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S. S. Kutateladze (ed.)

Vector Lattices and Integral Operators

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Contents

Foreword

The theory of vector lattices, stemming from the mid-thirties, is now at the stage where its main achievements are being summarized. The sweeping changes of the last two decades have changed its image completely. The range of its application was expanded and enriched so as to embrace diverse branches of the theory of functions, geometry of Banach spaces, operator theory, convex analysis, etc. Furthermore, the theory of vector lattices was impregnated with principally new tools and techniques from other sections of mathematics. These circumstances gave rise to a series of monographs treating separate aspects of the theory and oriented to specialists. At the same time, the necessity of a book intended for a wider readership, reflecting the modern diretions of research became clear. The present book is meant to be an attempt at implementing this task. Although oriented to readers making their first acquaintance with vector-lattice theory, it is composed so that the main topics dealt with in the book reach the current level of research in the field, which is of interest and import for specialists.

The monograph was conceived so as to be divisible into two parts that can be read independently of one another. The first part is mainly Chapter 1, devoted to the so-called Boolean-valued analysis of vector lattices. The term designates the application of the theory of Boolean-valued models by D. Scott, R. Solovay and P. Vopěnka to constructing a special realization of a (model of) vector lattice which allows one to treat elements of the lattice under study as reals. The starting point is an E. I. Gordon theorem claiming that the presentation of the field of reals in a Boolean-valued model constitutes a universally complete vector lattice (an extended K-space in the respective Russian terminology). Thus a huge part of the general theory of vector lattices admits of a straightforward derivation by interpreting the familiar properties of reals. This chapter also exposes Boolean-valued approaches to more advanced sections of vector-lattice theory such as lattice-normed spaces and fragments of positive operators, approaches being proposed by A. G. Kusraev and S. S. Kutateladze. The elementary exposition of the apparatus of model theory as well as that of vector-lattice theory makes it possible to present the material in such a form that for a reader-logician it appears as an introduction into new fields of applications of model theory whereas for a reader-analyst it is an introduction into applicable model theory. More advanced topics of the abstract theory of operators in vector lattices are treated in Chapter 5 by A. E. Gutman. His research into the properties of disjointness preserving operators is in many aspects motivated by Boolean-valued analysis, demonstrating the power of the latter.

The second part of the monograph consists of Chapters 2–4 and its Supplement. It deals with operator theory in spaces of measurable functions and is oriented to the reader who is interested in functional analysis and the theory of functions. The book treats the classes of operators that are explicitly or implicitly tied with the natural order relation between measurable functions.

The explicit connection is discussed, for instance, in considering regular operators which are differences of pairs of positive operators. The implicit connection relates to integral operators whose definition is given in the conventional terms of function theory. It turns out that the answer to the question of which operators are integral depends upon the theory of vector lattices not only in formulation but also in proof. Part of Chapter 2 and the first part of Chapter 4 address the answering of the just-mentioned question first raised by John von Neumann as long ago as the thirties. The book presents an original solution to the problem that was given by A. V. Bukhvalov (1974) and supplements it with the approaches that have appeared since then.

The theme of Chapter 3 and the second part of Chapter 4 is mainly concerned with studying the stability of different classes of operators defined in terms of order, where stability as regards multiplication by arbitrary continuous operators is treated. As a rule it is relatively easy to demonstrate that such composition does not always belong to the class considered initially. Thus the problem appears of describing subclasses of operators stable under the indicated operation. In Chapter 3 we study various modifications of the problem for regular and dominated operators. The main results in this direction are due to B. M. Makarov and V. G. Samarskiĭ. The technique of researching the topic turns out to be interwoven with the theory of p-absolutely summing operators and operator factorization theory of E. M. Nikishin-B. Maurey. This material related to modern Banach space theory is exposed in monograph form for the first time. The second part of Chapter 4 treats the same problem on composition but now for the class of integral operators.

In Chapter 4 the authors try their best to provide comprehensive information on the solution of problems raised in the book of P. Halmos and V. Sunder "Bounded Linear Operators on L^2 Spaces." The main results here are due to the contribution of V. B. Korotkov and W. Schachermayer. A separate supplement treats a related problem for the convolution operator which was settled by V. D. Stepanov.

The variety of addressed subjects and results determined the style of exposition. Some of the more elementary material freely accessible to the reader is presented without proofs. All principal results are however furnished with complete proofs. Commentaries appended to all chapters contain additional information and refer to the literature. The contributors assume the reader to be familiar with standard courses in the theory of functions and functional analysis.

The present collection is based on its predecessor in Russian which is enriched by Chapter 5 written by A. E. Gutman at my request. The Russian edition was a joint venture and a joint monograph by A. V. Bukhalov, V. B. Korotkov, A. G. Kusraev, B. M. Makarov and S. S. Kutateladze which was published in 1992. Tumultuous events in the former Soviet Union hinder the means of communications between the contributors. As a result, I became the only one of our team who had a chance of reading the whole manuscript in English. So, I solely bear full responsibility for all demerits of the present edition, pretending to none of its possible merits.

S. S. Kutateladze

CHAPTER 1

Nonstandard Theory of Vector Lattices

BY

A. G. Kusraev and S. S. Kutateladze

The theory of vector lattices appeared in early thirties of this century and is connected with the names of L. V. Kantorovich, F. Riesz, and H. Freudenthal. The study of vector spaces equipped with an order relation compatible as a rule with a given norm structure was evidently motivated by the general circumstances that brought functional analysis to life in those years. Here the general inclination to abstraction and "sociological" approach to studying functions, operations on functions, and equations related to them should be noted. A distinguishable circumstance was that the comparison of the elements could be added to the properties of functional objects under consideration. At the same time, the general concept of a Banach space ignored a specific aspect of the functional spaces—the existence of a natural order structure in them, which makes these spaces vector lattices.

Along with the theory of vector spaces, the theory of Banach algebras was developed almost at the same time. Although at the beginning these two theories advanced in parallel, soon their paths parted. Banach algebras were found to be effective in function theory, in the spectral theory of operators, and in other related fields. The theory of vector lattices developed more slowly and its achievements related to the characterization of various types of ordered spaces and to the description of operators between them were rather unpretentious and specialized.

In the middle of the seventies the renewed interest in the theory of vector lattices led to its fast development which was related to the general explosive development in functional analysis; there were also some specific reasons, the main due to the use of ordered vector space in the mathematical approach to the social phenomena, economics in particular. The scientific contribution and unique personality of L. V. Kantorovich also played important role in the development of the theory of order spaces and in the interplay and further synthesis of the theory with economics and optimization. Another, although less evident, reason for the interest in vector lattices was their unexpected role in the theory of nonstandard, Boolean-valued, models of set theory. Constructed by D. Scott, R. Solovay, and P. Vopěnka in connection with the well-known results by P. J. Cohen about the continuum hypothesis, these models proved to be inseparably linked with the theory of vector lattices. Indeed, it was discovered that the elements of such lattices serve as images of real numbers in a suitably selected Boolean-valued model. This fact not only gives a precise meaning to the initial idea that abstract ordered spaces are derived from real numbers, but also provides a new opportunity to infer common properties of vector lattices by using the fact that they, in a precise sense, depict the sublattices of the field $\mathbb{R}$. In fact, we grasp the opportunity while composing the present chapter.

1.1. Vector Lattices

Here we give a sketch of the basic concepts of the theory of vector lattices. One can find a more detailed presentation in [3, 4, 5, 9, 17, 22, 26, 27, 47, 52, 54, 55, 69].

1.1.1. Let $\mathbb{F}$ be a linearly ordered field. An *ordered vector space over* $\mathbb{F}$ is a pair $(E, \leq)$, where E is a vector space over $\mathbb{F}$ and $\leq$ is an order in E satisfying the following conditions:

(1) if $x \leq y$ and $u \leq v$ then $x + u \leq y + v$ whatever $x, y, u, v \in E$ might be;

(2) if $x \leq y$ then $\lambda x \leq \lambda y$ for all $x, y \in E$ and $0 \leq \lambda \in \mathbb{F}$.

Thus, in an ordered vector space we can sum inequalities and multiply them by all positive elements of the field $\mathbb{F}$. This circumstance is worded as follows: $\leq$ is an order compatible with vector space structure or, briefly, $\leq$ is a vector order.

Presetting a vector order on a vector space E over $\mathbb{F}$ is equivalent to indicating a set $E_+ \subset E$ (called the *positive cone* of E) with the following properties:

$$E_+ + E_+ \subset E_+, \quad \lambda E_+ \subset E_+ \ (0 \leq \lambda \in \mathbb{F}), \quad E_+ \cap -E_+ = \{0\}.$$

Moreover, the order $\leq$ and the cone E_+ are connected by the relation

$$x \leq y \leftrightarrow y - x \in E_+ \quad (x, y \in E).$$

The elements of E_+ are called *positive*.

1.1.2. A *vector lattice* is an ordered vector space that is also a lattice.

Thereby in a vector lattice there exists a least upper bound $\sup\{x_1, \dots, x_n\} := x_1 \vee \dots \vee x_n$ and a greatest lower bound $\inf\{x_1, \dots, x_n\} := x_1 \wedge \dots \wedge x_n$ for every finite set $\{x_1, \dots, x_n\} \subset E$. In particular, every element x of a vector lattice has the *positive part* $x^+ := x \vee 0$, the *negative part* $x^- := (-x)^+ := -x \wedge 0$, and the *modulus* $|x| := x \vee (-x)$.

The *disjointness relation* $\perp$ in a vector lattice E is defined by the formula

$$\perp := \{(x, y) \in E \times E \mid |x| \wedge |y| = 0\}.$$

A set

$$M^{\perp} := \{x \in E \mid (\forall y \in M) x \perp y\},$$

where M is an arbitrary nonempty set in E, is called a *band* (a *component* in the Russian literature) of E. A band of the form $\{x\}^{\perp\perp}$ with $x \in E$ is called *principal.* The collection $\mathfrak{B}(E)$ of all bands of E ordered by inclusion is a complete Boolean algebra under the Boolean operations

$$L \wedge K = L \cap K, \quad L \vee K = (L \cup K)^{\perp\perp}, \quad L^* = L^{\perp} \quad (L, K \in \mathfrak{B}(E)).$$

The algebra $\mathfrak{B}(E)$ is called the *base* of E.

An element $\mathbf{1} \in E$ is called an (*order*) *unity* or *order-unit* if $\{\mathbf{1}\}^{\perp\perp} = E$; i.e., if E lacks nonzero elements disjoint from $\mathbf{1}$. The set $\mathfrak{E}$ composed of all upper bounds of every order-unit in E is called the *order-unit filter* of E. Let $e \wedge (\mathbf{1} - e) = 0$ for some $0 \leq e \in E$. Then e is said to be a *unit element* (*relative to* $\mathbf{1}$). The set $\mathfrak{E}(\mathbf{1}) := \mathfrak{E}(E)$ of all unit elements with the order induced by E is a Boolean algebra. The lattice operations in $\mathfrak{E}(\mathbf{1})$ are taken from E and the Boolean complement has the form $e^* := \mathbf{1} - e$ ($e \in \mathfrak{E}(E)$).

Let K be a band of the vector lattice E. If there is an element $\sup\{u \in K \mid 0 \leq u \leq x\}$ in E then it is called the *projection* of x onto the band K and is denoted by $[K]x$ (or $\mathrm{Pr}_K\, x$). Given an arbitrary $x \in E$, we put $[K]x := [K]x^+ - [K]x^-$. The projection of an element $x \in E$ onto a band K exists if and only if x is representable as $x = y + z$ with $y \in K$ and $z \in K^{\perp}$. Furthermore, $y = [K]x$ and $z = [K^{\perp}]x$. Assume that every element $x \in E$ has a projection onto K, then the operator $x \mapsto [K]x$ ($x \in E$) is a linear idempotent and $0 \leq [K]x \leq x$ for all $0 \leq x \in E$, called a *band projection* or an *order projection.* The band projection onto a principal band is called *principal.* We say that E is a *vector lattice with the* (*principal*) *projection property* if every (principal) band in $\mathfrak{B}(E)$ is a projection band.

1.1.3. A linear subspace I of a vector lattice is called an *order ideal* or *o*-ideal (or, finally, just an *ideal*, when it is clear from the context what is meant) if the inequality $|x| \leq |y|$ implies $x \in I$ for arbitrary $x \in E$ and $y \in I$. If an ideal I

possesses the additional property $I^{\perp\perp} = E$ (or, which is the same, $I_{\perp} = \{0\}$) then it is referred to as an *order-dense ideal* of E (the term "foundation" is current in the Russian literature).

A *sublattice* is a subspace $E_0 \subset E$ such that $x \wedge y, x \vee y \in E_0$ for all $x, y \in E_0$. We say that a sublattice E_0 is *minorizing* if, for every $0 \neq x \in E_+$, there exists an element $x_0 \in E_0$ satisfying the inequalities $0 < x_0 \leq x$. We say that E_0 is a *majorizing* (or *massive*) sublattice if, for every $x \in E$, there exists $x_0 \in E_0$ such that $x \leq x_0$. Thus, E_0 is a minorizing (majorizing) sublattice if and only if

$$E_+ \setminus \{0\} = E_+ + E_{0+} \setminus \{0\} \quad (E = E_+ + E_0).$$

Henceforth, if the field $\mathbb{F}$ is not explicitly specified then we presume that a vector lattice is considered over the linearly ordered field $\mathbb{R}$ of real numbers. An *order interval* in E is a set of the form $[a, b] := \{x \in E \mid a \leq x \leq b\}$, where $a, b \in E$. A set in E is called (*order*) *bounded* (or *o-bounded*) if it is included in some order interval. We may introduce the following seminorm in the ideal $E(u) := \bigcup_{n=1}^{\infty}[-nu, nu]$ generated by the element $0 \leq u \in E$:

$$\|x\|_u := \inf\{\lambda \in \mathbb{R} \mid |x| \leq \lambda u\} \quad (x \in E(u)).$$

If $E(u) = E$ then we say that u is a *strong unity* or *strong order-unit* and E is a *vector lattice of bounded elements.* The seminorm $\|\cdot\|_u$ is a norm if and only if the lattice $E(u)$ is *Archimedean*, i.e., the order boundedness of the set $\{n|x| \mid n \in \mathbb{N}\}$ implies $x = 0$ for all $x \in E(u)$.

An element $x \geq 0$ of a vector lattice is called *discrete* if $[0, x] = [0, 1]x$; i.e., if $0 \leq y \leq x$ implies $y = \lambda x$ for some $0 \leq \lambda \leq 1$. A vector lattice E is called *discrete* if, for every $0 \neq y \in E$, there exists a discrete element $x \in E$ such that $0 < x \leq y$. If E lacks nonzero discrete elements then E is said to be *continuous.*

1.1.4. A *Kantorovich space* or, briefly, a *K-space* is a vector lattice over the field of real numbers such that every order bounded set in it has least upper and greatest lower bounds. Sometimes a more precise term, (*conditionally*) *order complete vector lattice*, is employed instead of K-space. If, in a vector lattice, least upper and greatest lower bounds exist only for countable bounded sets, then it is called a *K_σ-space.* Each K_σ-space and, hence, a K-space are Archimedean. We say that a K-space (K_σ-space) is *universally complete* or *extended* if every its subset (countable subset) composed of pairwise disjoint elements is bounded.

In a K-space, there exists a unique band projection onto every band. The set of all band projections of E is denoted by $\mathfrak{P}(E)$. Given projections π and ρ, we put $\pi \leq \rho$ if and only if $\pi x \leq \rho x$ for all $0 \leq x \in E$.

Theorem. *Let E be an arbitrary K-space. Then the operation of projecting onto bands determines the isomorphism $K \mapsto [K]$ of the Boolean algebras $\mathfrak{B}(E)$ and $\mathfrak{P}(E)$. If there is a unity in E then the mappings $\pi \mapsto \pi\mathbf{1}$ from $\mathfrak{P}(E)$ into $\mathfrak{C}(E)$ and $e \mapsto \{e\}^{\perp\perp}$ from $\mathfrak{C}(E)$ into $\mathfrak{B}(E)$ are isomorphisms of Boolean algebras too.*

1.1.5. The band projection π_u onto some principal band $\{u\}^{\perp\perp}$, where $0 \leq u \in E$, can be calculated by the following rule simpler than that indicated in 1.1.2:

$$\pi_u x = \sup\{x \wedge (nu) \mid n \in \mathbb{N}\}.$$

In particular, in a K_σ-space there is a unique projection of every element on every principal band.

Let E be a K_σ-space with unity $\mathbf{1}$. The band projection of unity onto the principal band $\{x\}^{\perp\perp}$ is called the *trace* of an element x and is denoted by e_x. Thus, $e_x := \sup\{\mathbf{1} \wedge (n|x|) \mid n \in \mathbb{N}\}$. The trace e_x serves both as a unity in $\{x\}^{\perp\perp}$ and as a unit element in E. Given a real λ, we denote by e_λ^x the *trace* of the positive part of $\lambda\mathbf{1} - x$, i.e., $e_\lambda^x := e_{(\lambda\mathbf{1}-x)^+}$. The so-defined function $\lambda \mapsto e_\lambda^x$ is called the *spectral function* or *characteristic* of an element x.

1.1.6. An *ordered algebra* over $\mathbb{F}$ is an ordered vector space E over $\mathbb{F}$ which is simultaneously an algebra over the same field and satisfies the following condition: if $x \geq 0$ and $y \geq 0$ then $xy \geq 0$ whatever $x, y \in E$ might be. To characterize the positive cone E_+ of an ordered algebra E, we must add to what was said in 1.1 the property $E_+ \cdot E_+ \subset E_+$. We say that E is a *lattice-ordered algebra* if E is a vector lattice and an ordered algebra simultaneously. A lattice-ordered algebra is an *f-algebra* if, for all $a, x, y \in E_+$, the condition $x \wedge y = 0$ implies that $(ax) \wedge y = 0$ and $(xa) \wedge y = 0$. An f-algebra is called *faithful* or *exact* if, for arbitrary elements x and y, $xy = 0$ implies $x \perp y$. It is easy to show that an f-algebra is faithful if and only if it lacks nonzero nilpotent elements. The faithfulness of an f-algebra is equivalent to absence of strictly positive element with nonzero square.

1.1.7. A *complex vector lattice* is defined to be the complexification $E \oplus iE$ (with i standing for the imaginary unity) of a real vector lattice E. Often it is additionally required that the modulus

$$|z| := \sup\{\mathrm{Re}(e^{i\theta}z) \mid 0 \leq \theta \leq \pi\}$$

exists for every element $z \in E \oplus iE$. In the case of a K-space or an arbitrary Banach lattice this requirement is automatically satisfied, since a complex K-space is the complexification of a real K-space. Speaking about order properties of a complex vector lattice $E \oplus iE$, we mean its real part E. The concepts of sublattice, ideal, band, projection, etc. are naturally translated to the case of a complex vector lattice by appropriate complexification.

1.1.8. The order of a vector lattice generates different kinds of convergence. Let $(\mathrm{A}, \leq)$ be an upward-directed set. A net $(x_\alpha) := (x_\alpha)_{\alpha \in \mathrm{A}}$ in E is called *increasing* (*decreasing*) if $x_\alpha \leq x_\beta$ ($x_\beta \leq x_\alpha$) for $\alpha \leq \beta$ ($\alpha, \beta \in \mathrm{A}$).

We say that a net (x_α) *o-converges* to an element $x \in E$ if there exists a decreasing net $(e_\alpha)_{\alpha \in \mathrm{A}}$ in E such that $\inf\{e_\alpha \mid \alpha \in \mathrm{A}\} = 0$ and $|x_\alpha - x| \leq e_\alpha$ ($\alpha \in \mathrm{A}$). In this event, we call x the *o-limit* of the net (x_α) and write $x = o\text{-}\lim x_\alpha$ or $x_\alpha \xrightarrow{(o)} x$. In a K-space, we also introduce the *upper* and *lower o-limits* of an order bounded net by the formulas

$$\limsup_{\alpha \in \mathrm{A}} x_\alpha := \overline{\lim_{\alpha \in \mathrm{A}}}\, x_\alpha := \inf_{\alpha \in \mathrm{A}} \sup_{\beta \geq \alpha} x_\beta,$$
$$\liminf_{\alpha \in \mathrm{A}} x_\alpha := \underline{\lim_{\alpha \in \mathrm{A}}}\, x_\alpha := \sup_{\alpha \in \mathrm{A}} \inf_{\beta \geq \alpha} x_\beta.$$

These objects are obviously connected as follows:

$$x = o\text{-}\lim_{\alpha \in \mathrm{A}} x_\alpha \leftrightarrow \limsup_{\alpha \in \mathrm{A}} x_\alpha = x = \liminf_{\alpha \in \mathrm{A}} x_\alpha.$$

We say that a net $(x_\alpha)_{\alpha \in \mathrm{A}}$ *converges relatively uniformly* or *converges with regulator* to $x \in E$ if there exist an element $0 \leq u \in E$ called the *regulator of convergence* and a numeric net $(\lambda_\alpha)_{\alpha \in \mathrm{A}}$ with the properties

$$\lim \lambda_\alpha = 0, \quad |x_\alpha - x| \leq \lambda_\alpha u \quad (\alpha \in \mathrm{A}).$$

The element x is called the *r-limit* of the net (x_α) and the notation $x = r\text{-}\lim_{\alpha\in \mathrm{A}} x_\alpha$ or $x_\alpha \overset{(r)}{\to} x$ is used. One can see that the relative uniform convergence is the norm convergence of the space $(E(u), \|\cdot\|_u)$.

The presence of o-convergence in a K-space justifies the definition of the sum for an infinite family $(x_\xi)_{\xi\in\Xi}$. Indeed, let $\mathrm{A} := \mathscr{P}_{\mathrm{fin}}(\Xi)$ be the set of all finite subsets of Ξ. Given $\alpha := \{\xi_1, \dots, \xi_n\} \in \mathrm{A}$, we denote $y_\alpha := x_{\xi_1} + \dots + x_{\xi_n}$. Thereby we obtain the net $(y_\alpha)_{\alpha\in\mathrm{A}}$ which is naturally ordered by inclusion. If there exists $x := o\text{-}\lim_{\alpha\in\mathrm{A}} y_\alpha$ then we call the element x the *o-sum* of the family (x_ξ) and denote it by

$$x = o\text{-}\sum_{\xi\in\Xi} x_\xi := \sum_{\xi\in\Xi} x_\xi.$$

It is evident that, for $x_\xi \geq 0$ ($\xi \in \Xi$), the o-sum of the family (x_ξ) exists if and only if the net $(y_\alpha)_{\alpha\in\mathrm{A}}$ is order bounded; in this case

$$o\text{-}\sum_{\xi\in\Xi} x_\xi = \sup_{\alpha\in\mathrm{A}} y_\alpha.$$

If the elements of the family (x_ξ) are pairwise disjoint then

$$o\text{-}\sum_{\xi\in\Xi} x_\xi = \sup_{\xi\in\Xi} x_\xi^+ - \sup_{\xi\in\Xi} x_\xi^-.$$

Every K-space is *o-complete* in the following sense: If a net $(x_\alpha)_{\alpha\in\mathrm{A}}$ satisfies the condition

$$\limsup |x_\alpha - x_\beta| := \inf_{\gamma\in\mathrm{A}} \sup_{\alpha,\beta\geq\gamma} |x_\alpha - x_\beta|$$

then there is an element $x \in E$ such that $x = o\text{-}\lim x_\alpha$.

1.1.9. Examples of vector lattices.

(1) Let $(E_\xi)_{\xi\in\Xi}$ be a family of vector lattices (f-algebras) over the same ordered field $\mathbb{F}$. Then the Cartesian product $E := \prod_{\xi\in\Xi} E_\xi$ endowed with the coordinatewise operations and order is a vector lattice (f-algebra) over the field $\mathbb{F}$. Furthermore, the lattice E is order complete, universally complete, or discrete if and only if all factors E_ξ possess the property. The base $\mathfrak{B}(E)$ is isomorphic to the product of the family of Boolean algebras $\mathfrak{B}(E_\xi)_{\xi\in\Xi}$. An element $e \in E$ is unity if and only if $e(\xi)$ is unity in E_ξ for all $\xi \in \Xi$. In particular, the collection $\mathbb{R}^\Xi$ ($\mathbb{C}^\Xi$) of real (complex) functions on a nonempty set Ξ exemplifies a universally complete discrete K-space (complex K-space).

(2) Every ideal and, hence, every order-dense ideal in a vector lattice (a K-space) is a vector lattice (a K-space). The base of a vector lattice is isomorphic to the base of its every order-dense ideal. In particular, $l^p(\Xi)$ is a K-space for every $1 \leq p \leq \infty$ (see (1)).

(3) Let N be an ideal in a vector lattice E. The quotient space $\widetilde{E} := E/N$ is a vector lattice as well, provided that the order in it is defined by the positive cone $\varphi(E_+)$, where $\varphi : E \to \widetilde{E}$ is the *canonical coset homomorphism*. The vector lattice $\widetilde{E}$ is Archimedean if and only if N is closed under relative uniform convergence. If E is an f-algebra and the o-ideal N is also a ring ideal then $\widetilde{E}$ is an f-algebra. If E is a K_σ-space and is sequentially o-closed then $\widetilde{E}$ is a K_σ-space and the homomorphism φ is sequentially o-continuous. The base of the vector lattice $\widetilde{E}$ is isomorphic to the complete Boolean algebra $\mathfrak{K}_\Delta := \{M^\Delta \mid M \in \mathscr{P}(E)\}$ of Δ-bands, where

$$\Delta := \{(x, y) \in E \times E \mid |x| \wedge |y| \in N\},$$
$$M^\Delta := \{x \in E \mid (\forall y \in M)\ (x,\ y) \in \Delta\}.$$

(4) Suppose that (T, Σ) is a *measure space*, i.e., T is a nonempty set and Σ is a σ-algebra of its subsets. Denote by $\mathscr{M}(T, \Sigma)$ the set of all real (complex) measurable functions on T with operations and order induced from $\mathbb{R}^T$ ($\mathbb{C}^T$). Take an arbitrary σ-complete ideal $\mathscr{N}$ in the algebra Σ. Let N be the set of functions $x \in \mathscr{M}(T, \Sigma)$ such that $\{t \in T \mid x(t) \neq 0\} \in \mathscr{N}$. Put

$$M(T, \Sigma, \mathscr{N}) := \mathscr{M}(T, \Sigma)/N.$$

Then $\mathscr{M}(T, \Sigma)$ and $M(T, \Sigma, \mathscr{N})$ are real (complex) K_σ-spaces and simultaneously f-algebras. Suppose that $\mu : \Sigma \to \mathbb{R} \cup \{+\infty\}$ is a countably-additive positive measure. The vector lattice $L^0(T, \Sigma, \mu) := M(T, \Sigma, \mu^{-1}(0))$ is a universally complete K_σ-space if the measure μ is finite or σ-finite. In general, the order completeness of the lattice $L^0(T, \Sigma, \mu)$ is connected with the direct sum property for the measure μ (see [12]). Here, for simplicity, we confine ourselves to the case of a σ-finite measure μ. The space $L^0(T, \Sigma, \mu)$ is continuous if and only if μ has no atoms. Recall that an *atom* of a measure μ is a set $A \in \Sigma$ such that $\mu(A) > 0$ and $A' \in \Sigma$, $A' \subset A$, implies either $\mu(A') = 0$ or $\mu(A') = \mu(A)$. The discreteness of $L^0(T, \Sigma, \mu)$ is equivalent to the fact that the measure μ is purely atomic, i.e., every set of nonzero measure contains an atom of μ. The equivalence class containing the identically unity function is an order and ring unit in $L^0(T, \Sigma, \mu)$.

The base of the K-space $L^0(T, \Sigma, \mu)$ is isomorphic to the Boolean algebra $\Sigma/\mu^{-1}(0)$ of measurable sets modulo sets of measure zero. By (2), the spaces $L^p(T, \Sigma, \mu)$ $(1 \le p \le \infty)$ are also K-spaces because they are order-dense ideals of $L^0(T, \Sigma, \mu)$.

(5) Let H be a complex Hilbert space and let $\mathfrak{A}$ be a strongly closed commutative algebra of selfadjoint bounded operators in H. Denote by B the set of all orthoprojections in H involved in the algebra $\mathfrak{A}$. Then B is a complete Boolean algebra. Let $\mathfrak{A}_\infty$ be the set of all selfadjoint densely defined operators a in H such that the spectral function $\lambda \mapsto e^a_\lambda$ $(\lambda \in \mathbb{R})$ of a takes its values in B. Further, let $\overline{\mathfrak{A}}_\infty$ be the set of densely defined normal operators in H such that if $a = u|a|$ is the polar decomposition of a then $|a| \in \mathfrak{A}_\infty$. We furnish the sets $\mathfrak{A}_\infty$ and $\overline{\mathfrak{A}}_\infty$ with the structure of an ordered vector space in a natural way. For $a, b \in \mathfrak{A}_\infty$, the sum $a + b$ and the product ab are defined as the unique selfadjoint extensions of the operators $h \mapsto ah + bh$ and $h \mapsto a(bh)$ $(h \in \mathscr{D}(a) \cap \mathscr{D}(b))$, where $\mathscr{D}(c)$ is the domain of c. Moreover, given $a \in \mathfrak{A}_\infty$, we set $a \ge 0$ if and only if $\langle ah, h\rangle \ge 0$ for all $h \in \mathscr{D}(a)$. The operations and order in $\overline{\mathfrak{A}}_\infty$ are defined by means of complexification.

The sets $\mathfrak{A}_\infty$ and $\overline{\mathfrak{A}}_\infty$ with the indicated operations and order are respectively a universally complete K-space and a complex universally complete K-space with base of unit elements B. Moreover, $\mathfrak{A}$ is the K-space of bounded elements in $\mathfrak{A}_\infty$.

(6) Let Q be a topological space and let $\mathscr{B}(Q, \mathbb{R})$ be the set of all Borel functions from Q into $\mathbb{R}$ endowed with the pointwise operations of addition and multiplication and with the pointwise order. Then $\mathscr{B}(Q, \mathbb{R})$ is a K_σ-space. Denote by N the set of Borel functions x such that $\{t \in Q \mid x(t) \neq 0\}$ is a *meager set* (i.e., a set of the first category). Let $B(Q) := B(Q, \mathbb{R})$ be the quotient space $\mathscr{B}(Q, \mathbb{R})/N$ with the operations and order induced from $\mathscr{B}(Q, \mathbb{R})$. Then $B(Q)$ is a K-space with base isomorphic to the Boolean algebra of Borel subsets of Q modulo sets of the first category. If Q is a *Baire space* (i.e., every nonempty open set in Q is nonmeager), then $\mathfrak{B}(B(Q))$ is isomorphic to the Boolean algebra of all regular open (or regular closed) subsets of Q. Both spaces $\mathscr{B}(Q, \mathbb{R})$ and $B(Q, \mathbb{R})$ are faithful f-algebras. The function identically equal to unity serves as an order and ring unity in these spaces. By replacing $\mathbb{R}$ with $\mathbb{C}$, we obtain the complex K-space $B(Q, \mathbb{C})$.

(7) Let Q be again a topological space and $C(Q)$ be the space of all continuous real functions on Q. Then $C(Q)$ is a sublattice and a subalgebra of $\mathscr{B}(Q, \mathbb{R})$. In particular, $C(Q)$ is a faithful Archimedean f-algebra. Generally speaking, $C(Q)$ is not a K-space. The order completeness of $C(Q)$ is connected with the so-called extremal disconnectedness of the space Q (see 1.12, 1.13). For a uniform topological space Q, the base of the vector lattice $C(Q)$ is isomorphic to the algebra of regular open sets.

Now let $\mathrm{LSC}(Q)$ be the set (of equivalence classes) of lower semicontinuous functions $x : Q \to \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$ such that $x^{-1}(-\infty)$ is nowhere dense and the interior of the set $x^{-1}([-\infty, \infty))$ is dense in Q. As usual, two functions are assumed equivalent if their values differ only on a meager set. The sum $x + y$ (the product xy) of elements $x,\ y \in \mathrm{LSC}(Q)$ is defined as the lower semicontinuous regularization of the pointwise sum $t \mapsto x(t) + y(t)$ $(t \in Q_0)$ (the pointwise product $t \mapsto x(t) \cdot y(t)$ $(t \in Q_0)$), where Q_0 is a dense subset of Q on which x and y are finite. Thereby $\mathrm{LSC}(Q)$ becomes a universally complete K-space and an f-algebra; moreover, the base of $\mathrm{LSC}(Q)$ is isomorphic to the algebra of regular open sets. Thus, if Q is Baire space then the K-spaces $B(Q)$ and $\mathrm{LSC}(Q)$ are isomorphic and if Q is uniform then $C(Q)$ is an (order) dense sublattice of $\mathrm{LSC}(Q)$.

1.1.10. Operators in vector lattices.

(1) Let E and F be vector lattices. A linear operator $U : E \to F$ is called ***positive*** if $U(E_+) \subset F_+$; U is ***regular*** if it is representable as a difference of two positive operators; and, finally, U is ***order bounded*** or ***o-bounded*** if U sends every o-bounded subset in E into an o-bounded subset in F. If F is a K-space then an operator is regular if and only if it is o-bounded. The set of all regular (positive) operators from E into F is denoted by $L^\sim(E, F)$ $(L^\sim(E, F)_+)$.

The Riesz-Kantorovich theorem. *If E is a vector lattice and F is some K-space then the space $L^\sim(E, F)$ of regular operators with cone $L^\sim(E, F)_+$ of positive operators is a K-space.*

Observe that if E is a K-space then $L^\sim(E) := L^\sim(E, E)$ with multiplication defined as superposition is a lattice-ordered algebra but not an f-algebra. The space of regular functionals is conventionally denoted by $E^\sim := L^\sim(E, \mathbb{R})$. The space $L^\sim(E, F)$ is discrete if and only if F and $E^\sim$ are discrete. No description for the base $\mathfrak{B}(L^\sim(E, F))$ in terms of the Boolean algebras $\mathfrak{B}(E)$ and $\mathfrak{B}(F)$ is known.

However there are some advances in this direction (see [5, 40, 44]).

(2) An operator $U : E \to F$ is called *order continuous* (or *o-continuous*) if for every net $(x_\alpha)_{\alpha \in \mathrm{A}}$ in E, the relation o-$\lim_{\alpha \in \mathrm{A}} x_\alpha = 0$ yields o-$\lim_{\alpha \in \mathrm{A}} Ux_\alpha = 0$. The set of all o-continuous regular operators furnished with the operations and order induced from $L^\sim(E, F)$ is denoted by $L_n^\sim(E, F)$. If $U \in L_n^\sim(E, F)$ then the band $\mathscr{N}(U)^\perp$, where $\mathscr{N}(U) = \{x \in E \mid U(|x|) = 0\}$, is called the *carrier* or *band of essential positivity* of the operator U. If $F = \mathbb{R}$ then we write $E_n^\sim$ rather than $L_n^\sim(E, \mathbb{R})$.

The space $L_n^\sim(E, F)$ is a band in $L^\sim(E, F)$ and consequently is a K-space. If $f \in E_n^\sim$ and E_f is the carrier of the functional f then the Boolean algebras $\mathfrak{B}(f) := \mathfrak{B}(\{f\}^{\perp\perp})$ and $\mathfrak{B}(E_f)$ are isomorphic. A functional f is a unity in $E_n^\sim$ if and only if $\mathscr{N}(f)^\perp = E$.

(3) Let E and F be again vector lattices. A linear operator $U : E \to F$ is a *lattice homomorphism* if $U(x \vee y) = Ux \vee Uy$ for all $x, y \in E$. It is clear that a lattice homomorphism preserves least upper and greatest lower bounds of finite nonempty sets and also preserves the modulus and positive and negative parts of every element. An injective lattice homomorphism is called a *lattice* (rarely *order*) *monomorphism*, *isomorphic embedding* and even *lattice isomorphism* from E into F. If a lattice homomorphism $U : E \to F$ is a bijection then we say that E and F are *lattice* (or *order*) *isomorphic* or that U provides an *latticial* or (*order*) *isomorphism* between E and F.

Latticially isomorphic vector lattices possess isomorphic bases. Such vector lattices are or not are (universally complete, discrete or continuous) K-spaces simultaneously.

(4) Consider a vector lattice E and some of its vector sublattices $D \subset E$. A linear operator U from D into E is said to be a *nonexpanding* operator (or a *stabilizer*) if $Ux \in \{x\}^{\perp\perp}$ for every $x \in D$. A nonextending operator may fail to be regular. A regular nonextending operator is an *orthomorphism*. Let $\mathrm{Orth}(E)$ denote the set of all orthomorphisms acting in E and let $\mathscr{Z}(E)$ be the o-ideal generated by the *identity operator* I_E in $L^\sim(E)$. The space $\mathscr{Z}(E)$ is often called the *center* of the vector lattice E. Now, define the space of all orthomorphisms $\mathrm{Orth}^\infty(E)$. First we denote by $\mathfrak{M}$ the collection of all pairs (D, π), where D is an order-dense ideal in E and π is an orthomorphism from D into E. Elements (D, π)

and (D', π') in $\mathfrak{M}$ are declared *equivalent* if the orthomorphisms π and π' coincide on the intersection $D \cap D'$. The quotient set of $\mathfrak{M}$ by the equivalence relation is exactly $\mathrm{Orth}^\infty(E)$. Identify every orthomorphism $\pi \in \mathrm{Orth}(E)$ with the corresponding equivalence class in $\mathrm{Orth}^\infty(E)$. Then $\mathscr{Z}(E) \subset \mathrm{Orth}(E) \subset \mathrm{Orth}^\infty(E)$. The set $\mathrm{Orth}^\infty(E)$ can be naturally furnished with the structure of an ordered algebra.

(a) Theorem. *If E is an Archimedean vector lattice then $\mathrm{Orth}^\infty(E)$ is a faithful f-algebra with unity I_E. Moreover, $\mathrm{Orth}(E)$ is an f-subalgebra in $\mathrm{Orth}^\infty(E)$ and $\mathscr{Z}(E)$ is an f-subalgebra of bounded elements in $\mathrm{Orth}(E)$.*

(b) Theorem. *Every Archimedean f-algebra E with unity $\mathbf{1}$ is algebraically and latticially isomorphic to the f-algebra of orthomorphisms. Moreover, the ideal $I(\mathbf{1})$ is mapped onto $\mathscr{Z}(E)$.*

If E is an Archimedean vector lattice then the base of each of the f-algebras $\mathrm{Orth}^\infty(E)$, $\mathrm{Orth}(E)$, and $\mathscr{Z}(E)$ is isomorphic to the base of E. If E is a K-space then $\mathrm{Orth}^\infty(E)$ is a universally complete K-space and $\mathrm{Orth}(E)$ is its order-dense ideal.

1.1.11. The space of continuous functions taking infinite values on a nowhere dense set plays an important role in the theory of vector lattices. To introduce this space, we need some auxiliary facts. Given a function $x : Q \to \overline{\mathbb{R}}$ and a number $\lambda \in \mathbb{R}$, we denote

$$\{x < \lambda\} := \{t \in Q \mid x(t) < \lambda\}, \quad \{x \leq \lambda\} := \{t \in Q \mid x(t) \leq \lambda\}.$$

Let Q be an arbitrary topological space, let Λ be a dense set in $\mathbb{R}$ and let $\lambda \mapsto G_\lambda$ $(\lambda \in \Lambda)$ be an increasing mapping from Λ into the set $\mathscr{P}(Q)$ ordered by inclusion. Then the following assertions are equivalent:

(1) *there exists a unique continuous function $x : Q \to \overline{\mathbb{R}}$ such that*

$$\{x < \lambda\} \subset G_\lambda \subset \{x \leq \lambda\} \quad (\lambda \in \Lambda);$$

(2) *for arbitrary $\lambda, \nu \in \Lambda$, the inequality $\lambda < \nu$ implies*

$$\mathrm{cl}(G_\lambda) \subset \mathrm{int}(G_\nu).$$

$\lhd$ The implication (1) $\Rightarrow$ (2) is trivial. Prove (2) $\Rightarrow$ (1). Given $t \in Q$, we put $x(t) := \inf\{\lambda \in \Lambda \mid t \in G_\lambda\}$. Thereby a function $x : Q \to \overline{\mathbb{R}}$ is determined, and we can easily verify that $\{x < \lambda\} \subset G_\lambda \subset \{x \leq \lambda\}$. It is clear also that

$$\{x < \lambda\} = \cup\{G_\nu \mid \nu < \lambda\}, \quad \{x \leq \lambda\} = \cap\{G_\nu \mid \lambda < \nu\}.$$

Observe that we have used only the isotonicity of the mapping $\lambda \mapsto G_\lambda$. Consider also the mappings

$$\lambda \mapsto \overset{\circ}{G}_\lambda := \operatorname{int}(G_\lambda), \quad \lambda \mapsto \overline{G}_\lambda := \operatorname{cl}(G_\lambda) \quad (\lambda \in \Lambda).$$

It is seen that these mappings increase too and thereby, in view of what was said above, there exist functions $y, z : Q \to \overline{\mathbb{R}}$ such that

$$\{y < \lambda\} \subset \overset{\circ}{G}_\lambda \subset \{y \leq \lambda\}, \quad \{z < \lambda\} \subset \overline{G}_\lambda \subset \{z \leq \lambda\} \quad (\lambda \in \Lambda).$$

By the definition of $\overline{G}_\lambda$, we have $G_\nu \subset \overline{G}_\lambda$ for $\nu < \lambda$. By virtue of the denseness of Λ in $\mathbb{R}$, for all $t \in Q$ and $\tau > x(t)$ there exist $\lambda, \nu \in \Lambda$ such that $x(t) < \nu < \lambda < \tau$, thus, $t \in G_\nu \subset \overline{G}_\lambda$ and $z(t) < \lambda < \tau$. Sending τ to $x(t)$, we obtain $z(t) \leq x(t)$. The same inequality is obvious for $x(t) = +\infty$ as well. Analogously, $\overset{\circ}{G}_\nu \subset G_\lambda$ for $\nu < \lambda$; consequently, $x(t) \leq y(t)$ for all $t \in Q$. Rewriting relations (2) as $\overline{G}_\nu \subset \overset{\circ}{G}_\lambda$ $(\nu < \lambda)$ and arguing as above, we again conclude that $y(t) \leq z(t)$ for all $t \in Q$. Thus, $x = y = t$. The continuity of x follows from the equalities

$$\{x < \lambda\} = \{y < \lambda\} = \cup\{\overset{\circ}{G}_\nu \mid \nu < \lambda, \ \nu \in \Lambda\},$$

$$\{x \leq \lambda\} = \{z \leq \lambda\} = \cap\{\overline{G}_\nu \mid \nu > \lambda, \ \nu \in \Lambda\},$$

since $\overline{G}_\nu$ is open and $\overset{\circ}{G}_\nu$ is closed for all $\nu \in \Lambda$. $\rhd$

1.1.12. Now let Q be a compact topological space. Recall that a compact space is called *extremally* (*quasiextremally*) *disconnected* or simply *extremal* (*quasiextremal*) if the closure of an arbitrary open set (open F_σ-set) in it is open or, which is equivalent, the interior of an arbitrary closed set (closed G_δ-set) is closed.

Let Q be a quasiextremal compact space, let Q_0 be an open dense F_σ-set in Q, and let $x_0 : Q \to \overline{\mathbb{R}}$ be a continuous function. There exists a unique continuous function $x : Q \to \overline{\mathbb{R}}$ such that $x(t) = x_0(t)$ $(t \in Q_0)$.

$\lhd$ Indeed, if $G_\lambda := \operatorname{cl}\{x_0 < \lambda\}$ then the mapping $\lambda \mapsto G_\lambda$ $(\lambda \in \mathbb{R})$ increases and satisfies condition (2) in 1.11. Consequently, there exists a continuous functions $x : Q \to \overline{\mathbb{R}}$ with the properties $\{x < \lambda\} \subset G_\lambda \subset \{x \leq \lambda\}$. It is easy to verify that $x \restriction Q_0 = x_0$. The function x is unique since Q_0 is dense in Q. $\rhd$

1.1.13. Denote by $C_\infty(Q)$ the set of all continuous functions $x : Q \to \overline{\mathbb{R}}$ that may take values $\pm\infty$ only on a nowhere dense set. Introduce some order on $C_\infty(Q)$ by putting $x \leq y$ if and only if $x(t) \leq y(t)$ for all $t \in Q$. Further, take $x, y \in C_\infty(Q)$ and put

$$Q_0 := \{|x| < \infty\} \cup \{|y| < \infty\}.$$

Then Q_0 is an open and dense F_σ-set in Q. According to 1.12, there exists a unique function $z : Q \to \overline{\mathbb{R}}$ such that $z(t) = x(t) + y(t)$ for $t \in Q_0$. This function z is considered to be the sum of the elements x and y. The product of two arbitrary elements is defined in a similar way. Identifying a number λ with the function identically equal to λ on Q, we obtain the product of an arbitrary $x \in C_\infty(Q)$ and $\lambda \in \mathbb{R}$.

It is easy to see that $C_\infty(Q)$ with the so-introduced operations and order is a vector lattice and simultaneously a faithful f-algebra. Below we observe that $C_\infty(Q)$ is a universally complete K_σ-space. The function identically equal to unity is a ring and lattice unity. The base of the vector lattice $C_\infty(Q)$ is isomorphic to the Boolean algebra of all regular open (closed) subsets of the compact space Q. If the compact space Q is extremal then $C_\infty(Q)$ is universally complete K-space whose base is isomorphic to the algebra of all clopen subsets in Q. The vector lattice $C(Q)$ of all continuous functions on Q is an order-dense ideal in $C_\infty(Q)$; thus, $C(Q)$ is a K-space (K_σ-space) if and only if such is $C_\infty(Q)$.

1.1.14. ***The Vulikh-Ogasawara theorem.*** *Let Q be the Stone space of a Boolean algebra B. Then Q is extremal (quasiextremal) if and only if B is complete (σ-complete).*

1.2. Boolean-Valued Models

In the section we briefly present necessary information on the theory of Boolean-valued models. Details may be found in [6, 33, 37, 48, 61, 62, 67, 68].

The most important feature of the method of Boolean-valued models consists in comparative analysis of standard and nonstandard (Boolean-valued) models which uses a special technique of descent and ascent. Moreover, it is often necessary to carry out some syntax comparison of formal texts. Therefore, before we launch into studying the descent and ascent technique, it is necessary to grasp a more clear idea of the status of mathematical objects in the framework of a formal set theory.

1.2.1. At present, the most widespread axiomatic foundation for mathematics is the Zermelo-Fraenkel set theory. We will briefly recall some of its concepts, outlining the details needed in the sequel. Observe that, speaking of a formal set theory, we will freely (because it is in fact unavoidable) adhere to the level of rigor accepted in mathematics and introduce abbreviations by means of the *definor*, *assignment operator*, $:=$ without specifying subtleties.

(1) The alphabet of the Zermelo-Fraenkel theory (ZF or ZFC if the presence of choice stressed, for short) comprises the symbols of variables; the parentheses (and); the propositional connectives (= the signs of propositional calculus) $\vee$, $\wedge$, $\to$, $\leftrightarrow$, and $\neg$; the quantifiers $\forall$ and $\exists$; the equality sign $=$; and the symbol of a special binary predicate of containment $\in$. In general, the domain of variation of the variables in the ZF theory is thought as the world or *universe* of sets. In other words, the universe of the ZF theory contains nothing but sets. We write $x \in y$ rather than $\in (x, y)$ and say that x is an *element* of y.

(2) The formulas of ZF are defined by means of a routine procedure. In other words, the formulas of ZF are finite texts resulting from the *atomic* formulas $x = y$ and $x \in y$, where x and y are variables of ZF, by reasonably placing parentheses, quantifiers, and propositional connectives. So, if φ_1 and φ_2 are formulas of ZF and x is a variable symbol then the texts $\varphi_1 \to \varphi_2$ and $(\exists\, x)(\varphi_1 \to (\forall y)\,\varphi_2) \vee \varphi_1$ are formulas of ZF, whereas $\varphi_1\, \exists x$ and $\forall\,(x \exists \varphi_1 \neg \varphi_2$ are not. We attach the natural meaning to the terms *free* and *bound variables* and the term *domain of action of a quantifier*. For instance, in the formula $(\forall x)(x \in y)$ the variable x is bound and the variable y is free, whereas in the formula $(\exists y)(x = y)$ the variable x is free and y is bound (for it is bounded by a quantifier). Henceforth, in order to emphasize that the only free variables in a formula φ are the variables $x_1, \dots, x_n$, we write $\varphi(x_1, \dots, x_n)$. Sometimes such a formula is considered as a "function"; in this event, it is convenient to write $\varphi(\cdot, \dots, \cdot)$ or $\varphi = \varphi(x_1, \dots, x_n)$, implying that $\varphi(y_1, \dots, y_n)$ is a formula of ZF obtained by replacing each free occurrence of x_k by y_k for $k := 1, \dots, n$.

(3) Studying ZF, it is convenient to use some expressive tools absent in its formal language. In particular, in the sequel it is worthwhile employing the concepts of class and definable class and also the corresponding symbols of *classifiers* like $A_\varphi := A_{\varphi(\cdot)} := \{x \mid \varphi(x)\}$ and $A_\psi := A_{\psi(\cdot, y)} := \{x \mid \psi(x, y)\}$, where φ and ψ are formulas of ZF and y is a distinguished collection of variables. If it is desirable to clarify or eliminate the appearing records then we may assume that use of classes and classifiers is connected only with the conventional agreement on introducing abbreviations. This agreement, sometimes called the *Church schema*, reads:

$$z \in \{x \mid \varphi(x)\} \leftrightarrow \varphi(z),$$
$$z \in \{x \mid \psi(x,y)\} \leftrightarrow \psi(z,y).$$

Working within ZF, we will employ some notations that are widely spread in mathematics. Some of them are as follows:

$$(\exists! z)\,\varphi(z) := (\exists z)\,\varphi(z) \wedge ((\forall x)\,(\forall y)\,(\varphi(x) \wedge \varphi(y) \rightarrow x = y));$$
$$x \neq y := \neg x = y, \quad x \notin y := \neg x \in y;$$
$$\varnothing := \{x \mid x \neq x\};$$
$$\{x,y\} := \{z \mid z = x \vee z = y\}, \quad \{x\} := \{x,x\},$$
$$(x,y) := \{x, \{x,y\}\};$$
$$(\forall x \in y)\,\varphi(x) := (\forall x)\,(x \in y \rightarrow \varphi(x));$$
$$(\exists x \in y)\,\varphi(x) := (\exists x)\,(x \in y \wedge \varphi(x));$$
$$\cup x := \{z \mid (\exists y \in x)\, z \in y\};$$
$$\cap x := \{z \mid (\forall y \in x)\, z \in y\};$$
$$x \subset y := (\forall z)\,(z \in x \rightarrow z \in y);$$
$$\mathscr{P}(x) := \text{the class of all subsets of } x := \{z \mid z \subset x\};$$
$$\mathbf{V} := \text{the class of all sets} := \{x \mid x = x\}.$$

Note also that in the sequel we accept more complicated descriptions in which much is presumed:

$$\mathrm{Funct}(f) := f \text{ is a function;}$$
$$\mathrm{dom}(f) := \text{the domain of definition of } f;$$
$$\mathrm{im}(f) := \text{the range of } f;$$
$$\varphi \vdash \psi := \varphi \rightarrow \psi := \psi \text{ is derivable from } \varphi;$$
$$\text{a class } A \text{ is a set} := A \in \mathbf{V} := (\exists x)\,(\forall y)\,(y \in A \leftrightarrow y \in x).$$

Such simplifications will be used in rendering more complicated formulas without special stipulations. For instance, instead of some rather complicated formulas of ZF we simply write

$$f : x \to y \equiv \text{“}f \text{ is a function from } x \text{ to } y;\text{”}$$
$$\text{“}E \text{ is a } K\text{-space;”}$$
$$U \in \mathscr{L}(X, Y) \equiv \text{“}U \text{ is a bounded operator from } X \text{ to } Y.\text{”}$$

1.2.2. In ZFC, we accept the usual axioms and derivation rules of a first-order theory with equality which fix the standard means of classical reasoning (syllogisms, the law of the excluded middle, modus ponens, generalization, etc.). Moreover, we accept the following *special* or *proper* axioms:

(1) The axiom of extensionality

$$(\forall x)(\forall y)(x \subset y \land y \subset x \to x = y).$$

(2) The axiom of union

$$(\forall x)(\exists y)(y = \cup x).$$

(3) The axiom of the powerset

$$(\forall x)(\exists y)(y = \mathscr{P}(x)).$$

(4) The axiom of replacement

$$(\forall x)((\forall y)(\forall z)(\forall u)\,\varphi(y, z) \land \varphi(y, u) \to z = u) \to (\exists v)(v = \{z \mid (\exists y \in x)\,\varphi(y, z)\}).$$

(5) The axiom of foundation

$$(\forall x)(x \neq \varnothing \to (\exists y \in x)(y \cap x = \varnothing)).$$

(6) The axiom of infinity

$$(\exists \omega)(\varnothing \in \omega) \land (\forall x \in \omega)(x \cup \{x\} \in \omega).$$

(7) The axiom of choice

$$(\forall F)\,(\forall x)\,(\forall y)\,((x \neq \varnothing \wedge F : x \to \mathscr{P}(y))$$
$$\to ((\exists f)\, f : x \to y \wedge (\forall z \in x)\, f(z) \in F(z)).$$

Grounding on the above axiomatics, we acquire a clear idea of the class of all sets, the *von Neumann universe* $\mathbf{V}$. As the initial object of all constructions we take the empty set. The elementary step of introducing new sets consists in taking the union of the powersets of the sets already available. Transfinitely repeating these steps, we exhaust the class of all sets. More precisely, we assign $\mathbf{V} := \cup_{\alpha \in \mathrm{On}} \mathbf{V}_\alpha$, where On is the class of all ordinals and

$$\mathbf{V}_0 := \varnothing,$$
$$\mathbf{V}_{\alpha+1} := \mathscr{P}(\mathbf{V}_\alpha),$$
$$\mathbf{V}_\beta := \bigcup_{\alpha<\beta} \mathbf{V}_\alpha \quad (\beta \text{ is a limit ordinal}).$$

The class $\mathbf{V}$ is the standard model of the ZFC theory.

1.2.3. Now, we describe the construction of a Boolean-valued universe. Let B be a complete Boolean algebra. Given an ordinal α, put

$$\mathbf{V}_\alpha^{(B)} := \{x \mid \mathrm{Funct}(x) \wedge (\exists \beta)\,(\beta < \alpha \wedge \mathrm{dom}(x) \subset \mathbf{V}_\beta^{(B)} \wedge \mathrm{im}(x) \subset B)\}.$$

Thus, in more detail we have

$$\mathbf{V}_0^{(B)} := \varnothing,$$
$$\mathbf{V}_{\alpha+1}^{(B)} := \{x \mid x \text{ is a function with domain in } \mathbf{V}_\alpha^{(B)} \text{ and range in } B\};$$
$$\mathbf{V}_\alpha^{(B)} := \bigcup_{\beta<\alpha} \mathbf{V}_\beta^{(B)} \quad (\beta \text{ is a limit ordinal}).$$

The class

$$\mathbf{V}^{(B)} := \bigcup_{\alpha \in \mathrm{On}} \mathbf{V}_\alpha^{(B)}$$

is a *Boolean-valued universe*. An element of the class $\mathbf{V}^{(B)}$ is a *B-valued set*. It is necessary to observe that $\mathbf{V}^{(B)}$ consists only of functions. In particular, $\varnothing$ is the function with domain $\varnothing$ and range $\varnothing$. Hence, the "lower" levels of $\mathbf{V}^{(B)}$ are organized as follows:

$$\mathbf{V}_0^{(B)} = \varnothing, \quad \mathbf{V}_1^{(B)} = \{\varnothing\}, \quad \mathbf{V}_2^{(B)} = \{\varnothing, (\{\varnothing\}, b) \mid b \in B\}.$$

It is worth stressing that $\alpha \leq \beta \to \mathbf{V}_\alpha^{(B)} \subset \mathbf{V}_\beta^{(B)}$ holds for all ordinals α and β. Moreover, the following *induction principle* is valid in $\mathbf{V}^{(B)}$:

$$(\forall x \in \mathbf{V}^{(B)})\,((\forall y \in \operatorname{dom}(x))\ \varphi(y) \to \varphi(x)) \to (\forall x \in \mathbf{V}^{(B)})\,\varphi(x),$$

where φ is a formula of ZFC.

1.2.4. Take an arbitrary formula $\varphi = \varphi(u_1, \dots, u_n)$ of the ZFC theory. If we replace the elements $u_1, \dots, u_n$ by elements $x_1, \dots, x_n \in \mathbf{V}^{(B)}$ then we obtain some statement about the objects $x_1, \dots, x_n$. It is to this statement that we intend to assign some *truth-value.* Such a value $[\![\psi]\!]$ must be an element of the algebra B. Moreover, it is naturally desired that the theorems of ZFC be true, i.e., attain the greatest truth-value, unity.

We must obviously define truth-values by double induction, taking into consideration the way in which formulas are built up from atomic formulas and assigning truth-values to the above formulas $x \in y$ and $x = y$, where $x, y \in \mathbf{V}^{(B)}$ in accord with the way in which $\mathbf{V}^{(B)}$ is constructed.

It is clear that if φ and ψ are evaluated formulas of ZFC and $[\![\varphi]\!] \in B$ and $[\![\psi]\!] \in B$ are their truth-values then we should put

$$\begin{aligned}
[\![\varphi \wedge \psi]\!] &:= [\![\varphi]\!] \wedge [\![\psi]\!],\\
[\![\varphi \vee \psi]\!] &:= [\![\varphi]\!] \vee [\![\psi]\!],\\
[\![\varphi \to \psi]\!] &:= [\![\varphi]\!] \Rightarrow [\![\psi]\!],\\
[\![\neg\varphi]\!] &:= [\![\varphi]\!]^*,\\
[\![(\forall x)\,\varphi(x)]\!] &:= \bigwedge_{x \in \mathbf{V}^{(B)}} [\![\varphi(x)]\!],\\
[\![(\exists x)\,\varphi(x)]\!] &:= \bigvee_{x \in \mathbf{V}^{(B)}} [\![\varphi(x)]\!],
\end{aligned}$$

where the right-hand sides involve Boolean operations corresponding to the logical connectives and quantifiers on the left-hand sides: $\wedge$ is the taking of an infimum, $\vee$

is the taking of a supremum, $*$ is the taking of the complement of an element, and the operation $\Rightarrow$ is introduced as follows: $a \Rightarrow b := a^* \vee b$ $(a, b \in B)$. Only such definitions provide the value "unity" for the classical tautologies.

We turn to evaluating the atomic formulas $x \in y$ and $x = y$ for $x, y \in \mathbf{V}^{(B)}$. The intuitive idea consists in the fact that a B-valued set y is a "(lattice) fuzzy set," i.e., a "set that contains an element z in $\mathrm{dom}(y)$ with probability $y(z)$." With this in mind and intending to preserve the logical tautology of $x \in y \leftrightarrow (\exists z \in y)(x = z)$ as well as the axiom of extensionality, we arrive at the following definition by recursion:

$$[\![x \in y]\!] := \bigvee_{z \in \mathrm{dom}(y)} y(z) \wedge [\![z = x]\!],$$

$$[\![x = y]\!] := \bigwedge_{z \in \mathrm{dom}(x)} x(z) \Rightarrow [\![z \in y]\!] \wedge \bigwedge_{z \in \mathrm{dom}(y)} y(x) \Rightarrow [\![z \in x]\!].$$

1.2.5. Now we are able to attach some meaning to formal expressions of the form $\varphi(x_1, \dots, x_n)$, where $x_1, \dots, x_n \in \mathbf{V}^{(B)}$ and φ is a formula of ZFC; i.e., we can define exactly in which sense the set-theoretic proposition $\varphi(u_1, \dots, u_n)$ is valid for elements $x_1, \dots, x_n \in \mathbf{V}^{(B)}$. Namely, we say that the *formula* $\varphi(x_1, \dots, x_n)$ *is valid inside* $\mathbf{V}^{(B)}$ or the *elements* $x_1, \dots, x_n$ *possess the property* φ if $[\![\varphi(x_1, \dots, x_n)]\!] = \mathbf{1}$. In this event, we write $\mathbf{V}^{(B)} \models \varphi(x_1, \dots, x_n)$.

It is easy to convince ourselves that the axioms and theorems of the first-order predicate calculus are valid in $\mathbf{V}^{(B)}$. In particular,

(1) $[\![x = x]\!] = \mathbf{1}$,

(2) $[\![x = y]\!] = [\![y = x]\!]$,

(3) $[\![x = y]\!] \wedge [\![y = z]\!] \leq [\![x = z]\!]$,

(4) $[\![x = y]\!] \wedge [\![z \in x]\!] \leq [\![z \in y]\!]$,

(5) $[\![x = y]\!] \wedge [\![x \in z]\!] \leq [\![y \in z]\!]$.

It is worth observing that for each formula φ we have

$$\mathbf{V}^{(B)} \models x = y \wedge \varphi(x) \to \varphi(y),$$

i.e., in detailed notation

(6) $[\![x = y]\!] \wedge [\![\varphi(x)]\!] \leq [\![\varphi(y)]\!]$.

1.2.6. In a Boolean-valued universe $\mathbf{V}^{(B)}$, the relation $[\![x = y]\!] = \mathbf{1}$ in no way implies that the functions x and y (considered as elements of $\mathbf{V}$) coincide. For example, the function equal to zero on each layer $\mathbf{V}_\alpha^{(B)}$, where $\alpha \geq 1$, plays the role of the empty set in $\mathbf{V}^{(B)}$. This circumstance may complicate some constructions in the sequel. In this connection, we pass from $\mathbf{V}^{(B)}$ to the *separated Boolean-valued universe* $\overline{\mathbf{V}}^{(B)}$ often preserving for it the same symbol $\mathbf{V}^{(B)}$; i.e., we put $\mathbf{V}^{(B)} := \overline{\mathbf{V}}^{(B)}$. Moreover, to define $\overline{\mathbf{V}}^{(B)}$, we consider the relation $\{(x, y) \mid [\![x = y]\!] = \mathbf{1}\}$ in the class $\mathbf{V}^{(B)}$ which is obviously an equivalence. Choosing an element (a representative of least rank) in each class of equivalent functions, we arrive at the separated universe $\overline{\mathbf{V}}^{(B)}$. Note that

$$[\![x = y]\!] = \mathbf{1} \rightarrow [\![\varphi(x)]\!] = [\![\varphi(y)]\!]$$

is valid for an arbitrary formula φ of ZF and elements x and y in $\mathbf{V}^{(B)}$. Therefore, in the separated universe we can calculate the truth-values of formulas paying no attention to the way of choosing representatives. Furthermore, working with the separated universe, for the sake of convenience, one often considers (exercising due caution) a concrete representative rather than a class of equivalence as it is customary, for example, while dealing with function spaces.

1.2.7. The most important properties of a Boolean-valued universe $\mathbf{V}^{(B)}$ are stated in the following three principles:

(1) The transfer principle. All theorems of ZFC are true in $\mathbf{V}^{(B)}$; i.e., the *transfer principle*, written in symbols as

$$\mathbf{V}^{(B)} \models \text{a theorem of ZFC},$$

is valid.

The transfer principle is established by rather laboriously checking that all axioms of ZFC have truth-value $\mathbf{1}$ and the rules of derivation preserve the truth-values of formulas. Sometimes, the transfer principle is worded as follows: "$\mathbf{V}^{(B)}$ is the Boolean-valued model of ZFC."

(2) The maximum principle. For each formula φ of ZFC there exists $x_0 \in \mathbf{V}^{(B)}$ for which

$$[\![(\exists x)\, \varphi(x)]\!] = [\![\varphi(x_0)]\!].$$

In particular, if it is true in $\mathbf{V}^{(B)}$ that there is an x for which $\varphi(x)$ then there is an element x_0 in $\mathbf{V}^{(B)}$ (in the sense of $\mathbf{V}$!) for which $[\![\varphi(x_0)]\!] = \mathbf{1}$. In symbols,

$$\mathbf{V}^{(B)} \models (\exists x)\, \varphi(x) \rightarrow (\exists x_0)\, \mathbf{V}^{(B)} \models \varphi(x_0).$$

In other words, the *maximum principle*

$$(\exists x_0 \in \mathbf{V}^{(B)})\, [\![\varphi(x_0)]\!] = \bigvee_{x \in \mathbf{V}^{(B)}} [\![\varphi(x)]\!]$$

is valid for each formula φ of the ZFC theory.

The last equality accounts for the origin of the term "maximum principle." The proof of the principle represents a simple application of

(3) The mixing principle. Let $(b_\xi)_{\xi \in \Xi}$ be a *partition of unity* in B, i.e. a family of elements of the Boolean-valued algebra B such that

$$\bigvee_{\xi \in \Xi} b_\xi = 1, \quad (\forall \xi, \eta \in \Xi)(\xi \neq \eta \to b_\xi \wedge b_\eta = 0).$$

For every family of elements $(x_\xi)_{\xi \in \Xi}$ of the universe $\mathbf{V}^{(B)}$ and a partition of unity $(b_\xi)_{\xi \in \Xi}$ there exists a (unique) mixing of (x_ξ) with probabilities (b_ξ), i.e. an element x of the separated universe $\mathbf{V}^{(B)}$ such that $b_\xi \leq [\![x = x_\xi]\!]$ for all $\xi \in \Xi$.

The *mixing* of x of a family (x_ξ) with respect to (b_ξ) is denoted as follows:

$$x = \mathrm{mix}_{\xi \in \Xi}(b_\xi x_\xi) = \mathrm{mix}\{b_\xi x_\xi \mid \xi \in \Xi\}.$$

1.2.8. The comparative analysis mentioned at the beginning of the subsection presumes that there is a close interconnection between the universes $\mathbf{V}$ and $\mathbf{V}^{(B)}$. In other words, we need a rigorous mathematical apparatus which would allow us to find out the interplay between the interpretations of one and the same fact in the two models $\mathbf{V}$ and $\mathbf{V}^{(B)}$. The base for such apparatus is constituted by the operations of canonical embedding, descent, and ascent to be presented below. We start with the canonical embedding of the von Neumann universe. Given $x \in \mathbf{V}$, we denote by the symbol $x^\wedge$ the *standard name* of x in $\mathbf{V}^{(B)}$; i.e., the element defined by the following recursion schema:

$$\varnothing^\wedge := \varnothing, \quad \mathrm{dom}(x^\wedge) := \{y^\wedge \mid y \in x\}, \quad \mathrm{im}(x^\wedge) := \{\mathbf{1}\}.$$

Observe some properties of the mapping $x \mapsto x^\wedge$ needed in the sequel.

(1) For an arbitrary $x \in \mathbf{V}$ and formula φ of ZF we have

$$[\![(\exists y \in x^{\wedge})\,\varphi(y)]\!] = \bigvee\{[\![\varphi(z^{\wedge})]\!] \mid z \in x\},$$
$$[\![(\forall y \in x^{\wedge})\,\varphi(y)]\!] = \bigwedge\{[\![\varphi(z^{\wedge})]\!] \mid z \in x\}.$$

(2) If x and y are elements of $\mathbf{V}$ then, by transfinite induction, we establish

$$x \in y \leftrightarrow \mathbf{V}^{(B)} \models x^{\wedge} \in y^{\wedge},$$
$$x = y \leftrightarrow \mathbf{V}^{(B)} \models x^{\wedge} = y^{\wedge}.$$

In other words, the standard name can be considered as an embedding of $\mathbf{V}$ into $\mathbf{V}^{(B)}$. Moreover, it is beyond a doubt that the standard name sends $\mathbf{V}$ onto $\mathbf{V}^{(2)}$, which fact is demonstrated by the next proposition:

(3) The following assertion holds:

$$(\forall u \in \mathbf{V}^{(2)})\,(\exists! x \in \mathbf{V})\ \mathbf{V}^{(B)} \models u = x^{\wedge}.$$

(4) A formula is called *bounded* or *restricted* if each bound variables in it is restricted by a bounded quantifier; i.e., a quantifier ranging over a specific set. The latter means that each bound variable x is restricted by a quantifier of the form $(\forall x \in y)$ or $(\exists x \in y)$ for some y.

The restricted transfer principle. For each bounded formula φ of ZFC and every collection $x_1, \dots, x_n \in \mathbf{V}$ the following equivalence holds:

$$\varphi(x_1, \dots, x_n) \leftrightarrow \mathbf{V}^{(B)} \models \varphi(x_1^{\wedge}, \dots, x_n^{\wedge}).$$

Henceforth, working in the separated universe $\overline{\mathbf{V}}^{(B)}$, we agree to preserve the symbol $x^{\wedge}$ for the distinguished element of the class corresponding to x.

(5) Observe as an example that the restricted transfer principle yields the following assertions:

$$\text{“}\Phi \text{ is a correspondence from } x \text{ to } y\text{”}$$
$$\leftrightarrow \mathbf{V}^{(B)} \models \text{“}\Phi^{\wedge} \text{ is a correspondence from } x^{\wedge} \text{ to } y^{\wedge}\text{;”}$$
$$\text{“}f \text{ is a function from } x \text{ to } y\text{”} \leftrightarrow \mathbf{V}^{(B)} \models \text{“}f^{\wedge} \text{ is a function from } x^{\wedge} \text{ to } y^{\wedge}\text{”}$$
$$(\text{moreover, } f(a)^{\wedge} = f^{\wedge}(a^{\wedge}) \text{ for every } a \in x).$$

Thus, the standard name can be considered as a covariant functor of the category of sets (or correspondences) in $\mathbf{V}$ to the appropriate subcategory of $\mathbf{V}^{(2)}$ in the separated universe $\mathbf{V}^{(B)}$.

1.2.9. Given an arbitrary element x of the (separated) Boolean-valued universe $\mathbf{V}^{(B)}$, we define the *descent* $x{\downarrow}$ of x as

$$x{\downarrow} := \{y \in \mathbf{V}^{(B)} \mid [\![y \in x]\!] = \mathbf{1}\}.$$

We list the simplest properties of the descent procedure:

(1) The class $x{\downarrow}$ is a set, i.e., $x{\downarrow} \in \mathbf{V}$ for each $x \in \mathbf{V}^{(B)}$. If $[\![x \neq \varnothing]\!] = \mathbf{1}$ then $x{\downarrow}$ is a nonempty set.

(2) Let $z \in \mathbf{V}^{(B)}$ and $[\![z \neq \varnothing]\!] = \mathbf{1}$. Then for every formula φ of ZFC we have

$$[\![(\forall x \in z)\, \varphi(x)]\!] = \bigwedge\{[\![\varphi(x)]\!] \mid x \in z{\downarrow}\},$$
$$[\![(\exists x \in z)\, \varphi(x)]\!] = \bigvee\{[\![\varphi(x)]\!] \mid x \in z{\downarrow}\}.$$

Moreover, there exists $x_0 \in z{\downarrow}$ such that $[\![\varphi(x_0)]\!] = [\![(\exists x \in z)\, \varphi(x)]\!]$.

