For homogeneously-primitive elements this follows from the solidity of U_A; in other cases it is obvious.

The above immediately implies

Theorem 16. *To each regular subalgebra* $\widetilde{\mathscr{X}} \subset \mathscr{X}$ *there is a unique partition of unity into three summands*

$$\mathbf{1} = x + y + z \quad (x, y, z \in \widetilde{\mathscr{X}}), \tag{22}$$

with the following properties:

1) *the band* $\widetilde{\mathscr{X}_z}$ *has no primitive elements;*

2) *the band* $\widetilde{\mathscr{X}_y}$ *is zero or discrete; its atoms are trivially-primitive;*

3) *if* $x > \mathbf{0}$ *then there is a unique partition*

$$x = x_1 + x_2 + \cdots, \tag{23}$$

where $x_1, x_2, \dots$ *are homogeneously-primitive elements with pairwise distinct certificates; the number of summands in the formula* (23) *is finite or countable;*

4) *if* $y > \mathbf{0}$ *then there is a unique partition*

$$y = y_1^* + y_2^* + \cdots, \tag{23*}$$

where $y_1^*, y_2^*, \dots$ *are primitively-discrete elements with pairwise distinct certificates; the number of summands in the formula* (23) *is finite or countable.*

The partitions (22), (23), and (23*) are related with the subalgebra $\widetilde{\mathscr{X}}$ canonically. The addends x, y, and z in the formula (22) may be equal zero (clearly, not simultaneously).

In the case when $z = \mathbf{0}$, we agree to call $\widetilde{\mathscr{X}}$ *semiprimitive.* An equivalent definition reads as follows: A semiprimitive subalgebra is such that

admits some partition of unity in primitive addends. Part of these addends is given by the formula (23); the remaining result from further splitting the elements y_i^* is the formula (23*). In this event y takes the form

$$y = \sum_n y_n, \tag{24}$$

where $y_1, y_2, \dots$ are trivially-primitive elements, the atoms of $\widetilde{\mathscr{X}}$. Note that this partition is not invariant since atoms are not always discernable and may swap places under automorphisms. Therefore, the partition that is mentioned in the second definition of semiprimitive subalgebra is not canonical in general.

Theorem 17. *For a subalgebra $\widetilde{\mathscr{X}} \subset \mathscr{X}$ to be semiprimitive it is necessary and sufficient that $\overline{\mathscr{X}_0}$ be a discrete subalgebra.*

PROOF. We start with proving NECESSITY. If $\widetilde{\mathscr{X}}$ is semiprimitive then $z > \mathbf{0}$. Denote by u either of x_n, y_n. Clearly, for all $b \in \mathscr{X}_0$ we observe one of the following two cases:

A) $u\, d\, b$,

B) $u \leq b$.

Put

$$B^u \equiv \{b \in \mathscr{X}_0,\ \ b \geq u\}$$

and $c_u \equiv \inf B^u$. Clearly, $c_u \in \overline{\mathscr{X}_0}$ and c_u is an atom of this subalgebra. Since $z = \mathbf{0}$ and

$$\mathbf{1} = \sum_n x_n + \sum_n y_n,$$

all possible c_u comprise a COMPLETE system of atoms; i.e., $\overline{\mathscr{X}_0}$ is a discrete subalgebra.

Now, on assuming $\overline{\mathscr{X}_0}$ discrete, prove that $\widetilde{\mathscr{X}}$ is semiprimitive (SUFFICIENCY).

Take an arbitrary $u \in \widetilde{\mathscr{X}}^+$ and choose $y \in \widetilde{\mathscr{X}}_u^+$ so that the band $\widetilde{\mathscr{X}_y}$ is continuous or consists of a sole atom y. We further find an atom b in $\overline{\mathscr{X}_0}$ such that $y_0 \equiv b \wedge y > \mathbf{0}$. Note that for all $\bar{b}_{(n,i)}$ we have $\bar{b}_{(n,i)} \geq y_0$. Hence, y_0 is a primitive element. From arbitrariness of u and the inequality $\mathbf{0} < y \leq u$ we may conclude that $z = \mathbf{0}$ and $\widetilde{\mathscr{X}}$ is semiprimitive.

Lemma 9. *The discreteness of $\overline{\mathscr{X}_0}$, as well as the semiprimitivity of $\widetilde{\mathscr{X}}$, amounts to the discreteness of the narrower regular subalgebra fully generated by the elements of the form $\bar{b}_{(n,1)}$.*

This follows from the fact that the elements $\bar{b}_{(n,0)}$ are pairwise disjoint by the remark on Theorem 10.

Consider an example. Assume that in a normable separable continuous algebra $\mathscr{X}$ we have distinguished two pairs of homogeneously-primitive elements u_1', u_2' and u_1'', u_2'' such that $u_1', u_2', u_1'', u_2'' > \mathbf{0}$, $u_1' + u_2' + u_1'' + u_2'' = \mathbf{1}$. Assume further that these elements have the following certificates:

$$\begin{aligned} u_1' :& \quad A_1' \equiv \{((1,0),0),\ ((\aleph_0,1),0),\ ((\aleph_0,1),1)\},\\ u_2' :& \quad A_2' \equiv \{((2,0),0),\ ((\aleph_0,1),1)\},\\ u_1'' :& \quad A_1'' \equiv \{((1,0),0),((\aleph_0,1),1)\},\\ u_2'' :& \quad A_2'' \equiv \{((2,0),0),\ ((\aleph_0,1),0),\ ((\aleph_0,1),1)\}. \end{aligned}$$

Let $\widetilde{\mathscr{X}'}_1$, $\widetilde{\mathscr{X}'}_2$, $\widetilde{\mathscr{X}''_1}$, and $\widetilde{\mathscr{X}''_2}$ be the subalgebras of the bands $\mathscr{X}_{u_1'}$, $\mathscr{X}_{u_2'}$, $\mathscr{X}_{u_1''}$, and $\mathscr{X}_{u_2''}$ that correspond to these certificates. Consider another two subalgebras (now in $\mathscr{X}$): $\widetilde{\mathscr{X}'}$, the direct sum of $\widetilde{\mathscr{X}'}_1$ and $\widetilde{\mathscr{X}'}_2$; and $\widetilde{\mathscr{X}''}$, the direct sum of $\widetilde{\mathscr{X}''_1}$ and $\widetilde{\mathscr{X}''_2}$. These subalgebras are semiprimitive and have the same type coincident with the union $A_1' \cup A_2' = A_1'' \cup A_2''$. However, $\widetilde{\mathscr{X}'}$ and $\widetilde{\mathscr{X}''}$ are not isomorphic since they have essentially different (canonical!) decompositions into primitive bands (cf. Fig. 10). Note that these subalgebras are isomorphic individually, since they are continuous, separable, and normable.

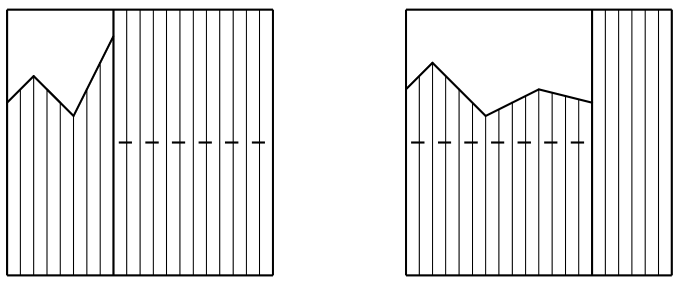

Fig. 10

We see that the semiprimitive subalgebras cannot be classified "by certificates" and require more sophisticated invariants.

Each semiprimitive subalgebra decomposes canonically into the bands of two types: a) homogeneously-primitive and b) primitively-discrete. The bands of the first type have the form $\widetilde{\mathscr{X}_{x_1}}, \widetilde{\mathscr{X}_{x_2}}, \dots$, where x_i are the addends in the formula (22). The primitively-discrete bands are $\widetilde{\mathscr{X}_{y_1^*}}, \widetilde{\mathscr{X}_{y_2^*}}, \dots$, where y_i^* are the addends in the formula (23*). Each primitively-discrete element is a sum of trivially-primitive elements each with the same certificate: the elements y_i^* are maximal in the sense that they result from uniting all y_i (the formula (24)) with the given certificate A_i; we will say that the element y_i^* *belongs* to the certificate

A_i; in this event different y_i^* belong to different certificates. Let A be a nonempty subset of H. The *weight* of this set we call

1) the number 1 provided that among $x_1, x_2, \dots$ there is a homogeneously-primitive nonzero element with certificate A (such an element is unique);

2) the cardinal k provided that among $y_1^*, y_2^*, \dots$ there is a nonzero element (also unique) belonging to the certificate A and consisting of k disjoint trivially-primitive addends (having certificate A);

3) the number 0 in the remaining cases (in particular, when A is not a certificate at all).

Denote the weight of A by $k(A)$. Thus, we have defined on 2^H some cardinal valued function related to the subalgebra which we will call the *weight* function and denote by $k[\widetilde{\mathscr{X}}, \mathscr{X}]$. This function is canonically related to the subalgebra. It uniquely characterizes the structure of the partition of unity into the elements $x_1, x_2, \dots, y_1^*, y_2^*, \dots$. Consider two normable BAs of the same weight $\mathscr{X}'$ and $\mathscr{X}''$. Then the same set H corresponds to each of them.

Theorem 18. *For semiprimitive subalgebras $\widetilde{\mathscr{X}'} \subset \mathscr{X}'$ and $\widetilde{\mathscr{X}''} \subset \mathscr{X}''$ be isomorphic it is necessary and sufficient that the weight functions $k[\widetilde{\mathscr{X}'}, \mathscr{X}']$ and $k[\widetilde{\mathscr{X}''}, \mathscr{X}'']$ coincide.*

As regards NECESSITY, this theorem is as obvious as the previous. Prove SUFFICIENCY. Denote by $\mathfrak{A} \equiv \mathfrak{A}[\widetilde{\mathscr{X}}, \mathscr{X}]$ the set of $A \subset H$ satisfying $k[\widetilde{\mathscr{X}'}, \mathscr{X}'](A) \neq 0$ or, which is the same, $k[\widetilde{\mathscr{X}''}, \mathscr{X}''](A) \neq 0$. The partitions of unity, described in Theorem 16, may be transformed by taking $\mathfrak{A}$ as an index set. Thus,

$$\mathbf{1}_{\mathscr{X}'} = \sum_{A \in \mathfrak{A}} u'_A, \quad \mathbf{1}_{\mathscr{X}''} = \sum_{A \in \mathfrak{A}} u''_A,$$

where the elements u'_A and u''_A are other than zero,, trivially-primitive, and primitively-discrete. In the first case u'_A and u''_A have a common certificate A; in the second, belong to this certificate. Clearly, $\widetilde{\mathscr{X}}_{u'_A}$ and $\widetilde{\mathscr{X}}_{u''_A}$ are isomorphic as subalgebras of $\mathscr{X}'_{u'_A}$ and $\mathscr{X}''_{u''_A}$. The partitions of Theorem 16 are internal with respect to the subalgebras $\widetilde{\mathscr{X}'}$ and $\widetilde{\mathscr{X}''}$. Hence, the "partial" isomorphisms for each $A \in \mathfrak{A}$ may be pasted together into a single isomorphism $\Phi : \mathscr{X}' \longrightarrow \mathscr{X}''$ sending $\widetilde{\mathscr{X}'}$ to $\widetilde{\mathscr{X}''}$. The proof of the theorem is complete.

We now give a few examples showing that the class of semiprimitive subalgebras is rather large.

1. Assume that $\tau(\widetilde{\mathscr{X}}) < \tau(\mathscr{X})$, and $\mathscr{X}$ is homogeneous. In this event the degree of nonsaturation of each band may equal only to $\tau(\mathscr{X})$

whereas the saturation is impossible in principle. Therefore, the external standard partition consists of the sole nonzero element $b_{(\tau(\mathscr{X}),1)}$. The subalgebra $\overline{\mathscr{X}_0}$ is trivial, while the subalgebra $\widetilde{\mathscr{X}}$ is semiprimitive by Theorem 17.

2. Assume that $\tau(\widetilde{\mathscr{X}}) \leq \tau(\mathscr{X}) \leq \aleph_k$ and k is a natural number. Only finitely many elements $\bar{b}_{(m,1)}$ are now available and the subalgebra they generate it clearly discrete. By Lemma 9 the subalgebra $\widetilde{\mathscr{X}}$ is semiprimitive.

The second example includes all separable algebras which are of the utmost interest for mathematical analysis and probability theory. Separable algebras occur in the isomorphism problem for measurable functions which we have mentioned in the beginning of this subsection. Note in passing that always separable are the subalgebras $\overline{\mathscr{X}_0}$ playing the key role in the general scheme.

In regard to homogeneous algebras, the case of $\tau(\widetilde{\mathscr{X}}) = \tau\mathscr{X} \geq \aleph_\omega$ is the only one in which Theorem 13 is applied to classification in full strength. In the other cases, as we have seen, it is possible to deal with simpler invariants and avoid recalling the algebraic type.

4.5 Isomorphism of Boolean measures and random variables (measurable functions)

Assume given two NBAs $\{\mathscr{X}_1, \mu_1\}$ and $\{\mathscr{X}_2, \mu_2\}$ together with a σ-algebra of sets Σ of some space T. (These μ_1 and μ_2 are probability measures.) Suppose that there are two σ-epimorphisms Ψ_1 and Ψ_2 from Σ to $\mathscr{X}_1$ and $\mathscr{X}_2$ respectively. We pose the following question: Under which conditions is there an isomorphism $\Phi : \mathscr{X}_1 \longrightarrow \mathscr{X}_2$ sending Ψ_1 to Ψ_2, i.e., such that the diagram commutes

or, in other words, the equality holds $\Psi_2 = \Phi \circ \Psi_1$? When does there exist a measure preserving isomorphism with this property? ("Measure preservation" means that $\mu_2 = \mu_1 \circ \Phi^{-1}$.)

Put $I_1 \equiv \ker \Psi_1$ and $I_2 \equiv \ker \Psi_2$. We have two σ-ideals in the same σ-algebra Σ. Introduce the canonical isomorphisms:

$$\widehat{\Psi}_1 : \Sigma|_{I_1} \longrightarrow \mathscr{X}_1, \quad \widehat{\Psi}_2 : \Sigma|_{I_2} \longrightarrow \mathscr{X}_2.$$

If

$$e_i \in \Sigma, \quad \widehat{e_i} \equiv e_i +_2 I_i \in \Sigma|_{I_i} \quad (i = 1, 2)$$

then $\Psi_i(e_i) = \widehat{\Psi}_i(\widehat{e_i})$. Consider the equality

$$I_1 = I_2. \tag{25}$$

Theorem 19. 1) *The equality* (25) *is necessary and sufficient for existence of an isomorphism* $\Phi : \mathscr{X}_1 \longrightarrow \mathscr{X}_2$ *sending* Ψ_1 *to* Ψ_2.

2) *There exists a measure preserving isomorphism* $\Phi : \mathscr{X}_1 \longrightarrow \mathscr{X}_2$ *sending* Ψ_1 *to* Ψ_2 *if and only if the equality*

$$\mu_2 \Psi_2(e) = \mu_1 \Psi_1(e) \tag{26}$$

holds for all $e \in \Sigma$.

In this event it suffices that the equality (26) *holds for all* e *belonging to some semiring of sets which* σ*-generates* Σ.

Proof. If such an isomorphism Φ exists then

$$I_1 \equiv \ker \Psi_1 = \ker(\Phi^{-1} \circ \Psi_2) = \ker \Psi_2 \equiv I_2,$$

because $\Phi^{-1}(\mathbf{0}) = \mathbf{0}$. In the case when Φ is measure preserving, for all e we also have the equality (26)

$$\mu_2 \Psi_2(e) = \mu_2 \Phi(\Psi_1(e)) = \mu_1 \Psi_1(e).$$

Assume now that $I_1 = I_2 = I$ and construct an isomorphism Φ. We do this by means of the equality

$$\Phi \equiv \widehat{\Psi}_2 \circ \widehat{\Psi}_1^{-1}.$$

This is an isomorphism from $\mathscr{X}_1$ onto $\mathscr{X}_2$ (since $\widehat{\Psi}_1$ and $\widehat{\Psi}_2$ are isomorphisms). By definition, $\widehat{\Psi}_2 = \Phi \circ \widehat{\Psi}_1$. Given $\mathscr{E} \in \Sigma$, we have

$$\Psi_2(e) = \widehat{\Psi}_2(\widehat{e}) = \Phi(\widehat{\Psi}_1(\widehat{e})) = \Phi(\Psi_1(e))$$

($\widehat{e} \equiv e +_2 I$). It is clear that we have constructed a sought isomorphism. If (26) holds then for all x we have

$$\mu_2 \Phi(x) = \mu_2 \widehat{\Psi}_2(\widehat{\Psi}_1^{-1}(x)) = \mu_2 \Psi_2(e),$$

where e is an arbitrary representative of the coset $\widehat{\Psi}^{-1}(x)$, i.e., $e \in \Psi^{-1}(x)$. By (26) we have

$$\mu_2\Psi_2(e) = \mu_1\Psi_1(e) = \mu_1 x.$$

Hence, $\mu_2\Phi(x) = \mu_1(x)$ for all $x \in \mathscr{X}_1$.

If $K \subset \Sigma$ is a semiring of sets σ-generating Σ then it is clear that the equality (26), holding for all $e \in K$, must hold also on the whole σ-algebra Σ because Ψ_1 and Ψ_2 are σ-continuous epimorphisms. The proof of the theorem is complete.

In place of a σ-algebra of sets there might appear whatever σ-complete BA in the role of Σ.

Among Boolean measures (homomorphisms given on algebras of sets) the most important are the spectral measures generated by resolutions of the identity (cf. p. 290). Theorem 19 relates also to these homomorphisms; in particular, it applies to the above-mentioned classification problem for measurable functions or, in other words, random variables.

A measurable function *mod* 0 is as was emphasized many times before the same as a resolution of the identity of an NBA $\{\mathscr{X}, \mu\}$ the metric structure of some measure space. Two of such resolutions $\mathfrak{f}'$ and $\mathfrak{f}''$ are algebraically (metrically) isomorphic if there is an algebraic (metric) isomorphism Φ sending one resolution to the other:

$$\Phi(e_\lambda^\pm(\mathfrak{f}')) = e_\lambda^\pm(\mathfrak{f}'') \quad (-\infty \le \lambda \le +\infty).$$

It is convenient for us now to assume that $\mathfrak{f}'$ and $\mathfrak{f}''$ are resolutions of the identity in DIFFERENT BAs $\mathscr{X}'$ and $\mathscr{X}''$. Clearly, the isomorphism Φ must also possess the property

$$\Phi(\widetilde{\mathscr{X}'}) = \widetilde{\mathscr{X}''},$$

where $\widetilde{\mathscr{X}'}$ and $\widetilde{\mathscr{X}''}$ are regular subalgebras fully generated by the spectral families $e_\lambda^\pm(\mathfrak{f}')$ and $e_\lambda^\pm(\mathfrak{f}'')$ respectively. Therefore, the problem of constructing an isomorphism Φ splits into two: we must first construct an isomorphism Φ_0 between subalgebras and then we have to extend it to an isomorphism $\Phi : \mathscr{X}' \longrightarrow \mathscr{X}''$. This plan relates to the problem of algebraic isomorphism as well as to its metric counterpart. We start with the "algebraic" case.

We assume the BAs $\mathscr{X}'$ and $\mathscr{X}''$ normable; we also need some probability measures μ' and μ'' on these algebras. The resolutions of the identity $\mathfrak{f}'$ and $\mathfrak{f}''$ generate the spectral measures Ψ' and Ψ'' on the Borel σ-algebra $\mathscr{B}$ of the real axis. As a σ-generating semiring mentioned in Theorem 19, we take the system of all segments. The forthcoming theorem is immediate from the preceding.

Theorem 20. *The equality*

$$\ker \Psi' = \ker \Psi'' \tag{27}$$

is necessary and sufficient for existence of an individual isomorphism $\Phi_0 : \widetilde{\mathscr{X}}' \longrightarrow \widetilde{\mathscr{X}}''$ *sending* $\mathfrak{f}'$ *to* $\mathfrak{f}''$. *For existence of a measure preserving isomorphism* Φ_0, *i.e. such that*

$$\mu''\Phi_0(x) = \mu'\Phi_0(x) \quad (x \in \mathscr{X}'),$$

it is necessary and sufficient that

$$\mu'(e_\lambda^-(\mathfrak{f}')) = \mu''(e_\lambda^-(\mathfrak{f}'')) \tag{28}$$

for all λ.

Recall that

$$e_\lambda^-(\mathfrak{f}') \equiv \Psi'(\Delta_\lambda^-),\ \ e_\lambda^-(\mathfrak{f}'') \equiv \Psi''(\Delta_\lambda^-),\ \ \mathscr{X}' = \Psi'(\mathscr{B}),\ \ \mathscr{X}'' = \Psi''(\mathscr{B}).$$

Clearly, we may replace (28) with the following

$$\mu'(e_\lambda^+(\mathfrak{f}')) = \mu''(e_\lambda^+(\mathfrak{f}'')). \tag{28*}$$

Consider two Lebesgue–Stieltjes "measures" m' and m'' on the Borel algebra $\mathscr{B}$ which are generated by the respective distribution functions

$$F'(\lambda) \equiv \mu'(e_\lambda^-(\mathfrak{f}')), \quad F''(\lambda) \equiv \mu''(e_\lambda^-(\mathfrak{f}'')).$$

In other words, $m' \equiv \mu' \circ \Psi'$ and $m'' \equiv \mu'' \circ \Psi''$. The condition (27) means exactly that m' and m'' are equivalent (absolutely continuous with respect to one another). The equality (28) proclaims their coincidence.

However, we are interested not only in the existence of an individual isomorphism but also in its EXTENDIBILITY to an isomorphism between $\mathscr{X}'$ and $\mathscr{X}''$. For this purpose, the equivalence and even coincidence of m' and m'' are insufficient.

Assume again given a resolution of the identity $\mathfrak{f}$ and the regular subalgebra $\widetilde{\mathscr{X}} \subset \mathscr{X}$ it generates.[17] Consider the external standard partition $\{b_t\}$ and relate to each t the Hellinger type (metric type) $\rho_t(\mathfrak{f}) \equiv \rho_t$ corresponding to the distribution function

$$F_t(\lambda) = \mu(e_\lambda^-(\mathfrak{f}) \wedge b_t).$$

If $b_t = \mathbf{0}$ then the type ρ_t is zero by definition. Clearly, each type ρ_t is invariant under isomorphisms of resolutions of the identity. The entire

[17] This means that the subalgebra is generated by the set of the elements of the form $e_\lambda^\pm(\mathfrak{f})$.

collection $\{\rho_t\}$ is precisely the complete system of invariants which is canonically related to $\mathfrak{f}$. This system is independent of the choice of μ. If we are given two BAs $\mathscr{X}'$ and $\mathscr{X}''$ and two resolutions of the identity $\mathfrak{f}'$ and $\mathfrak{f}''$ of these algebras then we denote the corresponding types by ρ_t' and ρ_t'' and the standard partitions, by $\{b_t'\}$ and $\{b_t''\}$.

Theorem 21. *Assume that $\mathscr{X}'$ and $\mathscr{X}''$ are normable algebras of the same weight and $\mathfrak{f}'$ and $\mathfrak{f}''$ are resolutions of the identity of these algebras. For these resolutions to be algebraically isomorphic it is necessary and sufficient that for all $t \in T(\mathscr{X}') \equiv T(\mathscr{X}'')$ the equality holds*

$$\rho_t(\mathfrak{f}') = \rho_t(\mathfrak{f}''). \tag{29}$$

As before, it suffices to prove only the sufficiency part. By Theorem 20 to each t there is an individual isomorphism $\Phi_t^{-1} : [\widetilde{\mathscr{X}'}]_{b_t'} \longrightarrow [\widetilde{\mathscr{X}''}]_{b_t''}$. (It is clear that under the condition (29) the equalities $b_t' = \mathbf{0}$ and $b_t'' = \mathbf{0}$ may hold only simultaneously.) These isomorphisms are such that

$$\Phi_t^{-1}(e_\lambda^-(\mathfrak{f}') \wedge b_t') = e_\lambda^-(\mathfrak{f}'') \wedge b_t''$$

for all λ and t.

Consider the regular subalgebras $\widetilde{\widetilde{\mathscr{X}'}}$ and $\widetilde{\widetilde{\mathscr{X}''}}$. Given $x \in \widetilde{\widetilde{\mathscr{X}'}}$, put $\Phi_1(x) \equiv \sum_t \Phi_t^1(x \wedge b_t')$.[18] It is easy to check that Φ_1 is an isomorphism of $\widetilde{\widetilde{\mathscr{X}'}}$ onto $\widetilde{\widetilde{\mathscr{X}''}}$. Moreover, for all λ we have

$$\Phi_1(e_\lambda^-(\mathfrak{f}')) = \sum e_\lambda^-(\mathfrak{f}'') \wedge b_t'' = e_\lambda^-(\mathfrak{f}''),$$

and

$$\Phi_1(b_t') = b_t''$$

for all t. Hence, $\Phi_1(\widetilde{\mathscr{X}'}) = \widetilde{\mathscr{X}''}$ and we may apply Theorem 12. The isomorphism Φ_1 extends to an isomorphism $\Phi : \mathscr{X}' \longrightarrow \mathscr{X}''$, and the resolutions of the identity $\mathfrak{f}'$ and $\mathfrak{f}''$ are algebraically isomorphic.

Therefore, intending to define a measurable *mod* 0 function to within an algebraic isomorphism, we must prescribe the collection of distribution functions $\{F_t\}$. Sometimes this collection may be replaced with a single function. This is so if all b_t equal to $\mathbf{0}$ for all except for the sole element $b_{(\tau(\mathscr{X}),1)}$ which equals 1. This case must happen if $\mathscr{X}$ is a homogeneous algebra of uncountable weight (recall that the subalgebra $\widetilde{\mathscr{X}}$ is always separable). Thus, we have proved the following

---

[18]If $x \in \widetilde{\widetilde{\mathscr{X}'}}$ then $x \wedge b_t' \in [\widetilde{\mathscr{X}'}]_{b_t'}$.

Theorem 22. *For resolutions of the identity $\mathfrak{f}'$ and $\mathfrak{f}''$ of homogeneous BAs $\mathscr{X}$ and $\mathscr{X}''$ satisfying the condition*

$$\tau(\mathscr{X}') = \tau(\mathscr{X}'') > \aleph_0$$

to be algebraically isomorphic it is necessary and sufficient that the distribution functions

$$F'(\lambda) = \mu'(e_\lambda^-(\mathfrak{f}')), \quad F''(\lambda) = \mu''(e_\lambda^-(\mathfrak{f}''))$$

generate equivalent Lebesgue–Stieltjes "measures."

(Here, as elsewhere, μ' and μ'' stand for some distinguished measures on $\mathscr{X}'$ and $\mathscr{X}''$.)

Thus, in the case of uncountable weight the isomorphism problem for measurable functions is solved in a simpler fashion. The reason behind this is clear: it is easier to extend an individual isomorphism between separable subalgebras in the case of uncountable weight. ("There is a room to extend.")[19]

We now discuss the application of Theorem 22 to separable BAs. In this event we may assume that $\mathscr{X}'$ and $\mathscr{X}''$ are the metric structures of some Lebesgue–Rokhlin spaces. We also assume that $\mathscr{X}' = \mathscr{X}''$.

Let $\{\Omega, \mathscr{E}, m\}$ be a Lebesgue–Rokhlin space, and let $\{\mathscr{X}, \mu\}$ be its metric structure (i.e., $\widehat{\mathscr{E}} = \mathscr{X}$, $\widehat{m} = \mu$). The resolution of the identity $\mathfrak{f}$ is generated by some measurable function f. To the subalgebra $\widetilde{\mathscr{X}}$ there corresponds a measurable partition whose elements are the level sets of f (cf. 7.5.7). The elements of the standard partition b_t also may be interpreted in terms of the function f. Namely, each element $b_{(n,0)}$ is the coset $\widehat{E}_{(n,0)}$, where $E_{(n,0)}$ is the "MAXIMAL *mod* 0" measurable set with the property that each value of f at a point of $E_{(n,0)}$ is attained in fact at exactly n points. The sets $E_{(n,0)}$ may be chosen to be pairwise disjoint; as $E_{(\aleph_0,1)}$ we may take the complement to all $E_{(n,0)}$. Then $\widehat{E}_{(\aleph_0,1)} = b_{(\aleph_0,1)}$. For $k > \aleph_0$ we have $b_{(k,1)} = \mathbf{0}$.

Clearly, the functions F_t, playing the key role in classification of subalgebras, may be determined by the formulas

$$F_{(n,i)}(\lambda) = m(E^\lambda \cap E_{(n,i)}),$$

where $E^\lambda \equiv \{\omega \in \Omega \mid f(\omega) < \lambda\}$, $\lambda \in [-\infty, +\infty]$.

The metric types of the corresponding Lebesgue–Stieltjes "measures" comprise a complete system of invariants by means of which we may

[19] D. A. Vladimirov and A. A. Samorodnitskiĭ [1].

classify measurable functions. More exactly, if these systems coincide for two functions f_1 and f_2 then there is an automorphism Φ of the metric structure $\widehat{\mathscr{E}} = \mathscr{X}$ such that

$$\Phi(e^{\pm}_{\lambda}(\mathfrak{f}_1)) = e^{\pm}_{\lambda}(\mathfrak{f}_2).$$

Using the particular features of Lebesgue–Rokhlin spaces, to this isomorphism we may find a "pointwise" measurable one-to-one mapping φ of the space Ω which is connected with Φ by the conditions

$$\Phi(\widehat{e}) = \widehat{\varphi^{-1}(e)}$$

and such that the function f_2 coincides m-almost everywhere with $f_1 \circ \varphi$.

As was mentioned, the classification problem for measurable functions on Lebesgue–Rokhlin spaces was posed and solved by V. A. Rokhlin. Prior to this, he also gave a complete classification for measurable partitions of Lebesgue–Rokhlin spaces or, which is equivalent, of σ-subalgebras. In the articles by V. A. Rokhlin there was considered the measure preserving pointwise transformations of Lebesgue–Rokhlin spaces corresponding to metric isomorphisms. We will address such a problem somewhat later.

The same Theorem 19 may be applied to spectral homomorphisms of a more general form. Namely, we may pass from an individual random variables (resolutions of the identity) to RANDOM PROCESSES, the families of the form $\{\mathfrak{f}_\xi\}_{\xi\in\Xi}$. We will consider the corresponding two-parameter families

$$\{e^{(\xi)}_{\lambda} \equiv e^{-}_{\lambda}(\mathfrak{f}_\xi)\}_{\xi\in\Xi}, \quad \lambda \in [-\infty, +\infty]. \tag{30}$$

Consider some (infinite, in general) cartesian product of the extended real axis:

$$\mathbf{R} \equiv \mathbf{R}_\xi \equiv (\overline{\mathbb{R}})^{\xi} \equiv \{r \mid r = \{r_\xi\}_{\xi\in\Xi},\ r_\xi \in \overline{\mathbb{R}}\}.$$

Recall that the STANDARD HALF-SPACES in $\mathbf{R}$ are given by conditions of the form

$$L^{\lambda-}_{\xi} \equiv \{r \mid r_\xi < \lambda\}. \tag{31}$$

We denote the σ-algebra they generate by $\mathbf{B}^{[\xi]} \equiv \mathbf{B}$.[20]

Using Theorem 5* of Chapter 6, construct the σ-homomorphism Ψ : $\mathbf{B} \longrightarrow \mathscr{X}$ that is determined for the sets (31) (for $\Delta_i \equiv (-\infty, \lambda)$) from the formula

$$\Psi(L^{\lambda-}_{\xi}) \equiv e^{(\xi)}_{\lambda}. \tag{32}$$

[20]This σ-algebra also contains all $L^{\lambda+}_{\xi}$.

The image $\Psi(\mathbf{B}) \equiv \widetilde{\mathscr{X}}$ is a regular subalgebra of $\mathscr{X}$ fully generated by the elements $e_\lambda^{(\xi)}$. This also gives rise to the countably additive "measure" m on $\mathbf{B}$:

$$m(e) \equiv \mu\Psi(e).$$

(This "measure" may be obtained also by using the celebrated Kolmogorov Theorem on the simultaneous extension of consistent distributions.)