(3) Let Φ be a correspondence from X to Y in $\mathbf{V}^{(B)}$. Thus, Φ, X, and Y are elements of $\mathbf{V}^{(B)}$ and, moreover, $[\![\Phi \subset X \times Y]\!] = 1$. There is a unique correspondence $\Phi{\downarrow}$ from $X{\downarrow}$ to $Y{\downarrow}$ such that

$$\Phi{\downarrow}(A{\downarrow}) = \Phi(A){\downarrow}$$

for every nonempty subset A of the set X inside $\mathbf{V}^{(B)}$. The correspondence $\Phi{\downarrow}$ from $X{\downarrow}$ to $Y{\downarrow}$ involved in the above proposition is called the *descent* of the correspondence Φ from X to Y in $\mathbf{V}^{(B)}$.

(4) The descent of the superposition of correspondences inside $\mathbf{V}^{(B)}$ is the superposition of their descents:

$$(\Psi \circ \Phi){\downarrow} = \Psi{\downarrow} \circ \Phi{\downarrow}.$$

(5) If Φ is a correspondence inside $\mathbf{V}^{(B)}$ then

$$(\Phi^{-1}){\downarrow} = (\Phi{\downarrow})^{-1}.$$

(6) Let I_X be the identity mapping inside $\mathbf{V}^{(B)}$ of the set $X \in \mathbf{V}^{(B)}$. Then

$$(I_X){\downarrow} = I_{X{\downarrow}}.$$

(7) Suppose that $X, Y, f \in \mathbf{V}^{(B)}$ are such that $[\![f : X \to Y]\!] = \mathbf{1}$, i.e., f is a mapping from X to Y inside $\mathbf{V}^{(B)}$. Then $f\!\downarrow$ is a unique mapping from $X\!\downarrow$ to $Y\!\downarrow$ for which

$$[\![f\!\downarrow\!(x) = f(x)]\!] = \mathbf{1} \quad (x \in X\!\downarrow).$$

By virtue of assertions (1)–(7), we can consider the descent operation as a functor from the category of B-valued sets and mappings (correspondences) to the category of usual (i.e., in the sense of $\mathbf{V}$) sets and mappings (correspondences).

(8) Given $x_1, \dots, x_n \in \mathbf{V}^{(B)}$, denote by $(x_1, \dots, x_n)^B$ the corresponding ordered n-tuple inside $\mathbf{V}^{(B)}$. Assume that P is an n-ary relation on X inside $\mathbf{V}^{(B)}$; i.e., $X, P \in \mathbf{V}^{(B)}$ and $[\![P \subset X^{n^{\wedge}}]\!] = 1$ $(n \in \omega)$. Then there exists an n-ary relation P' on $X\!\downarrow$ such that

$$(x_1, \dots, x_n) \in P' \leftrightarrow [\![(x_1, \dots, x_n)^B \in P]\!] = \mathbf{1}.$$

Slightly abusing notation, we denote the relation P' by the same symbol $P\!\downarrow$ and call it the *descent* of P.

1.2.10. Let $x \in \mathbf{V}$ and $x \subset \mathbf{V}^{(B)}$; i.e., let x be some set composed of B-valued sets or, in other words, $x \in \mathscr{P}(\mathbf{V}^{(B)})$. Put $\varnothing\!\uparrow := \varnothing$ and

$$\operatorname{dom}(x\!\uparrow) = x, \quad \operatorname{im}(x\!\uparrow) = \{\mathbf{1}\}$$

if $x \neq \varnothing$. The element $x\!\uparrow$ (of the separated universe $\mathbf{V}^{(B)}$, i.e., the distinguished representative of the class $\{y \in \mathbf{V}^{(B)} \mid [\![y = x\!\uparrow]\!] = \mathbf{1}\}$) is called the *ascent* of x.

(1) The following equalities hold for every $x \in \mathscr{P}(\mathbf{V}^{(B)})$ and every formula φ:

$$[\![(\forall z \in x\!\uparrow)\, \varphi(z)]\!] = \bigwedge_{y \in x} [\![\varphi(y)]\!],$$

$$[\![(\exists z \in x\!\uparrow)\, \varphi(z)]\!] = \bigvee_{y \in x} [\![\varphi(y)]\!].$$

Introducing the ascent of a correspondence $\Phi \subset X \times Y$, we have to bear in mind a possible difference between the domain of departure X and the domain of definition $\operatorname{dom}(\Phi) := \{x \in X \mid \Phi(x) \neq \varnothing\}$. This difference is inessential for our further goals; therefore, we assume that, speaking of ascents, we always consider everywhere-defined correspondences; i.e., $\operatorname{dom}(\Phi) = X$.

(2) Let $X, Y, \Phi \in \mathbf{V}^{(B)}$, and let Φ be a correspondence from X to Y. There exists a unique correspondence $\Phi\!\uparrow$ from $X\!\uparrow$ to $Y\!\uparrow$ inside $\mathbf{V}^{(B)}$ such that

$$\Phi\!\uparrow(A\!\uparrow) = \Phi(A)\!\uparrow$$

is valid for every subset A of the set $\mathrm{dom}(\Phi)$ if and only if Φ is *extensional*; i.e., satisfies the condition

$$y_1 \in \Phi(x_1) \to [\![x_1 = x_2]\!] \le \bigvee_{y_2 \in \Phi(x_2)} [\![y_1 = y_2]\!]$$

for $x_1, x_2 \in \mathrm{dom}(\Phi)$. In this event, $\Phi\!\uparrow = \Phi'\!\uparrow$, where $\Phi' := \{(x,y)^B \mid (x,y) \in \Phi\}$. The element $\Phi\!\uparrow$ is called the *ascent* of the initial correspondence Φ.

(3) The superposition of extensional correspondences is extensional. In addition, the ascent of the superposition is equal to the superposition of the ascents (inside $\mathbf{V}^{(B)}$): On the condition that $\mathrm{dom}(\Psi) \supset \mathrm{im}(\Phi)$ we have

$$\mathbf{V}^{(B)} \models (\Psi \circ \Phi)\!\uparrow = \Psi\!\uparrow \circ \Phi\!\uparrow.$$

Note that if Φ and Φ^{-1} are extensional then $(\Phi\!\uparrow)^{-1} = (\Phi^{-1})\!\uparrow$. However, in general, the extensionality of Φ in no way guarantees the extensionality of Φ^{-1}.

(4) It is worth mentioning that if an extensional correspondence f is a function from X to Y then its ascent $f\!\uparrow$ is a function from $X\!\uparrow$ to $Y\!\uparrow$. Moreover, the extensionality property can be stated as follows:

$$[\![x_1 = x_2]\!] \le [\![f(x_1) = f(x_2)]\!] \quad (x_1, x_2 \in X).$$

Given a set $X \subset \mathbf{V}^{(B)}$, we denote by the symbol $\mathrm{mix}\, X$ the set of all mixings of the form $\mathrm{mix}(b_\xi x_\xi)$, where $(x_\xi) \subset X$ and (b_ξ) is an arbitrary partition of unity. The following assertions are referred to as the *rules of canceling arrows* or the "*descent-ascent*" and "*ascent-descent*" *rules.*

(5) Let X and X' be subsets of $\mathbf{V}^{(B)}$ and $f : X \to X'$ be an extensional mapping. Suppose that $Y, Y', g \in \mathbf{V}^{(B)}$ are such that $[\![Y \ne \varnothing]\!] = [\![g : Y \to Y']\!] = \mathbf{1}$. Then the following relations are valid:

$$X\!\uparrow\downarrow = \mathrm{mix}\, X;$$

$$f\!\uparrow\downarrow = f;$$

$$Y\!\downarrow\uparrow = Y;$$

$$g\!\downarrow\uparrow = g.$$

1.2.11. Suppose that $X \in \mathbf{V}$, $X \neq \varnothing$; i.e., X is a nonempty set. Let the letter ι denote the standard name embedding $x \mapsto x^{\wedge}$ $(x \in X)$. Then $\iota(X)\uparrow = X^{\wedge}$ and $X = \iota^{-1}(X^{\wedge}\downarrow)$. Making use of the above relations, we can extend the descent and ascent operations to the case in which Φ is a correspondence from X to $Y\downarrow$ and $[\![\Psi$ is a correspondence from $X^{\wedge}$ to $Y]\!] = \mathbf{1}$, where $Y \in \mathbf{V}^{(B)}$. Namely, we put $\Phi\!\uparrow := (\Phi \circ \iota)\uparrow$ and $\Psi\!\downarrow := \Psi\!\downarrow \circ\, \iota$. In this case, $\Phi\!\uparrow$ is called the *modified ascent* of the correspondence Φ and $\Psi\!\downarrow$ is called the *modified descent* of the correspondence Ψ. (If the context excludes ambiguity then we simply speak of ascents and descents using simple arrows.) It is easy to see that $\Psi\!\uparrow$ is a unique correspondence inside $\mathbf{V}^{(B)}$ satisfying the relation

$$[\![\Phi\!\uparrow(x^{\wedge}) = \Phi(x)\uparrow]\!] = \mathbf{1} \quad (x \in X).$$

Similarly, $\Psi\!\downarrow$ is a unique correspondence from X to $Y\downarrow$ satisfying the equality

$$\Psi\!\downarrow(x) = \Psi(x^{\wedge})\downarrow \quad (x \in X).$$

If $\Phi := f$ and $\Psi := g$ are functions then the indicated relations take the form

$$[\![f\!\uparrow(x^{\wedge}) = f(x)]\!] = \mathbf{1}, \quad g\!\downarrow(x) = g(x^{\wedge}) \quad (x \in X).$$

1.2.12. (1) A *Boolean set* or a *set with B-structure* or just a *B-set* is a pair (X, d), where $X \in \mathbf{V}$, $X \neq \varnothing$, and d is a mapping from $X \times X$ to the Boolean algebra B which satisfies the following conditions for arbitrary $x, y, z \in X$:

(a) $d(x, y) = 0 \leftrightarrow x = y$;

(b) $d(x, y) = d(y, x)$;

(c) $d(x, y) \leq d(x, z) \vee d(z, y)$.

An example of a B-set is given by any $\varnothing \neq X \subset \mathbf{V}^{(B)}$ if we put

$$d(x, y) := [\![x \neq y]\!] = [\![x = y]\!]^{*} \quad (x,\ y \in X).$$

Another example is a nonempty X with the "discrete B-metric" d; i.e., $d(x, y) = \mathbf{1}$ if $x \neq y$ and $d(x, y) = \mathbf{0}$ if $x = y$.

(2) Let (X, d) be some B-set. There exist an element $\mathscr{X} \in \mathbf{V}^{(B)}$ and an injection $\iota : X \to X' := \mathscr{X}\downarrow$ such that $d(x, y) = [\![\iota x \neq \iota y]\!]$ $(x, y \in X)$ and every element $x' \in X'$ admits the representation $x' = \operatorname{mix}_{\xi \in \Xi}(b_{\xi}\iota x_{\xi})$, where $(x_{\xi})_{\xi \in \Xi} \subset X$

and $(b_\xi)_{\xi\in\Xi}$ is a partition of unity in B. The element $\mathscr{X} \in \mathbf{V}^{(B)}$ is referred to as the *Boolean-valued realization* of the B-set X. If X is a discrete B-set then $\mathscr{X} = X^\wedge$ and $\iota x = x^\wedge$ $(x \in X)$. If $X \subset \mathbf{V}^{(B)}$ then $\iota\uparrow$ is an injection from $X\uparrow$ to $\mathscr{X}$ (inside $\mathbf{V}^{(B)}$).

A mapping f from a B-set (X, d) to a B-set (X', d') is said to be *nonexpanding* if $d(x, y) \geq d'(f(x), f(y))$ for all $x, y \in X$.

(3) Let X and Y be some B-sets, $\mathscr{X}$ and $\mathscr{Y}$ be their Boolean-value realizations, and ι and $\varkappa$ be the corresponding injections $X \to \mathscr{X}\downarrow$ and $Y \to \mathscr{Y}\downarrow$. If $f : X \to Y$ is a nonexpanding mapping then there is a unique element $g \in \mathbf{V}^{(B)}$ such that $[\![g : \mathscr{X} \to \mathscr{Y}]\!] = 1$ and $f = \varkappa^{-1} \circ g\downarrow \circ \iota$. We also accept the notations $\mathscr{X} := \mathscr{F}^\sim(X)$ and $g := \mathscr{F}^\sim(f)$.

(4) We present an example of a B-set important for the sequel. Let E be a vector lattice and $B := \mathfrak{B}(E)$. Set

$$d(x, y) := \{|x - y|\}^{\perp\perp} \quad (x, y \in E).$$

One can easily check that d meets the conditions (b, c) of 1.2.12. At the same time, (a) of 1.2.12(1) is valid only for an Archimedean E (see 1.1.3).

Thus, (E, d) is a B-set if and only if the vector lattice E is Archimedean.

1.2.13. Grounding on the results of 1.2.9, we can define the descent of an algebraic system. For the sake of simplicity, we confine ourselves to the case of finite signature. Let $\mathfrak{A}$ be an algebraic system of finite signature inside $\mathbf{V}^{(B)}$. In more detail, it means that there are elements A, $f_1, \dots, f_n$, $P_1, \dots, P_m \in \mathbf{V}^{(B)}$ and natural numbers $a(f_1), \dots, a(f_n)$, $a(P_1), \dots, a(P_m)$ that satisfy the following conditions (all inside $\mathbf{V}^{(B)}$):

$$A \neq \varnothing, \quad P_k \subset A^{a(P_k)^\wedge} \quad (k := 1, \dots, m),$$
$$f_l : A^{a(f_l)^\wedge} \to A \quad (l := 1, \dots, n),$$
$$\mathfrak{A} := (A, f_1, \dots, f_n, P_1, \dots, P_m).$$

Executing the descent of the set A, functions $f_1, \dots, f_n$, and relations $P_1, \dots, P_m$ by the rules of 1.1.9, we obtain some algebraic system $\mathfrak{A}\downarrow = (A\downarrow, f_1\downarrow, \dots, P_m\downarrow)$ of a similar type called the *descent* of $\mathfrak{A}$. Thus, the descent of an algebraic system $\mathfrak{A}$ is the descent of the underlying set A furnished with descended operations and predicates.

1.3. Real Numbers in Boolean-Valued Models

Boolean-valued analysis stems from the assertion due to D. Scott that the image of the field of real numbers in a Boolean-valued model represents a universally complete K-space (of measurable functions). Depending on which Boolean algebra B (the algebra of measurable sets, open regular sets, or projections in a Hilbert space) forms the base for constructing a Boolean-valued model $\mathbf{V}^{(B)}$, we obtain different K-spaces (the spaces of measurable functions, semicontinuous functions, or selfadjoint operators). Thereby there appears a remarkable opportunity for transferring all the treasure-trove of knowledge about real numbers to a profusion of classical objects of analysis. This will constitute the topic of the section:

1.3.1. By the field of real numbers we mean an algebraic system that satisfies the axioms of an Archimedean ordered field (with distinct zero and unity) and the axiom of completeness. Recall the following two well-known assertions:

(1) *There exists a field* $\mathbb{R}$ *of real numbers which is unique to within isomorphism.*

(2) *If* $\mathbf{P}$ *is an Archimedean ordered field then there is an isomorphic embedding* h *of the field* $\mathbf{P}$ *into* $\mathbb{R}$ *such that the image* $h(\mathbf{P})$ *is a subfield of* $\mathbb{R}$ *containing the subfield of rational numbers. In particular,* $h(\mathbf{P})$ *is dense in* $\mathbb{R}$.

Successively applying the transfer and maximum principles to (1), we find an element $\mathscr{R} \in \mathbf{V}^{(B)}$ for which $[\![\mathscr{R}$ is a field of real numbers$]\!] = \mathbf{1}$. Moreover, if an arbitrary $\mathscr{R}' \in \mathbf{V}^{(B)}$ satisfies the condition $[\![\mathscr{R}'$ is a field of real numbers$]\!] = \mathbf{1}$ then it also satisfies $[\![$the ordered fields $\mathscr{R}$ and $\mathscr{R}'$ are isomorphic$]\!] = \mathbf{1}$. In other words, there exists a field $\mathscr{R}$ of real numbers in the model $\mathbf{V}^{(B)}$ and such a field is unique to within an isomorphism.

Note also that $\varphi(x)$ formally presenting the expressions of the axioms of an Archimedean ordered field is bounded; therefore, $[\![\varphi(\mathbb{R}^{\wedge})]\!] = \mathbf{1}$, i.e., $[\![\mathbb{R}^{\wedge}$ is an Archimedean ordered field$]\!] = \mathbf{1}$. "Pulling" assertion (2) through the transfer principle, we conclude that $[\![\mathbb{R}^{\wedge}$ is isomorphic to a dense subfield of the field $\mathscr{R}]\!] = \mathbf{1}$. In this regard, we further assume that $\mathscr{R}$ is a field of real numbers in the model $\mathbf{V}^{(B)}$ and $\mathbb{R}^{\wedge}$ is its dense subfield. It is easy to note that the elements $0 := 0^{\wedge}$ and $1 := 1^{\wedge}$ are the zero and unity of the field $\mathscr{R}$.

Observe that the equality $\mathscr{R} = \mathbb{R}^{\wedge}$ is not valid in general. Indeed, the axiom of completeness for $\mathbb{R}$ is not a bounded formula and may thus fail for $\mathbb{R}^{\wedge}$ inside $\mathbf{V}^{(B)}$.

Now, consider the descent $\mathscr{R}{\downarrow}$ of the algebraic system $\mathscr{R}$. In other words, we consider the descent of the underlying set of the system $\mathscr{R}$ together with descended operations and order. For simplicity, we denote the operations and order in $\mathscr{R}$ and $\mathscr{R}{\downarrow}$ by the same symbols $+$, $\cdot$, and $\leq$. In more detail, we introduce summation, multiplication, and order in $\mathscr{R}{\downarrow}$ by the formulas

$$\begin{gathered} z = x + y \leftrightarrow [\![z = x + y]\!] = \mathbf{1}, \\ z = x \cdot y \leftrightarrow [\![z = x \cdot y]\!] = \mathbf{1}, \\ x \leq y \leftrightarrow [\![x \leq y]\!] = \mathbf{1} \\ (x,\ y,\ z \in \mathscr{R}{\downarrow}). \end{gathered}$$

Also, we can introduce multiplication by real numbers in $\mathscr{R}{\downarrow}$ by the rule

$$y = \lambda x \leftrightarrow [\![\lambda^{\wedge} x = y]\!] = \mathbf{1} \quad (\lambda \in \mathbb{R},\ x, y \in \mathscr{R}{\downarrow}).$$

1.3.2. *The Gordon theorem.* *Let $\mathscr{R}$ be the ordered field of real numbers in the model $\mathbf{V}^{(B)}$. Then $\mathscr{R}{\downarrow}$ (with the descended operations and order) is a universally complete K-space with order-unit 1. Moreover, there exists an isomorphism χ of the Boolean algebra B onto the base $\mathfrak{P}(\mathscr{R}{\downarrow})$ such that the following equivalences hold:*

$$\begin{gathered} \chi(b)x = \chi(b)y \leftrightarrow b \leq [\![x = y]\!], \\ \chi(b)x \leq \chi(b)y \leftrightarrow b \leq [\![x \leq y]\!] \end{gathered}$$

for all $x, y \in \mathscr{R}{\downarrow}$ and $b \in B$.

$\lhd$ We omit elementary verification of the fact that $\mathscr{R}{\downarrow}$ is a vector space over $\mathbb{R}$ and an ordered set. Show that the operations and order in $\mathscr{R}{\downarrow}$ agree and the necessary exact bounds exist. Take elements $x, y \in \mathscr{R}{\downarrow}$ such that $x \leq y$. It means that

$$\mathbf{V}^{(B)} \models \text{“}x \text{ and } y \text{ are real numbers and } x \leq y.\text{”}$$

Let $u := x + z$, $v := y + z$, $x' := \lambda x$, and $y' := \lambda y$, where $z \in \mathscr{R}{\downarrow}$ and $\lambda \in \mathbb{R}$, $\lambda \geq 0$. By the definition of the operations and order in $\mathscr{R}{\downarrow}$, we have $\mathbf{V}^{(B)} \models$ “x', y', u, and v are real numbers; moreover, $u = x + z$, $v = y + z$, $x' = \lambda^{\wedge} x$, and $y' = \lambda^{\wedge} y$.”

the inequality $\lambda \geq 0$ implies $\mathbf{V}^{(B)} \models \lambda^\wedge \geq 0^\wedge = 0$. Using the requested properties of numbers inside $\mathbf{V}^{(B)}$, we obtain $\mathbf{V}^{(B)} \models$ "$u \leq v$ and $x' \leq y'$." Thereby $u \leq v$ and $x' \leq y'$.

Suppose that a set $A \subset \mathscr{R}{\downarrow}$ is bounded above by an element $y \in \mathscr{R}{\downarrow}$. By definition, it means that $[\![x \leq y]\!] = \mathbf{1}$ for every element $x \in A$. Then $\mathbf{V}^{(B)} \models$ "$A{\uparrow}$ is a set of numbers bounded above by the number y" or, in view of 1.2.10(1),

$$[\![(\forall x \in A{\uparrow})(x \leq y)]\!] = \bigwedge_{x \in A} [\![x \leq y]\!] = \mathbf{1}.$$

The completeness of the field $\mathscr{R}$ yields

$$[\![(\exists a \in \mathscr{R})(a = \sup(A{\uparrow}))]\!] = \mathbf{1}.$$

Employing the maximum principle, we find $a \in \mathbf{V}^{(B)}$ such that $[\![a \in \mathscr{R}]\!] = [\![a = \sup(A{\uparrow})]\!] = \mathbf{1}$. Thereby $a \in \mathscr{R}{\downarrow}$ and if $z \in \mathscr{R}{\downarrow}$ is an upper bound of A then, as was already shown, $[\![z$ is an upper bound of $A{\uparrow}]\!] = \mathbf{1}$; therefore, $[\![a \leq z]\!] = \mathbf{1}$ or $a \leq z$. Consequently, a is the supremum of the set A in $\mathscr{R}{\downarrow}$. Incidentally, we have established that $a = \sup(A)$ if and only if $[\![a = \sup(A{\uparrow})]\!] = \mathbf{1}$. In particular, given arbitrary $x,\ x_1,\ x_2 \in \mathscr{R}{\downarrow}$, we have $x = x_1 \vee x_2$ if and only if

$$[\![x = \sup\{x_1, x_2\} = x_1 \vee x_2]\!] = \mathbf{1},$$

since $[\![\{x_1, x_2\}{\uparrow} = \{x_1, x_2\}]\!] = \mathbf{1}$. Of course, an analogous assertion is valid for greatest lower bounds. Now, take an arbitrary set $A \subset \mathscr{R}{\downarrow}$ of positive pairwise disjoint elements. We can see from the above remarks and 1.2.10 that

$$[\![(\forall x_1 \in A{\uparrow})(\forall x_2 \in A{\uparrow}) x_1 \wedge x_2 = 0]\!] = \bigwedge_{x_1, x_2 \in A} [\![x_1 \wedge x_2 = 0]\!] = \mathbf{1}.$$

Hence, the numeric set $A{\uparrow}$ (inside $\mathbf{V}^{(B)}$) consists of pairwise disjoint positive elements. For such a set we have only the following two possibilities: either $[\![A{\uparrow} = \{0\}]\!] = \mathbf{1}$ and then $A \subset A{\uparrow}{\downarrow} = \{0\}$, or $[\![A{\uparrow} = \{0, a\}]\!] = \mathbf{1}$ for some $0 \leq a \in \mathscr{R}{\downarrow}$ (by the maximum principle!) and then $[\![\sup(A{\uparrow}) = a]\!] = \mathbf{1}$. As was mentioned above, the latter relation is equivalent to the equality $a = \sup A$. Now, we can conclude that $\mathscr{R}{\downarrow}$ is a universally complete K-space. Recalling that $1 := 1^\wedge$

is the unity of the field $\mathscr{R}$ inside $\mathbf{V}^{(B)}$ and employing the formulas of 1.2.9(2) and 1.2.4, we find

$$\mathbf{1} = [\![(\forall x \in \mathscr{R})(x \wedge 1 = 0 \to x = 0)]\!] = \bigwedge_{x \in \mathscr{R}\downarrow} [\![x \wedge 1 = 0]\!] \Rightarrow [\![x = 0]\!].$$

Hence, we see that $[\![x \wedge 1 = 0]\!] \leq [\![x = 0]\!]$ for each $x \in \mathscr{R}\downarrow$. If $x \wedge 1 = 0$ then $[\![x \wedge 1 = 0]\!] = \mathbf{1}$ and so $[\![x = 0]\!] = \mathbf{1}$, i.e., $x = 0$. Thereby 1 is the unity of the K-space $\mathscr{R}\downarrow$.

Now, introduce some mapping $\chi : B \to \mathfrak{P}(\mathscr{R}\downarrow)$. Take an arbitrary element $b \in B$ and put $\chi(b)x := \operatorname{mix}\{bx, b^*0\}$ for $x \in \mathscr{R}\downarrow$. In other words, the element $\chi(b)x \in \mathscr{R}\downarrow$ is uniquely determined by the following relations (see 1.2.7(3)):

$$b \leq [\![\chi(b)x = x]\!], \quad b^* \leq [\![\chi(b)x = 0]\!].$$

It implies that $\pi := \chi(b) : \mathscr{R}\downarrow \to \mathscr{R}\downarrow$ is an extensional mapping. Indeed, the following inequalities hold for $x, y \in \mathscr{R}\downarrow$ (see 1.2.5(3)):

$$[\![x = y]\!] \wedge b \leq [\![x = y]\!] \wedge [\![x = \pi x]\!] \wedge [\![y = \pi y]\!] \leq [\![\pi x = \pi y]\!],$$
$$b^* \leq [\![\pi x = 0]\!] \wedge [\![\pi y = 0]\!] \leq [\![\pi x = \pi y]\!].$$

If $\rho := \pi\uparrow$ then $[\![\rho : \mathscr{R} \to \mathscr{R}]\!] = \mathbf{1}$ by 1.2.10(4) and $\rho = \operatorname{mix}\{bIE_{\mathscr{R}}, b^*0\}$. Since $\mathbf{0}$ and $I_{\mathscr{R}}$ are idempotent positive linear mappings from $\mathscr{R}$ to $\mathscr{R}$, such is π. Moreover, $[\![(\forall x \in \mathscr{R}_+)\rho x \leq x]\!] = \mathbf{1}$; therefore, $\pi x \leq x$ for all $x \in \mathscr{R}\downarrow_+$. Thus, $\pi = \chi(b)$ is a band projection. Since ρ is positive, we have $[\![x \leq y \to \rho x \leq \rho y]\!] = \mathbf{1}$ for $x, y \in \mathscr{R}\downarrow$ and hence

$$[\![x \leq y]\!] \leq [\![\rho x \leq \rho y]\!] = [\![\pi x \leq \pi y]\!].$$

Assume $\pi x \leq \pi y$. Then

$$b = [\![\pi x \leq \pi y]\!] \wedge [\![\pi x = x]\!] \wedge [\![\pi y = y]\!] \leq [\![x \leq y]\!].$$

Conversely, if we assume that $b \leq [\![x \leq y]\!]$ then $b \leq [\![\pi x \leq \pi y]\!]$. Moreover,

$$b^* \leq [\![\pi x = 0]\!] \wedge [\![\pi y = 0]\!] \wedge [\![0 \leq 0]\!] \leq [\![\pi x \leq \pi y]\!];$$

consequently, $[\![\pi x \leq \pi y]\!] = \mathbf{1}$ or $\pi x \leq \pi y$.

Thereby we have established the second of the required equivalences. The first ensues from that by virtue of the formula $u = v \leftrightarrow u \leq v \wedge v = u$.

It remains to demonstrate that the mapping χ is an isomorphism between the Boolean algebras B and $\mathfrak{P}(\mathscr{R}{\downarrow})$. Take an arbitrary band projection $\pi \in \mathfrak{P}(\mathscr{R}{\downarrow})$ and put $b := [\![\pi{\uparrow} = I_{\mathscr{R}}]\!]$. The fact that a band projection is extensional (and hence the ascent $\pi{\uparrow}$ of π is well-defined) follows from the above-established equivalences, because

$$c = [\![x = y]\!] \rightarrow \chi(c)x = \chi(c)y \rightarrow \pi\chi(c)x = \pi\chi(c)y$$
$$\rightarrow \chi(c)\pi x = \chi(c)\pi y \rightarrow c \leq [\![\pi x = \pi y]\!].$$

Since π is idempotent, $\pi{\uparrow}$ as well is an idempotent mapping in $\mathscr{R}$; i.e., either $\pi{\uparrow} = I_{\mathscr{R}}$ or $\pi = \mathbf{0}$. Hence, we derive $b^* = [\![\pi \neq I_{\mathscr{R}}]\!] = [\![\pi = 0]\!]$ and thereby $\pi{\uparrow} = \operatorname{mix}\{bI_{\mathscr{R}}, b^*(\mathbf{0})\}$. The mixing is unique; therefore, $\pi{\uparrow} = \chi(b){\uparrow}$, i.e., $\pi = \chi(b)$. Thus, χ is a bijection between B and $\mathfrak{P}(\mathscr{R}{\downarrow})$.

Let b_1, $b_2 \in B$ and $\rho_k := \chi(b_k)$ $(k := 1, 2)$. Recalling that $\rho_k = \operatorname{mix}\{b_k I_{\mathscr{R}}, b_k^* \mathbf{0}\}$, we derive

$$[\![\chi(b_1 \wedge b_2){\uparrow} = I_{\mathscr{R}}]\!] = b_1 \wedge b_2 = [\![\rho_1 = I_{\mathscr{R}} \wedge \rho_2 = I_{\mathscr{R}}]\!] = [\![\rho_1 \circ \rho_2 = I_{\mathscr{R}}]\!],$$
$$[\![\chi(b_1 \wedge b_2){\uparrow} = 0]\!] = (b_1 \wedge b_2)^* = [\![\rho_1 = \mathbf{0} \vee \rho_2 = \mathbf{0}]\!] = [\![\rho_1 \circ \rho_2 = \mathbf{0}]\!].$$

Thus,

$$[\![\chi(b_1 \wedge b_2){\uparrow} = \rho_1 \circ \rho_2 = (\chi(b_1) \wedge \chi(b_2)){\uparrow}]\!] = \mathbf{1}$$

and hence

$$\chi(b_1 \wedge b_2) = \chi(b_1) \wedge \chi(b_2).$$

In particular, $0 = \chi(b) \wedge \chi(b^*)$, for $\chi(0) = 0$. Given elements $\rho := \chi(b){\uparrow}$ and $\rho' := \chi(b^*)$, we have $[\![\rho, \rho' \in \{\mathbf{0}, I_{\mathscr{R}}\};\ \rho = \mathbf{0}$ or $\rho' = \mathbf{0}$; and ρ and ρ' do not vanish simultaneously$]\!] = \mathbf{1}$. Hence, we see that $[\![\rho + \rho' = I_{\mathscr{R}}]\!] = \mathbf{1}$ and thereby $\chi(b) + \chi(b^*) = I_{\mathscr{R}\downarrow}$. Taking stock of the above, we conclude that χ preserves greatest lower bounds and complements; i.e., χ is an isomorphism. $\triangleright$

1.3.3. *The universally complete K-space $\mathscr{R}{\downarrow}$ is also a faithful f-algebra with ring unity* 1; *moreover, for every $b \in B$ the projection $\chi(b)$ is the operator of multiplication by the order-unit $\chi(b)1$.*

$\lhd$ The multiplicative structure on $\mathscr{R}{\downarrow}$ was defined in 1.3.1. As in 1.3.2, we establish that $\mathscr{R}{\downarrow}$ is a faithful f-algebra. Take $x \in \mathscr{R}{\downarrow}$ and $b \in B$. By the definition of the projection $\chi(b)$, we have $b \le [\![\chi(b)x = x]\!]$ and $b^* \le [\![\chi(b^*)x = 0]\!]$. Applying these relations to $x := 1$ and appealing to the definition of multiplication in $\mathscr{R}{\downarrow}$, we obtain $b \le [\![x = x \cdot 1 = x \cdot \chi(b)1]\!]$ and $b^* \le [\![0 = x \cdot 0 = x \cdot \chi(b)1]\!]$. Thereby

$$[\![\chi(b)x = x \cdot \chi(b)1]\!] \ge [\![\chi(b)x = x]\!] \wedge [\![x = x \cdot \chi(b)1]\!] \ge b.$$

In a similar way, $b^* \le [\![\chi(b)x = \chi(b)1 \cdot x]\!]$. Hence, $[\![\chi(b)x = x \cdot \chi(b)1]\!] = \mathbf{1}$. $\rhd$

We see from the above that the mapping $b \mapsto \chi(b)1$ $(b \in B)$ is a Boolean isomorphism between B and the algebra $\mathfrak{C}(\mathscr{R}{\downarrow})$ of order-units. This isomorphism is denoted by the same letter χ. Thus, depending on the context, $x \mapsto \chi(b)x$ is either a band projection or the operator of multiplication by the order-unit $\chi(b)$.

1.3.4. Henceforth, $\mathscr{R}$ denotes the field of real numbers in the model $\mathbf{V}^{(B)}$. We will clarify the meaning of the exact bounds and order limits in the K-space $\mathscr{R}{\downarrow}$.

(1) *Let $(b_\xi)_{\xi\in\Xi}$ be a partition of unity in B and let $(x_\xi)_{\xi\in\Xi}$ be a family in $\mathscr{R}{\downarrow}$. Then*

$$\operatorname*{mix}_{\xi\in\Xi}(b_\xi x_\xi) = o\text{-}\sum_{\xi\in\Xi} \chi(b_\xi)x_\xi.$$

$\lhd$ If $x := \operatorname{mix}_{\xi\in\Xi}(b_\xi x_\xi)$ then $b_\xi \le [\![x = x_\xi]\!]$ $(\xi \in \Xi)$ (see 1.2.7(3)). According to 1.3.2, $\chi(b_\xi)x_\xi = \chi(b_\xi)x$ for all $\xi \in \Xi$. Summing the last relations over ξ, we arrive at what was required. $\rhd$

(2) *The following equivalences hold for a nonempty set $A \subset \mathscr{R}{\downarrow}$ and arbitrary $a \in \mathscr{R}$ and $b \in B$:*

$$b \le [\![a = \sup(A{\uparrow})]\!] \leftrightarrow \chi(b)a = \sup \chi(b)(A),$$
$$b \le [\![a = \inf(A{\uparrow})]\!] \leftrightarrow \chi(b)a = \inf \chi(b)(A).$$

$\lhd$ We will prove only the first equivalence. The equality

$$\chi(b)a = \sup\{\chi(b)x \mid x \in A\}$$

holds if and only if $b \le [\![x \le a]\!]$ for all $x \in A$ and for each $y \in \mathscr{R}{\downarrow}$ the relation $(\forall x \in A)(b \le [\![x \le y]\!])$ implies $b \le [\![a \le y]\!]$ (see 1.3.2).

Using the rules for calculating the truth-values for quantifiers (see 1.2.4), we can represent the conditions in question in the following equivalent form:

$$b \leq [\![(\forall x \in A{\uparrow})\, x \leq a]\!],$$
$$b \leq [\![(\forall y \in R)\,(A{\uparrow} \leq y \to a \leq y)]\!].$$

This system of inequalities is equivalent to the formula $b \leq [\![a = \sup(A{\uparrow})]\!]$. $\triangleright$

(3) *Let* A *be an upward-filtered set and* $s : \mathrm{A} \to \mathscr{R}{\downarrow}$ *be a net in* $\mathscr{R}{\downarrow}$. *Then* $\mathrm{A}^\wedge$ *is filtered upward and* $\sigma := s{\uparrow} : \mathrm{A}^\wedge \to R$ *is a net in* $\mathscr{R}$ *(inside* $\mathbf{V}^{(B)}$*); moreover,*

$$b \leq [\![x = \lim \sigma]\!] \leftrightarrow \chi(b)x = o\text{-}\lim \chi(b) \circ s$$

for arbitrary $x \in \mathscr{R}{\downarrow}$ *and* $b \in B$.

$\triangleleft$ The assertion "A is an upward-filtered set" can be written down as a bounded formula. By virtue of the restricted transfer principle 1.2.8(4), we have $\mathbf{V}^{(B)} \models$ "$\mathrm{A}^\wedge$ is an upward-filtered set." The equality $\chi(b)x = o\text{-}\lim \chi(b) \circ s$ means that there exists a net $d : \mathrm{A} \to \mathscr{R}{\downarrow}$ for which the following system of conditions is compatible:

$$\alpha \leq \beta \to d(\alpha) \leq d(\beta) \quad (\alpha, \beta \in \mathrm{A}), \quad \inf_{\alpha \in \mathrm{A}} d(\alpha) = 0,$$
$$|\chi(b)x - \chi(b)s(\alpha)| \leq d(\alpha) \quad (\alpha \in \mathrm{A}).$$

Taking account of the easy formula $[\![s(\mathrm{A}){\uparrow} = \sigma(\mathrm{A}^\wedge)]\!] = \mathbf{1}$ and putting $\delta := d{\uparrow}$, we see that the indicated system of conditions is equivalent to the following system of inequalities:

$$b \leq [\![\inf \sigma(\mathrm{A}^\wedge) = 0]\!],$$
$$b \leq [\![(\forall \alpha, \beta \in \mathrm{A}^\wedge)\,(\alpha \leq \beta \to \sigma(\alpha) \leq \sigma(\beta))]\!],$$
$$b \leq [\![(\forall \alpha \in \mathrm{A}^\wedge)\,(|x - \sigma(\alpha)| < \delta(\alpha))]\!],$$

whose short form is just the relation $b \leq [\![x = \lim \sigma]\!]$. $\triangleright$

(4) *Suppose that* A *and* $\sigma \in \mathbf{V}^{(B)}$ *are such that* $[\![\mathrm{A}$ *is filtered upward and* $\sigma : \mathrm{A} \to \mathscr{R}]\!] = \mathbf{1}$. *Then* $\mathrm{A}{\downarrow}$ *is an upward-filtered set and hence the mapping* $s := \sigma{\downarrow} : \mathrm{A}{\downarrow} \to \mathscr{R}{\downarrow}$ *is a net in* $\mathscr{R}{\downarrow}$. *Moreover,*

$$b \leq [\![x = \lim \sigma]\!] \leftrightarrow \chi(b)x = o\text{-}\lim \chi(b) \circ s$$

for arbitrary $x \in \mathscr{R}{\downarrow}$ *and* $b \in B$.

$\lhd$ The proof is similar to that of (3). $\rhd$

(5) *Let f be a mapping from a nonempty set to $\mathscr{R}{\downarrow}$ and $g := f{\uparrow}$. Then*

$$b \leq [\![x = \sum_{\xi \in \Xi^{\wedge}} g(\xi)]\!] \leftrightarrow \chi(b)x = \sum_{\xi \in \Xi} \chi(b)f(\xi)$$

for arbitrary $x \in \mathscr{R}{\downarrow}$ and $b \in B$.

$\lhd$ First of all observe that the required equivalence holds for a finite set $\Xi_0 \subset \Xi$. Afterwards, apply (3) to the net $s : \mathscr{P}_{\text{fin}}(\Xi) \to \mathscr{R}{\downarrow}$, where $\mathscr{P}_{\text{fin}}(\Xi)$ is the set of finite subsets of Ξ and $s(\theta) := \sum_{\xi \in \theta} f(\xi)$, and employ the relation $[\![\mathscr{P}_{\text{fin}}(\Xi)^{\wedge} = \mathscr{P}_{\text{fin}}(\Xi^{\wedge})]\!] = \mathbf{1}$ (see [37]). $\rhd$

1.3.5. *The following relations hold for every element $x \in \mathscr{R}{\downarrow}$:*

$$e_x := \chi([\![x \neq 0]\!]), \quad e_\lambda^x = \chi([\![x < \lambda^{\wedge}]\!]) \quad (\lambda \in \mathbb{R}).$$

$\lhd$ A real number t is distinct from zero if and only if the supremum of the set $\{1 \wedge (n|t|) \mid n \in \omega\}$ is equal to 1. Consequently, for $x \in \mathscr{R}{\downarrow}$ the transfer principle yields $b := [\![x \neq 0]\!] = [\![1 = \sup A]\!]$, where $A \in \mathbf{V}^{(B)}$ is determined by the formula $A := \{1 \wedge (n|x|) \mid n \in \omega^{\wedge}\}$. If $C := \{1 \wedge (n|x|) \mid n \in \omega\}$ then we prove that $[\![C{\uparrow} = A]\!] = \mathbf{1}$ using the second formula of 1.2.10(1) and the representation $\omega^{\wedge} = (\iota\omega){\uparrow}$ of 1.2.11. Hence, $[\![\sup(A) = \sup(C{\uparrow})]\!] = \mathbf{1}$. Employing 1.3.4(2), we derive

$$b = [\![\sup(C{\uparrow}) = 1]\!] = [\![\sup(C) = 1]\!] = [\![e_x = 1]\!].$$

On the other hand, $[\![e_x = 0]\!] = [\![e_x = 1]\!]^* = b^*$. Now, according to 1.3.2, we can write down

$$\chi(b)e_x = \chi(b)1 = \chi(b), \quad \chi(b^*)e_x = 0 \to \chi(b)e_x = e_x.$$

Finally, $\chi(b) = e_x$.

Take $\lambda \in \mathbb{R}$ and put $y := (\lambda 1 - x)^+$. Since $[\![\lambda^{\wedge} = \lambda 1]\!] = \mathbf{1}$, we have $[\![y = (\lambda^{\wedge} - x)^+]\!] = \mathbf{1}$. Consequently, $e_\lambda^x = e_y = \chi([\![y \neq 0]\!])$. It remains to observe that inside $\mathbf{V}^{(B)}$ the number $y = (\lambda^{\wedge} - x) \vee 0$ is distinct from zero if and only if $\lambda^{\wedge} - x > 0$, i.e., $[\![y \neq 0]\!] = [\![x < \lambda^{\wedge}]\!]$. $\rhd$

1.3.6. Theorem. *Let E be an Archimedean vector lattice, let $\mathscr{R}$ be the field of real numbers in the model $\mathbf{V}^{(B)}$, and let j be an isomorphism of B onto the base $\mathfrak{B}(E)$. Then there exists an element $\mathscr{E} \in \mathbf{V}^{(B)}$ satisfying the following conditions:*

(1) $\mathbf{V}^{(B)} \models$ "*$\mathscr{E}$ is a vector sublattice of $\mathscr{R}$ considered as a vector lattice over $\mathbb{R}^{\wedge}$;*"

(2) *$E' := \mathscr{E}{\downarrow}$ is a vector sublattice of $\mathscr{R}{\downarrow}$ invariant under each band projection $\chi(b)$ $(b \in B)$ and such that every set of positive pairwise disjoint sets in it has a supremum;*

(3) *there is an o-continuous lattice isomorphism $\iota : E \to E'$ such that $\iota(E)$ is a minorizing sublattice in $\mathscr{R}{\downarrow}$;*

(4) *for every $b \in B$ the band projection in $\mathscr{R}{\downarrow}$ generated by the set $\iota(j(b))$ coincides with $\chi(b)$.*

$\triangleleft$ Assign $d(x,y) := j^{-1}(\{|x-y|\}^{\perp\perp})$. Let $\mathscr{E}$ be the Boolean-valued realization of the B-set (E,d) and $E' := \mathscr{E}{\downarrow}$ (see 1.2.12(4)). By 1.2.12(2), without loss of generality we may assume that $E \subset E'$, $d(x,y) = [\![x \neq y]\!]$ $(x, y \in E)$, and $E' = \operatorname{mix} E$. Further, furnish E' with a vector lattice structure. To this end, take a number $\lambda \in \mathbb{R}$ and elements $x, y \in E'$ of the form $x := \operatorname{mix}(b_\xi x_\xi)$ and $y := \operatorname{mix}(b_\xi y_\xi)$, where $(x_\xi) \subset E$, $(y_\xi) \subset E$, and (b_ξ) is a partition of unity in B, and define

$$x + y := \operatorname{mix}(b_\xi(x_\xi + y_\xi)),$$
$$\lambda x := \operatorname{mix}(b_\xi(\lambda x_\xi)),$$
$$x \le y \leftrightarrow x = \operatorname{mix}(b_\xi(x_\xi \wedge y_\xi)).$$

Inside $\mathbf{V}^{(B)}$, define the summation $\oplus$, multiplication $\odot$, and order $\sqsubseteq$ in the set $\mathscr{E}$ as the ascents of the corresponding objects in E'. More precisely, the operations $\oplus : \mathscr{E} \times \mathscr{E} \to \mathscr{E}$ and $\odot : \mathscr{E} \times \mathbb{R}^{\wedge} \to \mathscr{E}$ and the predicate $\sqsubseteq \subset \mathscr{E} \times \mathscr{E}$ are defined by the relations

$$[\![x \oplus y = x + y]\!] = \mathbf{1},$$
$$[\![\lambda \odot x = \lambda x]\!] = \mathbf{1} \quad (x, y \in E',\ \lambda \in \mathbb{R}),$$
$$[\![x \sqsubseteq y]\!] = \bigvee\{[\![x = x']\!] \wedge [\![y = y']\!] \mid x', y' \in E',\ x' \le y'\}.$$

Thus, we can claim that $\mathscr{E}$ is a vector lattice over the field $\mathbb{R}^{\wedge}$ and, in particular, a lattice-ordered group inside $\mathbf{V}^{(B)}$. Also, it is clear that the Archimedean axiom is valid on $\mathscr{E}$, since E' is an Archimedean lattice.

Note that if $x \in E_+$ then $\{x\}^{\perp\perp} = d(x,0) = [\![x \neq 0]\!]$, i.e., $\{x\}^{\perp} = [\![x = 0]\!]$. Consequently, we have

$$[\![x = 0]\!] \vee [\![y = 0]\!] = \{x\}^{\perp} \vee \{y\}^{\perp} = \mathbf{1}_B$$

for disjoint $x, y \in E$. Hence, we easily derive that $[\![\mathscr{E}$ is linearly ordered$]\!] = \mathbf{1}$, for

$$[\![(\forall x \in \mathscr{E})(\forall y \in \mathscr{E})\ (|x| \wedge |y| = 0 \to x = 0 \vee y = 0)]\!] = \mathbf{1}.$$

It is well known that an Archimedean linearly ordered group is isomorphic to an additive subgroup of the field of real numbers. Applying this assertion to $\mathscr{E}$ inside $\mathbf{V}^{(B)}$, without loss of generality we may assume that $\mathscr{E}$ is an additive subgroup of the field $\mathscr{R}$. Furthermore, we suppose that $1 \in \mathscr{E}$, because otherwise $\mathscr{E}$ can be replaced by the isomorphic group $e^{-1}\mathscr{E}$ with $0 < e \in \mathscr{E}$. The multiplication $\odot$ represents a continuous $\mathbb{R}^{\wedge}$-bilinear mapping from $\mathbb{R}^{\wedge} \times \mathscr{E}$ to $\mathscr{E}$. Let $\beta : \mathscr{R} \times \mathscr{R} \to \mathscr{R}$ be its extension by continuity. Then β is $\mathscr{R}$-bilinear and $\beta(1,1) = 1^{\wedge} \odot 1 = 1$. Consequently, β coincides with the usual multiplication in $\mathscr{R}$; i.e., $\mathscr{E}$ is a vector sublattice of the field $\mathscr{R}$ considered as a vector lattice over $\mathbb{R}^{\wedge}$. Thereby $E' \subset \mathscr{R}{\downarrow}$.

The fact that E' is minorizing in $\mathscr{R}{\downarrow}$ obviously ensues from the fact that $[\![\mathscr{E}$ is dense in $\mathscr{R}]\!] = \mathbf{1}$. Prove that E is minorizing in E'.

It follows from the properties of the isomorphism χ (see 1.3.2) that

$$\chi(b)\iota x = 0 \leftrightarrow j(b) \leq \{x\}^{\perp} \leftrightarrow x \in j(b^{\perp}),$$

whatever $b \in B$ and $x \in E_+$ might be. Hence, $\chi(b)$ is the band projection onto the band in $\mathscr{R}{\downarrow}$ generated by the set $\iota(j(b))$. Moreover, if $\chi(b)x = 0$ for all $x \in E_+$ then $b = \{0\}$. Thus, for every $b \in B$ we can find a positive element $y \in E$ for which $y = \chi(b)y$. Now, take $0 < z \in E'$. The representation $z = o\text{-}\sum_{\xi\in\Xi} \chi(b_\xi)x_\xi$ is valid, where (b_ξ) is a partition of unity in B and $(x_\xi) \subset E_+$. We see that $\chi(b_\xi)x_\xi \neq 0$ at least for one index ξ. Let $\pi := \chi(b_\xi) \circ \chi([\![x_\xi \neq 0]\!])$ and y be a strictly positive element in E such that $y = \pi y$. Then for $x_0 := y \wedge x_\xi$ we have

$$0 < x_0 \leq \pi x_\xi \leq \chi(b_\xi)x_\xi \leq z$$

and $x_0 \in E$. Thereby E is minorizing in E'. $\triangleright$

1.3.7. The element $\mathscr{E} \in \mathbf{V}^{(B)}$ arising in Theorem 1.3.6 is called the ***Boolean-valued realization*** of the vector lattice E. Thus, the Boolean-valued realizations of Archimedean vector lattices are vector sublattices of the field $\mathscr{R}$ of real numbers considered as a vector lattice over the field $\mathbb{R}^{\wedge}$.

Now, we indicated some corollaries to 1.3.2 and 1.3.6, with the same notations B, E, E', $\mathscr{E}$, ι, and $\mathscr{R}$.

(1) *For every $x' \in E'$ there exist a family $(x_\xi) \subset E$ and a partition of unity (π_ξ) in $\mathfrak{P}(\mathscr{R}{\downarrow})$ such that*

$$x' = o\text{-}\sum_{\xi \in \Xi} \pi_\xi \iota x_\xi.$$

(2) *For arbitrary $x \in \mathscr{R}{\downarrow}$ and $\varepsilon > 0$ there is $x_\varepsilon \in E'$ such that $|x - x_\varepsilon| \leq \varepsilon \mathbf{1}$.*

$\lhd$ This is a consequence of the fact that $[\![\mathscr{E}$ is dense in $\mathscr{R}]\!] = \mathbf{1}$. $\rhd$

(3) *If $h : \mathscr{E} \to \mathscr{R}{\downarrow}$ is a lattice isomorphism and for every $b \in B$ the order projection onto the band in $\mathscr{R}{\downarrow}$ generated by the set $h(j(b))$ coincides with $\chi(b)$ then there exists $a \in \mathscr{R}{\downarrow}$ such that $hx = a \cdot \iota(x)$ $(x \in E)$.*

$\lhd$ Indeed, if $E_0 := \operatorname{im} \iota$ and $h_0 := h \circ \iota^{-1}$ then the isomorphism $h_0 : E_0 \to \mathscr{R}{\downarrow}$ is extensional; therefore, for $\tau := h_0{\uparrow}$ we have $[\![$the mapping $\tau : \mathscr{E} \to \mathscr{R}$ is isotonic, injective, and additive$]\!] = \mathbf{1}$. Consequently, h_0 is continuous and has the form $\tau(\alpha) = a \cdot \alpha$ $(\alpha \in \mathscr{R})$, where a is a fixed element in $\mathscr{R}{\downarrow}$. Hence, we derive that $h_0(y) = a \cdot y$ $(y \in E_0)$ or $h(x) = a \cdot \iota(x)$ $(x \in E)$. $\rhd$

(4) *If there exists an order-unit $\mathbf{1}$ in E then the isomorphism ι is uniquely determined by the extra requirement that $\iota\mathbf{1} = 1$.*

(5) *If E is a K-space then $\mathscr{E} = \mathscr{R}$, $E' = \mathscr{R}{\downarrow}$, and $\iota(E)$ is an order-dense ideal of the K-space $\mathscr{R}{\downarrow}$. Moreover, $\iota^{-1} \circ \chi(b) \circ \iota$ is the band projection onto $j(b)$ for every $b \in B$.*

$\lhd$ If E is order complete then so is the lattice E'. From 1.3.4(2) we see that the order completeness of E' is equivalent to the axiom on existence of exact bounds for bounded sets in $\mathscr{E}$. By 1.3.1, $\mathscr{E} = \mathscr{R}$ and $E' = \mathscr{R}{\downarrow}$. Let $e \in E_+$, $y \in \mathscr{R}{\downarrow}$, and $|y| \leq \iota e$. Since $\iota(E)$ is a minorizing sublattice in $\mathscr{R}{\downarrow}$, we have $y^+ = \sup \iota(A)$, where $A := \{x \in E_+ \mid \iota x \leq y^+\}$. But the set A is bounded in E by the element e;

therefore, $\sup A \in E$ and $y^+ = \iota(\sup A) \in \iota E$. Similarly, $y^- \in \iota(E)$ and, finally, $y \in \iota(E)$. $\triangleright$

(6) *The image $\iota(E)$ coincides with the whole $\mathscr{R}{\downarrow}$ if and only if E is a universally complete K-space.*

$\triangleleft$ If E is a K-space then $\mathscr{E} = \mathscr{R}$ by (5) and, hence, $\mathscr{R}{\downarrow} = \mathscr{E}{\downarrow} = \operatorname{mix} \iota(E)$. However, for the universally complete K-space E we have $\operatorname{mix} \iota(E) = \iota(E)$. The converse is obvious. $\triangleright$

(7) *Universally complete K-spaces are isomorphic if and only if their bases are isomorphic.*

$\triangleleft$ If E and F are universally complete K-spaces and the Boolean algebras $\mathfrak{B}(E)$ and $\mathfrak{B}(F)$ are isomorphic then E and F are isomorphic to the same K-space $\mathscr{R}{\downarrow}$ by (6). On the other hand, if h is an isomorphism from E onto F then the mapping $K \mapsto h(K)$ $(K \in \mathfrak{B}(E))$ is an isomorphism between the bases. $\triangleright$

(8) *Let E be a universally complete K-space with unity* $\mathbf{1}$. *Then we can uniquely define the multiplication in E so as to make E into an f-algebra and* $\mathbf{1}$, *into a ring unity.*

$\triangleleft$ By (6) and (4), we may assume that $E = \mathscr{R}{\downarrow}$ and $\mathbf{1} = 1$. The existence of the required multiplication in E follows from 1.3.3. Assume that there is another multiplication $\odot : E \times E \to E$ in E and $(E, +, \odot, \leq)$ is a faithful f-algebra with unity $\mathbf{1}$. The faithfulness of the f-algebra implies that $\odot$ is an extensional mapping. But then the ascent $\times := \odot{\uparrow}$ is a multiplication in $\mathscr{R}$. By virtue of uniqueness of the multiplicative structure in $\mathscr{R}$, we conclude that $\times = \cdot$. Hence, we derive that $\odot$ coincides with the original multiplication in E (see 1.3.3). $\triangleright$

1.3.8. Now, we dwell upon the questions of extension and completion of Archimedean vector lattices.

A *universal completion* or *maximal extension* of an Archimedean vector lattice E is defined to be a universally complete K-space $mE := \mathscr{R}{\downarrow}$, where $\mathscr{R}$ is (some realization of) the field of real numbers in the model $\mathbf{V}^{(B)}$ ($\mathscr{R}$ is unique to within isomorphism!) and $B := \mathfrak{B}(E)$. We can see from Theorem 1.3.6 that there exists an isomorphism $\iota : E \to mE$; moreover, the sublattice $\iota(E)$ is minorizing in mE and $\iota(E)^{\perp\perp} = mE$. Such properties determine a universal completion to within an isomorphism which makes it possible to speak of *the* universal completion of a space. More precisely, the following assertion is valid:

(1) *Let E be an Archimedean vector lattice and F be a universally complete K-space. Assume that h is an isomorphism from E onto the minorizing lattice F and $h(E)^{\perp\perp} = F$. Then there exists an isomorphism $\varkappa$ from F onto mE such that $\iota = \varkappa \circ h$.*

$\lhd$ We easily derive from the hypothesis that the mapping $j : b \mapsto j(b) := h(b)^{\perp\perp}$ is an isomorphism from $B := \mathfrak{B}(E)$ onto $\mathfrak{B}(F)$. According to 1.3.7(5, 6), there is an isomorphism k from F onto mE such that $k^{-1} \circ \chi(b) \circ k$ is the projection onto the component $j(b)$ (for each $b \in B$). Apply 1.3.6(3) to $F_0 := h(E)$ and $g := \iota \circ h^{-1} : F_0 \to \mathscr{R}{\downarrow}$. There exists an element $a \in \mathscr{R}{\downarrow}$ such that $g(x) = a \cdot k(x)$, $x \in F_0$. Put $\varkappa(x) = a \cdot k(x)$ $(x \in F)$. Then $\iota = \varkappa \circ h$. $\rhd$

(2) *For every Archimedean vector lattice E there exists a K-space oE unique to within an isomorphism and an o-continuous lattice isomorphism $\iota : E \to oE$ such that*

$$\sup\{\iota x \mid x \in E,\ \iota x \le y\} = y = \inf\{\iota x \mid x \in E,\ \iota x \ge y\}$$

for every element $y \in oE$.

$\lhd$ Let oE be the order ideal in $mE := \mathscr{R}{\downarrow}$ generated by the set $\iota(E)$, where $\iota : E \to mE$ is the same as in (1). We preserve the same notation for the isomorphism from E into oE determined by the embedding ι. Then (oE, ι) is the sought pair. Indeed, $\iota(E)$ is a minorizing and simultaneously massive lattice in the K-space oE (see 1.3.6(3)); therefore, the required representation in terms of suprema and infima are valid for every $y \in oE$. The order continuity of ι as well is an immediate consequence of the fact that $\iota(E)$ is a minorizing sublattice. Assume that some pair (E', ι') satisfies the indicated conditions. Then $\iota'(E')$ is a minorizing and massive sublattice in oE; hence, we can easily derive that the bases $\mathfrak{B}(E')$ and $\mathfrak{B}(oE)$ are isomorphic. Thereby such are the bases $\mathfrak{B}(E')$ and $\mathfrak{B}(E)$, and 1.3.7(7) implies that the K-spaces mE and mE' are isomorphic; so without loss of generality we may assume that $E' \subset mE$. Further, arguing as in (1) and using 1.3.6(3), we conclude that $\iota' \circ \iota^{-1}$ is the restriction to E of the operator $m_a : \mathscr{R}{\downarrow} \to \mathscr{R}{\downarrow}$ of multiplication by some element $a \in \mathscr{R}{\downarrow}$. It is easily seen that $m_a(oE) = E'$; i.e., m_a establishes an isomorphism between the K-spaces oE and E'. $\rhd$

Suppose that F is a K-space and $A \subset F$. Denote by dA the set of all $x \in F$ representable as $o\text{-}\sum_{\xi\in\Xi} \pi_\xi a_\xi$, where $(a_\xi)_{\xi\in\Xi} \subset A$ and $(\pi_\xi)_{\xi\in\Xi}$ is a partition of unity in $\mathfrak{P}(F)$. Let rA be the set of all elements $x \in F$ of the form $x = r\text{-}\lim_{n\to\infty} a_n$, where (a_n) is an arbitrary sequence in A convergent with regulator.

(3) *The formula $oE = rdE$ holds for an Archimedean vector lattice E.*

$\lhd$ See 1.3.6(1, 2). $\rhd$

1.3.9. Interpreting the concept of a convergent numeric net inside $\mathbf{V}^{(B)}$ and employing 1.3.4(3) and 1.3.7(5), we obtain useful tests for o-convergence in a K-space E with unity $\mathbf{1}$.

Theorem. *Let $(x_\alpha)_{\alpha\in\mathrm{A}}$ be an order bounded net in E and $x \in E$.*

The following assertions are equivalent:

(1) *the net (x_α) o-converges to the element x;*

(2) *for every number $\varepsilon > 0$ the net $(e_\varepsilon^{y(\alpha)})_{\alpha\in\mathrm{A}}$ of unit elements, where $y(\alpha) := |x - x_\alpha|$, o-converges to $\mathbf{1}$;*

(3) *for every number $\varepsilon > 0$ there exists a partition of unity $(\pi_\alpha)_{\alpha\in\mathrm{A}}$ in the Boolean-valued algebra $\mathfrak{P}(E)$ such that*

$$\pi_\alpha|x - x_\beta| \le \varepsilon\mathbf{1} \quad (\alpha, \beta \in \mathrm{A},\ \beta \ge \alpha);$$

(4) *for every number $\varepsilon > 0$ there exists an increasing net $(\rho_\alpha)_{\alpha\in\mathrm{A}} \subset \mathfrak{P}(E)$ of projections such that*

$$\rho_\alpha|x - x_\beta| \le \varepsilon\mathbf{1} \quad (\alpha, \beta \in \mathrm{A},\ \beta \ge \alpha).$$

$\lhd$ Without loss of generality we may assume that E is an order-dense ideal of the universally complete K-space $\mathscr{R}\!\downarrow$ (see 1.3.7(5)).

(1) $\Leftrightarrow$ (2): It suffices to consider the case $y_\alpha = x_\alpha$ $(\alpha \in \mathrm{A})$, i.e., $(x_\alpha) \subset E_+$ and $x_\alpha \xrightarrow{(o)} 0$.

Let σ be the modified ascent of the mapping $s : \alpha \to x_\alpha$. Then $[\![\sigma$ is a net in $R_+]\!] = \mathbf{1}$. By 1.2.4(3), $o\text{-}\lim s = 0$ if and only if $[\![\lim \sigma = 0]\!] = \mathbf{1}$. We can rewrite the last equality in equivalent form:

$$\mathbf{1} = [\![(\forall \varepsilon \in \mathbb{R}^\wedge)(\varepsilon > 0 \to (\exists \alpha \in \mathrm{A}^\wedge)(\forall \beta \in \mathrm{A}^\wedge)(\beta \ge \alpha \to x_\beta < \varepsilon))]\!].$$

Calculating the Boolean truth-values for the quantifiers, we find another equivalent form

$$(\forall \varepsilon > 0)(\exists (b_\alpha)_{\alpha\in\mathrm{A}} \subset B)\ \Big(\bigvee_{\alpha\in\mathrm{A}} b_\alpha = \mathbf{1} \wedge (\forall \beta \in \mathrm{A})\ (\beta \ge \alpha \to [\![x_\beta < \varepsilon^\wedge]\!] \ge b_\alpha)\Big)$$

which in turn amounts to the following:

$$(\forall\varepsilon > 0)\Big(\bigvee_{\alpha\in\mathrm{A}}\bigwedge_{\substack{\beta\in\mathrm{A}\\ \beta\geq\alpha}} [\![x_\beta < \varepsilon^\wedge]\!] = \mathbf{1}\Big).$$

Since $\chi([\![x_\beta < \varepsilon^\wedge]\!]) = e_\varepsilon^{x_\beta}$ (see 1.3.5), we see from the above that $x_\alpha \xrightarrow{(o)} x$ if and only if

$$\liminf_{\alpha\in\mathrm{A}} e_\varepsilon^{x_\alpha} = \bigvee_{\alpha\in\mathrm{A}}\bigwedge_{\substack{\beta\in\mathrm{A}\\ \beta\geq\alpha}} e_\varepsilon^{x_\beta} = \mathbf{1}$$

for every $\varepsilon > 0$, i.e., $e_\varepsilon^{x_\alpha} \xrightarrow{(o)} \mathbf{1}$ for every $\varepsilon > 0$.

(1) $\leftrightarrow$ (3): Arguing as in (1) $\to$ (2), we find that the relation $o\text{-}\lim x_\alpha = x$ is equivalent to the following:

$$(\forall\varepsilon > 0)(\exists(c_\alpha)_{\alpha\in\mathrm{A}} \subset B)\Big(\bigvee_{\alpha\in\mathrm{A}} c_\alpha = 1 \wedge (\forall\beta\in\mathrm{A})(\beta\geq\alpha \to c_\alpha \leq [\![|x_\alpha - x| \leq \varepsilon^\wedge]\!])\Big).$$

By virtue of the exhaustion principle for Boolean algebras, there exist a partition of unity $(d_\xi)_{\xi\in\Xi}$ in B and a mapping $\delta : \Xi \to \mathrm{A}$ such that $d_\xi \leq c_{\delta(\xi)}$ $(\xi\in\Xi)$. Put $b_\alpha := \bigvee\{d_\xi \mid \alpha = \delta(\xi)\}$ if $\alpha\in\delta(\Xi)$ and $b_\alpha = 0$ if $\alpha\notin\delta(\Xi)$. We see that $(b_\alpha)_{\alpha\in\mathrm{A}}$ is a partition of unity and $b_\alpha \leq c_\alpha$ $(\alpha\in\mathrm{A})$. Thus, if $x_\alpha \to x$ then for every $\varepsilon > 0$ there is a partition of unity (b_α) such that

$$b_\alpha \leq [\![|x - x_\beta| \leq \varepsilon^\wedge]\!] \quad (\alpha,\beta\in\mathrm{A},\ \beta\geq\alpha).$$

As follows from 1.3.2, the latter means that

$$\pi_\alpha|x - x_\beta| \leq \varepsilon\mathbf{1} \quad (\alpha,\beta\in\mathrm{A},\ \beta\geq\alpha),$$

where $\pi_\alpha := \chi(b_\alpha)$. Since (π_α) is a partition of unity in $\mathfrak{P}(E)$, necessity is proven.

To prove sufficiency, observe that if the indicated conditions are satisfied and $a := \limsup|x_\alpha - x|$ then

$$\pi_\alpha a \leq \bigvee_{\beta\geq\alpha} |x_\beta - x| \leq \varepsilon\pi_\alpha\mathbf{1}$$

for all $\alpha\in\mathrm{A}$. Consequently,

$$0 \leq a = \sum \pi_\alpha a \leq \varepsilon\sum\pi_\alpha 1 = \varepsilon\mathbf{1}.$$

Since $\varepsilon > 0$ is arbitrary, we have $a = 0$ and $o\text{-}\lim x_\alpha = x$.

(3) $\leftrightarrow$ (4): We only have to put $\rho_\alpha := \bigvee\{\pi_\beta \mid \beta\in\mathrm{A},\ \alpha\leq\beta\}$ in (3). $\triangleright$

1.3.10. Let $\mathscr{C}$ be the field of complex numbers in the model $\mathbf{V}^{(B)}$. Then the algebraic system $\mathscr{C}{\downarrow}$ represents the complexification of the K-space $\mathscr{R}{\downarrow}$. In particular, $\mathscr{C}{\downarrow}$ is a complex universally complete K-space and a complex algebra.

$\lhd$ Since $\mathbb{C} = \mathbb{R} \oplus i\mathbb{R}$ is equivalent to a bounded formula, we have $[\![\mathbb{C}^\wedge = \mathbb{R}^\wedge \oplus \mathbb{R}^\wedge]\!] = \mathbb{1}$ (see 1.2.8(4)), where i is the imaginary unity and the element $i^\wedge$ is denoted by the same letter i. From 1.3.1 we see that $[\![\mathbb{C}^\wedge$ is a dense subfield of the field $\mathscr{C}]\!] = 1$ and, in particular, $[\![i$ is the imaginary unity of the field $\mathscr{C}]\!] = 1$. If $z \in \mathscr{C}{\downarrow}$ then z is a complex number inside $\mathbf{V}^{(B)}$; therefore,

$$[\![(\exists! x \in \mathscr{R})(\exists! y \in \mathscr{R})\, z = x + iy]\!] = 1.$$

The maximum principle implies that there is a unique pair of elements $x, y \in \mathbf{V}^{(B)}$ such that

$$[\![x, y \in \mathscr{R}]\!] = [\![z = x + iy]\!] = 1.$$

Hence, we obtain $x, y \in \mathscr{R}{\downarrow}$, $z = x + iy$, and thereby $\mathscr{C}{\downarrow} = \mathscr{R}{\downarrow} \oplus i\mathscr{R}{\downarrow}$. Appealing to 1.3.2 and 1.3.4 completes the proof. $\rhd$

1.4. Boolean-Valued Analysis of Vector Lattices

In this section, we show that the most important structure properties of vector lattices such as representability by means of function spaces, the spectral theorem, functional calculus, etc. are the images of properties of the field of real numbers in an appropriate Boolean-valued model.

1.4.1. We start with several useful remarks to be used below without further specifications. Take a K_σ-space E. By the realization theorem 1.3.6, we can assume that E is a sublattice of the universally complete K-space $\mathscr{R}{\downarrow}$, where, as usual, $\mathscr{R}$ is the field of real numbers in the model $\mathbf{V}^{(B)}$ and $B := \mathfrak{B}(E)$. Moreover, the ideal $\widehat{E} := I(E)$ generated by the set E in $\mathscr{R}{\downarrow}$ is an order-dense ideal of $\mathscr{R}{\downarrow}$ and an o-completion of E. The unity $\mathbb{1}$ of the lattice E is also a unity in $\mathscr{R}{\downarrow}$. The exact bounds of countable sets in E are inherited from $\mathscr{R}{\downarrow}$. In more detail, if the least upper (greatest lower) bound x of a sequence $(x_n) \subset E$ exists in $\mathscr{R}{\downarrow}$ then x is also the least upper (greatest lower) bound in E, provided that $x \in E$. Thus, it does not matter whether the o-limit (o-sum) of a sequence in E is calculated in E or $\mathscr{R}{\downarrow}$, provided the result belongs to E. The same is true for the r-limit and r-sums. In particular, we can claim that if $x \in E$ then the trace e_x and the spectral function (characteristic) e^x_λ of an element x calculated in $\mathscr{R}{\downarrow}$ are an element of $B := \mathfrak{C}(E)$ and a mapping from $\mathbb{R}$ to B respectively.

1.4.2. *Theorem.* *The following assertions hold for the spectral function of an element of an arbitrary K_σ-space with unity* $\mathbf{1}$:

(1) $s \le t \to e^x_s \le e^x_t \quad (s, t \in \mathbb{R})$;

(2) $\bigvee_{t \in \mathbf{P}} e^x_t = \mathbf{1}, \quad \bigwedge_{t \in \mathbf{P}} e^x_t = \mathbf{0}$;

(3) $\bigvee\{e^x_s \mid s \in \mathbf{P},\ s < t\} = e^x_t \quad (t \in \mathbb{R})$;

(4) $x \le y \to (\forall t \in \mathbf{P})\, e^y_t \le e^x_t$;

(5) $e^{x+y}_t = \bigvee\{e^x_r \wedge e^y_s \mid r, s \in \mathbf{P},\ r + s = t\} \quad (t \in \mathbf{P})$;

(6) $x \ge 0 \wedge y \ge 0 \to e^{x \cdot y}_t = \bigvee\{e^x_r \wedge e^y_s \mid r, s \in \mathbf{P}_+,\ rs = t\} \quad (t \in \mathbf{P},\ t > 0)$;

(7) $e^{-x}_t = \bigvee\{\mathbf{1} - e^x_{-s} \mid s \in \mathbf{P},\ s < t\} \quad (t \in \mathbf{P})$;

(8) $x = \inf(A) \leftrightarrow \left(e^x_t = \bigvee_{a \in A} e^a_t\right) \quad (t \in \mathbb{R})$;

(9) $e^{x \vee y}_t = e^x_t \wedge e^y_t \quad (t \in \mathbf{R})$;

(10) $c = \mathfrak{C}(\mathbf{1}) \to e^{cx}_t = (\mathbf{1} - c) \wedge e^x_t \quad (t \in \mathbb{R},\ t > 0)$;

$c \in \mathfrak{C}(\mathbf{1}) \to e^{cx}_t = c \wedge e^x_t \quad (t \in \mathbb{R},\ t \le 0)$.

Here $\mathbf{P}$ is an arbitrary dense subfield of the field $\mathbb{R}$. (In (6) and (8) we assume that the needed product and infimum exist.)

$\lhd$ According to remarks of 1.4.1, without loss of generality we may assume that the K_σ-space under consideration coincides with $\mathscr{R}{\downarrow}$. But then the required relations can be easily derived from the elementary properties of numbers with the help of 1.3.5.

Prove, for instance, (2), (6), and (8). First of all observe that $\mathbf{P}^\wedge$ is a dense subfield of the field $\mathscr{R}$ inside $\mathbf{V}^{(B)}$ Take $x \in \mathscr{R}{\downarrow}$ and consider the two formulas $\varphi(x) := (\exists t \in \mathbf{P}^\wedge)(x < t)$ and $\psi(x) := (\forall t \in \mathbf{P}^\wedge)(x < t)$. For a real number x the formula $\varphi(x)$ is true and $\psi(x)$ is false. Consequently, the transfer principle implies $[\![\varphi(x)]\!] = \mathbf{1}$ and $[\![\psi(x)]\!] = \mathbf{0}$. Calculating the Boolean truth-values for the quantifiers by the rules of 1.2.8(1) yields

$$\bigvee_{t \in \mathbf{P}} [\![x < t^\wedge]\!] = \mathbf{1}, \quad \bigwedge_{t \in \mathbf{P}} [\![x < t^\wedge]\!] = \mathbf{0}$$

which is equivalent to (2) by 1.3.5.

Take positive elements $x, y \in \mathscr{R}\!\downarrow$ and a number $0 < t \in \mathbf{P}$. Then x, y, and $t^\wedge$ are real numbers in the model $\mathbf{V}^{(B)}$. Make use of the following property of numbers:

$$x \geq 0 \wedge y \geq 0 \to (xy < t^\wedge \leftrightarrow (\exists r, s \in \mathbf{P}_+^\wedge)(x < r \wedge y < s \wedge rs = t)).$$

Employing again the transfer principle and the rules of 1.2.8(1) for calculating the Boolean truth-values, we arrive at the relation

$$[\![xy < t^\wedge]\!] = \bigvee_{\substack{0 \leq r,s \in \mathbf{P} \\ rs=t}} [\![x < r^\wedge]\!] \bigwedge [\![y < s^\wedge]\!].$$

Hence, the required equality (6) ensues if we apply χ to both sides of the preceding equality (see 1.3.5).

Now, let A be a set in the considered K_σ-space. Then $A\!\uparrow$ is some set of real numbers inside $\mathbf{V}^{(B)}$ and the formula $\inf(A) < t \leftrightarrow (\exists a \in A\!\uparrow)(a < t)$ holds. Employing 1.3.4(2) and 1.2.10(1), we can write down the following chain of equivalences:

$$\begin{aligned} x = &\inf(A) \leftrightarrow [\![x = \inf(A\!\uparrow)]\!] = \mathbf{1} \leftrightarrow [\![(\forall t \in \mathbf{P}^\wedge) \\ &(x < t \leftrightarrow \inf(A\!\uparrow) < t)]\!] = \mathbf{1} \leftrightarrow (\forall t \in \mathbf{P})[\![x < t^\wedge]\!] \\ = &[\![(\exists a \in A\!\uparrow)(a < t^\wedge)]\!] \leftrightarrow (\forall t \in \mathbf{P})[\![x < t^\wedge]\!] = \bigvee_{a \in A} [\![a < t^\wedge]\!]. \end{aligned}$$

Appealing to 1.3.5 completes the proof of (8). ▷

1.4.3. Thus, to each element of a K_σ-space with unity there corresponds the spectral function, moreover, the operations transform in a rather definite way. This circumstance suggests that an arbitrary K_σ-space with unity can be realized as a space of "abstract spectral functions." We will expatiate upon this.

A *resolution of identity* in a Boolean algebra B is defined as a mapping $e : \mathbb{R} \to B$ satisfying the conditions

(1) $s \leq t \to e(s) \leq e(t) \quad (s, t \in \mathbb{R})$;

(2) $\bigvee_{t \in \mathbb{R}} e(t) = \mathbf{1}, \quad \bigwedge_{t \in \mathbb{R}} e(t) = \mathbf{0}$;

(3) $\bigvee_{s \in \mathbb{R}, s < t} e(s) = e(t) \quad (t \in \mathbb{R})$.

Let $\mathfrak{R}(B)$ be the set of all resolution of identity in B. Introduce some order by the formula

$$e' \leq e'' \leftrightarrow (\forall t \in \mathbb{R})(e''(t) \leq e'(t)) \quad (e', e'' \in \mathfrak{R}(B)).$$

Further, suppose that B is a σ-algebra and $\mathbf{P}$ is a dense countable subfield of $\mathbb{R}$.

By property (3), every resolution of identity is uniquely determined by its values on $\mathbf{P}$.

Given $e', e'' \in \mathfrak{R}(B)$, we can define the mapping

$$e : t \mapsto \bigvee\{e'(r) \wedge e''(s) \mid r, s \in \mathbf{P},\ r + s = t\} \quad (t \in \mathbf{P}),$$
$$e : t \mapsto \bigvee\{e(s) \mid s \in \mathbf{P},\ s < t\} \quad (t \in \mathbb{R})$$

which is obviously a resolution of identity in B. Putting $e' + e'' := e$, we obtain the structure of a commutative group in $\mathfrak{R}(B)$; moreover, the zero element $\mathbf{0}$ and the inverse element $-e$ have the form

$$\mathbf{0}(t) := \begin{cases} \mathbf{1}, & \text{if} \quad t > 0, \\ \mathbf{0}, & \text{if} \quad t \leq 0, \end{cases}$$
$$-e(t) := \bigvee\{1 - e(-s) \mid s \in \mathbf{P},\ s < t\}.$$

Finally, define the product of an element $e \in \mathfrak{R}(B)$ and a number $\alpha \in \mathbb{R}$ by the rules

$$(\alpha e)(t) := e(t/\alpha) \quad (\alpha > 0,\ t \in \mathbb{R}),$$
$$(\alpha e)(t) :\doteq (-e)(-t/\alpha) \quad (\alpha < 0,\ t \in \mathbb{R}).$$

1.4.4. Theorem. *Let B be a complete Boolean algebra. The set $\mathfrak{R}(B)$ with introduced operations and order represents a universally complete K-space. The mapping sending an element $x \in \mathscr{R}{\downarrow}$ to the resolution of identity $t \mapsto [\![x < t^{\wedge}]\!]$ ($t \in \mathbb{R}$) is an isomorphism between the K-spaces $\mathscr{R}{\downarrow}$ and $\mathfrak{R}(B)$.*

$\triangleleft$ Denote the indicated mapping from $\mathscr{R}{\downarrow}$ to $\mathfrak{R}(B)$ by the letter h. By Theorem 1.4.2, h preserves the operations and order. Moreover, h is one-to-one, since the equality $h(x) = h(y)$ means

$$[\![x < t^{\wedge}]\!] = [\![y < t^{\wedge}]\!] \quad (t \in \mathbb{R})$$

or (see 1.2.8(1))

$$[\![(\forall t \in \mathbb{R}^{\wedge})(x < t \leftrightarrow y < t)]\!] = 1$$

and thereby is equivalent to coincidence of two numbers x and y inside $\mathbf{V}^{(B)}$. By virtue of Theorem 1.3.2, it remains to establish that h is surjective. Take an arbitrary resolution of identity e in the Boolean algebra B. Let $\beta := (t_n)_{n\in\mathbb{Z}}$ be a partition of the real axis; i.e., $t_n < t_{n+1}$ $(n \in \mathbb{Z})$, $\lim_{n\to\infty} t_n = \infty$, and $\lim_{n\to-\infty} t_n = -\infty$. The disjoint sum

$$\bar{x}(\beta) := \sum_{n\in\mathbb{Z}} t_{n+1}(\chi(e(t_{n+1})) - \chi(e(t_n)))$$

exists in the universally complete K-space $\mathscr{R}\!\downarrow$; here χ is the isomorphism of B onto $\mathfrak{C}(\mathscr{R}\!\downarrow)$ (see 1.3.2 and 1.3.3). Denote by the letter A the set of all elements $\bar{x}(\beta)$. Every element of the form

$$x(\beta) := \sum_{n\in\mathbb{Z}} t_n(\chi(e(t_{n+1})) - \chi(e(t_n)))$$

is a lower bound of A. Therefore, there exists $x := \inf A := \inf\{x(\beta)\}$. It is easy to observe that

$$e_\lambda^{\bar{x}(\beta)} = \bigvee\{\chi(e(t_n)) \mid t_n < \lambda\}.$$

Hence, by 1.4.2(8), we infer

$$e_\lambda^x = \bigvee_{a\in A} e_\lambda^a = \bigvee_{t\in\mathbb{R},t<\lambda} \chi(e(t)) = \chi(e(\lambda)) \quad (\lambda \in \mathbb{R}).$$

Thereby $h(x) = e$ (see 1.3.5). ▷

1.4.5. We derive several important corollaries to the just-proven theorem.

(1) *A universally complete K-space E with unity $\mathbf{1}$ is isomorphic to the K-space $\mathfrak{R}(B)$, where $B = \mathfrak{C}(\mathbf{1})$. The isomorphism is established by the mapping $x \mapsto ((e_\lambda^x)_{\lambda\in\mathbb{R}})$ $(x \in E)$.*

◁ It suffices to compare 1.3.7(6) and 1.4.4. ▷

(2) ***The Freudenthal spectral theorem.*** *Let E be an arbitrary K_σ-space with unity $\mathbf{1}$. Every element $x \in E$ admits the representation*

$$x = \int_{-\infty}^{\infty} \lambda\, de_\lambda^x,$$

where the integral is understood to be the limit with regulator $\mathbf{1}$ *of the integral sums*

$$x(\beta) := \sum_{n\in\mathbb{Z}} \tau_n(e^x_{t_{n+1}} - e^x_{t_n}), \quad t_n < \tau_n < t_{n+1},$$

as $\delta(\beta) := \sup_{n\in\mathbb{Z}}(t_{n+1} - t_n) \to 0$.

$\lhd$ We can assume that $\mathscr{R}{\downarrow}$ is a universal completion of E and $E \subset \mathscr{R}{\downarrow}$. Let $x \in E$, $\beta := (t_n)_{n\in\mathbb{Z}}$ be a partition of $\mathbb{R}$, and $t_n < \tau_n < t_{n+1}$ $(n \in \mathbb{Z})$. Then

$$\begin{aligned} b_n &\le [\![t_n^\wedge \le x \le t_{n+1}^\wedge]\!] \wedge [\![t_n^\wedge \le \tau_n^\wedge \le t_{n+1}^\wedge]\!] \wedge [\![t_{n+1}^\wedge - t_n^\wedge \le \delta(\beta)^\wedge]\!] \\ &\le [\![|x - \tau_n^\wedge| \le \delta(\beta)^\wedge]\!]. \end{aligned}$$

Taking the equality $x(\beta) := \operatorname{mix}_{n\in\mathbb{Z}}(b_n\tau_n^\wedge)$ into account, we derive

$$[\![|x - x(\beta)| \le \delta(\beta)^\wedge]\!] = \mathbf{1} \quad \text{or} \quad |x - x(\beta)| \le \delta(\beta)\mathbf{1}.$$

It remains to recall the remarks of 1.4.1. $\rhd$

(3) *For an arbitrary σ-algebra B, the set $\mathfrak{R}(B)$ (with structure defined as in* 1.4.3) *is a universally complete K_σ-space with unity. Conversely, every universally complete K_σ-space E with unity is isomorphic to $\mathfrak{R}(B)$, where $B = \mathfrak{C}(E)$.*

$\lhd$ Let $\widehat{B}$ be an o-completion of the σ-algebra B. According to 1.4.4, $\mathfrak{R}(\widehat{B})$ is a universally complete K-space. The set $\mathfrak{R}(B)$ is contained in $\mathfrak{R}(\widehat{B})$. Moreover, from 1.4.2(4–7) and 1.4.4 we can see that the vector lattice structure and the exact bounds of countable sets in $\mathfrak{R}(B)$ are inherited from $\mathfrak{R}(\widehat{B})$. Consequently, $\mathfrak{R}(B)$ is a K_σ-space with unity. The same arguments imply that every countable set of pairwise disjoint elements in $\mathfrak{R}(B)$ is bounded.

Now, take an arbitrary K_σ-space E with unity and a universal completion $\widehat{E}$ of E. If $B = \mathfrak{C}(E)$ and $\widehat{B} := \mathfrak{C}(\widehat{E})$ then $\widehat{B}$ is an o-completion of B. By (1), the spaces $\widehat{E}$ and $\mathfrak{R}(\widehat{B})$ are isomorphic; moreover, $\mathfrak{R}(B)$ is the image of the subspace E by (2). $\rhd$

1.4.6. From 1.4.4 and 1.4.5 we can immediately derive some results on function realization of vector lattices.

(1) Theorem. *Let Q be the Stone space of a σ-algebra B. The vector lattices $C_\infty(Q)$ and $\mathfrak{R}(B)$ are isomorphic. In particular, $C_\infty(Q)$ is a universally complete K_σ-space with unity for every quasiextremal compact space Q.*

$\triangleleft$ Take $e \in \mathfrak{K}(B)$. Let G_t be a clopen set in Q corresponding to the element $e(t) \in B$. By 1.1.11, there exists a unique continuous function $\hat{e} : Q \to \overline{\mathbb{R}}$ such that

$$\{\hat{e} < t\} \subset G_t \subset \{\hat{e} \leq t\} \quad (t \in \mathbf{R}).$$

It follows from the relations 1.4.2(2) that the closed set $\cap\{G_t \mid t \in \mathbb{R}\}$ has empty interior and the open set $\cup\{G_t \mid t \in \mathbb{R}\}$ is dense in Q. Hence, the function is finite everywhere, except possibly the points of a nowhere dense set; therefore, $\hat{e} \in C_\infty(Q)$.

It is easy to check that the mapping $e \mapsto \hat{e}$ is the sought isomorphism. $\triangleright$

(2) Theorem. *Let Q be the Stone space of a complete Boolean algebra B, and let $\mathscr{R}$ be the field of real numbers in the model $\mathbf{V}^{(B)}$. The vector lattice $C_\infty(Q)$ is isomorphic to the universally complete K-space $\mathscr{R}{\downarrow}$. The isomorphism is established by assigning to an element $x \in \mathscr{R}{\downarrow}$ the function $\hat{x} : Q \to \overline{\mathbb{R}}$ by the formula*

$$\hat{x}(q) := \inf\{t \in \mathbb{R} \mid [\![x < t^{\wedge}]\!] \in q\} \quad (q \in Q).$$

$\triangleleft$ The proof is immediate from (1) and 1.4.4. $\triangleright$

(3) Theorem. *Let E be an Archimedean vector lattice and Q be the Stone space of the base $\mathfrak{B}(Q)$. Then E is isomorphic to a minorizing sublattice $E_0 \subset C_\infty(Q)$. Moreover, E is an order-dense ideal of $C_\infty(Q)$ (coincides with $C_\infty(Q)$) if and only if E is a K-space (a universally complete K-space).*

$\triangleleft$ See (2), 1.3.6, and 1.3.7(5, 6). $\triangleright$

1.4.7. In the sequel, we need the concept of integral with respect to a spectral measure. Suppose that (T, Σ) is a measure space; i.e., T is a nonempty set and Σ is a fixed σ-algebra of subsets of T. A *spectral measure* is defined to be an o-continuous Boolean homomorphism μ from Σ into the Boolean σ-algebra B. More precisely, a mapping $\mu : \Sigma \to B$ is a spectral measure if $\mu(T - A) = \mathbf{1} - \mu(A)$ $(A \in \Sigma)$ and

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \bigvee_{n=1}^{\infty} \mu(A_n)$$

for each sequence (A_n) of elements of Σ.

Let $B := \mathfrak{E}(E)$ be the Boolean algebra of unit elements of a K_σ-space E with a fixed unity $\mathbf{1}$. Take a measurable function $f : T \to \mathbb{R}$. Given an arbitrary partition of the real axis

$$\beta := (\lambda_k)_{k \in \mathbb{Z}}, \quad \lambda_k < \lambda_{k+1} \ (k \in \mathbb{Z}), \quad \lim_{n \to \pm\infty} \lambda_n = \pm\infty,$$

assign $A_k := f^{-1}([\lambda_k, \lambda_{k+1}))$ and compose the integral sums

$$\underline{\sigma}(f, \beta) := \sum_{-\infty}^{\infty} \lambda_k \mu(A_k), \quad \overline{\sigma}(f, \beta) := \sum_{-\infty}^{\infty} \lambda_{k+1} \mu(A_k),$$

where the sums are calculated in E. It is clear that

$$\underline{\sigma}(f, \beta) \leq \sum_{-\infty}^{\infty} f(t_k) \mu(A_k) \leq \overline{\sigma}(f, \beta)$$

for every choice of $t_k \in A_k$ $(k \in \mathbb{Z})$. Also, it is evident that $\underline{\sigma}(f, \beta)$ increases and $\overline{\sigma}(f, \beta)$ decreases as we refine the partition β. If there exists an element $x \in E$ such that $\sup\{\underline{\sigma}(f, \beta)\} = x = \inf\{\overline{\sigma}(f, \beta)\}$, where the exact bounds are calculated over all partitions $\beta := (\lambda_k)_{k \in \mathbb{Z}}$ of the real axis, then we say that the function f is *integrable with respect to the spectral measure* μ or *the spectral integral* $I_\mu(f)$ *exists*; in this event we write

$$I_\mu(f) := \int_T f d\mu := \int_T f(t) d\mu(t) := x.$$

1.4.8. *The spectral integral $I_\mu(f)$ exists for every bounded measurable function f. If E is a universally complete K_σ-space then every almost everywhere finite measurable function is integrable with respect to each spectral measure.*

$\triangleleft$ Note that $A_k \cap A_l = \varnothing$ $(k \neq l)$ and $\bigcup_{k \in \mathbb{Z}} A_k = T$; therefore, $(\mu(A_k))_{k \in \mathbb{Z}}$ is a resolution of identity in the Boolean algebra B. Putting $\delta := \sup_{k \in \mathbb{Z}}\{\lambda_{k+1} - \lambda_k\}$, we can write down

$$0 \leq \overline{\sigma}(f, \beta) - \underline{\sigma}(f, \beta) \leq \sum_{k \in \mathbb{Z}} \delta \mu(A_k) = \delta \mathbf{1}.$$

Consequently, a measurable function f is integrable with respect to μ if and only if $\overline{\sigma}(f, \beta)$ and $\underline{\sigma}(f, \beta)$ exist at least for one partition β. If f is bounded then the sums $\overline{\sigma}(f, \beta)$ and $\underline{\sigma}(f, \beta)$ contain at most finitely many nonzero summands. If $\overline{E}$ is a universally complete K-space and a measurable function f is arbitrary then the indicated sums also make sense, since in this case they involve at most countably many pairwise disjoint elements. $\triangleright$

1.4.9. Theorem. *Let $E := \mathscr{R}\downarrow$ and let μ be a spectral measure with values in $B := \mathfrak{C}(E)$. Then for every measurable function f the integral $I_\mu(f)$ is a unique element of the K-space E satisfying the condition*

$$[\![I_\mu(f) < \lambda^\wedge]\!] = \mu(\{f < \lambda\}) \quad (\lambda \in \mathbb{R}).$$

$\lhd$ Take an arbitrary number $\lambda \in \mathbb{R}$ and a partition of the real axis $\beta := (\lambda_k)_{k\in\mathbb{Z}}$ such that $\lambda_0 = \lambda$. If $b := [\![I_\mu(f) < \lambda^\wedge]\!]$ then

$$b = [\![(\exists t \in \mathbb{R}^\wedge)(I_\mu(f) < t \wedge t < \lambda^\wedge)]\!].$$

By the mixing principle, there exist a partition $(b_\xi)_{\xi\in\Xi}$ of the element b and a family $(t_\xi)_{\xi\in\Xi} \subset \mathbb{R}$ such that $t_\xi < \lambda$ and $b_\xi \leq [\![I_\mu(f) \leq f_\xi^\wedge]\!]$ for all ξ. Hence, applying 1.3.2, we derive

$$b_\xi \underline{\sigma}(f, \beta) \leq t_\xi b_\xi < \lambda b_\xi \quad (\xi \in \Xi)$$

and further

$$\lambda_k b_\xi \mu(A_k) \leq t_\xi b_\xi \mu(A_k) < \lambda b_\xi \mu(A_k) \quad (\xi \in \Xi,\ k \in \mathbb{Z}).$$

For $k \geq 1$ we have $\lambda_k > \lambda$; therefore, $b_\xi \mu(A_k) = 0$. Thereby

$$b = \bigvee_{\xi\in\Xi} b_\xi \leq \bigwedge_{k=1}^{\infty} \mu(A_k)^* = \mu\left(T - \bigcup_{k=1}^{\infty} A_k\right) = \mu(\{f < \lambda\}).$$

On the other hand, $b^* = [\![I_\mu(f) \geq \lambda^\wedge]\!]$ and, by 1.3.2, we again infer that $\lambda b^* \leq b^* I_\mu(f) \leq b^* \overline{\sigma}(f, \beta)$ or

$$\lambda b^* \mu(A_k) \leq b^* \lambda_k \mu(A_k) \quad (k \in \mathbb{Z}).$$

For $k < 0$ we have $\lambda_k < \lambda$; therefore, $b^* \mu(A_k) = 0$. Consequently,

$$b^* \leq \bigwedge_{k=-1}^{-\infty} \mu(A_k)^* = \mu\left(T - \bigcup_{k=-1}^{-\infty} A_k\right) = \mu(\{f \geq \lambda\}).$$

This implies $b \geq \mu(\{f < \lambda\})$ and we finally obtain $b = \mu(\{f < \lambda\})$.

Assume that

$$[\![x < \lambda^\wedge]\!] = \mu(\{f < \lambda\}) \quad (\lambda \in \mathbb{R})$$

for some $x \in \mathscr{R}\downarrow$. Then by what was established above we have $[\![x < \lambda^\wedge]\!] = [\![I_\mu(f) < \lambda^\wedge]\!]$ for all $\lambda \in \mathbb{R}$. This is equivalent to the relation

$$[\![(\forall \lambda \in \mathbb{R}^\wedge)(x < \lambda \leftrightarrow I_\mu(f) < \lambda]\!] = \mathbf{1}.$$

Hence, recalling that $\mathbb{R}^\wedge$ is dense in $\mathscr{R}$, we obtain the equality $[\![x = I_\mu(f)]\!] = \mathbf{1}$ or $x = I_\mu(f)$. $\rhd$

1.4.10. *Take a measurable function $f : T \to \mathbb{R}$ and a spectral measure $\mu : \Sigma \to B := \mathfrak{E}(E)$, where E is some K-space. If the integral $I_\mu(f) \in E$ exists then $\lambda \mapsto \mu(\{f < \lambda\})$ $(\lambda \in \mathbb{R})$ coincides with the spectral function of the element $I_\mu(f)$.*

$\lhd$ We have only to compare 1.4.9 with 1.3.5. $\rhd$

1.4.11. Theorem. *Let E be a universally complete K_σ-space and $\mu : \Sigma \to B_0 := \mathfrak{E}(E)$ be some spectral measure. The spectral integral $I_\mu(\cdot)$ represents a sequential o-continuous (linear, multiplicative, and latticial) homomorphism from the f-algebra $\mathscr{M}(T, \Sigma)$ of measurable functions into E.*

$\lhd$ Without loss of generality we may assume that $E \subset \mathscr{R}{\downarrow}$ and $\mathscr{R}{\downarrow}$ is an o-completion of E (see 1.3.7). Here $\mathscr{R}$ is the field of real numbers in $\mathbf{V}^{(B)}$, where B is a completion of the algebra B_0. It is obvious that the operator I_μ is linear and positive. Prove its sequential o-continuity. Take a decreasing sequence $(f_n)_{n\in\mathbb{N}}$ of measurable functions such that $\lim_{n\to\infty} f_n(t) = 0$ for all $t \in T$, and let $x_n := I_\mu(f_n)$ $(n \in \mathbb{N})$ and $0 < \varepsilon \in \mathbb{R}$. If we assign $A_n := \{t \in T \mid f_n(t) < \varepsilon\}$ then $T = \bigcup_{n=1}^{\infty} A_n$. By Proposition 1.4.10, we can write down

$$o\text{-}\lim_{n\to\infty} e_\varepsilon^{x_n} = o\text{-}\lim_{n\to\infty} \mu(A_n) = \bigvee_{n=1}^{\infty} \mu(A_n) = \mathbf{1}.$$

Appealing to the test for o-convergence 1.3.8(2), we obtain $o\text{-}\lim_{n\to\infty} x_n = 0$. Further, given arbitrary measurable functions $f, g : T \to \mathbb{R}$, we derive from 1.4.2(9) and 1.4.10 that

$$e_\lambda^{I(f\vee g)} = \mu(\{f \vee g < \lambda\}) = \mu(\{f < \lambda\}) \wedge \mu(\{g < \lambda\}) = e_\lambda^{I(f)} \wedge e_\lambda^{I(g)} = e_\lambda^{I(f)\vee I(g)}$$

(with $I := I_\mu$); consequently, $I(f \vee g) = I(f) \vee I(g)$. It means that I_μ is a lattice homomorphism. In a similar way, for $f \geq 0$ and $g \geq 0$ it follows from 1.4.2(6) and 1.4.10 that

$$\begin{aligned} e_\lambda^{I(f\cdot g)} &= \mu(\{f \cdot g < \lambda\}) = \mu\Big(\bigcup_{\substack{r,s\in E_+\\ rs=\lambda}} \{f < r\} \cap \{g < s\}\Big) \\ &= \bigvee_{\substack{r,s\in E_+\\ rs=\lambda}} \mu(\{f < r\}) \wedge \mu(\{g < s\}) = \bigvee_{\substack{r,s\in E_+\\ rs=\lambda}} e_r^{I(f)} \wedge e_s^{I(g)} = e_\lambda^{I(f)\cdot I(g)} \end{aligned}$$

for $\lambda \in E$, $\lambda > 0$. Thus, $I(f \cdot g) = I(f) \cdot I(g)$. The validity of the latter equality for arbitrary functions f and g ensues from the above-established properties of the spectral integral:

$$\begin{aligned} I(f \cdot g) &= I(f^+g^+) + I(f^-g^-) - I(f^+g^-) - I(f^-g^+) \\ &= I(f)^+I(g)^+ + I(f)^-I(g)^- - I(f)^+I(g)^- - I(f)^-I(g)^+ \\ &= I(f) \cdot I(g). \ \rhd \end{aligned}$$

1.4.12. Below we shall need a certain fact about representation of Boolean algebras (see [56; Theorem 29.1]).

The Loomis-Sikorski theorem. *Let Q be the Stone space of a Boolean σ-algebra B. Let $\mathfrak{B}_\sigma(Q)$ be the σ-algebra of subsets of Q generated by the set $\mathfrak{B}(Q)$ of all clopen sets, and let Δ be a σ-ideal in $\mathfrak{B}_\sigma(Q)$ composed of meager sets. Then the algebra B is isomorphic to the quotient-algebra $\mathfrak{B}_\sigma(Q)/\Delta$. If ι_0 is an isomorphism of B onto $\mathfrak{B}(Q)$ then the mapping*

$$\iota : b \mapsto [\iota_0(b)]_\Delta \quad (b \in B),$$

where $[A]_\Delta$ is the equivalence class of a set $A \in \mathfrak{B}_\sigma(Q)$ by the ideal Δ, is an isomorphism of the algebra B onto the algebra $\mathfrak{B}_\sigma(Q)/\Delta$.

1.4.13. *Let $e_1, \dots, e_n : \mathbb{R} \to B$ be a finite collection of spectral functions with values in a σ-algebra B. Then there exists a unique B-valued spectral measure μ defined on the Borel σ-algebra $\mathscr{B}(\mathbb{R}^n)$ of the space $\mathbb{R}^n$ such that*

$$\mu\left(\prod_{k=1}^{n}(-\infty, \lambda_k)\right) = \bigwedge_{k=1}^{n} e_k(\lambda_k)$$

for all $\lambda_1, \dots, \lambda_n \in \mathbb{R}$.

$\lhd$ Without loss of generality we may assume that $B = \mathfrak{B}(Q)$, where Q is the Stone space of B. According to 1.1.11(1), there are continuous functions $x_k : Q \to \overline{\mathbb{R}}$ such that $e_k(\lambda) = \mathrm{cl}\{x_k < \lambda\}$ for all $\lambda \in \mathbb{R}$ and $k := 1, \dots, n$. Put $f(t) = (x_1(t), \dots, x_n(t))$ if all $x_k(t)$ are finite and $f(t) = \infty$ if $x_k = \pm\infty$ at least for one index k. Thereby we have defined a continuous mapping $f : Q \to \mathbb{R}^n \cup \{\infty\}$ (the neighborhood filterbase of the point ∞ is composed of the complements to various balls with center the origin). It is clear that f is measurable with respect to the Borel algebras $\mathscr{B}(Q)$ and $\mathscr{B}(\mathbb{R}^n)$. Let $\mathfrak{B}_\sigma(Q)$, ι, and $[\,\cdot\,]_\Delta$ be the same as in 1.4.12.

Define the mapping $\mu : \mathscr{B}(\mathbb{R}^n) \to B$ by the formula

$$\mu(A) := \iota^{-1}([f^{-1}(A)]_\Delta \quad (A \in \mathscr{B}(\mathbb{R}^n)).$$

It is obvious that μ is a spectral measure. If $A := \prod_{k=1}^n(-\infty, \lambda_k)$ then

$$f^{-1}(A) = \bigcap_{k=1}^{n} \{x_k < \lambda_k\},$$

and hence $\mu(A) = e_1(\lambda_1) \wedge \cdots \wedge e_n(\lambda_n)$. If ν is another spectral measure with the same properties as μ then the set $\mathscr{B} := \{A \in \mathscr{B}(\mathbb{R}^n) \mid \nu(A) = \mu(A)\}$ is a σ-algebra containing all sets of the form

$$\prod_{k=1}^{n}(-\infty, \lambda_k) \quad (\lambda_1, \dots, \lambda_n \in \mathbb{R}).$$

Hence, $\mathscr{B} = \mathscr{B}(\mathbb{R}^n)$. $\triangleright$

1.4.14. Now, take an ordered collection of elements $x_1, \dots, x_n$ in a K_σ-space E with unity $\mathbf{1}$. Let $e^{x_k} : \mathbb{R} \to B := \mathfrak{C}(\mathbf{1})$ denote the spectral function of the element x_k. According to the above-proven assertion, there exists a spectral measure $\mu : \mathscr{B}(\mathbb{R}^n) \to B$ such that

$$\mu\left(\prod_{k=1}^{n}(-\infty, \lambda_k)\right) = \bigwedge_{k=1}^{n} e^{x_k}(\lambda_k).$$

We can see that the measure μ is uniquely determined by the ordered collection $\mathfrak{x} := (x_1, \dots, x_n) \in E^n$. For this reason, we write $\mu_\mathfrak{x} := \mu$ and say that $\mu_\mathfrak{x}$ is the spectral measure of the collection $\mathfrak{x}$. The following notations are accepted for the integral of a measurable function $f : \mathbb{R}^n \to \mathbb{R}$ with respect to the spectral measure $\mu_\mathfrak{x}$:

$$\hat{\mathfrak{x}}(f) := f(\mathfrak{x}) := f(x_1, \dots, x_n) := I_\mu(f).$$

If $\mathfrak{x} = (x)$ then we also write $\hat{x}(f) := f(x) := I_\mu(f)$ and call $\mu_x := \mu$ the spectral measure of x. Recall that the space $\mathscr{B}(\mathbb{R}^n, \mathbb{R})$ of all Borel functions in $\mathbb{R}^n$ is a universally complete K_σ-space and a faithful f-algebra.

1.4.15. Theorem. *The spectral measures of a collection* $\mathfrak{x} := (x_1, \dots, x_n)$ *and the element* $f(\mathfrak{x})$ *are connected by the relation*

$$\mu_{f(\mathfrak{x})} = \mu_{\mathfrak{x}} \circ f^{\leftarrow},$$

where $f^{\leftarrow} : \mathscr{B}(\mathbb{R}) \to \mathscr{B}(\mathbb{R}^n)$ *is the homomorphism acting by the rule* $A \mapsto f^{-1}(A)$. *In particular,*

$$(f \circ g)(\mathfrak{x}) = g(f(\mathfrak{x}))$$

for measurable functions $f \in \mathscr{B}(\mathbb{R}^n, \mathbb{R})$ *and* $g \in \mathscr{B}(\mathbb{R}, \mathbb{R})$ *whenever* $f(\mathfrak{x})$ *and* $g(f(\mathfrak{x}))$ *exist.*

$\lhd$ By 1.4.10, we have

$$\mu_{f(\mathfrak{x})}(-\infty, t) = e_t^{f(\mathfrak{x})} = [\![f(\mathfrak{x}) < t]\!] = \mu_{\mathfrak{x}} \circ f^{-1}(-\infty, t)$$

for every $t \in \mathbb{R}$. Hence, the spectral measures $\mu_f(\mathfrak{X})$ and $\mu_{\mathfrak{X}} \circ f^{\leftarrow}$ defined on $\mathscr{B}(\mathbb{R})$ coincide on the intervals of the form $(-\infty, t)$. Afterwards, reasoning in a standard manner, we conclude that the measures coincide everywhere. To prove the second part, it suffices to observe that $(g \circ f)^{\leftarrow} = f^{\leftarrow} \circ g^{\leftarrow}$ and apply what was established above twice. $\rhd$

1.4.16. Theorem. *For every ordered collection* $\mathfrak{x} := (x_1, \dots, x_n)$ *of a universally complete* K_σ*-space* E, *the mapping*

$$\hat{\mathfrak{x}} : f \mapsto \hat{\mathfrak{x}}(f) \quad (f \in \mathscr{B}(\mathbb{R}^n, \mathbb{R}))$$

is a unique sequentially o-continuous homomorphism of the f*-algebra* $\mathscr{B}(\mathbb{R}^n, \mathbb{R})$ *into* E *satisfying the conditions*

$$\hat{\mathfrak{x}}(dt_k) = x_k \quad (k := 1, \dots, n),$$

where $dt_k : (t_1, \dots, t_n) \mapsto t_k$ *stands for the* k*th coordinate function on* $\mathbb{R}^n$.

$\lhd$ As was established in 1.4.11, the mapping $f \mapsto \hat{\mathfrak{x}}(f)$ is a sequentially o-continuous homomorphism of f-algebras. Theorem 1.4.15 yields the equalities

$$\mu_{dt_k(\mathfrak{x})} = \mu_{\mathfrak{x}} \circ (dt_k)^{\leftarrow} = \mu_{x_k}.$$

Consequently, the elements $\hat{\mathfrak{x}}(dt_k) = dt_k(\mathfrak{x})$ and x_k coincide, for they have the same spectral function. If $h : \mathscr{B}(\mathbb{R}^n, \mathbb{R}) \to E$ is another homomorphism of f-algebras with the same properties as $\hat{\mathfrak{x}}(\cdot)$ then h and $\hat{\mathfrak{x}}(\cdot)$ coincide on all polynomials. Afterwards, we infer that h and $\hat{\mathfrak{x}}(\cdot)$ coincide on the whole $\mathscr{B}(\mathbb{R}^n, \mathbb{R})$ due to o-continuity. $\rhd$

1.4.17. Theorem. *An element $x \in E$ has the form $x = f(\mathfrak{x})$ with some $\mathfrak{x} \in E^n$ and $f \in \mathscr{B}(\mathbb{R}^n, \mathbb{R})$ if and only if* $\operatorname{im}(\mu_x) \subset \operatorname{im}(\mu_{\mathfrak{x}})$.

$\lhd$ Necessity follows from 1.4.15. Sufficiency is left to the reader as an exercise. $\rhd$

1.5. Fragments of Positive Operators

In the current section we demonstrate that the combination of Boolean-valued and infinitesimal methods is very fruitful in the theory of vector lattices and positive operators. It is not perfectly clear what combination is optimal and what synthetic nonstandard analysis is desired, since there are various possibilities of combining technical tools. Therefore, we dwell upon a concrete but important question of calculating fragments of positive operators which can be studied in considerable detail by systematically applying nonstandard methods.

1.5.1. First we formulate some basic statements for the reader's comfort. Given a set A in a K-space, we denote by $A^{\vee}$ the union of A and the suprema of all its nonempty finite sets. The symbol $A^{(\uparrow)}$ denotes the result of adjoining to A the suprema of all increasing nonempty nets in A. The symbols $A^{(\uparrow\downarrow)}$ and $A^{(\uparrow\downarrow\uparrow)}$ are interpreted in a natural way.

Let E be a vector lattice, let F be a K-space, and let U be a positive operator from E into F. Given an element $e \in E_+$, introduce an operator $\pi_e U$ by the formulas

$$(\pi_e U)x := \sup_{n \in \mathbb{N}} U(x \wedge ne) \quad (x \in E_+);$$

$$(\pi_e U)x := (\pi_e U)x^+ - (\pi_e U)x^- \quad (x \in E).$$

It is easy to see that $\pi_e U \in L^{\sim}(E, F)$. Moreover, $\pi_e U$ is a fragment of U and the mapping $U \mapsto \pi_e U$ $(U \geq 0)$ extends naturally to $L^{\sim}(E, F)$ to become a band projection. If $\rho \in \mathfrak{P}(F)$ then we denote the band projection $U \mapsto \rho U$ in the K-space $L^{\sim}(E, F)$ by the same letter ρ.

(1) *The Boolean algebra $\mathfrak{C}(U)$ of fragments can be reconstructed from fragments of the form $(\rho \circ \pi_e)U$ by the formula*

$$\mathfrak{C}(U) = \{(\rho \circ \pi_e)U \mid \rho \in \mathfrak{P}(F),\ e \in E_+\}^{\vee(\uparrow\downarrow\uparrow)}.$$

The set $\mathscr{P}$ of all band projections in the K-space $L^{\sim}(E, F)$ is *generating* provided that $Ux^+ = \sup\{(\pi U)x \mid \pi \in \mathscr{P}\}$ for all $U \in L^{\sim}(E, F)_+$ and $x \in E$. Take positive operators U and V in $L^{\sim}(E,F)$ and the principal band projection W of V onto $\{U\}^{\perp\perp}$.

(2) *If $\mathscr{E}$ is the order-unit filter of F then*

$$Wx = \sup_{e\in\mathscr{E}} \inf\{\pi Vy + \pi^{\perp}Vx \mid 0 \le y \le x,\ \pi \in \mathfrak{P}(F),\ \pi U(x-y) \le \varepsilon\}$$

for every $x \in E_+$.

(3) *If $\mathscr{P}$ is a generating set of band projections in $L^{\sim}(E,F)$ then*

$$Wx = \sup_{e\in\mathscr{E}} \inf\{(\pi P)^{\perp}Vx \mid \pi PUx \le \varepsilon,\ P \in \mathscr{P},\ \pi \in \mathfrak{P}(F)\}$$

for $x \in E_+$.

1.5.2. Now, we will substantiate the above and other analogous formulas. First we examine the case of functionals, employing the methods of infinitesimal analysis. We shall use the neoclassical stance due to E. Nelson. A more detailed exposition of necessary information can be found in [16, 37, 43] (see also [28, 50, 51, 53, 60]). Here we confine ourselves to the next brief remarks. Without special stipulations, we agree to work in the standard entourage; i.e., while using the theory of internal sets, all free variables in a formal expression are assumed to be standard. The sign $\approx$ has the routine meaning in a K-space F: $x \approx y$ for $x, y \in F$ stands for $(\forall^{\mathrm{st}} e \in \mathscr{E})\, |x-y| \le e$ ($\mathscr{E}$ is the order-unit filter of F). It is clear that if $F = \mathbb{R}$ then $x - y$ is *infinitesimal* in the conventional sense of nonstandard analysis [37].

Let E be a vector space over a dense subfield $\widehat{\mathbb{R}}$ of the field $\mathbb{R}$. Further, let $q : E \to \mathbb{R}$ be a sublinear functional and let A be a *generating* set for q; i.e.,

$$q(x) = \sup\{f(x) \mid f \in A\}\ (x \in E).$$

Denote by τ the topology of pointwise convergence on elements of E in $E^{\#} := L(E,\mathbb{R})$, the algebraic dual of E. By the classical Milman theorem, for $\widehat{\mathbb{R}} = \mathbb{R}$ we have $\operatorname{ext}(q) \subset \operatorname{cl}_{\tau}(A)$ for the set $\operatorname{ext}(q)$ of extreme points of the *subdifferential*

$$\partial q := \{f \in E^{\#} \mid (\forall x \in E)\ f(x) \le q(x)\}.$$

The conclusion, the Milman converse of the Kreĭn-Milman theorem, is also valid in the case under consideration.

1.5.3. Theorem. *Every extreme point of the subdifferential ∂q lies in the τ-closure of a generating set for q.*

$\triangleleft$ It is clear that τ is a locally convex topology in the vector space $E^{\#}$ over the field $\mathbb{R}$. Moreover, ∂q is τ-compact by Tychonoff's theorem. Denote by D the τ-closure of the convex hull of A. Obviously, $D := \mathrm{cl}_\tau(\mathrm{co}(A))$ is a τ-compact convex set. Assume that some element $\bar{f} \in \partial q$ does not lie in D. By the separation theorem, there is a τ-continuous linear functional φ over $E^{\#}$ such that

$$\sup\{\varphi(f) \mid f \in D\} = \varphi(f_0) < r < \varphi(\bar{f})$$

for $f_0 \in D$ and $r \in \mathbb{R}$. By the continuity of φ,

$$|\varphi(f)| \le t|f(x_1)| \vee \dots \vee |f(x_n)|$$

for some $x_1, \dots, x_n \in E$ and $t \in \mathbb{R}$ and for all $f \in E^{\#}$. Thereby, $\varphi(f) = \alpha_1 f(x_1) + \dots \alpha_n f(x_n)$ for suitable $\alpha_1, \dots, \alpha_n \in \mathbb{R}$. Working in the standard entourage, choose $\widehat{\alpha}_1, \dots, \widehat{\alpha}_n \in \mathbb{R}$ infinitely close to $\alpha_1, \dots, \alpha_n$. Observe also that $f(x_k) \in {}^{\mathrm{fin}}\mathbb{R}$ by the hypothesis that x_k is standard and the inequality $q(-x_k) \le f(x_k) \le q(x_k)$; i.e., $f(x_k)$ is a finite number for an arbitrary $f \in \partial q$ and $k := 1, \dots, n$. Put

$$x := \sum_{k=1}^{n} \widehat{\alpha}_k x_k.$$

Then

$$\varphi(f) = f(x) + \sum_{k=1}^{n} (\alpha_k - \widehat{\alpha}_k) f(x_k) \approx f(x)$$

for $f \in \partial q$, because $\alpha_k - \widehat{\alpha}_k$ is infinitesimal for $k = 1, \dots, n$. Hence, $\varphi(f) + \varepsilon \ge f(x)$ for every standard $\varepsilon > 0$. Thus, for such an $\varepsilon > 0$, the following estimates hold:

$$q(x) = \sup\{f(x) \mid f \in A\} \le \sup\{\varphi(f) + \varepsilon \mid f \in A\} \le \varphi(f_0) + \varepsilon.$$

Hence, ${}^\circ q(x) \le \varphi(f_0) < r$. On the other hand,

$$(\forall^{\mathrm{st}} \varepsilon > 0)\ r \le \varphi(\bar{f}) \le \bar{f}(x) + \varepsilon \le q(x) + \varepsilon.$$

Consequently, ${}^\circ q(x) \ge r > {}^\circ q(x)$, a contradiction. Thus, $D = \partial q$ and $\mathrm{cl}_\tau(A) \supset \mathrm{ext}(q)$ by the above-mentioned Kreĭn-Milman theorem. $\triangleright$

1.5.4. Fix some set $\mathscr{P}$ of band projections and the corresponding set $\mathscr{P}(f) := \{pf \mid p \in \mathscr{P}\}$ of the fragments of a positive functional f in a vector lattice E over a dense subfield of $\mathbb{R}$ (with unity).

The following assertions are equivalent:

(1) $\mathscr{P}(f)^{\vee(\uparrow\downarrow\uparrow)} = \mathfrak{C}(t)$;

(2) $\mathscr{P}$ *generates the fragments of* f;

(3) $(\forall x \in {}^{\circ}E)(\exists p \in \mathscr{P})\, pf(x) \approx f(x^{+})$;

(4) *a functional* g *in* $[0, f]$ *is a fragment of* f *if and only if*

$$\inf_{p \in \mathscr{P}} ((p^{\perp}g)(x) + p(f-g)(x)) = 0$$

for every $x \in E_{+}$;

(5) $(\forall g \in {}^{\circ}\mathfrak{C}(f))(\forall x \in {}^{\circ}E_{+})(\exists p \in \mathscr{P})\, |pf - g|(x) \approx 0$;

(6) $\inf\{|pf - g|(x) \mid p \in \mathscr{P}\} = 0$ *for every fragment* $g \in \mathfrak{C}(f)$ *and every positive element* $x \geq 0$;

(7) *for* $x \in E_{+}$ *and* $g \in \mathfrak{C}(f)$, *there exists an element* $p \in \mathscr{P}(f)^{\vee(\uparrow\downarrow)}$, *providing the equality* $|pf - g|(x) = 0$.

The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are beyond questions.

(3) $\Rightarrow$ (4): We shall work in the standard entourage. First of all we observe that validity of the required inequality for some functionals g and f such that $0 \leq g \leq f$ yields, for a standard $x \geq 0$, the existence of $p \in \mathscr{P}$ for which $p^{\perp}g(x) \approx 0$ and $p(f-g)(x) \approx 0$. Thereby,

$${}^{\circ}p(g \wedge (f-g))(x) \leq {}^{\circ}p(f-g)(x) = 0$$

and

$${}^{\circ}p^{\perp}((f-g) \wedge g)(x) \leq {}^{\circ}p^{\perp}g(x) = 0;$$

i.e., $g \wedge (f-g) = 0$.

Now, establish that, under conditions (3), the required equality is provided by the conventional criterion for disjointness:

$$\inf_{\substack{x_1 \geq 0, x_2 \geq 0 \\ x_1 + x_2 = x}} (g(x_1) + (f-g)(x_2)) = 0.$$

Fixing a standard x, find internal positive elements x_1 and x_2 such that $x_1 + x_2 = x$ and, moreover, $g(x_1) \approx 0$ and $f(x_2) \approx g(x_2)$. In virtue of (3), the fragment

g lies in the weak closure of $\mathscr{P}(f)$ by 1.5.3. In particular, there is an element $p \in \mathscr{P}$ for which $g(x_1) \approx pf(x_1)$ and $g(x_2) \approx pf(x_2)$. Thus, $p^\perp g(x_2) \approx 0$ since $p^\perp g \leq p^\perp f$. Finally, $p^\perp g(x) \approx 0$. Hence,

$$p(f-g)(x) = pf(x_2) + pf(x_1) - pg(x) \approx g(x_2) + g(x_1) - pg(x) \approx p^\perp g(x) \approx 0,$$

which guarantees the required inequality.

(4) $\Rightarrow$ (5): Making use of the identity

$$|pf - g|(x) = p^\perp g(x) + p(f-g)(x)$$

and choosing $p \in \mathscr{P}$ such that $p^\perp g(x) \approx 0$ and $p(f-g)(x) \approx 0$, we prove the claim.

The equivalence (5) $\Leftrightarrow$ (6) is obvious. The implications (5) $\Rightarrow$ (7) $\Leftrightarrow$ (1) can be checked by the arguments exposed in [5, 40].

1.5.5. *The set $\mathscr{P}$ of band projections is generating if and only if the following representations hold for arbitrary positive functionals f and g and for every point $x \geq 0$:*

$$(f \vee g)(x) = \sup_{p \in \mathscr{P}} (pf(x) + p^\perp g(x));$$

$$(f \wedge g)(x) = \inf_{p \in \mathscr{P}} (pf(x) + p^\perp g(x)).$$

$\lhd$ This is a straightforward consequence of 1.5.4. $\rhd$

1.5.6. *For positive functionals f and g and a generating set $\mathscr{P}$ of band projections, the following assertions are equivalent:*

(1) $g \in \{f\}^{\perp\perp}$;

(2) *for every finite* $x \in {}^{\mathrm{fin}}E := \{x \in E \mid (\exists \bar{x} \in {}^\circ E)\ \ |x| \leq \bar{x}\}$ *whenever* $pf(x) \approx 0$ *for* $p \in \mathscr{P}$;

(3) $(\forall x \in E_+)(\forall \varepsilon > 0)(\exists \delta > 0)(\forall p \in \mathscr{P})\ \ pf(x) \leq \delta \rightarrow pg(x) \leq \varepsilon$.

$\lhd$ (1) $\Rightarrow$ (2): Employing, for instance, the classical Robinson lemma [37], take an infinitely large natural $N \approx +\infty$ such that $Npf(x) \approx 0$ for a positive finite vector x. Observe that $g(x) \approx (g \wedge Nf)(x)$ for such N since g coincides with its principal band projection onto $\{f\}^{\perp\perp}$. Hence, we conclude that $pg(x) \approx 0$, considering the relations

$$pg(x) = p(g - g \wedge Nf)(x) + p(g \wedge Nf)(x) \leq (g - Nf)(x) + Npf(x). \ \rhd$$

By applying the Nelson algorithm, we see that (2) is equivalent to the following assertion:

(2′) $(\forall^{\mathrm{st}} x \in E)(\forall p \in \mathscr{P})\ pf(x) \approx 0 \rightarrow pg(x) \approx 0.$

Thus, by (2) $\Rightarrow$ (2′), it remains to establish that (2′) $\Rightarrow$ (1).

(2′) $\rightarrow$ (1): Take a functional h such that $h \wedge f = 0$. Given a standard $x \in E_+$, by virtue of 1.5.4(4), there is an element $p \in \mathscr{P}$ for which $ph(x) \approx 0$ and $p^\perp f(x) \approx 0$. By (2′), we have $p^\perp g(x) \approx 0$. Consequently,

$$(h \wedge g)(x) \leq {}^\circ(ph(x) + p^\perp g(x)) = 0.$$

Grounding on 1.5.4(4), we conclude that $h \wedge g = 0$; i.e., $g \in \{f\}^{\perp\perp}$ by the arbitrariness of h. $\triangleright$

1.5.7. Theorem. *Let f and g be positive functionals on E and let x be a positive element of E. The following representations hold for the principal band projection π_f onto $\{f\}^{\perp\perp}$:*

(1) $\pi_f g(x) \rightleftharpoons \inf{}^*\{{}^\circ pg(x) \mid p^\perp f(x) \approx 0,\ p \in \mathscr{P}\}$

(the sign $\rightleftharpoons$ means that the formula is exact; i.e., the equality is attained);

(2) $\pi_f g(x) = \sup_{\varepsilon>0} \inf\{pg(x) \mid p^\perp f(x) \leq \varepsilon,\ p \in \mathscr{P}\}$;

(3) $\pi_f g(x) \rightleftharpoons \inf{}^*\{{}^\circ g(y) \mid f(x-y) \approx 0,\ 0 \leq y \leq x\}$;

(4) $(\forall \varepsilon > 0)(\exists \delta > 0)(\forall p \in \mathscr{P})\ pf(x) < \delta \rightarrow \pi_f g(x) \leq p^\perp g(x) + \varepsilon$;
$(\forall \varepsilon > 0)(\forall \delta > 0)(\exists p \in \mathscr{P})\ pf(x) \leq \delta \wedge p^\perp g(x) \leq \pi_f g(x) + \varepsilon$;

(5) $(\forall \varepsilon > 0)(\exists \delta > 0)(\forall 0 \leq y \leq x)\ f(x-y) \leq \delta \rightarrow \pi_f g(x) \leq g(y) + \varepsilon$;
$(\forall \varepsilon > 0)(\forall \delta > 0)(\exists 0 \leq y \leq x)\ f(x-y) \leq \delta \wedge g(y) \leq \pi_f g(x) + \varepsilon$.

$\triangleleft$ Put $h := \pi_f g$ for brevity. It is clear that $h(x) \leq g(x)$ and so $pg(x) \geq ph(x)$. If $p^\perp f(x) \approx 0$ then $p^\perp h(x) \approx 0$ and thus $h(x) = {}^\circ ph(x) \leq {}^\circ pg(x)$. Consequently, every standard element of the external set $\{{}^\circ pg(x) \mid p \in \mathscr{P},\ p^\perp f(x) \approx 0\}$ dominates $h(x)$. By the transfer principle, we conclude that the left-hand side in (1) does not exceed the corresponding right-hand side. To prove that the equality in (1) is attained, we observe that $f \wedge (g - h) = 0$. Thus, by 1.5.5, $p^\perp f(x) \approx 0$ for some $p \in \mathscr{P}$ and so $pg(x) \approx ph(x)$. Considering that $h \in \{f\}^{\perp\perp}$ and grounding on 1.5.6(2), we derive that $p^\perp h(x) \approx 0$. Finally,

$$pg(x) \approx ph(x) + p^\perp h(x) = h(x).$$

Thereby, $h(x) = {}^\circ pg(x)$ and (1) is proven.

To prove (2), take $\delta > 0$ and working in the standard entourage deduce the following:

$$\begin{aligned}\inf\{pg(x) \mid p^{\perp}f(x) \le \varepsilon\} &\le \inf{}^*\{pg(x)+\delta \mid p^{\perp}f(x) \le \varepsilon\}\\ &\le \inf{}^*\{{}^\circ pg(x) \mid p^{\perp}f(x) \approx 0\}+\delta\\ &= h(x)+\delta.\end{aligned}$$

By the arbitrariness of δ, we conclude that

$$h(x) \ge \sup_{\varepsilon>0} \inf\{pg(x) \mid p^{\perp}f(x) \le \varepsilon\}.$$

Fixing a standard number $\delta > 0$ again, we obtain the internal property

$$\inf\{pg(x) \mid p^{\perp}f(x) \le \varepsilon\}+\delta \ge h(x)$$

for every infinitesimal $\varepsilon > 0$ grounding on (1). Indeed, the inequality $p^{\perp}f(x) \le \varepsilon$ yields the relation $p^{\perp}f(x) \approx 0$ and thus $pg(x)+\delta \ge {}^\circ pg(x) \ge h(x)$. By the Cauchy principle, the above-mentioned internal property holds for some strictly positive standard ε. By making use of the transfer principle, we finally derive

$$(\forall\delta)(\exists\varepsilon > 0)\ h(x)-\delta \le \inf\{pg(x) \mid p^{\perp}f(x) \le \varepsilon\}$$

which completes the proof of (2).

To check (3), we take the proof of (1) as a pattern. Namely, if $0 \le y \le x$ and $f(x-y) \approx 0$ then $h(x-y) \approx 0$, since $h(x) = h(y)+h(x-y) \le g(y)+h(x-y)$ and $h \in \{f\}^{\perp\perp}$; therefore, $h(x) \le {}^\circ g(y)$. To establish exactness in (3), we use the equality $f\wedge(g-h) = 0$. This implies $f(x-y) \approx 0$ and $h(y) \approx g(y)$ for some $y \in [0,x]$. Since $h \in \{f\}^{\perp\perp}$, we have $h(x) \approx h(y)$ by 1.5.7. Thus, $h(x) = {}^\circ g(y)$. Assertions (4) and (5) can be verified similarly by applying the Nelson algorithm. Carry out the calculations, for instance, for (5). To this end, decipher the contents of (3). It comprises, first, some inequality and, second, the exactness of the inequality. By analyzing the inequality, we deduce

$$\begin{aligned}&(\forall 0 \le y \le x)\ f(x-y) \approx 0 \to h(x) \le {}^\circ g(y)\\ \leftrightarrow&(\forall^{\mathrm{st}}\varepsilon > 0)(\forall 0 \le y \le x)\ f(x-y) \approx 0 \to h(x) \le g(y)+\varepsilon\\ \leftrightarrow&(\forall^{\mathrm{st}}\varepsilon > 0)(\forall 0 \le y \le x)(\exists^{\mathrm{st}}\delta > 0)\ (f(x-y) \le \delta \to h(x) \le g(y)+\varepsilon)\\ \leftrightarrow&(\forall^{\mathrm{st}}\varepsilon > 0)(\exists^{\mathrm{st}}\delta > 0)(\forall 0 \le y \le x)\ f(x-y) \le \delta \to h(x) \le g(y)+\varepsilon.\end{aligned}$$

Considering the assertion about exactness, we have

$$
\begin{aligned}
&(\exists y)\ (0 \le y \le x) \wedge f(x-y) \approx 0 \wedge h(x) = {}^\circ g(y)\\
\leftrightarrow &(\exists y)(0 \le y \le x) \wedge (\exists^{\mathrm{st}} \delta > 0)\ f(x-y) \le \delta \wedge (\forall^{\mathrm{st}} \varepsilon > 0)\ |h(x) - g(y)| \le \varepsilon\\
\leftrightarrow &(\forall^{\mathrm{st}} \varepsilon > 0)(\forall^{\mathrm{st}} \delta > 0)(\exists y)(0 \le y \le x \wedge f(x-y) \le \delta \wedge |h(x) - g(y)| \le \varepsilon).
\end{aligned}
$$

Referring twice to the transfer principle completes the proof. $\rhd$

1.5.8. Thus, we have described the tools for generating fragments of functionals and exposed the representations of principal band projections. The general case of positive operators can be analyzed by ascending into a Boolean-valued universe and descending the obtained results for functionals. We need the following auxiliary facts:

(1) *Let $f : A \times B \to F$ be an extensional mapping and let $f_D(a) := \sup f(a, D)$ for $a \in A$ and $D \subset B$. Then the mapping $f_D : A \to F$ is extensional too; moreover, $f_D{\uparrow} = f{\uparrow}_{D\uparrow}$.*

$\lhd$ In virtue of the general rules of ascent, we successively have for $a \in A$:

$$
\begin{aligned}
f_D(a) &= \sup f(a, D) = \sup f(a, D){\uparrow} = \sup f((\{a\} \times D){\uparrow})\\
&= \sup f{\uparrow}((\{a\} \times D){\uparrow}) = \sup f{\uparrow}(\{a\}{\uparrow} \times D{\uparrow})\\
&= \sup f{\uparrow}(\{a\} \times D{\uparrow}) = \sup f{\uparrow}(a, D{\uparrow}) = f{\uparrow}_{D\uparrow}(a).
\end{aligned}
$$

Since $f{\uparrow}$ is a function inside the Boolean-valued universe under consideration, we derive

$$
\begin{aligned}
&[\![a_1 = a_2]\!] \le [\![f{\uparrow}_{D\uparrow}(a_1) = f{\uparrow}_{D\uparrow}(a_2)]\!]\\
&\qquad = [\![\sup f{\uparrow}(a_1, D{\uparrow}) = \sup f{\uparrow}(a_2, D{\uparrow})]\!] = [\![f_D(a_1) = f_D(a_2)]\!]
\end{aligned}
$$

by the above-proven relation for $a_1, a_2 \in A$. Thus, f_D is an extensional function. Moreover,

$$
[\![f_D{\uparrow}(a) = f{\uparrow}_{D\uparrow}(a)]\!] = [\![f_D(a) = f{\uparrow}_{D\uparrow}(a)]\!] = \mathbf{1} \quad (a \in A). \ \rhd
$$

(2) Consider the standard name $E^\wedge$ of E in the separated Boolean-valued universe $\mathbf{V}^{(B)}$ over $B := \mathfrak{P}(F)$. Observe that $E^\wedge$ is a vector lattice over the standard name $\mathbb{R}^\wedge$ of the field $\mathbb{R}$. Moreover, $\mathbb{R}^\wedge$ is a dense subfield of $\mathscr{R}$ inside $\mathbf{V}^{(B)}$. As usual, $\mathscr{R} = F\uparrow$ is the field of real numbers inside $\mathbf{V}^{(B)}$. We execute ascents of mappings from E into F up to mapping from $E^\wedge$ into $\mathscr{R}$ inside $\mathbf{V}^{(B)}$ by the general rules. As is easily verified,

$$E^{\wedge\sim}\downarrow = L(E^\wedge, \mathscr{R})\downarrow = \{U\uparrow \mid U \in L^\sim(E,F)\}.$$

The descended structures make $E^{\wedge\sim}\downarrow$ into a K-space and even into a universally complete (extended) module over the orthomorphism algebra [37]. Moreover, we in fact arrive at the above-studied scalar situation. For the sake of completeness, let us explicate some necessary typical instances.

(3) Recall that, given $U \in L^\sim(E,F)$, the ascent $U\uparrow$ is defined by the rule $[\![U\uparrow x^\wedge = Ux]\!] = 1$ for $x \in E$. Moreover, $U\uparrow$ becomes a regular functional on $E^\wedge$; namely, an element of $E^{\wedge\sim}$ inside $V^{(B)}$. The mapping $U\uparrow \mapsto (PU)\uparrow$ $(U \in L^\sim(E,F))$ is extensional for $P \in \mathscr{P}$. Indeed, for $\pi \in B$ we have

$$\begin{aligned}\pi \leq [\![U_1\uparrow = U_2\uparrow]\!] &\to (\forall x \in E)\ \pi \leq [\![U_1\uparrow x^\wedge = U_2\uparrow x^\wedge]\!] \to (\forall x \in E)\ \pi U_1 x = \pi U_2 x \\ &\to (\forall x \in E)\ \pi P U_1 x = \pi P U_2 x \to \pi \leq [\![(PU_1)\uparrow = (PU_2)\uparrow]\!].\end{aligned}$$

In such a way the ascent $P\uparrow$ is defined to be the band projection in $E^{\wedge\sim}$ inside $\mathbf{V}^{(B)}$ acting by the rule $P\uparrow U\uparrow = (PU)\uparrow$ for $U \in L^\sim(E,F)$.

(4) It is worth observing that $(U \wedge V)\uparrow = U\uparrow \vee V\uparrow$ inside $\mathbf{V}^{(B)}$ for $U, V \in L^\sim(E,F)_+$. Indeed, recalling that $[\![(U \wedge V)\uparrow \leq U\uparrow \wedge V\uparrow]\!] = \mathbf{1}$, we derive

$$\begin{aligned}[\![(U \wedge V)\uparrow = U\uparrow \wedge V\uparrow]\!] &= [\![U\uparrow \wedge V\uparrow \leq (U \wedge V)\uparrow]\!] \\ &= [\![(\forall W \in E^{\wedge\sim})\ \ W \leq U\uparrow \wedge W \leq V\uparrow \to W \leq (U \wedge V)\uparrow]\!] \\ &= \bigwedge_{W \in L^\sim(E,F)_+} [\![W\uparrow \leq U\uparrow \wedge W\uparrow \leq V\uparrow \to W\uparrow \leq (U \wedge V)\uparrow]\!].\end{aligned}$$

Put

$$\pi := [\![W\uparrow \leq U\uparrow]\!] \wedge [\![W\uparrow \leq V\uparrow]\!].$$

Undoubtedly, we have $\pi W \leq \pi U$ and $\pi W \leq \pi V$. Thus, $\pi W \leq \pi(U \wedge V)$. Hence,

$$\begin{aligned}[\![W\uparrow \leq (U \wedge V)\uparrow]\!] &= [\![(\forall x \in E^{\wedge\sim}, x \geq 0)\, W\uparrow x \leq (U \wedge V)\uparrow x]\!] \\ &= \bigwedge_{x \in E_+} [\![Wx \leq (U \wedge V)x]\!] \geq \pi;\end{aligned}$$

i.e., the truth-value in question equals unity. In other words, the mapping $W \in L^{\sim}(E,F) \mapsto W{\uparrow} \in E^{\wedge\sim}{\downarrow}$ implements an isomorphism between the structures of $L^{\sim}(E,F)$ and $E^{\wedge\sim}{\downarrow}$. Thereby, V is a fragment of U if and only if $V{\uparrow}$ is a fragment of $U{\uparrow}$ inside $\mathbf{V}^{(B)}$.

1.5.9. *The following assertions are equivalent for a set $\mathscr{P}$ of band projections in $L^{\sim}(E,F)$ and for $U \in L^{\sim}(E,F)_+$:*

(1) $\mathscr{P}(U)^{\vee(\uparrow\downarrow\uparrow)} = \mathfrak{E}(U)$;

(2) *$\mathscr{P}$ generates the fragments of U;*

(3) *an operator $V \in [0,U]$ is a fragment of U if and only if*

$$\inf_{P\in\mathscr{P}} (P^{\perp}Vx + P(U-V)x) = 0$$

for every $x \in E_+$;

(4) $(\forall x \in {}^{\circ}E)(\exists P \in \mathscr{P}{\uparrow\downarrow})\ PUx \approx Ux^+$.

$\lhd$ First consider the ascent $\mathscr{P}{\uparrow}$ defined as $\mathscr{P}{\uparrow} := \{P{\uparrow} \mid P \in \mathscr{P}\}{\uparrow}$. By 1.5.8, $\mathscr{P}$ generates the fragments of U is and only if $\mathscr{P}{\uparrow}$ generates the fragments of $U{\uparrow}$ inside $\mathbf{V}^{(B)}$. This establishes (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) as a matter of fact.

At last, prove (2) $\Leftrightarrow$ (4). To this end, using the definitions and the rules of assigning truth-values, we successively derive for $x \in E$:

$$\begin{aligned}
&[\![U{\uparrow}x^{\wedge+} = \sup\{(PU{\uparrow})x^{\wedge} \mid P \in \mathscr{P}{\uparrow}]\!] = \mathbb{1}\\
&\quad\leftrightarrow [\![(\forall\varepsilon > 0)(\exists P \in \mathscr{P}{\uparrow})\ (PU{\uparrow})x^{\wedge} + \varepsilon \geq Ux^+]\!] = \mathbb{1}\\
&\quad\leftrightarrow (\forall\varepsilon \in \mathscr{E}) \bigvee_{P\in\mathscr{P}} [\![(Ux^+ - P{\uparrow}U{\uparrow}x^{\wedge}) \leq \varepsilon]\!] = \mathbb{1}\\
&\quad\leftrightarrow (\forall\varepsilon \in \mathscr{E}) \bigvee_{P\in\mathscr{P}} [\![(Ux^+ - PUx) \leq \varepsilon]\!] = \mathbb{1}\\
&\quad\leftrightarrow (\forall\varepsilon \in \mathscr{E})(\exists(P_\xi))(\exists(\pi_\xi))(\forall\xi)\ \ \pi_\xi(Ux^+ - P_\xi Ux) \leq \varepsilon\\
&\quad\leftrightarrow (\forall^{\mathrm{st}}\varepsilon \in \mathscr{E})(\exists(P_\xi))(\exists(\pi_\xi))(\forall\xi)\ \ \pi_\xi(Ux^+ - P_\xi Ux) \leq \varepsilon\\
&\quad\leftrightarrow (\exists(P_\xi))(\exists(\pi_\xi))(\forall\xi)(\forall^{\mathrm{st}}\varepsilon \in \mathscr{E})\ \ \pi_\xi(Ux^+ - P_\xi Ux) \leq \varepsilon\\
&\quad\leftrightarrow (\exists(P_\xi))(\exists(\pi_\xi))(\forall\xi)\ \ \pi_\xi(Ux^+ - P_\xi Ux) \approx 0.
\end{aligned}$$

Here we have used natural notations for a family (P_ξ) of elements of $\mathscr{P}$ and a partition (π_ξ) of unity in B. Mixing (P_ξ) with probabilities (π_ξ) as P, we arrive at the claim.

1.5.10. (1) *A set $\mathscr{P}$ is generating if and only if the following relations are valid for every $U, V \in L^{\sim}(E,F)_+$ and $x \in E$:*

$$(U \vee V)x = \sup_{P \in \mathscr{P}} \{(PU)x + (P^{\perp}V)x\};$$

$$(U \wedge V)x = \inf_{P \in \mathscr{P}} \{(PU)x + (P^{\perp}V)x\}.$$

$\triangleleft$ This fact is an obvious consequence of 1.5.9 (or 1.5.5 by means of $\mathbf{V}^{(B)}$). $\triangleright$

(2) *The set $\mathscr{P} := \{\pi_e \mid e \in E_+\}$ of band projections is generating. In particular, Proposition* 1.5.1(1) *holds.*

$\triangleleft$ It suffices to observe that if $e := x^+$ then $\pi_e U x = \pi_e U x^+ = U x^+$ and to apply 1.5.9(4). The second part of the assertion ensues from 1.5.9(1). $\triangleright$

1.5.11. *The following assertions are equivalent for positive operators U and V and a generating set $\mathscr{P}$ of band projections in $L^{\sim}(E,F)$:*

(1) $V \in \{U\}^{\perp\perp}$;

(2) $(\forall x \in {}^{\mathrm{fin}}E)(\forall P \in \mathscr{P})(\forall \pi \in B)\ \pi P U x \approx 0 \to \pi P V x \approx 0$;

(3) $(\forall x \in {}^{\mathrm{fin}}E)(\forall \pi \in B)\ \pi U x \approx 0 \to \pi V x \approx 0$;

(4) $(\forall x \geq 0)(\forall \varepsilon \in \mathscr{E})(\exists \delta \in \mathscr{E})(\forall P \in \mathscr{P})(\forall \pi \in B)\ \pi P U x \leq \delta \to \pi P V x \leq \varepsilon$;

(5) $(\forall x \geq 0)(\forall \varepsilon \in \mathscr{E})(\exists \delta \in \mathscr{E})(\forall \pi \in B)\ \pi U x \leq \delta \to \pi V x \leq \varepsilon$.