Assume now that we are given two families, two random processes $\{\mathfrak{f}'_\xi\}_{\xi\in\Xi}$ and $\{\mathfrak{f}''_\xi\}_{\xi\in\Xi}$ (with the same parameter set ξ). To them there correspond the homomorphisms Ψ' and Ψ'' as well as the "measures" m' and m''. Let the regular subalgebras $\widetilde{\mathscr{X}}'$ and $\widetilde{\mathscr{X}}''$ be generated respectively by the families

$$\{(e_\lambda^{(s)})' \equiv e_\lambda^-(\mathfrak{f}'_\xi)\} \quad \text{and} \quad \{(e_\lambda^{(s)})'' \equiv e_\lambda^-(\mathfrak{f}''_\xi)\}.$$

Theorem 23. *For the existence of an individual isomorphism $\Phi_0 : \widetilde{\mathscr{X}}' \longrightarrow \widetilde{\mathscr{X}}''$ sending one processes to the other it is necessary and sufficient that the "measures" m' and m'' be absolutely continuous with respect to one another; in the case of coincidence of m' and m'', such an isomorphism may be chosen measure preserving.*

For all $\xi_1, \xi_2, \ldots, \xi_s$ and $\lambda_1, \lambda_2, \ldots, \lambda_s$ the following holds

$$\begin{aligned} &\Psi''(L_{\xi_1}^{\lambda_1} \cap \cdots \cap L_{\xi_s}^{\lambda_s}) = (e_{\lambda_1}^{(\xi_1)})'' \wedge \cdots \wedge (e_{\lambda_s}^{(\xi)})'' \\ &= \Phi_0((e_{\lambda_1}^{(\xi_1)})' \wedge \cdots \wedge (e_{\lambda_s}^{(\xi_s)})') = \Phi_0(\Psi'(L_{\xi_1}^{\lambda_1} \cap \cdots \cap L_{\xi_s}^{\lambda_s})). \end{aligned} \tag{33}$$

The proof reduces to appealing to Theorem 19. The condition $m' = m''$ may be replaced with an equivalent but seemingly weaker condition, the simultaneous equalities

$$\mu'((e_{\lambda_1}^{(\xi_1)})' \wedge \cdots \wedge (e_{\lambda_s}^{(\xi_s)})') = \mu''((e_{\lambda_1}^{(\xi_1)})'' \wedge \cdots \wedge (e_{\lambda_s}^{(\xi_s)})'').$$

The problem of extending an individual isomorphism to an "actual" isomorphism is settled in much the same way as in the case of a single spectral family. Namely, we introduce the families $\{\rho'_t\}$ and $\{\rho''_t\}$ of metric types which correspond to the elements of the external standard partitions $\{b'_t\}$ and $\{b''_t\}$. As representatives of the types ρ'_t and ρ''_t we may take the functions

$$m'_t : \quad m'_t(e) \equiv \mu'(\Psi'(e) \wedge b'_t),$$

$$m''_t : \quad m''_t(e) \equiv \mu''(\Psi''(e) \wedge b''_t),$$

where $e \in \mathbf{B}$, μ' and μ'' are arbitrarily chosen probability measures on $\mathscr{X}'$ and $\mathscr{X}''$.

The next theorem differs slightly from Theorem 21.

Theorem 24. *Let $\mathscr{X}'$ and $\mathscr{X}''$ be normable BAs of the same weight with given resolutions of the identity $\{\mathfrak{f}'_\xi\}_{\xi\in\Xi}$ and $\{\mathfrak{f}''_\xi\}_{\xi\in\Xi}$ of these algebras. For existence of an algebraic isomorphism $\Phi : \mathscr{X}' \longrightarrow \mathscr{X}''$, satisfying*

$$\Phi(e^{\pm}_{\lambda_1}(\mathfrak{f}'_{\xi_1}) \wedge \cdots \wedge e^{\pm}_{\lambda_s}(\mathfrak{f}'_{\xi_s})) = e^{\pm}_{\lambda_1}(\mathfrak{f}''_{\xi_1}) \wedge \cdots \wedge e^{\pm}_{\lambda_s}(\mathfrak{f}''_{\xi_s})$$

for all $\lambda_1, \lambda_2, \dots, \lambda_s \in \overline{\mathbb{R}}$ and $\xi_1, \xi_2, \dots, \xi_s \in \xi$ it is necessary and sufficient that

$$\rho'_t = \rho''_t \tag{34}$$

for all $t \in T(\mathscr{X}') = T(\mathscr{X}'')$.

It might seem that, replacing the condition (34) in Theorem 24 with a stronger condition, the simultaneous equalities $m'_t = m''_t$, we would obtain a test for existence of a measure preserving isomorphism. Unfortunately, this is not so: we can obviously make an individual isomorphism Φ_0 measure preserving under this stronger condition; however, its further extension may fail to be measure preserving (as in Theorem 21). We now address this question in more detail; moreover, we will obtain some tests for algebraic isomorphism which differ from those listed above.

4.6 Metric isomorphism between subalgebras

The problem of metric isomorphism between subalgebras was already discussed in 9.4.5 (Theorems 20 and 23). It is much more complicated that the problem of conditions of algebraic isomorphism. For a separable algebra, this problem was considered by V. A. Rokhlin, as was mentioned above. Even in this simplest case the conditions for existence of a measure preserving isomorphism are rather bulky. It is clear that the coincidence of algebraic types is necessary for existence of a metric isomorphism between two subalgebras, since metrically isomorphic subalgebras must have the same type. Example 6 of 9.4.3 shows that neither coincidence of algebraic types nor coincidence of “measures” representing these types guarantees the existence of a metric isomorphism (in this example the type consists of a sole delta-function irrespectively of the measures of the elements b' and b''). It is in any case clear that the existence of metric isomorphism between the subalgebras $\widetilde{\mathscr{X}'} \subset \mathscr{X}'$ and $\widetilde{\mathscr{X}''} \subset \mathscr{X}''$ implies the equalities

$$\mu' e'_h = \mu'' e''_h$$

for all h, but this condition is far from sufficient.

The elements $e_{((n,0),0)} \equiv b_{(n,0)}$ admit a "secondary" partition in saturated addends of the form

$$b_{(n,0)} = \sum_{q \in Q_{(n,0)}} u_q \quad (\overline{u}_q = \overline{b}_{(n,0)},\ \ card\, Q_{(n,0)} = n);$$

however, the elements of these secondary partitions are not determined uniquely. This circumstance drives us to formulate all conditions in terms of $b_{(n,0)}$ rather than u_q. However, although the decomposition into saturated addends is not unique and "noncanonical," the collection of CONDITIONAL measures of these elements is related to the subalgebra canonically as we will demonstrate right away.

The next theorem in a "separable" version belongs in actuality to V. A. Rokhlin.[21] A more general case was settled by A. A. Samorodnitskiĭ [1].

Theorem 25. *Among the elements saturated by $\widetilde{\mathscr{X}}$ there is an element of greatest measure.*

PROOF. Denote by α_u the supremum of the measures of saturated elements in the band $\mathscr{X}_u$, and let $\mathbf{C}$ stand for the set of $u \in \widetilde{\mathscr{X}}$ for which this supremum is attained; i.e., has the greatest value.

Show that $\mathbf{C} = \widetilde{\mathscr{X}}$. Note first that $\mathbf{C}$ is *solid* and *d-regular.* Check solidity. Let $w \in \mathbf{C}, w' < w, w' \in \widetilde{\mathscr{X}}^+$. There is a saturated element $u \in \mathscr{X}_w$ satisfying $\mu u = \alpha_w$. Suppose that $\mu(u \wedge w') < \alpha_{w'}$. Then there is a saturated $v \in \mathscr{X}_{w'}$ satisfying $\mu v > \mu(u \wedge w')$. We now put $u_1 \equiv v + u \wedge (w - w')$. The addends of this sum are separated from one another by elements of the subalgebra; since each is saturated by itself then so is u_1. But then

$$\mu u_1 = \mu v + \mu(u \wedge (w - w')) > \mu(u \wedge w') + \mu(u \wedge (w - w')) = \mu(u \wedge w) = \alpha_w,$$

which is impossible. Thus, $\mu(u \wedge w') = \alpha_{w'}$, i.e., $w' \in \mathbf{C}$. The solidity of $\mathbf{C}$ is validated. Show now that $\mathbf{C}$ is d-regular. Let $w = \sum_n w_n$ and $w_n \in \mathbf{C}$. To each $n = 1, 2, \dots$ there is a saturated element $u_n \mathscr{X}_{w_n}$ satisfying $\mu u_n = \alpha_{w_n}$. The element $u \equiv \sum_n u_n$ is also saturated (the addends of the sum are pairwise separated by the elements of $\widetilde{\mathscr{X}}$). We have $\alpha_w \geq \mu u = \sum_n \mu u_n = \sum_n \alpha_{w_n}$. However, it is easy to see that $\alpha_w \leq \sum_n \alpha_{w_n}$ in any case, implying that $\alpha_w = \mu u,\ w \in \mathbf{C}$.

[21] V. A. Rokhlin [1]. This article discusses the "one-stratum" sets with respect to a measurable partition. The corresponding "Boolean" concept of "saturated band" was introduced later (D. A. Vladimirov [7]).

We now use the corollary to Theorem 2.4 and the remark on it. There is a unique decomposition of $\widetilde{\mathscr{X}}$ into disjoint bands $\widetilde{\mathscr{X}_1}$ and $\widetilde{\mathscr{X}_2}$ such that $\mathbf{C} = \widetilde{\mathscr{X}_1}$ and $\widetilde{\mathscr{X}_2} \cap \mathbf{C} = \{\mathbf{0}\}$. We are to prove that the second band $\widetilde{\mathscr{X}_2}$ consists of the sole zero. Assume to the contrary that $\widetilde{\mathscr{X}_2} \equiv \widetilde{\mathscr{X}_w}$, $w > \mathbf{0}$. To refute this, we need a series of lemmas. In these lemmas by w we always mean the just-defined element of $\widetilde{\mathscr{X}}^+$, the greatest element disjoint from $\mathbf{C}$.

Lemma 10. *To each saturated element $u \in \mathscr{X}_w^+$ there is another saturated element $u' \in \mathscr{X}_w$ with the property: if $v \in \widetilde{\mathscr{X}_w^+}$ then $\mu(v \wedge u') > \mu(v \wedge u)$.*

Proof. We split the proof in several steps.

I. Show first that *to each $v \in \widetilde{\mathscr{X}}_w^+$ there are $\overline{v} \in \widetilde{\mathscr{X}}_v^+$ and a saturated element $u_{\overline{v}} \in \mathscr{X}_{\overline{v}}^+$ satisfying*

$$u_{\overline{v}} \, d \, u, \quad \mu u_{\overline{v}} > \mu(u \wedge \overline{v}).$$

Indeed, from the main property of the band $\widetilde{\mathscr{X}_w}$ it follows that $\alpha_v > \mu(u \wedge v)$. There is a saturated element $u_1 \in \mathscr{X}_v$ satisfying $\mu u_1 > \mu(u \wedge v)$. The elements u_1 and $u \wedge v$ are saturated, and so there is a decomposition $v = v_1 + v_2$ such that $v_1, v_2 \in \widetilde{\mathscr{X}}$, $u - 1 \wedge v_1 = u \wedge v_1$, $(u_1 \wedge v_2) \, d \, (u \wedge v_2)$. Clearly, $\mu(u_1 \wedge v_2) > \mu(u \wedge v_2)$. It suffices to take $\overline{v} \equiv v_2$, $u_{\overline{v}} \equiv u_1 \wedge v_2$.

Denote by $\mathscr{D}$ the set of all $\overline{v} \in \widetilde{\mathscr{X}}$ such that there is a saturated element $u_{\overline{v}} \leq \overline{v}$ satisfying the conditions: $u_{\overline{v}} \, d \, u$, $\mu u_{\overline{v}} > \mu(u \wedge \overline{v})$. We have just checked that $\mathscr{D}$ is a nonempty minorant for $\widetilde{\mathscr{X}_w}$.

II. Take an arbitrary $w' \in \widetilde{\mathscr{X}}_w^+$ and, using the exhaustion principle, represent this element as the sum

$$w' = \sum_n \overline{v}_n \quad (\overline{v}_n \in \mathscr{D})$$

(this sum is finite or countable).

Put $u_{w'} \equiv \sum_n u_{\overline{v}_n}$. The addends in this sum are pairwise separated from one another by elements of the subalgebra; hence, $u_{w'}$ is saturated. Given n, note that $\mu u_{\overline{v}_n} > \mu(u \wedge \overline{v}_n)$. Hence, $\mu u_{w'} > \mu(u \wedge w')$. Thus, to each $w' \in \widetilde{\mathscr{X}}_w^+$ there corresponds some saturated element $u_{w'} \leq w'$, such that $\mu u_{w'} > \mu(u \wedge w')$ (the axiom of choice enables us to consider the FAMILY $\{u_{w'}\}_{w' \in \widetilde{\mathscr{X}}_w^+}$). Moreover, $u_{w'} \, d \, u$.

III. Introduce the sets

$$A_{w'} \equiv \{z \in \widetilde{\mathscr{X}_{w'}} \mid \mu(u_{w'} \wedge z) > \mu(u \wedge z)\},$$

$$A'_{w'} \equiv \widetilde{\mathscr{X}_{w'}} \setminus A_{w'} \quad (w' \in \widetilde{\mathscr{X}}_w^+).$$

These sets are both d-regular; moreover, $A_{w'}^{+}$ is nonempty. Hence (Theorem 2.5), the set $((A_{w'})^0)^+$ is nonempty too. Moreover (Theorem 2.4), the two sets $A_{w'}^0$ and $(A'_{w'})^0$ are complementary bands. The first of them is clearly nonzero. Put $q_{w'} \equiv \sup A_{w'}^0$.

The set Q of all $q_{w'}$ $(w' \in \widetilde{\mathscr{X}}_w^+)$ minorizes $\widetilde{\mathscr{X}_w}$, since we have shown that each nonzero band $\widetilde{\mathscr{X}_{w'}}$ $(w' \le w)$ includes some nonzero band of the form $A_{w'}^0 \equiv \widetilde{\mathscr{X}_{q_w}}$.

IV. Using the exhaustion principle again, arrange the decomposition

$$w = \sum_n q_n,$$

where $q_n \equiv q_{w'_n}$ are pairwise disjoint and other than zero while $\{w'_n\}$ is at most countable family in $\widetilde{\mathscr{X}_w}$. The main property of this decomposition is as follows: *if $z \in \widetilde{\mathscr{X}}_{q_n}^+$ then $\mu(u_{w'} \wedge z) > \mu(u \wedge z)$ $(n = 1, 2, \ldots)$.*

V. Finally, we put

$$u' \equiv \sum_n (u_{w'_n} \wedge q_n).$$

The addends of this sum are saturated; they are separated by the elements of $\widetilde{\mathscr{X}}$ and so u' is saturated. Furthermore, given $z \in \widetilde{\mathscr{X}}_w^+$, we have

$$\mu(u' \wedge z) = \sum_n \mu(u'_{w'_n} \wedge q_n \wedge z) > \sum_n \mu(u \wedge z \wedge q_n)$$

$$= \mu \sum_n (u \wedge z \wedge q_n) = \mu((u \wedge z) \wedge \sum_n q_n) = \mu(u \wedge z \wedge w) = \mu(u \wedge z).$$

The proof of the lemma is complete.

Lemma 11. *If $u_1, u_2 \in \mathscr{X}_w$ are saturated elements such that $\mu(z \wedge u_2) > \mu(z \wedge u_1)$ for all $z \in \widetilde{\mathscr{X}}_w^+$ then they are disjoint: $u_1 \, d \, u_2$.*

Proof. There is a decomposition $w = w_1 + w_2$ satisfying

$$w_1, w_2 \in \widetilde{\mathscr{X}}, \;\; u_1 \wedge w_1 = u_2 \wedge u_1, \;\; (u_1 \wedge w_2) \, d \, (u_2 \wedge w_2).$$

Clearly, $w_1 = \mathbf{0}$ since otherwise we would have $\mu(u_2 \wedge w_1) = \mu(u_1 \wedge w_1)$, $w_1 \in \widetilde{\mathscr{X}}_w^+$, which is impossible. Thus, $w = w_2$ and $u_1 \, d \, u_2$.

In particular, the element u' of the previous lemma is disjoint from u.

Lemma 12. *Let $u_1, u_2, \ldots, u_n \in \mathscr{X}_w$ be pairwise disjoint saturated elements such that*

$$\mu(u_1 \wedge z) < \mu(u_2 \wedge z) < \ldots < \mu(u_n \wedge z)$$

for all $z \in \widetilde{\mathscr{X}}^+_w$. *Then there is a saturated element* u_{n+1} *disjoint from each of the elements* $u_1, \dots, u_n$ *and such that*

$$\mu(u_1 \wedge z) < \dots < \mu(u_n \wedge z) < \mu(u_{n+1} \wedge z)$$

for all $z \in \widetilde{\mathscr{X}}^+_w$.

To prove it suffices to apply Lemmas 10 and 11 to u_n.

We are now in a position to complete the proof of the theorem. From Lemma 12 it follows that there is an infinite sequence of disjoint elements $\{u_n\}$ such that $u_n \le w$ $(n = 1, 2, \dots)$ and $\mu u_1 < \mu u_2 < \dots$. Clearly, this is impossible. The only conclusion reads: in fact, $w = \mathbf{0}$ and $\mathscr{X}_2$ is the zero band. The proof of the theorem is complete.

There is a saturated element u of greatest measure. In fact, there are many such elements but each has the same measure.

Remark 1. If the subalgebra $\widetilde{\mathscr{X}}$ lacks a nonzero element disjoint from all saturated elements of $\mathscr{X}$ then the saturated element u of Theorem 25 satisfies the condition: if $\widetilde{x} \in \widetilde{\mathscr{X}}$ and $\widetilde{x} \ge u$ then $\widetilde{x} = \mathbf{1}$.

Indeed, in the opposite case we would find a saturated element $v \le C\widetilde{x}, v > \mathbf{0}$.

Remark 2. In the proof of Theorem 25 (while checking the solidity of $\mathbf{C}$) we have established the following general fact: if a saturated element u has the greatest measure for the band $\mathscr{X}_{\widetilde{w}}$ $(\widetilde{w} \in \widetilde{\mathscr{X}})$ then for each $\widetilde{w}' \in \widetilde{\mathscr{X}}^+_{\widetilde{w}}$ its projection $u \wedge \widetilde{w}'$ plays the same role in the band $\mathscr{X}_{\widetilde{w}'}$: if $\mu u = \alpha_{\widetilde{w}}$ then $\mu(u \wedge w') = \alpha_{\widetilde{w}'}$.

Remark 3. Theorem 25 and the two previous remarks remain valid if applied not to the BA $\mathscr{X}$ but rather to the bands of it of the form $\mathscr{X}_b$. In this event, instead of $\widetilde{\mathscr{X}}$ we should consider its trace $[\widetilde{\mathscr{X}}]_b$. By Proposition 1° of 9.4.3 the concept of "saturation" for the elements of $\mathscr{X}_b$ retains the previous meaning.

The element whose existence is demonstrated in the last theorem we will call the *element of maximal type*. The set of all these elements is denoted by M or M_b if we imply the band $\mathscr{X}_b$ (cf. Remark 3). From Remark 2 it follows that for all $v \in \widetilde{\mathscr{X}}^+$ and $b \in \widetilde{\mathscr{X}}^+$ the set $M_{v \wedge b}$ consists of all elements of the form $u \wedge v$, $u \in M_b$.

Choose $u_1 \in M \equiv M_1$ arbitrarily; then take $u_2 \in M_{Cu_1}$, $u_3 \in M_{C(u_1+u_2)}$, and so on (it may happen that from some index on we will have $u_n = \mathbf{0}$). We thus come to the sequence $u_1, u_2, \dots$ consisting of pairwise disjoint entries. To each u_n there corresponds the function α_n determined from the equality

$$\alpha_n(v) \equiv \mu(u_n \wedge v) \quad (v \in \widetilde{\mathscr{X}}).$$

In particular, $\alpha_1(v) = \alpha_v$ as follows from Remark 2, while in the remaining cases $\alpha_n(v)$ is the greatest value of the measure of a saturated element of the band $\mathscr{X}_{v\wedge(u_1+\dots+u_{n-1})}$. The functions α_n are totally additive quasimeasures on $\widetilde{\mathscr{X}}$. Clearly, $\alpha_1(v) \geq \alpha_2(v) \geq \dots$ for all $v \in \widetilde{\mathscr{X}}$.

Lemma 13. *The functions α_n are independent of the choice of the sequence $u_1, u_2, \dots$. More exactly, if we are given two sequences $u'_1, u'_2, \dots$ and $u''_1, u''_2, \dots$ that are constructed by the above scheme then*

$$\alpha'_n(v) \equiv \mu(u'_n \wedge v) = \mu(u''_n \wedge v) \equiv \alpha''_n(v) \tag{35}$$

for all $n = 1, 2, \dots,\ v \in \widetilde{\mathscr{X}}$.

PROOF. Clearly, (35) holds for $n = 1$. Assume that the lemma is valid for all $k < n$; more exactly, $\alpha'_k = \alpha''_k$ for all $k < n$ and whatever $\widetilde{\mathscr{X}}$ and $\mathscr{X}$. Then, arbitrarily taking $v \in \widetilde{\mathscr{X}}^+$, arrange the decomposition

$$v = v_1 + \dots + v_s,$$

such that all $v_i \in \widetilde{\mathscr{X}}$ and for each i and every pair of indices $p, q \leq n$ we have either of the two:

a) $u'_p \wedge v_i = u''_q \wedge v_i$;

b) $(u'_p \wedge v_i)\, d\, (u''_q \wedge v_i)$.

(We have again applied Proposition 10° of 9.4.2.) The two possibilities are open:

1) $(v_i \wedge (u'_1 + \dots + u'_{n-1})\, d\, (u''_1 \wedge v_i + \dots + u''_{n-1} \wedge v_i)$,

2) there are indices $p, q \leq n$ satisfying $v_i \wedge u'_p = u''_q \wedge v_i$.

In case 1) by the main property of the elements u'_n and u''_n we have

$$\alpha'_n(v_i) \equiv \mu(v_i \wedge u'_n) = \mu(v_i \wedge u''_n) \equiv \alpha''_n(v_i).$$

If 2) holds then the same equality holds by the induction hypothesis (applied to the band $\mathscr{X}_{Cu'_p\wedge v_i}$ instead if $\mathscr{X}$). This hypothesis is applicable since the number of the functions we compare has diminished while all properties are retained. By summing over all $i = 1, 2, \dots, s$, infer

$$\alpha'_n(v) \equiv \mu(u'_q \wedge v) = \mu(u''_n \wedge v) \equiv \alpha''_n(v).$$

Thus, we have proved the equality (35).

We now summarize the above.

Theorem 26. *Let $\{\mathscr{X}, \mu\}$ be an NBA. To each regular subalgebra $\widetilde{\mathscr{X}} \subset \mathscr{X}$ there corresponds the unique sequence of totally additive*

quasimeasures $\alpha = \{\alpha_n\}_1^\infty$ *on* $\widetilde{\mathscr{X}}$ *with the following property: there is a disjoint sequence* $\{u_n\}_1^\infty$ *of elements saturated by the subalgebra* $\widetilde{\mathscr{X}}$ *such that*

1) $\mu(u_n \wedge v) = \alpha_n(v)$ *for all* $v \in \widetilde{\mathscr{X}}$ *and* $n = 1, 2, \dots$;

2) *for all* $v \in \widetilde{\mathscr{X}}$, *the element* $u_n \wedge v$ *has the greatest measure among the elements* $u \in \mathscr{X}_{C(u_1+\dots+u_{n-1})\wedge v}$ *saturated by* $\widetilde{\mathscr{X}}$.

This theorem is in fact proved. Inspection of the proof shows that the sequence $\{\alpha_n\}$ is related to the subalgebra canonically with respect to the measure preserving isomorphisms (i.e., the isomorphisms of the category of normed BAs). More exactly, if we are given some NBAs $\{\mathscr{X}', \mu'\}$ and $\{\mathscr{X}'', \mu''\}$ and $\widetilde{\mathscr{X}'}$ and $\widetilde{\mathscr{X}''}$ are their regular subalgebras to which there correspond the sequences $\{\alpha'_n\}$ and $\{\alpha''_n\}$; then to each measure preserving isomorphism $\Phi : \mathscr{X}' \longrightarrow \mathscr{X}''$ (i.e., $\mu'' = \mu' \circ \Phi^{-1}$) sending $\widetilde{\mathscr{X}'}$ to $\widetilde{\mathscr{X}''}$ the following holds: $\alpha''_n = \alpha'_n \circ \Phi_0^{-1}$ for all n ($\Phi_0 \equiv \Phi|_{\widetilde{\mathscr{X}}}$). We again consider the "homogeneous" elements of the external standard partition, i.e., the elements of the form $b_{(k,1)}$. To each cardinal $k \geq \aleph_0$ we assign the totally additive quasimeasure β_k on $\widetilde{\mathscr{X}}$ determined by the equality

$$\beta_k(v) = \mu(v \wedge b_{(k,1)})$$

(at most countable many among them are nonzero). This collection of quasimeasures is related to $\widetilde{\mathscr{X}}$ canonically (in the same sense).

Introduce the following notation for these collections:

$$\alpha[\widetilde{\mathscr{X}}, \mathscr{X}] \equiv \{\alpha_n\}_{n \leq \aleph_0}, \quad \beta[\widetilde{\mathscr{X}}, \mathscr{X}] = \{\beta_m\}_{\aleph_0 \leq m \leq \tau(\mathscr{X})}.$$

We will call the pair $(\alpha[\widetilde{\mathscr{X}}, \mathscr{X}], \beta[\widetilde{\mathscr{X}}, \mathscr{X}]) \equiv \delta[\widetilde{\mathscr{X}}, \mathscr{X}]$ the *characteristic kit* of the subalgebra. It is worth observing that this kit is essentially connected with a particular measure: if we pass to another measure α_n then β_n may change so as to become completely unrecognizable. Also, even if the measure is fixed, the elements u_i are determined in a manner far from uniqueness. However, the BANDS OF ESSENTIAL POSITIVITY of the quasimeasures α_n and β_m are determined uniquely and do not depend on the measure. For β_m this band is $\mathscr{X}_{\overline{b}_{(m,1)}}$, while for α_n it is $\mathscr{X}_{\overline{u}_n}$. The relation between the elements $\overline{u}_m$ and $\overline{b}_{(n,0)}$ is easy to understand. Namely, it is clear that $\overline{u}_1 \geq \overline{u}_2 \geq \dots$ (this follows from Remark 2). Choose the indices $1 = n_0 < n_1 < \dots$ so that $\overline{u}_{n_k} = \overline{u}_{n_k+1} = \dots = \overline{u}_{n_{k+1}-1} > \overline{u}_{n_{k+1}}$ for all k. Then it is easy to check that

$$\overline{b}_{(n_{k+1}-1,0)} = \overline{u}_{n_k} - \overline{u}_{n_{k+1}}, \ \overline{b}_{(\aleph_0,0)} = \bigwedge_{n \geq 1} \overline{u}_n.$$

We see that *from the bands of essential positivity of the quasimeasures α_n and β_n we may uniquely recover all $\overline{b}_t$ and vice versa.*[22]

Assume that we again are given normed BAs $\{\mathscr{X}', \mu'\}$ and $\{\mathscr{X}'', \mu''\}$, and their regular subalgebras $\widetilde{\mathscr{X}'} \subset \mathscr{X}'$ and $\widetilde{\mathscr{X}''} \subset \mathscr{X}''$. As before, we denote by one and two primes the object of the same type which are related to these subalgebras. Consider Problems (A) and (B) again, but now as regards MEASURE PRESERVING isomorphisms.

Theorem 27. *For an individual isomorphism $\Phi_0 : \widetilde{\mathscr{X}'} \longrightarrow \widetilde{\mathscr{X}''}$ to be extendible to a measure preserving isomorphism $\Phi : \mathscr{X} \longrightarrow \mathscr{X}''$ it is necessary and sufficient that the characteristic kit $\delta[\widetilde{\mathscr{X}'}, \mathscr{X}']$ goes to the analogous kit $\delta[\widetilde{\mathscr{X}''}, \mathscr{X}'']$ under Φ_0, i.e., the following holds*

$$\alpha''_n = \alpha'_n \circ \Phi_0^{-1}, \ \ \beta''_m = \beta'_m \circ \Phi_0^{-1} \tag{36}$$

for all n and m.

As always with such theorems, it is only sufficiency that needs proving. From the conditions (36) it follows that $\overline{b}''_t = \Phi_0(\overline{b}'_t)$, so that Φ_0 has an extension Φ in this event; we are to prove that it may be chosen to be measure preserving.

Introduce the elements u'_n and u''_n. They give rise to two disjoint systems in $\mathscr{X}'$ and $\mathscr{X}''$; moreover,

$$\alpha'_n(x') = \mu'(u'_n \wedge x') = \mu''(u''_n \wedge x'') = \alpha''_n(x''),$$

where x' is an arbitrary element of $\widetilde{\mathscr{X}'}$ and $x'' = \Phi_0(x')$, $n = 1, 2, \dots$. For brevity, we put $v'_m \equiv b'_{(m,1)}$ and $v''_m \equiv b''_{(m,1)}$. We have

$$\beta'_m(x') = \mu'(x' \wedge v'_m) = \mu''(x'' \wedge v''_m) = \beta''_m(x''),$$

where $x'' = \Phi_0(x')$ and $x' \in \widetilde{\mathscr{X}'}$ once again. In particular,

$$\mu' v'_m = \beta'_m(\overline{v}'_m) = \beta''_m(\overline{v}''_m) = \mu'' v''_m.$$

In much the same way $\mu' u'_n = \mu'' u''_n$.

As before, we introduce the "induced" isomorphisms:[23]

$$\widetilde{\Phi}_n \equiv F_{u''_n} \circ \Phi_0 \circ F_{u'_n}^{-1}, \quad \widetilde{\widetilde{\Phi}}_h \equiv F_{v''_n} \circ \Phi_0 \circ F_{v'_n}^{-1}.$$

[22] If $n_k - 1 < m < n_{k+1} - 1$ then $b_{(m,0)} = \mathbf{0}$.

[23] For simplicity we now denote the restricted isomorphism $\Phi_0|_{\widetilde{\mathscr{X}'_c}}$ by the same letter Φ_0 $(c = u'_n, v'_n)$.

They respectively send $[\widetilde{\mathscr{X}'}]_{u'_n}$ to $[\widetilde{\mathscr{X}''}]_{u''_n}$ and $[\widetilde{\mathscr{X}'}]_{v'_n}$ to $[\widetilde{\mathscr{X}''}]_{v''_n}$. Since Φ_0 sends α'_n to α''_n; therefore,

$$\mu''(F_{u''_n} \circ \Phi_0 \circ F_{u'_n}^{-1})(x') = \mu''(u''_n \wedge \Phi_0(\overline{x}'))$$

$$= \alpha''_n(\Phi_0(\overline{x}')) = \alpha'_n(\overline{x}') = \mu'(\overline{x}' \wedge u'_n) = \mu' x'$$

for all $[\widetilde{\mathscr{X}'}]_{u'_n}$.

By analogy,

$$\mu''(F_{v''_m} \circ \Phi_0 \circ F_{v'_m}^{-1})(x') = \mu' x'$$

for $x' \in [\widetilde{\mathscr{X}'}]_{v'_m}$.

Thus the induced isomorphisms are measure preserving. The elements u'_n and u''_n are saturated. Therefore, the isomorphism $\widetilde{\Phi}_n$ sends $\mathscr{X}'_{u'_n}$ to $\mathscr{X}''_{u''_n}$ while preserving measure. As far as the bands $\mathscr{X}'_{v'_m}$ are concerned, we may construct for them some measure preserving isomorphisms $\widetilde{\widetilde{\Phi}}{}^*_m$ to the corresponding bands $\mathscr{X}''_{v''_m}$. This was explained above in connection with Lemma 7. The isomorphisms $\widetilde{\widetilde{\Phi}}{}^*_m$ are constructed by extending $\widetilde{\widetilde{\Phi}}_m$. We are thus left with "pasting together" the mappings we have obtained. Put

$$\Phi(x) \equiv \sum_n \widetilde{\Phi}_n(x \wedge u'_n) + \sum_m \widetilde{\widetilde{\Phi}}{}^*_m(x \wedge v'_m).$$

This is a measure preserving isomorphism. As in Lemma 5, we demonstrate that $\Phi_0 = \Phi|_{\widetilde{\mathscr{X}'}}$.

Inspection of the proof yields another (necessary and sufficient) condition for extendibility of ALGEBRAIC individual isomorphism Φ_0: we need that for all n and m the bands of essential positivity of the quasimeasures $\alpha'_n \circ \Phi_0^{-1}$ and $\beta'_n \circ \Phi_0^{-1}$ coincide with the analogous bands of the quasimeasures α''_n and β''_n. This fact, as we have seen, is independent of the choice of μ' and μ''. In fact, the just-formulated test is simply another version of Theorem 11 from which it follows easily. The shortcoming of this version lies in mentioning the quasimeasures α_n and β_n which are purely metric characteristics of the subalgebra.

We return to the problem of metric classification of measurable functions. We have already found the conditions for two resolutions of the identity $\mathfrak{f}'$ and $\mathfrak{f}''$ to go to one another under some algebraic isomorphism (Theorem 21). We now require that such an isomorphism be measure preserving.