$\triangleleft$ We omit the proof since this fact will not be used in the sequel. $\triangleright$

1.5.12. Theorem. *Let E be a vector lattice and let F be a K-space with order-unit filter $\mathscr{E}$ and base B. Further, let U and V be positive operators in $L^{\sim}(E,F)$ and let W be the principal band projection of V onto $\{U\}^{\perp\perp}$. The following representations hold for a positive $x \in E$:*

(1) $Wx = \sup\limits_{\varepsilon \in \mathscr{E}} \inf\{\pi V y + \pi^{\perp} U x \mid 0 \leq y \leq x,\ \pi \in B,\ \pi U(x-y) \leq \varepsilon\}$;

(2) $Wx = \sup\limits_{\varepsilon \in \mathscr{E}} \inf\{(\pi P)^{\perp} V x \mid \pi P U x \leq \varepsilon,\ P \in \mathscr{P},\ \pi \in B\}$, *where $\mathscr{P}$ is a generating set of band projections in $L^{\sim}(E,F)$.*

$\triangleleft$ Descend into the Boolean-valued universe $\mathbf{V}^{(B)}$ over the Boolean algebra $B = \mathfrak{P}(F)$. Considering 1.5.8, we see that $W{\uparrow}$ serves as the principal band projection of $V{\uparrow}$ onto $\{U{\uparrow}\}^{\perp\perp}$ in $E^{\wedge\sim}$ inside $\mathbf{V}^{(B)}$, since the band $\{U{\uparrow}\}^{\perp\perp}$ inside $\mathbf{V}^{(B)}$

coincides with ascent of the image of the band $\{U\}^{\perp\perp}$ under the ascent of mappings. Now, working within $\mathbf{V}^{(B)}$ and employing the first part of 1.5.7, we derive for $x \in E_+$:

$$\begin{aligned}
&[\![(\forall \varepsilon > 0)(\exists \delta > 0)(\forall y \in E^\wedge)\\
&(0 \le y \le x^\wedge \wedge U\!\uparrow\!(x^\wedge - y) \le \delta) \to W\!\uparrow\! x^\wedge \le V\!\uparrow\! y + \varepsilon]\!] = \mathbf{1}\\
\leftrightarrow&(\forall \varepsilon \in \mathscr{E})(\exists \delta \in \mathscr{E})(\forall y \in E)\ [\![0 \le y^\wedge \le x^\wedge\\
&\wedge U\!\uparrow\!(x^\wedge - y^\wedge) \le \delta) \to W\!\uparrow\! x^\wedge \le V\!\uparrow\! y^\wedge + \varepsilon]\!] = \mathbf{1}\\
\leftrightarrow&(\forall \varepsilon \in \mathscr{E})(\exists \delta \in \mathscr{E})(\forall 0 \le y \le x)\ [\![U(x-y) \le \delta \to Wx \le Vy + \varepsilon]\!] = \mathbf{1}\\
\leftrightarrow&(\forall \varepsilon \in \mathscr{E})(\exists \delta \in \mathscr{E})(\forall 0 \le y \le x)\ [\![U(x-y) \le \delta]\!] \le [\![Wx \le Vy + \varepsilon]\!]\\
\leftrightarrow&(\forall \varepsilon \in \mathscr{E})(\exists \delta \in \mathscr{E})(\forall 0 \le y \le x)(\forall \pi \in B)\\
&[\![U(x-y) \le \delta]\!] \ge \pi \to [\![Wx \le Vy + \varepsilon]\!] \ge \pi\\
\leftrightarrow&(\forall \varepsilon \in \mathscr{E})(\exists \delta \in \mathscr{E})(\forall 0 \le y \le x)(\forall \pi \in B)\\
&\pi U(x-y) \le \delta \to \pi W x \le \pi V y + \varepsilon\\
\leftrightarrow&(\forall \varepsilon \in \mathscr{E})(\exists \delta \in \mathscr{E})(\forall 0 \le y \le x)(\forall \pi \in B)\\
&\pi U(x-y) \le \delta \to Wx \le \pi V y + \pi^\perp V x + \varepsilon.
\end{aligned}$$

Put $r(\delta) := \inf\{\pi V y + \pi^\perp V x \mid \pi U(x-y) \le \delta,\ \pi \in B,\ 0 \le y \le x\}$. With this notation, it is evident that

$$(\forall \varepsilon \in \mathscr{E})(\exists \delta \in \mathscr{E})\ Wx \le r(\delta) + \varepsilon \to Wx \le \sup\{r(\delta) \mid \delta \in \mathscr{E}\}.$$

Analogously, we derive from the second part of 1.5.7(5):

$$\begin{aligned}
&[\![(\forall \varepsilon > 0)(\forall \delta > 0)(\exists 0 \le y \le x^\wedge)\ U\!\uparrow\!(x^\wedge - y) \le \delta \wedge V\!\uparrow\! y \le W\!\uparrow\! x^\wedge + \varepsilon]\!] = \mathbf{1}\\
\leftrightarrow&(\forall \varepsilon \in \mathscr{E})(\forall \delta \in \mathscr{E}) \bigvee_{0 \le y \le x} [\![U(x-y) \le \delta \wedge Vy \le Wx + \varepsilon]\!] = \mathbf{1}\\
\leftrightarrow&(\forall \varepsilon \in \mathscr{E})(\forall \delta \in \mathscr{E})(\exists (y_\xi))(\exists (\pi_\xi))(\forall \xi) \pi_\xi U(x - y_\xi) \le \delta \wedge \pi_\xi V y_\xi \le \pi_\xi W x + \varepsilon
\end{aligned}$$

for a family (y_ξ) of elements of the interval $[0, x]$ and a partition (π_ξ) of unity in B. Obviously, we have

$$r(\delta) \le \pi_\xi V y_\xi + \pi_\xi^\perp V x$$

for all indicated parameters. Hence,

$$\pi_\xi r(\delta) \leq \pi_\xi V y_\xi \leq \pi_\xi W x + \varepsilon$$

for every ξ and thus $r(\delta) \leq Wx + \varepsilon$. By the arbitrariness of ε, we derive

$$\sup\{r(\delta) \mid \delta \in \mathscr{E}\} \leq Wx.$$

This fact, together with the above-proven reverse inequality, yields (1).

Formula (2) can be obtained by the same pattern as (1). We only ought to take it into account that $\mathscr{P}\!\uparrow := \{P\!\uparrow \mid P \in \mathscr{P}\}\!\uparrow$ generates the set of band projections in $E^{\wedge\sim}$ inside $\mathbf{V}^{(B)}$. Observe also the following useful identities:

$$(\pi P)^\perp Q = Q - \pi PQ = \pi Q - \pi PQ + \pi^\perp Q = \pi(Q - PQ) + \pi^\perp Q = (\pi P)^\perp Q + \pi^\perp Q.$$

Finally, calculating the truth-values of the variants of 1.5.7(4) translated into $\mathbf{V}^{(B)}$, we derive for a positive $x \in E_+$ that

$$(\forall \varepsilon \in \mathscr{E})(\exists \delta \in \mathscr{E})(\forall P \in \mathscr{P})(\forall \pi \in B)\, \pi PUx \leq \delta \to \pi P^\perp Vx + \pi^\perp Vx + \varepsilon \geq Wx,$$

$$(\forall \varepsilon \in \mathscr{E})(\forall \delta \in \mathscr{E})(\exists (P_\xi))(\exists (\pi_\xi))\ \pi_\xi P_\xi Ux \leq \delta \wedge \pi_\xi P_\xi^\perp Vx \leq \pi_\xi Wx + \varepsilon$$

for an appropriate family (P_ξ) of elements of $\mathscr{P}$ and a partition (π_ξ) of unity in B. $\triangleright$

1.6. Lattice-Normed Spaces

Many objects studied in functional analysis lead to spaces normed by means of a vector lattice. A lattice-normed space becomes a normed space after ascending into a Boolean-valued model. A discussion of the resulting interplay constitutes the content of the current section.

1.6.1. Consider a vector space X and a real vector lattice E. We will assume all vector lattices to be Archimedean without further stipulations. A mapping $p : X \to E_+$ is called an *(E-valued) vector norm* if it satisfies the following axioms:

(1) $p(x) = 0 \leftrightarrow x = 0$ $(x \in X)$,

(2) $p(\lambda x) = |\lambda| p(x)$ $(\lambda \in \mathbb{R},\ x \in X)$,

(3) $p(x + y) \leq p(x) + p(y)$ $(x, y \in X)$.

A vector norm p is said to be a *decomposable* or *Kantorovich norm* if

(4) for arbitrary $e_1, e_2 \in E_+$ and $x \in X$, the equality $p(x) = e_1 + e_2$ implies the existence of $x_1, x_2 \in X$ such that $x = x_1 + x_2$ and $p(x_k) = e_k$ $(k := 1, 2)$.

The triple (X, p, E) (simpler, X or (X, p) with the implied parameters omitted) is called a *lattice-normed space* if p is an E-valued norm on the vector space X. If the norm p is decomposable then the space (X, p) itself is called *decomposable.* Take a net $(x_\alpha)_{\alpha \in \mathrm{A}}$ in X. We say that (x_α) *o-converges* to an element $x \in X$ and write $x = o\text{-}\lim x_\alpha$ provided that there exists a decreasing net $(e_\gamma)_{\gamma \in \Gamma}$ in E such that $\inf_{\gamma \in \Gamma} e_\gamma = 0$ and, for every $\gamma \in \Gamma$, there exists an index $\alpha(\gamma) \in \mathrm{A}$ such that $p(x - x_\alpha) \le e_\gamma$ for all $\alpha \ge \alpha(\gamma)$. Let $e \in E_+$ be an element satisfying the following condition: for an arbitrary $\varepsilon > 0$, there exists an index $\alpha(\varepsilon) \in \mathrm{A}$ such that $p(x - x_\alpha) \le \varepsilon e$ for all $\alpha \ge \alpha(\varepsilon)$. Then we say that (x_α) *r-converges* to x and write $x = r\text{-}\lim x_\alpha$. We say that a set (x_α) is *o-fundamental* (*r-fundamental*) if the net $(x_\alpha - x_\beta)_{(\alpha,\beta) \in \mathrm{A} \times \mathrm{A}}$ o-converges (r-converges) to zero. A lattice-normed space is *o-complete* (*r-complete*) if every o-fundamental (r-fundamental) net in it o-converges (r-converges) to some element of the space.

Take a net $(x_\xi)_{\xi \in \Xi}$ and relate to it a net $(y_\alpha)_{\alpha \in \mathrm{A}}$, where $\mathrm{A} := \mathscr{P}_{\mathrm{fin}}(\Xi)$ is the collection of all finite subsets of Ξ and $y_\alpha := \sum_{\xi \in \alpha} x_\xi$. If $x := o\text{-}\lim y_\alpha$ exists then we call (x_ξ) *o-summable* with *sum* x and write $x = o\text{-}\sum_{\xi \in \Xi} x_\xi$.

A set $M \subset X$ is called *bounded in norm* or *norm-bounded* if there exists $e \in E_+$ such that $p(x) \le e$ for all $x \in M$. A space X is said to be *d-complete* if every bounded set of pairwise disjoint elements in X is o-summable.

Let F be an order-dense ideal in E. Then the set $Y := \{x \in X \mid p(x) \in F\}$ is a vector space. If q is the restriction of p to Y then (Y, q, F) is a lattice-normed space called the *restriction* of X with respect to F or *F-restriction* of X for short.

1.6.2. We call elements $x, y \in X$ *disjoint* and write $x \perp y$ whenever $p(x) \wedge p(y) = 0$. Obviously, the relation $\perp$ satisfies all axioms of disjointness (see 0.1.9 in [2]). The complete Boolean algebra $\mathfrak{B}(X) := \mathfrak{K}_1(X)$ is called the *base* of X. It is easy to see that a band $K \in \mathfrak{B}(X)$ is a subspace of X. In fact, $K = h(L) := \{x \in X \mid p(x) \in L\}$ for some band L in E. The mapping $L \mapsto h(L)$ is a Boolean homomorphism from $\mathfrak{B}(E)$ onto $\mathfrak{B}(X)$. We call a norm p (or the whole space X) *d-decomposable* provided that, for every $x \in X$ and disjoint $e_1, e_2 \in E_+$, there exist $x_1, x_2 \in X$ such that $x = x_1 + x_2$ and $p(x_k) = e_k$ $(k := 1, 2)$. Recall that, speaking of a Boolean algebra of projections in a vector space X, we always mean a set of commuting idempotent linear operators with the following Boolean operations:

$$\pi \vee \rho = \pi + \rho - \pi \circ \rho, \quad \pi \wedge \rho = \pi \circ \rho, \quad \pi^* = I_X - \pi.$$

Theorem. *Let $E_0 := p(X)^{\perp\perp}$ be a lattice with projections and let X be an d-decomposable space. Then there exist a complete Boolean algebra $\mathscr{B}$ of projections in X and an isomorphism h from $\mathfrak{P}(E_0)$ onto $\mathscr{B}$ such that*

$$\pi \circ p = p \circ h(\pi) \quad (\pi \in \mathfrak{P}(E_0)).$$

$\lhd$ The mapping $L \mapsto h(L)$ $(L \in \mathfrak{B}(E_0))$ implements an isomorphism between the Boolean algebras $\mathfrak{B}(E_0)$ and $\mathfrak{B}(X)$ since X is d-decomposable and it is possible to project onto the bands of E_0. Moreover, given $K \in \mathfrak{B}(X)$, the band $K^\perp$ is the algebraic complement of K; i.e., $K \cap K^\perp = \{0\}$ and $K + K^\perp = X$. Consequently, there exists a projection $\pi_K : X \to X$ onto the band K along $K^\perp$.

Put $\mathscr{B} := \{\pi_K \mid K \in \mathfrak{B}(X)\}$. Then $\mathscr{B}$ is a complete Boolean algebra isomorphic to $\mathfrak{B}(X)$. We associate with $\rho \in \mathfrak{P}(E_0)$ the projection $\pi_K \in \mathscr{B}$, where $K := h(\rho E_0)$, and the so-obtained mapping $\rho \mapsto \pi_K$ is denoted by the same letter h. Then h is an isomorphism of $\mathfrak{P}(E_0)$ onto $\mathscr{B}$.

Take $\pi \in \mathfrak{P}(E_0)$ and $x \in X$. By the definition of h, we have $h(\pi)x \in h(\pi E_0)$ or $p(h(\pi)x) \in \pi E_0$; therefore, $\pi^* p(h(\pi)x) = 0$. Thus, $\pi p h(\pi) = p h(\pi)$. Further, we observe that $p(x+y) = p(x) + p(y)$ for disjoint $x, y \in X$. Indeed, the inequality $p(x) \le p(x+y) + p(y)$ yields $p(x) \le p(x+y)$, since $p(x) \perp p(y)$. In a similar way, $p(y) \le p(x+y)$. But then $p(x) + p(y) = p(x) \vee p(y) \le p(x+y)$. For $x \in X$, we may write down

$$p(x) = p(h(\pi)x + h(\pi^*)x) = p(h(\pi)x) + p(h(\pi^*)x).$$

Making use of the above-proven equality $\pi p h(\pi^*) = 0$, we obtain

$$\pi p(x) = \pi p(h(\pi)x) \quad (x \in X);$$

i.e., $\pi p = \pi p h(\pi)$. Finally,

$$\pi p = \pi p h(\pi) = p h(\pi) \quad (\pi \in \mathfrak{B}(E_0)). \ \rhd$$

1.6.3. A decomposable o-complete lattice-normed space is a *Banach-Kantorovich space*. Assume that (Y, q, F) is a Banach-Kantorovich space and $F = q(Y)^{\perp\perp}$.

One can show that F is a K-space and $q(Y) = F_+$. By 1.6.2, the Boolean algebras $\mathfrak{P}(F)$ and $\mathfrak{P}(Y)$ can be identified and $\pi q = q\pi$ $(\pi \in \mathfrak{P}(F))$.

For every bounded family $(x_\xi)_{\xi\in\Xi}$ in Y and a partition $(\pi_\xi)_{\xi\in\Xi}$ of unity in $\mathfrak{P}(Y)$, there exists $x := o\text{-}\sum_{\xi\in\Xi}\pi_\xi x_\xi$. Moreover, x is a unique element satisfying the relations $\pi_\xi x = \pi_\xi x_\xi$ $(\xi \in \Xi)$.

$\lhd$ Put $e := \sup p(x_\xi)$. Then, for $\alpha, \beta \in \mathscr{P}_{\mathrm{fin}}(\Xi)$, we have

$$q(y_\alpha - y_\beta) = q\Big(\sum_{\xi\in\alpha\Delta\beta}\pi_\xi x_\xi\Big) \le \Big(\sum_{\xi\in\alpha\Delta\beta}\pi_\xi\Big)e \le e,$$

where $y_\gamma = \sum_{\xi\in\gamma}\pi_\xi x_\xi$ and $\alpha\Delta\beta$ is the symmetric difference of α and β. Hence, the net (y_α) is o-fundamental. Consequently, there exists an $x := o\text{-}\lim y_\alpha$. $\rhd$

The proposition particularly involves the d-completeness of Y. Moreover, from its definitions it is immediate that Y is also r-complete.

If $F = mF$ then the space Y is called *universally complete.* This is equivalent to the fact that every disjoint family in Y is o-summable. A space Y is called a *universal completion* of a lattice-normed space (X, p, E) provided that (1) $F = mE$ (and consequently Y is universally complete); (2) there is a linear isometry $\iota : X \to Y$; (3) if Z is a decomposable o-complete subspace of Y and $\operatorname{im}\iota \subset Z$ then $Z = Y$. Later (in 1.6.7) we demonstrate that every lattice-normed space possesses a universal completion.

1.6.4. Examples.

(1) Put $X := E$ and $p(x) := |x|$ $(x \in X)$. Then p is a decomposable norm.

(2) Let Q be a topological space and let Y be a normed space. Let $X := C_b(Q, Y)$ be the space of continuous bounded vector-valued functions from Q into Y. Put $E := C_b(Q, \mathbb{R})$. Given $f \in X$, we introduce a vector norm $p(f)$ as follows:

$$p(f) : t \mapsto \|f(t)\| \quad (t \in Q).$$

Then p is decomposable and X is r-complete if and only if Y is a Banach space.

(3) Let (Ω, Σ, μ) be a measure space with σ-finite measure, let Y be a normed space, and let E be an order-dense ideal in $M(\Omega, \Sigma, \mu)$. Denote by $M(\mu, Y)$ the space of equivalence classes of μ-measurable vector-valued functions acting from Ω into Y. As usual, vector-functions are equivalent if they take equal values at almost all points of Ω. If $z \in M(\mu, Y)$ is the equivalence class of a measurable function $z_0 : \Omega \to Y$ then denote by $p(z) := |z|$ the equivalence class of the measurable scalar function $t \mapsto \|z_0(t)\|$ $(t \in \Omega)$. By definition, assign

$$E(Y) := \{z \in M(\mu, Y) \mid p(z) \in E\}.$$

Then $(E(Y), p, E)$ is a lattice-normed space with decomposable norm. If Y is a Banach space then $E(Y)$ is a Banach-Kantorovich space and $M(\mu, Y)$ is a universal completion of it.

(4) Take the same E and Y as above and consider a norming space $Z \subset Y'$, i.e., a subspace such that

$$\|y\| = \sup\{\langle y, y'\rangle \mid y' \in Z,\ \|y'\| \leq 1\} \quad (y \in Y).$$

Here Y' stands for the dual space and $\langle \cdot, \cdot \rangle$ is the canonical duality bracket $Y \leftrightarrow Y'$. A vector-function $z : \Omega \to Y$ is said to be *Z-measurable* if the function $t \mapsto \langle z(t), y'\rangle$ $(t \in \Omega)$ is measurable for every $y' \in Z$. Denote by $\langle z, y'\rangle$ the equivalence class of the last function. Let $\mathscr{M}$ be the set of all Z-measurable vector-functions z for which the set $\{\langle z, y'\rangle \mid y' \in Z,\ \|y'\| \leq 1\}$ is bounded in $M(\Omega, \Sigma, \mu)$. Denote by $\mathscr{N}$ the set of all $z \in \mathscr{M}$ such that the measurable function $t \mapsto \langle z(t), y'\rangle$ equals zero almost everywhere for each $y' \in Z$; i.e., $\langle z', y\rangle = 0$. Given $z \in \mathscr{M}/\mathscr{N}$, we put

$$p(z) := |z| := \sup\{\langle u, y'\rangle \mid y' \in Z,\ \|y'\| \leq 1\},$$

where u_h is an arbitrary representative of the class z and the supremum is calculated in the K-space $M(\Omega, \Sigma, \mu)$. Now, we define the space

$$E_s(Y, Z) := \{z \in \mathscr{M}/\mathscr{N} \mid p(z) \in E\}$$

with the decomposable E-valued norm p. If Y is a Banach space then $E_s(Y, Z)$ is a Banach-Kantorovich space.

(5) Suppose that E is an order-dense ideal in the universally complete K-space $C_\infty(Q)$, where Q is an extremal compact set. Let $C_\infty(Q, Y)$ be the set of equivalence classes of continuous vector-valued functions u acting from comeager subsets of $\mathrm{dom}(u) \subset Q$ into a normed space Y. A set is said to be *comeager* if its complement is meager. Vector-valued functions u and v are equivalent if $u(t) = v(t)$ for $t \in \mathrm{dom}(u) \cap \mathrm{dom}(v)$. Given $z \in C_\infty(Q, Y)$, there exists a unique function $x_z \in C_\infty(Q)$ such that $\|u(t)\| = x_z(t)$ $(t \in \mathrm{dom}(u))$ whatever a representative u of the class z might be. Put $p(z) := |z| := x_z$ and

$$E(Y) := \{z \in C_\infty(Q) \mid p(z) \in E\}.$$

Let Z be the same as in (4). Denote by $\mathscr{M}$ the set of all $\sigma(Y, Z)$-continuous vector-functions $u : Q_0 := \mathrm{dom}(u) \to Y$ such that $\mathrm{dom}(u)$ is a comeager set in Q and the set $\{\langle u, y' \rangle \mid y' \in Z, \ \|y'\| \leq 1\}$ is bounded in the K-space $C_\infty(Q)$. Here $\langle u, y' \rangle$ is the unique continuous extension of the function

$$t \mapsto \langle u(t), y' \rangle \quad (t \in Q_0)$$

to the whole Q. Consider the quotient set $\mathscr{M}/\sim$, where $u \sim v$ means that $u(t) = v(t)$ $(t \in \mathrm{dom}(u) \cap \mathrm{dom}(v))$. Given $z \in \mathscr{M}/\sim$, we put

$$p(z) := \sup\{\langle u, y' \rangle \mid y' \in Z, \ \|y'\| \leq 1\},$$

$$E_s(Y, Z) := \{z \in \mathscr{M}/\sim \mid p(z) \in E\}.$$

We can naturally furnish the sets $C_\infty(Q, Y)$ and $\mathscr{M}/\sim$ with the structure of a module over the ring $C_\infty(Q)$. Moreover, $E(Y)$ and $E_s(Y, Z)$ are lattice-normed spaces with decomposable norm. If Y is a Banach space then $E(Y)$ and $E_s(Y, Z)$ are Banach-Kantorovich spaces. Moreover, $C_\infty(Q, Y)$ is a universal completion of $E(Y)$.

(6) Let (X, p, E) and (Y, q, F) be lattice-normed spaces. A linear operator $T : X \to Y$ is called *dominated* if there exists a positive operator $S : E \to F$ (called a *dominant* of T) such that

$$q(Tx) \leq S(p(x)) \quad (x \in X).$$

If F is a Kantorovich space and the norm p is decomposable then there exists a least element $|T|$ in the set of all dominants with respect to the order in the space $L^\sim(E, F)$ of regular operators. The mapping $T \mapsto |T|$ $(T \in M(X, Y))$ is a vector norm on the space $M(X, Y)$ of all dominated operators from X into Y. If Y is a Banach-Kantorovich space and the norm in X is decomposable then $M(X,Y)$ is a Banach-Kantorovich space with the above-mentioned dominant norm.

Distinguish the following two instances. First, take $E := \mathbb{R}$ and $Y := F$. Then X is a normed space and the fact that an operator $T : X \to F$ is dominated means that the set

$$\{Tx \mid x \in X,\ \|x\| \le 1\}$$

is bounded in F. The least upper bound of this set is called the *abstract norm* of T and is denoted by $|T|$ (the notation agrees with what was introduced above provided that the spaces F and $L^{\sim}(\mathbb{R}, F)$ are identified). In this situation we say that T is an operator with abstract norm.

Now, let E and F be order-dense ideals in the same K-space. An operator $T \in M(X,Y)$ is called *bounded* if $|T| \in \mathrm{Orth}(E,F)$. Denote by $\mathscr{L}_b(X,Y)$ the space of all bounded operators. It is clear that T belongs to $\mathscr{L}_b(X,Y)$ if and only if there exists $c \in mE = mF$ such that $c \cdot E \subset F$ and $q(Tx) \le cp(x)$ $(x \in X)$, where we mean the multiplicative structure in mE uniquely determined by the choice of unity (see 1.3.7(8)).

1.6.5. *Theorem.* *Let $(\mathscr{X}, \rho)$ be a Banach space in the model $\mathbf{V}^{(B)}$. Put $X := \mathscr{X}{\downarrow}$ and $p := \rho{\downarrow}$. The following assertions hold:*

(1) *$(X, p, \mathscr{R}{\downarrow})$ is a universally complete Banach-Kantorovich space.*

(2) *The space X can be furnished with the structure of a faithful unitary module over the ring $\mathscr{C}{\downarrow}$ so that*

(a) $(\lambda 1)x = \lambda x \quad (\lambda \in \mathbb{C},\ x \in X)$,

(b) $p(ax) = |a| p(x) \quad (a \in \mathscr{C}{\downarrow},\ x \in X)$,

(c) $b \le [\![x = 0]\!] \leftrightarrow \chi(b)x = 0 \quad (b \in B,\ x \in X)$, *where χ is an isomorphism from B onto $\mathfrak{P}(\mathscr{R}{\downarrow})$.*

$\lhd$ It is easy to show that X is a universally complete (extended) abelian group (see [37; 4.2.7]). Moreover, $A := \mathscr{C}{\downarrow}$ is a complex commutative algebra with unity 1 (see 1.3.9). We denote the sum operation in $\mathscr{X}$, X, $\mathscr{C}$, and A by the same sign $+$. Temporarily denote by $\odot$ the external composition law $\mathscr{C} \times \mathscr{X} \to \mathscr{X}$ of the complex vector space $\mathscr{X}$ as well as the multiplication in $\mathscr{C}$. Let $\cdot : A \times X \to X$ be the descent of the mapping $\odot$. Then $[\![a \cdot x = a \odot x]\!] = \mathbf{1}$ for all $a \in A$ and $x \in X$ (see 1.2.9(7)). Considering the axioms of a vector space to be valid for $\mathscr{X}$, inside the model we can write down

$$a \cdot (x + y) = a \odot (x + y) = a \odot x + a \odot y = a \cdot x + a \cdot y,$$

$$(a + b) \cdot x = (a + b) \odot x = a \odot x + b \odot x = a \cdot x + b \cdot x,$$

$$(ab) \cdot x = (ab) \odot x = a \odot (b \odot x) = a \cdot (b \cdot x),$$

$$\mathbf{1} \cdot x = \mathbf{1} \odot x = x \quad (a, b \in A;\ x, y \in X).$$

In view of the separatedness of $\mathbf{V}^{(B)}$ these relations imply that the operations $+$ and $\cdot$ determine the structure of a unitary A-module over X. Putting

$$\lambda x := (\lambda \mathbf{1}) \cdot x \quad (\lambda \in \mathbb{C},\ x \in X),$$

we obtain the structure of a complex vector space over X with equality (a). Since

$$\chi(b) = \mathbf{1} \to \chi(b) \odot x = x,$$

$$\chi(b) = \mathbf{0} \to \chi(b) \odot x = \mathbf{0}$$

in the model $\mathbf{V}^{(B)}$, by the definition of χ (see 1.3.2) we have

$$b \leq [\![\chi(b) \odot x = x]\!] = [\![\chi(b) \cdot x = x]\!],$$

$$b^* \leq [\![\chi(b) \odot x = 0]\!] = [\![\chi(b) \cdot x = 0]\!].$$

If we put $\chi(b)x = 0$ in the first relation then $b \leq [\![x = 0]\!]$. Conversely, if $b \leq [\![x = 0]\!]$ then

$$b \leq [\![x = 0]\!] \wedge [\![\chi(b)x = x]\!] \leq [\![\chi(b)x = 0]\!]$$

which, together with the second of the above relations, yields $\chi(b)x = 0$. Now we turn to studying Banach properties of the space $(\mathscr{X}, \rho)$. The subadditivity and homogeneity of the norm ρ can be written as

$$\rho \circ + \leq + \circ (\rho \times \rho), \quad \rho \circ \odot = \odot \circ (|\cdot| \times \rho),$$

where $\rho \times \rho : (x, y) \mapsto (\rho(x), \rho(y))$ and $|\cdot| \times \rho : (a, x) \mapsto (|a|, \rho(x))$. Taking into account the rules of descent for superposition (see 1.2.9(4)), we obtain

$$p \circ + \leq + \circ (p \times p), \quad p \circ (\cdot) = (\cdot) \circ (|\cdot| \times p).$$

It means that the operator $p : X \to A_+$ satisfies 1.6.1(3) and condition (b). But then 1.6.2(2) is also valid in view of (a). If $p(x) = 0$ for some $x \in X$ then $[\![\rho(x) = 0]\!] = \mathbf{1}$ in virtue of $[\![\rho(x) = p(x)]\!] = \mathbf{1}$ and so $[\![x = 0]\!] = \mathbf{1}$ or $x = 0$. Thus, p is a vector norm. The decomposability of p follows from property (b). Indeed, suppose that

$$c := p(x) = c_1 + c_2 \quad (x \in X;\ c_1, c_2 \in A_+).$$

There exist $a_1, a_2 \in A_+$ such that $a_k c = c_k$ $(k := 1, 2)$ and $a_1 + a_2 = 1$ (it suffices to put $a_k := c_k(c + (1 - e_c))^{-1}$, where e_c is the trace of c). If $x_k := a_k \cdot x$ $(k := 1, 2)$ then $x = x_1 + x_2$ and $p(x_k) = p(a_k x) = a_k p(x) = c_k$.

It remains to prove the o-completeness of X. Take an o-fundamental net $s : \mathrm{A} \to X$. If $\bar{s}(\alpha, \beta) := s(\alpha) - s(\beta)$ $(\alpha, \beta \in \mathrm{A})$ then $o\text{-}\lim p \circ \bar{s}(\alpha, \beta) = 0$. Let $\sigma : \mathrm{A}^\wedge \to \mathscr{X}$ be the modified ascent of s and $\bar{\sigma}(\alpha, \beta) := \sigma(\alpha) - \sigma(\beta)$ $(\alpha, \beta \in \mathrm{A}^\wedge)$. Then $\bar{\sigma}$ is the modified ascent of $\bar{s}$ and $\rho \circ \bar{\sigma}$ is the modified ascent of $p \circ \bar{s}$. By 1.3.5, we have $[\![\lim \rho \circ \bar{\sigma} = 0]\!] = 1$; i.e., $\mathbf{V}^{(B)} \models$ "σ is a fundamental net in $\mathscr{X}$." Since $\mathscr{X}$ is a Banach space inside $\mathbf{V}^{(B)}$, by the maximum principle, there exists $x \in X$ such that $[\![\lim \rho \circ \sigma_0 = 0]\!] = \mathbf{1}$, where $\sigma_0 : \mathrm{A}^\wedge \to \mathscr{X}$ is defined as $\sigma_0(\alpha) := \sigma(\alpha) - x$ $(\alpha \in \mathrm{A}^\wedge)$. The net $s_0 : \alpha \mapsto s(\alpha) - x$ $(\alpha \in \mathrm{A})$ is the modified descent of σ_0. Consequently, $o\text{-}\lim p \circ s_0 = 0$ and $o\text{-}\lim p(s(\alpha) - x) = 0$ by 1.3.4. $\triangleright$

The universally complete Banach-Kantorovich space

$$\mathscr{X}{\downarrow} := (\mathscr{X}, \rho){\downarrow} := (\mathscr{X}{\downarrow}, \rho{\downarrow})$$

is referred to as the *descent* of a Banach space $(\mathscr{X}, \rho)$.

1.6.6. Theorem. *For every lattice-normed space (X, p, E), there exists a Banach space $\mathscr{X}$ inside $\mathbf{V}^{(B)}$ unique up to a linear isometry, where $B \simeq \mathfrak{B}(p(X)^{\perp\perp})$, such that the descent $\mathscr{X}{\downarrow}$ is a universal completion of (X, p, E).*

$\triangleleft$ Without loss of generality we may assume that $E = p(X)^{\perp\perp} \subset mE = \mathscr{R}{\downarrow}$ and $B = \mathfrak{B}(E)$. Put

$$d(x, y) := p(x - y)^{\perp\perp} \quad (x, y \in X).$$

It is easy to verify that d is a B-metric on X. If we furnish the field $\mathbb{C}$ with the discrete B-metric d_0 then the addition $+ : X \times X \to X$ and the multiplication

$\cdot : \mathbb{C} \times X \to X$ become nonexpanding mappings. So is the vector norm p. All these assertions are almost obvious. Thus, for the multiplication we have

$$d(\alpha x, \beta y) = p(\alpha x - \beta y)^{\perp\perp} \le (|\alpha| p(x-y))^{\perp\perp} \vee (|\alpha - \beta| p(y))^{\perp\perp} \le d(x,y) \vee d_0(\alpha, \beta).$$

for $\alpha, \beta \in \mathbb{C}$ and $x, y \in X$.

Let $\mathscr{X}_0$ be a Boolean-valued realization of the B-set (X, d) (see 1.2.12(2)). Put $\rho_0 := \mathscr{F}^{\sim}(p)$, $\oplus := \mathscr{F}^{\sim}(+)$, and $\odot := \mathscr{F}^{\sim}(\cdot)$, where $\mathscr{F}^{\sim}$ is the immersion functor (see 1.2.12(2, 3)). The mappings $\oplus$ and $\odot$ determine in $\mathscr{X}_0$ the structure of a vector lattice over the field $\mathbb{C}^{\wedge}$ and the function $\rho_0 : \mathscr{X}_0 \times \mathscr{X}_0 \to \mathscr{R}$ is a norm. In virtue of the maximum principle, there exist elements $\mathscr{X}, \rho \in \mathbf{V}^{(B)}$ such that $[\![(\mathscr{X}, \rho)$ is a complex Banach space being a completion of the Banach space $(\mathscr{X}_0, \rho_0)]\!] = \mathbf{1}$. Moreover, we may assume that $[\![\mathscr{X}_0$ is a dense $\mathbb{C}^{\wedge}$-subspace of $\mathscr{X}]\!] = \mathbf{1}$. Let $\iota : X \to X_0 := X_0{\downarrow}$ be the canonical immersion (see 1.2.12(2)). Since $+$ is a nonexpanding mapping from $X \times X$ into X, the sum in X_0, i.e. $+ := \oplus{\downarrow}$, is uniquely determined by the equality $\iota \circ + = + \circ (\iota \times \iota)$, where $\iota \times \iota := (x, y) \mapsto (\iota x, \iota y)$ is the canonical immersion of the B-set $X \times X$. The last is equivalent to the additivity of ι. Analogously, for the operation $(\cdot) := \odot{\downarrow}$, we have $\iota \circ (\cdot) = (\cdot) \circ (\varkappa \times \iota)$, where

$$\varkappa \times \iota : (\lambda, x) \mapsto (\lambda^{\wedge}, \iota x) \quad (\lambda \in \mathbb{C},\ x \in X).$$

Thus, ι is a linear operator. Applying once again the same arguments to $p_0 := \rho_0{\downarrow}$, we obtain $\iota_E \circ \rho_0 = p_0 \circ \iota$, where ι_E is the canonical immersion of E. It means that ι is an isometry; i.e., preserves the vector norm. Consider a subspace $Y \subset \mathscr{X}{\downarrow}$, $\iota X \subset Y$, which is a universally complete Banach-Kantorovich space under the norm

$$q(y) := \rho{\downarrow}(y) \quad (y \in Y).$$

Since q is decomposable and Y disjointly complete, we have $X_0 \subset Y$. Indeed, $X_0 = \operatorname{mix}(\iota X)$ and, by condition (c) in 1.6.5(2), $x = \operatorname{mix}(b_\xi \iota x_\xi)$ for $x \in \mathscr{X}{\downarrow}$ if and only if $x = o\text{-}\sum \chi(b_\xi)\iota x_\xi$. On the other hand, Y is decomposable and d-complete; thus, Y is invariant under each projection $x \mapsto \chi(b)x$ and contains all sums of this form by 1.6.3. Analogously, $Y = \operatorname{mix} Y$. If $\mathscr{Y} := Y{\uparrow}$ then $[\![\mathscr{X}_0 \subset \mathscr{Y} \subset \mathscr{X}]\!] = \mathbf{1}$; moreover, $\mathscr{Y}{\downarrow} = Y$. Let $\sigma : \omega^{\wedge} \to \mathscr{Y}$ be a Cauchy sequence and let s be its modified descent. Then s is an o-fundamental sequence in Y, thus, there exists $y = \lim s$. It is seen from 1.3.4 that $[\![y = \lim \sigma]\!] = 1$. This fact establishes the completeness of $\mathscr{Y}$ and consequently the relations $\mathscr{X} = \mathscr{Y}$ and $X = Y$.

Let $\mathscr{Z}$ be a Banach space inside $\mathbf{V}^{(B)}$ such that $\mathscr{Z}{\downarrow}$ is a universal completion of the lattice-normed space X. If ι' is the corresponding embedding of X into $\mathscr{Z}{\downarrow}$ then $\iota' \circ \iota$ can be uniquely extended to a linear isometry of X_0 onto a disjointly complete subspace $Z_0 \subset \mathscr{Z}{\downarrow}$. The spaces $\mathscr{X}_0$ and $\mathscr{Z}_0 := Z_0{\uparrow}$ are isometric. But then their completions $\mathscr{X}$ and $\mathscr{Y} \subset \mathscr{Z}$ are isometric too. Since $\mathscr{Y}{\downarrow}$ is a Banach-Kantorovich space and $\iota' X \subset \mathscr{Y}{\downarrow} \subset \mathscr{Z}{\downarrow}$, we have $\mathscr{Y}{\downarrow} = \mathscr{Z}{\downarrow}$. Therefore, $Y = Z$ and thus $\mathscr{X}$ and $\mathscr{Z}$ are linearly isometric.

1.6.7. Corollaries.

(1) *Every lattice-normed space (X, p, E) possesses a universal completion (mX, p_m, mE) unique to within a linear isometry. Moreover, for arbitrary $x \in mX$ and $\varepsilon > 0$, there exist a family $(x_\xi)_{\xi\in\Xi}$ in X and a partition $(\pi_\xi)_{\xi\in\Xi}$ of unity in $\mathfrak{P}(mX)$ such that*

$$p_m\Big(x - o\text{-}\sum_{\xi\in\Xi} \pi_\xi \iota x_\xi\Big) \leq \varepsilon p(x).$$

(2) *A lattice-normed space is linearly isomorphic to an order-dense ideal of a universal completion of it if and only if it is decomposable and o-complete, i.e., is a Banach-Kantorovich space.*

$\lhd$ It is convenient to prove both assertions together. By making use of the notation from 1.6.6, we put $mX := \mathscr{X}{\downarrow}$ and $p_m := \rho{\downarrow}$. Then (mX, p_m, mE, ι) is a universal completion of X. Take an $x \in mX$. Without loss of generality, we may assume that $e := p_m(x)$ is a unity in mE. Since $\mathscr{X}_0$ is dense in $\mathscr{X}$, for every $\varepsilon > 0$, there is an element $x_\varepsilon \in \mathbf{V}^{(B)}$ such that

$$[\![x_\varepsilon \in \mathscr{X}_0]\!] = [\![\rho(x - x_\varepsilon) \leq \varepsilon^\wedge \cdot e]\!] = \mathbf{1}$$

by the maximum principle. Hence, $x_\varepsilon \in X_0$ and $p_m(x - x_\varepsilon) \leq \varepsilon e$. It remains to observe that $X_0 = \operatorname{mix}(\iota X)$; therefore, x_ε has the form

$$\sum_{\xi\in\Xi} \pi_\xi \iota x_\xi,$$

where $(x_\xi) \subset X$ and (π_ξ) is a partition of unity in $\mathfrak{P}(mX)$.

Obviously, an order dense ideal in a Banach-Kantorovich space is decomposable and o-complete. Conversely, let X be a decomposable o-complete lattice-normed space. One can show that $E_0 = p(X)^{\perp\perp}$ is a K-space. Therefore, we in no way loose generality on assuming E_0 to be an order-dense ideal in $\mathscr{R}{\downarrow}$. Let $x \in mX$ and $p_m(x) \in E_0$. By (1), there exists a sequence $(x_n) \subset X_0$ such that

$$p_m(x_n - x) \le \frac{1}{n}e, \quad p_m(x_n) \le \left(1 + \frac{1}{n}\right) e \quad (n \in \omega).$$

Hence, $x_n \in X$ and $x \in X$, since an o-complete space is d-complete and r-complete. Thereby,

$$X = \{x \in mX \mid p_m(x) \in E_0\};$$

i.e., X is a order dense ideal in mX.

It remains to establish uniqueness in assertion (1). Let (Y, q, mE, ι_0) be a universal completion of X. In view of 1.6.6 and assertion (2) we may assume that $Y = \mathscr{Y}{\downarrow}$, where $\mathscr{Y}$ is a Banach space inside $\mathbf{V}^{(B)}$. By Theorem 1.6.6, $[\![$there exists a linear isometry λ of the space $\mathscr{X}$ onto $\mathscr{Y}]\!] = 1$. But then $\lambda{\downarrow}$ is a linear isometry of $\mathscr{X}{\downarrow}$ onto $\mathscr{Y}{\downarrow}$. ▷

A disjointly complete space (Y, q, dE), where dE stands for a disjoint completion of E, is said to be a *disjoint completion* (*d-completion*) of a lattice-normed space (X, p, E) if there exists a linear isometry $\iota : X \to Y$ such that $Y = \operatorname{mix}(\iota X)$.

A Banach-Kantorovich space (Y, q, oE), together with a linear isometry $\iota : X \to Y$, is an *order completion* (*o-completion*) of a lattice-normed space (X, p, E) provided that every o-complete decomposable subspace $Z \subset Y$ containing ιX coincides with Y. If $E = mE$ then an o-completion of X is a universal completion of it (see 1.6.3). Given a subset $U \subset Y$, introduce the notation

$$rU := \{y := r\text{-}\lim_{n\to\infty} y_n \mid (y_n)_{n\in\mathbb{N}} \subset U\},$$

$$oU := \{y := o\text{-}\lim y_\alpha \mid (y_\alpha)_{\alpha\in\mathrm{A}} \subset U\},$$

$$dU := \left\{y := o\text{-}\sum_{\xi\in\Xi} \pi_\xi y_\xi \mid (y_\xi)_{\xi\in\Xi} \subset U\right\},$$

where A is an arbitrary directed set, (π_ξ) is an arbitrary partition of unity in $\mathfrak{P}(Y)$, and the limits and sum exist in Y.

(3) *For every lattice-normed space, there exists an o-completion (d-completion) unique to within a linear isometry.*

$\lhd$ Recall that $dE \subset oE \subset mE$. Put

$$Y := \{x \in mX \mid p_m(x) \in oE\}.$$

Then Y is an o-completion and $d\iota X$ is a d-completion of X. $\rhd$

We always assume that a lattice-normed space X is contained in an o-completion $\overline{X}$ of X.

(4) *For an o-completion $\overline{X}$ of a space X, the equality $\overline{X} = rdX$ holds. Moreover, if X is decomposable and $E_0 := p(X)^{\perp\perp}$ is a vector lattice with the principal projection property, then $\overline{X} = oX$.*

$\lhd$ The first part of the assertion follows from (1). Take an $x \in \overline{X}$ and find a net $(x_\alpha) \subset X$ o-converging to x. Endow X with the equivalence and preorder defined by the formulas

$$z \sim y \leftrightarrow p(x - z) = p(y - z),$$

$$z \leq y \leftrightarrow p(x - z) \geq p(y - z).$$

If E_0 is a lattice with the principal projection property then there exists a projection $\pi \in \mathfrak{P}(X)$ such that

$$\pi p(x - y) + \pi^* p(x - z) = p(x - y) \wedge p(x - z).$$

For $u := \pi y + \pi^* z$, we have

$$p(x - u) = p(x - y) \wedge p(x - z);$$

therefore, $y \prec u$ and $z \prec u$. Thus, the preordered set $(X, \prec)$ is directed upward. Hence, the quotient set $\mathrm{A} := X/{\sim}$ with the quotient order is an upward-directed ordered set. Now, consider a net $(x_\alpha)_{\alpha \in \mathrm{A}}$, where $x_\alpha \in \alpha$ $(\alpha \in \mathrm{A})$. The net $(p(x - x_\alpha))_{\alpha \in \mathrm{A}}$ decreases by construction. Put $e := \inf p(x - x_\alpha)$, where the infimum is calculated in oE. By the equality $\overline{X} = rdX$, given an $\varepsilon > 0$, there exist a family $(x_\xi) \subset X$ and a partition of unity $(\pi_\xi) \subset \mathfrak{P}(X)$ such that

$$p_m\Big(x - o\text{-}\sum \pi_\xi x_\xi\Big) \leq \varepsilon p_m(x).$$

Considering 1.6.2 and 1.6.3, we can write down

$$e = \sum \pi_\xi e \le \sum \pi_\xi p(x - x_\xi) = p\Big(x - o\text{-}\sum \pi_\xi x_\xi\Big) \le \varepsilon p(x).$$

Hence $e = 0$ and $x = o\text{-}\lim x_\alpha$. $\rhd$

(5) *A decomposable lattice-normed space is o-complete if and only if it is d-complete and r-complete.*

$\lhd$ Necessity was mentioned in 1.6.3. Sufficiency follows from (4). $\rhd$

(6) *Let (X, p, E) be a Banach-Kantorovich space, $E = p(X)^{\perp\perp}$, and $A := \mathrm{Orth}(E)$. Then there is a unique extension to X of the structure of a faithful unitary A-module such that the natural representation of A in X implements an isomorphism between Boolean algebras $\mathfrak{P}(E) \subset A$ and $\mathfrak{P}(X)$. Moreover,*

$$p(ax) = |a|p(x) \quad (x \in X,\ a \in A).$$

$\lhd$ We have to apply 1.6.5(2). In particular, by virtue of condition (c) in the mentioned subsection, the Boolean algebra $\mathfrak{P}(X)$ coincides with the set of the multiplication operators $x \mapsto \chi(b)x$ $(x \in X)$, where $b \in B$. $\rhd$

A Banach space $\mathscr{X}$ inside $\mathbf{V}^{(B)}$ is said to be a *Boolean-valued realization* for a lattice-normed space X if $\mathscr{X}{\downarrow}$ is a universal completion of X.

1.6.8. Theorem. *Let $\mathscr{X}$ and $\mathscr{Y}$ be Boolean-valued realizations for Banach-Kantorovich spaces X and Y normed by some universally complete K-space E. Let $\mathscr{L}^B(\mathscr{X}, \mathscr{Y})$ be the space of bounded linear operators from $\mathscr{X}$ into $\mathscr{Y}$ inside $\mathbf{V}^{(B)}$, where $B := \mathfrak{B}(E)$. The immersion mapping $T \mapsto T^\sim$ of the operators implements a linear isometry between the lattice-normed spaces $\mathscr{L}_B(X, Y)$ and $\mathscr{L}^B(\mathscr{X}, \mathscr{Y}){\downarrow}$.*

$\lhd$ By Theorem 1.6.7(2), without loss of generality we may assume that $E = \mathscr{R}{\downarrow}$, $X = \mathscr{X}{\downarrow}$, and $\mathscr{Y}{\downarrow} = Y$. Take a mapping $\mathscr{T} : \mathscr{X} \to \mathscr{Y}$ inside $\mathbf{V}^{(B)}$ and put $T := \mathscr{T}{\downarrow}$. Let ρ and θ be the norms of the Banach spaces $\mathscr{X}$ and $\mathscr{Y}$, let $p := \rho{\downarrow}$ and $q := \theta{\downarrow}$, and let $+$ stands for all summation operations in $\mathscr{X}$, $\mathscr{Y}$, X, and Y. The linearity and boundedness of $\mathscr{T}$ imply validity for the relations

$$\mathscr{T} \circ + = + \circ (\mathscr{T} \times \mathscr{T}), \quad \theta \circ \mathscr{T} \le k\rho,$$

where $0 \le k \in \mathscr{R}\downarrow$. The descent and ascent rules for the superposition allow us to write down the relations in the following equivalent form:

$$T \circ + = + \circ (T \times T), \quad q \circ T \le kp.$$

But this means that T is linear and bounded. Let K be the set constituted of $0 \le k \in \mathscr{R}\downarrow$ such that $q(Tx) \le kp(x)$ $(x \in X)$. Then $K\uparrow = \{k \in \mathscr{R}_+ \mid \theta \circ \mathscr{T} \le k\rho\}$ inside $\mathbf{V}^{(B)}$.

Appealing to 1.3.4(2), we derive

$$\mathbf{V}^{(B)} \models |T| = \inf K = \inf(K\uparrow) = \|\mathscr{T}\|.$$

Hence, the mapping $\mathscr{T} \mapsto \mathscr{T}\downarrow$ preserves the vector norm. To justify the linearity of the mapping, it suffices to check its additivity. Given $\mathscr{T}_1, \mathscr{T}_2 \in \mathscr{L}^B(\mathscr{X}, \mathscr{Y})\downarrow$, we have

$$\begin{aligned}(\mathscr{T}_1 + \mathscr{T}_2)\downarrow(x) &= (\mathscr{T}_1 + \mathscr{T}_2)(x) = \mathscr{T}_1 x + \mathscr{T}_2 x \\ &= \mathscr{T}_1\downarrow x + \mathscr{T}_2\downarrow x = (\mathscr{T}_1\downarrow + \mathscr{T}_2\downarrow)x\end{aligned}$$

inside $\mathbf{V}^{(B)}$ for every $x \in X$. Consequently, $(\mathscr{T}_1+\mathscr{T}_2)\downarrow = \mathscr{T}_1\downarrow+\mathscr{T}_2\downarrow$. So, the descent implement a linear isometry of $\mathscr{L}^B(\mathscr{X}, \mathscr{Y})\downarrow$ onto the space of all bounded linear extensional operators from X into Y. It remains to observe that every bounded linear operator from X into Y is nonexpanding, or which is the same, satisfies the inequality $[\![x = 0]\!] \le [\![Tx = 0]\!]$. Indeed, if $b := [\![x = 0]\!]$ then $\chi(b)x = 0$ by 1.6.5(2); therefore,

$$\chi(b)q(Tx) \le \chi(b)p(x) = p(\chi(b)x) = 0.$$

Hence, $q(\chi(b)Tx) = 0$ or $\chi(b)Tx = 0$ and, employing 1.6.5(2) again, we conclude that $b \le [\![Tx = 0]\!]$. $\triangleright$

1.6.9. A *normed* (*Banach*) *lattice* is a vector lattice E which is also a vector (Banach) space with norm monotone in the following sense: if $|x| \le |y|$ then $\|x\| \le \|y\|$ $(x, y \in E)$. If (X, p, E) is a lattice-normed space, where E is a normed lattice, then we can furnish X with the mixed norm

$$|\!|\!|x|\!|\!| := \|p(x)\| \quad (x \in X).$$

In this event, the normed space $(X, |\!|\!|\cdot|\!|\!|)$ is referred to as a *space with mixed norm.* By virtue of the inequality $|p(x) - p(y)| \le p(x - y)$ and monotonicity of the norm in E, the vector norm p is a continuous operator from $(X, |\!|\!|\cdot|\!|\!|)$ into E.

(1) *Let E be a Banach lattice. Then $(X, ⦀\cdot⦀)$ is a Banach space if and only if (X, p, E) is complete with respect to relative uniform convergence.*

$\lhd$ $\Leftarrow$: Take a fundamental sequence $(x_n) \subset X$. Without losing generality, we may assume that $⦀x_{n+1} - x_n⦀ \le 1/n^3$ $(n \in \mathbb{N})$. Put

$$e_n := p(x_1) + \sum_{k=1}^{n} kp(x_{k+1} - x_k) \quad (n \in \mathbb{N}).$$

Then we can estimate

$$\|e_{n+l} - e_n\| = \left\| \sum_{k=n+1}^{n+l} kp(x_{k+1} - x_k) \right\|$$
$$\le \sum_{k=n+1}^{n+l} k⦀x_{k+1} - x_k⦀ \le \sum_{k=n+1}^{n+l} \frac{1}{k^2} \underset{n,l\to\infty}{\longrightarrow} 0.$$

Thus, the sequence (e_n) is fundamental and so it possesses a limit $e = \lim\limits_{n\to\infty} e_n$. Since $e_{n+k} \ge e_n$ $(n, k \in \mathbb{N})$, we have $e = \sup e_n$. If $n \ge m$ then

$$mp(x_{n+l} - x_n) \le \sum_{k=n+1}^{n+l} kp(x_{k+1} - x_k) \le e_{n+l} - e_n \le e;$$

consequently, $p(x_{n+l} - x_n) \le (1/m)e$. It means that the sequence (x_n) is r-fundamental. By r-completeness, there exists $x := r\text{-}\lim x_n$. It is clear that $\lim\limits_{n\to\infty} ⦀x - x_n⦀ = 0$.

$\Rightarrow$: Suppose that a sequence $(x_n) \subset X$ is r-fundamental; i.e.,

$$p(x_n - x_m) \le \lambda_k e \quad (m, n, k \in \mathbb{N},\ m, n \ge k),$$

where $0 \le e \in E$ and $\lim\limits_{k\to\infty} \lambda_k = 0$. Then

$$⦀x_n - x_m⦀ \le \lambda_k \|e\| \to 0 \text{ as } k \to \infty;$$

consequently, there is $x := \lim\limits_{n\to\infty} x_n$. Since the vector norm is continuous (with respect to $⦀\cdot⦀$), we have

$$p(x - x_n) \le \lambda_k e \quad (n \ge k);$$

therefore, $x = r\text{-}\lim x_n$. $\rhd$

(2) Let F be an ideal in E. Put $Y := \{x \in X \mid p(x) \in E\}$ and $q := p \restriction Y$. The triple (Y, q, F) is called the ***F-restriction*** of the space X. If X is a Banach-Kantorovich space then so is Y. If X is r-complete and F is a Banach lattice then Y is a Banach space with mixed norm.

Consider a Banach space $(\mathscr{X}, \rho)$ inside $\mathbf{V}^{(B)}$ and an order-dense ideal F in $\mathscr{R}{\downarrow}$. The restriction of the space $\mathscr{X}{\downarrow}$ with respect to F is called the ***F-descent*** of $\mathscr{X}$ or the ***descent of $\mathscr{X}$ with respect to F*** and is denoted by $F{\downarrow}(\mathscr{X})$. More precisely, the F-descent of $\mathscr{X}$ is the triple $(F^{\downarrow}(\mathscr{X}), p, F)$, where

$$F^{\downarrow}(\mathscr{X}) := \{x \in \mathscr{X}{\downarrow} \mid \rho{\downarrow}(x) \in F\}, \quad p := (\rho{\downarrow}) \restriction F^{\downarrow}(\mathscr{X}).$$

(3) *If a Banach lattice E is an ideal in $\mathscr{R}$ then $E^{\downarrow}(\mathscr{X})$ is a Banach space with mixed norm.*

1.6.10. Consider several categories related to lattice-normed spaces.

Let $\mathrm{Ban}^{(B)}$ be the descent of the category of Banach spaces and bounded linear operators in the model $\mathbf{V}^{(B)}$. In more detail, the objects and morphisms of the category $\mathrm{Ban}^{(B)}$ are elements $\mathscr{X} \in \mathbf{V}^{(B)}$ and $\alpha \in \mathbf{V}^{(B)}$ such that $[\![\mathscr{X}$ is a Banach space $]\!] = \mathbf{1}$ and $[\![\alpha$ is a bounded linear operator$]\!] = \mathbf{1}$ (cf. [37]).

Define the category $\mathrm{BK}(E)$ as follows. We enlist in the class $\mathrm{Ob}\,\mathrm{BK}(E)$ all Banach-Kantorovich spaces (X, p) such that $\mathrm{im}(p) = E_+$. As morphisms in the class we take all bounded linear operators (see the definition in 1.6.4(6)). The composition in the indicated categories is defined as the superposition of operators.

(1) Theorem. *If E is a universally complete K-space and $B \simeq \mathfrak{B}(E)$ then the categories* $\mathrm{Ban}^{(\mathrm{B})}$ *and* $\mathrm{BK}(E)$ *are equivalent. The equivalence is established by the pair of immersion and descent functors dual to each other.*

$\lhd$ The proof is contained in 1.6.5, 1.6.6, 1.6.7(2), and 1.6.8. $\rhd$

Let us introduce the category $\mathrm{Ban}(B)$. Its objects are the pairs (X, h), where X is a Banach space and h is a Boolean isomorphism of B onto the complete Boolean algebra of projections with norm at most 1 acting in X. A morphism from (X, h) into (Y, g) is an bounded operator $T : X \to Y$ such that $T \circ h(b) = g(b) \circ T$ for every $b \in B$. Taking some liberty, we suppose that $B \subset \mathscr{L}(X)$ for every $X \in \mathrm{ObBan}(B)$ and say that a morphism T is ***commutes with projections in B*** or that T is ***B-linear***. In this sense we will understand the following inaccurate but convenient notation: $\pi T = T\pi$ $(\pi \in B)$. The composition in the category $\mathrm{Ban}(B)$ is defined as the conventional superposition of mappings.

(2) *The category* BK(E) *is a subcategory of* Ban(B) *provided that E is a Banach lattice.*

$\lhd$ It follows from 1.6.9 that Banach spaces with mixed norm are objects of BK(E). The presence of a complete Boolean algebra of projections B in each of the spaces follows from 1.6.2. It remains to demonstrate that morphisms of the category BK(E) commute with projections in B. The boundedness of an operator $T : X \to Y$ $(Y \in \mathrm{Ob}\,\mathrm{BK}(E))$ means that $q \circ T \le c \circ p$, where $c \in \mathrm{Orth}(E)$ and p and q are E-valued norms in X and Y respectively. This is equivalent to the relation

$$(\forall S \in \partial q)\ (S \circ T \in \partial(c \circ p)).$$

The mappings $c \circ p : X \to E$ and $q : Y \to E$ commute with projections in B (see 1.6.2), Hence, the operators S and ST are B-linear for every $S \in \partial q$; i.e., $ST\pi = \pi ST = S\pi T$ $(\pi \in B)$ (see [36, Theorem 2.3.15]). In particular, $S(\pi T - T\pi) = 0$ for all $S \in \partial q$, consequently; $\pi T = T\pi$. $\rhd$

Consider also a subcategory $\mathrm{Ban}^{(\mathrm{B})}(E)$ of the category $\mathrm{Ban}^{(\mathrm{B})}$, where E is a Banach lattice and an order-dense ideal in $\mathscr{R}{\downarrow}$. The classes of objects in the categories coincide. An element $\alpha \in \mathbf{V}^{(B)}$ is a morphism of the category $\mathrm{Ban}^{(\mathrm{B})}(E)$ if and only if $\alpha \in \mathrm{Mor}\,\mathrm{Ban}^{(\mathrm{B})}$ and $|\alpha{\downarrow}| \in \mathrm{Orth}(E)$. In more detail, a morphism of the category $\mathrm{Ban}^{(\mathrm{B})}(E)$ is a bounded operator inside $\mathbf{V}^{(B)}$ satisfying the condition

$$[\![\|\alpha x\| \le c\|x\|\ (x \in \mathrm{dom}\,\alpha)]\!] = \mathbf{1}$$

for some $c \in \mathrm{Orth}(E)$. Observe that the definition of the category $\mathrm{Ban}^{(\mathrm{B})}(E)$ involves an object E external for $\mathbf{V}^{(B)}$. Denote by $E^{\downarrow}$ the mapping which associate with the object $\mathscr{X} \in \mathrm{Ban}^{(\mathrm{B})}(E)$ an object $E^{\downarrow}(\mathscr{X}) := \{x \in \mathscr{X}{\downarrow} \mid |x| \in E\}$, and with a morphism $\alpha \in \mathrm{Ban}^{(\mathrm{B})}(E)$ such that $D(\alpha) = \mathscr{X}$, the restriction of the operator $\alpha{\downarrow}$ to the subspace $E^{\downarrow}(\mathscr{X})$.

If E is a K-space of bounded elements (i.e., the order ideal in $\mathscr{R}{\downarrow}$ generated by unity $\mathbf{1} \in \mathscr{R}{\downarrow}$) then we speak of the restricted descent rather than of the E-descent and call $E^{\downarrow}$ the *restricted descent functor.*

(3) Theorem. *The mapping $E^{\downarrow}$ of E-descent is a covariant functor from* $\mathrm{Ban}^{(\mathrm{B})}(E)$ *into* Ban(B). *The functor $E^{\downarrow}$ and the immersion functor establish equivalence between the categories* $\mathrm{Ban}^{(\mathrm{B})}(E)$ *and* BK(E).

$\triangleleft$ The first part of the theorem is contained in 1.6.8 and in (2). To complete the proof it suffices to observe the following: Let X and Y be universally complete Banach-Kantorovich spaces and let X_0 and Y_0 be their E-restrictions ($E \subset mE = \mathscr{R}{\downarrow}$). If $T_0 : X_0 \to Y_0$ is a bounded operator then there exists a unique extension $T : X \to Y$ of T_0 such that T is a bounded operator and $|T| = |T_0|$. If X and Y are realized as $\mathscr{X}{\downarrow}$ and $\mathscr{Y}{\downarrow}$ (see 1.6.6) then we can put $T = T_0{\uparrow}{\downarrow}$. Conversely, for an operator $T \in \mathscr{L}_b(X, Y)$ such that $|T| \in \operatorname{Orth} E$ its restriction $T_0 : T \restriction X_0$ belongs to $\mathscr{L}_b(X_0, Y_0)$. The correspondence $T_0 \mapsto T$ is a linear isometry between the Banach-Kantorovich spaces $\mathscr{L}_b(X_0, Y_0)$ and $\mathscr{L}_E(X, Y)$, where $\mathscr{L}_E(X, Y)$ is the $\operatorname{Orth}(E)$-restriction of $\mathscr{L}_b(X, Y)$. $\triangleright$

1.6.11. What was exposed in the preceding subsection gives rise to the following natural question: which Banach spaces are linearly isometric to E-descents and, in particular, to restricted descents of Banach spaces in a Boolean-valued model? It is clear that the phenomenon depends essentially on the geometry of a Banach space. Consider in short a particular case of the restricted descent needed in the sequel without launching into the topic.

(1) A Banach space $X \in \operatorname{Ban}(B)$ is said to be *B-cyclic* if the unit ball $B_X := \{x \in X \mid \|x\| \le 1\}$ is cyclic with respect to B. More precisely, X is B-cyclic if and only if, for every partition $(b_\xi)_{\xi\in\Xi} \subset B$ of unity and an arbitrary family $(x_\xi)_{\xi\in\Xi} \subset B_X$, there exists a unique element $x \in B_X$ such that $b_\xi x = b_\xi x_\xi$ for all ξ (recall our agreement that $B \subset \mathscr{L}(X)$).

Theorem. *A Banach space is linearly isometric to the restricted descent of some Banach space in the model $\mathbf{V}^{(B)}$ if and only if it is B-cyclic.*

$\triangleleft$ Necessity follows from the definitions and 1.6.10(3). Assume X to be a Banach space with B-cyclic unit ball B_X and $J : B \to \mathscr{B}$ to be the corresponding isomorphism of B onto the Boolean algebra of projections in $\mathscr{B}$. Let E be an ideal in the universally complete K-space of all B-valued resolutions of identity (see 1.4.4). Consider a finite-valued element $\alpha := \sum_{k=1}^{n} \lambda_k b_k$, where $\{b_1, \dots, b_n\}$ is a partition of unity in B, $\{\lambda_1, \dots, \lambda_n\} \subset \mathbb{R}$, and λb stands for the spectral function $e : \nu \mapsto e(\nu) \in B$ equal to zero for $\nu \le \lambda$ and unity for $\nu > \lambda$. Put

$$J(\alpha) := \sum_{k=1}^{n} \lambda_k J(b_k)$$

and observe that $J(\alpha)$ is a bounded linear operator in X. Calculate its norm:

$$\begin{aligned}\sup_{\|x\|\le 1}\|J(\alpha)x\| &= \sup_{l\le n}\sup\left\{\left\|\sum_{k=1}^{n}\lambda_k x_l\right\| \mid x_l\in b_l(X)\wedge\|x_l\|\le 1\right\}\\ &= \sup_{l\le n}\sup\{\|b_k x_l\|\cdot|\lambda_k| \mid x_l\in b_k(X)\wedge\|x_l\|\le 1\}\\ &= \max\{|\lambda_1|,\dots,|\lambda_n|\}.\end{aligned}$$

On the other hand, the norm $|\alpha|_\infty$ of α in the K-space of bounded elements in E coincides also with $\max\{|\lambda_1|,\dots,|\lambda_n|\}$; consequently, J is a linear isometry of the subspace E_0 of bounded elements in E into the algebra $\mathscr{L}(X)$ of bounded operators. It is also clear that $J(\beta\alpha)=J(\beta)J(\alpha)$ for all $\alpha,\beta\in E_0$. Since E_0 is dense with respect to the norm in E, J can be extended by continuity to an isometric isomorphism of the algebra E onto a closed subalgebra of $\mathscr{L}(X)$. By putting $x\alpha:=\alpha x:=J(\alpha)x$ for $x\in X$ and $\alpha\in E$, we obtain the structure of an E-unitary module on X; moreover,

$$\|\alpha x\|\le\|\alpha\|\cdot\|x\|\quad(\alpha\in E,x\in X).$$

Furthermore, $\alpha B_X+\beta B_X\subset B_X$ for $|\alpha|+|\beta|\le 1$.

Now, introduce the mapping $p:X\to E_+$ by the formula

$$p(x):=\inf\{\alpha\in E_+\mid x\in\alpha B_X\}\quad(x\in X),$$

where the infimum is calculated in the K-space E. If $p(x)=0$ then, for $\varepsilon>0$, there exist a partition $(b_\xi)\subset B$ of unity and a family $(\alpha_\xi)\subset E_+$ such that $b_\xi\alpha_\xi\le\varepsilon\cdot\mathbf{1}$ and $x\in\alpha_\xi B_X$ for all ξ. Afterward $b_\xi x\in b_\xi\alpha_\xi B_X\subset\varepsilon B_X$ and thus $x\in\varepsilon B_X$ by virtue of the B-cyclicity of the ball B_X. Since $\varepsilon>0$ is arbitrary, we have $x=0$. If $x\in\alpha B_X$ and $y\in\beta B_X$ for some $\alpha,\beta\in E_+$ then we can write down

$$x+y=\gamma(\gamma^{-1}x+\gamma^{-1}y)\in\gamma\left(\frac{\alpha}{\gamma}B_X+\frac{\beta}{\gamma}B_X\right)\subset\gamma B_X,$$

where $\gamma:=\alpha+\beta+\varepsilon\cdot\mathbf{1}$. Consequently, $p(x+y)\le\alpha+\beta+\varepsilon\mathbf{1}$, and passage to the infimum over all indicated α, β, and ε yields $p(x+y)\le p(x)+p(y)$. Further, the following equalities hold for $b\in B$ and $x\in X$:

$$\begin{aligned}bp(x)&=\inf\{b\alpha\mid 0\le\alpha\in E\wedge x\in\alpha B_X\}\\ &=\inf\{\alpha\in E_+\mid bx\in\alpha B_X\}=p(bx).\end{aligned}$$

Therefore, we have

$$p(\alpha x) = \sum_{k=1}^{n} b_k |\lambda_k| p(x) = |\alpha| \cdot p(x)$$

for $\alpha = \sum_{k=1}^{n} \lambda_k b_k$, where $\{b_1, \dots, b_n\}$ is a partition of unity in B. Hence, $p(\alpha x) = |\alpha| \cdot p(x)$ for all $\alpha \in E$. Thereby, (X, p, E) is a decomposable lattice-normed space. The disjoint completeness of X follows from the B-cyclicity of the ball B_X and the r-completeness of X is equivalent to the completeness with respect to the initial scalar norm, for $\|x\| = \|p(x)\|_\infty$ $(x \in X)$. The last relation immediately follows from the definitions of p and $\|\cdot\|_\infty$. Finally, we conclude that (X, p, E) is a Banach-Kantorovich space (see 1.6.7(5)). If $\overline{X}$ is a universal completion of X then $\overline{X}$ is linearly isometric to the descent of some Banach space in the model $\mathbf{V}^{(B)}$ by Theorem 1.6.6. At the same time, X is the restriction of $\overline{X}$ with respect to E. $\triangleright$

Let C-Ban(B) be the complete subcategory of the category Ban(B) whose objects comprise the class of all B-cyclic Banach spaces. Put $\mathrm{Ban}_\infty^{(\mathrm{B})} := \mathrm{Ban}^{(\mathrm{B})}(E)$ if E is an ideal of bounded elements in $\mathscr{R}{\downarrow}$; i.e., $E = \bigcup_{n=1}^{\infty} [-n\mathbf{1}, n\mathbf{1}]$.

(2) *Theorem.* *The restricted descent functor establishes equivalence between* $\mathrm{Ban}_\infty^{(\mathrm{B})}$ *and* C-Ban(B).

$\triangleleft$ This follows from 1.6.10(3) and (1). $\triangleright$

Comments

The bibliography below pretends to be complete in regard to neither vector lattices nor nonstandard analysis. It mainly includes the monographs and surveys with extensive bibliography. Original articles are cited either for priority reasons or when they are absent from the monographs or surveys.

1.1. (1) In the history of functional analysis, the rise of the theory of ordered vector spaces is commonly atributed to the contribution of G. Birkhoff, L. V. Kantorovich, M. G. Kreĭn, H. Nakano, F. Riesz, H. Freudenthal, et al. At present, the theory of ordered vector spaces and its applications constitute a vast field of mathematics representing, in fact, one of the main sections of contemporary functional analysis. The theory is well exposed in many monographs (see [2, 4, 5, 12, 17, 22, 26, 27, 31, 33, 36, 45–47, 52, 54, 55, 69, 70, 72]). Observe also the surveys [7–10] with rich reference lists. The necessary information on the theory of Boolean algebras is given in [15, 56, 66].

(2) The credit for finding the most important instance of ordered vector spaces, an order complete vector lattice or a K-space, is due to L. V. Kantorovich. The notion appeared in Kantorovich's first fundamental article on this topic [23], where he wrote, "In this note, I define a new type of space that I call a semiordered linear space. The introduction of such a space allows us to study linear operations of one abstract class (those with values in the space) as linear functionals."

Here L. V. Kantorovich stated an important methodological principle, the *heuristic transfer principle* for K-spaces. An exemplar application of this principle is Theorem 3 of [23] now referred to as the Hahn-Banach-Kantorovich theorem. It claims that the *Kantorovich principle* is valid in relation to the classical Dominated extension theorem; i.e., we can replace the reals in the standard Hahn-Banach theorem by elements of an arbitrary K-space and a linear functional by a linear operator with values in this K-space.

(3) In [24], L. V. Kantorovich laid grounds for the theory of regular operators in K-spaces. Also, the Riesz-Kantorovich theorem appeared in this article for the first time (see 1.1.10(1)). F. Riesz formulated an analogous assertion for the space of continuous linear functionals over the lattice $C[a, b]$ in his famous report at the International Congress in Bologna in 1928 and thereby enlisted in the cohort of the founders of the theory of ordered vector spaces.

(4) It is difficult to construct an example of a nonexpanding but o-unbounded operator (1.1.10(4), for the references see [8]). However, in the case of a universally complete K-space, employing $\mathbf{V}^{(B)}$, we can easily reduce this question to existence of a discontinuous automorphism of the group $(\mathbb{R}, +)$, i.e., an additive but not linear function from $\mathbb{R}$ to $\mathbb{R}$. Let E be a universally complete K-space and let $B := \mathfrak{B}(E)$. Take a Boolean algebra B such that $\mathbb{R}^\wedge \neq \mathscr{R}$. Then $\mathscr{R}$ is an infinite-dimensional space over $\mathbb{R}^\wedge$ inside $\mathbf{V}^{(B)}$. By the Kuratowski-Zorn lemma, there exist an $\mathbb{R}^\wedge$-linear but not $\mathscr{R}$-linear function $u : \mathscr{R} \to \mathscr{R}$ in the model $\mathbf{V}^{(B)}$. The operator $U_0 := u{\downarrow} : \mathscr{R}{\downarrow} \to \mathscr{R}{\downarrow}$ is linear, extensional, and o-unbounded. If ι is an isomorphism of E onto $\mathscr{R}{\downarrow}$ then $U := \iota^{-1} \circ U_0 \circ \iota$ is a nonexpanding o-unbounded operator.

1.2. (1) As was mentioned in the comment on 1.1(2), the heuristic transfer principle proposed by L. V. Kantorovich in connection with the concept of K-space was substantiated by the author as well as by his successors. Essentially, this principle turned out to be one of those profound ideas that playing an organizing and leading role in the formation of the new branch of analysis eventually led to a deep and elegant theory of K-spaces rich in various applications. At the very beginning of the development of the theory, attempts were made at formalizing the above heuristic argument. In this direction, there appeared the so-called theorems on relation preservation which claimed that if some proposition involving finitely many functional relations is proven for the reals then an analogous fact remains valid automatically for the elements of every K-space (see [27, 69]).

However, the inner mechanism responsible for the phenomenon of relation preservation still remained vague and the applicability range for such assertions are not found nor the general causes of numerous analogies and parallels with the classical function theory. The depth and universal character of Kantorovich's principle were discovered in the framework of Boolean-valued analysis.

(2) Boolean-valued analysis is a branch of functional analysis which uses a special model-theoretic technique, the Boolean-valued models of set theory. It is interesting to observe that the invention of the Boolean-valued models was not connected with the theory of ordered vector spaces. The necessary language and technical tools were available within mathematical logic in the early 1960s. However, there was no general idea to breathe life into the already-created mathematical apparatus and promote rapid progress in model theory. Such an idea appeared along with P. J. Cohen's discovery; in 1963 he established that the classical continuum-problem is absolutely unsolvable (in a rigorous mathematical sense). It was the Cohen forcing method whose comprehension gave rise to the Boolean-valued models of set theory. Their appearance is commonly associated with the names of P. Vopěnka, D. Scott, and R. Solovay (see [58, 61, 67, 68]).

(3) The forcing method splits naturally into two parts: general and special. The general part comprises the apparatus of Boolean-valued models of set theory, i.e., construction of a Boolean-valued universe $\mathbf{V}^{(B)}$ and interpretation of the set-theoretic propositions in $\mathbf{V}^{(B)}$. Here, a complete Boolean algebra B is arbitrary. The special part consists in constructing specific Boolean algebras B providing some special (usually, pathological and exotic) properties of the objects (for example, K-spaces) obtained from $\mathbf{V}^{(B)}$. Both parts are of independent interest, but their combination yields the most impressive results. In the present chapter, like in most investigations in Boolean-valued analysis, we use only the general part of the forcing method. The special part is widely employed for proving independence or consistency (see [6, 18, 63]). The further progress in Boolean-valued analysis will almost surely be connected with applying the forcing method at full strength.

(4) The material of 1.2.1–1.2.8 is traditional; for its detailed exposition see [6, 37, 63], see also [18, 48]. The methods presented in 1.2.9–1.2.11 as well as their variants are widely used in the study of Boolean-valued models. In [32, 42], they are translated into the descent-ascent technique which is most appropriate for the problems of analysis. This form is used in [37]. Immersion (2.10) of the sets with Boolean structure into a Boolean-valued universe was carried out in [32]. Such immersion relies upon the Solovay-Tennenbaum method which was proposed earlier for the immersion of complete Boolean algebras [59].

1.3. (1) The Boolean-valued status of the concept of K-space is established in Gordon's theorem 1.3.2 obtained in [13]. This fact can be interpreted as follows: a universally complete K-space is the interpretation of the field of reals in an appropriate Boolean-valued model. Moreover, it turns out that every theorem on reals (in the framework of ZFC) has an analog for the corresponding K-space. Theorems are transferred by means of precisely-defined procedures: ascent, descent, and canonical embedding, that is, algorithmically as a matter of fact. Thereby Kantorovich's assertion that "the elements of a K-space are generalized numbers" acquires a rigorous mathematical meaning in Boolean-valued analysis. On the other hand, Boolean-valued analysis makes rigorous the heuristic transfer principle which played an auxiliary piloting role in many investigations in the pre-Boolean-valued theory of K-spaces.

(2) If B in 1.3.2 is the algebra of measurable sets modulo sets of zero measure μ then $\mathscr{R}{\downarrow}$ is isomorphic to the universally complete K-space $L^0(\mu)$ of measurable functions. This fact (for the Lebesgue measure on an interval) has already been known to Scott and Solovay (see [58]). If B is a complete Boolean algebra of projections in a Hilbert space then $\mathscr{R}{\downarrow}$ is isomorphic to the space of selfadjoint operators $\mathfrak{A}(B)$ (see 1.1.9(5)). The two indicated particular cases of Gordon's theorem were intensively and fruitfully exploited by G. Takeuti (see [61] and the bibliography in [37]). The object $\mathscr{R}{\downarrow}$ for general Boolean algebras was also studied by T. Jech [19, 20] who in fact rediscovered Gordon's theorem. The difference is that in [19] a (complex) universally complete K-space with unity is defined by another system of axioms and is referred to as a complete Stone algebra. The interconnections between properties of numeric objects and the corresponding objects in the K-space $\mathscr{R}\downarrow$ indicated in 1.3.4 and 1.3.5 were actually obtained by E. I. Gordon [13, 14].

(3) The realization theorem 1.3.6 was obtained by A. G. Kusraev [34]. A close result (in other terms) is presented in T. Jech's article [21], where Boolean-value interpretation of the theory of linearly ordered sets is developed. Corollaries 1.3.7(7, 8) are well known (see [27, 69]). The concept of universal completion for a K-space was introduced in another way by A. G. Pinsker. He also proved existence of a universal completion unique to within isomorphism for an arbitrary K-space. Theorem 1.2.8(2) on order completion of an Archimedean vector lattice was proven by A. I. Yudin (see the corresponding references in [27, 69]). Assertion 1.2.8(3) was established by A. I. Veksler [64].

(4) Tests 1.3.9(2) and 1.3.9(4) for o-convergence (in the case of sequences) were obtained by L. V. Kantorovich and B. Z. Vulikh (see [27]). It was shown in 1.3.8 that, in fact, they are merely the interpretation of convergence properties of numeric nets (sequences).

(5) As was mentioned in the comment on 1.2(1), the first attempts of formalizing the Kantorovich heuristic principle resulted in theorems on relation preservation (see [27, 69]). The contemporary forms of such theorems, based on the method of Boolean-valued models, may be found in [14, 20] (see also [37]).

(6) Subsystems of the field $\mathscr{R}$ can be obtained not only by Boolean-valued realization of Archimedean vector lattices (see 1.3.6(1)). For instance, the following assertions are stated in [34]: (1) the Boolean-valued realization of an Archimedean lattice-ordered group is a subgroup of the additive group of $\mathscr{R}$; (2) an Archimedean f-ring contains two complementary components one of which has zero multiplication and is realized as in (1) and the other, as a subring of $\mathscr{R}$; (3) an Archimedean f-algebra contains two complementary components one of which is realized as in 1.3.6 and the other, as a sublattice and subalgebra of the field $\mathscr{R}$ considered as a lattice-ordered algebra over the field $\mathbb{R}^\wedge$ (see also [21]).

1.4. (1) The results of the section, with minor exception, are well known to the specialists in the theory of vector lattices. The novelty consists in the method of proving: all basic facts are derived by interpreting simple properties of the field of reals in a Boolean-valued model.

(2) The concepts of unity, unit element, and spectral functions were introduced by G. Freudenthal. He also established the spectral theorem 1.4.5(2) (see [27, 69]). Theorem 1.4.4 implies that for a complete Boolean algebra B the set $\mathfrak{R}(B)$ of resolution of identity is a universally complete K-space whose base is isomorphic to B. This fact is due to L. V. Kantorovich [27]. Theorem 1.4.5(1) was obtained by A. G. Pinsker (see [27]). The main result of Subsection 1.4.6, the realization of an arbitrary K-space as an order dense ideal in $C^{\infty}(Q)$, was established for the first time independently by B. Z. Vulikh and T. Ogasawara (see [27, 69]).

(3) It follows from 1.4.13 that every resolution of identity with values in a σ-algebra determines a spectral measure on the Borel σ-algebra of the real axis. This fact was indicated for the first time by V. I. Sobolev in [57]. However, he assumed that such a measure can be obtained by means of the Carathéodory extension method. D. A. Vladimirov showed that the Carathéodory extension of a complete Boolean algebra of countable type is possible if and only if the algebra is regular. Thus, the extension method of 1.4.13 grounded on the Loomis-Sikorski representations 1.4.12 for Boolean σ-algebras essentially differs from the Carathéodory extension. In [71], M. Wright obtained Assertion 1.4.13 (for $n = 1$) as a consequence of a version of the Riesz theorem for operators with values in a K-space.

(4) V. I. Sobolev was apparently the first who considered Borel functions defined on an arbitrary K_{σ}-space with unity (see [57, 69]). Theorems 1.4.15 and 1.4.17 presented here were obtained in [38, 39]. In [38, 39], there was also constructed the Borel functional calculus for (countable or uncountable) collections of elements of an arbitrary K-space. A Boolean-valued proof of Theorem 1.4.16 is given in [19].

(5) The method of Boolean-valued realization is also useful for studying linear operators in vector lattices. The comment on 1.1(4) reveals the simplest example; more profound results of this sort are exposed for example in [14, 32, 33, 38]. Similar methods are involved in analysis of nonlinear operators (see [33, 36]).

1.5. (1) Exposing the material of this section, we follow the articles [30, 44]. The main idea proposed in [44] is as follows: the fragments of a positive operator U are the extreme points of the order interval $[0, U]$. The latter set coincides with the subdifferential at zero (the supporting set) ∂P of the sublinear operator $P(x) := Ux^+$. Thereby study of the fragments of a positive operator reduces to description for the extremal structure of subdifferentials. Such a description for general sublinear operators was obtained for the first time in the article [41] by S. S. Kutateladze (for a detailed exposition see [36]). Observe that the approach developed in [41] solves, in particular, the problem on extremal extension of a positive operator (for the corresponding bibliography see [8, 36]).

(2) A formula like 1.5.1(1) was established for the first time by de Pagter (see [5]); however, it involved two essential constraints: F should have a total set of o-continuous functional, and E must be order complete. The first constraint was eliminated in [40] and the second, in [1, 30]. All these cases correspond to different generating sets of projections. The concept of generating set was introduced in [44].

(3) The projection formulas like 1.5.1(2, 3) appeared gradually. A piece of this history can be learned from [5, 72]. The general approach proposed in [44] allows one to obtain various projection formulas by taking concrete generating sets of band projections. For instance, if E is a K-space then the set $\{\bar{\pi} : \pi \in \mathfrak{P}(E)\}$ of band projections, where $\bar{\pi} : U \mapsto U \circ \pi$, is generating in $L^{\sim}(E, F)$.

(4) Making use of the remarks of the preceding subsection, we can derive from 1.5.1(1–3) the following assertions:

(a) *Let E and F be K-spaces, let U be a positive operator from E to F, let W be the principal band projection of a positive operator $V : E \to F$ onto the band $\{U\}^{\perp\perp}$, $x \in E_+$, and let $\mathscr{E}$ be the filter of weak order units in F. Then the following formulas hold (Kusraev and Strizhevskiĭ [40]):*

$$\mathfrak{E}(U) = \{\pi U \rho \mid \rho \in \mathfrak{P}(E),\ \pi \in \mathfrak{P}(F)\}^{\vee(\uparrow\downarrow\uparrow)};$$

$$(V - W)x = \inf_{e \in \mathscr{E}} \sup\{\pi V \rho x \mid \pi U \rho x \le e,\ \rho \in \mathfrak{P}(F),\ \pi \in \mathfrak{P}(F)\}$$

$$= \inf_{0<\varepsilon\in\mathbb{R}} \sup\{\pi V \rho x \mid \pi U \rho x \le \varepsilon U x,\ \rho \in \mathfrak{P}(E),\ \pi \in \mathfrak{P}(F)\}.$$

(b) *Let $0 \le \varphi \in E^{\sim}$, $f \in F_+$, $x \in E_+$, π_f be the projection onto the band $\{f\}^{\perp}$, and let W_0 and W be the principal band projections of a positive operator $V : E \to F$ onto the bands $\{\varphi \otimes f\}^{\perp\perp}$ and $(E^{\sim} \otimes F)^{\perp\perp}$. Then the* *following formulas hold* (*Aliprantis and Burkinshaw* [5]):

$$W_0 x = \sup_{0<\varepsilon\in\mathbb{R}} \inf\{\pi_f V_e : \varphi(x-e) \le \varepsilon,\ 0 \le e \le x\},$$

$$Wx = \sup_{\substack{0\le\varphi\in E^{\sim} \\ 0<\varepsilon\in\mathbb{R}}} \inf\{Ve : \varphi(x-e) \le \varepsilon,\ 0 \le e \le x\}.$$

1.6. (1) The concept of lattice-normed space appeared for the first time in the article [23] by L. V. Kantorovich. It is worth stressing that he introduced an unusual axiom of decomposability for an abstract norm, axiom (4) (see 1.6.1(4)). It is interesting to observe that axiom (4) was often omitted as inessential in the further research by other authors. Its profound importance was discovered in connection with Boolean-valued analysis (see [33]). In the same article, L. V. Kantorovich considered for the first time the operators in abstract normed spaces dominated by a positive linear or increasing sublinear operator. Later, such operators has been called differently: regular, majorized, and, in some particular cases, dominated operators or operators with abstract norm.

(2) Theorem 1.6.6 on Boolean-valued realization of lattice-normed spaces was obtained by A. G. Kusraev. This result, certain assertions of 1.6.7 and 1.6.8, and some other applications are presented in [33]. The assertion $\overline{X} = oX$ of 1.6.7(4) was obtained by A. E. Gutman.

(3) The first applications of the concepts of lattice-normed space and dominated operator related to solving operator equations by means of successive approximation (see [27, 69]). Close ideas and methods were used by many mathematicians (A. V. Bukhvalov, V. L. Levin, N. Dinculeanu, et al.) while studying spaces of vector-functions and operators therein. A. G. Kusraev and his students constructed an advanced theory of dominated operators in lattice-normed spaces and found a broad circle of various applications [35, 38–40].

(4) The spaces with mixed norm discussed in 1.6.9 were studied in [35]. The same article reveals various applications of the concept of mixed norm (in particular, Theorem 1.6.1) to the geometry of Banach spaces and the theory of linear operators. The restricted descent of 1.6.10 was earlier used by G. Takeuti in his study of C^*-algebras by means of Boolean-valued models.

(5) It is desirable to obtain analogs of the projection rules of 1.5.1(2, 3) and 1.5.2(1, 2) for general dominated operators (see Definition 1.6.4(6)). We have no such formulas available. However, the following formula is valid for the order continuous component U_n of a dominated operator $U : X \to Y$:

$$U_n x = o\text{-}\lim_{(\pi_\xi)\in\Pi} \sum_\xi U\pi_\xi x \quad (x \in X),$$

where Π is the set of all partitions of unity in the Boolean algebra $\mathfrak{P}(X)$ and all the necessary limits exist. Moreover, $|U_n| = |U|_n$ and $|U - U_n| = |U| - |U|_n$.

As to the other problems of the theory of dominated operators, see [35].

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CHAPTER 2

Operator Classes Determined by Order Conditions

BY

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Operator theory is part and parcel of functional analysis whose own name is not unlikely to be consonant with a simplest operator, a functional. The theory of vector and Banach lattices began with distinguishing and studying the order properties of functionals and operators.

At the International Mathematical Congress in Bologna in 1928, F. Riesz proposed a calculus for continuous linear functionals over the space $C[0,1]$ of continuous functions. His calculus made it possible to find the modulus and the positive and negative components of a functional which are in many respects similar to those of the usual modulus and the positive and negative parts of a real-valued function. F. Riesz's constructions relied upon considering the natural pointwise order relation between the functions of $C[0,1]$. In the 1930s, while propounding a general theory of vector lattices, L. V. Kantorovich developed a calculus for order bounded linear operators in vector lattices which covered F. Riesz's construction as a particular case. Also, L. V. Kantorovich applied the calculus to solving abstract functional equations.

Among the pathfinders of the theory of vector lattices, it is compulsory to name a Holland mathematician H. Freudenthal who obtained the fundamental (Freudenthal!) theorem on "integral representation" of an element of a vector lattice already in his first article of 1936 and a British mathematician S. Steen who defined the modulus of an operator in the same way as L. V. Kantorovich and indicated one of the most unexpected applications of the theory of vector lattices, the spectral theory of selfadjoint operators in Hilbert space. The corresponding references and historical information may be found in [3, 16, 18, 19, 28, 29, 40, 42, 46, 48].

The class of linear operators in vector lattices which is simplest in definition (but not in the depth of its properties) is the class of positive operators; i.e., the operators U acting from a vector lattice E into a vector lattice F and such that $0 \le x \in E$ implies $0 \le Ux \in F$. The deep study of specific spectral properties of these operators stems from the Perron-Frobenius theory of positive matrices. In the infinite-dimensional case, this theory found its generalization within the theory of positive operators in a space with a cone which originated from M. G. Kreĭn's works written in the 1940s and which was later developed by M. A. Krasnosel′skiĭ in the USSR and by a number of research groups abroad (see, for instance, [31, 32]). However, here we do not intend to expose results of the spectral theory and theory of operator semigroups with positive generators which gained a new strong impetus to further development in the last decade (see [3, 16, 18, 39, 48] and also the bibliography in [18]).

The set of positive operators is not a linear space, as opposed to the set of differences of positive operators called the space of regular operators. In 1.1.10, we have already stated the fundamental Riesz-Kantorovich theorem claiming that, under natural conditions, the space of regular operators is an order complete vector lattice and that a regular operator can be described as an order bounded operator, i.e., an operator carrying order bounded sets into order bounded sets. Henceforth, we will, as a rule, use a shorter term "regular operators," constantly presuming that these classes of operators are identified.

In function spaces, the most important class of operators closely connected with the class of regular operators is the class of integral operators. Speaking of integral operators, we mean the operators that are defined with the help of the usual Lebesgue integral, excluding singular integral operators defined by means of various summation methods for divergent integrals and the operators whose kernels represent distributions as is accepted in quantum mechanics. Integral operators constantly attract the attention of researchers (see [26, 30, 32]). In Chapter 4, this class of operators will be studied in more detail.

Also, there are many classes of operators defined in mixed terms of boundedness and various forms of convergence, norm, and order. The reader may find a survey and the corresponding bibliography in [18, 43]. Historically, the first class of such operators was apparently the class of operators with abstract norm introduced by L. V. Kantorovich in the late 1930s.

An operator U acting from a Banach space X into a K-space E is called an *operator with abstract norm* if the image $U(B_X)$ of the unit ball is order bounded in E. Having introduced such operators U, we may define the element $|\!|U|\!| \in E$

$$|\!|U|\!| = \sup\{|Ux| \mid x \in X,\ \|x\| \leq 1\}$$

that is called the "*abstract norm*" of U. We call such operators *dominated.* It is clear that the class of dominated operators is closely connected with the class of bounded operators with values in the Banach space L^∞. The specifics of this class are mainly determined by the properties of L^∞. In the case of operators in the Hilbert space L^2 a dominated operator is merely a Hilbert-Schmidt operator.

Chapters 2–4 are united by the idea of versatile study of the above-indicated important classes of operators. First, we solve the problem of finding the characteristic properties of such operators. In Chapter 2 which is mainly preliminary by nature, we consider the above problem for regular and dominated operators and in Chapter 4, for integral operators. Second, we study stability of the indicated classes of operators under the taking of composition with an arbitrary continuous operator. As a rule, it is sufficiently easy to show that such compositions do not always belong to the initial class of operators. Thereby it is worth posing the problem of describing the subclass of operators stable under the indicated operation. In Chapter 3, we consider various modifications of the problem for regular and dominated operators. As regards the technique of solution this topic turned out to be closely connected with the theory of p-absolutely summing operators.

The results of the theory of vector lattices can be divided into two parts, in particular, according to the following feature. Certain results are new only in the abstract situation of an arbitrary vector lattice of a particular class or in the case of specially constructed examples; whereas they have essential meaning for the concrete spaces, say, the L^p-spaces or turn into simple well-known assertions (such is, for example, the theory of regular K-spaces). The other results involve new ideas and are nontrivial just in the L^p-spaces; however, they admit extension to more general classes of Banach lattices, although such extensions often turn out to be rather involved. The main results of Chapters 2–4 belong to the latter category. In this connection, to avoid obscuring the exposition by technical details, in many cases we confine ourselves within L^p-generality, indicating possible generalizations in the Comments.

As a rule, we omit the proofs that are given in the second and third editions of the monograph [28] by L. V. Kantorovich and G. P. Akilov.

2.1. Ideal Spaces

We begin the chapter on regular operators with exposing the simplest facts of the theory of a wide class of the spaces generalizing the Lebesgue L^p-spaces, the so-called ideal spaces. The ideal spaces constitute the most important class of vector lattices comprising almost all (but $C[0,1]$) of their important particular instances: L^p, Orlicz spaces, Marcinkiewicz spaces, and Lorentz spaces. It is in terms of ideal spaces that the material on integral operators is further stated. Opening our exposition of operator theory with the theory of ideal spaces, we intend to emphasize an analogy of many properties of operator spaces with more elementary properties of the spaces of measurable functions.

2.1.1. We start with conventions on the type of a measure space. We denote by (T, Σ, μ) (sometimes, (S, Λ, ν)) a set T with a σ-algebra Σ of measurable subsets and a σ-finite measure μ on Σ. The reader disinclined to so great degree of generality may assume, without making what follows trivial, that (T, Σ, μ) is the interval $[0, 1]$ with the Lebesgue measure or a domain in the space $\mathbb{R}^n$ with the Lebesgue measure.

We denote by $L^0(T, \Sigma, \mu)$ or simply by L^0 the set of all measurable almost everywhere finite functions on (T, Σ, μ) taking real or complex values, where equivalent functions are identified as usual. Moreover, in proofs we assume the spaces real by default; however, it is important for applications to operator theory that similar results with obvious modifications remain true for complex spaces. Later we provide relevant specification.

The convergence in measure for a sequence of functions in L^0 is meant to be the convergence in measure on every set of finite measure; the following notation is used: $x_n \to x(\mu)$.

2.1.2. We are interested in the following order relation between functions in L^0. Given functions $x, y \in L^0$, we set $x \leq y$ if $x(t) \leq y(t)$ almost everywhere. It is clear that L^0 becomes an ordered set in which two arbitrary functions $x, y \in L^0$ have a *supremum* $x \vee y \in L^0$ and an *infimum* $x \wedge y \in L^0$ defined by the formulas

$$(x \vee y)(t) = \max(x(t), y(t)), \tag{1}$$

$$(x \wedge y)(t) = \min(x(t), y(t)). \tag{2}$$

Moreover, for every function $x \in L^0$, we can define its *positive part* x_+, *negative part* x_- and *modulus* $|x|$:

$$x_+ = x \vee 0, \quad x_- = (-x) \vee 0, \quad |x| = x_+ + x_-; \quad x = x_+ - x_-; \tag{3}$$

we certainly have

$$|x|(t) = |x(t)|. \tag{4}$$

With a given set $A \in \Sigma$ we associate the projection P_A in L^0 on letting

$$(P_A x)(t) = \chi_A(t) x(t). \tag{5}$$

We have written down the trivial formulas (1)–(5) for they are basic although not always acknowledged for almost all calculations in the theory of measurable functions. However, it becomes nontrivial to derive analogs of formulas (1)–(5) in the exposition of the calculus of order bounded operators in Section 2.2 with operators standing in place of x and y. Indeed, in this case we have to take troubles of insuring linearity for the results.

2.1.3. Observe the following two simple properties of the order in L^0 which demonstrate that the order and linear structure are properly compatible:

$$\text{for every } z \in L^0, \text{ the relation } x \leq y \text{ implies } x + z \leq y + z; \tag{6}$$

$$\text{if } x \geq 0 \text{ and } \lambda \geq 0 \text{ then } \lambda x \geq 0. \tag{7}$$

2.1.4. Now we turn to some less trivial facts. A set $M \subset L^0$ is called ***bounded above*** if there exists $y \in L^0$ such that $x \leq y$ for all $x \in M$. ***Boundedness below*** is defined analogously. A set $M \subset L^0$ is called ***order bounded*** (*o-bounded*, in short) if it is bounded above and below. This is equivalent to existence of $y \in L^0$ such that $|x| \leq y$ for all $x \in M$. What happens if we try to find a supremum of an infinite set $M \subset L^0$ which is bounded above? If M is countable then there obviously exists $y = \sup M$; moreover, the function y may be calculated by the formula

$$y(t) = \sup \{x(t) \mid x \in M\}. \tag{8}$$

However we often wish to work with the exact bounds of the sets of arbitrary cardinality. Observe that formula (8) is of little avail here. First, formula (8) may give a nonmeasurable function in the case. Second, while calculating by formula (8) we may obtain two measurable but nonequivalent functions by taking different representatives $x(t)$ in the class $x \in M$ of equivalent functions. To show that the difficulties are real and unremovable, consider the interval $[0,1]$ with the Lebesgue measure. In the first case, it suffices to take a Lebesgue nonmeasurable set $A \subset [0,1]$ and to consider as M so many copies of the zero element in L^0 as is the cardinality of A. It is clear that $\sup M = 0$. Enumerate the elements of M with the points of A. To the copy of zero with number $t \in A$ assign the characteristic function of the set $\{t\}$ which is obviously measurable and equivalent to zero. Then the function $y(t)$ in (8) equals $\chi_A(t)$, thus being nonmeasurable. In a similar way one can provide an example for the second case; moreover, it is clear that the same construction "spoils" the calculation of any uncountable supremum by (8). Nevertheless the following theorem is valid whose proof can be found in [28] (see Theorem I.6.17):

2.1.5. *Theorem.* *If a set $M \subset L^0$ is bounded above then there exists $y = \sup M \in L^0$. Moreover, there is a countable set $\{x_n\} \subset M$ such that $\sup M = \sup\{x_n\}$ (the last supremum can be calculated pointwise by formula* (8)) *A similar assertion is valid for the infimum of a set bounded below.*

REMARK. A similar assertion is valid not only in the space L^0 but also in the ordered set of all measurable functions possibly assuming infinite values on sets of positive measure.

2.1.6. We exhibit a nontrivial application of the theorem immediately. Given an arbitrary function $x \in L^0$, we as usual define its *support*

$$\operatorname{supp} x = \{t \in T \mid x(t) \neq 0\} \in \Sigma.$$

It is clear that $\operatorname{supp} x$ for an element $x \in L^0$ is defined to within a *negligible* set (i.e., a set of measure zero). The support of an arbitrary set $M \subset L^0$ cannot be defined as the union of supports of all functions in M for the same reasons that make formula (8) incorrect. However, we proceed differently. Consider the set $M_1 = \{\chi_A \mid A = \operatorname{supp} x,\ x \in M\}$ which is obviously bounded above in L^0 (for instance, by the function $\mathbf{1}$). Therefore, there exists $y = \sup M_1$ by Theorem 2.1.5. By definition, we assign

$$\operatorname{supp} M = \operatorname{supp} y.$$

It is easy to show that the set $\operatorname{supp} M$ possesses the natural properties of the support of M: (1) $\operatorname{supp} x \subset \operatorname{supp} M$ for all $x \in M$; (2) $\operatorname{supp} M$ is the least set to within a set of measure zero which possesses property (1).

2.1.7. We proceed with introducing notations. The notation $x_n \uparrow$ means that $x_n \geq x_m$ for $n \geq m$; $x_n \uparrow x$ means that $x_n \uparrow$ and $\sup x_n = x$ (or that $x_n \to x$ almost everywhere). The notations $x_n \downarrow$ and $x_n \downarrow x$ are defined analogously.

2.1.8. Now we turn to studying subspaces of L^0. An *ideal space* on (T, Σ, μ) is a linear subset E in L^0 such that

$$(x \in L^0,\ y \in E;\ |x| \leq |y|) \Rightarrow (x \in E); \tag{9}$$

i.e., with every function the set E contains its modulus and each function with smaller modulus.

Denote by E_+ the *cone of positive elements* or the *positive cone* of an ideal space E:

$$E_+ = \{x \in E \mid x \geqslant 0\}.$$

A norm on an ideal space E is said to be *monotone* if

$$(x, y \in E;\ |x| \leq |y|) \Rightarrow (\|x\| \leq \|y\|). \tag{10}$$

A *Banach ideal space* on (T, Σ, μ) is defined as an ideal space E endowed with a monotone norm making E into a Banach space.

Observe that the notions are natural and simple in formulation: practically all spaces whose definitions do not involve smoothness or analyticity are Banach ideal spaces. The basic example is L^p.

2.1.9. We use the notion of the support of a set of functions to define an important notion of an order-dense ideal (a foundation in the Russian literature). An ideal space F is said to be *order-dense* in an ideal space E if $F \subset E$ and $\operatorname{supp} F = \operatorname{supp} E$. In what follows we always suppose that an ideal space E is an order-dense ideal in L^0; i.e., $\operatorname{supp} E = T$. Observe that if F is an order-dense ideal in E then, for every $x \in E_+$, there exists a sequence $\{x_n\}$ in F_+ such that $x_n \uparrow x$. Moreover, there exists an increasing sequence of sets $\{A_n\}$ such that $x\chi_{A_n} \in F$ and $x\chi_{A_n} \uparrow x$.

A *band* in an ideal space E is an ideal space F included in E and such that, for every set $M \subset F$ possessing the supremum $y = \sup M$ in E, we have $y \in F$. It is clear that for every fixed set $A \in \Sigma$ the set

$$P_A E = \{x \in E \mid \operatorname{supp} x \subset A\} \tag{11}$$

is a band in E (this fact accounts for the use of the word "band": P_A bands over the set A). Moreover, the operator P_A defined by formula (5) is the *band projection* from E onto $P_A E$ (which is incorporated in the notation). It turns out that the converse holds and every band F has the form (11) with some A. Indeed, it suffices to set $A = \operatorname{supp} F$. The generalization of the notions of band and band projection are very essential for studying operator spaces.

Now we list the simplest properties of Banach ideal spaces which are important for the sequel.

2.1.10. *Proposition* [28, Lemma IV.3.2]. *If a sequence x_n converges to x in the norm of a Banach ideal space E then $x_n \to x(\mu)$; moreover, there exist a subsequence $\{x_{n_k}\}$, a function $r \in E_+$, and a numeric sequence $\varepsilon_k \downarrow 0$ such that $|x_{n_k} - x| \leq \varepsilon_k r$.*

2.1.11. We do not intend here to expatiate upon general properties of Banach ideal spaces. However, we observe that surprisingly many facts depend only on some rather simply formulated conditions; although these facts often have highly nontrivial proofs (see [3, 28, 47, 48]). We mean conditions (A) and (B) to be introduced below. The first is an abstract analog of the Lebesgue dominated convergence theorem and the second, known as the "Fatou property," of the Beppo Levy theorem or the Fatou lemma.

We say that a norm on a Banach ideal space E is *o-continuous* or that *condition* (A) holds in E ($E \in (A)$) provided that

$$(x_n \downarrow 0) \Rightarrow (\|x_n\| \to 0).$$

Let us introduce some convenient notation for dominated convergence almost everywhere in E. We say that a sequence $\{x_n\} \subset E$ *o-converges in* E to an x if $x_n \to x$ almost everywhere and there is a $y \in E$ such that $|x_n| \leq y$ for all n (in writing $x_n \xrightarrow{o} x$). It is easy to see that if $E \in (A)$ then $x_n \xrightarrow{o} x$ implies $x_n \to x$ in norm. It is clear that $L^p \in (A)$ for $p \in [1, \infty)$ and $L^\infty \notin (A)$. Condition (A) distinguishes "nice" spaces:

2.1.12. *Proposition* [28, Theorem IV.3.3]. *Suppose that a measure μ is separable (for instance, that μ is the Lebesgue measure on a measurable subset of the real axis or in $\mathbb{R}^n$ or μ reduces to countably many point masses; in the last case all ideal spaces are sequence spaces). A Banach ideal space E is separable if and only if E satisfies condition (A).*

It is clear that condition (A) can be easily verified in specific situations. This explains why the facts like Proposition 2.1.12 are useful.

2.1.13. We say that a norm on E is *o-semicontinuous* or that *condition* (C) holds in E ($E \in (C)$) provided that

$$(0 \leq x_n \uparrow x \in E) \Rightarrow (\sup \|x_n\| = \|x\|).$$

We say that a norm on a Banach ideal space E is *monotonically complete* or that *condition* (B) holds in E ($E \in (B)$) if

$$(0 \le x_n\uparrow,\ \sup\|x_n\| < \infty) \Rightarrow (\exists x \in E \mid x_n \uparrow x).$$

A Banach ideal space in which conditions (B) and (C) hold simultaneously is called *perfect*. A Banach ideal space in which conditions (A) and (B) hold simultaneously is a *Kantorovich-Banach space* or a *KB-space* for short.

All L^p spaces, $1 \le p < \infty$, are perfect and present KB-spaces for $1 \le p \le \infty$. The space c_0 of all vanishing sequences presents a Banach ideal space with condition (A) (and consequently (C)) and without (B). Condition (C) is satisfied in all examples of concrete spaces mentioned in the beginning of the chapter.

2.1.14. *Proposition* [28, Lemma IV.3.5]. *A Banach ideal space E is perfect if and only if the unit ball B_E is closed with respect to convergence in measure in L^0; i.e., if $x_n \in E$, $x \in L^0$, $\|x_n\| \le 1\,\forall n$, $x_n \to x(\mu)$ then $x \in E$ and $\|x\| \le 1$.*

2.1.15. The time has come to equip the theory of ideal spaces with the terminology of the general theory of vector lattices which was considered in Chapter 1. Recall that a real vector space E is a *vector lattice* if E is a (partially) ordered set in which, for every two elements $x, y \in E$, there exist their supremum $x \vee y$ and infimum $x \wedge y$ and the order and linear operations are related by axioms (6) and (7). A vector lattice is a *Kantorovich space* or a *K-space* if it is order complete; i.e., its every bounded above subset has a supremum. Formulas (1)–(4) show that L^0 is a vector lattice and Theorem 2.1.5 states that L^0 is a K-space.

We make a relevant observation simplifying the verification of the fact that a vector lattice E is a K-space. A set $M \subset E$ is called *directed in increasing order* or *upward-directed* if, for arbitrary $x, y \in M$, there is $z \in M$ such that $z \ge x, y$.

Proposition. *If, in a vector lattice E, every set which is bounded above and directed upward has a supremum then E is a K-space.*

$\lhd$ Let M be an arbitrary set bounded above. Denote by M_1 the set composed of suprema of all finite collections of elements in M. Obviously, $\sup M$ and $\sup M_1$ do exist or do not exist simultaneously and $\sup M = \sup M_1$ in the case of existence. By construction, the set M_1 is upward-directed. $\rhd$

The theory of vector lattices and K-spaces historically arose before the theory of general ideal spaces which began developing in the 1950s with research by

J. Diedonnè, G. G. Lorentz, I. Galperin, H. W. Ellis, A. C. Zaanen, W. A. J. Luxemburg et al. After the papers of Diedonnè, one of the most common terms for an ideal space is a *Kőthe space.* The term "ideal space" was coined within the school of M. A. Krasnosel′skiĭ on integral operators and equations (P. P. Zabreĭko, P. E. Sobolevskiĭ, Ya. B. Rutitskiĭ et al.) due to the fact that an ideal space is actually an o-ideal in L^0. The most important subclass of Banach ideal spaces is the class of so-called *symmetric* or *rearrangement invariant spaces* which includes most of the concrete Banach ideal spaces and is important for interpolation theory. This class was studied by G. G. Lorentz, S. Shimagaki, D. Boyd, W. A. J. Luxemburg, E. M. Semënov et al. (see [32, 33, 47]).

2.1.16. Recall some definitions of 1.1.2 and 1.1.3. An *ideal* in a vector lattice E is a linear subset F in E such that $(x \in E, y \in F; |x| \leq |y|) \Rightarrow (x \in F)$. It is clear that every ideal itself is a vector lattice. Further we also consider operator ideals where the term "ideal" is understood in a completely different sense. For that reason an ideal in the context of the above-presented definition is often referred to as an *order ideal* or briefly *o-ideal.*

Let E be a vector lattice. Elements $x, y \in E$ are called *disjoint* $(x \perp y)$ if $|x| \wedge |y| = 0$. An element $x \in E$ is called *disjoint from a set* $M \subset E$ $(x \perp M)$ if $x \perp y$ for all $y \in M$. *Disjoint subsets* in E are defined in an obvious manner. Now we introduce the taking of the *disjoint complement* which associates with a set $M \subset E$ the set

$$M^{\perp} := M^d = \{x \in E \mid x \perp M\}. \tag{12}$$

Assign $M^{dd} = (M^d)^d$. Observe that if M is a subset in an ideal space then M^d is the band over the set $T \setminus \operatorname{supp} M$ and M^{dd} is the band over $\operatorname{supp} M$.

An ideal F in a K-space E is called a *band* if, for every set M in F possessing the supremum $y = \sup M$ in E, we have $y \in F$. It is not difficult to show that, for every set M, the disjoint complement M^d is a band and so M^{dd} is a band too. The band M^{dd} is called the *band generated by* M. This operation has no constructive description which is no way deprives it of important applications (see, for instance, the proof of a criterion for integral representability in Chapter 4).

We associate canonically with every band F in a K-space E the *band projection* $[F]$ from E onto F. Given $x \in E_+$, we let

$$[F](x) = \sup\{y \in F_+ \mid y \leq x\}. \tag{13}$$

By the definition of K-space, the supremum exists in E and by the definition of band we have $[F](x) \in F$. Given an arbitrary $x \in E$, we assign

$$[F](x) = [F](x_+) - [F](x_-).$$

It is obvious that $[F]$ is a linear operator that maps E onto F and acts identically on F. Every element $x \in E$ is uniquely representable as $x = y + z$, where $y \in F$ and $z \in F^d$; moreover, $y = [F](x)$ and $z = [F^d](x)$.

2.1.17. A vector lattice equipped with a monotone norm (see (10)) is called a *normed lattice.* A normed lattice which is a Banach space is said to be a *Banach lattice.* In a Banach lattice, it is useful to consider analogs of conditions (A), (B), and (C) for arbitrary nets rather than for sequences. In this event the properties holding for sequences are called *sequential* and the analogs are denoted by (A_σ), (B_σ), and (C_σ). In Banach ideal spaces over spaces with σ-finite measure, the corresponding properties for sequences and nets are equivalent as follows from Theorem 2.1.5 (the remark on the theorem should be used too in the case of condition (B)). As for the general case, see [28, 42, 46, 48].

2.1.18. Finally we explain, as was promised at the beginning of the section, what to do in the case of spaces over the field of complex numbers. Let L^0 be the space of measurable functions with complex values. For a (complex) subspace E in L^0, the notion of ideal space can be introduced as above by means of (9) and the notion of Banach ideal space, by means of (10) provided that the modulus in the formulas is defined by the usual formula (4) (formulas (3) make no sense in the case of complex values). The collection of all real-valued functions in a complex ideal space E constitutes some ideal real space $\mathrm{Re}(E)$ (which is a Banach ideal space if so is E). Further, we attribute some properties and notions to E whenever they are pertinent to $\mathrm{Re}(E)$. Thus, one can define bands and band projections, condition (A), etc. In the next section, dealing with operators between ideal spaces we have to use the decomposition $E = \mathrm{Re}(E) \oplus i\,\mathrm{Re}(E)$ and decompose an operator on $\mathrm{Re}(E)$ into its real and imaginary components (for more details, see [42, § II.11, IV.1]). We will pay no attention to these issues henceforth.

2.2. The Space of Regular Operators

In the present monograph, we as a rule are interested in some entire operator spaces rather that in a sole operator. Even such a theorem as the criterion for integral representability is proved with the help of some construction over operator spaces. Therefore, we begin with formulating the main facts on the space of regular operators in the form of a specific calculus for the class of operators which is similar to the calculus of measurable functions in form but is considerably deeper than that in content. The main ideas in studying regular operators were proposed by L. V. Kantorovich and extended by his disciples B. Z. Vulikh and A. G. Pinsker. At the late 1930s the ideas were taken up by the Japanese school of the theory of vector lattices where the eminent role of H. Nakano should be distinguished. During the last fifteen years, the theory is enjoying its Renaissance (see the monographs [3, 42, 48] and surveys [16, 18]).

2.2.1. As was already mentioned, we want to attach some meaning to the formulas similar to (1)–(5) of Section 2.1 in the case of operators. First of all, we specify an appropriate operator space in which this can be done. All the operators and functionals to be considered are linear.

Let E and F be vector lattices. An operator $U : E \to F$ is called *order bounded* (*o-bounded*) if it sends o-bounded sets in E into o-bounded sets in F. Denote by $L^{\sim}(E, F)$ the set of all o-bounded operators.

An operator $U : E \to F$ is called *positive* if $0 \leq x \in E$ implies $Ux \geqslant 0$. An operator $U : E \to F$ is called *regular* if $U = U_1 - U_2$, with $U_1, U_2 : E \to F$ positive operators. The set of all regular operators is denoted by $L^r(E, F)$.

2.2.2. Proposition. *If E is a vector lattice and F is a K-space then an operator is o-bounded if and only if it is regular; i.e.,*

$$L^{\sim}(E, F) = L^r(E, F).$$

$\lhd$ Every positive operator is obviously o-bounded. Moreover, the difference of o-bounded operators is o-bounded too, which proves the o-boundedness of every regular operator. We postpone the proof of the converse until 2.2.9. $\rhd$

The coincidence of the class of order bounded and the class of regular operators plays a fundamental role in the theory under consideration. Therefore, unless the contrary is specified, we henceforth assume that F is a K-space and thus $L^{\sim}(E, F) = L^r(E, F)$. Certainly, this covers the case in which E and F are ideal spaces. In Chapter 3, we prefer the shorter term "regular operator" to the term "order bounded operator."

Thus, let F be a K-space.

2.2.3. In virtue of Proposition 2.2.2, it is obvious that the set $L^{\sim}(E,F)$ is linear (however, we make no use of this fact for the time being). Introduce some order into the set. Write $U \geqslant 0$ if U is a positive operator. Write $U \geqslant V$ if $U - V \geqslant 0$ (in other words, if $Ux \geqslant Vx\ \forall x \in E_+$). It is easily seen that the order in $L^{\sim}(E,F)$ satisfies conditions (6) and (7) of 2.1.

The following theorem, obtained in full generality by L. V. Kantorovich, plays a fundamental role (in the case of functionals on $C[0,1]$ it had been earlier established by F. Riesz).

2.2.4. *The Riesz-Kantorovich theorem.* *The set $L^{\sim}(E,F)$ is a K-space. Moreover, for all $U, V \in L^{\sim}(E,F)$ and every $x \in E_+$, the following formulas hold:*

$$(U \vee V)x = \sup\{Ux_1 + Vx_2 \mid x_1, x_2 \geqslant 0,\ x = x_1 + x_2\}; \tag{1}$$

$$(U \wedge V)x = \inf\{Ux_1 + Vx_2 \mid x_1, x_2 \geqslant 0,\ x = x_1 + x_2\}; \tag{2}$$

$$U_+x = \sup\{Uy \mid 0 \leq y \leq x\}; \tag{3}$$

$$U_-x = -\inf\{Uy \mid 0 \leq y \leq x\}; \tag{4}$$

$$|U|x = \sup\{|Uy| \mid |y| \leq x\}; \tag{5}$$

$$|U|x = \sup\left\{\sum_{i=1}^{n} |Ux_i| \mid x_1, \dots, x_n \geqslant 0,\ x = \sum_{i=1}^{n} x_i,\ n \in \mathbb{N}\right\}; \tag{6}$$

$$|Ux| \leq |U|(|x|)\ \forall x \in E. \tag{7}$$

We begin the proof of the Riesz-Kantorovich theorem with the following lemma.

2.2.5. *Lemma.* *Let E be a vector lattice, let X be a vector space, and let U be an operator on E_+ with values in X which satisfies the conditions:*

$$U(x+y) = Ux + Uy \quad \forall x, y \in E_+; \tag{8}$$

$$U(\lambda x) = \lambda Ux; \quad \lambda \geq 0,\ x \in E_+. \tag{9}$$

Then U admits a unique linear extension to the whole vector lattice E.

$\lhd$ The uniqueness of extension is obvious from the formula $x = x_+ - x_-$; therefore, it remains to prove existence. For every $x \in E$, we let

$$Wx = Ux_+ - Ux_-.$$

Check that W is the sought extension. First of all, establish that

$$Wz = Wx - Wy \tag{10}$$

for $z = x - y$; $x, y \in E_+$. From $z_+ - z_- = x - y$ it follows that $z_+ + y = x + z_-$. Then $Uz_+ + Uy = Ux + Uz_-$ by (8), whence

$$Wz = Uz_+ - Uz_- = Ux - Uy = Wx - Wy;$$

i.e., we have (10). By (10) and (8) it is obvious that the operator W is additive on E. The additivity of W implies that $W(-x) = -Wx$ for all $x \in E$, which guarantees the homogeneity of W in virtue of (9). $\rhd$

2.2.6. We return to the proof of the Riesz-Kantorovich theorem.

$\lhd$ To establish that some ordered vector space X is a vector lattice, it suffices to show that, for every $x \in X$, there exists $|x| = x \vee (-x)$. With this available, it remains to use the formulas

$$x_+ = \frac{1}{2}(x + |x|),\ x_- = \frac{1}{2}(|x| - x),$$
$$x \vee y = x + (y - x)_+,\ x \wedge y = -|(-x) \vee (-y)|$$

which are straightforward from the definitions.

Verify that the operator $|U|$ given by (5) is the modulus of U with respect to the order in $L^\sim(E, F)$. To this end, first prove that the operator $|U|$ defined by (5) only on E_+ satisfies the hypotheses of Lemma 2.2.5. First of all, observe that formula (5) is correct in view of $U \in L^\sim(E, F)$; i.e., the supremum in (5) exists (for $x \in E_+$ fixed, the set $\{y \in E \mid |y| \leq x\}$ is bounded and thus $\{|Uy| \mid |y| \leq x\}$ is bounded in F; it remains to use the fact that F is a K-space). Condition (9) is satisfied for $|U|$ trivially. Verify that $|U|$ is additive on $E+$. Take $x_1, x_2 \in E+$ and prove that

$$|U|(x_1 + x_2) = |U|x_1 + |U|x_2. \tag{11}$$

Simplify formula (5). In virtue of the associativity of bounds, we have

$$\begin{aligned} |U|x &= \sup\{|Uy| \mid |y| \le x\} \\ &= \sup\{(Uy) \vee (U(-y)) \mid |y| \le x\} = \sup\{Uy \mid |y| \le x\}. \end{aligned} \tag{12}$$

If y_1 and y_2 are such that $|y_1| \le x_1$ and $|y_2| \le x_2$ then $|y_1 + y_2| \le x_1 + x_2$ and

$$Uy_1 + Uy_2 = U(y_1 + y_2) \le |U|(x_1 + x_2),$$

whence

$$|U|x_1 + |U|x_2 \le |U|(x_1 + x_2).$$

Show the reverse inequality. Let $|y| \le x_1 + x_2$. Set

$$y_1 = x_1 \wedge y_+ - x_1 \wedge y_-, \quad y_2 = y - y_1.$$

It is easy to check that $|y_1| \le x_1$, $|y_2| \le x_2$, whence

$$|Uy| = |U(y_1 + y_2)| \le |Uy_1| + |Uy_2| \le |U|x_1 + |U|x_2$$

which shows (11).

In view of 2.2.5, the operator $|U|$ admits a unique linear extension onto E which is denoted again by $|U|$. We have not proved yet that $|U|$ is the modulus of U with respect to the order in $L^\sim(E, F)$. Check that $|U| = U \vee (-U)$. By formula (10), it is obvious that $|U| \ge \pm U$ and that $W \ge \pm U$ implies $W \ge |U|$, which yields the sought relation.

Thus, we have established that $L^\sim(E, F)$ is a vector lattice and the modulus of an operator in this vector lattice can be calculated by formula (5). Formulas (1)–(4) and (7) follow easily from (5) provided that the relations in the beginning of the subsection are used. Prove that the vector lattice $L^\sim(E, F)$ is order complete. To this end, it suffices to prove the existence of a least upper bound of an arbitrary set bounded above. Without loss of generality, we may assume that the set is directed upward (see Proposition 2.1.15). Thus, let M be an upward-directed set of operators and let $W \in L^\sim(E, F)$ be an operator such that $W \ge U$ for all $U \in M$. Given $x \in E_+$, assign

$$Vx = \sup\{Ux \mid U \in M\}. \tag{13}$$

Since $Ux \leq Wx$ for all $U \in M$ and all $x \in E_+$, the supremum in (13) exists. It is easy to verify the possibility of applying Lemma 2.2.5 to the operator V because M is directed upward. It is clear that the extension of V to E given by Lemma 2.2.5 belongs to $L^\sim(E,F)$; indeed, $U \leq V \leq W \in L^\sim(E,F)$ for every $U \in M$. It is obvious that $V = \sup M$. Thus, we have verified that $L^\sim(E,F)$ is a K-space.

It remains to prove formula (6). By (7), the right-hand side of (6) is greater than or equal to the left-hand side. To prove the reverse inequality, fix an $x \in E_+$ and take an y such that $|y| \leq x$. Then

$$Uy = Uy_+ - Uy_- \leq |Uy_+| + |Uy_-| + |U(x - |y|)|$$

and the claim follows from (12).

2.2.7. REMARK. All suprema and infima in formulas (1)–(6) must be understood in the sense of the order of L^0 rather than pointwise (see 2.1.4) since the bounds of uncountable sets are calculated here. We return to this question in Chapter 4 while studying integral operators.

2.2.8. REMARK. Formula (13) for calculating the supremum fails if the set M is not upward-directed. An extension of formulas (1) and (2) to an arbitrary set of operators is given for instance in [28, p. 360] and [46, pp. 228–230].

2.2.9. THE COMPLETION OF THE PROOF OF PROPOSITION 2.2.2.

$\lhd$ By the Riesz-Kantorovich theorem, every order bounded operator $U \in L^\sim(E,F)$ is representable as $U = U_+ - U_-$, where $U_+, U_- \geq 0$. $\rhd$

2.2.10. We will touch the questions as to which classical operators are o-bounded and which are not. Confine ourselves to the case $E = L^p$, $1 \leq p \leq \infty$. First of all, o-bounded operators must be defined on the whole space, thus excluding densely defined partial differential operators. However, the resolvents often turn out to be o-bounded. For instance, the resolvents of second order elliptic operators are positive (integral) operators. Integral operators (with the integral understood to be proper) are o-bounded in suitable pairs of spaces (anyway they are such if we take L^0 as the whole space). We postpone a more detailed discussion of the question until Chapter 4 devoted to integral operators.

The operators of conditional expectation are positive operators acting in each of the L^p spaces, $1 \leq p \leq \infty$. The operators of weighted substitution, having the form

$$(Ux)(t) = y(t)x(\varphi(t))$$

where $y \in L^0$ and φ is a measurable transformation of the interval $[0,1]$, are positive provided that $y \geq 0$. If $y \in L^\infty$ and φ is an automorphism then these operators act in every L^p. The operators of weighted substitution appear in describing the set of isometries in L^p and in more general spaces, in spectral theory and ergodic theory (see [18, 2.2]).

Below we prove that every continuous operator acting from L^1 into L^q or from L^p into L^∞ is o-bounded ($1 < q \leq \infty$, $1 \leq p < \infty$).

Until recently there have appeared no proofs of the claim that some operator fails to be o-bounded. This happens apparently since that no application is envisaged for such a fact. In a series of papers (see references in [18, 19]), A. V. Bukhvalov related the question of the absence of o-boundedness of an operator with the question of the continuity of extension of the classical operators of analysis to the case of vector-valued functions. In this connection, a sufficiently general theorem on the absence of o-boundedness for singular integral operators was stated in [12] and proved in [16]. In [4], an exceptionally simple approach was developed for the proof of some stronger fact for operators of the above-mentioned class: namely, they fail to belong to the closure in the uniform norm of the set of regular operators.

2.2.11. We now return to the operator calculus constructed in Theorem 2.2.4. It is possible to achieve a useful refinement by observing that the bounds in certain formulas can be calculated over disjoint elements. This fact was proved by Yu. A. Abramovich for the analogs of formulas (1)–(5) in [1] and for the case of formula (6) (which result is slightly weaker) in [37].

Proposition. *If E is a K-space then the bounds in formulas* (1), (2), *and* (6) *can be calculated over all disjoint sum partitions of $x \in E_+$ (that is, $x_1 \perp x_2$ in* (1), (2); *$x_i \perp x_j$ $(i \neq j)$ in* (6)):

$$(U \vee V)x = \sup\{Ux_1 + Vx_2 \mid x = x_1 + x_2;\ x_1, x_2 \in E_+,\ x_1 \perp x_2\}; \tag{14}$$

$$(U \wedge V)x = \inf\{Ux_1 + Vx_2 \mid x = x_1 - x_2;\ x_1, x_2 \in E_+,\ y_1 \perp x_2\}; \tag{15}$$

$$|U|x = \sup\left\{\sum_{i=1}^{n} |Ux_i| \mid x = \sum_{i=1}^{n} x_i;\ x_i \geq 0;\ x_i \perp x_j\,(i \neq j);\ n \in \mathbb{N}\right\}. \tag{16}$$

◁ For simplicity, confine ourselves to the case in which E is an ideal space. A proof in the general case differs only in some technical details connected with applying principal band projections.

Verify formula (15). First let $U \wedge V = 0$. We need to check that the infimum on the right-hand side of (15) equals zero. By hypothesis, we have $(U \wedge V)(2x) = 0$. Consider an arbitrary representation $2x = x_1 + x_2$; $x_1, x_2 \geq 0$. Set

$$A = \{t \mid x_1(t) \geq x_2(t)\}, \quad B = \{t \mid x_2(t) > x_1(t)\}.$$

Then $T = A \cup B$ and $A \cap B = \varnothing$. Since $U, V \geq 0$, we have

$$Ux_1 + Vx_2 \geq U(x_1 \chi_A) + V(x_2 \chi_B) \geq U(x\chi_A) + V(x\chi_B).$$

Taking the infimum in the last inequality over all representations $2x = x_1 + x_2$, infer that the infimum on the right-hand side in (15) equals zero.

Now let operators U and V be arbitrary. The identity

$$(U - U \wedge V) \wedge (V - U \wedge V) = 0$$

holds in every vector lattice. From the already-proved particular case of formula (15) we infer

$$\begin{aligned} 0 &= ((U - U \wedge V) \wedge (V - U \wedge V))(x) \\ &= \inf\{U(x_1) - (U \wedge V)(x_1) + V(x_2) - (U \wedge V)(x_2) \mid \\ &\qquad x = x_1 + x_2;\ x_1, x_2 \geq 0;\ x_1 \perp x_2\} \\ &= \inf\{U(x_1) + V(x_2) - (U \wedge V)(x) \mid x = x_1 + x_2;\ x_1, x_2 \geq 0,\ x_1 \perp x_2\} \\ &= \inf\{U(x_1) + V(x_2) \mid x = x_1 + x_2;\ x_1, x_2 \geq 0,\ x_1 \perp x_2\} - (U \wedge V)(x) \end{aligned}$$

whence (15) follows in the general case. Formula (14) is ensured by (15) in view of the relations indicated at the beginning of 2.2.6. Finally, it is obvious that in formula (16) the left-hand side is greater or equal to the right-hand side. The reverse inequality can be obtained by the following calculation based on (15):

$$\begin{aligned} |U|x &= (U \vee (-U))(x) = \sup\{Ux_1 - Ux_2 \mid x = x_1 + x_2,\ x_1, x_2 \geq 0,\ x_1 \perp x_2\} \\ &\leq \sup\{|Ux_1| + |Ux_2| \mid x = x_1 + x_2;\ x_1, x_2 \geq 0,\ x_1 \perp x_2\}. \ \triangleright \end{aligned}$$

Remark. Formulas (6) and (16) for $|U|$ are useful in view of the fact that the supremum in the formulas is taken over an upward-directed set. This allows one to establish a connection between order operations and the norm in many cases. We are going to use this observation in the near future while proving Theorem 2.2.16.

2.2.12. The subspace of o-continuous operators plays an important role in the K-space $L^{\sim}(E, F)$.

An operator $U : E \to F$ is called *o-continuous* if $x_n \xrightarrow{o} 0$ in E implies $Ux_n \xrightarrow{o} 0$ in F (recall that the o-convergence of an ideal space is the dominated convergence almost everywhere). The set of all o-continuous regular operators is denoted by $L_n^{\sim}(E, F)$.

The following proposition can often simplify the verification of the fact that a given operator belongs to the class $L_n^{\sim}(E, F)$:

Proposition. *Let $U \in L^{\sim}(E, F)$. The following assertions are equivalent:*

(1) $U \in L_n^{\sim}(E, F)$;

(2) if $x_n \downarrow 0$ then $Ux_n \xrightarrow{o} 0$ in F.

If E is an ideal space then assertions (1) and (2) are also equivalent to the following:

(3) if $x \in E_+$ and $A_n \downarrow \varnothing$ then $U(x\chi_{A_n}) \xrightarrow{o} 0$.

$\lhd$ The proof can be found in [46] (Lemmas VIII.3.1 and VIII.3.2). $\rhd$

2.2.13. *Proposition* [46, Theorem VII.3.3]. *The space $L_n^{\sim}(E, F)$ is a band in the K-space $L_n^{\sim}(E, F)$.*

2.2.14. It often happens that

$$L^{\sim}(E, F) = L_n^{\sim}(E, F). \tag{17}$$

Proposition. *If E is a Banach ideal space with condition (A) then equality (17) holds.*

$\lhd$ Apply (2) of the proposition in 2.2.12. Let $U \in L^{\sim}(E, F)$ and let $x_n \downarrow 0$. By $E \in (A)$, we have $\|x_n\| \to 0$. Demonstrate that $|U|x_n \downarrow 0$. Since $|Ux_n| \le |U|x_n$, this will yield $Ux_n \xrightarrow{o} 0$ in F.

In virtue of the positivity of the operator $|U|$, we have $0 \le |U|x_n \downarrow$. Since $\|x_n\| \to 0$, in view of Proposition 2.1.10 we may assume that $|x_n| \le \varepsilon_n r$, where $r \in E_+$, $\varepsilon_n \to 0$, by passing to a subsequence (without losing generality in view of the monotonicity of the sequence $\{|U|x_n\}$). Then

$$|U|x_n \le \varepsilon_n Ur \xrightarrow{o} 0,$$

whence we conclude that $|U|x_n \downarrow 0$. $\rhd$

The following statement can be proved by a similar technique:

2.2.15. *Proposition.* *If E and F are Banach lattices then*

$$L^r(E,F) \subset \mathscr{L}(E,F). \tag{18}$$

$\lhd$ It suffices to observe that the second part of Proposition 2.1.10 holds in an arbitrary Banach lattice [46, Lemma IV.3.2]. $\rhd$

If we additionally suppose that F is a K-space then (7) immediately implies the following inequality for the norms:

$$\|U\| \leq \| \, |U| \, \|. \tag{19}$$

Generally speaking, inclusion (18) and inequality (19) are strict.

Henceforth the following theorem will be of use.

2.2.16. *Theorem.* *Let one of the following conditions be valid:*

(a) *$E = L^1$ and F is a Banach lattice being a K-space with conditions (B) and (C) (in terms of nets);*

(b) *E is an arbitrary Banach lattice and $F = L^\infty$.*

Then

$$L^\sim(E,F) = \mathscr{L}(E,F), \tag{20}$$

$$\|U\| = \| \, |U| \, \|. \tag{21}$$

REMARK. Conditions (B) and (C) for nets are satisfied in an arbitrary perfect Banach ideal space (see 2.1.17).

$\lhd$ Case (a): Let $U \in \mathscr{L}(L^1, F)$. Prove that $U \in L^\sim(L^1, F)$ and $\| \, |U| \, \| \leq \|U\|$. To this end, we use formula (6) for calculating the modulus of an operator. Simultaneously we prove existence of the supremum on the right-hand side of the formula and estimate the norm. Fix an $x \in L^1_+$ with $\|x\| \leq 1$. Consider the set M of all finite sums of the form

$$y = \sum_{i=1}^{n} |Ux_i|,$$

where

$$x = \sum_{i=1}^{n} x_i, \quad x_i \geq 0 \quad (i = 1, \ldots, n).$$

It is obvious that M is directed upward. Moreover, it is norm bounded in F:

$$\|y\| \leq \sum_{i=1}^{n} \|Ux_i\| \leq \|U\| \sum_{i=1}^{n} \|x_i\| = \|U\| \, \|x\| \leq |U|. \tag{22}$$

Namely, in the last inequality we made use of the characteristic property of L^1 expressed as the additivity of the norm for positive summands. In virtue of condition (B) in F, we conclude that the set M has a supremum in F, which by (6) means the existence of $|U|x$ for an arbitrary $x \in L^1$; i.e., $U \in L^{\sim}(L^1, F)$. Since $F \in (C)$, inequality (22) also yields the estimate $\| \, |U|x\| \leq \|U\|$ for the norm, which implies (21).

Case (b): Let $U \in \mathscr{L}(E, L^\infty)$. Make use of the fact that conditions (B) and (C) hold in L^∞ and the norm is an AM-*norm*:

$$\|x \vee y\| = \max(\|x\|, \|y\|); \quad x, y \in L^\infty_+.$$

If $x_1, \dots, x_n$ are such that $|x_1|, \dots, |x_n| \leq x \in E_+$ then

$$\| \, |Ux_1| \vee \cdots \vee |Ux_n| \, \|_{L^\infty} = \max(\|Ux_1\|, \dots, \|Ux_n\|) \leq \|U\| \, \|x\|.$$

Hence by (5) there exists $|U|x \in L^\infty$ and $\| \, |U|x\| \leq \|U\| \, \|x\|$. $\rhd$

If we require that equality (20) is valid together with equality (21) for the norms then the validity of either of the conditions (a) or (b) is essentially necessary. From this point of view, it is interesting that Yu. A. Abramovich succeeded in constructing a pair of some Banach ideal spaces E and F, where E is not isomorphic to L^1 and F is not isomorphic to a sublattice in L^∞, such that equality (20) holds together with equivalence of the norms. This cannot happen in the scale of the L^p spaces: equality (20) (with equivalence of the norms) holds for $E = L^p$ and $F = L^q$ if and only if either $p = 1$ or $q = \infty$. The reader may find the corresponding references in [18, § 2.6]).

2.2.17. Let E be a Banach lattice and let F be a Banach lattice presenting a K-space. Introduce the following norm on the operator space $L^{\sim}(E, F)$:

$$\|U\|_r = \| \, |U| \, \|_{\mathscr{L}(E,F)},$$

which is called the *regular norm of* U.

Proposition. *The space* $(L^\sim(E,F), \|\cdot\|_r)$ *is a Banach lattice.*

$\lhd$ The validity of all properties of a norm, except completeness, is obvious. In virtue of the completeness criterion for a normed lattice [28, Theorem X.3.2], it suffices to prove that if $0 \le U_n \uparrow$ is a Cauchy sequence in $L^\sim(E,F)$ then there exists $U = \sup U_n \in L^\sim(E,F)$. Since the regular norm is stronger than the conventional operator norm, $\{U_n\}$ is a Cauchy sequence in $\mathscr{L}(E,F)$ which converges to some operator $U \in \mathscr{L}(E,F)$. Then, for every $x \in E_+$, we have $U_n x \to Ux$ in norm and $U_n x \uparrow$, whence $Ux = \sup Ux_n$ and thus $U = \sup U_n$; $U \ge 0$. $\rhd$

2.2.18. Substituting the scalar field for F in the definitions of the above classes of operators, we arrive at the definitions of the K-space of regular functionals $E^\sim$ and of the band $E_n^\sim$ of o-continuous functionals. In the case of an arbitrary vector lattice E, we ought to define the spaces $L_n^\sim(E,F)$ and $E_n^\sim$ by making use of arbitrary nets instead of sequences [3, 28, 46, 48]. For an ideal space on a space with σ-finite measure, Theorem 2.1.5 demonstrates that definitions with sequences and with nets lead to the same result.

2.2.19. *Let E be a Banach lattice. Then $E^* = E^\sim$ with the following equality for the norms:* $\|f\| = \|\,|f|\,\|$ $\forall f \in E^*$; *i.e., E^* is a Banach lattice.*

$\lhd$ It suffices to verify the equality for the norms, which easily follows from formulas (5) and (7) and the monotonicity of the norm on E:

$$|f|(x) \le |f|(|x|) = \sup\{|f(y)| \mid |y| \le |x|\} \le \sup\{\|f\|\,\|y\| \mid |y| \le |x|\} = \|f\|\,\|x\|. \ \rhd$$

It is well known [3, 28, 46, 48] that conditions (B) and (C) for nets are satisfied in E^*.

Consider the *canonical embedding operator* $\pi : E \to E^{**}$:

$$\langle x, y\rangle = \langle y, \pi x\rangle; \quad x \in E,\ y \in E^*.$$

As is well known (see [46, § IX.7; 3, 48]), the operator is a *lattice homomorphism*; i.e., it satisfies the following condition:

$$|\pi x| = \pi(|x|), \quad x \in E.$$

It is obvious that every lattice homomorphism also preserves suprema and infima of finite collections of elements. Preserving infinite suprema and infima happens only in the case $E^* = E_n^\sim$ [46]. In this connection, of interest is the following

Proposition. *Let E be a Banach lattice, representing also a K-space with conditions (B) and (C). Let M be a subset in E. If $\pi(M)$ is o-bounded in E^{**} then M is o-bounded in E and*

$$\|\sup_E M\|_E = \|\sup_{E^{**}} \pi(M)\|_{E^{**}}, \tag{23}$$

where the index of the supremum sign means that the operation is fulfilled in the corresponding space.

$\lhd$ Without loss of generality, we may assume that M is directed upward. Since $\pi(M)$ is o-bounded, M is norm bounded and, by condition (B), there exists $\sup_E M \in E$ in E. Now we arrive at (23) since π preserves exact bounds of finite sets and is an isometry while condition (C) holds both in the spaces E and E^{**}. $\rhd$

2.2.20. Now we consider o-continuous functionals. Let E be an ideal space on (T, Σ, μ) which is also an order-dense ideal in L^0. Define the *dual space* E' as

$$E' = \left\{ y \in L^0 \mid \int |xy| \, d\mu < \infty \ \forall x \in E \right\}.$$

It is clear that the dual space is an ideal space. It may happen that $E' = \{0\}$ (for instance, if $E = L^0(0,1)$ or $E = L^p(0,1)$, $0 < p < 1$). However, we shall soon see that there are always many integral functionals in the case of a Banach ideal space and in particular $\operatorname{supp} E' = T$. If $E = L^p$ $(1 \le p \le \infty)$ then $E' = L^{p'}$ $(1/p + 1/p' = 1)$.

Given an $y \in E'$, we may construct some linear functional φ_y on E by the formula

$$\varphi_y(x) = \int x(t)y(t)\, d\mu(t), \quad x \in E. \tag{24}$$

By the Lebesgue dominated convergence theorem it is obvious that $\varphi_y \in E_n^{\sim}$.

Proposition [28, Theorem VI.1.1]. *Formula (24) provides the general form of an o-continuous functional on E. The mapping $y \in E' \mapsto \varphi_y \in E_n^{\sim}$ is an order and linear isomorphism of K-spaces; i.e., $y \ge 0$ if and only if $\varphi_y \ge 0$.*

Hence we may easily conclude that

$$|\varphi_y|(x) = \int x(t)|y(t)|d\mu(t) = \varphi_{|y|}(x), \quad x \in E.$$

The proposition shows that $E_n^{\sim}$ is exactly the set of those functionals on E which admit integral representation.

2.2.21. *Proposition* [28, Theorem VI.1.5]. *If E is a Banach ideal space then* $\operatorname{supp} E = \operatorname{supp} E'$ *and thus* $E_n^\sim$ *separates the points of* E.

Identify E' with $E_n^\sim$ by Proposition 2.2.20 and extend the norm from E^* to E' as follows:

$$\|y\| = \sup \left\{ \left| \int xy \, d\mu \right| \mid \|x\| \leq 1 \right\}, \quad y \in E'.$$

Now we formulate answers to some natural questions connected with the notion of dual space.

2.2.22. *Proposition* [28, Theorem VI.1.2]. *The dual space E' is a perfect Banach ideal space.*

Construct the *second dual* ideal space $E'' = (E')'$ over the ideal space E'. Then the following inclusion and inequality for the norms hold: $E \subset E''$, $\|x\|_{E''} \leq \|x\|_E$; moreover, the inclusion and the inequality can both be strict. For instance, if $E = c_0$ then $E'' = l^\infty$ and $E \neq E''$ (but the norms in the case are equal; in general, the strict inequality for the norms is connected with an "exotic" situation, namely with a Banach ideal space without condition (C) (see [28, Theorem VI.1.6]).

2.2.23. *Proposition* [28, Theorem VI.1.7]. *Banach ideal spaces E and E'' have the same elements and norms if and only if E is perfect.*

The question whether all functionals on a Banach ideal space E possess integral representation arouses a natural interest.

2.2.24. *Proposition* [28, Theorem VI.1.4]. *If E is a Banach ideal space then* $E^* = E_n^\sim$ *if and only if* $E \in (A)$ *(which means separability in the case when the measure μ is separable).*

$\lhd$ Sufficiency was already proved in Proposition 2.2.14 for the general case of operators. $\rhd$

The preceding proposition yields $(L^\infty)^* \neq (L^\infty)_n^\sim$ in the case $E = L^\infty$. The so-called Banach limits on L^∞ provide examples of the functionals on L^∞ that are not o-continuous (see, for instance, [28, II.4.2]). The functionals on E^* disjoint from the functionals of $E_n^\sim$ can be described by the generalized Yosida-Hewitt theorem [28, Theorem XI.4.6]. We return to the question in the comments on the chapter.

2.2.25. In many constructions of functional analysis, the taking of the *dual* of an operator is required. Here we address the question for regular operators.

Let E and F be a Banach lattices; moreover, let F be a K-space; let $U \in L^{\sim}(E,F)$. Then $U \in \mathscr{L}(E,F)$ and the dual operator $U^* \in \mathscr{L}(F^*,E^*)$ is defined. To begin with we consider its properties in the pair of the Banach lattices F^* and E^*.