Introduce the families of distribution functions

$$\widetilde{F}'_n : \quad \widetilde{F}'_n(s) \equiv \alpha_n(e_s^-(\mathfrak{f}')),$$

$$\widetilde{F}''_n : \quad \widetilde{F}''_n(s) \equiv \alpha_n(e_s^-(\mathfrak{f}''));$$
$$\widetilde{\widetilde{F}}'_n : \quad \widetilde{\widetilde{F}}'_n(s) \equiv \beta_m(e_s^-(\mathfrak{f}')),$$
$$\widetilde{\widetilde{F}}''_n : \quad \widetilde{\widetilde{F}}''_n(s) \equiv \beta_m(e_s^-(\mathfrak{f}'')), \ -\infty \le s \le +\infty.$$

Moreover, as before we let

$$F' : \ F'(s) \equiv \mu' e_s^-(\mathfrak{f}'), \quad F'' : \ F''(s) \equiv \mu'' e_s^-(\mathfrak{f}'').$$

Theorem 28. *For the two resolutions of the identity $\mathfrak{f}'$ and $\mathfrak{f}''$ to be metrically isomorphic it is necessary and sufficient that*

$$\widetilde{F}'_n(s) = \widetilde{F}''_n(s), \quad \widetilde{\widetilde{F}}'_m(s) = \widetilde{\widetilde{F}}''_m(s) \tag{37}$$

for all n, m, and s.

As regards necessity, this theorem is obvious; and so we prove the sufficiency of the conditions (37). Since it is easy to see that

$$F'(s) = \sum_n \widetilde{F}'_n(s) + \sum_n \widetilde{\widetilde{F}}'_n(s), \ F''(s) = \sum_n \widetilde{F}'_n(s) + \sum_n \widetilde{\widetilde{F}}''_n(s);$$

therefore, (37) implies $F' = F''$ and by Theorem 20 there is a measure preserving individual isomorphism $\Phi_0 : \widetilde{\mathscr{X}'} \longrightarrow \widetilde{\mathscr{X}''}$ sending $\mathfrak{f}'$ to $\mathfrak{f}''$:

$$e_s^-(\mathfrak{f}'') = \Phi_0(e_s^-(\mathfrak{f}')) \quad (-\infty \le s \le +\infty).$$

The conditions (37) may be written as the equalities

$$\alpha''_n(\Phi_0(e_s^-(\mathfrak{f}'))) = \alpha'_n(e_s^-(\mathfrak{f}')),$$
$$\beta''_m(\Phi_0(e_s^-(\mathfrak{f}'))) = \beta'_m(e_s^-(\mathfrak{f}')).$$

The families $\{e_s^-(\mathfrak{f}')\}$ and $\{e_s^-(\mathfrak{f}'')\}$ fully generate the subalgebras $\widetilde{\mathscr{X}'}$ and $\widetilde{\mathscr{X}''}$. Hence, for all $x' \in \widetilde{\mathscr{X}'}$ we have

$$\alpha''_n(\Phi_0(x')) = \alpha'_n(x'), \quad \beta''_m(\Phi_0(x')) = \beta'_m(x').$$

By Theorem 27 the mapping Φ_0 extends to a measure preserving isomorphism $\Phi : \mathscr{X}' \longrightarrow \mathscr{X}''$.

This theorem belongs mostly to V. A. Rokhlin who considered the separable case by using the technique of “Lebesgue spaces” (i.e., Lebesgue–Rokhlin spaces) (cf. V. A. Rokhlin [2]). Seemingly, the statement of the problem also belongs to him.

If the algebras $\mathscr{X}'$ and $\mathscr{X}''$ are homogeneous and have the same weight then the conditions (37) are essentially simplified and for existence of

a metric isomorphism between $\mathfrak{f}'$ and $\mathfrak{f}''$ it suffices only that the distribution functions coincide:

$$F'(s) = F''(s) \quad (-\infty \le s \le +\infty)$$

(D. A. Vladimirov and A. A. Samorodnitskiĭ [1]).

Using Theorem 28 and replacing the equalities (37) with a weaker condition that the corresponding Lebesgue–Stieltjes "measures" are equivalent, we arrive at another version of the theorem on algebraic isomorphism between resolutions of the identity. We leave it to the reader to find the exact formulation and ponder over the details of the proof.

We now turn to Problem (B). What are the conditions under which subalgebras are metrically isomorphic? The following test is almost obvious: *there must exist an individual measure preserving isomorphism* $\Phi_0 : \widetilde{\mathscr{X}'} \longrightarrow \widetilde{\mathscr{X}''}$ *sending one characteristic kit to the other.* Then such an isomorphism may be extended by Theorem 27 to a measure preserving isomorphism $\Phi : \mathscr{X}' \longrightarrow \mathscr{X}''$. However, this formulation is far from satisfactory since it reduces the problem of existence of an isomorphism to an analogous problem for another isomorphism. It is desirable to indicate an invariant that allows us to classify subalgebras in much the same way as this was done in the algebraic case in Theorem 13.

Introduce the densities $\frac{d\alpha_n}{d\mu}$ and $\frac{d\beta_m}{d\mu}$ (the conditional measures of the elements u_n and v_m) and the corresponding spectral families

$$\Big\{e_\lambda^-\Big(\frac{d\alpha_n}{d\mu}\Big)\Big\}, \quad \Big\{e_\lambda^-\Big(\frac{d\beta_m}{d\mu}\Big)\Big\}.$$

If one of these densities equals zero then

$$e_\lambda^-(\ldots) = \begin{cases} \mathbf{0}, & \lambda \le 0, \\ \mathbf{1}, & \lambda > 0. \end{cases} \tag{38}$$

All these families together fully generate a regular (always separable) subalgebra $\widehat{\mathscr{X}}$ of $\widetilde{\mathscr{X}}$. Thus, there arises another kit $\delta[\widehat{\mathscr{X}}, \widetilde{\mathscr{X}}]$ consisting of the quasimeasures $\widehat{\alpha_n}$ and $\widehat{\beta_m}$. Consider their densities $\frac{d\widehat{\alpha_n}}{d\mu}$ and $\frac{d\widehat{\beta_m}}{d\mu}$ and the corresponding spectral families. We assume that they are determined also for $m > \tau(\widetilde{\mathscr{X}})$ and $m < \tau(\mathscr{X})$; however, in this event, $\widehat{\beta_m}$ is clearly a zero quasimeasure and the corresponding spectral family is defined in accord with (38).

It is convenient for us to enumerate all available densities in a uniform manner as it was done earlier. Introduce the index set

$$\Xi \equiv \Xi_0 \cup \Xi_0' \cup \Xi_1 \cup \Xi_1'.$$

The elements $\xi \in \Xi$ are ordered pairs of the form (k, i), where $i = 0, 1$ and k is a cardinal; moreover, let

$$\Xi_0 \equiv \{(n,0) \mid n = 1,2,\ldots\},\ \Xi_1 \equiv \{(n,1) \mid n = 1,2,\ldots\},$$

$$\Xi_0' \equiv \{(m,0) \mid \aleph_0 \le m \le \tau(\mathscr{X})\},\ \Xi_1' \equiv \{(m,1) \mid \aleph_0 \le m \le \tau(\mathscr{X})\}.$$

Put

$$e_\lambda^\xi \equiv \begin{cases} e_\lambda^-(\dfrac{d\alpha_n}{d\mu}), & \xi = (n,0) \in \Xi_0, \\ e_\lambda^-(\dfrac{d\beta_m}{d\mu}), & \xi = (m,0) \in \Xi_0', \\ e_\lambda^-(\dfrac{d\widehat{\alpha_n}}{d\mu}), & \xi = (n,1) \in \Xi_1, \\ e_\lambda^-(\dfrac{d\widehat{\beta_m}}{d\mu}), & \xi = (m,1) \in \Xi_1'. \end{cases}$$

We will also denote by $\mathfrak{f}_\xi$ the resolution of the identity corresponding to the spectral family $\{e_\lambda^\xi\}$. The families of the form $\{e_\lambda^\Xi\},\ \xi \in \Xi_0 \cup \xi_0'$, fully generate $\widehat{\mathscr{X}}$ as was mentioned above.

We again consider the product of the real axes $\mathbf{R} \equiv \mathbf{R}_\Xi \equiv \overline{\mathbb{R}}^\Xi$ and the collection of all possible half-spaces in it (cf. the formula (31))

$$L_\xi^{c-} = \{r \mid r_\xi < c\},\ -\infty \le c \le +\infty,\ \xi \in \Xi.$$

The resultant σ-algebra $\mathbf{B}_{\mathbf{R}} \equiv \mathbf{B}$ includes the σ-subalgebra $\mathbf{B}^0$ that is generated by the half-spaces in the system (31) which correspond to the elements $\xi_0 \in \Xi_0 \cup \Xi_0'$. Define a σ-homomorphism $\Psi : \mathbf{B} \longrightarrow \mathscr{X}$ by the conditions

$$\Psi(L_\xi^{\lambda-}) \equiv e_\lambda^\xi \quad (\xi \in \Xi).$$

We also determine the quasimeasure $m \equiv m[\widetilde{\mathscr{X}}, \mathscr{X}]$ by the rule $me \equiv \mu\Psi(e)$. Clearly, $\Psi(\mathbf{B}) = \Psi(\mathbf{B}^0) = \widehat{\mathscr{X}}$.

Returning to the main theme of this subsection, we consider the two normed BAs $\{\mathscr{X}', \mu'\}$ and $\{\mathscr{X}'', \mu''\}$ and their regular subalgebras $\widetilde{\mathscr{X}'} \subset \mathscr{X}'$ and $\widetilde{\mathscr{X}''} \subset \mathscr{X}''$. As before, to denote the objects that are canonically related to the subalgebras we will use one or two primes; this concerns the homomorphism Ψ as well: Ψ' corresponds to $\widetilde{\mathscr{X}'}$, while Ψ'' corresponds to $\widetilde{\mathscr{X}''}$. If there is a measure preserving isomorphism $\Phi : \mathscr{X}' \longrightarrow \mathscr{X}''$ sending $\widetilde{\mathscr{X}'}$ to $\widetilde{\mathscr{X}''}$ then it is clear that

$$\Psi'' = \Phi_0 \circ \Psi',\ m''(e) = \mu'\Psi(e) = m'(e).$$

Therefore, the quasimeasure $m \equiv m[\widetilde{\mathscr{X}}, \mathscr{X}]$ is an invariant. It is easy to see that this invariant suffices for the metric classification of subalgebras.

Indeed, if $m' = m''$ then the restrictions of these quasimeasures to $\mathbf{B}^0$ coincide too, and so, by Theorem 23, there is a measure preserving isomorphism $\Phi_0 : \widehat{\mathscr{X}'} \longrightarrow \widehat{\mathscr{X}''}$ sending each $\mathfrak{f}'_\xi$ to the corresponding $\mathfrak{f}''_\xi$. In particular, this holds for $\xi \in \Xi_1 \cup \xi'_1$; hence, Φ_0 extends to a measure preserving isomorphism $\widetilde{\Phi} : \widetilde{\mathscr{X}'} \longrightarrow \widetilde{\mathscr{X}''}$ (Theorem 27). The situation is exactly the same for $\xi \in \Xi_0 \cup \Xi'_0$, which implies further extendibility of the individual isomorphism $\widetilde{\Phi}$ to a measure preserving isomorphism $\Phi : \mathscr{X}' \longrightarrow \mathscr{X}''$. Clearly, $\Phi(\widetilde{\mathscr{X}'}) = \widetilde{\mathscr{X}''}$ and $\Phi(\widehat{\mathscr{X}'}) = \widehat{\mathscr{X}''}$. We have thus proved

Theorem 29. *The equality*

$$m' \equiv m[\widetilde{\mathscr{X}'}, \mathscr{X}'] = m[\widetilde{\mathscr{X}''}, \mathscr{X}''] \equiv m'' \tag{39}$$

is necessary and sufficient for the existence of a metric isomorphism $\Phi : \mathscr{X}' \longrightarrow \mathscr{X}''$ sending $\widetilde{\mathscr{X}'}$ to $\widetilde{\mathscr{X}''}$.

(The equality (39) implies also the preservation of the spaces which the quasimeasures m' and m'' belong to, i.e., coincidence of the weights $\tau(\mathscr{X}')$ and $\tau(\mathscr{X}'')$. The set Ξ is the same for both subalgebras.)

As was mentioned, the condition $m' = m''$ amounts to the simultaneous equalities

$$\begin{aligned} &\mu'(e^-_\lambda(\mathfrak{f}'_{\xi_1}) \wedge \dots \wedge e^-_\lambda(\mathfrak{f}'_{\xi_s})) \\ &= \mu''(e^-_\lambda(\mathfrak{f}''_{\xi_1}) \wedge \dots \wedge e^-_\lambda(\mathfrak{f}''_{\xi_s})), \end{aligned} \tag{40}$$

$$\xi_1, \xi_2, \dots, \xi_s \in \Xi,\ \ \lambda_1, \lambda_2, \dots, \lambda_s \in \mathbb{R},\ \ s = 1, 2, \dots .$$

Therefore, the last theorem may be equivalently rephrased as follows:

Theorem 30. *The simultaneous equalities* (40) *provide a necessary and sufficient condition for the existence of a metric isomorphism between the subalgebras $\widetilde{\mathscr{X}'}$ and $\widetilde{\mathscr{X}''}$.*

The equalities (40) proclaim the coincidence of finite-dimensional distributions of two random processes. To each subalgebra $\widetilde{\mathscr{X}}$ there canonically corresponds the random process $\mathfrak{f} \equiv \mathfrak{f}[\widetilde{\mathscr{X}}, \mathscr{X}]$; the existence of a metric isomorphism between subalgebras means that the processes $\mathfrak{f}[\widetilde{\mathscr{X}'}, \mathscr{X}']$ and $\mathfrak{f}[\widetilde{\mathscr{X}''}, \mathscr{X}'']$ have the same finite-dimensional distributions.

Note finally that the subalgebra $\widetilde{\mathscr{X}}$ is in general significantly larger than the subalgebra $\widehat{\mathscr{X}}$ generated by $\mathfrak{f}_\xi$. This follows for instance from the fact that $\widehat{\mathscr{X}}$ is always separable whereas the weight $\tau(\widetilde{\mathscr{X}})$ may be arbitrary.

If the BA $\mathscr{X}$ is separable itself then the situation becomes essentially simpler: among the element v_m there may happen only one nonzero,

namely, $v_{\aleph_0} \equiv b_{(\aleph_0,1)}$; moreover,

$$\beta_{\aleph_0}(v) = \mu v - \sum_k \alpha_k(v) \quad (v \in \widetilde{\mathscr{X}})$$

so that the characteristic kit is determined only from the single collection $\alpha_1, \alpha_2, \ldots$. The statements of Theorems 27–29 may be simplified correspondingly.

4.7 Concluding remarks

We have already mentioned that the "inverse" problem is not always solvable of recovering a subalgebra from the invariants of this chapter. Not each set $A \subset H$ may serve as the certificate of some subalgebra and not each Hellinger type of the space X_A as the algebraic type of a subalgebra. For instance, the type supported at a point $\chi \in X_A$ which has at least one nonzero coordinate χ_h may fail to be the algebraic type of any subalgebra.

The conditions 1), 2), and 3) of 9.4.4, as was mentioned above, in fact characterize the certificate of a primitive subalgebra. Therefore, it is easy to determine from the form of a weight function k whether or not it corresponds to some semiprimitive subalgebra and recover this subalgebra up to isomorphism. This is the case in which the "inverse" problem is uniquely solvable; we leave the precise formulations to the reader. The general case remains unexplored.

5. Isomorphism of systems of subalgebras

In this subsection we discuss the isomorphism problem for systems of subalgebras. Assume that, given some complete BAs $\mathscr{X}'$ and $\mathscr{X}''$ we have distinguished two families of regular subalgebras $\{\widetilde{\mathscr{X}'_\gamma}\}_{\gamma\in\Gamma}$ and $\{\widetilde{\mathscr{X}''_\gamma}\}_{\gamma\in\Gamma}$. The index set Γ is the same for both subalgebras. Put

$$\widetilde{\mathscr{X}'} \equiv \overline{\mathscr{X}'\langle\bigcup_{\gamma\in\Gamma}\widetilde{\mathscr{X}'_\gamma}\rangle}, \quad \widetilde{\mathscr{X}''} \equiv \overline{\mathscr{X}''\langle\bigcup_{\gamma\in\Gamma}\widetilde{\mathscr{X}''_\gamma}\rangle}.$$

We are interested in the conditions under which

a) there is an individual isomorphism $\widetilde{\Phi} : \widetilde{\mathscr{X}'} \longrightarrow \widetilde{\mathscr{X}''}$ satisfying $\widetilde{\Phi}(\widetilde{\mathscr{X}'_\gamma})$ for all $\gamma \in \Gamma$;

b) there is an isomorphism $\Phi : \mathscr{X}' \longrightarrow \mathscr{X}''$ with the same property.

In these formulations we talk about an ALGEBRAIC isomorphism; however, we still consider only normable algebras. If we distinguish some probability measures on $\mathscr{X}'$ and $\mathscr{X}''$ then we encounter the problems analogous to Problems a) and b) but related now to METRIC isomorphisms.

Clearly, the above-formulated problems have as particular cases the problems we addressed in the previous subsection. It is also clear that the new problems are harder than the previous. We will confine exposition to only one particular case of Problem a). Assume that $\widetilde{\mathscr{X}'} = \mathscr{X}'$ and $\widetilde{\mathscr{X}''} = \mathscr{X}''$ and all subalgebras $\widetilde{\mathscr{X}'_\gamma}$ and $\widetilde{\mathscr{X}''_\gamma}$ are *separable*. In this event it is Theorem 23 of this chapter that gives a key to solution. The point is that, as we know (Theorem 6.7), each separable subalgebra is generated by some resolution of the identity. Let the subalgebras $\widetilde{\mathscr{X}'_\gamma}$ and $\widetilde{\mathscr{X}''_\gamma}$ be respectively generated by the resolutions of the identity $\mathfrak{f}'_\gamma$ and $\mathfrak{f}''_\gamma$. In the space $\mathbf{R} \equiv \overline{\mathbb{R}}^\Gamma$ there are defined two measures m' and m'' on the σ-algebra $\mathbf{B}$. The coincidence of these measures means by Theorem 23 that there is a metric isomorphism $\Phi : \mathscr{X}' \longrightarrow \mathscr{X}''$ sending $\mathfrak{f}'_\gamma$ to $\mathfrak{f}''_\gamma$ and also $\widetilde{\mathscr{X}'_\gamma}$ to $\widetilde{\mathscr{X}''_\gamma}$ for all $\gamma \in \Gamma$. The equivalence of m' and m'' means the existence of an algebraic isomorphism with analogous properties. That is how Problem a) is solved in our case.

We now consider a SOLE normed BA $\{\mathscr{X}, \mu\}$ and a family $\{\widetilde{\mathscr{X}_\gamma}\}_{\gamma\in\Gamma}$. The normed BA $\{\mathscr{X}, \mu\}$ may be represented as the image of the σ-algebra $\mathbf{B}$ under a σ-continuous mapping Ψ. In this event each of the resolutions of the identity $\mathfrak{f}_\gamma$ is the image of the appropriate "coordinate," i.e., the projection $\pi_\gamma \equiv r_\gamma$ $(r \in \mathbb{R})$. Each subalgebra $\widetilde{\mathscr{X}_\gamma}$ is the image of the "coordinate" subalgebra $\mathbf{B}_\gamma$ that is generated by the "cylinders" of the form $\pi_\gamma^{-1}(e)$, where e is a Borel subset of the real axis.[24] On the metric structure $\widehat{\mathbf{B}} \equiv \mathbf{B}|_{I_m}$ (with I_m the σ-ideal of m-negligible sets) we may naturally define the isomorphism $\widehat{\Psi} : \widehat{\mathbf{B}} \longrightarrow \mathscr{X}$. Clearly, $\widehat{\Psi}$ is a metric isomorphism from the normed BA $\{\widehat{\mathbf{B}}, \widehat{m}\}$ to the normed BA $\{\mathscr{X}, \mu\}$. Passing to the quotient algebra by the ideal I_m, we obtain the regular subalgebras $\widehat{\mathbf{B}}_\gamma$ which goes under $\widehat{\Psi}$ to the corresponding $\mathscr{X}_\gamma$. On the subalgebras $\widehat{\mathbf{B}}_\gamma$ there are given the "marginal" measures $\widehat{m}_\gamma$, the "traces" of $\widehat{m}$.

For simplicity, we will assume that all $\mathscr{X}_\gamma$ are CONTINUOUS subalgebras. It is easy to see that we may specially choose the resolutions of the identity in this case. It is well known that to each γ there is a metric isomorphism $T_\gamma : E_0 \longrightarrow \mathscr{X}_\gamma$; as $\mathfrak{f}_\gamma$ we may take the resolutions of the identity that correspond "*mod* 0 to the function" $\widehat{f}$: $f(t) = t,\ t \in [0, 1]$, under this isomorphism.[25] Then it is easy to check that the "measure" m turns out to be supported in the cube $\mathbf{R}' \equiv [0, 1]^\Gamma$ (in the sense that

[24]Thus we have define a "straightforwardly given" random process with the same finite-dimensional distributions as the process $\mathfrak{f}$.
[25]That is $e_t^{\pm}(\mathfrak{f}_\gamma) = T_\gamma \widehat{\chi}_{[0,1]}$ for $t \in [0, 1]$. For $\mathscr{X}_{\gamma_1} = \mathscr{X}_{\gamma_2}$ we assume $T_{\gamma_1} = T_{\gamma_2}$.

the outer measure of this cube equals 1). To each collection of sets $\mathscr{E} \subset \mathbf{B}$ we assign its "trace" onto $\mathbf{R}'$: $\mathscr{E}' \equiv \{e \cap \mathbf{R}' \mid e \in \mathscr{E}\}$. Then the system $\mathbf{B}'$ becomes a σ-algebra of subsets of the cube $\mathbf{R}'$, while I'_m is a σ-ideal of this σ-algebra. If we define the "measure" m' on $\mathbf{B}'$ by the rule $m'(e \cap \mathbf{R}') = me$ then I'_m turns out the σ-ideal of m'-negligible sets. Clearly, the metric structure $\{\widehat{\mathbf{B}}', \widehat{m}\}$, $\widehat{\mathbf{B}}' = \mathbf{B}|_{I_m}$, is isomorphic to the normed BA $\{\widehat{\mathbf{B}}, \widehat{m}\}$, and thus to the NBA $\{\mathscr{X}, \mu\}$; we define a measure preserving isomorphism U by the rule $U(\widehat{e}) = \widehat{e}'$ (recall that the elements $\widehat{e} \in \widehat{\mathbf{B}}$ are classes of sets).

We now return back to the "coordinate" subalgebras $\mathbf{B}_\gamma$. These may be described in another manner. Recall that the σ-algebra $\mathbf{B}$ was primarily defined as the σ-algebra in $\mathbf{R}$ generated by all standard half-spaces $L_\gamma^{\lambda-}$; if γ is distinguished then the corresponding part of this set of half-spaces generates the σ-algebra $\mathbf{B}_\gamma$ which transforms by factorization to the regular subalgebra $\widehat{\mathbf{B}}_\gamma \subset \widehat{\mathbf{B}}$:

$$\widehat{\mathbf{B}}_\gamma \equiv \mathbf{B}_\gamma +_2 I_m.$$

In much the same way the σ-algebra $\mathbf{B}'$ may be characterized as generated by all intersections $L_\gamma^{\lambda-} \cap \mathbf{R}'$; distinguishing γ, we obtain the σ-algebras $\mathbf{B}'_\gamma$ and then the regular subalgebras $\widehat{\mathbf{B}}'_\gamma \subset \widehat{\mathbf{B}}'$ by factorization:

$$\widehat{\mathbf{B}}'_\gamma \equiv \mathbf{B}'_\gamma +_2 I_{m'}.$$

The subalgebras $\widehat{\mathbf{B}}'_\gamma$ consist of the elements of the form $\widehat{\pi_\gamma^{-1}(b) \cap \mathbf{R}'}$, where b is a Borel subset of $[0,1]$. Clearly, $\widehat{\mathbf{B}}'_\gamma = U(\widehat{\mathbf{B}}_\gamma)$. By our choice of the resolution of the identity $\mathfrak{f}_\gamma$, the value of the "marginal" measure $\widehat{m}'_\gamma$ at $\widehat{\pi_\gamma^{-1}(b) \cap \mathbf{R}'}$ is always equal to the Lebesgue "measure" of b, i.e.,

$$\widehat{m}_\gamma(\widehat{\pi_\gamma^{-1}(b) \cap \mathbf{R}'}) \equiv m'(\pi_\gamma^{-1}(b) \cap \mathbf{R}') = l(b).$$

Slightly abusing the language, we may say that each measure $\widehat{m}_\gamma$ is Lebesgue measure translated from the Boolean algebra E_0 by the mapping $\widehat{b} \longrightarrow \widehat{\pi_\gamma^{-1}(b) \cap \mathbf{R}'}$.

Our speculations are summarized by the following theorem which gives a universal way of realizing a system of subalgebras. We first change the notations: we let I^Γ stand for the main cube; in the sequel we also agree that

$$\mathbf{B}' \equiv \mathscr{B},\ \widehat{\mathbf{B}}' \equiv \widehat{\mathscr{B}}_m,\ \mathbf{B}'_\gamma \equiv \mathscr{B}_\gamma,\ \widehat{\mathbf{B}}'_\gamma \equiv (\widehat{\mathscr{B}}_m)_\gamma.$$

(If Γ is finite or countable then $\mathscr{B}$ is the Borel algebra of $\mathbf{R}$.) Finally, by l_γ we denote the "Lebesgue" quasimeasure on $\mathscr{B}_\gamma$ that is uniquely

determined by the condition:

$$l_\gamma\left(L_\gamma^{t\pm}\cap I^\Gamma\right)=t,\quad t\in[0,1].$$

Theorem 31. *Assume that $\{\mathscr{X},\mu\}$ is a probability BA, and the regular subalgebras $\widetilde{\mathscr{X}_\gamma}$ are continuous and separable; $\mathscr{X}=\overline{\mathscr{X}\langle\bigcup_{\gamma\in\Gamma}\widetilde{\mathscr{X}_\gamma}\rangle}$. There exist a σ-additive quasimeasure m on $\mathscr{B}$ and a σ-epimorphism $\Psi:\mathscr{B}\longrightarrow\mathscr{X}$ such that*

(a) *$m=\mu\circ\Psi$;*

(b) *for all $\gamma\in\Gamma$ the following holds*

$$\Psi(\mathscr{B}_\gamma)=\widetilde{\mathscr{X}_\gamma},\quad m|_{\mathscr{B}_\gamma}=l_\gamma.$$

We see that under the conditions of the theorem there exists a metric isomorphism $\widehat{\Psi}$ from the normed BA $\{\widehat{\mathscr{B}}_m,\widehat{m}\}$ to the normed BA $\{\mathscr{X},\mu\}$ sending each subalgebra $(\widehat{\mathscr{B}}_m)_\gamma$ to the corresponding $\widetilde{\mathscr{X}_\gamma}$. Therefore, we may say that the system of subalgebras $\{(\widehat{\mathscr{B}}_m)_\gamma\}$ actually realizes $\{\widetilde{\mathscr{X}_\gamma}\}$. The marginal measures $\widehat{m}_\gamma\equiv\widehat{m}|_{(\widehat{\mathscr{B}}_m)_\gamma}$ may be called "Lebesgue" by the following reason. The cube I^Γ, as a product of segments, is furnished with the "genuine" Lebesgue quasimeasure ("product-measure"). We may consider it on the σ-algebra $\mathscr{B}$ together with m. Despite these quasimeasures may even be singular, their restrictions to $\mathscr{B}_\gamma$ coincide by the claim b) of the theorem.

We distinguish the case in which Γ is finite or countable. Then, after completion, the measurable space $\{I^\Gamma,\mathscr{B},m\}$ becomes a Lebesgue–Rokhlin space and the subalgebras $\mathscr{B}_\gamma$ will be related with the coordinate partitions which are Rokhlin-measurable in this event. Thus, each finite or countable system of partitions may be viewed as a system of coordinate partitions of the cube of an appropriate dimension. We have assumed the subalgebras $\widetilde{\mathscr{X}_\gamma}$ continuous which amounts to the requirement of nonatomicity of the quotient measures corresponding to these partitions. However, it is possible to free Theorem 31 from the assumption that the subalgebras are continuous by making the formulation slightly more complicated. We leave implementation of this possibility to the reader.

Chapter 10

INDEPENDENCE

The problem of independence of subalgebras has already been discussed in the preceding chapters. The concept of independence is one of the most important in the entire theory of Boolean algebras; in particular, it plays a key role in the problems concerning the structure of BAs. In the "metric" version the idea of independence is basic for probability theory. A significant part of probability theory (limit theorems, laws of large numbers, etc.) is devoted to independent random variables or, which is equivalent, to independent subalgebras.

We discriminate between the three types of independence. A system of subalgebras of a BA $\mathscr{X}$ may be algebraically independent; independent with respect to a given particular probability measure; and, finally, independent with respect to SOME measure residing on this algebra. In the last case we speak about *metric independence*. Theorem 7 of the previous chapter demonstrates the role of the concept of metric independence in regard to the simplest situation of two subalgebras.

1. A system of two subalgebras

1.1 Standard pairs of subalgebras

In this subsection we study the following model situation. Assume that in a separable continuous NBA $\{\mathscr{X}, \mu\}$ we have distinguished two subalgebras $\mathscr{X}_1$ and $\mathscr{X}_2$ satisfying

$$\mathscr{X} = \overline{\mathscr{X}\langle \mathscr{X}_1, \mathscr{X}_2 \rangle}.$$

Assume further that these subalgebras are regular and continuous. All properties of such a pair of subalgebras are conveniently seen in the simplest model that we will describe right away.

Alongside $\mathscr{X}$, the subalgebras $\mathscr{X}_1$ and $\mathscr{X}_2$ are also separable. Therefore, we may apply Theorem 9.31, with $\Gamma = \{1, 2\}$. We draw the following conclusion:

To each pair $\{\mathscr{X}_1, \mathscr{X}_2\}$ of continuous separable regular subalgebras of an NBA $\{\mathscr{X}, \mu\}$ fully generating $\mathscr{X}$, there corresponds a "Borel measure," a countably additive quasimeasure m on the Borel algebra $\mathscr{B}$ of the square I^2 such that the metric structures of $\{\widehat{\mathscr{B}}, \widehat{m}\}$ and the normed BA $\{\mathscr{X}, \mu\}$ are isomorphic (as NBAs). In this event, the isomorphism between these NBAs sends the subalgebra $\widehat{\mathscr{B}}_i \subset \widehat{\mathscr{B}}$ $(i = 1, 2)$ to the subalgebra $\mathscr{X}_i \subset \mathscr{X}$.[1]

This isomorphism generates a σ-homomorphism $\Psi : \mathscr{B} \longrightarrow \mathscr{X}$. Theorem 9.31 again makes it clear that the quasimeasure m coincides with the Lebesgue quasimeasure l on the subalgebras $\mathscr{B}_1$ and $\mathscr{B}_2$. After factorization, we obtain on $\widehat{\mathscr{B}}_1$ and $\widehat{\mathscr{B}}_2$ the "traces" of the measure $\widehat{m}$ coincident with the traces of Lebesgue measure $\widehat{l}$. (The rigorous meaning here is as follows: although $\widehat{\mathscr{B}}$ is not necessarily included in E_0^2, the subalgebras $\widehat{\mathscr{B}}_1$ and $\widehat{\mathscr{B}}_2$ may nonetheless be treated as subalgebras in E_0^2; moreover, the same measure is induced on them from $\widehat{\mathscr{B}}$ and from E_0^2.) We will denote these traces (we have called them "marginal" measures or "projections") by $\widehat{m}_1$ and $\widehat{m}_2$ (or $\widehat{l}_1$ and $\widehat{l}_2$).

The quasimeasure m bears all information about the interlocation of the pair of subalgebras $\mathscr{X}_1$ and $\mathscr{X}_2$. If we replace it with some equivalent quasimeasure m' (in the sense of absolute continuity) then this amounts to passing to a new measure μ' on $\mathscr{X}$.

Therefore, if we talk only about the algebraic properties of the pair $\{\mathscr{X}_1', \mathscr{X}_2''\}$, which are not related to a choice of a particular measure μ then we may study these properties on using not the quasimeasure m itself but rather its "metric type" (Hellinger type). We know already that the homomorphism Ψ and, all the more, the quasimeasure m are in fact given not on $\mathscr{B}$ but rather on the larger σ-algebra

$$\widetilde{\mathscr{B}} = \mathscr{B} +_2 \ker \Psi$$

and this algebra is the common domain of definition of all quasimeasures equivalent to m.

Thus, this σ-algebra, or, which is equivalent, the σ-ideal $\ker \Psi$ is a true carrier of all information about the algebraic properties of the pair

[1] We explain it to make sure that the subalgebras $\mathscr{X}_1$ and $\mathscr{X}_2$ are absolutely arbitrary; they must only be continuous. It is not excluded that $\mathscr{X}_1 = \mathscr{X}_2 = \mathscr{X}$, in which case the quasimeasure m is supported in the diagonal of the square.