Proposition. *The operator U^* belongs to $L^{\sim}(F^*,E^*)$ and $|U^*| \le |U|^*$.*

$\triangleleft$ If $f' \in F^*_+$ and $x \in E_+$ then

$$\begin{aligned}\langle x, |U^* f'|\rangle &= \sup\{\langle y, U^* f'\rangle \mid |y| \le x\} = \sup\{\langle Uy, f'\rangle \mid |y| \le x\} \\ &= \sup\{\langle |U|(|y|), f'\rangle \mid |y| \le x\} = \langle |U|x, f'\rangle = \langle x, |U|^* f'\rangle. \ \triangleright\end{aligned}$$

Example 5.9 in [3] demonstrates that the inequality of the moduli in the statement of the proposition can be strict. It is important that the pathology disappears in the case of order continuous operators.

Observe that $U \le V$ yields $U^* \le V^*$, which is of use below in proving Proposition 2.2.26. In the proof we also make use of the following agreement convenient in what follows: if the operator U^* acts from $F_n^{\sim}$ into $E_n^{\sim}$ then we identify $F_n^{\sim}$ with the ideal space F' and $E_n^{\sim}$ with the ideal space E' by Proposition 2.2.20 and employ the same notation U^* for the operator induced by U^* and acting from F' into E'.

2.2.26. Proposition. *Let E and F be Banach ideal spaces. If $U \in L_n^{\sim}(E,F)$ then*

$$U^*(F_n^{\sim}) \subset E_n^{\sim}; \tag{25}$$

$$|U^*|f' = |U|^* f' \ \forall f' \in F_n^{\sim}. \tag{26}$$

$\triangleleft$ First we prove inclusion (25). Take an $f' \in F_n^{\sim}$ and demonstrate that $U^* f' \in E_n^{\sim}$. If $x_n \downarrow 0$ in E then $Ux_n \xrightarrow{o} 0$ since $U \in L_n^{\sim}(E,F)$. Then

$$\langle x_n, U^* f'\rangle = \langle Ux_n, f'\rangle \to 0,$$

which means that $U^* f' \in E_n^{\sim}$.

Henceforth, it is convenient to identify $E_n^{\sim}$ with E' and $F_n^{\sim}$ with F' by making use of Proposition 2.2.20. Then we can consider U^* as a regular operator from F' into E'; moreover, $|U^*| \le |U|^*$ by 2.2.25.

To prove (26), we first check that $U^* \in L_n^\sim(F', E')$. If $y_n \downarrow 0$ in F' then, for every $x \in E_+$, we have

$$\langle x, |U|^* y_n \rangle = \langle |U|x, y_n \rangle \to 0.$$

Since $|U|^* y_n \downarrow$ by the positivity of $|U|^*$; therefore, $|U|^* y_n \downarrow 0$. Hence,

$$|U^* y_n| \le |U^*| y_n \le |U|^* y_n \downarrow 0;$$

i.e., $U^* y_n \xrightarrow{o} 0$ in E'.

Now we can apply the already-proved part of the proposition to the operator U^* and obtain the operator $U^{**} : E'' \to F''$; moreover, it is clear that $U^{**}|_E = U$. Since E is an order-dense ideal in E'', we have $|U|x = |U^{**}|x$ for all $x \in E$. By 2.2.25, $|U^{**}| \le |U^*|^*$. Thus, the inequality $|U| = |U^{**}| \le |U^*|^*$ implies that $|U|^* \le |U^*|^{**} = |U^*|$. Comparing the inequality with $|U^*| \le |U|^*$, we arrive at (26). $\rhd$

2.2.27. Corollary. *Let E and F are Banach ideal spaces with condition (A) and let $U \in L^\sim(E, F)$. Then $U^* \in L^\sim(E, F)$ and $|U|^* = |U^*|$.*

$\lhd$ It is straightforward from Propositions 2.2.24 and 2.2.26. $\rhd$

2.3. Spaces of Vector-Functions

Henceforth we need functions with values in a Banach space while representing operators. We call such functions *vector-functions* or *vector-valued functions*, keeping in mind that usually vectors in infinite-dimensional spaces are meant. In this case, the various definitions of measurability appear. As a result, problems arise on comparing different definitions of measurability of a vector-function to one another and to the classical definition of measurability for a function of two variables. Moreover, operators of some classes are described as operators admitting representation by means of vector-functions in the corresponding spaces of vector-functions whose norms are calculated iteratively. Such spaces are said to be spaces with mixed norm. In the present section, we briefly present necessary facts in this direction. We refer the reader to [21–23] for a more detailed exposition of the material connected with measurability.

2.3.1. A function of the form

$$\vec{f}(t) = \sum_{i=1}^{n} \chi_{A_i}(t) x_i \quad (x_i \in X,\ A_i \in \Sigma,\ A_i \cap A_j = \varnothing\ (i \neq j)) \tag{1}$$

is called *finite-valued.* A function $\vec{f} : T \to X$ is called *measurable* if there is a sequence $\{\vec{f}_n\}$ of measurable finite-valued functions such that $\|\vec{f}_n(t) - \vec{f}(t)\|_X \to 0$ for almost all $t \in T$. Without loss of generality we may assume that

$$\|\vec{f}_n(t)\|_X \leq \|\vec{f}(t)\|_X$$

almost everywhere (see [22, Theorem 6.2]).

Now let Y be a linear set of functionals in X^* which is total over X. We say that a function $\vec{f} : T \to X$ is *Y-scalarly measurable* provided that the functions $t \to \langle \vec{f}(t), x' \rangle$ are measurable for every $x' \in Y$. Two such functions $\vec{f}$ and $\vec{g}$ are called *Y-scalarly equivalent* if $\langle \vec{f}(t), x' \rangle = \langle \vec{g}(t), x' \rangle$ almost everywhere for each $x' \in Y$ (the negligible set depending generally on x'). The following theorem describes a connection between measurability and scalar measurability:

Theorem. *A function $\vec{f} : T \to X$ is measurable if and only if the following assertions are valid:*

(a) *$\vec{f}$ is Y-scalarly measurable for every (some) $Y \subset X^*$;*

(b) *$\vec{f}$ is almost separable-valued; i.e., the set $\vec{f}(T)$ is separable with respect to the norm on X after deleting a negligible set in T.*

Thus, measurability and scalar measurability coincide for functions with values in a separable space, while failing to coincide in the general case. A discussion of the interplay in general Banach spaces and Banach lattices can be found in [21].

Introduce two spaces of measurable vector-functions according to the definitions given above (see 1.6.4(3,4)).

2.3.2. Let E be an ideal space and let X be a Banach space. We denote by $E(X)$ the space of all measurable functions $\vec{f} : T \to X$ such that the numeric function $|\vec{f}| = \|\vec{f}(\cdot)\|_x$ belongs to E. Equivalent functions in $E(X)$ are identified. If E is a Banach ideal space then the norm

$$\|\vec{f}\| = \||\vec{f}|\|_E$$

makes $E(X)$ into a Banach space. Observe that the function $\|\vec{f}(\cdot)\|_X$ is automatically measurable by Theorem 2.3.1.

A typical example of a space like $E(X)$ is $L^p(X)$ with the norm

$$\|\vec{f}\| = \left(\int \|\vec{f}(t)\|_X^p d\mu(t)\right)^{1/p}.$$

The scope of the theory of $E(X)$ spaces includes a wide range of problems: questions of geometry of Banach spaces; boundedness of operators in the spaces; applications to representation of operators; solution to various classes of equations and so on.

Here we dwell upon only one structure question of the theory of the spaces, namely, on the question whether the tensor product $E \otimes X$ is dense in $E(X)$.

The *algebraic tensor product* $E \otimes X$ is identified with the set of vector-functions of the form

$$\vec{f}(t) = \sum_{i=1}^{n} e_i(t)x_i \quad (e_i \in E,\ x_i \in X).$$

It is clear that $E \otimes X$ is a linear subset in $E(X)$ whose connection with the initial spaces E and X is rather visual.

Theorem. *Let E be a Banach ideal space and let X be an infinite-dimensional Banach space. The following assertions are equivalent:*

(1) *$E \otimes X$ is dense in $E(X)$ with respect to the norm;*

(2) *condition (A) is satisfied in E;*

(3) *the set of finite-valued functions* (1) *with $\chi_{A_i} \in E$ is dense in $E(X)$ with respect to the norm.*

$\lhd$ (2) $\Rightarrow$ (3) by 2.3.1; (3) $\Rightarrow$ (1) is obvious. It is these claims that we use in applications. So here we confine ourselves only to what was said above although the basic nontrivial implication is (1) $\Rightarrow$ (2) [13]. The difficulty lies in the fact that the validity of (1) is assumed only for a single fixed infinite-dimensional Banach space X. $\rhd$

2.3.3. Now we turn to defining some space of scalarly measurable vector-functions which is similar to $E(X)$. We suppose that Y is a *norming* subspace (for X) in X^*; i.e.,

$$\|x\|_X = \sup\{|\langle x, x'\rangle| \mid x' \in Y,\ \|x'\| \le 1\} \quad \forall x \in X.$$

In the case under consideration, the matter is somewhat more complicated than the definition of $E(X)$. The reason is that the function $\|\vec{f}(t)\|_X$ is not correctly defined under identifying Y-scalarly equivalent functions in a way natural for such spaces. Indeed, we have the pointwise supremum

$$\|\vec{f}(t)\|_X = \sup\{|\langle \vec{f}(t), x'\rangle| \mid x' \in Y,\ \|x'\| \leq 1\}$$

of an uncountable of measurable functions for which it is easy to provide analogs of the pathological constructions in Section 2.1. In this connection, we define

$$|\vec{f}| = \sup\{|\langle \vec{f}(\cdot), x'\rangle| \mid x' \in Y,\ \|x'\| \leq 1\}, \tag{2}$$

where the supremum is understood with respect to the order of L^0. The function $|\vec{f}|$ is almost everywhere finite. Indeed, by the remark on Theorem 2.1.5, the supremum in (2) is attained on some countable set $\{x'_n\}$ of functionals for which the estimate $|\langle \vec{f}(t), x'_n\rangle| \leq \|\vec{f}(t)\|$ holds on a common set of full measure.

We denote by $E_s(X, Y)$ the set of all Y-scalarly measurable functions $\vec{f} : T \to X$ such that $|\vec{f}| \in E$. We assume that Y-scalarly equivalent functions are identified. If E is a Banach ideal space then the norm of $E_s(X, Y)$ is introduced as follows:

$$\|\vec{f}\| = \|\, |\vec{f}|\, \|_E.$$

Generally speaking, this normed space is not complete. However it follows from Theorem 2.4.7 (see below) that the most important example $E_s(X^*) = E_s(X^*, X)$ gives a Banach space.

2.3.4. For a function $\vec{f} \in L^1_s(X, Y)$, the integral is defined in the weak sense as some element of Y^*:

$$\left\langle \int \vec{f}(t)\, d\mu(t), y \right\rangle := \int \langle \vec{f}(t), y\rangle\, d\mu(t), \quad y \in Y.$$

In the case $\vec{f} \in L^1_s(X^*)$ we obviously have $\int \vec{f}(t)\, d\mu(t) \in X^*$.

For a function $\vec{f} \in L^1(X)$, the Bochner integral [21, 23] $\int \vec{f}(t)\, d\mu(t) \in X$ is defined. Its definition agrees certainly with that of weak integral:

$$\left\langle \int \vec{f}(t)\, d\mu(t), x' \right\rangle = \int \langle \vec{f}(t), x'\rangle\, d\mu(t), \quad x' \in X^*.$$

2.3.5. In the present book, we are mainly interested in integral operators in the classical sense; namely, in operators acting between spaces of measurable functions and having measurable functions of two variables as kernels. However, we cannot ignore the representation with the help of vector-functions since it is natural to compare various approaches to proving integral representability. Indeed, if $X = F$ is a Banach ideal space of functions of a variable s then every vector-function $\vec{f} : T \to X = F$ generates a function of two variables by the formula

$$\Phi(s,t) = [\vec{f}(t)](s). \tag{3}$$

Here, the function $\Phi(s,t)$ may fail to be measurable as a function of two variables even in the simplest cases (for instance, for the measure space $[0,1]$ with the Lebesgue measure and $X = L^2(0,1)$). Indeed, W. Sierpinski constructed the well-known example of a subset of a square which is Lebesgue nonmeasurable and has at most two common points with every straight line. The characteristic function $\Phi_0(s,t)$ of the set is nonmeasurable as a function of two variables whereas vector-function (3) is the zero function since $\forall t\ \Phi_0(s,t) = 0$ almost everywhere. By adding Φ_0 to nonzero measurable vector-functions, we obtain similar examples with nonzero vector-functions. However, there are no principally different examples: each function in (3) can be improved so as to become measurable provided that $\vec{f}$ is a measurable vector-function.

2.3.6. Lemma. *Let F be a Banach ideal space on (S, Λ, ν). For every measurable function $\vec{f} : T \to F$, there is a measurable function $K(s,t)$ on $S \times T$ such that, for almost all $t \in T$, the following equality holds for almost all $s \in S$: $K(s,t) = [\vec{f}(t)](s)$.*

$\lhd$ Approximate $\vec{f}$ by a sequence $\{\vec{f}_n\}$ of finite-valued functions in the sense of norm convergence almost everywhere in F:

$$\|\vec{f}_n(t) - \vec{f}(t)\|_F \to 0 \text{ almost everywhere,}$$
$$\|\vec{f}_n(t)\| \leq \|\vec{f}(t)\| \text{ almost everywhere.}$$

Every function $\vec{f}_n$ generates a measurable function $K_n(s,t)$ by formula (3). Restricting T and S, if necessary, we may assume that $\|\vec{f}(\cdot)\|_F \in L^1(T,\mu)$ and $F \subset L^1(S,\nu)$. Then $\|K_n - K_m\| \to 0$ in the space $L^1(S \times T)$ constructed for the product $\nu \otimes \mu$ of measures. By the completeness of the space, there exists a function $K \in L^1(S \times T)$ such that $\|K_n - K\| \to 0$. By Fubini's theorem, the function K possesses the required property. $\triangleright$

The lemma shows that formula (3) determines an embedding of the space of measurable vector-functions into the space of measurable functions on the product assuming some reservation about improvement of a function of two variables. Henceforth, we simply speak of the embedding defined by (3).

Studying questions about analytical representation of operators, we often need to prove the coincidence of integrals of different types. As a rule it can be easily established by passing to the weak integral.

2.3.7. Further, it will be convenient to compare properties of vector-functions and functions of two variables related to one another by (3) within the framework of the theory of spaces with mixed norm. It is all the more appropriate since the spaces with mixed norm allow one to describe the containment of integral operators in some important classes in terms of the properties of their kernels.

Let E be a Banach ideal space on (T, Σ, μ) and let F be a Banach ideal space on (S, Λ, ν) with condition (C). Denote by $E[F]$ the space of all measurable functions K on $S \times T$ satisfying the following conditions:

(1) the function $s \to K(s, t)$ belongs to F for almost all $t \in T$;

(2) the function $|K|(t) = \|K(\cdot, t)\|_F$ belongs to E.

A nontrivial Theorem XI.1.2 in [28] stemming from the works of A. C. Zaanen, W. A. J. Luxemburg, and Yu. I. Gribanov demonstrates that condition (C) in F provides measurability for the function $|K|$. So, it is clear that $E[F]$ is a linear space and thus an ideal space on $S \times T$. If E is a Banach ideal space then the formula

$$\|K\|_{E[F]} = \|\, |K| \,\|_E$$

makes $E[F]$ into a Banach space named a *space with mixed norm.* Lemma 2.3.6 shows that formula (3) implements an isometric embedding of $E(F)$ into $E[F]$ as a closed subspace and a Banach sublattice. The following statement answers the question whether the spaces coincide.

Proposition. *Let the measure μ be not purely atomic. The following assertions are equivalent:*

(1) $E(F) = E[F]$ *(under embedding* (3)*);*

(2) *F is a Banach ideal space with condition (A).*

The case of an atomic measure is of no interest because then all functions are measurable. The basic part of the proposition was proved by H. W. Ellis in [24] (see also [8]).

Considerably greater difficulties appear in considering formula (3) for scalarly measurable functions. We study this question in Chapter 4 and its solution will be grounded on some criterion for integral representability of operators.

2.3.8. We complete the section with the statement of the generalized Kolmogorov-Nagumo theorem disclosing essential difficulties that arise in study of the spaces with mixed norm if we pass from the metric in L^p to the norm of an arbitrary Banach ideal space other than L^p. These difficulties become apparent in a broad range of questions from seeking into geometry of spaces with mixed norm to estimating singular integrals and proving Sobolev's embedding theorems.

By Fubini's theorem, we have

$$\Big(\int\limits_{S\times T} |K(s,t)|^p \, d(\nu\otimes\mu)(s,t)\Big)^{\frac{1}{p}} = \Big(\int\limits_S \Big(\int\limits_T |K(s,t)|^p \, d\mu(t)\Big)^{\frac{1}{p}p} d\nu(s)\Big)^{\frac{1}{p}}$$
$$= \Big(\int\limits_T \Big(\int\limits_S |K(s,t)|^p \, d\nu(s)\Big)^{\frac{1}{p}p} d\mu(t)\Big)^{\frac{1}{p}}.$$

This means that

$$\| \, \|K(s,t)\|_{L^p,t} \|_{L^p,s} = \| \, \|K(s,t)\|_{L^p,s} \|_{L^p,t}. \tag{4}$$

The equality gives grounds for the methods of proof well known as "detaching a variable." It turns out that equality (4) is characteristic of the L^p-norm.

Theorem (the generalized Kolmogorov-Nagumo theorem [15]). *Let E and F be Banach ideal spaces such that*

$$\| \, \|K(s,t)\|_{E,t} \|_{F,s} \sim \| \, \|K(s,t)\|_{F,s} \|_{E,t}$$

on the set of functions

$$K(s,t) = \sum_{k=1}^{n} e_k(t) f_k(s),$$

where $\{e_k\}$ is an arbitrary collection of pairwise disjoint functions in E_+ and $\{f_k\}$ is an arbitrary collection of pairwise disjoint functions in F_+. Then either there exist $p \in [1, \infty)$ and weights w_1 (on T) and w_2 (on S) such that

$$E = L^p(w_1\, d\mu), \quad F = L^p(w_2\, d\nu)$$

(the equality is set-theoretic with equivalence of the norms) or the norms in E and F are equivalent to an AM-norm; i.e., there exists a $C > 0$ such that

$$\| \sup_{i=1,\dots,n} |x_i| \,\| \le C \max_{i=1,\dots,n} \|x_i\|$$

for every finite collection of elements (the latter means that E and F are some "almost" weighted L^∞ spaces).

2.4. Dominated Operators

The material for the section was selected so as to achieve two goals. The first is to expose the fundamentals of the theory of dominated operators to an extend necessary for their use as the object and tools of research in Chapter 3. The second is to present at least in surveying form the material on representation of dominated operators by means of measurable vector-functions as an ideologically desirable line of presentation which is parallel to the exposition of the theory of integral operators in Chapter 4.

2.4.1. Let X be a Banach space and let E be an ideal space. An operator $U : X \to E$ is called *dominated* if the image of the unit ball in X is order bounded in E. Under the assumptions, there is some element $|U| \in E$ defined as

$$|U| = \sup\{|Ux| \mid x \in X,\ \|x\| \le 1\}.$$

The element $|U|$ is called the *abstract norm* of an operator U. The linear space of all dominated operators is denoted by $M(X, E)$. It is clear that every operator $U \in \mathscr{L}(X, L^\infty)$ is dominated as an operator taking values in an arbitrary ideal space including L^∞. Moreover, the example is universal. More precisely, let $U \in M(X, E)$ and set $g = |U|$. We assume that $1/0 = 0$. Introduce the operator $V : X \to L^\infty$ by the formula

$$Vx = \frac{1}{g} Ux. \tag{1}$$

It is clear that $V \in \mathscr{L}(X, L^\infty)$ and many properties of U are determined by the properties of V. Theorems on integral representation in various statements are among such properties. On the other hand, the spectral properties of U, for instance, have no simple expression in terms of the properties of V. Moreover, if we consider the whole space $M(X, E)$ and the properties of the latter as a whole rather than an individual dominated operator then the matter certainly does not reduce to the case L^∞.

If E is a Banach ideal space then the norm on $M(X, E)$ is introduced as follows:

$$\|U\|_M = \| \,|U|\, \|_E.$$

It is easy to demonstrate [9] that $M(X, E)$ thus becomes a Banach space.

Consider one of the most important examples of the spaces of dominated operators; namely, the space of Hilbert-Schmidt operators. Recall that an operator U acting from one Hilbert space H_1 into another H_2 is called a *Hilbert-Schmidt operator* $((U \in \mathfrak{S}_2(H_1, H_2))$ if the set of its s-numbers $\{\lambda_n\}$ is square summable and the Hilbert-Schmidt norm is defined by the equality

$$\sigma_2(U) = \left(\sum |\lambda_n|^2\right)^{1/2}.$$

It is well known that if $H_1 = L^2(S, \nu)$ and $H_2 = L^2(T, \mu)$ then an operator U is a Hilbert-Schmidt operator if and only if U is an integral operator

$$(Ux)(t) = \int K(s,t)x(s)\,d\nu(s)$$

with kernel $K(s,t)$ satisfying the following condition:

$$\sigma_2(U) = \left(\iint |K(s,t)|^2\,d\nu(s)\,d\mu(t)\right)^{1/2} < \infty.$$

By Fubini's theorem it is obvious that the preceding condition is equivalent to

$$g = \left(\int |(K(s,\cdot)|^2\,d\nu(s)\right)^{1/2} \in L^2(T, \mu).$$

It is clear that $|U| \geq g$. If we take the set $M = \{|Ux| \mid x \in L^2, \|x\| \leq 1\}$ then it becomes evident that $|U|$ is the supremum of M in the K-space L^2 and g is the pointwise supremum of some function set in M. It is easy to verify that these two suprema coincide in the separable case. The same is true in the general case but we become able to prove this fact only at the end of the section. Anyway, we have $g = |U|$ and thus *an operator U belongs to $\mathfrak{S}_2(L^2, L^2)$ if and only if $U \in M(L^2, L^2)$*; moreover, $\sigma_2(U) = \|U\|_M$.

We give one more example. An integral operator $U \in \mathscr{L}(L^2, L^2)$ is a *Carleman operator* if

$$g(t) = \left(\int |K(s,t)|^2\, d\nu(s)\right)^{1/2} < \infty \text{ almost everywhere.}$$

As above, one can verify that $g = |U|$ and thus the fact that U is a Carleman operator is equivalent to $U \in M(L^2, L^0)$.

2.4.2. One of the problems of the theory of dominated operators is as follows: Given a dominated operator, exhibit a space which is as narrow as possible and contains the abstract norm of the operator. In the theory of operators in Banach spaces, an important role is played by the elementary coincidence of the classes of continuous and bounded operators. A similar but less elementary theorem is true for dominated operators.

We begin with a criterion for the o-boundedness of a set in L^0.

Lemma. *A set $M \subset L^0$ is o-bounded in L^0 if and only if the following condition is satisfied:*

(+) $\lambda_n x_n \xrightarrow{o} 0$ for every number sequence $\lambda_n \to 0$ and every sequence $\{x_n\} \subset M$.

$\triangleleft$ If M is bounded then there is a function $y \in L^0$ such that $|x| \le y$ for all $x \in M$ whence $|\lambda_n x_n| \le |\lambda_n| y \to 0$ almost everywhere.

Conversely, let condition (+) be satisfied. Without loss of generality we may assume that M consists of nonnegative functions and is directed upward. By the remark on Theorem 2.1.5, there exists $y = \sup M$; moreover, there is a sequence $\{x_n\} \subset M$ such that $x_n \uparrow y$ in virtue of the conditions imposed on M. If $y(t) < \infty$ almost everywhere then the set M is bounded. Suppose that $y(t) = +\infty$ on a set A with $\mu(A) > 0$ and arrive at a contradiction. To this end, let $x_n \uparrow +\infty$ on A. By applying the Egorov theorem on uniform convergence to the sequence $\{1/(1+x_n)\}$, we conclude that there is a set $B \subset A$, $\mu(B) > 0$, satisfying

$$(\forall C > 0)(\exists n_c \in \mathbb{N}) x_n(t) \geq C \ \forall t \in B \, \forall n \geq n_c.$$

Set $C = m$ ($m \in \mathbb{N}$) and construct a sequence $n_1 < n_2 < \dots < n_m < \dots$ such that $x_n(t) \geq m$ for all $t \in B$ for $n \geq n_m$. Now we let $\lambda_m = 1/m \to 0$ and obtain

$$\lambda_m x_{n_m}(t) \geq \lambda_m m = 1; \ t \in B, \ m \in \mathbb{N},$$

which contradicts $\lambda_m x_{n_m} \to 0$ almost everywhere in virtue of $(+)$. $\triangleright$

REMARK. A similar criterion of o-boundedness is true in an arbitrary KB-space.

$\triangleleft$ For simplicity, let E be a Banach ideal space presenting a KB-space. If a set M satisfies condition $(+)$ with respect to the o-convergence on E then there is $y = \sup M \in L^0$ by the preceding lemma. As above, we may assume that there is $\{x_n\} \subset M : x_n \uparrow y$. Since $E \in (A)$, the set $\{x_n\}$ is bounded in the norm in view of $(+)$. Recalling that $E \in (B)$, we conclude that $y \in E$. $\triangleright$

Now we present two corollaries to Lemma 2.4.2.

2.4.3. The first of the results is connected with the o-continuity of operators and will be used in Chapter 4.

Proposition. *Let E and F be ideal spaces and $F = L^0$ or F is a KB-space. Then every o-continuous operator from E into F is o-bounded and thus is contained in the class $L_n^\sim(E, F)$.*

2.4.4. *Proposition.* *If $E = L^0$ or E is a KB-space then $U \in M(X, E)$ if and only if $\|x_n\| \to 0$ implies $Ux_n \xrightarrow{o} 0$ in the ideal space E.*

2.4.5. To begin with we obtain some analytical representation of dominated operators by means of scalarly measurable vector-functions. The result immediately follows from a powerful theorem by von Neumann and Maharam on existence of a lifting [22, 27]. The lifting property is a specific feature of the space of bounded measurable functions allowing one to choose a representative with special properties in every equivalence class of functions, which makes it possible to define correctly a value of a function at a point and thereby to define point functionals.

Denote by $\mathscr{L}^\infty = \mathscr{L}^\infty(T, \mu)$ the space of bounded measurable functions in which no identification of equivalent function was executed.

The lifting theorem. *There exists a mapping $\rho : \mathscr{L}^\infty \to \mathscr{L}^\infty$ called a lifting of $\mathscr{L}^\infty$, which possesses the following properties:*

(a) *$\rho(f)(t) = f(t)$ almost everywhere;*

(b) *$f(t) = g(t)$ almost everywhere yields $\rho(f)(t) = \rho(g)(t)$ for every t;*

(c) *ρ is linear;*

(d) *$\rho(\mathbf{1})(t) = 1$ for every t;*

(e) *$|\rho(f)(t)| = \rho(|f|)(t)$ for every t.*

2.4.6. Lemma. *Let $\{f_\alpha\} \subset \mathscr{L}^\infty$ and let $\rho(f_\alpha) = f_\alpha$ for every α. Then*

(1) *the pointwise supremum $f_\infty(t) = \sup_\alpha\{f_\alpha(t)\}$ is measurable;*

(2) *the function f_∞ is almost everywhere finite if and only if there exists $f = \sup f_\alpha$ in the K-space $L^0(T,\mu)$; in the last case $f(t) = f_\infty(t)$ almost everywhere.*

$\triangleleft$ First, suppose that $\sup f_\alpha = f \in L^\infty$. Then $f(t) \geq f_\alpha(t)$ almost everywhere for every α. Thus, $\rho(f)(t) \geq f_\infty(t)$ for every t. On the other hand, there exists a sequence $\{f_{\alpha_n}\}$ such that $f(t) = \sup\{f_{\alpha_n}(t)\}$ almost everywhere, whence $f(t) \leq f_\infty(t)$ almost everywhere and thereby $f_\infty(t) = f(t)$ almost everywhere and the function f_∞ is measurable.

In the general case, consider

$$f_\alpha^n(t) = f_\alpha(t) \wedge n\mathbf{1} \quad (n \in \mathbb{N}).$$

By properties (d) and (e) of a lifting, we obtain $\rho(f_\alpha^n)(t) = f_\alpha^n(t)$ for every t. As was proved, the functions $\sup_\alpha\{f_\alpha^n(t)\}$ are measurable for every n. Since

$$f_\infty(t) = \sup_n \sup_\alpha \{f_\alpha^n(t)\},$$

assertion (1) is proved. From the above reasoning one can easily derive the validity of (2). $\triangleright$

2.4.7. Theorem. *An operator $U : X \to L^0(T,\mu)$ is dominated if and only if there exists a function $\vec{f} \in L_s^0(X^*)$ such that*

$$(Ux)(t) = \langle x, \vec{f}(t)\rangle, \quad x \in X. \tag{2}$$

Moreover,

$$|U| = |\vec{f}|. \tag{3}$$

If E is a Banach ideal space then $U \in M(X,E)$ if and only if $\vec{f} \in E_s(X^*)$; moreover, $\|U\|_M = \|\vec{f}\|$.

$\triangleleft$ If an operator U is defined by formula (2) then it is obvious that $U \in M(X, L^0)$ and $|U| = |\vec{f}|$. Now let $U \in M(X, L^0)$. By 2.4.1, we may assume that $U \in \mathscr{L}(X, L^\infty)$ and $|U| = 1$. Using a lifting, define a functional on X by the formula

$$\varphi_t(x) = \rho(Ux)(t)$$

for every $t \in T$. Since

$$|\varphi_t(x)| = |\rho(Ux)(t)| = \rho(|Ux|)(t) \leq \rho(|U|)(t) = 1 = |U|$$

for all $t \in T$ and $x \in B_X$, we have $\varphi_t \in X^*$. On introducing the function $\vec{f} : t \mapsto \varphi_t \in X^*$, we conclude that

$$\rho(Ux)(t) = \langle x, \vec{f}(t) \rangle \tag{4}$$

and consequently the function $\vec{f}$ is X-scalarly measurable. Lemma 2.4.6 shows that the function $\vec{f}$ constructed by means of a lifting possesses in addition the following properties: the function $t \mapsto \|\vec{f}(t)\|_{X^*}$ is measurable and $|\vec{f}| = \|\vec{f}(\cdot)\|_{X^*} = |U|$. $\triangleright$

REMARK. The property just mentioned means that, for every function $\vec{f} \in E_s(X^*)$, there is a function $\vec{g} \in E_s(X^*)$ X-equivalent to the former (i.e., representing of the same element in the space of vector-functions); moreover, the function $t \to \|\vec{g}(t)\|_{X^*}$ is measurable and $|\vec{f}| = \|\vec{g}(\cdot)\|_{X^*}$.

2.4.8. Corollary. *An operator U belongs to $\mathscr{L}(X, L^\infty)$ if and only if there exists a function $\vec{f} \in L_s^\infty(X^*)$ such that*

$$(Ux)(t) = \langle x, \vec{f}(t) \rangle, \quad x \in X.$$

Moreover, $\|U\|_{\mathscr{L}(X,L^\infty)} = \operatorname{vrai\,sup}|\vec{f}|$ $(= \operatorname{vrai\,sup} \|\vec{f}(\cdot)\|_{X^*}$ *for a "good" representative of $\vec{f}$ by the preceding remark*).

2.4.9. By duality, we obtain the following fact:

Theorem. *The general form of an operator $U \in \mathscr{L}(L^1, X^*)$ is given by the formula*

$$Ue = \int e(t)\, \vec{f}(t)\, d\mu(t), \quad e \in L^1, \tag{5}$$

where $\vec{f} \in L_s^\infty(X^*)$; moreover, $\|U\| = \operatorname{vrai\,sup}|\vec{f}|$. *The integral in* (5) *is understood in the weak sense:*

$$\langle Ue, x \rangle = \int e(t) \langle x, \vec{f}(t) \rangle\, d\mu(t), \quad x \in X.$$

2.4.10. We thus obtained some analytical representations that are valid for wide classes of operators. However, the representation with the help of a scalarly measurable function is bad: the corresponding operators possess neither compactness properties nor some other specific operator properties. In this connection, the question whether the vector-functions in the representations could be taken measurable has been considered since the end of the 1930s. For the operators with values in L^∞, the answer is formulated in terms of equimeasurability (A. Grothendieck [25]; see also the articles [44, 45] by D. A. Vladimirov and the article [10] connecting the two approaches). For the operators on L^1, the answer is given by the following

2.4.11. *The Dunford-Pettis theorem.* *Let $U \in \mathscr{L}(L^1, X)$. Given $A \in \Sigma$, denote by U_A the operator $U_A(e) = U(e\chi_A)$. The following assertions are equivalent:*

(1) *the operator U is representable by means of a measurable function $\vec{f} \in L^\infty(X)$; i.e.,*

$$Ue = \int e(t)\,\vec{f}(t)\,d\mu(t); \tag{6}$$

(2) *for every $A_0 \in \Sigma$, $\mu(A_0) > 0$, there exists an $A \in \Sigma$, $A \subset A_0$, $\mu(A) > 0$, such that the operator U_A is compact;*

(3) *for every $A_0 \in \Sigma$, $\mu(A_0) > 0$, there exists an $A \in \Sigma$, $A \subset A_0$, $\mu(A) > 0$, such that the operator U_A is weakly compact.*

$\lhd$ It is clear that (2) $\Rightarrow$ (3).

(1) $\Rightarrow$ (2): Approximate $\vec{f}$ by a sequence of finite-valued functions $\{\vec{f}\}$ such that $\|\vec{f}_n(t) - \vec{f}(t)\|_X \to 0$ almost everywhere. By a vector version of the Egorov theorem [21, 22, 23], there is a subset $A \subset A_0$, $A \in \Sigma$, $\mu(A) > 0$, such that $\|\vec{f}_n(t) - \vec{f}(t)\|_X \to 0$ uniformly on A. Assign

$$U_n e = \int_A e(t)\,\vec{f}_n(t)\,d\mu(t).$$

In virtue of the estimate

$$\begin{aligned}\|U_n(e) - U(e\chi_A)\|_X &\le \int_A |e(t)|\,\|\vec{f}_n(t) - \vec{f}(t)\|_X\,d\mu(t)\\ &\le \|e\|_{L^1}\|(\vec{f}_n - \vec{f})\chi_A\|_{L^\infty(X)},\end{aligned}$$

we have $U_n \to U_A$ in the norm of $\mathscr{L}(L^1, X)$. Since the operators U_n are finite-dimensional, we obtain the compactness of U_A.

To prove (3) $\Rightarrow$ (1), we need a criterion for the weak compactness of a set in L^1.

Lemma [23, Theorem IV.8.9]. *A subset $M \subset L^1(T, \mu)$ is relatively weakly compact if and only if any of the following assertions is valid:*

(a) *the set M is norm bounded and uniformly integrable; i.e.,*

$$\sup\left\{ \int\limits_{A^n} |x(t)|\, d\mu(t) \mid x \in M \right\} \to 0, \quad A_n \downarrow \varnothing;$$

(b) *if $y_n \xrightarrow{o} 0$ in $L^\infty(T, \mu)$ then*

$$\sup\left\{ \int |x(t)\, y_n(t)|\, d\mu(t) \mid x \in M \right\} \to 0.$$

$\lhd$ Observe that (a) $\Rightarrow$ (b) by the Egorov theorem. $\rhd$

(3) $\Rightarrow$ (1): Clearly we may suppose that the operator U itself is compact since "global" representation can be trivially pasted from "local" ones. Consider the operator $U^* \in \mathscr{L}(X^*, L^\infty)$ which is weakly compact too. By (4), the operator admits the representation

$$\rho(U^* x')(t) = \langle x', \vec{f}(t) \rangle, \quad x' \in X^*,$$

where $\vec{f} \in L_s^\infty(X^{**}, X^*)$ and $\|U^*\| = \|\, |\vec{f}|\, \|_{L^\infty}$. Demonstrate that $\vec{f} \in L^\infty(X)$ (we assume that X is canonically embedded in X^{**}).

Observe that if a net x'_α tends to zero in the weak topology $\sigma(X^*, X)$ then $\rho\big(U^* x'_\alpha\big)(t) \to 0$ for every $t \in T$. First we establish that $U^* x'_\alpha \to 0$ in the weak topology $\sigma(L^\infty, L^\infty)^*$. Take $\varphi \in (L^\infty)^*$. Since U is weakly compact, we have $U^{**}((L^\infty)^*) \subset X$; whence $U^{**}\varphi \in X$ and

$$\big\langle U^* x'_\alpha, \varphi \big\rangle = \big\langle x'_\pi, U^{**}\varphi \big\rangle \to 0.$$

With every point $t \in T$ we associate the positive functional $\psi_t(y) = \rho(y)(t)$, $y \in L^\infty$. It is clear that $\psi_t \in (L^\infty)^*$. Thus,

$$\rho\big(U^* x'_\alpha\big)(t) = \psi_t\big(U^* x'_\alpha\big) \to 0.$$

Hence $\langle x'_\alpha, \vec{f}(t)\rangle \to 0$ for every $t \in T$; consequently, $\vec{f}(t) \in X$.

From now on, without loss of generality we assume that $\mathbf{1} \in L^1$. Indeed, a countable set of measurable functions given on sets of some partition of T can be joined into a measurable function.

So, the function $\vec{f} : T \to X$ is X-scalarly measurable. By Theorem 2.3.2, if we prove that the function is separably-valued then we obtain the measurability of $\vec{f}$, thus proving the theorem.

So, it therefore suffices to prove that the image $U(L^1)$ is separable. To this end, it sufficed in turn to establish that the image of the order interval $[0, \mathbf{1}] = \{e : 0 \le e \le \mathbf{1}\}$ under the mapping U is compact in X. We use the fact that all order intervals are weakly compact in L^1 (as in every Banach ideal space with property (A), see [3, 28]; by the way, for the case of L^1 one can directly apply the above lemma). Then it suffices to prove that $e_n \to 0$ in $\sigma(L^1, L^\infty)$ yields $\|Ue_n\| \to 0$. Moreover, we can pass to the Banach sublattice generated by $\{e_n\}$, which is itself an L-space and in addition separable. Then $U(L^1)$ is separable; therefore, we may assume that X is separable. Suppose that $\|Ue_n\| \not\to 0$. By passing to a subsequence if necessary, we find $\varepsilon > 0$ such that $\|Ue_n\| > \varepsilon$ $\forall n$ and so there exist $x'_n \in X^*$, $\|x'_n\| \le 1$, such that

$$\left|\langle Ue_n, x'_n\rangle\right| > \varepsilon.$$

We have

$$\langle Ue_n, x'_n\rangle = \langle e_n, U^* x'_n\rangle.$$

By the separability of X, we have $x'_{n_k} \to x'$ in $\sigma(X^*, X)$ for a subsequence x'_{n_k} so that $U^* x'_{n_k} \to U^* x'$ almost everywhere as was mentioned in the beginning of the proof. Moreover, the sequence $\{U^* x'_n\}$ is order bounded in L^∞. By the criterion for weak compactness of a set in L^1 (see the lemma above), we have

$$\sup_n \left|\langle e_n, U^* x'_{n_k} - U^* x'\rangle\right| \underset{k\to\infty}{\longrightarrow} 0.$$

The inequality

$$|\langle e_{n_k}, U^* x'\rangle| \ge \left|\langle e_{n_k}, U^* x'_{n_k}\rangle\right| - \left|\langle e_{n_k}, U^* x'_{n_k} - U^* x'\rangle\right|$$

yields

$$\overline{\lim}\, |\langle e_n, U^* x'\rangle| \ge \underline{\lim} \left|\langle Ue_n, x'_n\rangle\right| \ge \varepsilon.$$

This contradicts the fact that $e_n \to 0$ in the weak topology.

Observe a number of corollaries to the proved theorems for the cases in which we succeed in obtaining the strong measurability of a representative vector-function or existence of a measurable kernel with the help of the results of Section 2.3.

2.4.12. *Proposition.* *Every operator $V \in \mathscr{L}(X, L^\infty)$ admits the representation*

$$(Vx)(t) = \langle x, \vec{f}(t) \rangle$$

and every operator $U \in \mathscr{L}(L^1, X^*)$ admits the representation

$$Ue = \int e(t)\,\vec{f}(t)\,d\mu(t)$$

with $\vec{f} \in L^\infty(X^*)$ in each of the following situations:

(a) *X is a reflexive Banach space;*

(b) *X^* is separable.*

$\lhd$ Assertion (b) is straightforward from Theorem 2.3.2. Assertion (a) in the case of $\mathscr{L}(L^1, X^*)$ follows from the fact that every operator with values in a reflexive Banach space is obviously weakly compact and from Theorem 2.4.11. The case $\mathscr{L}(X, L^\infty)$ is settled by duality. $\rhd$

2.4.13. Proposition 2.4.12 does not open a straightforward opportunity to obtain theorems on integral representability for corresponding classes of operators by measurable kernels from spaces with mixed norm in full generality. We can obtain such theorems only in Chapter 4 by making use of another technique. In Chapter 3 we however need particular cases of the representation theorems for L^p.

We can obtain these results right now; therefore, we formulate them:

Proposition. (1) *Let $U \in M(L^p(S,\nu), L^q(T,\mu))$; $1 \le p < \infty$, $1 \le q \le \infty$. Then the following representation holds:*

$$(Ux)(t) = \int K(s,t)\,x(s)\,d\nu(s), \quad x \in L^p(S,\nu),$$

with $K \in L^q[L^{p'}]$ $(1/p + 1/p' = 1)$; moreover,

$$|U| = \|K(s,\cdot)\|_{L^{p'},s}, \quad \|U\|_M = \|K\|_{L^q[L^p]}.$$

(2) *Let* $U \in \mathscr{L}(L^1(T,\mu)), L^p(S,\nu))$, $1 < p \le \infty$. *Then the following integral representation holds:*

$$(Ue)(s) = \int K(s,t)\, e(t)\, d\mu(t), \quad e \in L^1(T,\mu),$$

with $K \in L^\infty[L^{p'}]$; *moreover,*

$$\|U\| = \operatorname*{vrai\,sup}_t \|K(s,t)\|_{L^{p'},s}.$$

$\lhd$ For $1 < p < \infty$, the L^p space is reflexive, and both claims follow immediately from 2.4.12. Passage from vector integrals to pointwise integrals can be realized by verifying the equality in the weak sense. The unsettled cases are covered by the results of Chapter 4. $\rhd$

Comments

The sources [3, 16, 18, 29, 42, 46, 48] are basic for the theory of regular operators in Banach lattices. It the text of the chapter, we consciously avoided the most general statements that involve arbitrary vector lattices. In this case certain subtleties are revealed sometimes. In particular, if the range is a vector lattice but not a K-space, then there may be a difference between the classes of regular and order bounded operators.

2.1. Ideal spaces constitute a subclass of the class of vector lattices. The theory of ideal spaces began developing later than that of vector lattices and independently of the latter for the time being. The synthesis of these theories occurred in the 1960s in the works of W. A. J. Luxemburg, A. C. Zaanen, and G. Ya. Lozanovskiĭ (see the bibliography in [19, 48]).

2.2. Comments on this main section of Chapter 2 are in [16, 18, 48]. We only observe that the dual operator was studied by U. Krengel and Yu. Synnatzscke who obtained much more general results than those stated in 2.2.25 and 2.2.26 (see [3]).

In the principal text, we mentioned the generalized Yosida-Hewitt theorem for functionals describing the band in $E^\sim$ complementary to $E^\sim_n$ as the band of singular functionals; i.e., functionals vanishing on some order-dense ideal. The correspondent material is in detail exposed in [48, Chapter 12] and [28, Theorem

X.3.6]. Applications to various problems of analysis are given in [28, § X.5], [18, § 4.2], and [20].

Here we consider an operator variant of the Yosida-Hewitt theorem for general vector lattices. To state this, we recall the definitions of some classes of operators with slightly changing the notations of [28] so as to make them closer to [48].

Let E be a vector lattice and let F be a K-space. Then the space $L_n^\sim(E,F)$ of o-continuous operators is defined by nets and the analogous space of sequentially o-continuous operators is denoted by $L_{n\sigma}^\sim(E,F)$. Denote by $L_n^\sim(E,F)$ the band complementary to $L_n^\sim(E,F)$ in the K-space $L^\sim(E,F)$. A member of $L_{sn}^\sim$ is called a *singular normal operator*. The space $L_{sn\sigma}^\sim = (L_{n\sigma}^\sim)^d$ and the term *singular σ-normal operator* are introduced similarly. It is clear that

$$L^\sim = L_n^\sim \oplus L_{ns}^\sim, \quad L^\sim = L_{n\sigma}^\sim \oplus L_{sn\sigma}^\sim.$$

Usefulness of the versions of the Yosida-Hewitt theorem for functionals consists in characterizing functionals disjoint from o-continuous ones as those vanishing on massive sets. We introduce corresponding definitions in the case of operators.

An operator $U \in L^\sim(E,F)$ is said to be *singular* provided that there exists an order-dense ideal $G \subset E$ such that $U|_G = 0$. An operator $U \in L^\sim(E,F)$ is called *strongly singular* if, for every nonzero band $G \subset E$, there exists a nonzero band $G_0 \subset G$ such that $U|_{G_0} = 0$. The set of all singular operators is denoted by $L_s^\sim(E,F)$ and the set of all strongly singular operators, by $L_{ss}^\sim(E,F)$. In [46], a member of $L_{sn}^\sim$ is called *antinormal*, a member of $L_s^\sim$, *abnormal*. In the case of $F = \mathbb{R}$, the class of strongly singular functionals was introduced by G. Ya. Lozanovskiĭ [36] who called their members *localized functionals*. It is clear that the classes $L_s^\sim$ and $L_{ss}^\sim$ are ideals in $L^\sim$ and

$$L_{ss}^\sim(E,F) \subset L_s^\sim(E,F), \quad L_s^\sim(E,F) \subset L_{sn}^\sim(E,F). \tag{1}$$

As is well known [46], $L_s^\sim(E,F)$ is an order-dense ideal in $L_{sn}^\sim(E,F)$. Elucidation of the conditions for equality to hold in (1) is an important matter as regards applications.

We say that the generalized Yosida-Hewitt theorem holds for a vector lattice E and a K-space F if

$$L_{sn}^\sim(E,F) = L_s^\sim(E,F). \tag{2}$$

It is well known that in the case of functionals the Yosida-Hewitt theorem holds provided that E possesses the Egorov property (see below) or if there is an order-dense ideal in E with sufficiently many o-continuous functionals (see [48]; observe that the second case reduces in fact to the first as is shown below). The requirement on E cannot be omitted as the instance of $E = C[0,1]$ demonstrates. We generalize this result to operators.

We say that the *Egorov property* is fulfilled in a vector lattice E (all vector lattices are supposed Archimedean) if the diagonal sequence theorem holds in its every order interval: for an $x_0 \geq 0$ and a double-sequence $\{x_{nk}\}$ in E such that $x_0 \geq x_{nk} \downarrow_k 0\ \forall n \in \mathbb{N}$, there exists a sequence $y_m \downarrow 0$ in E such that $y_m \geq x_{n,k(n,m)}$ for an arbitrary pair $(n,m) = \mathbb{N} \times \mathbb{N}$ for some $k = k(n,m)$. It is obvious that the Egorov property is fulfilled in every ideal space on a space with σ-finite measure.

Theorem (the generalized Yosida-Hewitt theorem for operators). *Let E be a vector lattice possessing the Egorov property and let F be a K-space of countable type. Then equality* (2) *holds.*

$\lhd$ We sketch the proof. What we need is to verify that

$$U \in L_s^{\sim}(E,F) \tag{3}$$

for every operator $U \in L_{sn}^{\sim}(E,F)$. Without loss of generality, we may assume that $U \geq 0$. Denote by P the band projection onto $L_n^{\sim}(E,F)$. It is known [3] that

$$PU(x) = \inf\{\lim_{\alpha} U(x_\alpha) \mid 0 \leq x_\alpha \uparrow x\}, \quad x \in E_+. \tag{4}$$

By the definition of $L_{sn}^{\sim}(E,F)$ we have

$$PU(x) = 0\ \forall x \in E. \tag{5}$$

If we succeed in demonstrating that the infimum in (4) is attained at some net $x_\alpha \uparrow x$ for every $x \in E_+$ then we would obtain an order-dense ideal on which U equals zero; i.e., (3) would be verified. However, we failed to succeed in finding a direct proof of the fact; therefore, we use the band projection P_σ onto $L_{n\sigma}^{\sim}(E,F)$ which acts by a formula similar to (4) with sequences standing for nets. Represent the operator U as

$$U = P_\sigma U + (I - P_\sigma)U.$$

By making use of the Egorov property of E and the fact that F is of a countable type, one can easily prove that, for every operator V, the value $P_\sigma V(x)$ is attained at some sequence. Since $P_\sigma V = 0$ for $V = (I - P_\sigma)U$, hence it follows that $(I - P_\sigma)U$ equals zero on some order-dense ideal.

The operator $P_\sigma U$ belongs to $L^\sim_{n\sigma}(E, F) \cap L^\sim_{ns}(E, F)$. The proof of Theorem 87.6 in [48] in which the case of functionals is discussed can be word for word translated to the case of operators (only the countable type of F is used). In virtue of the result, we have $P_\sigma U \in L^\sim_s(E, F)$.

Thus, $U \in L^\sim_s(E, F)$. $\triangleright$

Observe the following: *For F to be of countable type is necessary for validity of the above theorem.*

$\triangleleft$ Indeed, let $E = L^\infty(0,1)$ be an ideal space (and thus the Egorov property is fulfilled). By the realization theorem for a K-space of bounded elements, $L^\infty(0,1)$ is isomorphic to the K-space $C(Q)$, where Q is an extremally disconnected compact set. Define some positive operator $U : L^\infty(0,1) = C(Q) \to l^\infty(Q)$ by the formula

$$(Ux)(t) = \{x(t) \mid t \in Q\}.$$

It is obvious that U is strictly positive and thus $U \notin L^\sim_s(C(Q), l^\infty(Q))$. On the other hand, by (4) one can easily verify that $PU = 0$ whence $U \in L^\sim_{ns}(C(Q), l^\infty(Q))$. $\triangleright$

REMARK. The statement of the generalized Yosida-Hewitt theorem can be slightly improved. Namely, it suffices to require that E possesses an order-dense ideal E_0 that admits a decomposition into arbitrarily many bands with the Egorov property. An instance of such a vector lattice is provided by a vector lattice E possessing an order-dense ideal with sufficiently many o-continuous functionals. To prove this fact, we can take the order completion of such an ideal, which also has a total set of o-continuous functionals and is realizable as an ideal space (see [28]).

The results of the subsection are exposed for the first time. They were obtained jointly by A. V. Bukhvalov and M. Ya. Yakubson. For detailed exposition and development of the presented results see [5] by A. Basile, A. V. Bukhvalov, and M. Ya. Yakubson.

2.3. The material on Banach-space-valued functions and on analytical representation of operators by such functions can be found in [21, 23]. The spaces $E(X)$ were introduced at the beginning of the 1950s in the articles on the theory of ideal spaces. Theorem 2.3.3 stems from the article of B. S. Mityagin and

A. S. Shwartz [38] in which, however, only the case $X = l^2$ was considered in proving the nontrivial implication (1) $\Rightarrow$ (2). The general situation demanded a quite different technique related to o-continuous projections which was developed by A. V. Bukhvalov in [13].

The cited definition of the spaces of scalarly measurable functions was given and used in [7, 11] although it in essence dates back to the Grothendieck article [25]. The priority references on the material of Subsections 2.3.6 and 2.3.7 see in [8, 21, 24].

The generalized Kolmogorov-Nagumo in a nontrivial isomorphic setting was first appeared in the N. J. Nielsen article [41]. The general result, presented here, was proved in [15].

2.4. Dominated operators were first introduced by L. V. Kantorovich in the 1930s under the name "operators with abstract norm"; he also established the result of Proposition 2.4.4 (see [29] and the bibliography therein). S. Bochner, N. Dunford, B. J. Pettis, R. S. Phillips et al. began studying the $L^p(X)$ spaces in the 1930s in connection with analytical representation of operators (see the bibliography in [21, 23]). We refer to [21, 22, 23] as regards the plentiful literature about the history of Theorem 2.4.11 and its corollaries.

The reader could observe that various objects were denoted by the same symbol $|\cdot|$ in Section 2.4. This relates to the fact that we deal with different examples of lattice-normed spaces which were also introduced by L. V. Kantorovich at the end of the 1930s (see [28, 29, 46]). The ideology of lattice-normed spaces was consistently applied to constructing the whole theory of vector-functions spaces and some classes of operators (including dominated operators) by A. V. Bukhvalov [17] (see also [6–11]) The general theory of such spaces with various connections and applications is now developed by A. G. Kusraev and his students (see, for instance, [34, 35] and also Section 1.6).

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CHAPTER 3

Stably Dominated and Stably Regular Operators

BY

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This chapter is mainly devoted to studying those properties of Banach lattices and operators on them whose description requires the vector lattice structure as well as the pure Banach space structure. In particular, we study the properties of the operators acting between vector lattices and remaining to be order bounded under multiplication by arbitrary Banach endomorphisms in these lattices. In spite of being natural, such classes of operators have attracted very little attention of mathematicians. At present, the theory of p-absolutely summing operators plays a key role in solving the arising problems; in Sections 3.1–3.3 and 3.5, we briefly expose those results of the theory which are most important for us. The reader interested in a more complete presentation of the theory will refer to the monographs [42, 32, 45] and the articles [20, 22, 39].

3.1. p-Absolutely Summing Operators

Throughout this section the letters X, Y, and Z (possibly, with indices) stand for Banach spaces and the letters E and F, for Banach lattices.

3.1.1. Definition. Let $0 < p < \infty$. An operator $U \in \mathscr{L}(X,Y)$ is called *p-absolutely summing* if there exists a constant C such that

$$\left(\sum_{k=1}^{n} \|Ux_k\|^p\right)^{1/p} \leq C \sup\left\{\left(\sum_{k=1}^{n} |\langle x_k,\, x'\rangle|^p\right)^{1/p} \mid x' \in X^*,\ \|x'\| \leq 1\right\} \tag{1}$$

for every $n \in \mathbb{N}$ and arbitrary $x_1, \dots, x_n \in X$.

The infimum of numbers C satisfying (1) is denoted by $\pi_p(U)$ and is called the *p-absolutely summing norm* of the operator U. It is easy to verify that

$$\left(\sum_{k=1}^{n} \|Ux_k\|^p\right)^{1/p} \leq \pi_p(U) \sup\left\{\left(\sum_{k=1}^{n} |\langle x_k,\, x'\rangle|^p\right)^{1/p} \mid x' \in X^*,\ \|x'\| \leq 1\right\} \tag{2}$$

for every $n \in \mathbb{N}$ and arbitrary $x_1, \dots, x_n \in X$. Inequality (2) immediately implies that $\|U\| \leq \pi_p(U)$.

We denote by $\Pi_p(X,Y)$ the set of all p-absolutely summing operators belonging to $\mathscr{L}(X,Y)$. For the sake of consistency, we assume that $\Pi_\infty(X,Y) = \mathscr{L}(X,Y)$, $\pi_\infty(U) = \|U\|$.

REMARK. From the definition of the p-absolutely summing norm of an operator it is straightforward that, for every $C > 0$, the set $\{U \in \Pi_p(X, Y) \mid \pi_p(U) \le C\}$ is closed in $\mathscr{L}(X, Y)$ under pointwise convergence.

3.1.2. Theorem. *Given $p \geq 1$ $(0 < p < 1)$, the function π_p is a norm (quasinorm) in $\Pi_p(X, Y)$. The space $\Pi_p(X, Y)$ is complete with respect to the norm (quasinorm). If $V \in \mathscr{L}(X_0, X)$, $W \in \mathscr{L}(Y, Y_0)$, and $U \in \Pi_p(X, Y)$ then $WUV \in \Pi_p(X_0, Y_0)$ and $\pi_p(WUV) \le \|W\| \cdot \|V\| \cdot \pi_p(U)$.*

$\lhd$ The completeness of $\Pi_p(X, Y)$ follows from Remark 3.1.1 and the lemma given below. The proof of the remaining statements is left to the reader. $\rhd$

Lemma. *Let L be a Hausdorff sequentially complete topological vector space, let X be a normed (quasinormed) space, $X \subset L$ and let the embedding of X into L be continuous. If the unit ball B of the space X is closed in L then X is complete.*

$\lhd$ Let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence in X. Then it is also a Cauchy sequence in L, thus convergent in L. Prove that the vector $x_0 = \lim x_n$ belongs to X and $x_n \to x_0$ in X. Fix an arbitrary $\varepsilon > 0$ and find N such that $x_n - x_m \in \varepsilon B$ for $n, m > N$. Passing to the limit as $m \to \infty$, conclude that $x_n - x_0 \in \varepsilon B$ for $n > N$ since B is closed in L. $\rhd$

REMARK. It is obvious that every restriction of an operator of the class Π_p is an operator of the same class. Moreover, if j is an isometric or isomorphic embedding of a space Y into a space Y_0 then operators $U \in \mathscr{L}(X, Y)$ and jU are or are not p-absolutely summing simultaneously. In this sense the containment of an operator in the class Π_p does not depend on the arrival set Y of the operator (in contrast to its containment in, for instance, the space of nuclear operators; the matter will be discussed later). This property of the class Π_p is called *injectivity.*

3.1.3. Theorem. *If $0 < p < q$ then $\Pi_p(X, Y) \subset \Pi_q(X, Y)$ and the inequality $\pi_q(U) \le \pi_p(U)$ holds for every operator $U \in \Pi_p(X, Y)$.*

We omit the simple direct proof of the theorem which leans on the Hölder inequality. Another proof will be given further (see 3.1.8, Corollary 4).

As is well known (see [4, 30]), for $0 < p < 1$, the set $\Pi_p(X, Y)$ is stable, i.e., $\Pi_q(X, Y) = \Pi_p(X, Y)$ for $0 < q < p < 1$. Therefore, the operators belonging to the classes Π_p for $p < 1$ will be also called 0-absolutely summing and the set of such operators will be denoted $\Pi_0(X,Y)$ alongside with $\Pi_p(X,Y)$.

3.1.4. A simple and important criterion for an operator to belong to the class Π_p is given in the next theorem.

Theorem. *Let $U \in \mathscr{L}(X,Y)$, let $(T, \mathfrak{A}, \mu)$ be a measure space, let $\vec{f} \in L_s{}^\infty(T, \mu, X^*)$, and let $\|\vec{f}\| = \operatorname{Vrai\,sup}\{\|\vec{f}(t)\| \mid t \in T\}$ (see 2.4.7). If*

$$\|Ux\| \le C\Big(\int\limits_T |\langle x, \vec{f}(t)\rangle|^p \, d\mu(t)\Big)^{1/p}$$

for every $x \in X$ then $U \in \Pi_p(X,Y)$ and $\pi_p(U) \le C(\mu(T))^{1/p}\|\vec{f}\|$.

$\triangleleft$ Let $x_1, \dots, x_n \in X$. Then

$$\begin{aligned}\sum_{k=1}^n \|Ux_k\|^p &\le C^p \int\limits_T \sum_{k=1}^n |\langle x_k, \vec{f}(t)\rangle|^p \, d\mu(t) \\ &\le C^p \mu(T) \sup\left\{\sum_{k=1}^n |\langle x_k, x'\rangle|^p \mid \|x'\| \le \|\vec{f}\|\right\} \\ &= C^p \mu(T) \|\vec{f}\| \sup\left\{\sum_{k=1}^n |\langle x_k, x'\rangle|^p \mid \|x'\| \le 1\right\}. \ \triangleright\end{aligned}$$

3.1.5. Examples.

(a) Henceforth a finite regular Borel measure on subsets of a compact space K will be called a *Radon measure. Let μ be such a measure and let i_p be the identity embedding of $C(K)$ into $L^p(K,\mu)$. Then $i_p \in \Pi_p(C(K), L^p(K,\mu))$ and $\pi_p(i_p) = (\mu(K))^{1/p}$.*

$\triangleleft$ Indeed,

$$\begin{aligned}\sum_{k=1}^n \|i_p x_k\|^p &= \int\limits_K \sum_{k=1}^n |x_k(t)|^p \, d\mu(t) \le \mu(K) \sup\left\{\sum_{k=1}^n |x_k(t)|^p \mid t \in K\right\} \\ &\le \mu(K) \sup\left\{\sum_{k=1}^n |\langle x_k, x'\rangle|^p \mid x' \in C^*(K), \ \|x'\| \le 1\right\}\end{aligned}$$

whence it follows that $i_p \in \Pi_p(C(K), L^p(K,\mu))$ and $\pi_p(i_p) \le (\mu(K)))^{1/p}$. Otherwise, $\pi_p(i_p) \ge \|i_p\| = (\mu(K))^{1/p}$. $\triangleright$

As will be seen below, every p-absolutely summing operator is presentable as product of a restriction of the operator i_p and some continuous operator.

(b) *Let $(T, \mathfrak{A}, \mu)$ be an arbitrary measure space, let $g \in L^p(T,\mu)$, and let $M_g : L^\infty(T,\mu) \to L^p(T,\mu)$ be the multiplication operator:*

$$M_g(x) = gx \quad (x \in L^\infty(T,\mu)).$$

Then $M_g \in \Pi_p(L^\infty(T,\mu), L^p(T,\mu))$ and $\pi_p(M_g) = \|g\|_p$.

$\triangleleft$ This result reduces to example (a) by realizing $L^\infty(T,\mu)$ as a space of continuous functions and considering the measure $\tilde{\mu}$ with density $|g|^p$ with respect to the measure μ. However, it is easy to obtain the proof of the claim of the example directly if we observe that it is sufficient to establish inequality (1) only for vectors in some set everywhere dense in X and check that in this case inequality (1) is valid for a function with finite range. $\triangleright$

(c) *Each dominated operator $U \in \mathscr{L}(X, L^p(T,\mu))$ belongs to the space $\Pi_p(X, L^p(T,\mu))$; moreover, $\pi_p(U) \le \|U\|_M$.*

$\triangleleft$ The statement follows from example (b) and the fact that U can be represented as the product $U = M_g V$, where $V \in \mathscr{L}(X, L^\infty(T,\mu))$, $\|V\| \le 1$, and M_g is a multiplication operator. $\triangleright$

(d) *If $U \in \mathscr{L}^\sim(L^\infty(T',\mu'), L^p(T,\mu))$ then $U \in \Pi_p(L^\infty(T',\mu'), L^p(T,\mu))$ and $\pi_p(U) \le \|\,|U|\,\|$.*

$\triangleleft$ This follows from example (c) since the operator U is dominated and $\|U\|_M = \|\,|U|\,\|$. $\triangleright$

(e) *Let $J : l^1 \to l^2$ be the identity embedding. Then $J \in \Pi_0(l^1, l^2)$; moreover, $\pi_p(J) = 1$ for $2 \le p < \infty$ and $\pi_p(J) \le B_p$ for $0 < p \le 2$, where B_p is the constant in the Khinchin inequality* (3.S.1).

$\triangleleft$ Assume that $0 < p < 2$, $\{r_k\}_{k=1}^\infty$ is the sequence of Rademacher functions, and $x = (x_1, x_2, \dots) \in l^1$. We have

$$\|Jx\| = \left(\sum_{k=1}^{\infty} |x_k|^2\right)^{1/2} = \left(\int_0^1 \left|\sum_{k=1}^{\infty} x_k r_k(t)\right|^2 dt\right)^{1/2}$$

$$\le B_p \left(\int_0^1 \left|\sum_{k=1}^{\infty} x_k r_k(t)\right|^p dt\right)^{1/p} = B_p \left(\int_0^1 |\langle x, \bar{f}(t)\rangle|^p \, dt\right)^{1/p},$$

where $\bar{f}(t) = \{r_k(t)\}_{k=1}^{\infty} \in l^{\infty} = (l^1)^*$. To complete the proof, we are left with appealing to Theorem 3.1.4. $\triangleright$

3.1.6. The next simple theorem is a convenient technical tool often involved in proving that an operator belongs to the class Π_p.

Theorem. *Let $p \ge 1$ and $U \in \mathscr{L}(X, Y)$. Then $U \in \Pi_p(X, Y)$ if and only if $UV \in \Pi_p(l^{p'}, Y)$ for every operator $V \in \mathscr{L}(l^{p'}, X)$. Moreover,*

$$\pi_p(U) = \sup\{\pi_p(UV) \mid \|V\| \le 1\} \tag{3}$$

(here $p' = p/(1-p)$ and, for $p = 1$, the symbol $l^{p'}$ stands for the space c_0).

$\triangleleft$ If $U \in \Pi(X, Y)$, the containment $UV \in \Pi(l^{p'}, Y)$ together with the inequality

$$\pi_p(UV) \le \pi_p(U) \cdot \|V\| \tag{4}$$

follows from 3.1.2. Let now $UV \in \Pi_p(l^{p'}, Y)$ for every $V \in \mathscr{L}(l^{p'}, X)$. Since the mapping $V \mapsto UV$ from $\mathscr{L}(l^{p'}, X)$ into $\Pi_p(l^{p'}, Y)$ is obviously closed, it is continuous in virtue of the closed graph theorem and, therefore,

$$\alpha = \sup\{\pi_p(UV) \mid \|V\| \le 1\} < \infty.$$

Given an arbitrary collection $x_1, x_2, \dots, x_n$ of vectors in X, we define the operator $V : l^{p'} \to X$ by the equality

$$V(t) = \sum_{k=1}^{n} t_k \cdot x_k,$$

where $t = (t_k)_{k=1}^{\infty} \in l^{p'}$. It can be easily verified that

$$\|V\| = \sup\left\{\left(\sum_{k=1}^{n} |\langle x_k, x'\rangle|^p\right)^{1/p} \mid \|x'\| \le 1\right\}.$$

In addition, $V(e_k) = x_k$ for $k = 1, 2, \dots, n$ (e_k are the canonical basis vectors for $l^{p'}$). Thus,

$$\begin{aligned}\left(\sum_{k=1}^{n} \|Ux_k\|^p\right)^{1/p} &= \left(\sum_{k=1}^{n} \|UVe_k\|^p\right)^{1/p} \\ &\leq \pi_p(UV) \sup\left\{\left(\sum_{k=1}^{n} |\langle e_k, t'\rangle|^p\right)^{1/p} \,\middle|\, t' \in l^p,\ \|t'\| \leq 1\right\} \\ &= \pi_p(UV) \leq \alpha\|V\| \\ &= \alpha \sup\left\{\left(\sum_{k=1}^{n} |\langle x_k, x'\rangle|^p\right)^{1/p} \,\middle|\, x' \in X^*,\ \|x'\| \leq 1\right\}.\end{aligned}$$

Consequently, $U \in \Pi_p(X, Y)$ and $\pi_p(U) \leq \alpha$. This fact together with inequality (4) provides equality (3).

3.1.7. The Pietsch inequality. Here we establish one of the most important characteristics of the operators of the class Π_p. By K we denote the unit ball of the dual space of X equipped with the weak* topology.

Theorem. *For $U \in \Pi_p(X, Y)$, there exists a probability Radon measure ν over K ($K \subset X^*$) such that*

$$\|Ux\| \leq \pi_p(U)\left(\int_K |\langle x, x'\rangle|^p \, d\nu(x')\right)^{1/p} \quad (x \in X).$$

$\lhd$ Let $\Phi \subset C(K)$ be the set of all functions φ of the form

$$\varphi(x') = \sum_{k=1}^{n} \|Ux_k\|^p - \pi_p^p(U) \sum_{k=1}^{n} |\langle x_k, x'\rangle|^p,$$

where $x' \in K$ and $x_1, x_2, \dots, x_n$ are arbitrary vectors in X. Observe that

$$\inf_K \varphi(x') \leq 0.$$

As easily seen, Φ is a convex cone disjoint from the (open) cone G consisting of strictly positive functions. Let f ($f \neq 0$) be a functional separating these cones:

$$f(\varphi) \leq 0 \leq f(\psi) \text{ for all } \varphi \in \Phi \text{ and } \psi \in G. \tag{5}$$

Since the functional f is positive, it can be represented by some measure ν that may be assumed to be a probability measure without loss of generality. On taking as φ in inequality (5) the function

$$\varphi(x') = \|Ux\|^p - \pi_p^p(U)|\langle x, x'\rangle|^p,$$

we obtain

$$f(\varphi) = \|Ux\|^p - \pi_p^p(U)\int\limits_K |\langle x, x'\rangle|^p\,d\nu \leq 0,$$

and so

$$\|Ux\| \leq \pi_p(U)\Big(\int\limits_K |\langle x, x'\rangle|^p\,d\nu\Big)^{1/p}. \ \triangleright$$

3.1.8. Corollaries to the Pietsch inequality. As far as the support of the measure ν in the Pietsch inequality is concerned, we only know that it is included in K. Is it always possible to choose the measure in such a way that it is supported by a given closed subset Q of the compact set K? For $p \geq 1$, this is possible provided that the closure of the absolutely convex hull of Q coincides with K. In this case, the cones Φ and G introduced in the proof of the Pietsch inequality (if considered as cones of functions on Q) are still disjoint since the greatest lower bounds of the functions in Φ over the sets K and Q coincide in view of concavity of these functions; therefore, the proof of the Pietsch inequality given in Subsection 3.1.7 remains valid if we replace K with Q.

In particular, if $p \geq 1$ and $X = C(S)$, where S is a compact space, then (by identifying the points of S with the functionals generated by unit masses loaded in the points) we can take Q as coinciding with S. Thus, we proved

Corollary 1. *If $p \geq 1$ and $U \in \Pi_p(C(S), Y)$ then there exists a probability Radon measure ν on S such that*

$$\|Ux\| \leq \pi_p(U)\Big(\int\limits_S |x(s)|^p\,d\nu(s)\Big)^{1/p} \quad (x \in C(S)).$$

We list other corollaries to the Pietsch inequality.

Corollary 2. *Every p-absolutely summing operator maps a weakly convergent (and weakly fundamental) sequence into a norm convergent sequence.*

$\lhd$ It follows immediately from the Pietsch inequality and Lebesgue's dominated convergence theorem. $\rhd$

Corollary 3. *Every p-absolutely summing operator defined on a reflexive space is compact.*

Corollary 4. *If $0 < p < q$ and $U \in \Pi_p(X,Y)$ then $U \in \Pi_q(X,Y)$; moreover, $\pi_q(U) \le \pi_p(U)$.*

$\lhd$ Since

$$\|Ux\| \le \pi_p(U)\Big(\int_K |\langle x, x'\rangle|^p \, d\nu(x')\Big)^{1/p}$$

and

$$\Big(\int_K |\langle x, x'\rangle|^p \, d\nu(x')\Big)^{1/p} \le \Big(\int_K |\langle x, x'\rangle|^q \, d\nu(x')\Big)^{1/q},$$

the claim follows from Theorem 3.1.4. $\rhd$

3.1.9. Canonical factorization. The Pietsch inequality allows one to implement factorization of an operator in the class Π_p in a canonical way. Preserving the notation of 3.1.7, we may assert that *an operator U can be represented as the product $V\tilde{\imath}_p\jmath$*, where $\jmath$ is the canonical embedding of the space X into $C(K)$ $((\jmath x)(x') = \langle x, x'\rangle,\ x' \in K,\ x \in X)$, $\tilde{\imath}_p$ is the restriction of the identity embedding $\tilde{\imath}_p$ of the space $C(K)$ into $L^p(K,\nu)$ to the set $X_\infty = \jmath(X)$, and V is some operator defined on the closure of $\tilde{\imath}_p\jmath(X)$ in $L^p(K,\nu)$. Moreover, $\|\jmath\| = 1$, $\pi_p(\tilde{\imath}_p) = 1$, and $\|V\| = \pi_p(U)$.