$\{\mathscr{X}_1, \mathscr{X}_2\}$. We call the so-constructed realization of the pair $\{\mathscr{X}_1, \mathscr{X}_2\}$ *basic.*

Note that in the situation we now consider when the pair of subalgebras has the basic realization, the quasimeasure m can contain no loads at points, i.e., it is continuous. This is connected with the fact that *an NBA, fully generated by a pair of continuous regular subalgebras, is itself continuous.* Indeed, we have mentioned in due time (Chapter 7, Section 6) a more general fact: a complete BA with atoms can possess no continuous regular subalgebras.

The basic realization serves often as means for an initial definition of a pair of subalgebras. Intending to have a pair with the particular properties, we choose a Borel quasimeasure m and consider as $\mathscr{X}$ the metric structure $\widehat{\mathscr{B}}$ and as $\mathscr{X}_1$ and $\mathscr{X}_2$, the subalgebras $\widehat{\mathscr{B}}_1$ and $\widehat{\mathscr{B}}_2$ corresponding to the coordinate partitions.

Much of what was said is valid in the nonseparable case. If $\mathscr{X}_1$ and $\mathscr{X}_2$ are homogeneous and have weights $\tau_1 \equiv \tau(\mathscr{X}_1)$ and $\tau_2 \equiv \tau(\mathscr{X}_2)$ then we may construct the realization of the pair $\{\mathscr{X}_1, \mathscr{X}_2\}$ by replacing the square I^2 with the "cube" of dimension $\tau_1 + \tau_2$. The details of this construction are left to the reader.

Returning to the separable case, we list the main properties of the pair $\{\mathscr{X}_1, \mathscr{X}_2\}$ which will be of interest for us in the sequel.

(A) The subalgebras $\mathscr{X}_1$ and $\mathscr{X}_2$ are algebraically independent.

(M_μ) The subalgebras $\mathscr{X}_1$ and $\mathscr{X}_2$ are μ-independent (with respect to the probability measure μ).

(M) The subalgebras $\mathscr{X}_1$ and $\mathscr{X}_2$ are metrically independent.

(N) The subalgebras $\mathscr{X}_1$ and $\mathscr{X}_2$ do not saturate bands.

In terms of the basic realization, these read:

(A) $m(b_1 \cap b_2) > 0$ for all $b_1 \in \mathscr{B}_1$ and $b_2 \in \mathscr{B}_2$ and $mb_1, mb_2 > 0$.

(M_μ) $m(b_1 \cap b_2) = mb_1 mb_2$ for all $b_1 \in \mathscr{B}_1$ and $b_2 \in \mathscr{B}_2$.

(M) There is a quasimeasure m' equivalent to m such that $m'(b_1 \cap b_2) = m'b_1 m'b_2$ for all $b_1 \in \mathscr{B}_1$ and $b_2 \in \mathscr{B}_2$.

(N) The "canonical measures" corresponding to the elements of coordinate partitions into vertical and horizontal segments are purely continuous for almost all these segments.

The last formulation needs clarification: completing the measure m, we come to a Lebesgue–Rokhlin space. The coordinate partitions will be

measurable and admit canonical measures. The words "almost all" in regard to segments comprising the partitions are related to the quotient measures corresponding to these partitions.

If the conditions (A) and (N) are fulfilled then we say that $\{\mathscr{X}_1, \mathscr{X}_2\}$ is a *standard pair.* If the condition (A) is replaced with a stronger condition (M_μ) then we say that $\{\mathscr{X}_1, \mathscr{X}_2\}$ is a *principal pair.* This means exactly that the quasimeasure m, determining the basic realization, coincides with the Lebesgue quasimeasure; the NBA $\{\mathscr{X}, \mu\}$ admits an isomorphism onto E_0^2 sending the subalgebras $\mathscr{X}_1$ and $\mathscr{X}_2$ to the subalgebras corresponding to the coordinate partitions ("subalgebras of vertical and horizontal cylinders").

Finally, the condition (M) tells us that the quasimeasure m is *equivalent* to the Lebesgue quasimeasure. The pair $\{\mathscr{X}_1, \mathscr{X}_2\}$ is in this case (algebraically) isomorphic to a principal pair. In this event the condition (N) holds as well: were it violated the quasimeasure m would fail to be equivalent to the Lebesgue quasimeasure and the obstacles are the "conditional atoms" on the segments of the coordinate partition. For instance, if we add to the Lebesgue quasimeasure l some probability quasimeasure l_1 that is supported in the diagonal of the square and distributed uniformly along this diagonal then, taking

$$m = \frac{1}{2}(l + l_1),$$

we see that the pair of subalgebras $\widehat{\mathscr{B}}_1$ and $\widehat{\mathscr{B}}_2$ is algebraically independent, fully generates $\widehat{\mathscr{B}}$, but fails to be metrically independent.

Thus, the metric independence of subalgebras means that the quasimeasure m is equivalent to the Lebesgue quasimeasure; i.e., it is representable as

$$m(e) = \iint\limits_e f\, dl \equiv \iint\limits_e f(x_1, x_2)\, dx_1 dx_2,$$

where f is a Lebesgue measurable almost everywhere strictly positive density.

We now show that the conditions (A) and (N) taken jointly do not imply (M). We will exhibit an example of a standard pair such that the quasimeasure m is singular with respect to the Lebesgue quasimeasure. (In the preceding example the condition (N) was absent.)

We approach the problem of constructing the quasimeasure m with the needed properties as follows: Distinguish a homomorphism F of the square I^2 onto some planar domain G and define m by the equality $me \equiv l(F(e))$ (with l the Lebesgue quasimeasure). Moreover, the choice of m is subordinate to the following conditions:

1) The "marginal" quasimeasures $m_1 \equiv m|_{\mathscr{B}_1}$ and $m_2 \equiv m|_{\mathscr{B}_2}$ must coincide with the traces of the Lebesgue quasimeasure l;

2) The conditional canonical measures on the vertical and horizontal segments must have no atoms;

3) The subalgebras $\widehat{\mathscr{B}}_1$ and $\widehat{\mathscr{B}}_2$ must be algebraically independent; i.e., $m(b_1 \cap b_2) > 0$ for all sets $b_1, b_2 \in \mathscr{B}$, composed of vertical (respectively, horizontal) segments such that $m_1 b_1 \equiv l b_1 > 0$ and $m_2 b_2 \equiv l b_2 > 0$;

4) The quasimeasure m is singular with respect to the Lebesgue quasimeasure; i.e., it is supported in an l-negligible set.

Similar "singular homomorphisms" are well studied in the one-dimensional situation. Let f_0 be one of these functions. Namely, assume that

1) f_0 is continuous, singular, and strictly increasing on $(-\infty, +\infty)$;

2) $f_0(0) = 0$;

3) $f_0(x+1) = f_0(x) + 1$ for all x.

It is well known that such functions exist. Put $g_0 \equiv f_0^{-1}$, so obtaining a function with the same properties. Arrange the two families:

$$f_a: \quad f_a(x) \equiv f_0(x) - a;$$

$$g_c: \quad g_c(y) \equiv f_0^{-1}(y+c) \equiv g_0(y+c).$$

We now define a homomorphism F of the unit square I^2 to some G by the formula

$$F(u,v) \equiv (g_u(v), v) \equiv (g_0(u+v), v). \tag{1}$$

The image of the square is the set G given by the inequalities

$$g_0(y) \le x \le g_1(y), \;\; 0 \le y \le 1.$$

We now put $me \equiv l(F(e))$ for all $e \in \mathscr{B}$ (where l is the planar Lebesgue quasimeasure). We must show that the subalgebras $\widehat{\mathscr{B}}_1$ and $\widehat{\mathscr{B}}_2$ comprise a standard pair.

a) The condition 1) is obvious for m_2, whereas this condition is easy to check for m_1 (it helps that $f_0(x+1) = f_0(x) + 1$).

b) The continuity of the canonical "conditional measures" (the condition 2)). It is easier here to replace the square I^2 and the coordinate partitions with an isomorphic space: the set G, the Lebesgue quasimeasure l, and the partitions by the lines

$$(1) \quad y = const; \qquad (2) \quad x = g_0(y+c).$$

It is nothing left to prove for the first family. Consider the family (2). If there is a "conditional atom" in one of the curves (2) then these atoms must comprise a while vertical line since the curves (2) go from the other

by vertical translations preserving the quasimeasure l. But then such a vertical line can never be negligible.

c) The algebraic independence of subalgebras (the condition 3)). Let b_1 and b_2 be the sets in the condition 3).

Assume that

$$V_i \equiv F(b_i) \ \ (i = 1, 2), \quad E \equiv V_1 \cap V_2 = F(b_1 \cap b_2)$$

(cf. Fig. 11).

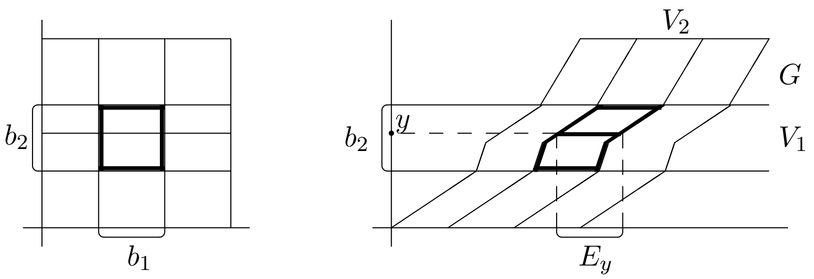

Fig. 11

Denote the linear Lebesgue quasimeasure by l_1, and let φ stand for the Lebesgue–Stieltjes quasimeasure generated by the function g. Noting that the horizontal section E_y of E is calculated by the formula

$$E_y \equiv g(b_1 + y) \quad (y \in b_2),$$

infer that

$$l_1 E_y = \varphi(b_1 + y) = \int \chi_{b_1}(s - y)\, d\varphi_s;$$

$$m(b_1 \cap b_2) = l(E) = \int\limits_{b_2} l_1 E_y\, dy = \int\limits_{b_2} dl_{1y} \int \chi_{b_1}(s - y)\, d\varphi_s$$

$$= \int d\varphi_s \int \chi_{b_2}(y)\chi_{b_1}(s - y)\, dl_{1y} = \int \chi_{b_2} * \chi_{b_1}\, d\varphi > 0.$$

(The convolution $\chi_{b_2} * \chi_{b_1}$ is continuous and differ from the identically zero function; the quasimeasure φ is strictly positive at intervals, its singularity notwithstanding.)

d) We now prove singularity. Find the quasimeasure of the set P_h, the square of side $h > 0$. Put

$$P_h = [x_1, x_1 + h] \times [x_2, x_2 + h].$$

We have

$$mP_h = l(F(P_h)) = \int\limits_{x_2}^{x_2+h} [g_0(\eta + x_1 + h) - g_0(\eta + x_1)]\, d\eta$$

$$= [g_0(x_2 + x_1 + (1 + \theta(h))h) - g_0(x_2 + x_1 + \theta(h))]h,$$

where $\theta(h) \in (0, 1)$. Taking an arbitrary sequence $h_n \downarrow 0$ and denoting $\theta(h_n) \equiv \theta_n$, infer

$$\frac{mP_{h_n}}{h_n^2} = \frac{g_0(x_2 + x_1 + (1 + \theta_n)h_n) - g_0(x_2 + x_1)}{h_n(1 + \theta_n)}(1 + \theta_n)$$

$$- \frac{g_0(x_2 + x_1 + \theta_n h_n) - g_0(x_2 + x_1)}{h_n \theta_n}\theta_n \equiv A_n(1 + \theta_n) - B_n\theta_n.$$

Assume that the derivative $g_0'(x_2+x_1)$ exists and is finite. Then it is easy to see that $g_0'(x_2 + x_1) = \lim_n \frac{mP_{h_n}}{h_n^2}$. We now use the singularity of g_0. Its derivative equals zero l-almost everywhere. Therefore, on almost all straight lines of the form $x_1 + x_2 = const$ we have

$$\lim_n \frac{mP_{h_n}}{h_n^2} = 0, \tag{2}$$

for whatever sequence of the squares of the form

$$P_{h_n} = [x_1, x_1 + h_n] \times [x_2, x_2 + h_n],$$

which collapses to the point (x_1, x_2). The words "almost all" relate here to the linear Lebesgue quasimeasure on the axis orthogonal to the given family of straight lines. It is clear that (2) holds then also l-almost everywhere on the square I^2. This implies that m and l are singular with respect to one another.

This example shows that algebraic independence does not imply metric independence even in the most favorable circumstances when both subalgebras are regular (and, consequently, closed), continuous, and failing to saturate any band. Recalling Theorem 9.7, we may conclude that it is impossible in this example to construct the direct product $\widehat{m}_1 \times \widehat{m}_2$; as a rule, the direct products of automorphisms on the subalgebras $\widehat{\mathscr{B}}_1$ and $\widehat{\mathscr{B}}_2$ do not exist either.

2. A test for metric independence

2.1 The main theorems

We now address the main problem: Under which conditions do the regular subalgebras $\{\mathscr{X}_t\}_{t \in T}$ comprise a metrically independent system?

We assume that the algebra $\mathscr{X}$ is normable, and let μ stand for a distinguished probability measure. Clearly, the condition of algebraic independence is necessary; the fact that it fails to be sufficient follows from the above example. We will always assume this condition fulfilled.

We outline possible approaches to solving the problem we have posed. Denote by μ_t the trace (projection) of the "basic" measure μ onto the subalgebra $\mathscr{X}_t$; i.e., $\mu_t \equiv \mu|_{\mathscr{X}_t}$. (Another term is a "marginal measure.") Consider two possible situations in which the question is answered in the affirmative.

I. It may happen that the measures μ_t admit the direct product, i.e., there exists a measure on $\mathscr{X}$ whose projections are μ_t (like μ) and such that our subalgebras are independent with respect to $\widetilde{\mu}$. (This is the most favorable situation since in this case the random variables measurable with respect to μ have the same individual characteristics (integrals, moments, etc.) with respect to the new measure as they have with respect to the old measure μ.)

From Theorem 9.7 it follows that if our system consists of two subalgebras then the existence of a direct product is not only sufficient but also necessary for metric independence. Obviously, the same holds in the case of an arbitrary finite system of subalgebras.

II. If the system $\{\mathscr{X}_t\}$ is infinite then we may encounter the following situation: metric independence is available; the subalgebras are independent with respect to some measure ν whereas the direct product does not exist; the measure ν has the traces on the subalgebras $\mathscr{X}_t$ which differ from μ_t; the attempt to "multiply" the marginal measures μ_t results in failure. This means that in this case there is no measure on $\mathscr{X}$ with respect to which the subalgebras are independent and which has μ_t as its projections (although the measure ν does exist).

So, the first approach to constructing a sought pair consists in attempting to "multiply" the measures μ_t. We consider the subalgebra $\mathscr{X}^0$ comprising all elementary polynomials of the form

$$u = x_{t_1} \wedge x_{t_2} \wedge \dots \wedge x_{t_s} \ \ (x_{t_i} \in \mathscr{X}_{t_i}, \ \ s = s(u))$$

and their disjoint sum

$$v = u_1 + u_2 + \dots + u_n$$

and define the quasimeasure μ^0 on it by the formula

$$\mu^0 u \equiv \mu_{t_1} x_{t_1} \cdot \mu_{t_2} x_{t_2} \cdot \ldots \cdot \mu_{t_s} x_{t_s},$$

$$\mu^0 v = \mu^0 u_1 + \mu^0 u_2 + \dots + \mu^0 u_n.$$

Soundness is easy; after noting this, we must justify the possibility of extending μ^0 from $\mathscr{X}^0$ to some measure on $\overline{\mathscr{X}^0}$. We will use Theorems 10 and 11 of Chapter 7.

If the system of subalgebras is finite then the above approach is in fact the only possible: if it does not lead to the destination then this destination is impossible to reach at all (as in the example of the previous subsection).

The other approach consists in constructing an equicontinuous continuous group of automorphisms of $\mathscr{X}$. If there are such groups on each $\mathscr{X}_t$ then we may try to multiply them while preserving equicontinuity. In much the same way as in the proof of Theorem 9.7, this will imply the independence of the subalgebras under study with respect to the invariant measure.

In the sequel we reserve more room to the first of the methods we have just described. Let us start considering finitely many subalgebras. All essential features reveal themselves already in the case of a PAIR $\{\mathscr{X}_1, \mathscr{X}_2\}$.

Thus, we assume given two regular subalgebras of an NBA $\{\mathscr{X}, \mu\}$, say, $\mathscr{X}_1$ and $\mathscr{X}_2$. Let $\mathscr{X}^0$ stand for the subalgebra they generate which consists of all possible disjoint sums of elements of the form $x_1 \wedge x_2$, where $x_1 \in \mathscr{X}_1$ and $x_2 \in \mathscr{X}_2$. Denote by θ the pair of disjoint partitions of unity:

$$\tau_1 \equiv \{x_i^1\}_{i=1}^{n(\theta)}, \ \ \tau_2 \equiv \{x_i^2\}_{i=1}^{n(\theta)}, \ \ \theta \equiv \{\tau_1, \tau_2\},$$

where $x_i^1 \in \mathscr{X}_1^+$ and $x_i^2 \in \mathscr{X}_2^+$ (a "kit of partitions"). Put

$$\mu_{i_1 i_2} \equiv \mu(x_{i_1}^1 \wedge x_{i_2}^2), \ \ i_1, i_2 = 1, 2, \dots, n(\theta).$$

In Theorem 7.12 we have mentioned the class $\mathscr{K}$ of continuous real functions Φ on $[0, +\infty)$ with the properties:

$$\Phi(0) = 0, \quad \frac{\Phi(u)}{u} \uparrow +\infty \quad \text{as} \quad u \longrightarrow +\infty.$$

It turns out useful once again.

Theorem 1. *For the metric independence of subalgebras $\mathscr{X}_1$ and $\mathscr{X}_2$ it is necessary and sufficient that there are functions $\Phi_1, \Phi_2 \in \mathscr{K}$ satisfying*

$$\mathscr{D}^{\Phi_1}(\mathscr{X}_1, \mathscr{X}_2) \equiv \sup_\theta \sum_{i_1, i_2 = 1}^{n(\theta)} \Phi_1\Big(\frac{\mu x_{i_1}^1 \cdot \mu x_{i_2}^2}{\mu_{i_1 i_2}}\Big) \mu_{i_1 i_2} < +\infty, \tag{$\mathscr{D}$}$$

$$\mathscr{D}_{\Phi_2}(\mathscr{X}_1, \mathscr{X}_2) \equiv \sup_{\theta} \sum_{i_1, i_2 = 1}^{n(\theta)} \Phi_2\Big(\frac{\mu_{i_1 i_2}}{\mu x^1_{i_1} \cdot \mu x^2_{i_2}}\Big) \mu x^1_{i_1} \cdot \mu x^2_{i_2} < +\infty$$

(the supremum is taken oven all kits θ).

PROOF. Let $\mathscr{X}_1$ and $\mathscr{X}_2$ be independent with respect to some probability measure. Then by Theorem 9.7 we may work with the direct product $\widetilde{\mu}$ of $\mu_1 \equiv \mu|_{\mathscr{X}_1}$ and $\mu_2 \equiv \mu|_{\mathscr{X}_2}$. Temporarily putting $\mu^0 \equiv \mu|_{\mathscr{X}^0}$ and $\widetilde{\mu}^0 \equiv \widetilde{\mu}|_{\mathscr{X}^0}$, note that μ^0 and $\widetilde{\mu}^0$ extend from $\mathscr{X}^0$ to $\mathscr{X}$ with preservation of countable additivity. By Theorem 7.12 this means existence of the functions Φ_1 and Φ_2 with the sought properties. The theorem is proved as regards necessity.

SUFFICIENCY. Define $\widetilde{\mu}^0$ first only on $\mathscr{X}^0$ by the rule

$$\widetilde{\mu}^0(x^1_1 \wedge x^1_2 + \dots + x^s_1 \wedge x^s_2) \equiv \sum_{i=1}^{s} \mu x^i_1 \cdot \mu x^i_2$$

for $x^i_1 \in \mathscr{X}_1$ and $x^i_2 \in \mathscr{X}_2$ (the soundness of this definition is immediate).

The first of the conditions ($\mathscr{D}$) guarantees in view of the same Theorem 7.12 that there exists a countably additive quasimeasure $\widetilde{\mu}$ extending $\widetilde{\mu}^0$ and representable as the integral

$$\widetilde{\mu}(x) = \int_x \mathfrak{q}\, d\mu,$$

where $\mathfrak{q} \in \mathfrak{L}^{\Phi_1}$ (an Orlicz class).

The second of the conditions ($\mathscr{D}$) means conversely that the initial measure μ is connected with $\widetilde{\mu}$ by the formulas

$$\mu x = \int \mathfrak{q}_*\, d\widetilde{\mu},$$

where $\mathfrak{q}_* \in \mathfrak{L}^{\Phi_2}$. It is then clear that $\widetilde{\mu}$ is also a measure and $\mathfrak{q}_* = \frac{1}{\mathfrak{q}}$. We see that $\mathscr{X}_1$ and $\mathscr{X}_2$ are independent with respect to $\widetilde{\mu}$. The proof of the theorem is complete.

Under the conditions ($\mathscr{D}$) the densities $\frac{d\widetilde{\mu}}{d\mu} \equiv \mathfrak{q}$ and $\frac{d\mu}{d\widetilde{\mu}} \equiv \mathfrak{q}_*$ belong to the Orlicz classes $\mathfrak{L}^{\Phi_1}(\mu)$ and $\mathfrak{L}^{\phi_2}(\widetilde{\mu})$ respectively. Moreover,

$$\int \Phi_1(\mathfrak{q})\, d\mu \le \mathscr{D}^{\Phi_1}, \quad \int \Phi_2(\mathfrak{q}_*)\, d\widetilde{\mu} \le \mathscr{D}^{\Phi_2}.$$

The estimates of this sort may be helpful. We will return to this matter later. Now, we pose the following question: What will happen if we take only the first of the inequalities ($\mathscr{D}$)? In this case we may assert

the existence of a quasimeasure $\widetilde{\mu}$ with respect to which $\mathscr{X}_1$ and $\mathscr{X}_2$ are independent (in the previous sense). Intending to find a genuine measure, we might have to reject the whole band that annihilates $\widetilde{\mu}$. On the remaining part, the traces of the subalgebras $\mathscr{X}_1$ and $\mathscr{X}_2$ will be $\widetilde{\mu}$-independent and $\widetilde{\mu}$ will be a measure. To give an example of this situation is an easy matter, and we leave this to the reader.

Exactly the same theorem holds for an arbitrary finite collection $\mathscr{X}_1, \mathscr{X}_2, \dots, \mathscr{X}_m$ of algebraically independent regular subalgebras. We confine exposition to its statement since the proof contains no new ideas.

We now consider the kits of the form $\theta = \{\tau_1, \tau_2, \dots, \tau_m\}$, where

$$\tau_i = \{x_k^i\}_{k=1}^{n(\theta)}, \ x_k^i \in \mathscr{X}_i \quad (k = 1, 2, \dots, n(\theta); \ i = 1, 2, \dots, m),$$

$\sum_k x_k^i = \mathbf{1}$ for all $i = 1, 2, \dots, m$. Put

$$\mu_{i_1 i_2 \dots i_m} \equiv \mu(x_{i_1}^1 \wedge x_{i_2}^2 \wedge \dots \wedge x_{i_m}^m),$$

$$x_{i_k}^k \in \tau_k \ (k = 1, 2, \dots, m; \ i_1, i_2, \dots, n(\theta)),$$

$$\mathfrak{q}_{i_1 i_2 \dots i_m} \equiv \frac{\mu x_{i_1}^1 \cdot \mu x_{i_2}^2 \cdot \dots \cdot \mu x_{i_m}^m}{\mu_{i_1 i_2 \dots i_m}}, \quad \mathfrak{q}^*_{i_1 i_2 \dots i_m} \equiv \frac{1}{\mathfrak{q}_{i_1 i_2 \dots i_m}}.$$

Theorem 2. *For the metric independence of $\mathscr{X}_1, \dots, \mathscr{X}_m$ it is necessary and sufficient that there exist functions $\Phi_1, \Phi_2 \in \mathscr{K}$ satisfying*

$$\mathscr{D}^{\Phi_1} \equiv \sup_\theta \sum_{i_1, i_2, \dots, i_m = 1}^{n(\theta)} \Phi_1(\mathfrak{q}_{i_1 i_2 \dots i_m}) \mu_{i_1 i_2 \dots i_m} < +\infty,$$

$$(\mathscr{D}')$$

$$\mathscr{D}_{\Phi_2} \equiv \sup_\theta \sum_{i_1, i_2, \dots, i_m = 1}^{n(\theta)} \Phi_2(\mathfrak{q}^*_{i_1 i_2 \dots i_m}) \mu x_{i_1}^1 \cdot \mu x_{i_2}^2 \cdot \dots \cdot \mu x_{i_m}^m < +\infty$$

(with the supremum taken over all kits θ).

As before, under the conditions $(\mathscr{D}')$ the subalgebras turn out independent with respect to the measure $\widetilde{\mu}$ that is uniquely determined from the formula

$$\widetilde{\mu}(x^1 \wedge x^2 \wedge \dots \wedge x^m) = \mu x^1 \cdot \mu x^2 \cdot \dots \cdot \mu x^m, \ x^i \in \mathscr{X}_i \ (i = 1, 2, \dots, m).$$

Moreover, the densities $\mathfrak{q} \equiv \frac{d\widetilde{\mu}}{d\mu}$ and $\mathfrak{q}_* \equiv \frac{d\mu}{d\widetilde{\mu}}$ belong respectively to the Orlicz classes $\mathfrak{L}^{\Phi_1}(\mu)$ and $\mathfrak{L}^{\Phi_2}(\widetilde{\mu})$ and satisfy the same inequalities as before.

In many important cases we may take the supremum in the inequalities $(\mathscr{D})$ and $(\mathscr{D}')$ not over all kits θ but rather over those of them that consist of partitions into the elements of the SAME MEASURE: $\mu x_k^i = \frac{1}{n(\theta)}$ for all i and k. It happens so for instance if the subalgebras $\mathscr{X}_i$ are continuous. In this event we may replace $\mathscr{X}^0$ with a narrower subalgebra $\mathscr{X}^{00}$ that is generated by the elements of the form $x^1 \wedge x^2 \wedge \dots \wedge x^m$, where

$$x^i \in \mathscr{X}_i,\ \ \mu x^1 = \mu x^2 = \dots = \mu x^m = \frac{1}{n} \quad (n = 1, 2, \dots).$$

The point is that in this case $\overline{\mathscr{X}^o} = \overline{\mathscr{X}^{00}} = \overline{\mathscr{X}\langle \mathscr{X}_1, \dots, \mathscr{X}_m \rangle}$. Another case in which it suffices to use the kits of the just-mentioned particular type is the case in which the subalgebras $\mathscr{X}_i$ are discrete and have the form $\mathscr{X}^{(\tau)}$, where τ is a partition into elements of the same measure $p = \frac{1}{n}$. In these cases, the conditions $(\mathscr{D})$ and $(\mathscr{D}')$ have a simpler form and contain only the numbers $\mu_{i_1 i_2 \dots i_m}$ characterizing the kit:

$$\overline{\mathscr{D}}^{\Phi_1}(\mathscr{X}_1, \mathscr{X}_2) \equiv \sup_\theta \sum_{i_1, i_2 = 1}^{n} \Phi_1\Big(\frac{1}{n^2 \mu_{i_1 i_2}}\Big) \mu_{i_1 i_2} < +\infty,$$

$$(\overline{\mathscr{D}})$$

$$\overline{\mathscr{D}}_{\Phi_2}(\mathscr{X}_1, \mathscr{X}_2) \equiv \sup_\theta \sum_{i_1, i_2 = 1}^{n} \frac{1}{n^2} \Phi_2\Big(n^2 \mu_{i_1 i_2}\Big) < +\infty.$$

(Throughout $n = n(\theta)$.)

$$\overline{\mathscr{D}}^{\Phi_1}(\mathscr{X}_1, \mathscr{X}_2, \dots, \mathscr{X}_m)$$
$$\equiv \sup_\theta \sum_{i_1, i_2, \dots, i_m = 1}^{n} \Phi_1\Big(\frac{1}{n^m \mu_{i_1, i_2, \dots, i_m}}\Big) \mu_{i_1 i_2 \dots i_m} < +\infty,$$

$$(\overline{\mathscr{D}}')$$

$$\overline{\mathscr{D}}_{\Phi_2}(\mathscr{X}_1, \mathscr{X}_2, \dots, \mathscr{X}_m) \equiv \sup_\theta \sum_{i_1, i_2, \dots, i_m = 1}^{n} \frac{1}{n^m} \Phi_2(n^m \mu_{i_1, i_2, \dots, i_m}) < +\infty.$$

We now turn to the case of an infinite algebraically independent system of subalgebras $\{\mathscr{X}_t\}_{t \in T}$. As was mentioned, the measure $\widetilde{\mu}$ may fail to exist even in the event of metric independence. Therefore, the previous theorem fails as regards necessity. However, its conditions still remain sufficient.

Assume that $T' \equiv \{t_1, t_2, \dots, t_m\} \subset T$, $m = m(T')$, and t_i are pairwise distinct.

Theorem 3. *For the metric independence of a family of subalgebras $\{\mathscr{X}_t\}$ it is sufficient that there exist functions $\Phi_1, \Phi_2 \in \mathscr{K}$ such that*

$$\widehat{\mathscr{D}}^{\Phi_1} \equiv \sup_{T' \subset T} \mathscr{D}^{\Phi_1}(\mathscr{X}_{t_1}, \mathscr{X}_{t_2}, \dots, \mathscr{X}_{t_m}) < +\infty,$$

$$(\mathscr{D}_\infty)$$

$$\widehat{\mathscr{D}}_{\Phi_2} \equiv \sup_{T' \subset T} \mathscr{D}_{\Phi_2}(\mathscr{X}_{t_1}, \mathscr{X}_{t_2}, \dots, \mathscr{X}_{t_m}) < +\infty.$$

In this event, the subalgebras $\mathscr{X}_t$ are independent with respect to the measure $\widetilde{\mu}$, the direct product of the measures μ_t which exist in these circumstances, and the densities $\mathfrak{q} \equiv \frac{d\widetilde{\mu}}{d\mu}$ and $\mathfrak{q}_* \equiv \frac{d\mu}{d\widetilde{\mu}} = \frac{1}{\mathfrak{q}}$ belong to the Orlicz classes $\mathfrak{L}^{\Phi_1}(\mu)$ and $\mathfrak{L}^{\Phi_2}(\widetilde{\mu})$, while

$$\int \Phi_1(\mathfrak{q})\, d\mu \le \widehat{\mathscr{D}}^{\Phi_1},$$

$$\int \Phi_2(\mathfrak{q}_*)\, d\widetilde{\mu} \le \widehat{\mathscr{D}}_{\Phi_2}.$$

If the subalgebras $\mathscr{X}_t$ are continuous or generated by partitions of unity into the elements of the same measure $p = \frac{1}{n}$ then it is possible to replace $\mathscr{D}^{\Phi_1}$ with $\overline{\mathscr{D}}^{\Phi_1}$ and $\mathscr{D}_{\phi_2}$ with $\overline{\mathscr{D}}_{\phi_2}$ in the inequalities $(\mathscr{D}_\infty)$.

This theorem, in much the same way as all previous, is a consequence of Theorem 7.11.

Remark. Since Theorem 3 speaks only about sufficient conditions, as μ_t there may appear whatever measures on the subalgebras $\mathscr{X}_t$ rather than only the marginal measures $\mu|_{\mathscr{X}_t}$.

Beyond a doubt, the most interesting case is as follows:

$$\Phi_1(u) = u^{p_1}, \ \ \Phi_2(u) = u^{p_2} \quad (p_1, p_2 > 1).$$

For instance, the conditions $(\overline{\mathscr{D}}')$ now take the form

$$\sup_\theta \sum_{i_1, i_2, \dots, i_m = 1}^{n} \mu_{i_1 i_2 \dots i_m}^{-p_1+1} = O(n^{mp_1}),$$

$$\sup_\theta \sum_{i_1, i_2, \dots, i_m = 1}^{n} \mu_{i_1 i_2 \dots i_m}^{p_2} = O(n^{-m(p_2-1)}).$$

The densities $\mathfrak{q}$ and $\mathfrak{q}_*$ will belong to $\mathfrak{L}^{p_1}(\mu)$ and $\mathfrak{L}^{p_2}(\widetilde{\mu})$, while the estimates for the norms are obvious.

As regards the case $p_1, p_2 = +\infty$, it is not embraced formally by these theorems but in fact is trivial: If

$$0 < a \le \frac{\mu_{i_1 i_2 \dots i_m}}{\mu x^1_{i_1} \dots \mu x^m_{i_m}} \le b < +\infty$$

then we observe metric independence and $|\mathfrak{q}_*| \le b\mathbf{1}^\bullet$, $|\mathfrak{q}| \le \frac{1}{a}\mathbf{1}^\bullet$.

We illustrate what was said above by the classical example. Let an NBA $\mathscr{X}$ be homogeneous and separable. It thus has the form

$$\mathscr{X} = \prod_{n=1}^{\infty} \mathscr{X}_n,$$

where $\mathscr{X}_n$ are simple subalgebras like $\mathscr{X}_n = \{u_n, Cu_n, \mathbf{0}, \mathbf{1}\}$ and, moreover, $\mu u_n \mu C u_n = \frac{1}{2}$. In other words, it is the tossing of a symmetric coin. As a realization space for our BA $\mathscr{X}$ we may take the Cantor discontinuum of countable type, the countable power of the discrete two-point space $\{0, 1\}$. The points of this space are binary sequences $\chi = (\chi_1, \chi_2, \dots)$. Each of this points is a history of an infinite coin tossing: if $\chi_i = 1$ then it means that we observe heads at trial i. Each element u_n is the coset containing $\{\chi \mid \chi_n = 1\}$. Therefore, the number is the probability of the event that "at trial n we observe heads and the rest of the outcomes are arbitrary." We may assume that this probability is $\frac{1}{2}$; i.e., the coin is symmetric. However, we do not assume μ-independence; but we do assume algebraic independence; this means that we ascribe nonzero probabilities to all events

$$u_{n_1} \wedge u_{n_2} \wedge \dots \wedge u_{n_i} \wedge Cu_{n'_1} \wedge \dots \wedge Cu_{n'_j}$$

("at trials $n_1, \dots, n_i$ we have heads and at trials $n'_1, \dots, n'_j$ we have tails, while the rest of the outcomes are arbitrary"). Assigning these probabilities determines the measure μ in this case; it is these numbers that comprise the set of all $\mu_{i_1 i_2 \dots i_m}$ we encounter in our theorems. Let us elaborate this. Clearly, the numbers $\mathscr{D}^{\cdots}_{\cdots}(\mathscr{X}_{t_1}, \dots, \mathscr{X}_{t_m})$ increase as the collection $T' = \{t_1, t_2, \dots, t_m\}$ is enlarged. Hence, in our case when $T = \mathbb{N}$, we may take not all $T' \subset \mathbb{N}$ but rather the intervals $\{1, 2, \dots, m\}$. To each $m = 1, 2, \dots$ there now corresponds the unique kit of partitions θ_m which consists of

$$\tau_1 = \{u_1, Cu_1\}, \dots, \tau_m = \{u_m, Cu_m\}.$$

Let $u^i_1 \equiv u_i$ and $u^i_2 \equiv Cu_i$. The number

$$\mu_{i_1 i_2 \dots i_m} \equiv \mu(u^1_{i_1} \wedge \dots \wedge u^m_{i_m})$$

is then probability of the following event: "If $k \le m$ then at trial i we have heads if $i_k = 1$ and tails if $i_k = 0$; at trial $m+1$ and the subsequent trials the outcomes are arbitrary." If these subalgebras were μ-independent then all $\mu_{i_1 \dots i_m}$ would have the value $\frac{1}{2^m}$, however we do not assume this. The natural hypothesis that the coin is symmetric plays no role in the sequel. Since we have a sole kit θ; therefore,

$$\overline{\mathscr{D}}^{\Phi_1}(\mathscr{X}_1, \mathscr{X}_2, \dots, \mathscr{X}_m) = \sum_{i_1, i_2, \dots, i_m = 1}^{2} \Phi_1\Big(\frac{1}{2^m \mu_{i_1 i_2 \dots i_m}}\Big) \mu_{i_1 i_2 \dots i_m},$$

$$\overline{\mathscr{D}}_{\Phi_2}(\mathscr{X}_1, \mathscr{X}_2, \dots, \mathscr{X}_m) = \sum_{i_1, i_2, \dots, i_m = 1}^{2} \frac{1}{2^m} \Phi_2(2^m \mu_{i_1 i_2 \dots i_m}),$$

and the condition $(\mathscr{D}_\infty)$ takes the form

$$\sup_m \sum_{i_1, i_2, \dots, i_m = 1}^{2} \Phi_1\Big(\frac{1}{2^m \mu_{i_1 i_2 \dots i_m}}\Big) \mu_{i_1 i_2 \dots i_m} < +\infty,$$

$$\sup_m \sum_{i_1, i_2, \dots, i_m = 1}^{2} \frac{1}{2^m} \Phi_2(2^m \mu_{i_1 i_2 \dots i_m}) < +\infty. \qquad (\mathscr{D}^0_\infty)$$

Under this condition, we may multiply the measures μ_i that are determined by $\mu_i(u) \equiv \mu_i(Cu) \equiv \frac{1}{2}$, and their direct product $\widetilde{\mu}$ is a measure on $\mathscr{X}$ while the subalgebras $\mathscr{X}_n$ are μ-independent.[2] In fact, the talk is about the question whether the initially prescribed probability is equivalent to the Bernoulli probability $\beta_{\frac{1}{2}\frac{1}{2}}$.

Our conditions look simplest if $\Phi_1(u) = u^{p_1}$ and $\Phi_2 = u^{p_2}$; i.e.,

$$\sum_{i_1, i_2, \dots, i_m = 1}^{2} \frac{1}{\mu^{p_1 - 1}_{i_1 i_2 \dots i_m}} = O(2^{2mp_1}),$$

$$\sum_{i_1, i_2, \dots, i_m = 1}^{2} \mu^{p_2}_{i_1 i_2 \dots i_m} = O\Big(\frac{1}{2^{2m(p_2 - 1)}}\Big).$$

Under this condition, there exists a probability measure $\widetilde{\mu}$ (which, slightly abusing the language, we may identify with the Bernoulli probability $\beta_{\frac{1}{2}\frac{1}{2}}$), and the densities $\mathfrak{q} = \frac{d\widetilde{\mu}}{d\mu}$ and $\mathfrak{q}_* = \frac{d\mu}{d\widetilde{\mu}}$ belong to the corresponding classes $\mathfrak{L}^{p_1}(\mu)$ and $\mathfrak{L}^{p_2}(\widetilde{\mu})$.

[2]The measures μ_i do not necessarily coincide with the marginal measures (since the coin may be asymmetric).

We will call the constants $\mathscr{D}^\Phi, \mathscr{D}_\Phi$, etc. of the preceding theorems "$\mathscr{D}$-numbers."

Inspecting all versions of our conditions of the "type ($\mathscr{D}$)," we may see the principal feature: among $\mu_{i_1 i_2 \dots i_m}$ there must be neither very small nor very large numbers.

The sum of these numbers is equal to 1 always (for all m). For metric independence we need that the addends do not differ greatly from one another "on average." The ideal case is that in which they are all equal; this is the case of μ-independence. If the system of $\mu_{i_1 i_2 \dots i_m}$ has a strong "disparity": there are many large (hence, many small) numbers then metric independence is absent whereas algebraic independence might be available as it was in the example of the previous subsection. If these numbers are not much dispersed and tend to their mean value then we may hope that some measure is available with respect to which our subalgebras are independent. This prompts us to treat this family as a RANDOM VARIABLE. Clearly, it is not the numbers $\mu_{i_1 i_2 \dots i_m}$ themselves, having a fixed sum, that matter but rather their interrelations.

We now consider in more detail the space in which there is given a sequence of simple subalgebras $\mathscr{X}_n \equiv \{u_n, Cu_n, \mathbf{0}, \mathbf{1}\}$. In this event we have a sole kit for each $m = 1, 2, \dots$ and the family $\{\mu_{i_1 i_2 \dots i_m}\}$ is determined uniquely. (As before,

$$\mu_{i_1 i_2 \dots i_m} \equiv \mu(u^1_{i_1} \wedge \dots \wedge u^m_{i_m}), \quad u^i_1 \equiv u_i, \; u^i_2 \equiv Cu_i.)$$

We view each family of the form

$$\Lambda_m \equiv \{2^m \mu_{i_1 i_2 \dots i_m} \equiv \lambda_{i_1 i_2 \dots i_m}\}$$

as a random variable on the probability space $X^{(m)} \equiv \{0, 1\}^m$ (the mth power of the two-point space). The basic measure on this finite space is Bernoulli: the load at every point $\chi = (\chi_1, \chi_2, \dots, \chi_m)$ is equal to $\frac{1}{2^m}$. Correspondingly, the distribution function $F_m(t)$ is the number of points $\chi = (\chi_1, \chi_2, \dots, \chi_m)$, with $\lambda_{i_1 i_2 \dots i_m} < t$, divided by 2^m.

The key role belongs to the behavior of these functions as $t \longrightarrow +\infty$ and $t \longrightarrow +0$. (Clearly, $F_m(t) = 0$ for $t \le 0$ and $F_m(t) = 1$ for $t > 0$.) Alongside F_m we also introduce the functions F^*_m as follows:

$$F^*_m(t) \equiv \sum \mu_{\chi_1 \chi_2 \dots \chi_m} \quad (t > 0),$$

where the sum is taken over the points $\chi = (\chi_1, \chi_2, \dots, \chi_m)$ such that $\lambda_{i_1 i_2 \dots i_m} > \frac{1}{t}$. For $t \le 0$ we put $F^*_m(t) = 0$. It is easy to derive the formulas that connect F_m and F^*_m:

$$F^*_m(t) = \int\limits_{(\frac{1}{t}, +\infty)} \tau \, dF_m(\tau), \;\; F_m(t) = \int\limits_{(\frac{1}{t}, +\infty)} \tau \, dF^*_m(\tau). \tag{3}$$

Return to the conditions ($\mathscr{D}^0_\infty$). They may be rewritten as

$$\sup_m \int\limits_{(-\infty,+\infty)} \Phi_1(\tau)\, dF^*_m(\tau) < +\infty,$$

$$\sup_m \int\limits_{(-\infty,+\infty)} \Phi_2(\tau)\, dF_m(\tau) < +\infty.$$

The possibility of choosing $\Phi_1, \Phi_2 \in \mathscr{K}$ with this property means exactly the uniform convergence with respect to $m = 1, 2, \ldots$ in the formulas

$$\left.\begin{aligned} F^*_m\Big(\frac{1}{t}\Big) &\equiv \int\limits_{(t,+\infty)} \tau\, dF_m(\tau) \longrightarrow 0, \\ F_m\Big(\frac{1}{t}\Big) &\equiv \int\limits_{(t,+\infty)} \tau\, dF^*_m(\tau) \longrightarrow 0. \end{aligned}\right\} \text{ as } t \longrightarrow +\infty \tag{4}$$

(It is the Vallée-Poussin Theorem.) This means equicontinuity of the functions F^*_m and F_m at zero and as $t \longrightarrow +\infty$.[3]

The equalities (3) allow us to deal only with the functions F_m (or only with F^*_m) and to write down (4) as

$$F_m(\tau) \rightrightarrows 1 \quad \text{as} \quad \tau \longrightarrow +\infty, \quad F_m(\tau) \rightrightarrows 0 \quad \text{as} \quad \tau \longrightarrow +0, \tag{5}$$

or, equivalently,

$$F^*_m(\tau) \rightrightarrows 1 \quad \text{as} \quad \tau \longrightarrow +\infty, \quad F^*_m(\tau) \rightrightarrows 0 \quad \text{as} \quad \tau \longrightarrow +0. \tag{5'}$$

(The sign $\rightrightarrows$ denotes uniform convergence with respect to m.)

If we put $\rho_0(\tau) \equiv \sup_m F_m(\tau)$ and $\rho_\infty(\tau) \equiv \inf_m F_m(\tau)$ then the conditions (5) take the form

$$\lim_{\tau \longrightarrow +\infty} \rho_\infty(\tau) = 1, \quad \lim_{\tau \longrightarrow +0} \rho_0(\tau) = 0.$$

We thus come to the following

[3]It is well known that "equi-integrability," expressed by the formula $\int_{(t,+\infty)} \tau\, dG_\xi(\tau) \rightrightarrows 0$, with G_ξ a distribution function, is equivalent to uniform convergence in the relation $G_\xi(\tau) \rightrightarrows 1$ as $\tau \longrightarrow +\infty$ (for instance, cf. J. Neveu [1, p. 80]).

Theorem 4. *The conditions* (5) *are sufficient for metric independence of the subalgebras* $\mathscr{X}_n \equiv \{u_n, Cu_n, \mathbf{0}, \mathbf{1}\}$; *they are also necessary for the existence in the NBA* $\{\mathscr{X}, \mu\}$ *of the product measure* $\widetilde{\mu}$, *where*

$$\widetilde{\mu} = \bigotimes \prod_{n=1}^{\infty} \mu_n$$

and $\mu_n u = \mu_n C u_n = \frac{1}{2}$ $(n = 1, 2, \dots)$.

We now discuss the role of each of the conditions (5) in turn. The first relates to the behavior of the function F_m about $+\infty$; whereas the second, about zero. Both are important, each in its own right. Recall that the problem we are solving consists in extending the quasimeasure ν_0 that is given as

$$\nu_0(x_1 \wedge x_2 \wedge \dots \wedge x_n) = \mu_1 x_1 \cdot \mu_2 x_2 \cdot \ldots \cdot \mu_n x_n, \ \ x_n \in \mathscr{X}_n \ \ (n = 1, 2, \dots).$$

The behavior of F_m about zero is important for ν_0 to admit a countably additive extension to a quasimeasure on $\mathscr{X}$. The presence of many "small" addends in $\sum \mu \dots$ is an obstacle to the possibility of extension. This obstacle is removed by the second of the conditions (5) (in the general case, the first condition ($\mathscr{D}$)). In the case when this sum has may "large" addends, the extension of a quasimeasure ν_0, even if existent, may degenerate, i.e., vanish on a whole band. The antidote here is provided by the first of the conditions (5) (respectively, the second of the conditions ($\mathscr{D}$)).

We now consider a more general situation. Let

$$\mu_n u_n \equiv p_n, \ \ \mu_n C u_n \equiv q_n = 1 - p_n, \quad 0 < p_n < 1 \quad (n = 1, 2, \dots).$$

(Coin tossing in which we change the coin at each trial.)

In this event we still may pose the same question: Does there exist the product measure $\widetilde{\mu}$ on $\{\mathscr{X}, \mu\}$? It is easy to define for this problem the distribution functions analogous to F_m and F_m^* we have already considered above and to prove a theorem that will generalize Theorem 4. We leave this task to the reader.

The metric independence of the subalgebras $\mathscr{X}_m$ means that the measure $\widetilde{\mu}$ may be defined on the NBA $\{\mathscr{X}, \mu\}$ AT LEAST FOR ONE SEQUENCE $\{p_n\}$. It seems interesting to find necessary and sufficient conditions for this to happen in terms of the initial family $\{\mu_{i_1 \dots i_m}\}$.

Among the available results on the problem of metric independence, we must mention a nontrivial theorem belonging in fact to H. Araki and E. J. Woods.[4] Consider the product of countably many simple

[4] H. Araki and E. J. Woods [1]. The formulation below was communicated to the author by A. M. Vershik.

subalgebras: $\mathscr{X} = \prod_{n\in\mathbb{N}} \mathscr{X}_n$, where $\mathscr{X}$ is an NBA. To each set $\Delta \subset \mathbb{N}$ we assign the two regular subalgebras:

$$\mathscr{X}_\Delta \equiv \prod_{n\in\Delta} \mathscr{X}_n, \quad \mathscr{X}'_\Delta \equiv \prod_{n\in\mathbb{N}\setminus\Delta} \mathscr{X}_n.$$

If $\mathscr{X}_\Delta$ and $\mathscr{X}'_\Delta$ are metrically independent for every Δ then so is the whole system $\{\mathscr{X}_\mu\}$.

The problem that we consider in this subsection is a version of the martingale problem. For instance, in the case of two subalgebras the sought density $\mathfrak{q} = \frac{d\tilde{\mu}}{d\mu}$ is the limit of the martingale $\{\mathfrak{q}_\theta, \mathscr{X}^\theta\}$, where $\theta = \{\tau_1, \tau_2\}$ and $\mathscr{X}^\theta$ is the subalgebra generated by the elements of the form $x' \wedge x'', x' \in \tau_1, x'' \in \tau_2$,

$$\mathfrak{q}_\theta = \sum_{x'\in\tau_1,\, x''\in\tau_2} \frac{\mu x' \mu x''}{\mu(x' \wedge x'')} x'^\bullet \wedge x''^\bullet.$$

We now bear in mind the context of Theorem 1: we assume that $\tau_1 \equiv \tau_1^{(n)}$ and $\tau_2 \equiv \tau_2^{(n)}$ are fundamental sequences of partitions in $\mathscr{X}_1$ and $\mathscr{X}_2$. Then, for instance the first inequality of $(\mathscr{D})$ implies the convergence of our martingale as well as the existence of a totally additive quasimeasure with density $\mathfrak{q} \in \mathfrak{L}^{\Phi_1}$.

However, the general theory of martingales guarantees convergence without the conditions $(\mathscr{D})$. In our case, $\int \mathfrak{q}_\theta \, d\mu = 1$, $\mathfrak{q}_\theta > \mathbf{0}$ for all θ, which suffices for $\mathfrak{q}_\theta$ to converge in $\mathfrak{S}_{\mathscr{X}}$ to some element $\mathfrak{q} \in \mathfrak{L}_{\mathscr{X}}(\mu)$. (This result belongs to J. Doob.) However, if the first inequality of $(\mathscr{D})$ fails for all $\Phi_1 \in \mathscr{K}$ then $\mathfrak{q}$ will never be a density: there will be no convergence in $\mathfrak{L}$ and we will have with certainty that $\int \mathfrak{q} \, d\mu < 1$. Moreover, rather feasible is the possibility of degeneration: $\int \mathfrak{q} \, d\mu = 0$. It is exactly what happens in the example of Section 1.

2.2 Possible applications

The subalgebras $\mathscr{X}_n$ are mostly generated by random variables, the resolutions of the identity $\mathfrak{f}_1, \mathfrak{f}_2, \ldots$. In this case each of the subalgebras $\mathscr{X}_n$ is separable and generated by the spectral family $\{e^{\pm}_\lambda(\mathfrak{f}_n)\}$. Independence of subalgebras with respect to a measure μ is the same stochastic independence of random variables that plays such a significant role in probability theory. Here μ is a distinguished initial probability that occupies the first place by some reasons.

Assume now that μ-independence is absent whereas metric independence is present. This means that the random variables will become independent after passing to another measure ν. Recall that all these

measures are mutually continuous: if, for instance, $\mu x_n \longrightarrow 0$ then $\nu x_n \longrightarrow 0$ and conversely. Hence, metric independence, if established, enables us to apply to the sequence $\mathfrak{f}_n$ many theorems that were demonstrated for independent random variables. The talk is primarily about the theorems in which the values of the measure do not appear but which speak about almost everywhere convergence ((o)-convergence) or convergence in measure. True enough, these theorems usually involve the numerical characteristics of random variables: moments, variances, expectations, etc. However if we are in the context of the theorem of this subsection when we may take $\widetilde{\mu} = \bigotimes \prod \mu_k$ as ν then the individual characteristics will not vary on substituting $\widetilde{\mu}$ for μ (which suffices in many cases).

We give a few examples illustrating this observation. Here go several classical theorems which treat the sequences of μ-independent random variables in their commonest versions.

Let $\{\mathfrak{f}_n\}$ be such a sequence, and let $S_n = \sum_{k=1}^{n} \mathfrak{f}_k$, while $\mathscr{X}_n$ is the regular subalgebra of the main BA $\mathscr{X}$ which is fully generated by $\mathfrak{f}_n$.

In this environment the following theorems hold:

1. The zero-one law. The "residual" subalgebra

$$\bigcap_{n=1}^{\infty} \overline{\mathscr{X}\langle \mathscr{X}_n, \mathscr{X}_{n+1}, \dots \rangle}$$

is trivial (i.e., it consists only of $\mathbf{0}$ and $\mathbf{1}$).

2. The strong law of large numbers. By MS_n we as usual denote the expectation of the sum S_n, i.e., the integral $\int S_n \, d\mu$; by $\mathscr{D}\mathfrak{f}_n$, the variance of $\mathfrak{f}_n$:

$$\mathscr{D}\mathfrak{f}_n \equiv \int \mathfrak{f}_n^2 \, d\mu - \Big(\int \mathfrak{f}_n \, d\mu \Big)^2.$$

Let $b_n \uparrow +\infty$ and $\sum_{n=1}^{\infty} \frac{\mathscr{D}\mathfrak{f}_n}{b_n^2} < +\infty$. Then

$$\frac{S_n - MS_n}{b_n} \longrightarrow 0$$

"almost surely," i.e. in the sense of (o)-convergence in the universally complete K-space over the BA $\mathscr{X}$.

3. The law of the iterated logarithm. Let $\mathfrak{f}_n$ be identically distributed random variables;

$$\int \mathfrak{f}_n \, d\mu = 0; \quad \int \mathfrak{f}_n^2 \, d\mu \equiv \sigma^2 > 0.$$

Then

$$\overline{\lim} \frac{S_n}{\sqrt{2\sigma^2 n \ln\ln n}} = \mathbf{1}$$

(the upper limit is taken in the same K-space).

In these theorems (as in many other results of this sort), the condition of μ-independence may be replaced with the condition of Theorem 3 guaranteeing metric independence. Substituting $\widetilde{\mu}$ for μ, we obtain independence, while the numbers MS_n, $\mathscr{D}\mathfrak{f}_n$, and σ remain the same.

The matter is more involved with the Central Limit Theorem which concerns the behavior of the distribution functions that are essentially related to a particular measure.

Needless to say that the conditions of our theorems are difficult to verify; to estimate the $\mathscr{D}$-numbers becomes sometimes an uneasy problem. Note however that, assuming the subalgebras μ-independent (in order to apply the technique of probability theory), this hypothesis is usually justified by some physical reasons or experimental data. In these cases, treating the variables that have a clear physical meaning and considered to be primary, a rigorous mathematical proof seems to be unlikely; independence is understood to be an experimental fact reflecting the absence of any perceptible correlation between the phenomena under study. In reality, we should have no doubts however that some correlation exists but is practically intractable and so we may treat the variable in question as practically independent. But then the problem arises of STABILITY of the classical theorem similar to the three stated above. They are established for independent variables but applied to "almost" independent. Is this sound? The precise statement of the problem needs some numerical or other characteristics that enable us to estimate the "degree of independence." The $\mathscr{D}$-numbers $\mathscr{D}^{\Phi}, \mathscr{D}_{\Phi}$, etc., appearing in Theorems 1–3, are instances of these characteristics. Even in case we do not know how to estimate them precisely, Theorems 1–3 speak about some stability of the phenomena related to independence. Instead of ideal independent random variables we practically deal with perturbed and deformed variables. The question is: In which sense must this deformation be "small" to change the qualitative picture? One of the possible answers is prompted by our theorems: the deformation must be such that the $\mathscr{D}$-numbers remain finite for at least some Φ_1 and Φ_2. The distribution functions F_m in Theorem 4 also may be inserted into the formulation of the corresponding limit theorems as well as the functions ρ_0, ρ_∞.

The commonest approach to the limit theorems in the modern research in probability theory stems from S. N. Bernstein and involves the replacement of the condition of independence with various versions of "weak dependence" ("regularity," "mixing," etc.). We do not address

these questions here; the above theorem rests on the entirely different principles.

2.3 Exact and approximate independence

We have already remarked that the practical independence of events, random variables, or subalgebras is asserted on the grounds of some physical reasons in the absence of any perceptible correlation between the observable phenomena. In reality, we can never be completely convinced that any correlation is absolutely excluded. Most tools of probability theory rest on the hypothesis of independence of the variables under consideration, whereas they are applied to "almost independent" variables. Experience corroborates the legitimacy of this approach; however, there is no theoretical justification of this practice as far as we know. One of the possible approaches consists in using the $\mathscr{D}$-numbers which may be inserted into formulas so that in the ideal case when all these numbers equal 1 we obtain the classical formulas that were established for independent variables. We have already mentioned this approach.

Another approach which seems to us more promising consists in treating the whole space M of probability measures on an NBA $\mathscr{X}$ as a probability space. Inspecting some process (say, coin tossing), we actually never know exactly what events (elements of $\mathscr{X}$) are in fact independent and what are not. However, we assume the dependence between some events hardly probable or even improbable at all: it is these events that we view as independent. In other words, we in fact always bear in mind some probability α on the space of measures and our hypotheses about probable events as regards their dependence or independence are determined by this measure α. Not attempting at pursuing this approach further in more detail, we confine exposition to a model example that illustrates the matter satisfactorily.

We will still begin with the general situation: $\mathscr{X}$ is an NBA (algebra of events); M is the set of all probability measures on $\mathscr{X}$, and A is the set of probability Borel quasimeasures on M. Each quasimeasure $\alpha \in A$ has the barycenter

$$\mu_\alpha : \quad \mu_\alpha(x) = \int\limits_M \mu(x)\, d\alpha_\mu. \tag{6}$$

Clearly, μ_α is a probability measure on $\mathscr{X}$. If α is supported in the singleton $\{\overline{\mu}\}$ then $\mu_\alpha = \overline{\mu}$.

The typical situation is as follow: The support of α lies in a small convex neighborhood ω about the point μ_α; all the rest of $\mu \in \omega$ differ slightly from μ_α, which implies proximity of all possible probabilistic

characteristics of the random variables related to the measures μ and μ_α.

The formula (6) is in essence the total probability formula: $\mu(x)$ under the integral sign is the conditional probability of an event x which is calculated in the HYPOTHESIS that the events, comprising $\mathscr{X}$, happen with probability μ. The measure μ itself is a random variable distributed with probability α. Therefore, if we know only the quasimeasure α then we should look at $\mu_\alpha(x)$ as the "genuine" probability of the event x. (Here we have a compound trial: first, we test x and, second, we choose μ at random with probability α.)

So, we know the probabilities of the events $x \in \mathscr{X}$ not exactly but only with some probability. However, we usually have the following hypothesis: some measure $\mu^0 \in M$ seems to us as occupying a privileged position, it is exactly this measure we are inclined to view as a "genuine probability." This hypothesis rests as a rule on the grounds of two kinds. It may be corroborated by experimental data (approximately, of course). Moreover, we may be enticed by the μ^0-independence of some important events, subalgebras, or random variables which opens an opportunity to use the theory available for independent entities. However, this hypothesis is not fully exact. Most likely, the "genuine probability" differs slightly from μ^0 but remains unknown precisely. The words "most likely" are decoded as follows: the distribution α is supported in a small neighborhood about μ^0; hence, there is no great error in substituting μ_α for μ^0 and vice versa. We will elaborate this in more detail on confining exposition to a simple model. Denote by X the space of binary n-tuples: $X \equiv X_{\{1,2,\dots,n\}} \equiv \{0,1\}^n$. The elements of X are the tuples $\chi = (\chi_1, \dots, \chi_n)$, with $\chi_i = 0, 1$. As $\mathscr{X}$ we take the algebra 2^X of all subsets of X. This boolean comprises all events that may occur in coin tossing when we toss the coin n times in each trial. Each measure $\mu \in M$ is the family

$$\{\mu_\chi\}_{\chi \in X},\ \ \mu_\chi > 0,\ \ \sum \mu_\chi = 1.$$

We may say that M is a simplex of dimension $2^n - 1$. (More exactly, the interior of the simplex; there are nongenuine measures on the boundary since some values μ_χ may be zero.) A privileged place is occupied in this case by the measure μ^0 for which every μ^0_χ equals $\frac{1}{2^n}$. This is the BARYCENTER of M. This measure corresponds to the ideal situation in which the coin is absolutely symmetric and the outcomes of separate trials are μ^0-independent. The last assertion means precisely that so are the generators of our algebra, the elements x_k that are determined from the condition

$$x_k \equiv Q_k \equiv \{\chi \mid \ \chi_k = 1\}$$

("at trial k the coin shows heads and at the other trials it falls arbitrarily"). Indeed, if $\widetilde{x}_k$ is either x_k or $Cx_k \equiv \{\chi \mid \chi_k = 0\}$ and $m \leq n$, then

$$\mu^0(\widetilde{x}_{i_1} \wedge \widetilde{x}_{i_2} \wedge \dots \wedge \widetilde{x}_{i_m}) = \frac{1}{2^m} \quad (1 \leq i_1 < i_2 < \dots < i_m \leq n).$$

Taking u and v, $0 < u < 1 < v$, arbitrarily, distinguish the subset of M that is given as follows:

$$M_{uv} \equiv \Big\{\mu \mid \ (\forall \chi \in X) \ u \leq \frac{\mu_\chi}{\mu^0_\chi} \equiv 2^n \mu_\chi \leq v\Big\}.$$

Driving u and v nearer and nearer, we arrive at narrower and narrower neighborhoods of the point $\mu^0 \in M$.

We agree on the following notation. If f is a random variable on $\mathscr{X}$ (i.e., simply an arbitrary real function on X) then

$m(f) \equiv m_\mu(f) \equiv \int f \, d\mu$ is the expectation of f with respect to μ;

$\sigma^2(f) \equiv \sigma^2_\mu(f) \equiv \int (f - m(f))^2 \, d\mu$ is the variance of f with respect to μ.

If μ is replaced with μ^0 or μ_α then we reflect this by an subscript, writing $m_0(f), m_\alpha(f), \dots$, etc. Given $\mu \in M_{uv}$, we easily derive the following estimates:

1) *If* $\gamma(f) \equiv m(f) - m_0(f)$ *then*

$$|\gamma(f)| \leq (v-1) m_0(|f|), \ \ |\gamma(f)| \leq \frac{v-u}{uv} m(|f|).$$

2) *If* $\delta(f) \equiv \sigma^2(f) - \sigma^2_0(f)$ *then*

$$|\delta(f)| \leq (m_0(f^2) + 2m_0^2(|f|))(v-u) + m_0^2(|f|)(v-u)^2.$$

Moreover, if $v - u \leq 1$ *and* $|f| \leq C$ *then*

$$|\delta(f)| \leq 4C^2(v-u).$$

As was mentioned in the beginning, we are interested in the "measure" α concentrated about the point μ^0. Therefore, we need some numerical characteristics of this concentration. We will confine exposition to only one of these characteristics. Put

$$h_\alpha \equiv \inf_{0<u\leq 1\leq v<+\infty} \{v - u + \alpha(M \setminus M_{uv})\}.$$

Clearly, the equality $h_\alpha = 0$ means that the support of α is the singleton $\{\mu^0\}$. We may now demonstrate some estimates whose derivation is not a craft matter.

3) *If* $\tau(f) \equiv m_\alpha(f) - m_0(f)$ *then*

$$|\tau(f)| \le \Big| \int\limits_{M\setminus M_{uv}} m_\mu(f)\, d\alpha_\mu \Big| + |m_0(f)| \Big(\alpha\Big(M \setminus M_{uv}\Big) + v - u\Big).$$

Moreover, if $|f| \le C$ *then*
3′) $|\tau(f)| \le 2Ch_\alpha$.
If in this event α *is supported in some* $M_{u_0v_0}$ *then*
3″) $|\tau(f)| \le Km_0(|f|)h_\alpha$, *where* $K \equiv 1 + v_0 - u_0$.
4) *If* $\eta(f) \equiv \sigma_0^2(f) - \sigma_\alpha^2(f)$ *and* $|f| \le C$ *then*

$$|\eta(f)| \le 3C^2 h_\alpha.$$

If in this event α *is supported in some* $M_{u_0v_0}$ *then*

$$|\eta(f)| \le Lm_0^2(|f|)h_\alpha,$$

where L *is a constant related with* u_0 *and* v_0.

From the identity $\sigma_\alpha^2(f) = \sigma_0^2(f) + \eta(f)$ it follows that

$$\sigma_\alpha(f) = \sigma_0(f)\sqrt{1 + \frac{\eta(f)}{\sigma_0^2(f)}} = \sigma_0(f)\Big[1 + \frac{\eta(f)}{2\sigma_0^2(f)} + \gamma_1\Big];$$

moreover, for γ_1 the estimates hold

a) $|\gamma_1| \le \frac{9}{8}C^4 \frac{h_\alpha^2}{\sigma_0^4(f)}$, if $|f| \le C$,

b) $|\gamma_1| \le \frac{1}{8}L^2 \frac{m_0^4(|f|)}{\sigma_0^4(f)} h_\alpha^2$, if α is supported in M_{uv} (where L is the same constant as before).

If we put $\gamma \equiv \sigma_\alpha(f) - \sigma_0(f)$ then from a) we infer that

$$|\gamma| \le L\frac{m_0^2(|f|)}{2\sigma_0(f)}h_\alpha + \frac{L^2}{8} \cdot \frac{m_0^4(|f|)}{\sigma_0^3}h_\alpha^2,$$

while from b) we find that

$$|\gamma| \le P\Big(\frac{m_0(|f|)}{\sigma_0(f)}h_\alpha + \frac{m_0^4(|f|)}{\sigma_0^3}h_\alpha^2\Big),$$

where P, like L, is a constant dependent on α. All constants in these formulas may be estimated. The above inequalities enable us to translate the results about μ^0 into the theorems in which the role of the main measure is allotted to μ_α. We have stated them for the only purpose: to show the role of the coefficient h_α which presents to most primitive numerical characteristics of the "concentration" of the quasimeasure α about the center μ^0.