$\lhd$ We first define the operator V on the functions of the form $\imath_p\jmath(x)$ by the equality $V(\tilde{\imath}_p\jmath(x)) = Ux$. As easily seen, the definition of V is correct and

$$\|V(\tilde{\imath}_p\jmath(x))\| = \|Ux\| \le \pi_p(u)\|\tilde{\imath}_p\jmath(x)\|_p.$$

Therefore, $\|V\| \le \pi_p(U)$. Extending V to the closure X_p of the set $(\imath_p\jmath)(X)$, we obtain the following commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{U} & Y \\ {\scriptstyle j}\downarrow & & \uparrow{\scriptstyle V} \\ X_\infty & \xrightarrow{\tilde{i}_p} & X_p \\ \cap & & \cap \\ C(K) & \xrightarrow{i_p} & L^p(K,\nu). \end{array}$$

Moreover, $\|j\| = 1$ and since $\pi_p(U) \le \|j\|\pi_p(\tilde{i}_p)\|V\| \le \pi_p(\tilde{i}_p)\|V\| \le \|V\| \le \pi_p(U)$, we have $\pi_p(\tilde{i}_p) = 1$, $\|V\| = \pi_p(U)$. $\triangleright$

The factorization demonstrates in particular that *every p-absolutely summing operator is weakly compact.* This fact together with Corollary 2 of 3.1.8 implies in turn that *the product of a pair of p-absolutely summing operators is a compact operator.*

3.1.10. Canonical factorization (refinement). The shortcoming of the described factorization is the fact that the operator V is defined not on the whole space $L^p(K,\nu)$ but only on a subspace X_p of it about which little is known. However, in some important cases we may assume the operator V to be defined on the whole $L^p(K,\nu)$. Now we list some of the cases.

(a) $p \ge 1$ and $X = C(S)$. Grounding on the inequality in Corollary 1 of 3.1.8 and arguing as in constructing the diagram in 3.1.9, we infer that the operator U admits the factorization

$$\begin{tikzcd} C(S) \arrow[rr, "U"] \arrow[dr, "i_p"'] & & Y \\ & L^p(S,\nu). \arrow[ur, "V"'] & \end{tikzcd}$$

Constructing the operator V, we now take as X_p the closure of $C(S)$, i.e. the whole space $L^p(S,\nu)$.

(b) $p = 2$. In this case, the subspace X_2 is complemented in $L^2(K,\nu)$, so, by using an orthogonal projection, we can extend the operator V on the whole space $L^2(K,\nu)$, preserving the norm of V.

(c) The space Y is a $\mathscr{P}_1$-space. By definition, this means that every Y-valued operator defined on an arbitrary subspace of an arbitrary Banach space

Z possess a norm-preserving extension to the whole space Z. Examples of $\mathscr{P}_1$-spaces are provided by the spaces l^∞, $L^\infty(T,\mu)$ and the space $l^\infty(\Omega)$ of all bounded functions defined on a set Ω and endowed with the sup-norm.

3.1.11. *Let $U \in \mathscr{L}(X,Y)$. For U to be in $\Pi_p(X,Y)$ it necessary and sufficient that $U^{**} \in \Pi_p(X^{**},Y^{**})$. In addition, $\pi_p(U) = \pi_p(U^{**})$.*

$\triangleleft$ Sufficiency is obvious. Necessity is easily obtainable from the local reflexivity principle (see 3.S.4). $\triangleright$

Considering the operator J in Example 3.1.5(e), we see that, although $J \in \Pi_0(l^1,l^2)$, its adjoint operator is p-absolutely summing for no $p < \infty$ since it is defined on a reflexive space and is not compact.

3.2. p-Absolutely Summing Operators in Hilbert Space

Throughout this section the symbols H and H_1 stand for Hilbert spaces and $\{r_k\}_{k=1}^\infty$, the sequence of Rademacher functions.

3.2.1. It is well known (see, for instance, [8]) that an operator $U \in \mathscr{L}(H,H_1)$ is compact if and only if it can be represented as

$$Ux = \sum_{k=1}^{\infty} s_k(x,e_k)e'_k \quad (x \in H),$$

where $\{e_k\}_{k=1}^\infty$ and $\{e'_k\}_{k=1}^\infty$ are orthonormal systems in the spaces H and H_1 and $\{s_k\}_{k=1}^\infty$ is a nonincreasing scalar sequence that tends to zero. If

$$\sum_{k=1}^{\infty} s_k^2 < +\infty$$

then U is called a *Hilbert-Schmidt operator.* We set

$$\sigma_2(U) = \left(\sum_{k=1}^{\infty} s_k^2\right)^{1/2}$$

(see 2.4.1).

The following theorem is easily verified (see [8]):

Theorem. *An operator $U \in \mathscr{L}(H, H_1)$ is a Hilbert-Schmidt operator if and only if the sum $\sum_\alpha \|Ue_\alpha\|^2$ is finite for some (or, equivalently, for an arbitrary) complete orthonormal system $\{e_\alpha\}_{\alpha \in A}$ in H. In this event,*

$$\sigma_2(U) = \Big(\sum_\alpha \|Ue_\alpha\|^2\Big)^{1/2}.$$

3.2.2. Theorem. *An operator $U \in \mathscr{L}(H, H_1)$ is a Hilbert-Schmidt operator if and only if $U \in \Pi_2(H, H_1)$. Moreover, $\sigma_2(U) = \pi_2(U)$.*

$\lhd$ If U is a Hilbert-Schmidt operator then (see 3.2.1)

$$Ux = \sum_{k=1}^{\infty} s_k (x, e_k) e'_k, \tag{1}$$

where $\{e_k\}_{k=1}^\infty$ and $\{e'_k\}_{k=1}^\infty$ are orthonormal systems and $\sum_{k=1}^\infty s_k^2 < \infty$. Consequently,

$$\|Ux\| = \left(\sum_{k=1}^{\infty} s_k^2 |(x, e_k)|^2\right)^{1/2} = \left(\int_0^1 \left|\sum_{k=1}^{\infty} s_k (x, e_k) r_k(t)\right|^2 dt\right)^{1/2}. \tag{2}$$

By putting

$$\bar{f}(t) = \sum_{k=1}^{\infty} s_k r_k(t) e_k, \tag{3}$$

we can rewrite inequality (2) as

$$\|Ux\| = \left(\int_0^1 |(x, \bar{f}(t))|^2 \, dt\right)^{1/2}.$$

Since

$$\|\bar{f}(t)\| \le \left(\sum_{k=1}^{\infty} s_k^2\right)^{1/2} = \sigma_2(U)$$

for all $t \in (0,1)$, we have (see 3.1.4)

$$U \in \Pi_2(H, H_1), \quad \pi_2(U) \le \sigma_2(U). \tag{4}$$

On the other hand, if $U \in \Pi_2(H, H_1)$ then the operator U is compact (see 3.1.8, Corollary 3) and therefore it is representable as (1). Since $\|Ue_k\| = s_k$ ($k = 1, 2, \dots$), we have

$$\left(\sum_{k=1}^{n} s_k^2\right)^{1/2} = \left(\sum_{k=1}^{n} \|Ue_k\|^2\right)^{1/2}$$
$$\leq \pi_2(U) \sup \left\{ \left(\sum_{k=1}^{n} |(e_k, x')|^2\right)^{1/2} \mid x' \in H,\ \|x'\| \leq 1 \right\} \leq \pi_2(U)$$

for every $n \in \mathbb{N}$. Hence, U is a Hilbert-Schmidt operator and $\sigma_2(U) \leq \pi_2(U)$, which together with inequality (4) provides the equality $\sigma_2(U) = \pi_2(U)$. $\rhd$

3.2.3. In the current and subsequent subsections we establish that all classes of absolutely p-summing operators in a Hilbert space are pairwise coincident. In the theorem below, the symbol A_p stands for the constant in the Khinchin inequality (3.S.1).

Theorem. *If $0 < p \leq 2$ then $\Pi_p(H, H_1) = \Pi_2(H, H_1)$; moreover, $\pi_p(U) \leq A_p^{-1} \pi_2(U)$ for every operator $U \in \Pi_2(H, H_1)$.*

$\lhd$ Let $U \in \Pi_2(H, H_1)$. By using representation (1) and the Khinchin inequality, we obtain

$$\|Ux\| = \left\| \sum_{k=1}^{\infty} s_k (x, e_k) r_k \right\|_{L^2(0,1)} \leq A_p^{-1} \left\| \sum_{k=1}^{\infty} s_k (x, e_k) r_k \right\|_{L^p(0,1)}$$
$$= A_p^{-1} \left(\int_0^1 |(x, \bar{f}(t))|^p \, dt \right)^{1/p},$$

where $\bar{f}(t)$ is the vector-valued function defined by equality (3).

Theorem 3.1.4 implies that $U \in \Pi_p(H, H_1)$ and

$$\pi_p(U) \leq A_p^{-1} \sup\{\|\bar{f}(t)\| \mid t \in (0,1)\} \leq A_p^{-1} \sigma_2(U). \ \rhd$$

Corollary. *Since $A_1 = 1/\sqrt{2}$, we have $\pi_1(U) \leq \sqrt{2} \pi_2(U)$.*

3.2.4. The our next goal is to demonstrate that $\Pi_p(H, H_1) = \Pi_2(H, H_1)$ not only for $0 < p \le 2$ but also for $2 < p < \infty$. By using the notion of 2-cotype space (see 3.S.2), we can obtain the following more general result.

Theorem. *Let $2 \le p < \infty$ and let Y be a 2-cotype space. Then $\Pi_p(X, Y) = \Pi_2(X, Y)$; moreover, $\pi_2(U) \le C_2(Y) B_p \pi_p(U)$ for every operator $U \in \Pi_p(X, Y)$ (here B_p is the constant in the Khinchin inequality* (3.S.1) *and $C_2(Y)$ is the 2-cotype constant for the space Y).*

$\lhd$ Let $U \in \Pi_p(X, Y)$. By 3.1.7, we have

$$\|Ux\| \le \pi_p(U) \Big(\int\limits_K |\langle x, x' \rangle|^p \, d\nu(x') \Big)^{1/p}, \tag{5}$$

where $K = \{x' \in X^* \ \|x'\| \le 1\}$ and ν is a probability Radon measure on K.

Given vectors $x_1, x_2, \dots, x_n \in X$, we have

$$\begin{aligned} \left(\sum_{k=1}^n \|Ux_k\|^2 \right)^{1/2} &\le C_2(Y) \int\limits_0^1 \left\| \sum_{k=1}^n r_k(t) U x_k \right\| dt \\ &\le C_2(Y) \left(\int\limits_0^1 \left\| U \left(\sum_{k=1}^n r_k(t) x_k \right) \right\|^p dt \right)^{1/p}, \end{aligned}$$

which together with inequality (5) gives

$$\begin{aligned} \left(\sum_{k=1}^n \|Ux_k\|^2 \right)^{1/2} &\le C_2(Y) \pi_p(U) \left(\int\limits_0^1 \int\limits_K \left| \left\langle \sum_{k=1}^n r_k(t) x_k, x' \right\rangle \right|^p d\nu(x') \, dt \right)^{1/p} \\ &= C_2(Y) \pi_p(U) \left(\int\limits_K \left(\int\limits_0^1 \left| \sum_{k=1}^n \langle x_k, x' \rangle r_k(t) \right|^p dt \right) d\nu(x') \right)^{1/p}. \end{aligned}$$

Estimating the innermost integral on the right-hand side with the help of the Khinchin inequality, we derive

$$\begin{aligned} \left(\sum_{k=1}^n \|Ux_k\|^2 \right)^{1/2} &\le C_2(Y) \pi_p(U) \left(\int\limits_K B_p^p \left(\sum_{k=1}^n |\langle x_k, x' \rangle|^2 \right)^{p/2} d\nu(x') \right)^{1/p} \\ &\le C_2(Y) B_p \pi_p(U) \sup \left\{ \left(\sum_{k=1}^n |\langle x_k, x' \rangle|^2 \right)^{1/2} \ \Big| \ x' \in K \right\}. \ \rhd \end{aligned}$$

Observe that if Y is a Hilbert space then we can obtain a more precise estimate $\pi_2(U) \leq B_p\pi_p(U)$ for $\pi_2(U)$ by making use of the identity

$$\sum_{k=1}^{n} \|y_k\|^2 = \int_0^1 \left\| \sum_{k=1}^{n} r_k(t) y_k \right\|^2 dt \quad (y_1, \dots, y_n \in Y).$$

Corollary. $\Pi_p(H, H_1) = \Pi_2(H, H_1)$ *for every* p, $0 < p < \infty$.

The adjoint of a Hilbert-Schmidt operator is a Hilbert-Schmidt operator again; therefore, an operator acting between Hilbert spaces is p-absolutely summing simultaneously with its adjoint in contrast to the general case (see 3.1.11).

3.3. Nuclear Operators

3.3.1. Definitions. An operator $U \in \mathscr{L}(X, Y)$ is called *nuclear* if there exist vectors $y_n \in Y$ and functionals $x'_n \in X^*$ $(n = 1, 2, \dots)$ such that

$$Ux = \sum_{n=1}^{\infty} \langle x, x'_n \rangle y_n \quad (x \in X), \tag{1}$$

and

$$\sum_{n=1}^{\infty} \|x'_n\| \cdot \|y_n\| < \infty. \tag{2}$$

Let the symbol $x' \otimes y$, where $x' \in X^*$ and $y \in Y$, denote the rank-one operator $x \mapsto \langle x, x' \rangle y$. The nuclearity of an operator U means that it can be represented as an absolutely convergent series of rank-one operators

$$U = \sum_{n=1}^{\infty} x'_n \otimes y_n. \tag{3}$$

Representation (1) satisfying condition (2) or, which is the same, representation (3) will be referred as *nuclear representation of* U. The greatest lower bound of sums (2) over all nuclear representations of U is called the *nuclear norm of* U and is denoted by $\nu(U)$. The set of all nuclear operators in $\mathscr{L}(X, Y)$ is denoted by $N(X, Y)$.

3.3.2. Properties of nuclear operators.

(a) *A nuclear operator is compact.*

$\lhd$ Let $\sum_{n=1}^{\infty} x'_n \otimes y_n$ be a nuclear representation of U. Then the finite rank operators

$$U_k = \sum_{n=1}^{k} x'_n \otimes y_n$$

approximate U in operator norm. $\rhd$

(b) *If $V \in \mathscr{L}(X_0, X)$, $W \in \mathscr{L}(Y, Y_0)$, and $U \in N(X,Y)$ then $WUV \in N(X_0, Y_0)$ and $\nu(WUV) \leq \|W\| \cdot \|V\| \cdot \nu(U)$.*

(c) *If $U \in N(X,Y)$ then $U^* \in N(Y^*, X^*)$ and $\nu(U^*) \leq \nu(U)$.*

(d) *$N(X,Y) \subset \Pi_1(X,Y)$; moreover, $\pi_1(U) \leq \nu(U)$ for every nuclear operator U.*

$\lhd$ Let $U \in N(X,Y)$ and $\varepsilon > 0$. Fix a nuclear representation

$$\sum_{n=1}^{\infty} x'_n \otimes y_n$$

of U such that

$$\sum_{n=1}^{\infty} \|x'_n\| \cdot \|y_n\| \leq \nu(U) + \varepsilon.$$

Without loss of generality we may assume that $\|x'_n\| = 1$ for $n = 1, 2, \dots$. Then

$$\|Ux\| \leq \sum_{n=1}^{\infty} |\langle x, x'_n \rangle| \cdot \|y_n\| = \int_K |\langle x, x' \rangle| \, d\mu(x') \quad (x \in X),$$

where $K = \{x' \in X^* \mid \|x'\| = 1\}$ and μ is the measure generated by the masses $\|y_n\|$ at the points x'_n $(n = 1, 2, \dots)$. By making use of 3.1.4 with the identity mapping $\bar{f}$ on K, we obtain $U \in \Pi_1(X,Y)$ and $\pi_1(U) \leq \nu(U) + \varepsilon$. Consequently, $\pi_1(U) \leq \nu(U)$ in view of the arbitrariness of ε. $\rhd$

(e) *Let H and H_1 be Hilbert spaces and let $U \in \mathscr{L}(H, H_1)$. An operator U is nuclear if and only if it is compact and the numbers s_k in the representation*

$$Ux = \sum_{k=1}^{\infty} s_k (x, e_k) e'_k \quad (x \in H)$$

satisfy the condition $\sum_{k=1}^{\infty} s_k < \infty$.

◁ Sufficiency of the condition is obvious. Check it necessity. Let

$$U = \sum_{n=1}^{\infty} x'_n \otimes y_n$$

be a nuclear representation of U. Then

$$s_k e'_k = Ue_k = \sum_{k=1}^{\infty} (e_k, x'_n) y_n$$

and

$$s_k = (Ue_k, e'_k) = \sum_{n=1}^{\infty} (e_k, x'_n)(y_n, e'_k) \le \sum_{n=1}^{\infty} |(e_k, x'_n)| \cdot |(y_n, e'_k)|.$$

Consequently,

$$\begin{aligned}
\sum_{k=1}^{\infty} s_k &\le \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |(e_k, x'_n)| \cdot |(y_n, e'_k)| \\
&\le \sum_{n=1}^{\infty} \left(\sum_{k=1}^{\infty} |(e_k, x'_n)|^2 \right)^{1/2} \left(\sum_{k=1}^{\infty} |(y_n, e'_k)|^2 \right)^{1/2} \\
&\le \sum_{n=1}^{\infty} \|x'_n\| \cdot \|y_n\| < \infty. \ \triangleright
\end{aligned}$$

(f) *The function $U \mapsto \nu(U)$ is a norm on $N(X, Y)$. The set $N(X, Y)$ endowed with the norm becomes a Banach space.*

3.3.3. Example. *Let $(T, \mathfrak{A}, \mu)$ be a measure space and let $\vec{f} \in L^1(T, \mu; Y)$. Then the operator $U : L^\infty(T, \mu) \to Y$ defined by the equality*

$$Ux = \int_T x(t) \vec{f}(t)\, d\mu(t) \quad (x \in L^\infty(T, \mu))$$

is nuclear; moreover, $\nu(U) \le \|\vec{f}\|_{L^1(T, \mu; Y)}$.

◁ As was pointed out in 2.3.2, the set of functions assuming finitely many values is dense in $L^1(T, \mu; Y)$. Therefore, $\vec{f}$ can be represented as an absolutely convergent series of such functions: $\vec{f} = \sum_{n=1}^{\infty} \vec{f}_n$. Fix an arbitrary $\varepsilon > 0$. Replacing, if necessary, $\vec{f}_1$ with a partial sum of sufficiently large index of the series $\sum_{n=1}^{\infty} \vec{f}_n$, we may assume that

$$\sum_{n=1}^{\infty} \|\vec{f}_n\| \le \|\vec{f}\| + \varepsilon.$$

Let

$$\vec{f}_n = \sum_{j=1}^{m_n} \chi_{E_{n,j}} \otimes y_{n,j},$$

where $E_{n,j} \cap E_{n,i} = \varnothing$ for $i \neq j$. Then

$$Ux = \sum_{n=1}^{\infty} \sum_{j=1}^{m_n} \int\limits_{E_{n,j}} x(t)\, d\mu(t) y_{n,j} \quad (x \in L^{\infty}(T, \mu))$$

is a nuclear representation of the operator U; moreover,

$$\nu(U) \le \sum_{n=1}^{\infty} \sum_{j=1}^{m_n} \|\chi_{E_{n,j}}\|_{L^1(T,\mu)} \|y_{n,j}\| = \sum_{n=1}^{\infty} \|\vec{f}_n\| \le \|\vec{f}\| + \varepsilon.$$

This fact yields the estimate $\nu(U) \le \|\vec{f}\|$ in view of the arbitrariness of ε. $\rhd$

3.3.4. Lemma. *Let $(T, \mathfrak{A}, \mu)$ be a space with finite measure; let i be the identity embedding of $L^{\infty}(T,\mu)$ into $L^1(T,\mu)$; let Z be a reflexive space, and let $W \in \mathscr{L}(L^1(T,\mu), Z)$ and $W_1 \in \mathscr{L}(Z, L^{\infty}(T,\mu))$. Then*

(1) the operator $U = Wi$ is nuclear and $\nu(U) \le \|W\|\mu(T)$;

(2) the operator $V = iW_1$ is nuclear and $\nu(V) \le \|W_1\|\mu(T)$.

$\lhd$ Since the space Z is reflexive, the operator W is weakly compact. Therefore, by Theorem 2.4.14, it can be represented as

$$Wx = \int\limits_T x(t) \vec{f}(t)\, d\mu(t) \quad (x \in L^1(T,\mu)),$$

where $\vec{f} \in L^{\infty}(T,\mu,Z)$, $\|W\| = \|\vec{f}\|_{L^{\infty}(T,\mu,Z)}$. Consequently, the operator U is nuclear as was established in 3.3.3. Moreover,

$$\nu(U) \le \|\vec{f}\|_{L^1(T,\mu,Z)} \le \|\vec{f}\|_{L^{\infty}(T,\mu,Z)} \mu(T) = \|W\|\mu(T).$$

Passing to studying the operator V, we see that the operator V^* is representable in the form $W_0 i$, where W_0 is the restriction of the operator W_1^* to $L^1(T,\mu)$. Therefore, the operator V^* is nuclear by the first claim of the lemma and the inequality $\nu(V^*) \le \|W_1\|\mu(T)$. Consequently, the operator V^{**} is nuclear too (see 3.2.2(c)) and so the operator V is nuclear as well since it is representable in the form PV^{**}, where P is the canonical projection of $(L^1(T,\mu))^{**}$ onto $L^1(T,\mu)$ (see [18]). Moreover, $\|P\| = 1$ and therefore

$$\nu(V) = \nu(PV^{**}) \le \nu(V^{**}) \le \nu(V^*) \le \|W_1\|\mu(T). \ \triangleright$$

3.3.5. Lemma. *Let $(T, \mathfrak{A}, \mu)$ be a space with probability measure; let i be the identity embedding of $L^\infty(T,\mu)$ into $L^2(T,\mu)$; let H be a Hilbert space; let $W : L^2(T,\mu) \to H$ be a Hilbert-Schmidt operator, and let $W_0 = Wi$. Then $W_0 \in \Pi_1(L^\infty(T,\mu), H)$; moreover, $\pi_1(W_0) \le \pi_2(W)$.*

$\triangleleft$ Since $W \in \Pi_1(L^2(T,\mu), H)$ (see 3.2.3), the containment $W_0 \in \Pi_1(L^\infty(T,\mu), H)$ is obvious and the only difficulty is the proving of the claimed estimate for $\pi_1(W_0)$. Let τ be a partition of T into pairwise disjoint subsets $E_1, \dots, E_N$ of positive measure, and let P_τ be the projection corresponding to the partition ("conditional expectation"):

$$P_\tau x = \sum_{j=1}^{N} (\mu(E_j))^{-1} \int\limits_{E_j} x\, d\mu \chi_{E_j} \quad (x \in L^\infty(T,\mu)).$$

Estimate the nuclear norm of the operator $W_0 P_\tau$. Since W is a Hilbert-Schmidt operator, it can be represented as

$$Wx = \sum_{k=1}^{\infty} s_k (x, e_k) e'_k \quad (x \in L^2(T,\mu)),$$

where $\{e_k\}_{k=1}^{\infty}$ and $\left\{e'_k\right\}_{k=1}^{\infty}$ are orthonormal systems and

$$\left(\sum_{k=1}^{\infty} s_k^2\right)^{1/2} = \pi_2(W).$$

Therefore,

$$W_0 P_\tau x = \sum_{k=1}^{\infty} s_k \left[\sum_{j=1}^{N} (\mu(E_j))^{-1} \int\limits_{E_j} x \, d\mu \int\limits_{E_j} e_k \, d\mu \right] e'_k = \sum_{j=1}^{N} (\mu(E_j))^{-1} \int\limits_{E_j} x \, d\mu y_j,$$

where

$$y_j = \sum_{k=1}^{\infty} s_k \int\limits_{E_j} e_k \, d\mu e'_k.$$

The representation of $W_0 P$ yields the inequality

$$\nu(W_0 P_\tau) \leq \sum_{j=1}^{N} \|y_j\|. \tag{4}$$

Since

$$\|y_j\|^2 = \sum_{k=1}^{\infty} s_k^2 \left| \int\limits_{E_j} e_k \, d\mu \right|^2 \leq \sum_{k=1}^{\infty} s_k^2 \mu(E_j) \int\limits_{E_j} |e_k|^2 \, d\mu,$$

we have

$$\begin{aligned}
\sum_{j=1}^{N} \|y_j\| &\leq \sum_{j=1}^{N} (\mu(E_j))^{1/2} \left(\sum_{k=1}^{\infty} s_k^2 \int\limits_{E_j} |e_k|^2 \, d\mu \right)^{1/2} \\
&\leq \left(\sum_{j=1}^{N} \mu(E_j) \right)^{1/2} \left(\sum_{j=1}^{N} \sum_{k=1}^{\infty} s_k^2 \int\limits_{E_j} |e_k|^2 \, d\mu \right)^{1/2} \\
&= \left(\sum_{k=1}^{\infty} s_k^2 \right)^{1/2} = \pi_2(W).
\end{aligned}$$

This fact together with inequality (4) yields the estimate $\nu(W_0 P_\tau) \leq \pi_2(W)$.

Supposing the set of partitions to be naturally ordered, we see that the operator W_0 is the pointwise limit of the net $\{W_0 P_\tau\}_\tau$. By making use of the remark in 3.1.1 and the inequality of 3.3.2(d), we obtain

$$\pi_1(W_0) \leq \sup_\tau \pi_1(W_0 P_\tau) \leq \sup_\tau \nu(W_0 P_\tau) \leq \pi_2(W). \ \rhd$$

3.3.6. Multiplication theorem. Now we are in a position to obtain an important result which links 2-absolutely summing operators with nuclear operators.

Theorem. *The product of two 2-absolutely summing operators is a nuclear operator. More precisely, if $U \in \Pi_2(X,Y)$ and $\widetilde{U} \in \Pi_2(Y,Z)$ then $\widetilde{U}U \in N(X,Z)$ and $\nu(\widetilde{U}U) \le \pi_2(U)\pi_2(\widetilde{U})$.*

$\lhd$ Consider the canonical factorization of the operators U and $\widetilde{U}$ (see 3.1.10(b)):

$$U : X \xrightarrow{j} C(K) \xrightarrow{i_2} L^2(K,\nu) \xrightarrow{V} Y,$$

$$\widetilde{U} : Y \xrightarrow{\tilde{j}} C(\widetilde{K}) \xrightarrow{\tilde{i}_2} L^2(\widetilde{K},\tilde{\nu}) \xrightarrow{\widetilde{V}} Z.$$

Recall that $\|V\| = \pi_2(U)$ and $\|\widetilde{V}\| = \pi_2(\widetilde{U})$. Then

$$\widetilde{U}U : X \xrightarrow{j} C(K) \xrightarrow{j_2} L^2(K,\nu) \xrightarrow{W} L^2(\widetilde{K},\tilde{\nu}) \xrightarrow{\widetilde{V}} Z,$$

where $W = \tilde{i}_2\tilde{j}V$ and $\pi_2(W) \le \|V\| = \pi_2(U)$. By applying Lemma 3.3.5 to the operator $W_0 = Wi_2$, we see that $\pi_1(W_0) \le \pi_2(W) \le \pi_2(U)$. Therefore (see 3.1.10(a)), the operator W_0 admits the canonical representation

$$W_0 : C(K) \xrightarrow{i_1} L^1(K,\nu') \xrightarrow{V_0} L^2(\widetilde{K},\tilde{\nu})$$

since the operator is given on the space of continuous functions; in addition, $\|V_0\| = \pi_1(W_0) \le \pi_2(U)$. Now, Lemma 3.3.4 implies that $W_0 \in N(C(K), L^2(\widetilde{K},\tilde{\nu}))$ and $\nu(W_0) \le \|V_0\| \le \pi_2(U)$. Therefore,

$$\nu(\widetilde{U}U) = \nu(\widetilde{V}W_j) \le \|j\|\nu(W_0)\|\widetilde{V}\| \le \pi_2(U)\pi_2(\widetilde{U}). \ \rhd$$

3.3.7. Here we demonstrate that the nuclearity property of an operator depends essentially on the arrival set of the operator. More precisely, we exhibit a nonnuclear operator U on a Hilbert space H such that the operator jU turns out nuclear for every isometric embedding j of H into l^∞. Thus, the nuclearity of an operator can be lost in narrowing the arrival set (compare with the remark in 3.1.2).

$\lhd$ Indeed, let U be a nonnuclear Hilbert-Schmidt operator in a Hilbert space H and let j be an isometric embedding of H into l^∞. Since $U \in \Pi_1(H,H)$ and l^∞ is a $\mathscr{P}_1$-space, the operator jU (see 3.1.10(c)) admits the canonical factorization

$$jU : H \xrightarrow{j} C(K) \xrightarrow{i_1} L^1(K,\nu) \xrightarrow{V} l^\infty.$$

By Lemma 3.3.4, the operator i_1j is nuclear and so the operator jU is nuclear too. $\rhd$

3.3.8. Trace of a nuclear operator in a Hilbert space.

(a) Lemma. *Let H be a Hilbert space, let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis for H, let $U : H \to H$ be a nuclear operator, and let*

$$U = \sum_{n=1}^{\infty} x'_n \otimes y_n$$

be a nuclear representation of U. Then the series

$$\sum_{n=1}^{\infty} (y_n, x'_n), \tag{5}$$

$$\sum_{n=1}^{\infty} (Ue_n, e_n) \tag{6}$$

converge absolutely and their sums coincide.

$\lhd$ The absolute convergence of series (5) is obvious. To prove absolute convergence of series (6), we observe that

$$(Ue_n, e_n) = \sum_{k=1}^{\infty} (e_n, x'_k)(y_k, e_n) \quad (n = 1, 2, \dots). \tag{7}$$

Therefore,

$$\begin{aligned}\sum_{n=1}^{\infty} |(Ue_n, e_n)| &\le \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \left|(e_n, x'_k)(y_k, e_n)\right| \\ &\le \sum_{k=1}^{\infty} \left(\sum_{n=1}^{\infty} \left|(e_n, x'_k)\right|^2 \right)^{1/2} \left(\sum_{n=1}^{\infty} |(y_k, e_n)|^2 \right)^{1/2} \\ &= \sum_{k=1}^{\infty} \|x'_k\| \cdot \|y_k\| < \infty.\end{aligned}$$

Moreover, by summing equalities (7), we obtain

$$\sum_{n=1}^{\infty} (Ue_n, e_n) = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} (e_n, x'_k)(y_k, e_n) = \sum_{k=1}^{\infty} (y_k, x'_k). \ \rhd$$

(b) Definition. Let H be a Hilbert space, let $U \in N(H, H)$, and let

$$U = \sum_{k=1}^{\infty} x'_k \otimes y_k$$

be a nuclear representation of U. The sum

$$\sum_{k=1}^{\infty} (y_k, x'_k)$$

(independent of the choice of a nuclear representation of U by Lemma (a)) is called the *trace of the nuclear operator* U and is denoted by $\operatorname{trace}(U)$.

(c) *Lemma.* *If U is a nuclear operator in a Hilbert space then* $|\operatorname{trace}(U)| \leq \nu(U)$.

$\triangleleft$ Indeed, let $\sum_{k=1}^{\infty} x'_k \otimes y_k$ be a nuclear representation of U. Then, obviously,

$$|\operatorname{trace}(U)| \leq \sum_{k=1}^{\infty} |(y_k, x'_k)| \leq \sum_{k=1}^{\infty} \|x'_k\| \cdot \|y_k\|.$$

It remains to observe that $\nu(U)$ is the greatest lower bound of the right-hand sides of this inequality over all possible nuclear representations of U. $\triangleright$

3.3.9. As was established in 3.1.8, every p-absolutely summing operator defined on a reflexive space is compact. In many cases the indicated result can be essentially strengthened. Not addressing this question in full generality, we confine ourselves to the next

Theorem. *Let $1 < p < \infty$. Then the set of finite rank operators is dense in the space $\Pi_1(L^p(T, \mu), Y)$.*

$\triangleleft$ Consider the canonical factorization of the operator $U \in \Pi_1(L^p(T, \mu), Y)$:

$$\begin{tikzcd}
U : L^p(T,\mu) \arrow[r, "j_0"] \arrow[dr, "j"'] & X_\infty \arrow[r, "\tilde{i}_1"] \arrow[d, phantom, "\cap"] & X_p \arrow[r, "V"] \arrow[d, phantom, "\cap"] & Y \\
 & C(K) \arrow[r, "i_1"] & L^1(K, \nu). &
\end{tikzcd}$$

Since the space $L^p(T, \mu)$ is reflexive, the operator $W = i_1 j$ is nuclear by Lemma 3.3.4. Consider the nuclear representation of W:

$$W = \sum_{k=1}^{\infty} x'_k \otimes f_k,$$

where

$$x'_k \in L^{p'}(T,\mu), \quad f_k \in L^1(K,\nu), \quad \sum_{k=1}^{\infty} \|x'_k\| \cdot \|f_k\| < \infty.$$

Without loss of generality we may assume that $\|f_k\| = 1$ ($k = 1, 2, \dots$). Then, after putting $W_0 = i_1 j_0$, we have

$$\|W_0(x)\| = \|W(x)\| \leq \sum_{k=1}^{\infty} \left|\langle x, x'_k \rangle\right| \tag{8}$$

for every $x \in L^p(T,\mu)$. This inequality allows us to approximate the operator W as well as the operator U by finite rank operators.

To do this, we construct a new factorization of W_0. Take the numbers $\alpha_k > 0$ such that

$$\sum_{k=1}^{\infty} \alpha_k^{-1} \|x'_k\| \leq \infty$$

and $\alpha_k \to 0$. Define the operators $A : L^p(T,\mu) \to c_0$ and $\Delta : c_0 \to l^1$ by the equalities

$$A(x) = \left\{\alpha_k \langle x, x'_k / \|x'_k\| \rangle\right\}_{k=1}^{\infty} \quad (x \in L^p(T,\mu)),$$
$$\Delta(t) = \left\{\alpha_k^{-1} \|x'_k\| t_k\right\}_{k=1}^{\infty} \quad \left(t = \{t_k\}_{k=1}^{\infty} \in c_0\right).$$

Note that the operator A is compact and the operator Δ is nuclear. On the range L of the operator ΔA, we define the operator $V_0 : L \to X_1$ by the equality $V_0(z) = W_0(x)$ for $z = \Delta A(x)$. With the help of inequality (8) it is easy to verify that the definition is correct and

$$\|V_0(z)\| = \|W_0(x)\| \leq \sum_{k=1}^{\infty} \left|\langle x, x'_k \rangle\right| = \|\Delta A(x)\| = \|z\|.$$

Extending V_0 onto the closure $\overline{L}$ of the set L, we obtain the following factorization of the operator W_0:

$$\begin{tikzcd}
W_0 : L^p(T,\mu) \arrow[r, "A_0"] \arrow[dr, "A"'] & \widetilde{X} \arrow[r, "\widetilde{\Delta}"] \arrow[d, phantom, "\cap"] & \overline{L} \arrow[r, "V_0"] \arrow[d, phantom, "\cap"] & X_1 \\
& C_0 \arrow[r, "\Delta"] & l^1,
\end{tikzcd}$$

where $\widetilde{X}$ is the closure of the range of A in the space c_0; $\widetilde{\Delta}$ is the restriction of Δ to $\widetilde{X}$; and A_0 is the same operator A considered as a mapping from $L^p(T,\mu)$ into $\widetilde{X}$.

Since the operator A_0 is compact, there exist finite rank operators $A_n : L^p(T,\mu) \to \widetilde{X}$ such that $\|A_n - A_0\| \to 0$. Put $W_n = V_0\widetilde{\Delta}A_n$. Then

$$\pi_1(W_n - W_0) = \pi_1(V_0\widetilde{\Delta}(A_n - A_0)) \leq \|V_0\|\pi_1(\widetilde{\Delta})\|A_n - A_0\| \to 0.$$

Consequently, $\pi_1(U - VW_n) \leq \|V\|\pi_1(W_0 - W_n) \to 0$. $\triangleright$

3.4. Stably Dominated Operators

In this section we consider certain interrelations between the properties of operators acting in partially ordered spaces with the properties of p-absolutely summing operators. Recall that the letters X, Y, and Z stand for Banach spaces, the letters E and F stand for Banach lattices, and p' denotes the conjugate exponent to p.

3.4.1. (a) *Theorem.* *If $U \in \Pi_1(E,X)$, $V \in \mathscr{L}(E^*,E^*)$ then the operator VU^* is dominated; moreover, $\|VU^*\|_M \leq \pi_1(U)\|V\|$.*

$\triangleleft$ Since the space E^* satisfies conditions (B) and (C) (see 2.2.19), it suffices to verify that the inequality $\|\Phi\| \leq \pi_1(U)\cdot\|V\|$, with

$$\Phi = \max\{|VU^*x_1'|, |VU^*x_2'|, \dots, |VU^*x_n'|\},$$

is valid for arbitrary vectors $x_1', x_2', \dots, x_n' \in B_{X^*}$. It is clear that

$$\Phi = \left|\sum_{k=1}^{n} P_kVU^*x_k'\right|,$$

where $P_1, \dots, P_n$ are projections onto pairwise disjoint bands in E^*. Let ξ_0'' be a functional in E^{**} such that $\|\xi_0''\| = 1$ and

$$\left\langle \sum_{k=1}^{n} P_kVU^*x_k, \xi_0'' \right\rangle = \left\|\sum_{k=1}^{n} P_kVU^*x_k'\right\| = \|\Phi\|.$$

Then

$$\begin{aligned}\|\Phi\| &= \sum_{k=1}^{n}\langle P_kVU^*x_k', \xi_0''\rangle \leq \sum_{k=1}^{n}\|U^{**}V^*P_k^*\xi_0''\| \\ &\leq \pi_1(U^{**}V^*)\sup\left\{\sum_{k=1}^{n}|\langle \xi', P_k^*\xi_0''\rangle| \mid \xi' \in E^*,\ \|\xi'\| \leq 1\right\} \\ &\leq \pi_1(U^{**})\|V\|\sup\left\{\left\|\sum_{k=1}^{n}\varepsilon_kP_k\xi'\right\| \mid \xi' \in E^*,\ \|\xi'\| \leq 1,\ |\varepsilon_k| \leq 1\right\}.\end{aligned}$$

Since $\pi_1(U^{**}) = \pi_1(U)$ (see 3.1.11) and

$$\left\| \sum_{k=1}^{n} \varepsilon_k P_k \xi' \right\| \leq \|\xi'\|$$

in view of pairwise disjointness of the projections P_k, we obtain the sought estimate. $\triangleright$

(b) By repeating the arguments in the proof of the preceding theorem, we can convince ourselves that the following theorem is valid:

Theorem. *If a Banach lattice E satisfies conditions (B) and (C) (see 2.2.13), $U \in \mathscr{L}(X, E)$ and $U^* \in \Pi_1(E^*, X^*)$ then the operator U is dominated and $\|U\|_M \leq \pi_1(U^*)$.*

3.4.2. In the case $E = L^p(T, \mu)$, the claims in 3.4.1 can be made more precise.

(a) ***Theorem.*** *Let $1 \leq p < \infty$. If $U \in \Pi_p(L^{p'}(T, \mu), X)$ then the operator U^* is dominated and $\|U^*\|_M \leq \pi_p(U)$.*

$\triangleleft$ The proof of the theorem is similar to that of Theorem 3.4.1(a) from which our theorem appears as a particular case with $p = 1$. Therefore, in what follows we suppose that $p > 1$. It suffices to verify that the inequality $\|\Phi\|_p \leq \pi_p(U)$, where $\Phi = \max_{1 \leq k \leq n} |U^* x'_k|$, is valid for arbitrary vectors $x'_1, \dots, x'_n \in B_{X^*}$. It is clear that

$$\Phi = \left| \sum_{k=1}^{n} \chi_{e_k} U^* x'_k \right|,$$

where $e_1, \dots, e_n$ are pairwise disjoint subsets of T. Then

$$\|\Phi\|_p^p = \sum_{k=1}^{n} \|\chi_{e_k} U^* x'_k\|_p^p.$$

Choose functions $\xi_k \in L^{p'}(T, \mu)$ such that

$$\|\chi_{e_k} U^* x'_k\|_p = \langle \xi_k, \chi_{e_k} U^* x'_k \rangle, \quad \|\xi_k\|'_p = 1 \quad (k = 1, 2, \dots, n).$$

We see that

$$\|\Phi\|_p = \left(\sum_{k=1}^{n} \left|\langle \xi_k, \chi_{e_k} U^* x'_k\rangle\right|^p\right)^{1/p} \le \left(\sum_{k=1}^{n} \|U\chi_{e_k}\xi_k\|_{p'}^p\right)^{1/p}$$

$$\le \pi_p(U) \sup\left\{ \left(\sum_{k=1}^{n} |\langle \chi_{e_k}\xi_k, \eta\rangle|^p\right)^{1/p} \;\middle|\; \eta \in L^p(T,\mu),\ \|\eta\|_p \le 1\right\}$$

$$\le \pi_p(U) \sup\left\{ \left(\sum_{k=1}^{n} \|\chi_{e_k}\eta\|_p^p\right)^{1/p} \;\middle|\; \eta \in L^p(T,\mu),\ \|\eta\|_p \le 1\right\}.$$

Since

$$\sum_{k=1}^{n} \|\chi_{e_k}\eta\|_p^p \le \|\eta\|_p^p,$$

we obtain the claimed estimate. $\rhd$

(b) By duality, the preceding theorem yields

Theorem. *Let $1 \le p < \infty$. If $U \in \mathscr{L}(X, L^p(T,\mu))$ and $U^* \in \Pi_p(L^{p'}(T,\mu), X^*)$ then $U \in M(X, L^p(T,\mu))$; moreover, $\|U\|_M \le \pi_p(U^*)$.*

(c) In the case $0 < p < 1$ when the dual operator to the operator $U : X \to L^p(T,\mu)$ fails to exist, the preceding theorem can be supplemented with the following assertion.

Let $0 < p < 1$; let $U : X \to L^p(T,\mu)$ be a finite rank operator, and let $V \in \mathscr{L}(Z, X)$. If $V^* \in \Pi_p(X^*, Z^*)$ then $\|UV\|_M \le \|U\|\pi_p(V^*)$.

$\lhd$ The operator U, being a finite rank operator, is representable as

$$Ux = \sum_{k=1}^{N} \langle x, x'_k\rangle f_k,$$

where $x'_1, \dots, x'_N \in X^*$ and $f_1, \dots, f_N \in L^p(T,\mu)$. By putting

$$\varphi(t) = \sum_{k=1}^{N} f_k(t) x'_k \quad (t \in T),$$

we obtain

$$(Ux)(t) = \langle x, \varphi(t)\rangle \quad (t \in T).$$

Therefore,

$$\sup\{|UVz| \mid z \in Z,\ \|z\| \le 1\} = \sup\{|\langle z, V^*\varphi(\cdot)\rangle| \mid z \in Z,\ \|z\| \le 1\} = \|V^*\varphi(\cdot)\|$$

and, by 3.1.7, we have

$$\begin{aligned}\|UV\|_M &= \Big(\int_T \|V^*\varphi(t)\|^p \, d\mu(t)\Big)^{1/p} \\ &\le \pi_p(V^*)\Big(\int_T \int_K |\langle \varphi(t), x''\rangle|^p \, d\nu(x'')\, d\mu(t)\Big)^{1/p},\end{aligned}$$

where K is $\{x'' \in X^{**} \mid \|x''\| \le 1\}$ with the topology $\sigma(X^{**}, X^*)$ and ν is a probability Radon measure on K. Since the function

$$x'' \mapsto \int_T |\langle \varphi(t), x''\rangle|^p \, d\mu(t)$$

is continuous on K, we have

$$\begin{aligned}&\sup\Big\{\int_T |\langle\varphi(t), x''\rangle|^p \, d\mu(t) \mid x'' \in K\Big\} \\ &= \sup\Big\{\int_T |\langle x, \varphi(t)\rangle|^p \, d\mu(t) \mid x \in X,\ \|x\| \le 1\Big\};\end{aligned}$$

therefore,

$$\begin{aligned}\|UV\|_M &\le \pi_p(V^*) \sup\Big\{\int_T |\langle x, \varphi(t)\rangle|^p \, d\mu(t) \mid x \in X,\ \|x\| \le 1\Big\} \\ &= \pi_p(V^*)\|U\|. \ \triangleright\end{aligned}$$

3.4.3. As is easily verified, Theorems (a) and (b) in 3.4.2 cannot be converted. Thus, the operator of multiplication $M_g : L^\infty(T,\mu) \to L^2(T,\mu)$ by a function $g \in L^2(T,\mu)$ is obviously dominated but its dual is not p-absolutely summing for any $p < \infty$ since it is defined on a reflexive space but fails to be compact (see 3.1.8, Corollary 3).

However, the assertions of theorems in 3.4.2 can be strengthened, which makes it possible to achieve validity of the converse theorems. Henceforth the following notion will be convenient for us:

DEFINITION. An operator $U \in \mathscr{L}(X,E)$ is called *stably dominated* if the operator VU is dominated for every operator $V \in \mathscr{L}(E,E)$.

The operator M_g in the above-exhibited example is obviously dominated but not stably dominated.

3.4.4. Theorem. *Let $1 \le p \le \infty$. An operator $U \in \mathscr{L}(X, L^p(T,\mu))$ is stably dominated if and only if $U^* \in \Pi_p(L^{p'}(T,\mu), X^*)$. Moreover,*

$$\pi_p(U^*) = \sup\{\|VU\|_M \mid V \in \mathscr{L}(L^p(T,\mu), L^p(T,\mu)),\ \|V\| \le 1\}. \qquad (1)$$

$\triangleleft$ Sufficiency together with the inequality

$$\sup\{\|VU\|_M \mid V \in \mathscr{L}(L^p(T,\mu), L^p(T,\mu)),\ \|V\| \le 1\} \le \pi_p(U^*) \qquad (2)$$

follows from 3.4.2(a).

To prove necessity, we first convince ourselves that the quantity

$$C_0 = \sup\{\|VU\|_M \mid V \in \mathscr{L}(L^p(T,\mu), L^p(T,\mu)),\ \|V\| \le 1\}$$

is finite. Indeed, the mapping $V \mapsto VU$ acting from $\mathscr{L}(L^p(T,\mu), L^p(T,\mu))$ into $M(X, L^p(T,\mu))$ satisfies the hypotheses of the closed graph theorem and consequently is continuous. The real C_0 is none other than the norm of the mapping.

Let $\xi'_1, \dots, \xi'_n \in L^{p'}(T,\mu)$. To estimate the sum $\sum_{k=1}^n \|U^*\xi'_k\|^p$, we actually use the fact that the functions ξ'_k belong to the range of the canonical basis under some operator W_0 defined on $l^{p'}$. The dual of the operator W defined below is none other than a modification of W_0 which acts in the space $L^{p'}(T,\mu)$.

Let $e_1, \dots, e_n$ be pairwise disjoint subsets of a set T of positive measure. Define the operator $W : L^p(T,\mu) \to L^p(T,\mu)$ by the equality

$$W(\xi) = \sum_{k=1}^n \chi_{e_k} \langle \xi, \xi'_k \rangle (\mu(e_k))^{-1/p} \quad (\xi \in L^p(T,\mu)).$$

Then

$$W^*(\xi') = \sum_{k=1}^n (\mu(e_k))^{-1/p} \xi'_k \int_{e_k} \xi' \, d\mu \quad (\xi' \in L^{p'}(T,\mu))$$

and $W^*\eta'_k = \xi'_k$, where $\eta'_k = (\mu(e_k))^{-1/p'}\chi_{e_k}$ (for $p = 1$ we put $\eta'_k = \chi_{e_k}$), $k = 1, \dots, n$. It is clear that

$$\|W\| = \sup\left\{\left(\sum_{k=1}^{n} |\langle \xi, \xi'_k\rangle|^p\right)^{1/p} \mid \xi \in L^p(T,\mu),\ \|\xi\| \le 1\right\}. \tag{3}$$

Fix an arbitrary positive ε and find vectors $x_1, \dots, x_n \in X$ such that

$$\|U^*\xi'_k\| \le (1+\varepsilon)\langle x_k, U^*\xi'_k\rangle, \quad \|x_k\| = 1, \quad k = 1, 2, \dots, n.$$

Then

$$\begin{aligned}\left(\sum_{k=1}^{n} \|U^*\xi'_k\|^p\right)^{1/p} &\le (1+\varepsilon)\left(\sum_{k=1}^{n} \big|\big\langle x_k, U^*\xi'_k\big\rangle\big|^p\right)^{1/p} \\ &\le (1+\varepsilon)\left(\sum_{k=1}^{n} |\langle x_k, U^*W^*\eta_k\rangle|^p\right)^{1/p} \\ &\le (1+\varepsilon)\left(\sum_{k=1}^{n} |\langle WUx_k, \eta_k\rangle|^p\right)^{1/p} \\ &\le (1+\varepsilon)\left(\sum_{k=1}^{n} |\langle a, \eta_k\rangle|^p\right)^{1/p},\end{aligned}$$

where $a = \sup\{|WUx| \mid x \in X,\ \|x\| \le 1\}$. Consequently,

$$\begin{aligned}\left(\sum_{k=1}^{n} \|U^*\xi'_k\|^p\right)^{1/p} &\le (1+\varepsilon)\left(\sum_{k=1}^{n} \int_{e_k} a^p\, d\mu\right)^{1/p} \\ &= (1+\varepsilon)\|WU\|_M \le (1+\varepsilon)C_0\|W\|.\end{aligned}$$

This fact together with equality (3) provides the relations

$$U^* \in \Pi_p(L^{p'}(T,\mu), X^*), \quad \pi_p(U^*) \le (1+\varepsilon)C_0.$$

In view of the arbitrariness of ε it thus follows that $\pi_p(U^*) \le C_0$, whence we obtain equality (1) on use made of equality (2). $\rhd$

Corollary. *Let H be a Hilbert space and let $U : H \to L^2(T,\mu)$ be a Hilbert-Schmidt operator. Then $\|U\|_M = \pi_2(U)$.*

$\lhd$ By the preceding theorem and 3.2.2, we have

$$\|U\|_M \le \pi_2(U^*) = \sigma_2(U) = \pi_2(U).$$

On the other hand, $\pi_2(U) \le \|U\|_M$ by 3.1.5(c). $\rhd$

3.4.5. Theorem. *Let $1 \le p < \infty$. An operator $U \in \mathscr{L}(L^{p'}(T,\mu), X)$ is p-absolutely summing if and only if the operator U^* is stably dominated. Moreover,*

$$\pi_p(U) = \sup\{\|W^*U^*\|_M \mid W \in \mathscr{L}(L^{p'}(T,\mu), L^{p'}(T,\mu)),\ \|W\| \le 1\}.$$

$\lhd$ This theorem follows from 3.4.4 by duality. $\rhd$

3.4.6. Corollaries. (a) *Let $U \in \mathscr{L}(X,Y)$. Then $U \in \Pi_p(X,Y)$ $(U^* \in \Pi_p(Y^*,X^*))$ if and only if the operator V^*U^* (WU) is dominated for every operator $V \in \mathscr{L}(l^{p'}, X)$ $(W \in \mathscr{L}(Y, l^p)$, respectively).*

$\lhd$ The proof consists in comparing Theorem 3.4.5 (3.4.4) with Theorem 3.1.6. $\rhd$

(b) *Let $1 < p < \infty$ and let $U \in \mathscr{L}(L^{p'}(T',\mu'), L^p(T,\mu))$. The following assertions are equivalent:*

$$(\alpha)\ U \in M(L^{p'}(T',\mu'), L^p(T,\mu)); \quad (\alpha')\ U^* \in M(L^{p'}(T,\mu), L^p(T',\mu'));$$
$$(\beta)\ U \in \Pi_p(L^{p'}(T',\mu'), L^p(T,\mu)); \quad (\beta')\ U^* \in \Pi_p(L^{p'}(T,\mu), L^p(T',\mu')).$$

$\lhd$ The proof follows from validity of the implications $(\alpha) \Rightarrow (\beta) \Rightarrow (\alpha') \Rightarrow (\beta') \Rightarrow (\alpha)$, the first and the third of which follow from 3.1.5(c) and the rest, from 3.4.2. $\rhd$

Note that if we know a general form of an operator in $M(L^{p'}(T',\mu'), L^p(T,\mu))$ then we know a general form of an operator in $\Pi_p(L^{p'}(T',\mu'), L^p(T,\mu))$ too.

3.4.7. Invariantly order bounded sets.

DEFINITION. We say that a set $A \subset L^p(T,\mu)$ is *invariantly order bounded* if the V-image of A is order bounded in $L^p(T,\mu)$ for every operator $V \in \mathscr{L}(L^p(T,\mu), L^p(T,\mu))$.

Observe that for $p = 1$ every order bounded set will be also invariantly order bounded, since every operator in $\mathscr{L}(L^1(T,\mu), L^1(T,\mu))$ is regular.

Theorem. *Let $1 < p < \infty$ and let A be a norm bounded subset in $L^p(T, \mu)$. The set A is invariantly order bounded if and only if there exists an operator $U \in \Pi_p(L^{p'}(T', \mu'), L^p(T, \mu))$ such that $A \subset U(B)$, where B is the unit ball of the space $L^{p'}(T', \mu')$.*

$\triangleleft$ Sufficiency follows from Corollary 3.4.6(b) and 3.4.4. To prove necessity, consider the set $X = \operatorname{lin}(\widetilde{A})$, where $\widetilde{A}$ is the closed absolutely convex hull of A. Make X into a Banach space by taking the Minkowski functional of the set $\widetilde{A}$ as a norm in X (the completeness of X follows from Lemma 3.1.2). Since the identity embedding $i : X \to L^p(T, \mu)$ is stably dominated, we have $i^* \in \Pi_p(L^{p'}(T, \mu), X^*)$. Consequently, the operator i^* admits the canonical factorization

$$\begin{array}{l} i^* : L^{p'}(T,\mu) \xrightarrow{j} X_\infty \xrightarrow{\tilde{i}_p} X_p \xrightarrow{V} X^* \\ \qquad\qquad\qquad\quad\;\; \cap \qquad\;\; \cap \\ \qquad\qquad\qquad\; C(K) \xrightarrow{i_p} L^p(K,\nu). \end{array}$$

Thus, $i^* \in V\alpha$, where $\alpha = \tilde{i}_p j \in \Pi_p(L^{p'}(T, \mu), X_p)$. Therefore,

$$A^* \subset \alpha^*(\|V\| \cdot B_{X_p^*}) \subset \|V\|\alpha^*\varphi(B_{L^{p'}(K,\nu)}),$$

where φ is the canonical homomorphism of $L^{p'}(K, \nu)$ onto X_p^*. Since the operator $U = \|V\|\alpha^*\varphi$ is dominated by Corollary 3.4.6(b), it is p-absolutely summing as well. $\triangleright$

Corollary. *A set $A \subset L^2(T, \mu)$ is invariantly order bounded if and only if there exists a Hilbert-Schmidt operator $U : l^2 \to L^2(T, \mu)$ such that $A \subset U(B)$, where B is the unit ball of l^2.*

Observe that the corollary (certainly, together with the theorem) gives a description for invariantly order bounded sets purely in terms of Banach spaces. In particular, a set $A \subset L^2(0, 1)$ and its isometric image in the space l^2 are (or are not) invariantly order bounded simultaneously whereas the orders in the spaces $L^2(0, 1)$ and l^2 are quite different.

3.4.8. Here we prove two auxiliary assertions that are necessary for the further study of invariantly order bounded sets in a space.

(a) Lemma. *Let H_0 be a Hilbert space and let $x_1, \dots, x_n \in H_0$. If* $\dim H_0 \geq n$ *and*

$$\sup\left\{\sum_{k=1}^{n} |(x_k, x')|^2 \mid \|x'\| \leq 1\right\} \leq 1$$

then there are nonnegative numbers $\alpha_1, \dots, \alpha_N$ and a family of orthonormal systems $\{e_k^{(i)}\}_{k=1}^{n}$ in H_0 $(i = 1, 2, \dots, N)$ such that

$$\sum_{i=1}^{N} \alpha_i = 1$$

and

$$x_k = \sum_{i=1}^{N} \alpha_i e_k^{(i)}, \quad k = 1, 2, \dots, n.$$

$\lhd$ Without loss of generality we may assume that $\dim H_0 = n$. Let $\{g_k\}_{k=1}^{n}$ be an orthonormal basis for H_0 and let $A : H_0 \to H_0$ be the operator defined by the equality

$$Ax = \sum_{k=1}^{n} (x, g_k) x_k \quad (x \in H_0).$$

Then

$$\|A\| \leq 1, \quad A g_k = x_k \quad (k = 1, 2, \dots, n).$$

Since the set of unitary operators is the set of extreme points of the unit ball in $\mathscr{L}(H_0, H_0)$, the set A is a convex combination of unitary operators Q_i:

$$A = \sum_{k=1}^{N} \alpha_i Q_i, \quad \sum_{k=1}^{N} \alpha_i = 1, \quad \alpha_i \geq 0.$$

Consequently,

$$x_k = A g_k = \sum_{k=1}^{N} \alpha_i Q_i g_k \quad (k = 1, 2, \dots, n).$$

Thus,

$$\{Q_i g_k\}_{k=1}^{n} \quad (i = 1, 2, \dots, N)$$

is the sought family of orthonormal systems. $\rhd$

(b) Lemma. *Let H be an infinite-dimensional space; let $H_0 \subset H$,* $\operatorname{codim} H_0 = m < \infty$; *let $V \in \mathscr{L}(H,Y)$, and let $K_0 > 0$. If*

$$\sum_{k\geq 1} \|Ve_k\|^2 \leq K_0^2$$

for every orthonormal system $\{e_k\}_{k\geq 1} \subset H_0$ then there exists a number K such that

$$\sum_{k\geq 1} \|Vh_k\|^2 \leq K^2$$

for every orthonormal system $\{h_k\}_{k\geq 1} \subset H$.

$\lhd$ Let $\{h_k\}_{k=1}^{\infty}$ be an arbitrary orthonormal system in H; let $g_1, \dots, g_m$ be a orthonormal basis for $H_0^{\perp}$, and let P be the orthogonal projection onto H_0. Fix a natural n and put $x_k = Ph_k$ $(k = 1, 2, \dots, n)$. Since the vectors $x_1, \dots, x_n$ satisfy the hypothesis of Lemma (a), they are representable as

$$x_k = \sum_{i=1}^{N} \alpha_i e_k^{(i)} \quad (k = 1, 2, \dots, n),$$

where $\alpha_i \geq 1$, $\sum_{k=1}^{N} \alpha_i = 1$, and $\{e_k^{(i)}\}_{k=1}^{n}$ are orthonormal systems in H_0 $(i = 1, 2, \dots, N)$. Thus,

$$h_k = \sum_{j=1}^{m} (h_k, g_j) g_j + x_k$$

and

$$Vh_k = \sum_{j=1}^{m} (h_k, g_j) V g_j + \sum_{i=1}^{N} \alpha_i V e_k^{(i)} \quad (k = 1, 2, \dots, n).$$

Consequently,

$$\|Vh_k\| \leq \|V\| \sum_{j=1}^{m} |(h_k, g_j)| + \sum_{i=1}^{N} \alpha_i \|Ve_k^{(i)}\|.$$

By using the triangle inequality for the norm in l^2, we obtain

$$\left(\sum_{k=1}^{n}\|Vh_k\|^2\right)^{1/2} \leq \left(\sum_{k=1}^{n}\left[\|V\|\sum_{j=1}^{m}|(h_k,g_j)| + \sum_{i=1}^{N}\alpha_i\|Ve_k^{(i)}\|\right]^2\right)^{1/2}$$
$$\leq \|V\|\sum_{j=1}^{m}\left(\sum_{k=1}^{n}|(h_k,g_j)|^2\right)^{1/2} + \sum_{i=1}^{N}\alpha_i\left(\sum_{k=1}^{n}\|Ve_k^{(i)}\|^2\right)^{1/2}$$
$$\leq \|V\|\left(\sum_{j=1}^{m}\|g_j\|^2\right)^{1/2} + K_0.$$

So, we may put $K := \|V\|m + K_0$. ▷

3.4.9. Definition. Let H be a Hilbert space; let $\mathscr{E} = \{e_k\}_{k=1}^{\infty}$ be an orthonormal system in H, and let $A \subset H$. Put

$$c_k = c_k(A, \mathscr{E}) = \sup\{|(x, e_k)| \mid x \in A\}.$$

The numbers c_k are called the *gauges* of A with respect to the system $\mathscr{E}$.

Theorem (see [53, 54]). *Let A be a norm bounded subset in $L^2(T,\mu)$. The following assertions are equivalent:*

(1) *the set A is invariantly order bounded;*

(2) $\sum_{k=1}^{\infty} c_k^2(A,\mathscr{E}) < \infty$ *for every orthonormal system* $\mathscr{E} = \{e_k\}_{k=1}^{\infty} \subset L^2(T,\mu)$, *where $c_k(A,\mathscr{E})$ the gauges of A with respect to the system $\mathscr{E}$;*

(3) *there exists a number $K > 0$ such that*

$$\sum_{k=1}^{\infty} c_k^2(A,\mathscr{E}) \leq K^2$$

for every orthonormal system $\mathscr{E} \subset L^2(T,\mu)$.

◁ (1) ⇒ (2): According to Corollary 3.4.7, there exists a Hilbert-Schmidt operator $U : l^2 \to L^2(T,\mu)$ such that $A \subset U(B)$, where B is the unit ball of the space l^2. Then

$$c_k(A,\mathscr{E}) \leq \sup\{|(Uy, e_k)| \mid y \in B\} = \|U^* e_k\|.$$

Since U^* is a Hilbert-Schmidt operator, we have

$$\sum_{k=1}^{\infty} c_k^2(A, \mathscr{E}) \leq \sum_{k=1}^{\infty} \|U^* e_k\|^2 < \infty.$$

(2) $\Rightarrow$ (3): Let (3) be false. Then by Lemma 3.4.8(b) we see that

$$\begin{gathered}\text{given a subspace } H_0 \subset H, \ \operatorname{codim} H_0 < \infty, \\ \text{there exists an orthonormal system} \\ G = \{g_k\}_{k=1}^{\infty} \subset H_0, \text{ such that } \sum_{k=1}^{\infty} c_k^2(A, G) \geq 1. \end{gathered} \tag{4}$$

Fix a finite system $\{e_k\}_{k=1}^{n_1} \subset H$ such that

$$\sum_{k=1}^{n_1} c_k^2\big(A, \{e_k\}_{k=1}^{n_1}\big) \geq 1$$

and put

$$H_0 = \big(\operatorname{lin}(\{e_k\}_{k=1}^{n_1})\big)^{\perp}.$$

By virtue of (4), there exists a finite orthonormal system $\{e_k\}_{k=n_1+1}^{n_2} \subset H_0$ such that

$$\sum_{k=n_1+1}^{n_2} c_k^2\big(A, \{e_k\}_{k=n_1+1}^{n_2}\big) \geq 1.$$

Continuing the construction by induction, we obtain the orthonormal system $\mathscr{E} = \{e_k\}_{k=1}^{\infty}$ in H such that

$$\sum_{k=1}^{\infty} c_k^2(A, \mathscr{E}) = \infty,$$

which is impossible by condition (2).

(3) $\Rightarrow$ (1): Consider the set $X = \operatorname{lin}(\widetilde{A})$, where $\widetilde{A}$ is the closure of the absolutely convex hull of A and make it into a Banach space by taking the Minkowski functional of the set $\widetilde{A}$ as a norm in X (the completeness of X follows from Lemma 3.1.2). Prove that the identity embedding $j : X \to L^2(T, \mu)$ is a stably dominated operator,

which obviously ensures the invariant order boundedness of A. By Theorem 3.4.4, it is sufficient to verify that $j^* \in \Pi_2(L^2(T,\mu), X^*)$. We check that

$$\left(\sum_{k=1}^{n} \|j^* x_k\|^2\right)^{1/2} \leq K \sup\left\{\left(\sum_{k=1}^{n} |(x_k, x')|^2\right)^{1/2} \mid \|x'\|_2 \leq 1\right\}$$

for arbitrary functions $x_1, \dots, x_n \in L^2(T,\mu)$. In view of homogeneity of the preceding inequality, we may assume that

$$\sup\left\{\left(\sum_{k=1}^{n} |(x_k, x')|^2\right)^{1/2} \mid \|x'\|_2 \leq 1\right\} \leq 1.$$

By making use of the result and notation of Lemma 3.4.8(a), we have

$$x_k = \sum_{i=1}^{N} \alpha_i e_k^{(i)}, \quad k = 1, 2, \dots, n,$$

where $\mathscr{E}_i = \{e_k^{(i)}\}_{k=1}^{n}$ $(i = 1, 2, \dots, N)$ are orthonormal systems. Consequently,

$$\|j^* x_k\| \leq \sum_{i=1}^{N} \alpha_i \|j^* e_k^{(i)}\|$$

and

$$\begin{aligned}
\left(\sum_{k=1}^{n} \|j^* x_k\|^2\right)^{1/2} &\leq \left(\sum_{k=1}^{n}\left(\sum_{i=1}^{N} \alpha_i \|j^* e_k^{(i)}\|\right)^2\right)^{1/2} \\
&\leq \sum_{i=1}^{N} \alpha_i \left(\sum_{k=1}^{n} \|j^* e_k^{(i)}\|^2\right)^{1/2} = \sum_{i=1}^{N} \alpha_i \left(\sum_{k=1}^{N} c_k^2(A, \mathscr{E}_i)\right)^{1/2} \\
&\leq \sum_{i=1}^{N} \alpha_i K = K. \ \rhd
\end{aligned}$$

Corollary. *If $A \subset l^2$ and the set $U(A)$ is order bounded for every unitary operator $U : l^2 \to l^2$ then A is an invariantly order bounded set.*

$\lhd$ The hypothesis of the corollary is equivalent to item (2) of the theorem. $\rhd$

3.4.10. V. N. Sudakov's theorem. The following result shows that each set that fails to be invariantly order bounded in $L^2(0,1)$ can be "rotated" so that it fails to be order bounded even in the space $L^0(0,1)$. We start with proving the following

Lemma. *Let $c_1, \dots, c_p$ be a collection of positive numbers and let $\varphi_1, \dots, \varphi_N$ be a collection of step functions on $(0,1)$. Then there are a constant $\alpha > 0$ and an orthonormal function system $f_1, \dots, f_p \in L^2(0,1)$ possessing the following properties:*

(1) each of the functions f_k assumes only two nonzero values $\pm h_k$, where $h_k = \alpha c_k^{-1}$ and $\int_0^1 f_k(t)\,dt = 0$;

(2) $f_k \perp \varphi_j$ for all $k = 1, 2, \dots, p$; $j = 1, 2, \dots, N$;

(3) the sets on which the functions f_k take nonzero values are pairwise disjoint.

$\triangleleft$ Suppose that some functions f_k are constructed. Let

$$e_k = \{t \in (0,1) \mid f_k(t) \neq 0\}$$

and let δ_k be the measure of e_k, $k = 1, 2, \dots, p$. Then

$$\sum_{k=1}^{p} \delta_k = 1$$

and

$$\alpha^2 c_k^{-2} \delta_k = 1 \text{ for } k = 1, 2, \dots, p.$$

Consequently,

$$\alpha = \left(\sum_{k=1}^{p} c_k^2 \right)^{1/2}, \quad \delta_k = c_k^2 \left(\sum_{k=1}^{p} c_k^2 \right)^{-1}. \tag{5}$$

Now we turn out to constructing the functions f_k on assuming that the values α and δ_k were chosen by formulas (5). Let each of the $\varphi_1, \dots, \varphi_N$ be constant on each of the sets $A_1, \dots, A_M$ making up a partition of $(0,1)$ and let $e_1, \dots, e_p$ be another partition of $(0,1)$ which consists of some sets of measures $\delta_1, \dots, \delta_p$ respectively. Put $e_{kj} = e_k \cap A_j$ and divide each of the sets e_{kj} onto two parts e_{kj}^+

and e_{kj}^- of the same measure. Define a function f_k as

$$f_k(t) = \begin{cases} h_k, & \text{if } t \in \bigcup\limits_{j=1}^{M} e_{kj}^+, \\ -h_k, & \text{if } t \in \bigcup\limits_{j=1}^{M} e_{kj}^-, \\ 0, & \text{if } t \notin e_k. \end{cases}$$

It is clear that the so-defined functions f_k satisfy all requirements of the lemma. $\rhd$

In what follows we for brevity shall use the symbol $|A|$ to denote the set $\{|x| \mid x \in A\}$. All spaces under consideration may be either real or complex.

Theorem (see [53, 54]). *Suppose that a set $A \subset L^2(0,1)$ is norm bounded but not invariantly order bounded. Then there exists a unitary operator $U : L^2(0,1) \to L^2(0,1)$ such that $\sup|U(A)| = +\infty$ almost everywhere on $(0,1)$.*

$\lhd$ We split the proof of the theorem into three steps.

I. Let $\mathscr{E} = \{e_n\}_{n=1}^{\infty}$ be an orthonormal system in $L^2(0,1)$ and let $c_k = c_k(A, \mathscr{E})$ be the gauges of A with respect to $\mathscr{E}$. Prove that if $\sum_{k=1}^{\infty} c_k^2 = \infty$ then there exists an orthonormal system of functions $\{f_k\}_{k=1}^{\infty} \subset L^2(0,1)$ such that

(α) the equality

$$\sup\left\{ \left|\sum_{k=1}^{\infty}(x, f_k)\right| \mid x \in A \right\} = \infty$$

is valid almost everywhere on $(0,1)$;

(β) $\operatorname{lin}(\{f_k\}_{k=1}^{\infty})$ possesses an infinite-dimensional orthogonal complement.