We will illustrate this observation by the example of the Central Limit Theorem. Assume given the random variables $f_1, f_2, \dots, f_n$ independent with respect to the "ideal" measure μ^0. Assume moreover that

$$m_0(f_i) = 0, \ \ \sigma_0(f_i) = 1 \quad (i = 1, 2, \dots, n).$$

Then it is easy to prove for the "genuine" measure μ_α that for all t we have

$$\mu_\alpha \left\{ \frac{S_n - m_\alpha(S_n)}{\sigma_\alpha(S_n)} < t \right\} = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{t} e^{-\frac{x^2}{2}}\, dx + \rho, \tag{7}$$

where $S_n \equiv f_1 + f_2 + \dots + f_n$, $|\rho| \le C\Big(\frac{1}{\sqrt{n}} + h_\alpha\Big)$ and the constant may be estimated. (Evidently, the proof bases on the celebrated Berry–Esseen Theorem.) We give these results as a simplest illustration of the approach we suggested. Clearly, the key point here is the estimation of h_α. It is possible even not to know the quasimeasure α itself; it is the coefficient h_α that matters.

In conclusion we exhibit a rather concrete example. The quasimeasure α describes the process of n-times coin tossing in the following environment. Assume that the mechanism of tossing is absolutely ideal and any physical correlation between different trials is completely excluded. But the coin is selected at a start in some array; then we toss it n times. Let p be the probability of heads. The initial array contains only different coins (no two of them are the same). Therefore, we must view p as a random variable taking values in the interval $(0, 1)$. The distribution law of this variable determines the quasimeasure α. It is supported in the set of Bernoulli measures. In this case we have an opportunity to estimate h_α. We mention only the simplest (although not very real) situation in which p a random variable distributed on the interval $[\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon]$ $(\varepsilon > 0)$. Note that, despite the tossing mechanism is ideal, the outcomes of separate trials are not μ_α-independent. In this problem the coefficient h_α is estimated as follows: $h_\alpha \le K\varepsilon n$, where K is some constant we may estimate in principle.

The further inspection of this example drives us quickly to the conclusion: the least values of the correction ρ are attained at $h \asymp \frac{1}{\varepsilon^{2/3}}$ and, moreover, $\rho \asymp \varepsilon^{\frac{1}{3}}$. Increase in the number of tosses will lead to the growth of ρ, so worsening the estimate.

This example lacks the μ_α-independence of separate trials. It is easy to grasp: each outcome adds information about the measure μ; i.e., about the coin we actually use in the game. For instance, if the coin shows tails twice then most likely it will lie the same way in the third trial. (Indeed, we know that it cannot be symmetric.) The cause of

dependence here is not the physical correlation between separate trials but rather asymmetry of the coin. If the series continues for a sufficiently long time (n is large) the probability p, characterizing the coin, is determined more and more precisely from the outcomes of the trials and the trials themselves become "more and more independent." As n increases, more and more applicable becomes the classical Central Limit Theorem concerning the conventional Bernoulli trials. And so we see the reasons for the estimate for ρ to worsen with the growth of n, since by the classical theorem the distribution of the sums $f_1 + \dots + f_n$ must differ in the limit from that which the formula (7) yields as $\rho \longrightarrow 0$.

The above scheme grounds on the view of M as a probability space. This is customary in statistics while desiring to make a conclusion about an unknown "genuine" measure hidden in the background of these results (the "Bayes approach"). In both cases the central question reads: What is the a priori probability α?

In essence the scheme we have described considers the events happening in the probability space $M \times X$ in which X is naturally embedded. We determine the probability on this space from the conditions

$$P(E \times x) \equiv \int\limits_E \mu x \, d\alpha_\mu,$$

with $E \subset M$. The projections of this probability are exactly α and μ_α. We may speak of the conditional probabilities of the events $x \mid E$ and $E \mid x$. These are defined by the formulas:

$$P(x \mid E) = \frac{1}{\alpha(E)} \int\limits_E \mu x \, d\alpha_\mu,$$

$$P(E \mid x) = \frac{1}{\int\limits_M \mu x \, d\alpha_\mu} \cdot \int\limits_E \mu x \, d\alpha_\mu.$$

For instance, the latter gives the probability of the inclusion $\mu \in E$ given the event $x \in \mathscr{X}$ (the "Bayes formula"). In the case when $\mathscr{X}$ is an arbitrary BA, we may consider the probability space of the type $\Pi_0 \times R$, where R is some realization space for $\mathscr{X}$, say, $\mathfrak{Q}(\mathscr{X})$.

Many interesting problems crop up here but they lie certainly aside the main topic of this book. We now emphasize the most principal thing: the whole triple $\{\mathscr{X}, M, A\}$ matters in studying an NBA in much the same way as the dual and the second dual are inseparable from a normed vector space in functional analysis.

Appendix A
Prerequisites to Set Theory and General Topology

1. General remarks

As mentioned in the Introduction, this book is intended to the reader acquainted with the basics of "naive" set theory. The modern (axiomatic and "model-theoretic") set theory is not evoked in fact.[1] We advice the reader to peruse the Introduction to the book "Topology" by K. Kuratowski. It is also helpful to consult other sources; for instance, the celebrated books by F. Hausdorff, P. S. Aleksandroff, or I. P. Natanson as regards ordinal numbers and transfinite induction.

A few remarks about notation and terminology are now in order. We use the following symbols of set-theoretic operations and relations:

$$\cup, \cap, \setminus, \in, \notin, \subset, \supset .$$

We abstain from using the notations like $\subseteq$, on presuming that the symbol $\subset$ does not exclude the equality of sets. The empty set is denoted by $\varnothing$; the cardinality of E, by $card\, E$.

The class of all x satisfying some property is denoted by

$$\{x \mid \ \ldots\}.$$

To denote a function[2] we usually take a single letter: f, g, φ and so on. In this event $f(x)$ is the value of f at a point x; the parentheses are sometimes omitted. We practice only the definition of mappings by the formulas of the type $x \longrightarrow f(x)$. At last, to denote a function on some set A, we use the record like $\{x_\alpha\}_{\alpha \in A}$. It this case we speak of a *family* $\xi \equiv \{x_\alpha\}_{\alpha \in A}$ and call A the "index set." A particular case of a family is a simple sequence $\{x_n\}_{n=1}^{\infty}$ (the role of A is played by the set of natural numbers). The meaning of the terms "system," "totality," "collection," "tuple," etc. is always clear from the context (we usually imply the set of members of some family).

[1] For instance, cf. T. Jech [1] and E. I. Gordon and S. F. Morozov [1].
[2] In this book the words "function," "mapping," and "operator" are viewed as synonyms.

The inverse of a function f is denoted by f^{-1}. Irrespective of whether or not the inverse of a function f exists, the symbol $f^{-1}(e)$ always stands for the *inverse image* of a set e as well as $f(e)$ symbolizes the *image* of e. The restriction of a function f to some e is denoted by $f|_e$.

Assume given a family of sets $\{E_\alpha\}_{\alpha\in A}$. The *product* of this family is the set of all families $\varepsilon \equiv \{e_\alpha\}_{\alpha\in A}$ satisfying $e_\alpha \in E_\alpha$ for every $\alpha \in A$. This product is denoted by $\prod_{\alpha\in A} E_\alpha$.

In the case when $A = \{1, 2\}$, we obtain the *product of two sets* which is denoted by $E_1 \times E_2$; this is the collection of all ordered pairs (e_1, e_2), where $e_1 \in E_1$ and $e_2 \in E_2$. If $E_1 = E_2 = E$ then we write E^2 instead of $E_1 \times E_2$ and speak about the "product of two copies of E"; the analogous terminology applies to the general case in which all E_α $(\alpha \in A)$ coincide.

To each value of the index $\alpha_0 \in A$ we assign the "projection" π_{α_0} that sends an arbitrary family $\varepsilon \equiv \{e_\alpha\}$ treated as a point in the product E_α to the element

$$\pi_{\alpha_0}(\varepsilon) \equiv e_{\alpha_0} \in E_{\alpha_0}.$$

The values of these mappings are called the "projections" or "coordinates" of ε.

2. Partially ordered sets

A *partial ordering* or *partial order*[3] on a nonempty set $\mathscr{X}$ is a subset $P \subset \mathscr{X}^2$ satisfying the following axioms:

I. $(x, x) \in P$ for all $x \in \mathscr{X}$.

II. If $(x, y) \in P$ and $(y, x) \in P$ then $x = y$.

III. If $(x, y) \in P$ and $(y, z) \in P$ then $(x, z) \in P$.

Thus, a partial order is a RELATION between the elements of $\mathscr{X}$. The axioms I and III express the REFLEXIVITY and TRANSITIVITY of this relation; the axiom II tells us that it is ANTISYMMETRIC. As a rule, we write $x \le y$ or $y \ge x$ rather than $(x, y) \in P$. Other similar symbols may replace $\le$; for instance, $\prec$. The axioms I–III may be rewritten as

I. $x \le x$ for all $x \in \mathscr{X}$.

II. If $x \le y$ and $y \le x$ then $x = y$.

III. If $x \le y$ and $y \le z$ then $x \le z$.

The formula $x < y$ (or $y > x$) means by definition that $x \le y$ and $x \ne y$. The expressions of the form $a \le b, a < b$, etc. are called *inequalities.*

A *partially ordered set* or *poset* is some set $\mathscr{X}$ furnished with some partial order P on $\mathscr{X}$; i.e., the pair $\{\mathscr{X}, P\}$. Most often we denote a partially ordered set by the same letter $\mathscr{X}$ as the underlying set; accordingly, we call the members of $\mathscr{X}$ "elements of a partially ordered set." This abuse of the language is very common in mathematics but is excusable only in the case when $\mathscr{X}$ is equipped with a single order.

If every two elements in a partially ordered set $\mathscr{X}$ are compatible, i.e., either $(x, y) \in P$ or $(y, x) \in P$ for all $x, y \in \mathscr{X}$; then we say that $\mathscr{X}$ is *linearly ordered.* As an example, we may take every set of real numbers furnished with the conventional order.

Likewise on the real axis, the set of all x satisfying the inequality $a \le x \le b$ is called an *interval* and we denote it by $[a, b]$.

[3]The word "partial" is often omitted.

Let $\mathscr{X}$ be equipped with some partial order. Each subset $\mathscr{X}_0 \subset \mathscr{X}$ may also be furnished with some partial order; to this end, we put

$$P_0 \equiv \mathscr{X}_0^2 \cap P.$$

It is easy to see that the axioms I–III are satisfied. The so-defined partial order P_0 is said to be *induced* by P or *induced from outside*. Practically speaking, this means that all inequalities have the same sense in $\mathscr{X}_0$ as they have in $\mathscr{X}$. It might happen that $\mathscr{X}_0$ is linearly ordered in the induced order; in this event we call $\mathscr{X}_0$ a *chain*. An element x_0 of a partially ordered set $\mathscr{X}$ is *maximal* provided that the inequality $x \geq x_0$ implies $x = x_0$.

A significant role in mathematics belongs to the following

Kuratowski–Zorn Lemma.[4] *Assume that a partially ordered set $\mathscr{X}$ possesses the following property: to each chain $\mathscr{X}_0 \subset \mathscr{X}$ there is some $y \in \mathscr{X}$ such that $x \leq y$ for all $x \in \mathscr{X}_0$. Then, to whatever element $x \in \mathscr{X}$, there is a maximal element x_0 satisfying the inequality $x_0 \geq x$.*

The Kuratowski–Zorn Lemma often replaces the principle of mathematical induction in proofs, enabling us to avoid ordinal numbers. However, it is sometimes more natural to use the Zermelo Theorem or transfinite numbers which leads faster to the aim (for instance, cf. the proof of the theorem about the structure of a homogeneous BA in Chapter 9 of this book).

3. Topologies

An important example of a partially ordered set is provided by the system of topologies [5] on some set R.

By a *topology* we routinely mean a class τ of subsets of R that is closed under all unions and finite intersections. The members of τ are called *open* and their complements, *closed* sets in R. The system τ' of all closed sets also uniquely determined the topology of R; the role of the latter may be performed by each class of sets closed under all intersections and finite unions. The pair $\{R, \tau\}$ is a *topological space* denoted by the same letter R.

We say that a topology τ_1 is *stronger* than a topology τ_2 whenever $\tau_1 \supset \tau_2$ (or, which is the same, $\tau_1' \supset \tau_2'$). In this event we say also that τ_1 *majorizes*, or *dominates* τ_2, or τ_2 is *weaker* than τ_1, etc. For each nonempty set T of topologies on R, there always exists a unique WEAKEST topology among those dominating every member of T; in much the same way, there is a unique STRONGEST topology among those dominated by every member of T.

We say that x is an *interior* point of a set V or that V is a *neighborhood* about x whenever there is an open set G satisfying $x \in G \subset V$.

Let $\mathfrak{V}(x)$ stand for the collection of all neighborhoods of x. If $\mathfrak{V}_0$ is such a set of neighborhoods that to each $V \in \mathfrak{V}(x)$ there is some $V_0 \in \mathfrak{V}_0$ satisfying $V_0 \subset V$ then we say that $\mathfrak{V}_0$ is a *base of neighborhoods* about x.

Assume that to each point $x \in R$ there corresponds some base of neighborhoods $\mathfrak{V}_0(x)$; then the inclusion $G \in \tau$ means that to each point $x \in G$ there is some $V \in \mathfrak{V}_0(x)$ satisfying $V \subset G$.

---

[4]K. Kuratowski [1] and M. Zorn [1]. This proposition is equivalent to the axiom of choice and also to the so-called "Hausdorff maximality principle" (J. Kelley [2]).

[5]N. Bourbaki [1]; K. Kuratowski [2]; and J. Kelley [2].

We may thus uniquely recover the topology τ from available bases of neighborhoods about all points. This idea is often used for the INITIAL INTRODUCTION OF A TOPOLOGY. Assume given a family $\{\mathfrak{V}_0(x)\}$ whose every member $\mathfrak{V}_0(x)$ is some class of subsets containing the point x. Assume further that the following hold:

I. If $V_1, V_2 \in \mathfrak{V}_0(x)$ then there is a set $V \in \mathfrak{V}_0(x)$ lying in $V_1 \cap V_2$.

II. Given $V \in \mathfrak{V}_0(x)$, we may find $V' \in \mathfrak{V}_0(x)$ so that each class $\mathfrak{V}_0(x')$, $x' \in V'$, contains at least one set included in V.

Then there is a unique topology τ with respect to which each system $\mathfrak{V}_0(x)$ is a base of neighborhood about x. If to each point x there is a COUNTABLE base of neighborhoods then we say that the space under consideration satisfies the *first axiom of countability*. The *second axiom of countability* requires that there exists a countable system $\mathfrak{G}$ of open sets such that each open set is a union of some family of elements of $\mathfrak{G}$.

A topological space is called *separated* or *Hausdorff* provided that to each pair of distinct points x_1 and x_2 there exist disjoint neighborhoods of x_1 and x_2 (the "Hausdorff axiom").

If R_0 is a subset of a topological space $\{R, \tau\}$ then the system τ_0 of all sets of the form $G \cap R_0$, $G \in \tau$, is some topology on R_0 that is called *natural* or *induced* (from outside). The set R_0, furnished with the topology τ_0, is called a *subspace* of the original topological space.

A class of sets $\sigma \subset \tau$ is called a *cover* or *covering* of E provided that

$$E \subset \bigcup_{P \in \sigma} P.$$

A topological space is called *compact* if E is separated and each open cover of E includes a finite part that is a cover of E too. Such a space is shortly called a *compactum*.[6] For a separated space to be compact it is necessary and sufficient that the intersection of every centered system of its closed subsets be nonempty (a system of sets is centered whenever it has the *finite intersection property*; i.e., the intersection of each finite subsystem is nonempty). Each closed subset of a compact space is compact too in the induced topology. Note finally that every compact space is not only separated but possesses the important *normality* property: every two closed disjoint sets are included in open disjoint sets.

Assume given the two topological spaces $\{R_1, \tau_1\}$ and $\{R_2, \tau_2\}$. A mapping f from R_1 to R_2 is called *continuous* with respect to the topologies τ_1 and τ_2 provided that $f^{-1}(G) \in \tau_1$ for all $G \in \tau_2$. If the topologies are implied then we simply speak about a continuous mapping.

Consider some family $\{\{R_\alpha, \tau_\alpha\}\}_{\alpha \in A}$ of topological spaces, and put

$$R \equiv \prod_{\alpha \in A} R_\alpha.$$

We may consider various topologies on R: the most important are those that guarantee the continuity of all mappings π_α $(\alpha \in A)$. Mostly we use the weakest of these topologies; it is called *Tychonoff*. The set R, equipped with the Tychonoff topology, constitutes the *product* of the topological spaces R_α. The topology of this product

[6] Or a *bicompactum*, the latter is the historically first term coined by P. S. Aleksandroff, the founder of the theory of compact spaces.

is the weakest among those in which all sets of the form $\pi_\alpha^{-1}(G)$, $G \in \tau_\alpha$, are open. Assume that $\{R_\alpha\}$ is finite; $A = \{1, 2, \dots, m\}$. By a *continuous function* in m variables we always mean an arbitrary function on the product of the topological space $R_1, R_2, \dots, R_m$ and continuous with respect to the Tychonoff topology. (The values of the function may belong to an arbitrary topological space.) In particular, those continuous functions are most important that send a topological space or its "powers" $R^2, R^3, \dots$ to R; in this event we usually speak about *continuous operations* on R.

As an example we may take the multiplication operation or, in the "additive" terminology, the addition operation on some group Γ furnished with a topology. The continuity of this operation (as a mapping from Γ^2 to Γ) together with the continuity of the taking of the inverse means by definition that Γ is a *topological group* (L. S. Pontryagin [1]; J. Kelley [2]). A more general concept is that of "uniform space" (J. Kelley [2]).

In this book we mostly pay attention to partially ordered sets of a special form (Boolean algebras) which are furnished with various topologies. These topologies must be properly compatible with order; our constant preference is the joint consideration of order-theoretic and topological properties.

Appendix B
Basics of Boolean Valued Analysis

1. General remarks

Boolean valued analysis[1] is a branch of functional analysis which uses a special model-theoretic technique that is embodied in the Boolean valued models of set theory. The term was coined by G. Takeuti. The invention of Boolean valued models was not connected with the theory of Boolean algebras but rather has revealed a gemstone among the diverse applications of the latter. It was the celebrated Cohen forcing method for solving the continuum problem whose comprehension gave rise to the Boolean valued models of set theory. Their appearance is commonly associated with the names of D. Scott, R. Solovay, and P. Vopěnka.

Boolean valued analysis consists primarily in comparative analysis of a mathematical object or idea simultaneously in some standard and some Boolean valued models which is accomplished by a special technique of ascending and descending.

2. Boolean valued models

Now we briefly present necessary information on the theory of Boolean valued models. All details may be found in a book by A. G. Kusraev and S. S. Kutateladze[2] and the literature cited therein.

The universe of discourse of Boolean valued analysis is a Boolean valued model of $\mathbb{ZFC}$. To sketch its structure, we start with a complete BA B. Given an ordinal α, put

$$\mathbf{V}_\alpha^{(B)} := \{x \mid x \text{ is a function} \wedge (\exists\beta)(\beta < \alpha \wedge \operatorname{dom}(x) \subset \mathbf{V}_\beta^{(B)} \wedge \operatorname{im}(x) \subset B)\}.$$

Thus, in more detail we have

$$\mathbf{V}_0^{(B)} := \varnothing,$$

$$\mathbf{V}_{\alpha+1}^{(B)} := \{x \mid x \text{ is a function with domain in } \mathbf{V}_\alpha^{(B)} \text{ and range in } B\};$$

$$\mathbf{V}_\alpha^{(B)} := \bigcup_{\beta<\alpha} \mathbf{V}_\beta^{(B)} \quad (\alpha \text{ is a limit ordinal}).$$

---

[1]This appendix is compiled by S. S. Kutateladze.

[2]A. G. Kusraev and S. S. Kutateladze [2].

The class

$$\mathbf{V}^{(B)} := \bigcup_{\alpha\in\mathrm{On}} \mathbf{V}_\alpha^{(B)}$$

is a *Boolean valued universe*. An element of the class $\mathbf{V}^{(B)}$ is a *B-valued set*. It is necessary to observe that $\mathbf{V}^{(B)}$ consists only of functions. In particular, $\varnothing$ is the function with domain $\varnothing$ and range $\varnothing$. Hence, the "lower" levels of $\mathbf{V}^{(B)}$ are organized as follows:

$$\mathbf{V}_0^{(B)} = \varnothing, \quad \mathbf{V}_1^{(B)} = \{\varnothing\}, \quad \mathbf{V}_2^{(B)} = \{\varnothing, (\{\varnothing\}, b) \mid b \in B\}.$$

It is worth stressing that $\alpha \le \beta \to \mathbf{V}_\alpha^{(B)} \subset \mathbf{V}_\beta^{(B)}$ for all ordinals α and β. Moreover, the following *induction principle* is valid in $\mathbf{V}^{(B)}$:

$$(\forall x \in \mathbf{V}^{(B)})\,((\forall y \in \mathrm{dom}(x))\ \varphi(y) \to \varphi(x)) \to (\forall x \in \mathbf{V}^{(B)})\,\varphi(x),$$

where φ is a formula of $\mathbb{ZFC}$.

Take an arbitrary formula $\varphi = \varphi(u_1, \dots, u_n)$ of $\mathbb{ZFC}$. If we replace the elements $u_1, \dots, u_n$ by elements $x_1, \dots, x_n \in \mathbf{V}^{(B)}$ then we obtain some statement about the objects $x_1, \dots, x_n$. It is to this statement that we intend to assign some *truth-value*. Such a value $[\![\psi]\!]$ must be an element of the algebra B. Moreover, it is naturally desired that the theorems of $\mathbb{ZFC}$ be true, i.e., attain the greatest truth-value, unity.

We must obviously define the truth-value of a well-formed formula by double induction, on considering the way in which this formula is built up from the atomic formulas $x \in y$ and $x = y$, while assigning truth-values to the latter when x and y range over $\mathbf{V}^{(B)}$ on using the recursive definition of this universe.

It is clear that if φ and ψ are evaluated formulas of $\mathbb{ZFC}$ and $[\![\varphi]\!] \in B$ and $[\![\psi]\!] \in B$ are their truth-values then we should put

$$[\![\varphi \wedge \psi]\!] := [\![\varphi]\!] \wedge [\![\psi]\!],$$

$$[\![\varphi \vee \psi]\!] := [\![\varphi]\!] \vee [\![\psi]\!],$$

$$[\![\varphi \to \psi]\!] := [\![\varphi]\!] \to [\![\psi]\!],$$

$$[\![\neg\varphi]\!] := C[\![\varphi]\!],$$

$$[\![(\forall x)\,\varphi(x)]\!] := \bigwedge_{x\in\mathbf{V}^{(B)}} [\![\varphi(x)]\!],$$

$$[\![(\exists x)\,\varphi(x)]\!] := \bigvee_{x\in\mathbf{V}^{(B)}} [\![\varphi(x)]\!],$$

where the right-hand sides involve the Boolean operations corresponding to the logical connectives and quantifiers on the left-hand sides: $\wedge$ is the meet, $\vee$ is the join, C is the complementation, while the implication $\to$ is introduced as follows: $a \to b := Ca \vee b$ for $a, b \in B$. Only such definitions provide the value "unity" for the classical tautologies.

We turn to evaluating the atomic formulas $x \in y$ and $x = y$ for $x, y \in \mathbf{V}^{(B)}$. The intuitive idea consists in the fact that a B-valued set y is a "(lattice) fuzzy set," i.e., a "set that contains an element z in $\mathrm{dom}(y)$ with probability $y(z)$." Keeping this in mind and intending to preserve the logical tautology of $x \in y \leftrightarrow (\exists z \in y)\,(x = z)$ as well as the axiom of extensionality, we arrive at the following definition by recursion:

$$[\![x \in y]\!] := \bigvee_{z\in\mathrm{dom}(y)} y(z) \wedge [\![z = x]\!],$$

$$[\![x = y]\!] := \bigwedge_{z\in\mathrm{dom}(x)} x(z) \to [\![z \in y]\!] \wedge \bigwedge_{z\in\mathrm{dom}(y)} y(z) \to [\![z \in x]\!].$$

Now we can attach some meaning to formal expressions of the form $\varphi(x_1, \dots, x_n)$, where $x_1, \dots, x_n \in \mathbf{V}^{(B)}$ and φ is a formula of $\mathbb{ZFC}$; i.e., we may define the exact sense in which the set-theoretic proposition $\varphi(u_1, \dots, u_n)$ is valid for the assignment of $x_1, \dots, x_n \in \mathbf{V}^{(B)}$.

Namely, we say that the *formula* $\varphi(x_1, \dots, x_n)$ *is valid inside* $\mathbf{V}^{(B)}$ or the *elements* $x_1, \dots, x_n$ *possess the property* φ if $[\![\varphi(x_1, \dots, x_n)]\!] = \mathbf{1}$. In this event, we write $\mathbf{V}^{(B)} \vDash \varphi(x_1, \dots, x_n)$.

It is easy to convince ourselves that the axioms and theorems of the first-order predicate calculus are valid in $\mathbf{V}^{(B)}$. In particular,

(1) $[\![x = x]\!] = \mathbf{1}$,

(2) $[\![x = y]\!] = [\![y = x]\!]$,

(3) $[\![x = y]\!] \wedge [\![y = z]\!] \le [\![x = z]\!]$,

(4) $[\![x = y]\!] \wedge [\![z \in x]\!] \le [\![z \in y]\!]$,

(5) $[\![x = y]\!] \wedge [\![x \in z]\!] \le [\![y \in z]\!]$.

It is worth observing that for each formula φ we have

$$\mathbf{V}^{(B)} \models x = y \wedge \varphi(x) \to \varphi(y),$$

i.e., in detailed notation

(6) $[\![x = y]\!] \wedge [\![\varphi(x)]\!] \le [\![\varphi(y)]\!]$.

3. Principles of Boolean valued analysis

In a Boolean valued universe $\mathbf{V}^{(B)}$, the relation $[\![x = y]\!] = \mathbf{1}$ in no way implies that the functions x and y (considered as elements of $\mathbf{V}$) coincide. For example, the function equal to zero on each layer $\mathbf{V}_\alpha^{(B)}$, where $\alpha \ge 1$, plays the role of the empty set in $\mathbf{V}^{(B)}$. This circumstance may complicate some constructions in the sequel.

In this connection, we pass from $\mathbf{V}^{(B)}$ to the *separated Boolean valued universe* $\overline{\mathbf{V}}^{(B)}$ often preserving for it the same symbol $\mathbf{V}^{(B)}$; i.e., we put $\mathbf{V}^{(B)} := \overline{\mathbf{V}}^{(B)}$. Moreover, to define $\overline{\mathbf{V}}^{(B)}$, we consider the relation $\{(x, y) \mid [\![x = y]\!] = \mathbf{1}\}$ on the class $\mathbf{V}^{(B)}$ which is obviously an equivalence. Choosing an element (a representative of the least rank) in each class of equivalent functions, we arrive at the separated universe $\overline{\mathbf{V}}^{(B)}$. Note that

$$[\![x = y]\!] = \mathbf{1} \to [\![\varphi(x)]\!] = [\![\varphi(y)]\!]$$

is valid for an arbitrary formula φ of $\mathbb{ZFC}$ and elements x and y in $\mathbf{V}^{(B)}$.

Therefore, in the separated universe we can calculate the truth-values of formulas paying no attention to the way of choosing representatives. Furthermore, working with the separated universe, for the sake of convenience we (exercising due caution) often consider some particular representative of an equivalence class rather than the whole class as it is customary, for example, while dealing with function spaces.

The most important properties of a Boolean valued universe $\mathbf{V}^{(B)}$ are stated in the following three principles:

(1) Transfer Principle. *All theorems of* $\mathbb{ZFC}$ *are true in* $\mathbf{V}^{(B)}$; *in symbols,*

$$\mathbf{V}^{(B)} \models \text{a theorem of } \mathbb{ZFC}.$$

The transfer principle is established by laboriously checking that all axioms of $\mathbb{ZFC}$ have truth-value $\mathbf{1}$ and the rules of inference preserve the truth-values of formulas. Sometimes, the transfer principle is worded as follows: "$\mathbf{V}^{(B)}$ is a Boolean valued model of $\mathbb{ZFC}$."

(2) Maximum Principle. *For each formula* φ *of* $\mathbb{ZFC}$ *there exists* $x_0 \in \mathbf{V}^{(B)}$ *for which*

$$[\![(\exists x)\,\varphi(x)]\!] = [\![\varphi(x_0)]\!].$$

In particular, if it is true in $\mathbf{V}^{(B)}$ that there is an x for which $\varphi(x)$ then there is an element x_0 in $\mathbf{V}^{(B)}$ (in the sense of $\mathbf{V}$) for which $[\![\varphi(x_0)]\!] = \mathbf{1}$. In symbols,

$$\mathbf{V}^{(B)} \models (\exists x)\,\varphi(x) \to (\exists x_0)\,\mathbf{V}^{(B)} \models \varphi(x_0).$$

Thus, the *maximum principle* reads:

$$(\exists x_0 \in \mathbf{V}^{(B)})\,[\![\varphi(x_0)]\!] = \bigvee_{x \in \mathbf{V}^{(B)}} [\![\varphi(x)]\!]$$

for each formula φ of $\mathbb{ZFC}$.

The last equality accounts for the origin of the term "maximum principle." The proof of the maximum principle is a simple application of the following

(3) Mixing Principle. Let $(b_\xi)_{\xi\in\Xi}$ be a *partition of unity* in B, i.e. a family of elements of a Boolean valued algebra B such that

$$\bigvee_{\xi\in\Xi} b_\xi = \mathbf{1}, \quad (\forall \xi, \eta \in \Xi)\,(\xi \neq \eta \to b_\xi \wedge b_\eta = 0).$$

For each family of elements $(x_\xi)_{\xi\in\Xi}$ *of the universe* $\mathbf{V}^{(B)}$ *and each partition of unity* $(b_\xi)_{\xi\in\Xi}$ *there exists a (unique) mixing of* (x_ξ) *by* (b_ξ); *i.e. an element* x *of the separated universe* $\mathbf{V}^{(B)}$ *such that* $b_\xi \leq [\![x = x_\xi]\!]$ *for all* $\xi \in \Xi$.

The *mixing* x of a family (x_ξ) by (b_ξ) is denoted as follows:

$$x = \mathrm{mix}_{\xi\in\Xi}(b_\xi x_\xi) = \mathrm{mix}\{b_\xi x_\xi \mid \xi \in \Xi\}.$$

4. Ascending and descending

The comparative analysis mentioned above presumes that there is some close interconnection between the universes $\mathbf{V}$ and $\mathbf{V}^{(B)}$. In other words, we need a rigorous mathematical apparatus that would allow us to find out the interplay between the interpretations of one and the same fact in the two models $\mathbf{V}$ and $\mathbf{V}^{(B)}$. The base for such apparatus is constituted by the operations of canonical embedding, descent, and ascent to be presented below. We start with the canonical embedding of the von Neumann universe. Given $x \in \mathbf{V}$, we denote by the symbol $x^\wedge$ the *standard name* of x in $\mathbf{V}^{(B)}$; i.e., the element defined by the following recursion schema:

$$\varnothing^\wedge := \varnothing, \quad \mathrm{dom}(x^\wedge) := \{y^\wedge \mid y \in x\}, \quad \mathrm{im}(x^\wedge) := \{\mathbf{1}\}.$$

Observe some properties of the mapping $x \mapsto x^\wedge$ we need in the sequel.

(1) For an arbitrary $x \in \mathbf{V}$ and a formula φ of $\mathbb{ZFC}$ we have

$$[\![(\exists y \in x^{\wedge})\, \varphi(y)]\!] = \bigvee\{[\![\varphi(z^{\wedge})]\!] : z \in x\},$$

$$[\![(\forall y \in x^{\wedge})\, \varphi(y)]\!] = \bigwedge\{[\![\varphi(z^{\wedge})]\!] : z \in x\}.$$

(2) If x and y are elements of $\mathbf{V}$ then, by transfinite induction, we establish

$$x \in y \leftrightarrow \mathbf{V}^{(B)} \models x^{\wedge} \in y^{\wedge},$$

$$x = y \leftrightarrow \mathbf{V}^{(B)} \models x^{\wedge} = y^{\wedge}.$$

In other words, the standard name can be considered as an embedding of $\mathbf{V}$ into $\mathbf{V}^{(B)}$. Moreover, it is beyond a doubt that the standard name sends $\mathbf{V}$ onto $\mathbf{V}^{(2)}$, which fact is demonstrated by the next proposition:

(3) The following holds:

$$(\forall u \in \mathbf{V}^{(2)})\, (\exists! x \in \mathbf{V})\ \mathbf{V}^{(B)} \models u = x^{\wedge}.$$

(4) A formula is called *bounded* or *restricted* if each bound variable in it is restricted by a bounded quantifier; i.e., a quantifier ranging over a specific set. The latter means that each bound variable x is restricted by a quantifier of the form $(\forall x \in y)$ or $(\exists x \in y)$ for some y.

Restricted Transfer Principle. *For each bounded formula φ of $\mathbb{ZFC}$ and every collection $x_1, \dots, x_n \in \mathbf{V}$ the following equivalence holds:*

$$\varphi(x_1, \dots, x_n) \leftrightarrow \mathbf{V}^{(B)} \models \varphi(x_1^{\wedge}, \dots, x_n^{\wedge}).$$

Henceforth, working in the separated universe $\overline{\mathbf{V}}^{(B)}$, we agree to preserve the symbol $x^{\wedge}$ for the distinguished element of the class corresponding to x.

(5) Observe by way of example that the restricted transfer principle yields the following assertions:

$$\text{“}\Phi \text{ is a correspondence from } x \text{ to } y\text{”}$$

$$\leftrightarrow \mathbf{V}^{(B)} \models \text{“}\Phi^{\wedge} \text{ is a correspondence from } x^{\wedge} \text{ to } y^{\wedge}\text{”};$$

$$\text{“}f \text{ is a function from } x \text{ to } y\text{”} \leftrightarrow \mathbf{V}^{(B)} \models \text{“}f^{\wedge} \text{ is a function from } x^{\wedge} \text{ to } y^{\wedge}\text{”}$$

(moreover, $f(a)^{\wedge} = f^{\wedge}(a^{\wedge})$ for all $a \in x$). Thus, the standard name can be considered as a covariant functor of the category of sets (or correspondences) in $\mathbf{V}$ to an appropriate subcategory of $\mathbf{V}^{(2)}$ in the separated universe $\mathbf{V}^{(B)}$.

(6) A set X is *finite* if X coincides with the image of a function on a finite ordinal. In symbols, this is expressed as $\operatorname{fin}(X)$; hence,

$$\operatorname{fin}(X) := (\exists\, n)(\exists\, f)(n \in \omega \wedge f \text{ is a function} \wedge \operatorname{dom}(f) = n \wedge \operatorname{im}(f) = X)$$

(as usual $\omega := \{0, 1, 2, ...\}$). Obviously, the above formula is not bounded. Nevertheless there is a simple transformation rule for the class of finite sets under the canonical embedding. Denote by $\mathscr{P}_{\mathrm{fin}}(X)$ the class of all finite subsets of X:

$$\mathscr{P}_{\mathrm{fin}}(X) := \{Y \in \mathscr{P}(X) \mid \operatorname{fin}(Y)\}.$$

For an arbitrary set X the following holds:

$$\mathbf{V}^{(B)} \models \mathscr{P}_{\mathrm{fin}}(X)^{\wedge} = \mathscr{P}_{\mathrm{fin}}(X^{\wedge}).$$

Given an arbitrary element x of the (separated) Boolean valued universe $\mathbf{V}^{(B)}$, we define the *descent* $x{\downarrow}$ of x as

$$x{\downarrow} := \{y \in \mathbf{V}^{(B)} \mid [\![y \in x]\!] = \mathbf{1}\}.$$

We list the simplest properties of descending:

(1) The class $x{\downarrow}$ is a set, i.e., $x{\downarrow} \in \mathbf{V}$ for each $x \in \mathbf{V}^{(B)}$. If $[\![x \neq \varnothing]\!] = \mathbf{1}$ then $x{\downarrow}$ is a nonempty set.

(2) Let $z \in \mathbf{V}^{(B)}$ and $[\![z \neq \varnothing]\!] = \mathbf{1}$. Then for every formula φ of $\mathbb{ZFC}$ we have

$$[\![(\forall x \in z)\, \varphi(x)]\!] = \bigwedge \{[\![\varphi(x)]\!] \mid x \in z{\downarrow}\},$$

$$[\![(\exists x \in z)\, \varphi(x)]\!] = \bigvee \{[\![\varphi(x)]\!] \mid x \in z{\downarrow}\}.$$

Moreover, there exists $x_0 \in z{\downarrow}$ such that $[\![\varphi(x_0)]\!] = [\![(\exists x \in z)\, \varphi(x)]\!]$.

(3) Let Φ be a correspondence from X to Y in $\mathbf{V}^{(B)}$. Thus, Φ, X, and Y are elements of $\mathbf{V}^{(B)}$ and, moreover, $[\![\Phi \subset X \times Y]\!] = \mathbf{1}$. There is a unique correspondence $\Phi{\downarrow}$ from $X{\downarrow}$ to $Y{\downarrow}$ such that

$$\Phi{\downarrow}(A{\downarrow}) = \Phi(A){\downarrow}$$

for every nonempty subset A of the set X inside $\mathbf{V}^{(B)}$. The correspondence $\Phi{\downarrow}$ from $X{\downarrow}$ to $Y{\downarrow}$ involved in the above proposition is called the *descent* of the correspondence Φ from X to Y in $\mathbf{V}^{(B)}$.

(4) The descent of the composite of correspondences inside $\mathbf{V}^{(B)}$ is the composite of their descents:

$$(\Psi \circ \Phi){\downarrow} = \Psi{\downarrow} \circ \Phi{\downarrow}.$$

(5) If Φ is a correspondence inside $\mathbf{V}^{(B)}$ then

$$(\Phi^{-1}){\downarrow} = (\Phi{\downarrow})^{-1}.$$

(6) Let Id_X be the identity mapping inside $\mathbf{V}^{(B)}$ of a set $X \in \mathbf{V}^{(B)}$. Then

$$(\mathrm{Id}_X){\downarrow} = \mathrm{Id}_{X{\downarrow}}.$$

(7) Suppose that $X, Y, f \in \mathbf{V}^{(B)}$ are such that $[\![f : X \to Y]\!] = \mathbf{1}$, i.e., f is a mapping from X to Y inside $\mathbf{V}^{(B)}$. Then $f{\downarrow}$ is a unique mapping from $X{\downarrow}$ to $Y{\downarrow}$ for which

$$[\![f{\downarrow}(x) = f(x)]\!] = \mathbf{1} \quad (x \in X{\downarrow}).$$

By virtue of (1)–(7), we can consider the descent operation as a functor from the category of B-valued sets and mappings (correspondences) to the category of the usual sets and mappings (correspondences) (i.e., in the sense of $\mathbf{V}$).

(8) Given $x_1, \dots, x_n \in \mathbf{V}^{(B)}$, denote by $(x_1, \dots, x_n)^B$ the corresponding ordered n-tuple inside $\mathbf{V}^{(B)}$. Assume that P is an n-ary relation on X inside $\mathbf{V}^{(B)}$; i.e., $X, P \in \mathbf{V}^{(B)}$ and $[\![P \subset X^{n^\wedge}]\!] = \mathbf{1}$, where $n \in \omega$. Then there exists an n-ary relation P' on $X{\downarrow}$ such that

$$(x_1, \dots, x_n) \in P' \leftrightarrow [\![(x_1, \dots, x_n)^B \in P]\!] = \mathbf{1}.$$

Slightly abusing notation, we denote the relation P' by the same symbol $P{\downarrow}$ and call it the *descent* of P.

Let $x \in \mathbf{V}$ and $x \subset \mathbf{V}^{(B)}$; i.e., let x be some set composed of B-valued sets or, in other words, $x \in \mathscr{P}(\mathbf{V}^{(B)})$. Put $\varnothing{\uparrow} := \varnothing$ and

$$\mathrm{dom}(x{\uparrow}) = x, \quad \mathrm{im}(x{\uparrow}) = \{\mathbf{1}\}$$

if $x \neq \varnothing$. The element $x{\uparrow}$ (of the separated universe $\mathbf{V}^{(B)}$, i.e., the distinguished representative of the class $\{y \in \mathbf{V}^{(B)} \mid [\![y = x{\uparrow}]\!] = \mathbf{1}\}$) is called the *ascent* of x.

(1) The following equalities hold for every $x \in \mathscr{P}(\mathbf{V}^{(B)})$ and every formula φ:

$$[\![(\forall z \in x{\uparrow})\, \varphi(z)]\!] = \bigwedge_{y \in x} [\![\varphi(y)]\!],$$

$$[\![(\exists z \in x{\uparrow})\, \varphi(z)]\!] = \bigvee_{y \in x} [\![\varphi(y)]\!].$$

Introducing the ascent of a correspondence $\Phi \subset X \times Y$, we have to bear in mind a possible difference between the domain of departure X and the domain $\mathrm{dom}(\Phi) := \{x \in X \mid \Phi(x) \neq \varnothing\}$. This difference is inessential for our further goals; therefore, we assume that, speaking of ascents, we always consider everywhere-defined correspondences; i.e., $\mathrm{dom}(\Phi) = X$.

(2) Let $X, Y, \Phi \in \mathbf{V}^{(B)}$, and let Φ be a correspondence from X to Y. There exists a unique correspondence $\Phi{\uparrow}$ from $X{\uparrow}$ to $Y{\uparrow}$ inside $\mathbf{V}^{(B)}$ such that

$$\Phi{\uparrow}(A{\uparrow}) = \Phi(A){\uparrow}$$

is valid for every subset A of the set $\mathrm{dom}(\Phi)$ if and only if Φ is *extensional*; i.e., satisfies the condition

$$y_1 \in \Phi(x_1) \to [\![x_1 = x_2]\!] \leq \bigvee_{y_2 \in \Phi(x_2)} [\![y_1 = y_2]\!]$$

for $x_1, x_2 \in \mathrm{dom}(\Phi)$. In this event, $\Phi{\uparrow} = \Phi'{\uparrow}$, where $\Phi' := \{(x, y)^B \mid (x, y) \in \Phi\}$. The element $\Phi{\uparrow}$ is called the *ascent* of the initial correspondence Φ.

(3) The composite of extensional correspondences is extensional. Moreover, the ascent of a composite is equal to the composite of the ascents (inside $\mathbf{V}^{(B)}$): On assuming that $\mathrm{dom}(\Psi) \supset \mathrm{im}(\Phi)$ we have

$$\mathbf{V}^{(B)} \models (\Psi \circ \Phi){\uparrow} = \Psi{\uparrow} \circ \Phi{\uparrow}.$$

Note that if Φ and Φ^{-1} are extensional then $(\Phi{\uparrow})^{-1} = (\Phi^{-1}){\uparrow}$. However, in general, the extensionality of Φ in no way guarantees the extensionality of Φ^{-1}.

(4) It is worth mentioning that if an extensional correspondence f is a function from X to Y then its ascent $f{\uparrow}$ is a function from $X{\uparrow}$ to $Y{\uparrow}$. Moreover, the extensionality property can be stated as follows:

$$[\![x_1 = x_2]\!] \leq [\![f(x_1) = f(x_2)]\!] \quad (x_1, x_2 \in X).$$

Given a set $X \subset \mathbf{V}^{(B)}$, we denote by the symbol $\mathrm{mix}\, X$ the set of all mixings of the form $\mathrm{mix}(b_\xi x_\xi)$, where $(x_\xi) \subset X$ and (b_ξ) is an arbitrary partition of unity. The following propositions are referred to as the *rules for cancelling arrows* or the "*descending-ascending*" and "*ascending-descending*" *rules*.

(5) Let X and X' be subsets of $\mathbf{V}^{(B)}$ and $f : X \to X'$ be an extensional mapping. Suppose that $Y, Y', g \in \mathbf{V}^{(B)}$ are such that $[\![\, Y \neq \varnothing]\!] = [\![\, g : Y \to Y']\!] = \mathbf{1}$. Then

$$X{\uparrow\downarrow} = \mathrm{mix}\, X, \quad Y{\downarrow\uparrow} = Y; \quad f{\uparrow\downarrow} = f, \quad g{\downarrow\uparrow} = g.$$

(6) From (6) follows the useful relation:

$$\mathscr{P}_{\mathrm{fin}}(X{\uparrow}) = \{\theta{\uparrow} \mid \theta \in \mathscr{P}_{\mathrm{fin}}(X)\}{\uparrow}\,.$$

Suppose that $X \in \mathbf{V}$, $X \neq \varnothing$; i.e., X is a nonempty set. Let the letter ι denote the standard name embedding $x \mapsto x^{\wedge}$ $(x \in X)$. Then $\iota(X)\uparrow = X^{\wedge}$ and $X = \iota^{-1}(X^{\wedge}\downarrow)$. Using the above relations, we may extend the descent and ascent operations to the case in which Φ is a correspondence from X to $Y\downarrow$ and $[\![\Psi$ is a correspondence from $X^{\wedge}$ to $Y]\!] = \mathbf{1}$, where $Y \in \mathbf{V}^{(B)}$. Namely, we put $\Phi\Uparrow := (\Phi \circ \iota)\uparrow$ and $\Psi\Downarrow := \Psi\downarrow \circ \iota$. In this case, $\Phi\Uparrow$ is called the *modified ascent* of the correspondence Φ and $\Psi\Downarrow$ is called the *modified descent* of the correspondence Ψ. (If the context excludes ambiguity then we simply speak of ascents and descents using simple arrows.) It is easy to see that $\Psi\Uparrow$ is a unique correspondence inside $\mathbf{V}^{(B)}$ satisfying the relation

$$[\![\Phi\Uparrow(x^{\wedge}) = \Phi(x)\uparrow]\!] = \mathbf{1} \quad (x \in X).$$

Similarly, $\Psi\Downarrow$ is a unique correspondence from X to $Y\downarrow$ satisfying the equality

$$\Psi\Downarrow(x) = \Psi(x^{\wedge})\downarrow \quad (x \in X).$$

If $\Phi := f$ and $\Psi := g$ are functions then these relations take the form

$$[\![f\Uparrow(x^{\wedge}) = f(x)]\!] = \mathbf{1}, \quad g\Downarrow(x) = g(x^{\wedge}) \quad (x \in X).$$

(1) A *Boolean set* or a *set with B-structure* or just a *B-set* is a pair (X, d), where $X \in \mathbf{V}$, $X \neq \varnothing$, and d is a mapping from $X \times X$ to the Boolean algebra B such that for all $x, y, z \in X$ the following hold:

(a) $d(x, y) = \mathbf{0} \leftrightarrow x = y$;

(b) $d(x, y) = d(y, x)$;

(c) $d(x, y) \leq d(x, z) \vee d(z, y)$.

An example of a B-set is given by each $\varnothing \neq X \subset \mathbf{V}^{(B)}$ if we put

$$d(x, y) := [\![x \neq y]\!] = C[\![x = y]\!] \quad (x,\ y \in X).$$

Another example is a nonempty X with the "discrete B-metric" d; i.e., $d(x, y) = \mathbf{1}$ if $x \neq y$ and $d(x, y) = \mathbf{0}$ if $x = y$.

(2) Let (X, d) be some B-set. There exist an element $\mathscr{X} \in \mathbf{V}^{(B)}$ and an injection $\iota : X \to X' := \mathscr{X}\downarrow$ such that $d(x, y) = [\![\iota x \neq \iota y]\!]$ $(x, y \in X)$ and every element $x' \in X'$ admits the representation $x' = \operatorname{mix}_{\xi \in \Xi}(b_\xi \iota x_\xi)$, where $(x_\xi)_{\xi \in \Xi} \subset X$ and $(b_\xi)_{\xi \in \Xi}$ is a partition of unity in B. The element $\mathscr{X} \in \mathbf{V}^{(B)}$ is referred to as the *Boolean valued realization* of the B-set X. If X is a discrete B-set then $\mathscr{X} = X^{\wedge}$ and $\iota x = x^{\wedge}$ $(x \in X)$. If $X \subset \mathbf{V}^{(B)}$ then $\iota\uparrow$ is an injection from $X\uparrow$ to $\mathscr{X}$ (inside $\mathbf{V}^{(B)}$).

A mapping f from a B-set (X, d) to a B-set (X', d') is said to be *nonexpanding* if $d(x, y) \geq d'(f(x), f(y))$ for all $x, y \in X$.

(3) Let X and Y be some B-sets, $\mathscr{X}$ and $\mathscr{Y}$ be their Boolean valued realizations, and ι and $\varkappa$ be the corresponding injections $X \to \mathscr{X}\downarrow$ and $Y \to \mathscr{Y}\downarrow$. If $f : X \to Y$ is a nonexpanding mapping then there is a unique element $g \in \mathbf{V}^{(B)}$ such that $[\![g : \mathscr{X} \to \mathscr{Y}]\!] = \mathbf{1}$ and $f = \varkappa^{-1} \circ g\downarrow \circ \iota$. We also accept the notations $\mathscr{X} := \mathscr{F}^{\sim}(X) := X^{\sim}$ and $g := \mathscr{F}^{\sim}(f) := f^{\sim}$.

(4) Moreover, the following are valid:

(1) $\mathbf{V}^{(B)} \models f(A)^{\sim} = f^{\sim}(A^{\sim})$ for $A \subset X$;

(2) If $g : Y \to Z$ is a contraction then $g \circ f$ is a contraction and $\mathbf{V}^{(B)} \models (g \circ f)^{\sim} = g^{\sim} \circ f^{\sim}$;

(3) $\mathbf{V}^{(B)} \models$ "$f^{\sim}$ is injective" if and only if f is a B-isometry;

(4) $\mathbf{V}^{(B)} \models$ "$f^\sim$ is surjective" if and only if $\bigvee\{d(f(x), y) \mid x \in X\} = \mathbf{1}$ for all $y \in Y$.

Recall that a *signature* is a 3-tuple $\sigma := (F, P, \mathfrak{a})$, where F and P are some (possibly, empty) sets and $\mathfrak{a}$ is a mapping from $F \cup P$ to ω. If the sets F and P are finite then σ is a *finite signature*. In applications we usually deal with algebraic systems of finite signature.

An *n-ary operation* and an *n-ary predicate* on a B-set A are contractive mappings $f : A^n \to A$ and $p : A^n \to B$ respectively. By definition, f and p *are contractive mappings* provided that

$$d(f(a_0, \dots, a_{n-1}), f(a_0', \dots, a_{n-1}')) \leq \bigvee_{k=0}^{n-1} d(a_k, a_k'),$$

$$d_s\Big(p(a_0, \dots, a_{n-1}), p(a_0', \dots, a_{n-1}')\Big) \leq \bigvee_{k=0}^{n-1} d(a_k, a_k')$$

for all $a_0,\ a_0', \dots, a_{n-1},\ a_{n-1}' \in A$, where d is the B-metric of A, and d_s is the *symmetric difference* on B; i.e., $d_s(b_1, b_2) := b_1 \triangle b_2$.

Clearly, the above definitions depend on B and it would be cleaner to speak of B-operations, B-predicates, etc. We adhere to a simpler practice whenever it entails no confusion.

An *algebraic B-system* $\mathfrak{A}$ of signature σ is a pair (A, ν), where A is a nonempty B-set, the *underlying set* or *carrier* or *universe* of $\mathfrak{A}$, and ν is a mapping such that (a) $\mathrm{dom}(\nu) = F \cup P$; (b) $\nu(f)$ is an $\mathfrak{a}(f)$-ary operation on A for all $f \in F$; and (c) $\nu(p)$ is an $\mathfrak{a}(p)$-ary predicate on A for all $p \in P$.

It is in common parlance to call ν the *interpretation* of $\mathfrak{A}$ in which case the notation f^ν and p^ν are common substitutes for $\nu(f)$ and $\nu(p)$.

The signature of an algebraic B-system $\mathfrak{A} := (A, \nu)$ is often denoted by $\sigma(\mathfrak{A})$; while the carrier A of $\mathfrak{A}$, by $|\mathfrak{A}|$. Since $A^0 = \{\varnothing\}$, the nullary operations and predicates on A are mappings from $\{\varnothing\}$ to the set A and to the algebra B respectively. We agree to identify a mapping $g : \{\varnothing\} \to A \cup B$ with the element $g(\varnothing)$. Each nullary operation on A thus transforms into a unique member of A. Analogously, the set of all nullary predicates on A turns into the Boolean algebra B. If $F := \{f_1, \dots, f_n\}$ and $P := \{p_1, \dots, p_m\}$ then an algebraic B-system of signature σ is often written down as $(A, \nu(f_1), \dots, \nu(f_n),\ \nu(p_1), \dots, \nu(p_m))$ or even $(A, f_1, \dots, f_n,\ p_1, \dots, p_m)$. In this event, the expression $\sigma = (f_1, \dots, f_n,\ p_1, \dots, p_m)$ is substituted for $\sigma = (F, P, \mathfrak{a})$.

We now address the B-valued interpretation of a first-order language. Consider an algebraic B-system $\mathfrak{A} := (A, \nu)$ of signature $\sigma := \sigma(\mathfrak{A}) := (F, P, \mathfrak{a})$.

Let $\varphi(x_0, \dots, x_{n-1})$ be a formula of signature σ with n free variables. Assume given $a_0, \dots, a_{n-1} \in A$. We may readily define the truth-value $|\varphi|^{\mathfrak{A}}(a_0, \dots, a_{n-1}) \in B$ of a formula φ in the system $\mathfrak{A}$ for the given values $a_0, \dots, a_{n-1}$ of the variables $x_0, \dots, x_{n-1}$. The definition proceeds as usual by induction on the complexity of φ: Considering propositional connectives and quantifiers, we put

$$|\varphi \wedge \psi|^{\mathfrak{A}}(a_0, \dots, a_{n-1}) := |\varphi|^{\mathfrak{A}}(a_0, \dots, a_{n-1}) \wedge |\psi|^{\mathfrak{A}}(a_0, \dots, a_{n-1});$$

$$|\varphi \vee \psi|^{\mathfrak{A}}(a_0, \dots, a_{n-1}) := |\varphi|^{\mathfrak{A}}(a_0, \dots, a_{n-1}) \vee |\psi|^{\mathfrak{A}}(a_0, \dots, a_{n-1});$$

$$|\neg\varphi|^{\mathfrak{A}}(a_0, \dots, a_{n-1}) := C|\varphi|^{\mathfrak{A}}(a_0, \dots, a_{n-1});$$

$$|(\forall\, x_0)\varphi|^{\mathfrak{A}}(a_1, \dots, a_{n-1}) := \bigwedge_{a_0 \in A} |\varphi|^{\mathfrak{A}}(a_0, \dots, a_{n-1});$$

$$|(\exists\, x_0)\varphi|^{\mathfrak{A}}\,(a_1,\dots,a_{n-1}) := \bigvee_{a_0\in A} |\varphi|^{\mathfrak{A}}(a_0,\dots,a_{n-1}).$$

Now, the case of atomic formulas is in order. Suppose that $p \in P$ symbolizes an m-ary predicate, $q \in P$ is a nullary predicate, and $t_0,\dots,t_{m-1}$ are terms of signature σ assuming values $b_0,\dots,b_{m-1}$ at the given values $a_0,\dots,a_{n-1}$ of the variables $x_0,\dots,x_{n-1}$. By definition, we let

$$|\varphi|^{\mathfrak{A}}(a_0,\dots,a_{n-1}) := \nu(q),\ \text{ if } \varphi = q^{\nu};$$

$$|\varphi|^{\mathfrak{A}}(a_0,\dots,a_{n-1}) := Cd(b_0,b_1), \quad \text{if } \varphi = (t_0 = t_1);$$

$$|\varphi|^{\mathfrak{A}}(a_0,\dots,a_{n-1}) := p^{\nu}(b_0,\dots,b_{m-1}), \quad \text{if } \varphi = p^{\nu}(t_0,\dots,t_{m-1}),$$

where d is a B-metric on A.

Say that $\varphi(x_0,\dots,x_{n-1})$ is *valid* in $\mathfrak{A}$ at the given values $a_0,\dots,a_{n-1} \in A$ of $x_0,\dots,x_{n-1}$ and write $\mathfrak{A} \models \varphi(a_0,\dots,a_{n-1})$ provided that $|\varphi|^{\mathfrak{A}}(a_0,\dots,a_{n-1}) = \mathbf{1}_B$. Alternative expressions are as follows: $a_0,\dots,a_{n-1} \in A$ *satisfy* $\varphi(x_0,\dots,x_{n-1})$; or $\varphi(a_0,\dots,a_{n-1})$ holds true in $\mathfrak{A}$. In case $B := \{\mathbf{0},\mathbf{1}\}$, we arrive at the conventional definition of the validity of a formula in an algebraic system.

Recall that a closed formula φ of signature σ is a *tautology* if φ is valid on every algebraic **2**-system of signature σ.

Consider algebraic B-systems $\mathfrak{A} := (A,\nu)$ and $\mathfrak{D} := (D,\mu)$ of the same signature σ. The mapping $h : A \to D$ is a *homomorphism* of $\mathfrak{A}$ to $\mathfrak{D}$ provided that, for all $a_0,\dots,a_{n-1} \in A$, the following are valid:

(1) $d_B(h(a_1),h(a_2)) \le d_A(a_1,a_2)$;

(2) $h(f^{\nu}) = f^{\mu}$ if $\mathfrak{a}(f) = 0$;

(3) $h(f^{\nu}(a_0,\dots,a_{n-1})) = f^{\mu}(h(a_0),\dots,h(a_{n-1}))$ if $0 \ne n := \mathfrak{a}(f)$;

(4) $p^{\nu}(a_0,\dots,a_{n-1}) \le p^{\mu}(h(a_0),\dots,h(a_{n-1}))$, with $n := \mathfrak{a}(p)$.

A homomorphism h is called *strong* if

(5) $\mathfrak{a}(p) := n \ne 0$ for $p \in P$, and, for all $d_0,\dots,d_{n-1} \in D$ the following inequality holds:

$$p^{\mu}(d_0,\dots,d_{n-1})$$
$$\ge \bigvee_{a_0,\dots,a_{n-1}\in A} \{p^{\nu}(a_0,\dots,a_{n-1}) \wedge d_D(d_0,h(a_0)) \wedge \dots \wedge d_D(d_{n-1},h(a_{n-1}))\}.$$

If a homomorphism h is injective and (1) and (4) are fulfilled with equality holding; then h is said to be a *isomorphism from* $\mathfrak{A}$ *to* $\mathfrak{D}$. Undoubtedly, each surjective isomorphism h and, in particular, the identity mapping $\mathrm{Id}_A : A \to A$ are strong homomorphisms. The composite of (strong) homomorphisms is a (strong) homomorphism. Clearly, if h is a homomorphism and h^{-1} is a homomorphism too, then h is an isomorphism.

Note again that in the case of the two-element Boolean algebra $B := \{\mathbf{0},\mathbf{1}\}$ we come to the conventional concepts of homomorphism, strong homomorphism, and isomorphism.

Before giving a general definition of the descent of an algebraic system, consider the descent of the two-element Boolean algebra. Choose two arbitrary elements, 0, $1 \in \mathbf{V}^{(B)}$, satisfying $[\![0 \ne 1]\!] = \mathbf{1}_B$. We may for instance assume that $0 := \mathbf{0}_B^{\wedge}$ and $1 := \mathbf{1}_B^{\wedge}$.

(1) *The descent D of the two-element Boolean algebra $\{0,1\}^B \in \mathbf{V}^{(B)}$ is a complete Boolean algebra isomorphic to B. The formulas*

$$[\![\chi(b) = 1]\!] = b, \quad [\![\chi(b) = 0]\!] = Cb \quad (b \in B)$$

defines an isomorphism $\chi : B \to D$.

Consider now an algebraic system $\mathfrak{A}$ of signature $\sigma^\wedge$ inside $\mathbf{V}^{(B)}$, and let $[\![\mathfrak{A} = (A,\nu)^B]\!] = \mathbf{1}$ for some A, $\nu \in \mathbf{V}^{(B)}$. The *descent* of $\mathfrak{A}$ is the pair $\mathfrak{A}{\downarrow} := (A{\downarrow}, \mu)$, where μ is the function determined from the formulas:

$$\mu : f \mapsto (\nu{\downarrow}(f)){\downarrow} \quad (f \in F),$$

$$\mu : p \mapsto \chi^{-1} \circ (\nu{\downarrow}(p)){\downarrow} \quad (p \in P).$$

Here χ is the above isomorphism of the Boolean algebras B and $\{0,1\}^B{\downarrow}$.

In more detail, the modified descent $\nu{\downarrow}$ is the mapping with domain $\operatorname{dom}(\nu{\downarrow}) = F \cup P$. Given $p \in P$, observe $[\![\mathfrak{A}(p)^\wedge = \mathfrak{A}^\wedge(p^\wedge)]\!] = \mathbf{1}$, $[\![\nu{\downarrow}(p) = \nu(p^\wedge)]\!] = \mathbf{1}$ and so

$$\mathbf{V}^{(B)} \models \nu{\downarrow}(p) : A^{\mathfrak{a}(f)^\wedge} \to \{0,1\}^B.$$

It is now obvious that $(\nu{\downarrow}(p)){\downarrow} : (A{\downarrow})^{\mathfrak{a}(f)} \to D := \{0,1\}^B{\downarrow}$ and we may put $\mu(p) := \chi^{-1} \circ (\nu{\downarrow}(p)){\downarrow}$.

Let $\varphi(x_0, \dots, x_{n-1})$ be a fixed formula of signature σ in n free variables. Write down the formula $\Phi(x_0, \dots, x_{n-1}, \mathfrak{A})$ in the language of set theory which formalizes the proposition $\mathfrak{A} \models \varphi(x_0, \dots, x_{n-1})$. Recall that the formula $\mathfrak{A} \models \varphi(x_0, \dots, x_{n-1})$ determines an n-ary predicate on A or, which is the same, a mapping from A^n to $\{0,1\}$. By the maximum and transfer principles, there is a unique element $|\varphi|^{\mathfrak{A}} \in \mathbf{V}^{(B)}$ such that

$$[\![|\varphi|^{\mathfrak{A}} : A^{n^\wedge} \to \{0,1\}^B]\!] = \mathbf{1},$$

$$[\![|\varphi|^{\mathfrak{A}}(a{\uparrow}) = 1]\!] = [\![\Phi(a(0), \dots, a(n-1), \mathfrak{A})]\!] = \mathbf{1}$$

for all $a : n \to A{\downarrow}$. Instead of $|\varphi|^{\mathfrak{A}}(a{\uparrow})$ we will write $|\varphi|^{\mathfrak{A}}\ (a_0, \dots, a_{n-1})$, where $a_l := a(l)$. Therefore, the formula

$$\mathbf{V}^{(B)} \models \text{“}\varphi(a_0, \dots, a_{n-1}) \text{ is valid in } \mathfrak{A}\text{”}$$

holds true if and only if $[\![\Phi(a_0, \dots, a_{n-1}, \mathfrak{A})]\!] = \mathbf{1}$.

(2) Let $\mathfrak{A}$ be an algebraic system of signature $\sigma^\wedge$ inside $\mathbf{V}^{(B)}$. Then $\mathfrak{A}{\downarrow}$ is a universally complete algebraic B-system of signature σ. In this event,

$$\chi \circ |\varphi|^{\mathfrak{A}{\downarrow}} = |\varphi|^{\mathfrak{A}}{\downarrow}$$

for each formula φ of signature σ.

(3) Let $\mathfrak{A}$ and $\mathfrak{B}$ be algebraic systems of the same signature $\sigma^\wedge$ inside $\mathbf{V}^{(B)}$. Put $\mathfrak{A}' := \mathfrak{A}{\downarrow}$ and $\mathfrak{B}' := \mathfrak{B}{\downarrow}$. Then, if h is a homomorphism (strong homomorphism) inside $\mathbf{V}^{(B)}$ from $\mathfrak{A}$ to $\mathfrak{B}$ then $h' := h{\downarrow}$ is a homomorphism (strong homomorphism) of the B-systems $\mathfrak{A}'$ and $\mathfrak{B}'$.

Conversely, if $h' : \mathfrak{A}' \to \mathfrak{B}'$ is a homomorphism (strong homomorphism) of algebraic B-systems then $h := h'{\uparrow}$ is a homomorphism (strong homomorphism) from $\mathfrak{A}$ to $\mathfrak{B}$ inside $\mathbf{V}^{(B)}$.

Let $\mathfrak{A} := (A, \nu)$ be an algebraic B-system of signature σ. Then there are $\mathscr{A}$ and $\mu \in \mathbf{V}^{(B)}$ such that the following are fulfilled:

(1) $\mathbf{V}^{(B)} \models$ *“$(\mathscr{A}, \mu)$ is an algebraic system of signature $\sigma^\wedge$”;*

(2) *If $\mathfrak{A}' := (A', \nu')$ is the descent of $(\mathscr{A}, \mu)$ then $\mathfrak{A}'$ is a universally complete algebraic B-system of signature σ;*

(3) *There is an isomorphism $\imath$ from $\mathfrak{A}$ to $\mathfrak{A}'$ such that $A' = \operatorname{mix}(\imath(A))$;*

(4) *For every formula φ of signature σ in n free variables, the equalities hold*

$$|\varphi|^{\mathfrak{A}}(a_0, \dots, a_{n-1}) = |\varphi|^{\mathfrak{A}'}(\imath(a_0), \dots, \imath(a_{n-1}))$$

$$= \chi^{-1} \circ (|\varphi|^{\mathfrak{A}^\sim}){\downarrow}(\imath(a_0), \dots, \imath(a_{n-1}))$$

for all $a_0, \dots, a_{n-1} \in A$ and χ the same as above.

In closing, we apply the technique of Boolean valued analysis to the algebraic system that is most important for analysis, the system of real numbers.

By the transfer and maximum principles, there is an element $\mathscr{R} \in \mathbf{V}^{(B)}$ such that $\mathbf{V}^{(B)} \models$ "$\mathscr{R}$ is an ordered field of the reals." It is obvious that inside $\mathbf{V}^{(B)}$ the field $\mathscr{R}$ is unique up to isomorphism; i.e., if $\mathscr{R}'$ is another field of the reals inside $\mathbf{V}^{(B)}$ then $\mathbf{V}^{(B)} \models$ "$\mathscr{R}$ and $\mathscr{R}'$ are isomorphic."

It is an easy matter to show that $\mathbb{R}^\wedge$ is an Archimedean ordered field inside $\mathbf{V}^{(B)}$ and so we may assume that $\mathbf{V}^{(B)} \models$ "$\mathbb{R}^\wedge \subset \mathscr{R}$ and $\mathscr{R}$ is the (metric) completion of $\mathbb{R}^\wedge$." Regarding the unity 1 of $\mathbb{R}$, notice that $\mathbf{V}^{(B)} \models$ "$1 := 1^\wedge$ is an order unit of $\mathscr{R}$."

Consider the descent $\mathscr{R}{\downarrow}$ of the algebraic system $\mathscr{R} := (|\mathscr{R}|, +, \cdot\,, 0, 1, \leq)$. By implication, we equip the descent of the underlying set of $\mathscr{R}$ with the descended operations and order of $\mathscr{R}$. In more detail, the addition, multiplication, and order on $\mathscr{R}{\downarrow}$ appear in accord with the following rules:

$$x + y = z \leftrightarrow [\![x + y = z]\!] = \mathbf{1}, \quad xy = z \leftrightarrow [\![xy = z]\!] = \mathbf{1},$$

$$x \leq y \leftrightarrow [\![x \leq y]\!] = \mathbf{1}, \quad \lambda x = y \leftrightarrow [\![\lambda^\wedge x = y]\!] = \mathbf{1}$$

$$(x, y, z \in \mathscr{R}{\downarrow},\ \lambda \in \mathbb{R}).$$

Gordon Theorem. *Let $\mathscr{R}$ be the reals in $\mathbf{V}^{(B)}$. Assume further that $\mathscr{R}{\downarrow}$ stands for the descent $|\mathscr{R}|{\downarrow}$ of the underlying set of $\mathscr{R}$ equipped with the descended operations and order. Then the algebraic system $\mathscr{R}$ is a universally complete K-space.*

Moreover, there is a (canonical) isomorphism χ from the Boolean algebra B onto the base $H_{\mathscr{R}\downarrow}$ of $\mathscr{R}{\downarrow}$ such that the following hold:

$$\chi(b)x = \chi(b)y \leftrightarrow b \leq [\![x = y]\!],$$

$$\chi(b)x \leq \chi(b)y \leftrightarrow b \leq [\![x \leq y]\!]$$

for all $x, y \in \mathscr{R}{\downarrow}$ and $b \in B$.

References

Aleksandroff P. S.

1 An Introduction to the General Theory of Sets and Functions [in Russian], Gostekhizdat, Moscow (1948).

Adel′son-Vel′skiĭ G. M. and Shreĭder Yu. A.

1 "The Banach mean on groups," Uspekhi Mat. Nauk, **12**, No. 6, 131–136 (1957).

Akilov G. P. and Kutateladze S. S.

1 Ordered Vector Spaces [in Russian], Nauka, Novosibirsk (1978)

Antonovsiĭ M. Ya., Boltyanskiĭ V. G., and Sarymsakov T. A.

1 Topological Boolean Algebras [in Russian], Izdat. Akad. Nauk UzSSR, Tashkent (1963).

Araki H. and Woods E. J.

1 "Complete Boolean algebras of type I factors," Publ. Res. Inst. Math. Sci. Ser. A, **2**, No. 2, 157–242 (1966).

Bazhenov I. I.

1 "On a theorem of A. A. Lyapunov on the range of a vector measure," in: Optimizatsiya [in Russian], Inst. Mat. (Novosibirsk), Novosibirsk, 1986, No. 38(55), pp. 135–144.

2 "On convexity of the range of a vector measure," Vestnik Leningrad. Univ. Mat. Mekh. Astronom. Ser. 1, No. 3, 93–95 (1986).

Balcar B. and Franěk F.

1 “Independent families in complete Boolean algebras,” Trans. Amer. Math. Soc., **274**, No. 2, 607–618 (1982).

Balcar B. and Štěpánek P.

1 “Boolean matrices, subalgebras and automorphisms of complete Boolean algebras,” Fund. Math., **96**, 211–229 (1977).

Banach S.

1 “Sur la probleme de la mesure,” Fund. Math., **4**, 7–33 (1923).

2 Théorie des Opérations Linéaires, Warsaw (1932). (Monografie Matematyczne.)

Bernstein S. N.

1 “An attempt of axiomatic foundations of probability theory,” Soobshch. Khar′kovsk. Mat. Obshch. Ser. 2, **15**, 209–274 (1917).

Biktasheva V. A.

1 “Rotations of the spaces $\Lambda(\Psi)$,” Vestnik Leningrad. Univ. Mat. Mekh. Astronom., **13**, No. 3, 116–118 (1979).

Birkhoff G.

1 “Moore–Smith convergence in general topology,” Ann. of Math., **38**, 39–56 (1937).

2 Lattice Theory [Russian translation], Izdat. Inostr. Lit., Moscow (1952).

3 Lattice Theory [Russian translation], Nauka, Moscow (1984).

Bogolyubov N. N. and Krylov N. M.

1 “La théorie génerale de la mesure dans son application á l‘etude des systémes dynamiques de la mécanique non linéarie,” Ann. of Math., **38**, 65–113 (1937).

Boole G.

1 The Mathematical Analysis of Logic, Cambridge (1847).

2 An Investigation of the Laws of Thought, Walton and Maberly, London (1854).

Borovkov A. A.

1 Probability [in Russian], Nauka, Moscow (1976).

Bourbaki N.

1 General Topology. Basic Structures [Russian translation], Fizmatgiz, Moscow (1958).

2 Topological Vector Spaces [Russian translation], Izdat. Inostr. Lit., Moscow (1959).

Carathéodory C.

1 Mass und Integral und ihre Algebraisierung, Birkhäuser Verlag, Basel and Stuttgart (1956).

2 "Die Homomorphieen von Somen und die multiplikation von inhaltsfunktionen," Ann. Scuola Norm. Sup. Pisa Cl. Sci., No. 8, 105–130 (1939).

Chang C.-C.

1 "On the representation of α-complete Boolean algebras," Trans. Amer. Math. Soc., **85**, No. 1, 208–218 (1957).

Christensen J. P. R. and Herer W.

1 "On the existence of pathological submeasures and the construction of exotic topological groups," Math. Ann., **213**, 203–210 (1975).

Darmois G.

1 "Analyse générale des liaisons stochastiques. Étude particulière de l'analyse factoríelle linéaire," Rev. Inst. Internat. Statistique, No. 21, 2–8 (1953).

Davis M.

1 Applied Nonstandard Analysis, John Wiley & Sons, New York etc. (1977).

De Marr R.

1 "A martingale convergence theorem in vector lattices," Canad. J. Math., **18**, 424–432 (1966).

Dikanova Z. T.

1 "On some properties of the partially ordered space of continuous functions on a bicompactum," Uspekhi Mat. Nauk, **14**, No. 6, 165–172 (1959).

Doob J. L.

1 Stochastic Processes [Russian translation], Izdat. Inostr. Lit., Moscow (1956).

Dunford N. and Schwartz J. T.

1 Linear Operators. Vol. 1 [Russian translation], Izdat. Inostr. Lit., Moscow (1962).

2 Linear Operators. Vol. 2 [Russian translation], Mir, Moscow (1966).

Efimov B. A.

1 "On extremally disconnected bicompacta," Dokl. Akad. Nauk SSSR, **172**, 771–774 (1967).

2 "On extremally disconnected compact spaces and absolutes," Trudy Moskov. Mat. Obshch., **23**, 243–283 (1970).

Erdös P. and Tarski A.

1 "On families of mutually exclusive sets," Ann. of Math., **44**, 315–329 (1943).

Feller W.

1 An Introduction to Probability Theory and Its Applications. Vol. 2 [Russian translation], Mir, Moscow (1967).

Fikhtengol′ts G. M. and Kantorovich L. V.

1 "Sur les operations lineaires dans l'espace des fonctions bornees," Studia Math., **5**, 69–98 (1935).

Floyd E. E.

1 "Boolean algebras with pathological order topologies," Pacific J. Math., **5**, 687–689 (1955).

Freudenthal H.

1 "Teilweise geordnete Moduln," Proc. Acad. Amsterdam, **39**, 641–651 (1936).

Gaifman H.

1 "Concerning measures on Boolean algebras," Pacific J. Math., **14**, No. 1, 61–73 (1964).

Gelfand I. M., Kolmogorov A. N., and Yaglom A. M.

1 "On the general definition of the amount of information," Dokl. Akad. Nauk SSSR, **111**, 745–748 (1956).

Glivenko V. I.

1 A Course in Probability Theory [in Russian], GONTI, Moscow and Leningrad (1939).

2 Theorie Generale des Structures, Herman and Cie, Paris (1938).

3 "Fundamentals of the general theory of lattices," Uchen. Zap. Leningrad. Gos. Ped. Inst. Ser. Fiz.-Mat., **1**, 3–33 (1937).

Gordon E. I. and Morozov S. F.

1 Boolean-Valued Models of Set Theory [in Russian], Gor′kovsk. Univ., Gor′kiĭ (1982).

Greenleaf F. P.

1 Invariant Means on Topological Groups and Their Applications [in Russian], Mir, Moscow (1973).

Grigorchuk R. I.

1 "Degrees of growth of finitely generated groups and the theory of invariant means," Izv. Akad. Nauk SSSR Ser. Mat., **48**, No. 5, 939–985 (1984).

Hajian A. B. and Kakutani S.

1 "Weakly wandering sets and invariant measures," Trans. Amer. Math. Soc., **110**, 136–151 (1964).

Hales A.

1 "On the non-existence of free-complete Boolean algebras," Fund. Math., **54**, 44-66.

Halmos P. R.

1 Lectures on Boolean Algebras, Van Nostrand, Toronto (1963).

2 Lectures on Ergodic Theory [Russian translation], Izdat. Inostr. Lit., Moscow (1959).

3 Measure Theory, Springer-Verlag, New York (1974).

Halmos P. R. and von Neumann J.

1 "Operator methods in classical mechanics. II," Ann. of Math., **43**, 332–350 (1942).

Hodges J. L. and Horn A.

1 "On Maharam's conditions for measure," Trans. Amer. Math. Soc., **64**, 594–595 (1948).

Hopf E.

1 "Theory of measure and invariant integrals," Trans. Amer.Math. Soc., **34**, 373–393 (1937).

Horn A. and Tarski A.

1 "Measures on Boolean algebras," Trans. Amer. Math. Soc., **64**, 467–497 (1948).

Ionescu Tulcea A. and Ionescu Tulcea C.

1 Topics in the Theory of Lifting, Springer-Verlag, Berlin etc. (1969).

Jónsson B.

1 "A Boolean algebra without proper automorphisms," Proceedings Amer. Math. Soc., **2**, 766–770 (1951).

Jech T. J.

1 Lectures in Set Theory with Particular Emphasis on the Method of Forcing [Russian translation], Mir, Moscow, (1973).

Kaĭmanovich V. A. and Vershik A. M.

1 "Random walks on discrete groups: boundary and entropy," Ann. Probab., **11**, No. 3, 457–490 (1983).

Kakutani S.

1 "On equivalence of infinite product measures," Ann. of Math., **49**, 214–224 (1948).

2 "Ueber die Metrization der topologischen Gruppen," Proc. Imp. Acad. Japan, No. 12, 82–84 (1936).

Kakutani S. and Oxtoby J. C.

1 "Construction of a non-separable invariant extension of the Lebesgue measure space," Ann. of Math., **52**, 580–590 (1950).

Kalinin V. V.

1 "Dimension functions on Boolean logics. I," Izv. Akad. Nauk SSSR Ser. Mat., **160**, No. 9, 89–90 (1975).

2 "Dimension functions on Boolean logics. II," Izv. Akad. Nauk SSSR Ser. Mat., **161**, No. 10, 92–93 (1975).

Kalton N. J. and Roberts J. W.

1 "Uniformly exhaustive submeasures and nearly additive set functions," Trans. Amer. Math. Soc., **278**, 803–816 (1983).

Kantorovich L. V.

1 "Some theorems on almost everywhere convergence," Dokl. Akad. Nauk SSSR, **14**, 537–540 (1937).

2 "Sur les proprietes des espaces semiordonnes lineaires," C. R. Acad. Sci., **202**, 813–816 (1936).

Kantorovich L. V., Vulikh B. Z., and Pinsker A. G.

1 Functional Analysis in Partially Ordered Spaces [in Russian], Gostekhizdat, Moscow and Leningrad (1950).

Kappos D.

1 Strukturtheorie der Wahrscheinlichkeitsfelder und Räume, Springer Verlag, Berlin (1960).

Karp C. R.

1 "A note on representation of α-complete Boolean algebras," Proc. Amer. Math. Soc., **14**, 705–757 (1963).

Katětov M.

1 "Remarks on Boolean algebras," Colloq. Math., **2**, 229–235 (1952).

Kellerer H. G.

1 "Funktionen auf Produkträumen mit vorgegebenen Marginal-Funktionen," Math. Ann., No. 144, 323–344 (1961).

Kelley J. L.

1 General Topology [Russian translation], Nauka, Moscow (1968).

2 "Measures on Boolean algebras," Pacific J. Math., **9**, 1165–1177 (1959).

Kharazishvili A. B.

1 Invariant Extension of Lebesgue Measure [in Russian], Tbiliss. Univ., Tbilisi (1983).

Kingman J. F. C. and Robertson A. P.

1 "On a theorem of Lyapunov," J. Lond. Math. Soc. **43**, 347-351 (1968).

Kiseleva T. G.

1 "Partially ordered sets endowed with a uniform structure," Vestnik Leningrad. Univ. Mat. Mekh. Astronom. Ser. 1, No. 13, 51–57 (1967).

Kislyakov S. V.

1 "Free subalgebras of complete Boolean algebras and the spaces of continuous functions," Sibirsk. Mat. Zh., **14**, No. 3, 569–581 (1973).

Kluvanek I. and Knowles G.

1 "Liapunov decomposition of a vector measure." Math. Ann., **210**, 123-127 (1974).

Knowles G.

1 "On Liapunov vector measures," in: Measure Theory, Proceedings Conf. Oberwolfach 1975, Lect. Notes Math. **541**, 61-67 (1976).

2 "Lyapunov vector measures," SIAM J. Control, **13**, 294-303 (1975).

Kolmogorov A. N.

1 Grundbegriffe der Wahrscheinlichkeitsrechnung, Berlin (1933). The Principal Ideas of Probability [in Russian], ONTI, Moscow and Leningrad (1936).

2 "Structure of complete metric Boolean algebras," Uspekhi Mat. Nauk, **3**, No. 1(23), 212 (1948).

3 "Algebres de Boole metriques completes. VI," Zjazd Mat. Polskich., 22–30 (1948).

Kolmogorov A. N. and Fomin S. V.

1 Elements of Function Theory and Functional Analysis [in Russian], Nauka, Moscow (1989).

Koppelberg S.

1 "Free subalgebras of complete Boolean algebras," Notices Amer. Math. Soc., **20**, No. 4, 1–18 (1973).

Krylov A. N.

1 Lectures on Approximate Calculations [in Russian], Izdat. Tekhnich. Teor. Lit., Moscow and Leningrad (1950).

Kulakova V. G.

1 "Positive projections and conditional expectations," Zap. Nauchnych Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI), No. 47, 172–174 (1974).

Kuratowski K.

1 "Une méthode d'élimination des nombres transfinis des raisonnements mathématiques," Fund. Math., **5**, No. 3, 76–108 (1922).

2 Topology. Vol. 1 [Russian translation], Mir, Moscow (1966).

Kurosh A. G.

1 Lectures on General Algebra [in Russian], Nauka, Moscow (1962).

Kusraev A. G.

1 Elements of Boolean-Valued Analysis [in Russian], Inst. Mat. (Novosibirsk), Novosibirsk (1987).

2 Vector Duality and Its Applications [in Russian], Nauka, Novosibirsk (1985).

Kusraev A. G. and Kutateladze S. S.

1 Nonstandard Methods of Analysis [in Russian], Nauka, Novosibirsk (1990).

2 Boolean Valued Analysis, Kluwer Academic Piblishers, Dordhrecht (1999).

Kusraev A. G. and Malyugin S. A.

1 Some Questions of Vector Measure Theory [in Russian], Inst. Mat. (Novosibirsk), Novosibirsk (1988).

2 "A product and projective limit of vector measures," in: Contemporary Problems of Algebra and Analysis. Vol. 14 [in Russian], Trudy Inst. Mat. (Novosibirsk), Novosibirsk, 1987, pp. 132–152.

Liptser R. Sh. and Shiryaev A. N.

1 The Theory of Martingales [in Russian], Nauka, Moscow (1986).

Lorch E. R.

1 "Functions of self-adjoint transformations in Hilbert space," Acta Sci. Math. (Szeged), **7**, 136–146 (1935).

Lozanovskiĭ G. Ya.

"On the functions of elements of vector lattices" [in Russian], Izv. Vyssh. Uchebn. Zaved., Mat. **131**, No. 4, 45-54 (1973).

Lyapunov A. A.

1 "On totally additive vector-functions. I," Izv. Akad. Nauk SSSR Ser. Mat., **4**, No. 6, 465–478 (1940).

Lyubetskiĭ V. A. and Gordon E. I.

1 "Boolean extensions of uniform structures," in: Studies on Nonclassical Logics and Formal Systems [in Russian], Moscow, 1983, pp. 82–153.

MacNeille H. M.

1 "Partially ordered sets," Trans. Amer. Math. Soc., **42**, 416–460 (1937).

Mackey G. W.

1 Lectures on Mathematical Foundations of Quantum Mechanics [Russian translation], Mir, Moscow (1965).

Maharam D.

1 "An algebraic characterization of measure algebras," Ann. Math., **48**, 154–167 (1947).

2 "On homogeneous measure algebras," Proc. Nat. Acad. Sci. U. S. A., **28**, 108–111 (1942).

3 "Automorphisms of products of measure spaces," Proc. Amer. Math. Soc., **9**, 702–707 (1958).

Malyugin S. A.

1 "On sequential order topologies," Sibirsk. Mat. Zh., **31**, No. 1, 194–197 (1990).

Marczewski E. and Sikorski R.

1 "On isomorphism types of measure algebras," Fund. Math., **38**, 92–98 (1951).

Markov A. A.

1 "On existence of an integral invariant," Dokl. Akad. Nauk SSSR, **17**, 459–462 (1937).

Matthes K.

1 "Über die Ausdehnung von χ-Homomorphismen Boolescher Algebren. I and II," Z. Math. Logik Grundlag. Math., I: **6**, 97–105 (1960); II: **7**, 16–19 (1961).

2 "Über eine Schar von Regularitätsbedingungen fur Verbände," Math. Nachr., No. 22, 93–128 (1960).

McKenzie R.

1 "Automorphism groups of denumerable Boolean algebras," Canad. J. Math., **26**, 466-471 (1977).

McAlloon K.

1 "Consistency results about ordinal definability," Ann. Math. Logic, **2**, No. 4, 449–467 (1971).

Monk J. D. and Bonnet R. (eds.)

1 Handbook of Boolean Algebras, North-Holland, Amsterdam etc. (1987).

Monk J. D. and Rassbach W.

1 "The number of rigid Boolean algebras," Algebra Universalis, **9**, No. 2, 207–210 (1979).

Moore E. H. and Smith H. L.

1 "A general theory of limits," Amer. J. Math., **44**, 102–121 (1922).

Natanson I. P.

1 Theory of Functions of a Real Variable [in Russian], Nauka, Moscow (1974).

Neumann J., von

1 "Zur Allgemeiner Theorie des Mass," Fund. Math., **13**, 73–116 (1929).

2 "Algebraische Repräsentanten der Functionen 'bis auf eine Menge von Masse Null'" J. Reine Angew. Math., **165**, 109–115 (1931).

Neveu J.

1 Mathematical Foundations of Probability Theory [in French], Masson et Cie, Paris (1964).

Ogasawara T.

1 "Theory of vector lattices. I and II," J. Sci. Hiroshima Univ. Ser. A, I: **12**, 37–100 (1942); II: **13**, 41–161 (1944).

Ol′shanskiĭ A. Yu.

1 "To the question of existence of an invariant mean on a group," Uspekhi Mat. Nauk, **35**, No. 4, 199–200 (1980).

Ore O.

1 "On the foundation of abstract algebra. I," Ann. of Math., **36**, 406–437 (1935).

Parthasarathy K. R.

1 Introduction to Probability and Measure [Russian translation], Mir, Moscow (1983).

Pashenkov A. V.

1 "Representations of topological spaces on Boolean algebras of regular open sets," Mat. Sb., **91(133)**, No. 3(7), 293–309 (1973).

Pierce R. S.

1 "Some questions about complete Boolean algebras," Proc. Sympos. Pure Math., **2**, 129-140 (1961).

Pinsker A. G.

1 "Lattices equivalent to K-spaces," Dokl. Akad. Nauk SSSR, **99**, 503–505 (1954).

2 "A latticial characterization of function spaces," Uspekhi Mat. Nauk, **12**, No. 1(73), 226–229 (1957).

3 "On completing partially ordered spaces," Dokl. Akad. Nauk SSSR, **21**, 6–9 (1938).

4 "On some properties of universally complete K-spaces," Dokl. Akad. Nauk SSSR, **22**, 216–219 (1939).

Pkhakadze Sh. S.

1 "To the theory of Lebesgue measure," Trudy Tbiliss. Razmadze Inst. Mat., 25 (1958).

Plesner A. I.

1 Spectral Theory of Linear Operators [in Russian], Nauka, Moscow (1965).

Plesner A. I. and Rokhlin V. A.

1 "Spectral theory of linear operators. II," Uspekhi Mat. Nauk, **1**, No. 1(11), 71–191 (1946).

Pontryagin L. S.

1 Continuous Groups [in Russian], GITTL, Moscow (1954).

Popov V. A.

1 "Additive and subadditive functions on Boolean algebras," Sibirsk. Mat. Zh., **17**, No. 2, 331–355 (1976).

Potepun A. V.

1 On Order Convergence and Topology in Boolean Algebras and Vector Lattices [in Russian], Dis. Kand. Fiz.-Mat. Nauk, Leningrad (1975)

2 "Generalized regularity of K-lineals and the properties of order topology," Dokl. Akad. Nauk SSSR, **207**, No. 3, 541–543 (1972).

3 "On equivalence relations on Boolean algebras and vector lattices," Vestnik Leningrad. Univ. Mat. Mekh. Astronom. Ser. 1, No. 1, 140–142 (1973).

4 "Some forms of order convergence and topologies on partially ordered spaces," Sibirsk. Mat. Zh., **12**, No. 4, 819–827 (1971).

5 "Some properties for existence of measure on a Boolean algebra," submitted to VINITI on August, 29, 1986, No. 6265-I.

Purtseladze R. R.

1 "On open mappings of extremally disconnected compact sets," Vestnik Leningrad. Univ. Mat. Mekh. Astronom., No. 1, 117–118 (1981).

Rasiowa H. and Sikorski R.

1 The Mathematics of Metamathematics [Russian translation], Moscow (1972).

Rieger L. S.

1 "On free χ_ξ-complete Boolean algebras. (With an application to logic)," Fund. Math., **38**, 35–52 (1951).

2 “Some remarks on automorphisms of Boolean algebras,” Fund. Math., **38**, 209–216 (1951).

Riesz F. and Sz.-Nagy B.

1 Lectures on Functional Analysis [Russian translation], Izdat. Inostr. Lit., Moscow (1954).

Rokhlin V. A.

1 “On the principal ideas of measure theory,” Mat. Sb., **25**, No. 1, 107–150 (1949).

2 “The metric classification of measurable functions,” Uspekhi Mat. Nauk, **12**, No. 2(74), 169–174 (1957).

Romanovskiĭ I. V. and Sudakov V. N.

1 “On existence of independent partitions,” Trudy Mat. Inst. Steklov., **79**, 5–10 (1965).

Sacks S.

1 Theory of the Integral [Russian translation], Izdat. Inostr. Lit., Moscow (1949).

Samorodnitskiĭ A. A.

1 “Boolean exhaustion and the structure of a measure space,” Vestnik Syktyvkarsk. Univ., No. 1, 89–100 (1995).

2 Measure Theory [in Russian], Leningrad University Press, Leningrad (1990).

Sarymsakov T. A.

1 “Topological semifields and the principal ideas of probability,” Dokl. Akad. Nauk UzSSSR Ser. Fiz.-Mat. Nauk, No. 8, 5–9 (1965).

Sarymsakov T. A. et al.

1 Ordered Algebras [in Russian], Fan, Tashkent (1983).

Savel′ev L. Ya.

1 “Extension of measures by continuity,” Sibirsk. Mat. Zh., **5**, No. 3, 639–650 (1964).

2 “On continuous measures,” Dokl. Akad. Nauk SSSR, **160**, 44–45 (1965).

3 “On countably additive vector measures,” Dokl. Akad. Nauk SSSR, **167**, 758–759 (1966).

4 “On order topologies and continuous measures,” Sibirsk. Mat. Zh., **6**, No. 6, 1357–1364 (1965).

Schubert H.

1 Kategorien. Vol. 1 and 2, Springer Verlag, Berlin and New York (1970).

Semadeni Z.

1 Banach Spaces of Continuous Functions. Vol. 1, PWN-Polish Scientific Publishers, Warsaw (1971). (Monografie Matematyczne.)

Shamaev I. I.

1 “Countable extension of a vector-lattice-valued measure,” Sibirsk. Mat. Zh., **22**, No. 3, 197–203 (1981).

Shatunovskiĭ S. O.

1 An Introduction to Analysis [in Russian], Odessa (1923).

Sheffer H. M.

1 “A set of five independent postulates for Boolean algebras with application to logical constants,” Trans. Amer. Math. Soc., **14**, 481–488 (1913).

Shelah S.

1 “Why there are many non-isometric models for unsuperstable theories,” in: Proceedings of the International Congress of Mathematics, 1974, Vancouver, pp. 85–90.

Sikorski R.

1 Boolean Algebras [Russian translation], Mir, Moscow (1969).

Sirvint Yu. F.

1 “Convex sets and linear functionals in abstract space. I: Convex sets. II: Linear functionals,” Izv. Akad. Nauk SSSR Ser. Mat., **6**, I: 143–170; II: 189–226 (1942).

Skitovich V. P.

1 “On a certain property of the normal distribution,” Dokl. Akad. Nauk SSSR, **89**, 217–219 (1953).

Sobolev V. I.

1 "On a partially ordered measure of sets, measurable functions, and certain abstract integrals," Dokl. Akad. Nauk SSSR, **91**, 23–26 (1953).

Soler J.-L.

1 Notion de Liberté en Statistique Mathématique [Russian translation], Mir, Moscow (1972).

Solovay R. M. and Tennenbaum S.

1 "Iterated Cohen extensions and Suslin's problem," Ann. of Math., **94**, No. 2, 201–245 (1972).

Stone M. H.

1 "Applications of the theory of Boolean rings to general topology," Trans. Amer. Math. Soc., **41**, 375–481 (1937).

2 "Algebraic characterizations of special Boolean rings," Fundam. Math., **29**, 223–303 (1937).

3 "The theory of representations for Boolean algebras," Trans. Amer. Math. Soc., **40**, 37–111 (1936).

Sucheston L.

1 "On existence of finite invariant measures," Math. Z., **86**, 327–336 (1964).

Sudakov V. N.

1 "Geometric problems of the theory of infinite-dimensional probability distributions," Trudy Mat. Inst. Steklov., **141** (1976).

Sukonkin V. I.

1 "A certain example in the theory of sets," Sibirsk. Mat. Zh., **7**, No. 6, 1435 (1966).

Talagrand M.

1 "A simple example of pathological submeasure," Math. Ann., **252**, No. 2, 97–102 (1980).

Tsalenko M. Sh. and Shul′geĭfer E. G.

1 Fundamentals of the Theory of Categories [in Russian], Nauka, Moscow (1974).

Vallée-Poussin Ch.

1 "Sur l'integrale de Lebesgue," Trans. Amer. Math. Soc., **16** (1915).

Veksler A. I.

1 "On topological completeness of Boolean algebras in the sequential order topology," Sibirsk. Mat. Zh., **14**, No. 4, 726–737 (1973).

Vershik A. M.

1 "Decreasing sequences of measurable decompositions and their applications," Dokl. Akad. Nauk SSSR, **193**, No. 4, 748–751 (1970).

Vinokurov V. G.

1 "On additional representations of measure algebras," in: Probability and Statistics [in Russian], Tashkent, 1964, No. 1, pp. 126–129.

2 "On infinite products of Lebesgue spaces," Dokl. Akad. Nauk SSSR, **158**, 1247–1249 (1964).

3 "Representations of Boolean algebras and measure spaces," Mat. Sb., **56(98)**, No. 3, 374–391 (1962).

Vinokurov V. G., Rubshteĭn B. A., and Fëdorov A. L.

1 The Lebesgue Space and Its Measurable Partitions [in Russian], Tashkent (1985).

Vladimirov D. A.

1 Boolean Algebras [in Russian], Nauka, Moscow (1969).

2 "Invariant measures on Boolean algebras," Mat. Sb., **67(109)**, No. 3, 440–460 (1965).

3 "On completeness of a partially ordered space," Uspekhi Mat. Nauk, **15**, No. 2(92), 165–172 (1960).

4 "On existence of invariant measures on Boolean algebras," Dokl. Akad. Nauk SSSR, **157**, 764–766 (1964).

5 "On independence of subalgebras and partitions," Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI), **85**, 30–38 (1979).

6 "On normability of a Boolean algebra," Dokl. Akad. Nauk SSSR, **146**, 987–990 (1962).

7 "On the countable additivity of a Boolean measure," Vestnik Leningrad. Univ. Mat. Mekh. Astronom., **16**, No. 19, 5–15 (1961).

8 "Structure of complete Boolean algebras," in: Abstracts: The International Mathematical Congress, Moscow, 1966, No. 2, p. 36.

9 "Isomorphic subalgebras of probability Boolean algebras," Journal Math. Sci., New York 75, No.5, 1900–1902 (1995); translation from Zap. Nauchn. Semin. POMI 194, 44–47 (1992).

Vladimirov D. A. and Efimov B. A.

1 "On the cardinalities of extremally disconnected spaces and complete Boolean algebras," Dokl. Akad. Nauk SSSR, **194**, No. 6, 1247–1250 (1970).

Vladimirov D. A. and Samorodnitskiĭ A. A.

1 "The metric classification of measurable functions in nonseparable measure spaces," in: Optimizatsiya [in Russian], Inst. Mat. (Novosibirsk), Novosibirsk, 1986, No. 37(54), pp. 17–27.

Vladimirov D. A. and Shergin Yu. V.

1 "Characterization of L^p spaces in the class of normed vector lattices by means of the group of their isometric order automorphisms," Vestnik Leningrad. Univ. Mat. Mekh. Astronom. Ser. 1, No. 1, 118–120 (1991).

Vladimirov D. A. and Zenf P.

1 "Partitions of totally disconnected bicompacta," Z. Anal. Anwendungen, **2**, No. 3, 281–286 (1983).

Vulikh B. Z.

1 Introduction to the Theory of Partially Ordered Spaces [in Russian], Fizmatgiz, Moscow (1961).

2 "Concrete representations of partially ordered linear spaces," Dokl. Akad. Nauk SSSR, **58**, 733–736 (1947).

3 "On a Boolean measure," Uchen. Zap. Leningrad. Gos. Ped. Inst., **125**, 95–114 (1956).

Weber H.

1 "Die atomare Struktur topologischer Boolescher Ringe und s-beschränkter Inhalte," Studia Math., **74**, No. 1, 57–81 (1982).

2 "Topological Boolean rings. Decomposition of finitely additive set functions," Pacific J. Math., **110**, No. 2, 471–495 (1984).

Wright J. D. M.

1 “Stone-algebra-valued measures and integrals,” Proc. London Math. Soc., **19**, No. 1, 107–122 (1969).

2 “The solution of a problem of Sikorski and Matthes,” Bull. London Math. Soc., **3**, 52–54 (1971).

Yablonskiĭ S. V., Gavrilov G. P., and Kudryavtsev V. B.

1 Boolean Functions and Post Classes [in Russian], Nauka, Moscow (1966).

Yaglom I. M.

1 A Boolean Lattice and Its Models [in Russian], Sov. Radio, Moscow (1980).

Yosida K. and Hewitt E.

1 “Finitely additive measures,” Trans. Amer. Math. Soc., **72**, 46–66 (1952).

Yudin A. I.

1 “On completing partially ordered linear spaces,” Uchen. Zap. Leningrad Univ., **12**, 57–61 (1941).

Zaĭdenberg M. G.

1 “On the isometric classification of rearrangement-invariant spaces,” Dokl. Akad. Nauk SSSR, **234**, No. 2, 283–286 (1977).

Zhegalkin I. I.

1 “On the technique of computing sentences in symbolic logic,” Mat. Sb., **34**, 9–28 (1927).

Zorn M.

1 “A remark on a method in transfinite algebra,” Bull. Amer. Math. Soc., **41**, 667–670 (1935).

Index