Fix $x_k \in A$ such that $|(x_k, e_k)| > c_k/2$ $(k = 1, 2, \dots)$ and put

$$\eta_k(x) = \sum_{j=k+1}^{\infty} |(x, e_j)|^2.$$

We construct the sought system by induction. Let $m_1 = 1$; let f_1 be an arbitrary step function with $\|f_1\|_2 = 1$, and let

$$S_1 = \max_{0<t<1} c_1 |f_1(t)|.$$

Find $p_1 \geq 2$ such that

$$\frac{1}{2}\left(\sum_{k=m_1+1}^{m_1+p_1} c_k^2\right)^{1/2} > S_1 + 1.$$

Fix an arbitrary step function g_1, $\|g_1\|_2 = 1$, which is orthogonal to f_1. By applying the lemma to the numbers $c_{m_1+1}, \dots, c_{m_1+p_1}$ and to the functions f_1 and g_1, we obtain functions $f_{m_1+1}, \dots, f_{m_1+p_1}$. Now we choose $m_2 > m_1 > p_1$ so that $\eta_{m_2}(x_i) < 1/p_1^2$ for every $i = 1, 2, \dots, m_1 + p_1$. As $f_{m_1+p_1+1}, \dots, f_{m_2}$ we take an orthonormal system of step functions on $\Delta_1 = (0, 1/p_1)$ such that these functions are orthogonal to $f_1, \dots, f_{m_1+p_1}$ and g_1. Put

$$S_2 = \max_{0<t<1} \sum_{k=1}^{m_2} c_k |f_k(t)|.$$

Suppose that the natural $m_1, \dots, m_j$ and $p_1, \dots, p_{j-1}$ and the step functions $g_1, \dots, g_{j-1}$ and $f_1, \dots, f_{m_j}$ are already constructed so as

(1) $p_{j-1} < p_i, \quad m_i + p_i < m_{i+1}$ for $i = 1, 2, \dots, j-1$;

(2) $\eta_{m_{i+1}}(x_k) < 1/p_i^2$ for $k = 1, 2, \dots, m_i + p_i$ $(i = 1, 2, \dots, j-1)$;

(3) the functions $g_1, \dots, g_{j-1}$; $f_1, \dots, f_{m_j}$ form an orthonormal system;

(4) given $m_i < k \le m_i + p_i$, the functions $|f_k|$ take only the values 0 and h_k, their supports are pairwise disjoint and

$$c_k h_k = \left(\sum_{l=m_i+1}^{m_i+p_i} c_l^2 \right)^{1/2};$$

(5) the following inequality is valid:

$$\frac{1}{2} \left(\sum_{l=m_i+1}^{m_i+p_i} c_l^2 \right)^{1/2} > S_i + i + 1,$$

where

$$S_i = \max_{0<t<1} \sum_{k=1}^{m_i} c_k |f_k(t)|,$$

Now put

$$S_j = \max_{0<t<1} \sum_{k=1}^{m_j} c_k |f_k(t)|,$$

find a number $p_j > p_{j-1}$ such that

$$\frac{1}{2} \left(\sum_{l=m_j+1}^{m_j+p_j} c_l^2 \right)^{1/2} > S_j + j + 1,$$

and fix a step function g_j with $\|g_j\|_2 = 1$ which is orthogonal to all functions g_i for $i < j$ and f_k for $k \le m_j$. By applying the lemma to the numbers $c_{m_j+1}, \dots, c_{m_j+p_j}$ and the functions

$$g_1, \dots, g_j, \quad f_1, \dots, f_{m_j}, \tag{6}$$

we construct functions $f_{m_j+1}, \dots, f_{m_j+p_j}$. Choose now $m_{j+1} > m_j + p_j$ so that $\eta_{m_j+1}(x_n) < 1/p_j^2$ for $1 \le n \le m_j + p_j$. We take as $f_{m_j+p_j+1}, \dots, f_{m_{j+1}}$ an orthonormal system of step functions vanishing outside the interval $\Delta_j = (0, 1/p_j)$ so that these functions be orthogonal to all functions g_i for $i \le j$ and f_k for $k \le m_j + p_j$ that completes the induction step.

Thus, we constructed the orthonormal system $\{f_k\}_{k=1}^{\infty}$ satisfying (β) (because $f_k \perp g_j$ for all k and j). Prove that (α) is satisfied as well. Fix a natural j and consider the function

$$y_x = \sum_{k=1}^{\infty} (x, e_k) f_k,$$

where $x \in A$. We have

$$\begin{aligned} |y_x| &\ge \left| \sum_{k=m_j+1}^{m_{j+1}} (x, e_k) f_k \right| - \sum_{k=1}^{m_j} |(x, e_k)| \cdot |f_k| - \left| \sum_{k>m_{j+1}} (x, e_k) f_k \right| \\ &\ge \left| \sum_{k=m_j+1}^{m_j+p_j} (x, e_k) f_k \right| - \left| \sum_{k=m_j+p_j+1}^{m_{j+1}} (x, e_k) f_k \right| - S_j - \left| \sum_{k>m_{j+1}} (x, e_k) f_k \right|. \end{aligned}$$

The sum

$$\sum_{k=m_j+p_j+1}^{m_{j+1}} (x, e_k) f_k$$

is identically zero beyond the interval Δ_j. Therefore, beyond Δ_j we have

$$\begin{aligned} \sup_{x \in A} |y_x| &\ge \sup_{m_j < n < m_j+p_j} \left| \sum_{k=m_j+1}^{m_j+p_j} (x_n, e_k) f_k \right| - S_j - \max_{m_j < n < m_j+p_j} |\psi_n| \\ &\ge \frac{1}{2} \left(\sum_{k=m_j+1}^{m_j+p_j} c_k^2 \right)^{1/2} - S_j - \max_{m_j < n < m_j+p_j} |\psi_n|, \end{aligned}$$

where

$$\psi_n = \sum_{k>m_{j+1}} (x_n, e_k) f_k.$$

Hence, by (5), it follows

$$\sup_{x\in A}|y_x| \geq j+1-\max_{m_j<n<m_j+p_j}|\psi_n|. \tag{7}$$

Estimate the maximum on the left-hand side of inequality (7). By choosing m_{j+1}, we have $\|\psi_n\|_2^2 < 1/p_j^2$ for every $n \leq m_j + p_j$. Put

$$e_n = \{t \in (0,1) \mid |\psi_n(t)| \geq 1\}, \quad E_j = \bigcup_{n=m_j+1}^{m_j+p_j} e_n.$$

Since $\operatorname{mes}(e_n) \leq \|\psi_n\|_2^2 < 1/p_j^2$, we have $\operatorname{mes}(E_j) < 1/p_j$ and

$$\max_{m_j<n<m_j+p_j}|\psi_n(t)| \leq 1$$

for $t \notin E_j$. This fact together with inequality (7) provides

$$\sup_{x\in A}|y_x(t)| \geq j \text{ for } t \notin E_j \cup \Delta_j.$$

Hence the required result follows because j is arbitrary and $\operatorname{mes}(E_j \cup \Delta_j) \underset{j\to\infty}{\longrightarrow} 0$.

II. In addition we now suppose that A is a relatively compact set. By 3.4.9, there is an orthonormal basis $\{\tilde{e}_n\}_{n=1}^{\infty}$ for $L^2(0,1)$ such that $\sum_{n=1}^{\infty}\tilde{c}_n^2 = \infty$, where $\tilde{c}_n = \sup\{|(x,\tilde{e}_n)| \mid x \in A\}$. Since $\tilde{e}_n \to 0$, in view of the relative compactness of A we can choose a subsequence $\{\tilde{e}_{n_k}\}_{k=1}^{\infty}$ such that $2k^2\tilde{c}_{n_k} \leq 1$ for $k = 1,2,\dots$. Divide the system $\{\tilde{e}_n\}_{n=1}^{\infty}$ into two parts: the sequence $\{\tilde{e}_{n_k}\}_{k=1}^{\infty}$ and the complement to it denoted by $\{e_k\}_{k=1}^{\infty}$. By applying the result of the first step of the proof to this system, we obtain some orthonormal system $\{f_k\}$ that satisfies (α) and (β). Let $\{\tilde{f}_k\}_{k=1}^{\infty}$ be an orthonormal basis for the orthogonal complement to $\operatorname{lin}\big(\{f_k\}_{k=1}^{\infty}\big)$. Define the unitary operator $U : L^2(0,1) \to L^2(0,1)$ by the equalities

$$U(e_k) = f_k, \quad U(\tilde{e}_{n_k}) = \tilde{f}_k \quad (k = 1,2,\dots).$$

We now check that $\sup|U(A)| = \infty$ almost everywhere on $(0,1)$. Indeed, let P be the orthogonal projection onto $\overline{\operatorname{lin}}\big(\{f_k\}_{k=1}^{\infty}\big)$. Then

(1) $\sup|UP(A)| = \infty$ almost everywhere on $(0,1)$;

(2) the set $C = U(I-P)(A)$ is order bounded.

The first assertion is obvious. For proving the second, we use Corollary 3.4.6(d) and verify that the set C is contained in the image of the unit ball under some Hilbert-Schmidt operator. Indeed, if $x \in A$ then

$$U(I-P)x = \sum_{k=1}^{\infty} (x, \tilde{e}_{n_k})\tilde{f}_k = \sum_{k=1}^{\infty} \frac{1}{k} t_k \tilde{f}_k,$$

where

$$|t_k| = |k(x, \tilde{e}_{n_k})| \le \frac{1}{2k}, \quad \sum_{k=1}^{\infty} |t_k|^2 \le 1.$$

By putting

$$V(t) = \sum_{k=1}^{\infty} \frac{t_k}{k} \tilde{f}_k,$$

where $t = \{t_k\}_{k=1}^{\infty} \in l^2$, we see that V is a Hilbert-Schmidt operator and

$$C \subset \{V(t) \mid t \in l^2, \ \|t\| \le 1\}.$$

Therefore, $\Phi = \sup\{|y| \mid y \in C\} < \infty$ almost everywhere and consequently

$$|Ux| \ge |UPx| - |U(I-P)x| \ge |UPx| - \Phi$$

for $x \in A$. Hence, $\sup |U(A)| = \sup |UP(A)| = \infty$ almost everywhere.

III. Finally, let A be a set satisfying the hypotheses of the theorem and let $V : L^2(0,1) \to L^2(0,1)$ be an operator such that $\sup |V(A)| \notin L^2(0,1)$. Take a sequence $\{x_n\}_{n=1}^{\infty} \subset A$ such that $\|y_n\| \to \infty$, where

$$y_n = \max\{|Vx_1|, |Vx_2|, \dots, |Vx_n|\}.$$

Put $b_n = 1 + \|y_n\|^{1/2}$ and $\tilde{x}_n = x_n/b_n$. The set $\{\tilde{x}_1, \tilde{x}_2, \dots\}$ is relatively compact but not invariantly order bounded since $\|y_n/b_n\| \to \infty$ and

$$\max(|V\tilde{x}_1|, \dots, |V\tilde{x}_n|) = \max\left(\frac{1}{b_1}|Vx_1|, \dots, \frac{1}{b_n}|Vx_n|\right) \ge \frac{y_n}{b_n}.$$

Consequently, by what was proven above, there exists a unitary operator U such that $\sup\limits_n |U(\tilde{x}_n)| = \infty$ almost everywhere on $(0,1)$. Then

$$\sup |U(A)| \ge \sup_n |U(x_n)| \ge \sup_n |U(\tilde{x}_n)| = \infty. \ \rhd$$

Corollary. *Let the space $L^2(T,\mu)$ be separable and let $A \subset L^2(T,\mu)$ be a norm bounded set. The set A is invariantly order bounded if and only if the set $U(A)$ is order bounded for every unitary operator $U : L^2(T,\mu) \to L^2(T,\mu)$.*

◁ The space $L^2(T, \mu)$ treated as a Hilbert space and as a partially ordered space is isomorphic either to $L^2(0,1)$, or to l^2, or to the direct sum of $L^2(0,1)$ and l^2, or, finally, to the direct sum of $L^2(0,1)$ and l^2_n. In all the cases, the assertion of the corollary follows from Theorem 3.6.10 and Corollary 3.4.9. ▷

REMARK. Another proof of the corollary (in the complex case) ensures from the fact that every operator in a Hilbert space H is a linear combination of at most eight unitary operators.

◁ Obviously, it suffices to prove the preceding assertion only for a positive operator $V : H \to H$ such that $\|V\| \leq 1$. Assign $U = V + i(I - V^2)^{1/2}$, where I is the identity operator in H. As easily verified, U is a unitary operator and $U^* = V - i(I - V^2)^{1/2}$. Therefore, $V = (1/2)(U + U^*)$. ▷

3.4.11. Having an acquaintance with Theorem 3.4.10 may inspire oneself to transfer it on the arbitrary spaces $L^p(0,1)$ by replacing unitary operators with isometries. This is however impossible since the isometries are too "scarce" in $L^p(0,1)$ for $p \neq 2$.

Theorem. *Let* $1 \leq p < \infty$, $\varepsilon > 0$, *and* $A \subset L^p = L^p(0,1)$. *If* $\sup|A| \notin L^p$ *then there exists a positive operator* $U : L^p \to L^p$ *such that* $\|U - I\| < \varepsilon$ *and* $\sup|U(A)| = \infty$ *almost everywhere on* $(0,1)$.

First we prove the following two lemmas.

(a) ***Lemma.*** *Let* $(T, \mathfrak{A}, \mu)$ *be a measure space with finite measure* μ; *let* $A \subset L^0(T, \mu)$, *and let* $\sup|A| > C$ *almost everywhere on* T. *Given* $\delta > 0$, *there exist functions* $x_1, \dots, x_N$ *in* A *and a set* $e \subset T$ *such that*

$$\sup\{|x_1|, \dots, |x_N|\} > C \text{ on } T \setminus e \text{ and } \mu(e) < \delta.$$

We leave a simple proof of the lemma to the reader.

(b) ***Lemma.*** *Let* $1 \leq p < \infty$, $A \subset L^p = L^p(0,1)$, *and* $\sup|A| \notin L^p$. *Given* $C > 0$ *and* $\delta > 0$, *there exists a positive isometry* $U : L^p \to L^p$ *such that* $\sup|U(A)| > C$ *on a set of measure greater than* $1 - \delta$.

◁ First we suppose that A consists of nonnegative functions. Obviously, without loss of generality we may assume that the range of the function $\sup A$ consists of numbers $0 \leq \lambda_1 < \lambda_2 < \dots$. (Otherwise, we can replace A with the set consisting of the functions $\min(x, f)$, where $x \in A$, $f \leq \sup A$, $f \notin L^p(0,1)$, and the range of f is composed of the numbers λ_k.) Then with the help of a measure preserving automorphism of a segment one can make $h = \sup|A|$ nonincreasing. If $h(t) \equiv +\infty$ on some open segment $(0, a)$, where $0 < a < 1$, then as is easily verified, the sought isometry U can be given by the formula

$$(Ux)(t) = (\varphi'(t))^{1/p} x(\varphi(t)),$$

where the function φ takes the values 0, a, and 1 at the points 0, $1-\delta$, and 1 and is linear on the segments $[0, 1-\delta]$ and $[1-\delta, 1]$.

Now we consider the case in which $h(t) < +\infty$ on $(0,1)$. Since $h \notin L^p$, there exists an $a \in (0,1)$ such $h(a) > 0$. We shall look for a strictly increasing absolutely continuous function φ on $(0, 1-\delta]$ satisfying the conditions

(α) $\varphi(1-\delta) = a$;

(β) $(\varphi'(t))^{1/p} h(\varphi(t)) = Ct^{-1/p}$ for $0 < t \le 1-\delta$.

The preceding property implies

$$(\gamma) \qquad \int_s^{1-\delta} h^p(\varphi(t))\varphi'(t)\,dt = \int_s^{1-\delta} \frac{C^p}{t}\,dt = C^p \ln\frac{1-\delta}{s} \text{ for } 0 < s \le 1-\delta.$$

Put

$$H(u) = \int_u^a h^p(v)\,dv$$

for $0 < u \le a$. Considering (α), we can rewrite (γ) as

$$H(\varphi(s)) = C^p \ln((1-\delta)/s)$$

for $0 < s \le 1-\delta$. The above reasoning demonstrates that the function φ must be defined by the equality

$$\varphi(s) = H^{-1}(C^p \ln((1-\delta)/s)), \quad 0 < s \le 1-\delta.$$

Consequently, it is absolutely continuous since $|H'(u)| \ge h^p(a) > 0$, satisfies condition (β) and also condition (α) in view of $H(a) = 0$. Moreover, we have $\lim\limits_{s\to+0} \varphi(s) = 0$, since $\lim\limits_{u\to+0} \varphi(s) = +\infty$. Put $\varphi(0) = 0$ and $\varphi(1) = 1$ and assume the function φ linear on the segment $[1-\delta, 1]$. Thus we obtain the isometry

$$(Ux)(t) = (\varphi'(t))^{1/p} x(\varphi(t)) \quad (t \in (0,1);\ x \in L^p). \ \rhd$$

If the set A consists not only of nonnegative functions then it suffices to construct an isometry for the set $|A|$ in the above-indicated fashion.

Now we turn to proving the theorem.

$\triangleleft$ First, let A consist only of real functions; let $\sup A \notin L^p$, and let $\sup A < +\infty$ almost everywhere on $(0,1)$. Also assume that $\varepsilon < 1$.

By induction, construct sequences of positive numbers $\{\varepsilon_n\}_{n=1}^{\infty}$, $\{C_n\}_{n=1}^{\infty}$, a sequence $\{A_n\}_{n=1}^{\infty}$ of finite subsets of A such that $A_n \subset A_{n+1}$, and a sequence $\{U_n\}_{n=1}^{\infty}$ of positive isometries in L^p which, for $n \in N$, satisfy the following conditions:

(1) $2^n \varepsilon_n \|h_{n-1}\| < 2^{-n/p}$, where $h_0 = 0$ and $h_{n-1} = \sup |A_{n-1}|$, $\varepsilon_n \leq \varepsilon_1^n$;

(2) $\operatorname{mes}\{t \in (0,1) \mid (\sup V_{n-1}(A))(t) > C_n\} < 2^{-n}$, where $V_0 = I$ and $V_{n-1} = I + \varepsilon_1 U_1 + \ldots + \varepsilon_{n-1} U_{n-1}$;

(3) $\sup U_n(A_n) > (C_n + n)/\varepsilon_n$ outside the set e_n with $\operatorname{mes}(e_n) < 2^{-n}$.

Put $\varepsilon_1 = \varepsilon/2$ and choose a C_1 so that

$$\operatorname{mes}\{t \in (0,1) \mid (\sup(A))(t) > C_1\} < 1/2.$$

Let U_1 be a positive isometry such that $\sup U_1(A) > (C_1 + 1)/\varepsilon_1$ outside a set of measure less than $1/4$. Let A_1 be a finite subset of A such that $\sup U_1(A_1) > (C_1 + 1)/\varepsilon_1$ outside a set of measure less than $1/2$. We thus have a base for induction.

Suppose that some numbers $\varepsilon_1, \ldots, \varepsilon_{N-1}$; $C_1, \ldots, C_{N-1}$, finite sets $A_1 \subset A_2 \subset \ldots \subset A_{N-1} \subset A$, and positive isometries $U_1, \ldots, U_{N-1}$ that satisfy conditions (1)–(3) for $n = 1, 2, \ldots, N-1$ are constructed. Now construct ε_N, C_N, A_N, and U_N. Choose the number ε_N so that condition (1) be satisfied and C_N so that condition (2) be satisfied. By Lemma (b), there exists a positive isometry U_N such that $\sup U_N(A) > (C_N + N)/\varepsilon_N$ outside a set of measure less than 2^{-N-1}. By Lemma (a), there exists a finite set $B \subset A$ such that $\sup U_N(B) > (C_N + N)/\varepsilon_N$ outside a set e_N with $\operatorname{mes}(e_N) < 2^{-N}$. By putting $A_N = B \cup A_{N-1}$, we obtain validity of condition (3).

Now demonstrate that the operator $U = I + \varepsilon_1 U_1 + \varepsilon_2 U_2 + \ldots$ meets the claim. It is clear that $U \geq 0$ and $\|U - I\| \leq \varepsilon_1/(1-\varepsilon_1) < \varepsilon$. Estimate $\sup U(A)$ from below. Given an arbitrary $n \in \mathbb{N}$, we have

$$\sup U(A) \geq \sup U(A_n) \geq \varepsilon_n \sup U_n(A_n) - \sup V_{n-1}(A_n) - \sum_{j=n+1}^{\infty} \varepsilon_j U_j(h_n)$$
$$\geq \varepsilon_n \sup U_n(A_n) - \sup V_{n-1}(A) - \sum_{j=n+1}^{\infty} \varepsilon_j U_j(h_{j-1}) \tag{8}$$

(recall that $h_k = \sup |A_k|$, $h_1 \leq h_2 \leq \dots$). By condition (3), we have

$$\varepsilon_n \sup U_n(A_n) \geq C_n + n$$

outside a set e_n with $\operatorname{mes}(e_n) < 2^{-n}$. By condition (2), we have

$$\sup V_{n-1}(A) \leq C_n$$

outside a set e'_n with $\operatorname{mes}(e'_n) < 2^{-n}$. To estimate the sum

$$\sum_{j=n+1}^{\infty} \varepsilon_j U_j(h_{j-1}),$$

consider $E_j = \{t \in (0,1) \mid \varepsilon_j (U_j h_j)(t) > 2^{-j}\}$. Then

$$\operatorname{mes}(E_j) \leq (\varepsilon_j 2^j \|U_j h_{j-1}\|)^p = (\varepsilon_j 2^j \|h_{j-1}\|)^p.$$

By virtue of condition (1), we have $\operatorname{mes}(E_j) < 2^{-j}$. Thus, (8) implies that

$$\sup U(A) \geq C_n + n - C_n - \sum_{j=n+1}^{\infty} 2^{-j} > n - 1$$

everywhere on $(0,1)$ outside the set

$$e_n \cup e'_n \cup \bigcup_{j=n+1}^{\infty} E_j$$

of measure less than $3 \cdot 2^{-n}$. Hence $\sup U(A) = +\infty$ almost everywhere on $(0,1)$ in view of the arbitrariness of n.

If $\sup A = +\infty$ on a set e_0 of positive measure, then we can change the set A by "truncating" its functions properly. Indeed, let f be a positive almost everywhere finite function not summable on e_0. Put

$$\widetilde{A} = \{\min(x, f) \mid x \in A\}.$$

It is clear that $\sup \widetilde{A} < +\infty$ almost everywhere and $\sup \widetilde{A} \notin L^p$. Let U be a positive operator such that $\sup U(\widetilde{A}) \equiv +\infty$. Since $U(x) \geq U(\min(x, f))$, we have

$$\sup U(A) \geq \sup U(\widetilde{A}) \equiv +\infty.$$

If $\sup A \in L^p$ then we may replace the set A with the set $-A$. Finally, we have to apply the obtained result to one of the sets $\{\operatorname{Re} x \mid x \in A\}$ and $\{\operatorname{Im} x \mid x \in A\}$ in the complex case.

Corollary. *If a set $A \subset L^p(0,1)$ is not invariantly order bounded then, given $\varepsilon > 0$, there exists an operator $U : L^p(0,1) \to L^p(0,1)$ such that $\sup U(A) = \infty$ almost everywhere on $(0,1)$ and $\|U - I\| < \varepsilon$.*

$\triangleleft$ The case $\sup |A| \notin L^p(0,1)$ was considered in the theorem. Let now $\sup |A| \in L^p(0,1)$ and let $V : L^p(0,1) \to L^p(0,1)$ be an operator such that $\sup |V(A)| \notin L^p(0,1)$ and $\|V\| = 1$. Put $V_0 = I - \varepsilon_0 V$, assuming $\varepsilon_0 < \varepsilon/3$ and $\varepsilon_0 < 1$. Then obviously V_0 is an isomorphism and $\sup |V_0(A)| \notin L^p(0,1)$ since $\sup |V_0(A)| \geq \varepsilon_0 \sup |V(A)| - \sup |A|$. Let now $U_0 : L^p(0,1) \to L^p(0,1)$ be the operator in the theorem constructed for the set $V_0(A)$ and let $\|I - U_0\| < \varepsilon_0$. Put $U = U_0 V_0$. It is clear that

$$\|I - U\| \leq \|I - U_0\| + \|U_0\| \cdot \|I - V_0\| \leq \varepsilon_0 + \varepsilon_0(1 + \varepsilon_0) < \varepsilon. \ \triangleright$$

3.4.12. Tandori's theorem. This and next subsections are dedicated to results connected with the famous Menshov-Rademacher theorem that reads as follows:

If

$$\sum_{n=1}^{\infty} |a_n \ln n|^2 < +\infty$$

then the series

$$\sum_{n=1}^{\infty} a_n \varphi_n$$

converges almost everywhere on T for each orthonormal system $\{\varphi_n\}_{n=1}^{\infty} \subset L^2(T, \mu)$.

It turns out that if we confine ourselves to decreasing sequences $\{a_n\}_{n=1}^{\infty}$, then the condition $\sum_{n=1}^{\infty} |a_n \ln n|^2 < \infty$ in the Menshov-Rademacher theorem will be not only sufficient but also necessary (if arbitrary orthonormal systems are considered). The following Tandori's theorem (see [1, 57]) demonstrates the claim.

Theorem. *If $a_n \downarrow 0$ and*

$$\sum_{n=1}^{\infty} (a_n \ln n)^2 = \infty$$

then there exists an orthonormal system $\{f_n\}_{n=1}^{\infty} \subset L^2(0,1)$ such that the series

$$\sum_{n=1}^{\infty} a_n f_n$$

converges almost everywhere on $(0,1)$.

$\lhd$ Let $\{e_k\}_{k=1}^{\infty}$ be a sequence of pairwise disjoint subsets of $(0,1)$ of positive measure. Put $\varphi_k = (\operatorname{mes}(e_k))^{-1/2}\chi_{e_k}$ and show that the set A of the partial sums of the series $\sum_{k=1}^{\infty} a_k\varphi_k$ is not invariantly order bounded.

Consider the operator H defined with the help of the Hilbert matrix on the closed subspace $\overline{\operatorname{lin}}(\{\varphi_k\}_{k=1}^{\infty})$ and vanishing on the complement of the subspace. In other words,

$$H\left(\sum_{k=1}^{\infty} x_k\varphi_k\right) = \sum_{j=1}^{\infty}\Big(\sum_{k\neq j}\frac{x_k}{j-k}\Big)\varphi_j = \sum_{k=1}^{\infty} x_k\psi_k,$$

where

$$\psi_k = \sum_{j\neq k}\frac{1}{j-k}\varphi_j, \quad \sum_{k=1}^{\infty}|x_k|^2 < \infty$$

and $H(f) = 0$ for $f \perp \operatorname{lin}(\{\varphi_k\}_{k=1}^{\infty})$. Put $S = \sup|H(A)|$, i.e.,

$$S = \sup_n \left|\sum_{k=1}^{n} a_k\psi_k\right|.$$

Estimate S from below. If $t \in e_m$ then

$$\psi_k(t) = \frac{1}{m-k}\varphi_m(t)$$

for $k \neq m$. Therefore,

$$S(t) \geq \left| \sum_{k=1}^{m-1} a_k \psi_k(t) \right| = \left(\sum_{k=1}^{m-1} \frac{a_k}{m-k} \right) \varphi_m(t)$$
$$\geq a_m \left(1 + \frac{1}{2} + \cdots + \frac{1}{m-1} \right) \varphi_m(t) \geq a_m \ln m \varphi_m(t)$$

for $t \in e_m$ and $m \geq 2$. Consequently,

$$\int_0^1 S^2(t)\, dt \geq \sum_{m=1}^{\infty} \int_{e_m} S^2(t)\, dt \geq \sum_{m=2}^{\infty} (a_m \ln m)^2 = +\infty.$$

By Corollary 3.4.10, there exists a unitary operator $U : L^2(0,1) \to L^2(0,1)$ such that

$$\sup_n \left| U \left(\sum_{k=1}^{n} a_k \varphi_k \right) \right| = +\infty$$

almost everywhere on $(0,1)$. Thus, the series

$$\sum_{k=1}^{\infty} a_k U \varphi_k$$

diverges to ∞ almost everywhere on $(0,1)$ and so $\{U\varphi_k\}_{k=1}^{\infty}$ is a sought orthonormal system. ▷

3.4.13. The following generalization of the Menshov-Rademacher theorem is well known (see [32, p. 120]).

If a series

$$\sum_{k=1}^{\infty} f_k$$

converges unconditionally in the space $L^p(T,\mu)$ then the series

$$\sum_{k=1}^{\infty} \frac{f_k}{\ln(k+1)}$$

converges almost everywhere on T.

This fact implies that if a sequence $\{\varphi_k\}_{k=1}^{\infty} \subset L^p(T,\mu)$ is equivalent to the standard basis for the space l^p (i.e., the series $\sum_{k=1}^{\infty} a_k\varphi_k$ converges in norm if and only if $\sum_{k=1}^{\infty} |a_k|^p < \infty$) and

$$\sum_{k=1}^{\infty} |a_k \ln k|^p < \infty,$$

then the series

$$\sum_{k=1}^{\infty} a_k\varphi_k$$

converges almost everywhere on T. In the case when the sequence $\{a_k\}_{k=1}^{\infty}$ is decreasing, the condition

$$\sum_{k=1}^{\infty} |a_k \ln k|^p < \infty$$

turns out to be not only sufficient but necessary too. More precisely, the following generalization of Tandori's theorem is valid:

Theorem. *Let $1 < p < \infty$. If $a_k \downarrow 0$ and*

$$\sum_{k=1}^{\infty} (a_k \ln k)^p = \infty$$

then there exists a sequence $\{f_k\}_{k=1}^{\infty} \subset L^p(0,1)$ equivalent to the standard basis for l^p such that the series

$$\sum_{k=1}^{\infty} a_k f_k$$

diverges almost everywhere on $(0,1)$.

$\lhd$ The Hilbert matrix defines a continuous operator in l^p for $1 < p < \infty$. Thus, to prove the theorem we have to repeat the proof of Tandori's theorem given in 3.4.12 while putting

$$\varphi_k = (\mathrm{mes}(e_k))^{-1/p}\xi_{e_k}$$

and appealing to Corollary 3.4.11 instead of 3.4.10.

3.5. Coincidence of Some Classes of Operators in the Scale of the Banach Spaces L^p

In this section we prove some important theorems on properties of p-absolutely summing operators in the scale of Lebesgue's spaces, in particular, the famous Grothendieck theorem that gives grounds for various applications and provides an impetus to developing the theory of absolutely summing operators.

3.5.1. Two lemmas. **(a)** *Let $(T, \mathfrak{A}, \mu)$ be a space with probability measure.*
(1) *If $1 < q < 2$ and $U \in \mathscr{L}(X, L^2(T, \mu))$ then*

$$\|U\|_{X \to L^q} \leq \|U\|_{X \to L^1}^{(2-q)/q} \|U\|_{X \to L^2}^{(2q-2)/q}.$$

(2) *If $2 < p < \infty$ and $V \in \mathscr{L}(L^2(T, \mu), Y)$ then*

$$\|V\|_{L^p \to Y} \leq \|V\|_{L^\infty \to Y}^{(p-2)/p} \|V\|_{L^2 \to Y}^{2/p}.$$

$\lhd$ First assertion can be easily obtained with the help of the Hölder inequality. By putting $q = p'$ and applying the first assertion to the operator V^*, we deduce the second assertion of the lemma. $\rhd$

(b) *Let $U \in \Pi_2(C(K), Y)$. Then $\pi_4(U) \leq \sqrt{\pi_2(U)\|U\|}$.*

$\lhd$ To prove, we consider the canonical factorization of U (see 3.1.10(b)):

$$U : C(K) \xrightarrow{i_2} L^2(K, \nu) \xrightarrow{V} Y,$$

where $\|V\|_{L^2 \to Y} = \pi_2(U)$. Then

$$\pi_4(U) \leq \|V\|_{L^4 \to Y}, \quad \|U\| = \|V\|_{L^\infty \to Y}$$

and our assertion follows from the lemma (a) for $p = 4$. $\rhd$

3.5.2. Maurey's theorem. Now we are in a position to prove the following important theorem.

Theorem. *If Y is a 2-cotype space then*

$$\mathscr{L}(C(K), Y) = \Pi_2(C(K), Y);$$

moreover, $\pi_2(U) \leq (C_2(Y)B_4)^2\|U\|$, where $U \in \mathscr{L}(C(K), Y)$ and $B_4 = \sqrt[4]{3}$ is the constant in the Khinchin inequality (see 3.S.1).

◁ Let $U \in \mathscr{L}(C(K), Y)$. First we suppose that $\operatorname{rank}(U) < \infty$. Then $U \in \Pi_2(C(K), Y)$ and $\pi_2(U) \le C_2(Y)B_4\pi_4(U)$ by Theorem 3.2.4. By making use of Lemma 3.5.1(b), we see that

$$\pi_2(U) \le C_2(Y)B_4\pi_4(U) \le C_2(Y)B_4\sqrt{\pi_2(U)\|U\|}$$

and consequently

$$\pi_2(U) \le (C_2(Y)B_4)^2\|U\|.$$

The general case is settled by Remark 3.1.1 since the operator U can be approximated pointwise by finite rank operators with norms not exceeding $\|U\|$. ▷

Observe that the proven theorem remains valid if we replace the space of continuous functions with either the space $L^\infty(T, \mu)$ or a complemented subspace of the space of continuous functions, for instance, the space c_0.

3.5.3. Corollaries.

(a) *Let $1 \le p \le 2$ and let $(T, \mathfrak{A}, \mu)$ be a measure space. Then*

$$\mathscr{L}(C(K), L^p(T, \mu)) = \Pi_2(C(K), L^p(T, \mu))$$

for a 2-cotype space $L^p(T, \mu)$.

(b) *If X is a 2-cotype space then $\Pi_2(X, Y) = \Pi_1(X, Y)$.*

◁ Let $U \in \Pi_2(X, Y)$. According to 3.1.6, to prove $U \in \Pi_1(X, Y)$ it suffices to verify that $UV \in \Pi_1(c_0, Y)$ for every operator $V \in \mathscr{L}(c_0, X)$. Since $V \in \Pi_2(c_0, X)$ by Maurey's theorem, the operators U, V admit the canonical factorization

$$U : X \xrightarrow{j} C(K) \xrightarrow{i_2} L^2(K, \nu) \xrightarrow{U_1} Y, \quad V : c_0 \xrightarrow{\tilde{j}} C(\widetilde{K}) \xrightarrow{\tilde{i}_2} L^2(\widetilde{K}, \tilde{\nu}) \xrightarrow{V_1} X.$$

The operator $i_2 j V_1$, presenting a Hilbert-Schmidt operator, is an 1-absolutely summing (see 3.2.3). Consequently, the operator UV is also 1-absolutely summing. ▷

The proven assertion can be strengthened in the following way:

(c) *If X is a 2-cotype space then $\Pi_2(X, Y) = \Pi_0(X, Y)$; i.e., $\Pi_2(X, Y) = \Pi_p(X, Y)$ for every p, $0 < p < 2$.*

The reader can find the proof of the fact for instance in [42].

(d) *If X and Y are a 2-cotype spaces then $\Pi_2(X, Y) = \Pi_p(X, Y)$ for every $p \in (0, +\infty)$.*

Indeed, we have $\Pi_2(X,Y) = \Pi_p(X,Y)$ for $2 < p < \infty$ in view of 3.2.2 and $\Pi_2(X,Y) = \Pi_p(X,Y)$ for $1 \leq p \leq 2$ by Corollary (b) and, for $0 < p < 1$, by the preceding corollary.

Note that the last result generalizes Corollary 3.2.4.

3.5.4. Grothendieck's theorem. *Let H be a Hilbert space. Then*

$$\mathscr{L}(L^1(T,\mu),H) = \Pi_1(L^1(T,\mu),H).$$

$\triangleleft$ Since $L^1(T,\mu)$ is a 2-cotype space, in accord with 3.5.3(b) it suffices to verify that

$$\mathscr{L}(L^1(T,\mu),H) = \Pi_2(L^1(T,\mu),H).$$

To this end we prove (see 3.1.6) that the product UV is a Hilbert-Schmidt operator for arbitrary operators $U \in \mathscr{L}(L^1(T,\mu),H)$ and $V \in \mathscr{L}(l^2, L^1(T,\mu))$. Indeed, we have $V^* \in \Pi_2(L^\infty(T,\mu),l^2)$ by 3.5.2. Therefore, $(UV)^* = V^*U^*$ and, consequently, UV are Hilbert-Schmidt operators. $\triangleright$

By Grothendieck's theorem there is a constant C such that $\pi_1(U) \leq C\|U\|$ for every operator $U \in \mathscr{L}(L^1(T,\mu),H)$. The least of such constants is called the *Grothendieck constant* and denoted by K_G. As is easily verified, the Grothendieck constant does not depend on the choice of the measure space (provided the corresponding space L^1 is infinite-dimensional), but it turns out miraculously that the constant depends on the scalar field over which the space are considered; so it would be more precise to write $K_G^{\mathbb{R}}$ and $K_G^{\mathbb{C}}$ rather than K_G for real and complex cases respectively. In the present time, the following estimates for $K_G^{\mathbb{R}}$ and $K_G^{\mathbb{C}}$ are known (see [42, 45]):

$$\frac{4}{\pi} \leq K_G^{\mathbb{C}} \leq e^{1-\gamma} < 1{,}527 < \frac{\pi}{2} \leq K_G^{\mathbb{R}} \leq \frac{\pi}{2\ln(1+\sqrt{2})} < 1{,}782,$$

here γ is the Euler constant.

REMARK. The qualitative result obtained in Corollaries 3.5.3(a) and (b) can be supplemented with some estimates. Prove that, for $1 \leq p \leq 2$, the estimate $\pi_2(U) \leq K_G\|U\|$ holds for every operator $U \in \mathscr{L}(C(K), L^p(T,\mu))$ and $\pi_1(U) \leq K_G\pi_2(U)$ holds for every $U \in \Pi_2(L^p(T,\mu),Y)$.

$\lhd$ Let $V \in \mathscr{L}(l^2, C(K))$. Since the space $C^*(K)$ is isometric to the space $L^1(T_0, \mu_0)$ with a suitable choice of the measure space, we have $\pi_2(V^*) \le K_G\|V\|$. Therefore (see 3.4.1(b)),

$$\|UV\|_M \le \pi_1((UV)^*) \le K_G\|U\| \cdot \|V\|.$$

Consequently,

$$\pi_2(UV) \le \pi_p(UV) \le \|UV\|_M \le K_G\|U\| \cdot \|V\|$$

by 3.1.5(c). Now the sought estimate follows from 3.1.6.

If $U \in \Pi_2(L^p(T,\mu), Y)$ and $V \in \mathscr{L}(c_0, L^p(T,\mu))$ then

$$\pi_1(UV) \le \nu(UV) \le \pi_2(U)\pi_2(V) \le K_G\pi_2(U)\|V\|$$

by the multiplication theorem. The required result follows again from 3.1.6. $\rhd$

Recall that if U is a Hilbert-Schmidt operator, then a slightly better estimate is valid: $\pi_1(U) \le \sqrt{2}\pi_2(U)$ (see 2.3.2, the corollary).

3.5.5. *L. Schwartz's duality theorem* [51]. *Let H be a Hilbert space and let $U \in \mathscr{L}(X, H)$. If $U^* \in \Pi_q(H, X^*)$ for some $q < \infty$ then $U \in \Pi_0(X, H)$.*

$\lhd$ To prove $U \in \Pi_0(X, H)$ it suffices to verify that $U|_{X_0} \in \Pi_0(X_0, H)$, where $U|_{X_0}$ is the restriction of U to a separable subspace $X_0 \subset X$. So we may suppose that the space X is separable and $H = l^2$. Let ω_r, $0 < r < \infty$, be a family of isomorphic embeddings of l^2 into $L^r(0,1)$ satisfying the condition

$$\omega_r = j_{s,r}\omega_s \text{ for } 0 < r < s < \infty, \tag{1}$$

where $j_{s,r}$ is the identity embedding of $L^s(0,1)$ into $L^r(0,1)$. We shall construct such a family later, but now we apply it to completing the theorem.

Fix $r \in (0, q)$ and prove that $U \in \Pi_r(X, l^2)$. By virtue of condition (1), the operator $\omega_r U$ admits the factorization

$$\omega_r U : X \xrightarrow{U} l^2 \xrightarrow{\omega_q} L^q(0,1) \xrightarrow{j_{q,r}} L^r(0,1).$$

Clearly, the operator $\omega_q U$ is dominated since

$$(\omega_q U)^* = U^*\omega_q{}^* \in \Pi_q(L^{q'}(0,1), X^*)$$

(see 3.4.2(b)). Therefore, the operator $\omega_r U$ is dominated too and consequently it is r-absolutely summing (see 3.1.5)). Since ω_r is isomorphic embedding, the operator U is r-absolutely summing together with $\omega_r U$.

To complete the prove, it remains to construct the family $\{\omega_r\}_{0<r<\infty}$. Let $\{g_k\}_{k=1}^{\infty}$ be a sequence of standard independent Gaussian variables defined on $(0,1)$. Given $r \in (0,+\infty)$, define ω_r by the equality

$$\omega_r(\xi) = \sum_{k=1}^{\infty} \xi_k g_k \quad (\xi = \{\xi_k\}_{k=1}^{\infty} \in l^2).$$

Observe that $\omega_r(\xi)$ is a Gaussian random variable with mean zero and variance $\|\xi\|^2$. Therefore,

$$\|\omega_r(\xi)\|_r^r = \int_0^1 \left|\sum_{k=1}^{\infty} \xi_k g_k(\tau)\right|^r d\tau = \int_{-\infty}^{+\infty} |u|^r \, d\Phi\left(\frac{u}{\|\xi\|}\right),$$

where Φ is the distribution of a standard Gaussian random variable. Consequently,

$$\|\omega_r(\xi)\|_r = \|\xi\| \left(\int_{-\infty}^{+\infty} |t|^r \, d\Phi(t)\right)^{1/r} = C_r \|\xi\|,$$

where

$$C_r = \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} |t|^r e^{-t^2/2} \, dt\right)^{1/r}.$$

Thus, the operator ω_r is an isometric embedding of l^2 into $L^r(0,1)$ to within the multiplier C_r. Condition (1) holds obviously. $\triangleright$

3.5.6. To generalize Schwartz's duality theorem, we need the notion of a random variable distributed by the p-stable law (see, for instance, [24, § 5.7]). Recall that the distribution function F of a random variable is called *p-stable* for some p, $0 < p \leq 2$, if a linear combination $af + bg$ of two independent random variables f and g distributed by the law $F(x)$ will be distributed by the law $F(x/c)$, where $c = (|a|^p + |b|^p)^{1/p}$. We shall consider only symmetric random variables distributed by the p-stable law F. In this case, the characteristic function

$$\widehat{F}(s) = \int_{-\infty}^{+\infty} e^{ist} \, dF(t) \quad (s \in \mathbb{R})$$

of the law F has the form $\widehat{F}(s) = e^{-c|s|^p}$. The constant c can be replaced with 1 by the suitable change of variables. Henceforth, arguing about random variables distributed by the p-stable law, we shall mean only symmetric variables and suppose that $c = 1$. The distribution law of such variables will be denoted by F_p. It is well known (see [24, § 5.8, 5.9]) that

$$F_p \in C^1(\mathbb{R}), \quad F_p'(u) \underset{u\to\infty}{\sim} C_p|u|^{-1-p}$$

for $0 < p < 2$, where

$$C_p = \frac{\sin(\pi p/2)}{\pi}\Gamma(1+p).$$

Particularly,

$$\int_{-\infty}^{+\infty} |u|^r \, dF_p(u) < \infty$$

for $0 < r < p$.

Lemma. *Let $0 < r < p \le 2$ and let $\{h_n\}_{n=1}^\infty$ be a sequence of symmetric independent random variables on $(0,1)$ distributed by the p-stable law F_p. Then the equality*

$$\omega_r(\xi) = \sum_{k=1}^\infty \xi_k h_k, \text{ where } \xi = \{\xi_k\}_{k=1}^\infty \in l^p, \tag{2}$$

determines an isomorphic embedding ω_r of the space l^p into the space $L^r(0,1)$; moreover,

$$\|\omega(\xi)\|_r = C_{p,r}\|\xi\|_p \quad (\xi \in l^p), \tag{3}$$

where

$$C_{p,r} = \left(\int_{-\infty}^{+\infty} |u|^r \, dF_p(u)\right)^{1/r},$$

and

$$\omega_r = j_{s,r}\omega_s \text{ for } 0 < r < s < p, \tag{4}$$

where $j_{s,r}$ is the identity embedding of $L^s(0,1)$ into $L^r(0,1)$.

$\lhd$ By the definition of p-stable law, the distribution function of the random variable $\sum_{k=1}^{n} \xi_k h_k$ is $F_p(x/\sigma_n)$, where

$$\sigma_n = \left(\sum_{k=1}^{n} |\xi_k|^p \right)^{1/p}.$$

Therefore,

$$\left\| \sum_{k=1}^{n} \xi_k h_k \right\|_r = \left(\int_{-\infty}^{+\infty} |u|^r \, dF_p \left(\frac{u}{\sigma_n} \right) \right)^{1/r} = C_{p,r}\sigma_n.$$

This implies the convergence of series (2) in the space $L^r(0,1)$ and equality (3). Equality (4) is obvious. $\rhd$

3.5.7. *S. Kwapien's duality theorem* [21]. *Let $1 \le p < 2$ and $U \in \mathscr{L}(X, L^p(T,\mu))$. If $U^* \in \Pi_q(L^{p'}(T,\mu), X^*)$ for some $q < p$ then the containment $U \in \Pi_0(X, L^p(T,\mu))$ holds.*

$\lhd$ First of all we note that, as in Theorem 3.5.5, we can assume the space X to be separable and consequently the measure μ to be σ-finite. By Remark 3.1.1 we may consider the operators $P_\tau U$ rather than operator U, where P_τ are the projections ("conditional expectations") generated by at most countable partitions τ of the set T. This fact allows us to reduce the proof of the theorem to the case $L^p(T,\mu) = l^p$. The further reasoning proceeds on use made of Lemma 3.5.6 by the same scheme as the proof of Theorem 3.5.5. $\rhd$

3.5.8. *Corollary.* *Let $0 < p < q < s < 2$ and let $q \ge 1$. Then*

$$\Pi_q(L^{s'}(T,\mu), L^q(T_1,\mu_1)) = \Pi_p(L^{s'}(T,\mu), L^q(T_1,\mu_1)).$$

$\lhd$ Let $U \in \Pi_q(L^{s'}(T,\mu), L^q(T_1,\mu_1))$ and $V = U^*$. Applying Theorem 3.5.7 to the operator V $\quad (V^* = U \in \Pi_q(L^{s'}(T,\mu), L^q(T_1,\mu_1))$ and $q < s)$ we infer that $V = U^* \in \Pi_p(L^{q'}(T_1,\mu_1), L^s(T,\mu))$. Applying the same theorem to the operator U, we obtain the claim. $\rhd$

3.6. Nikishin-Maurey Factorization Theorems

In this section we consider the question concerning some special factorization for operators with values in the space $L^p(T,\mu)$ which expresses the fact that sometimes a mapping from a Banach space into $L^p(T,\mu)$ is "in the main" a mapping in the space $L^q(T,\mu)$ for some $q > p$.

3.6.1. DEFINITION. Let $U \in \mathscr{L}(X, L^p(T,\mu))$ with $0 < p < q \leq \infty$, $1/p = 1/q + 1/r$. We say that the operator U admits *strong factorization through the space* $L^q(T,\mu)$ if there exists a function $g \in L^r(T,\mu)$ such that $U = M_g V$, where $V \in \mathscr{L}(X, L^q(T,\mu))$, and $M_g : L^q(T,\mu) \to L^p(T,\mu)$ is the operator of multiplication by the function $g : M_g f = gf$ $(f \in L^q(T,\mu))$.

REMARK. Each dominated operator $U \in \mathscr{L}(X, L^p(T,\mu))$ admits strong factorization through $L^\infty(T,\mu)$. Indeed, let

$$g := \sup\{|Ux| \mid x \in X,\ \|x\| \leq 1\}.$$

We can obtain a factorization by defining the operator V as

$$Vx := \frac{Ux}{g} \quad (x \in X)$$

(we put $\frac{0}{0} := 0$).

The converse holds obviously: if an operator $U : X \to L^p(T,\mu)$ admits strong factorization through $L^\infty(T,\mu)$ then it is dominated.

3.6.2. Our next aim is to give necessary and sufficient conditions for strong factorization. Before achieving it in the next subsection, we are to establish the following important auxiliary fact.

Lemma. *Let X be a set; let $(T, \mathfrak{A}, \mu)$ be a measure space, and let A be the set of all finitely supported scalar families $\alpha = (\alpha_x)_{x\in X}$. Let now $0 < p < q < \infty$, $1/p = 1/q + 1/r$, and let $(f_x)_{x\in X}$ be a function family in $L^p(T,\mu)$. If*

$$\left(\int_T \Big(\sum_{x\in X} |\alpha_x f_x|^q\Big)^{p/q} d\mu\right)^{1/p} \leq \Big(\sum_{x\in X} |\alpha_x|^q\Big)^{1/q} \tag{1}$$

for every family $(\alpha_x)_{x\in X}$ in A then there exists a function $g \in L^r(T,\mu)$ such that

$$\|g\|_r \leq 1, \quad \int_T \left|\frac{f_x}{g}\right|^q d\mu \leq 1$$

for all $x \in X$.

$\lhd$ To prove, we use the Fan Ky lemma (see 3.S.5). Let $s = q/p$ and let K be a set of nonnegative functions in the unit ball of the space $L^{s'}(T,\mu)$ endowed with the topology $\sigma(L^{s'}(T,\mu), L^s(T,\mu))$. It is clear that K is a compact convex set. Consider the set Φ consisting of all functions φ_α that are defined on K and have the form

$$\varphi_\alpha(h) = \int\limits_T \left(\frac{\omega_\alpha}{h}\right)^s d\mu - \sum_{x\in X} |\alpha_x|^q,$$

where

$$\alpha = (\alpha_x)_{x\in X}, \quad \omega_\alpha = \Big(\sum_{x\in X} |\alpha_x f_x|^q\Big)^{p/q}, \quad h \in K.$$

We verify below that Φ satisfies the conditions of the Fan Ky lemma. Right now we finish the proof on the assumption that the above-indicated fact is true. By the Fan Ky lemma, there exists a function $h_0 \in K$ such that $\varphi_\alpha(h_0) \le 0$ for every $\alpha \in A$. Fix a point $x_0 \in X$ and put $\alpha^0 = (\alpha^0_x)_{x\in X}$, where $\alpha^0_{x_0} = 1$ and $\alpha^0_x = 0$ for $x \ne x_0$. Then $\omega_{\alpha_0} = |f_{x_0}|^p$ and consequently

$$\varphi_{\alpha^0}(h_0) = \int\limits_T \frac{|f_{x_0}|^q}{h_0^s}\, d\mu - 1 \le 0.$$

Put $g = h_0^{1/p}$. We have

$$\int\limits_T g^r\, d\mu = \int\limits_T h_0^{r/p}\, d\mu = \int\limits_T h_0^{s'}\, d\mu \le 1$$

and

$$\int\limits_T \left|\frac{f_{x_0}}{g}\right|^q d\mu = \int\limits_T \frac{|f_{x_0}|^q}{h_0^s}\, d\mu \le 1.$$

In view of the arbitrariness of x_0, this g is a sought function.

Now we verify that the set Φ satisfies the conditions of the Fan Ky lemma. Convexity of Φ is obvious. Convexity of the functions φ_α follows from convexity of the function $u \mapsto u^{-s}$ $(0 < u < \infty)$. Prove that the functions φ_α are lower semicontinuous; i.e., that the sets

$$K(\varphi_\alpha \le C) = \{h \in K \mid \varphi_\alpha(h) \le C\}$$

are closed for every $C \in \mathbb{R}$. Observe that the set $K(\varphi_\alpha \le C)$ is convex in view of convexity of φ_α. Therefore, it suffices to prove the closure of the set in the norm topology rather than in the topology $\sigma(L^{s'}(T,\mu), L^s(T,\mu))$. Since norm convergence implies convergence in measure, the closure of $K(\varphi_\alpha \le C)$ follows from Fatou's theorem. Last, verify that

$$\min\{\varphi_\alpha(h) \mid h \in K\} \le 0.$$

Put $h_\alpha = c_0 \omega_\alpha^{1/s'}$, where the constant c_0 is chosen so that

$$\int_T h_\alpha^{s'}\, d\mu = 1,$$

i.e.,

$$c_0^{s'} = \left(\int_T \omega_\alpha\, d\mu\right)^{-1}.$$

Then

$$\begin{aligned}\varphi_\alpha(h_\alpha) &= \int_T \left(\frac{\omega_\alpha}{h_\alpha}\right)^s d\mu - \sum_{x\in X} |\alpha_x|^q \\ &= c_0^{-s}\int_T \omega_\alpha\, d\mu - \sum_{x\in X} |\alpha_x|^q = \left(\int_T \omega_\alpha\, d\mu\right)^{q/p} - \sum_{x\in X} |\alpha_x|^q.\end{aligned}$$

By virtue of condition (1) we obtain $\varphi_\alpha(h_\alpha) \le 0$. $\triangleright$

3.6.3. *Theorem.* *Let $0 < p < q < \infty$, $1/p = 1/q + 1/r$; let $C_0 > 0$; let $(T, \mathfrak{A}, \mu)$ be a measure space, and let $U \in \mathscr{L}(X, L^p(T,\mu))$. The following assertions are equivalent:*

(1) *the operator U admits strong factorization through $L^q(T,\mu)$; that is, $U = M_g V$; moreover, $\|g\|_r \le 1$ and $\|V\| \le C_0$;*

(2) *for arbitrary vectors $x_1, \dots, x_n \in X$, the following inequality holds:*

$$\left(\int_T \left(\sum_{k=1}^n |Ux_k|^q\right)^{p/q} d\mu\right)^{1/p} \le C_0 \left(\sum_{k=1}^n \|x_k\|^q\right)^{1/q}.$$

◁ The implication (1) ⇒ (2) follows from the Hölder inequality ($s = q/p$):

$$\left(\int_T \left(\sum_{k=1}^n |Ux_k|^q\right)^{p/q} d\mu\right)^{1/p} = \left(\int_T |g|^p \left(\sum_{k=1}^n |Vx_k|^q\right)^{p/q} d\mu\right)^{1/p}$$

$$\leq \left(\int_T |g|^{ps'} d\mu\right)^{1/s'} \left(\int_T \sum_{k=1}^n |Vx_k|^q \, d\mu\right)^{1/q}$$

$$= \|g\|_r^p \left(\sum_{k=1}^n \|Vx_k\|^q\right)^{1/q} \leq \left(\sum_{k=1}^n \|V\|^q \|x_k\|^q\right)^{1/q}$$

$$\leq C_0 \left(\sum_{k=1}^n \|x_k\|^q\right)^{1/q}.$$

The implication (2) ⇒ (1) can be proven with the help of Lemma 3.6.2. Put

$$f_x = C_0^{-1}\|x\|^{-1}Ux \quad (x \in X, x \neq 0).$$

Then we have

$$\left(\int_T \left(\sum_{k=1}^n |\alpha_k f_{x_k}|^q\right)^{p/q} d\mu\right)^{1/p} = C_0^{-1} \left(\int_T \left(\sum_{k=1}^n \left|U\left(\frac{\alpha_k x_k}{\|x_k\|}\right)\right|^q\right)^{p/q} d\mu\right)^{1/p}$$

$$\leq C_0^{-1} C_0 \left(\sum_{k=1}^n \left\|\frac{\alpha_k x_k}{\|x_k\|}\right\|^q\right)^{1/q}$$

$$= \left(\sum_{k=1}^n |\alpha_k|^q\right)^{1/q}$$

for numbers $\alpha_1, \ldots, \alpha_n$ and vectors $x_1, \ldots, x_n \in X$ ($x_k \neq 0$). By Lemma 3.6.2, there exists a function $g \in L^r(T, \mu)$ such that

$$\int_T \left|\frac{C_0^{-1}Ux}{g\|x\|}\right|^q d\mu \leq 1,$$

i.e.,

$$\left(\int_T \left|\frac{Ux}{g}\right|^q d\mu\right)^{1/q} \leq C_0\|x\|$$

for every $x \in X$. By putting $Vx = \frac{Ux}{g}$, we obtain a sought factorization. ▷

REMARK. If an operator U is the pointwise limit of some family of operators each of which satisfies condition (2), i.e., admits strong factorization with a uniform estimate; then operator U satisfies condition (2) and consequently admits strong factorization too.

3.6.4. Theorem. *Let $1 \le p < q \le \infty$, $1/p = 1/q + 1/r$, $C > 0$, and let X be a Banach space. The following assertions are equivalent:*

(1) *given a Banach space Y, the equality $\Pi_p(X,Y) = \Pi_q(X,Y)$ holds; moreover, $\pi_p(U) \le C\pi_q(U)$ for every operator $U \in \Pi_q(X,Y)$;*

(2) *$\Pi_p(X,l^q) = \Pi_q(X,l^q)$; moreover, $\pi_p(U) \le C\pi_q(U)$ for every operator $U \in \Pi_q(X,l^q)$;*

(3) *for arbitrary operators $U \in \mathscr{L}(X^*, L^p(T,\mu))$ and $V \in \Pi_q(X,l^q)$ the estimate $\|UV^*\|_M \le C\pi_q(V)\|U\|$ holds;*

(4) *every operator $U \in \mathscr{L}(X^*, L^p(T,\mu))$ admits strong factorization through $L^q(T,\mu)$, that is, $U = MgV$; moreover, $\|g\|_r \le 1$ and $\|V\| \le C\|U\|$;*

(5) *for a finite collection of vectors $x_1, \dots, x_n \in X$, there exist collections of scalars $a_1, \dots, a_n$ and of vectors $\tilde{x}_1, \dots, \tilde{x}_n \in X$ such that $x_k = a_k\tilde{x}_k$ for $k = 1, 2, \dots, n$ and*

$$\sum_{k=1}^{n} |a_k|^r = 1,$$

$$\sup\left\{\left(\sum_{k=1}^{n} |\langle \tilde{x}_k, x'\rangle|^q\right)^{1/q} \mid x' \in X^*,\ \|x'\| \le 1\right\}$$
$$\le C\sup\left\{\left(\sum_{k=1}^{n} |\langle x_k, x'\rangle|^p\right)^{1/p} \mid x' \in X^*,\ \|x'\| \le 1\right\}.$$

The implication (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (3): Put $W = UV^*$. Then

$$\pi_p(W^*) = \pi_p(V^{**}U^*) \le \pi_p(V^{**})\|U^*\| = \pi_p(V)\|U\| \le C\pi_q(V)\|U\|.$$

By Theorem 3.4.2(b), the operator W is dominated; moreover,

$$\|W\|_M \le \pi_p(W^*) \le C\pi_q(V)\|U\|.$$

(3) $\Rightarrow$ (4): Let $\{x'_k\}_{k=1}^n \subset X^*$. Assuming $x'_k = 0$ for $k > n$, define the operator $V : X \to l^q$ by the equality

$$Vx = \{\langle x, x'_k\rangle\}_{k=1}^{\infty} \quad (x \in X).$$

It is clear that $V^* e_k = x'_k$, where e_k are the canonical basis vectors for $l^{q'}$. Since

$$\sup\{|Vx| \mid \|x\| \leq 1\} = \{\|x'_k\|\}_{k=1}^n,$$

we have

$$\|V\|_M = \left(\sum_{k=1}^n \|x'_k\|^q\right)^{1/q}$$

and consequently (see 3.1.5(c))

$$\pi_q(V) \leq \|V\|_M = \left(\sum_{k=1}^n \|x'_k\|^q\right)^{1/q}.$$

Now we find

$$S = \sup\{|UV^*\xi| \mid \xi \in l^{q'},\ \|\xi\| \leq 1\}.$$

Since

$$|UV^*\xi| = \left|\sum_{k=1}^n \xi_k UV^* e_k\right|,$$

where $\xi = \{\xi_k\}_{k=1}^n \in l^{q'}$, we have

$$S = \left(\sum_{k=1}^n |Ux'_k|^q\right)^{1/q}.$$

Consequently,

$$\left(\int_T \left(\sum_{k=1}^n |Ux'_k|^q\right)^{p/q} d\mu\right)^{1/p} = \|UV^*\|_M \leq C\|U\|\pi_q(V)$$

$$\leq C\|U\| \left(\sum_{k=1}^n \|x'_k\|^q\right)^{1/q}.$$

It remains only to refer to Theorem 3.6.3.

(4) $\Rightarrow$ (5): To prove this implication one ought to apply (4) to the operator $U : X^* \to l^p_n$ defined as

$$Ux = \{\langle x_k, x'\rangle\}_{k=1}^n \quad (x \in X).$$

(5) $\Rightarrow$ (1): Let $U \in \Pi_q(X,Y)$, $x_1, \dots, x_n \in X$. Representing $\{x_k\}_{k=1}^n$ in the manner indicated in (5), we obtain ($s = q/p$):

$$\begin{aligned}\left(\sum_{k=1}^n \|Ux_k\|^p\right)^{1/p} &= \left(\sum_{k=1}^n |\alpha_k|^p \|U\tilde{x}_k\|^p\right)^{1/p} \\ &\leq \left(\sum_{k=1}^n |\alpha_k|^{ps'}\right)^{1/ps'} \left(\sum_{k=1}^n \|U\tilde{x}_k\|^q\right)^{1/q} \\ &\leq \left(\sum_{k=1}^n |\alpha_k|^r\right)^{1/r} \pi_q(U) \sup\left\{\left(\sum_{k=1}^n |\langle\tilde{x}_k, x'\rangle|^q\right)^{1/q} \mid \|x'\| \leq 1\right\} \\ &\leq C\pi_q(U) \sup\left\{\left(\sum_{k=1}^n |\langle\tilde{x}_k, x'\rangle|^p\right)^{1/p} \mid \|x'\| \leq 1\right\} \ \triangleright .\end{aligned}$$

REMARK. For $0 < p < 1 \leq q$, Theorem 3.6.4 remains true together with the proof of all implications except (2) $\Rightarrow$ (3) provided that in items (3) and (4) one only considers finite rank operators U. For proving the implication (2) $\Rightarrow$ (3), we ought to refer to 3.4.2(c).

3.6.5. Strong factorization in the scale of Lebesgue spaces.

(a) *Theorem.* *Let $1 \leq p < q \leq s \leq \infty$ and $1/p = 1/q + 1/r$. If either $q \leq 2 \leq s$ or $q < s < 2$ then every operator $U \in \mathscr{L}(L^s(T', \mu'), L^p(T, \mu))$ is factorizable through $L^q(T, \mu)$; moreover, for $q \leq 2 \leq s$, there is a factorization $U = M_g V$ satisfying the estimate $\|g\|_r \leq 1$, $\|V\| \leq K_G\|U\|$, where K_G is the Grothendieck constant.*

$\triangleleft$ By Theorem 3.6.4, it suffices to prove that the equality

$$\Pi_p(L^{s'}(T', \mu'), l^q) = \Pi_q(L^{s'}(T', \mu'), l^q)$$

holds for indicated p, q, and s. For $q \leq 2 \leq s$, the equality and the estimate $\pi_1(U) \leq K_G \pi_2(U)$ for operators $U \in \Pi_2(L^{s'}(T', \mu'), l^q)$ were obtained in 3.5.3(b) and Remark 3.5.4. For $q < s < 2$, the equality was established in 3.5.8. $\triangleright$

(b) Corollary. *If $1 \le p < 2 \le s \le \infty$ and $1/p = 1/2 + 1/r$ then every operator $U \in \mathscr{L}(L^s(T', \mu'), L^p(T, \mu))$ is strongly factorizable through $L^2(T, \mu)$; that is, $U = MgV$ with the estimate*

$$\|g\|_r \le 1, \quad \|V\| \le K_G \|U\|.$$

(c) REMARK. In the case $1 \le p < 2 = s$, the estimate obtained in (b) can be improved. Indeed, the estimate $\pi_p(U) \le \pi_1(U) \le \sqrt{2}\pi_2(U)$ holds for each operator $U \in \Pi_2(L^2(T', \mu'), l^2)$ (see Corollary 3.2.3). By using the equivalence of items (2) and (4) in Theorem 3.6.4, we infer that the operator U admits strong factorization $U = M_g V$ with the estimate $\|g\|_r \le 1$, $\|V\| \le \sqrt{2}\|U\|$.

3.6.6. Strong factorization of regular operators.

Theorem. *Let $1 \le p < q \le \infty$, let $(T, \mathfrak{A}, \mu)$ and $(S, \mathfrak{B}, \nu)$ be arbitrary measure spaces, and let $U \in L^{\sim}(L^q(S, \nu), L^p(T, \mu))$. Then the operator U admits strong factorization through the space $L^q(T, \mu)$.*

$\lhd$ Verify condition (2) of Theorem 3.6.3. Take $x_1, \dots, x_n \in L^q(S, \nu)$. Then

$$\begin{aligned}
\left(\sum_{k=1}^n |Ux_k|^q\right)^{1/q} &= \sup\left\{\left|\sum_{k=1}^n a_k U x_k\right| \mid \sum_{k=1}^n |a_k|^{q'} \le 1\right\} \\
&\le \sup\left\{|U|\left(\left|\sum_{k=1}^n a_k x_k\right|\right) \mid \sum_{k=1}^n |a_k|^{q'} \le 1\right\} \\
&\le |U|\left(\left(\sum_{k=1}^n |x_k|^q\right)^{1/q}\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\left(\int_T \left(\sum_{k=1}^n |Ux_k|^q\right)^{p/q} d\mu\right)^{1/p} = \left\|\left(\sum_{k=1}^n |Ux_k|^q\right)^{1/q}\right\|_p \\
&\le \left\| |U| \left(\left(\sum_{k=1}^n |x_k|^q\right)^{1/q}\right)\right\|_p \le \| \, |U| \, \| \cdot \left\|\left(\sum_{k=1}^n |x_k|^q\right)^{1/q}\right\|_q \\
&\qquad = \| \, |U| \, \| \left(\sum_{k=1}^n \|x_k\|_q^q\right)^{1/q}. \rhd
\end{aligned}$$

3.6.7. Strong factorization of regular operators (continuation). We say that an operator $U : X \to L^0(T, \mu)$ *admits strong factorization through* $L^q(T, \mu)$ if there exist a function $g \in L^0(T, \mu)$ and an operator $V : X \to L^q(T, \mu)$ such that $U = M_g V$, where $M_g : L^q(T, \mu) \to L^0(T, \mu)$ is the operator of multiplication by the function g.

Lemma. *Let $(T, \mathfrak{A}, \mu)$ be a σ-finite measure space, let E be a Banach space, and let $U \in \mathscr{L}^{\sim}(E, L^0(T, \mu))$. Then the operator U admits strong factorization through $L^1(T, \mu)$, that is $U = M_g V$, where $V : E \to L^1(T, \mu)$ is a regular operator.*

$\lhd$ Set

$$B = \{y \in L^0(T, \mu) \mid \exists x \in E : \|x\| \leq 1, \ |y| \leq |U|(|x|)\}.$$

It is easy to verify that the set B is absolutely convex, closed, and bounded in $L^0(T, \mu)$. Put $Z = \text{lin}(B)$. If we assume B to be the unit ball, then Z becomes an ideal Banach space; moreover, the identity embedding $i : Z \to L^0(T, \mu)$ is continuous. Thus, the operator U admits the factorization

$$U : E \xrightarrow{U_0} Z \xrightarrow{i} L^0(T, \mu),$$

where U_0 is the operator defined by the equality $U_0 x = Ux$ $(x \in E)$. It is clear that the operator U_0 is regular and $\|U_0\| \leq 1$. Let g be a function from the dual space to Z which is strictly positive on T (see 2.2.21). Then the operator i is factorizable as follows:

$$i : Z \xrightarrow{M_g} L^1(T, \mu) \xrightarrow{M_h} L^0(T, \mu),$$

where M_g and M_h are the operators of multiplication by the functions g and $h = 1/g$ respectively. So, $U = M_h V$, where $V = M_g U_0$ is a sought factorization. $\rhd$

Theorem. *Let $(T, \mathfrak{A}, \mu)$ be a σ-finite measure space, $1 < q \leq \infty$, $U \in \mathscr{L}^{\sim}(L^q(S, \nu), L^0(T, \mu))$. Then the operator U admits strong factorization through the space $L^q(T, \mu)$.*

$\lhd$ To prove the theorem it suffices to successively apply the lemma and Theorem 3.6.6 to the operator U.

3.7. Stably Regular Operators

3.7.1. DEFINITION. Let $p, q \in [1, +\infty]$. An $U \in \mathscr{L}(L^p(T,\mu), L^q(T_1,\mu_1))$ is called ***right*** (***left***) ***stably regular*** if the operator UV (WU) is regular for every operator $V \in \mathscr{L}(L^p(T,\mu), L^p(T,\mu))$ (every operator $W \in \mathscr{L}(L^q(T_1,\mu_1), L^q(T_1,\mu_1))$ If the operator WUV is regular for all operators V and W in the above-indicated spaces then U is called (***bilaterally***) ***stably regular***.

We denote by $\mathscr{L}^{\sim}_{\mathrm{st}}(L^p(T,\mu), L^q(T_1,\mu_1))$ the set of all bilaterally stably regular operators in $\mathscr{L}(L^p(T,\mu), L^q(T_1,\mu_1))$.

Observe that the right stable regularity does not generally yields the left stable regularity and vice versa (see example 4.9.9).

3.7.2. Properties of stably regular operators. While speaking of L^p, L^q, etc. in the current and next subsections, we mean spaces that are constructed possibly over different measure spaces.

(a) *An operator $U \in \mathscr{L}(L^p, L^q)$ is right (left) stably regular if and only if the operator U^* is left (right) stably regular. An operator U^* is bilaterally stably regular together with U.*

We leave the reader with proving.

(b) *Let $p, r \in [1, +\infty]$, $q \in [1, \infty]$, $\max(r, q) \geq 2$, and $U_0 \in \mathscr{L}(L^q, L^r)$. If an operator $U : L^p \to L^q$ is left stably regular then the operator $U_0 U$ is left stably regular too.*

$\lhd$ Since U_0 is arbitrary, it suffices to verify the regularity of $U_0 U$. To this end, we prove that the image of every interval in L^p under the mapping $U_0 U$ is an order bounded set in L^r. Let $g \in L^p$, $g \geq 0$, and $I_g = \{x \in L^p \mid |x| \leq g\}$ and let $M_g : L^\infty \to L^p$ be the operator of multiplication by g. Since the operator U is left stably regular, the operator UM_g is stably dominated; thus, $(UM_g)^* \in \Pi_q(L^{q'}, (L^\infty)^*)$ by 3.4.4. Further, we have

$$(U_0 U M_g)^* = (UM_g)^* U_0^* \in \Pi_q((L^r)^*, (L^\infty)^*).$$

The condition $\max(r, q) \geq 2$ implies that at least one of the spaces $L^{q'}$ or $(L^r)^*$ has cotype 2 (see 3.S.2). Since $(L^\infty)^*$ is a 2-cotype space as well, by making use of 3.5.3(d) we see that

$$(U_0 U M_g)^* \in \Pi_1((L^r)^*, (L^\infty)^*).$$

Consequently, the operator U_0UM_g is dominated (see 3.4.1(b)) and thus the set $U_0U(I_g)$ is order bounded. ▷

(c) *Let $p \in (1, +\infty]$, $q, s \in [1, +\infty]$, $\min(p, q) \leq 2$, and $U_0 \in \mathscr{L}(L^s, L^p)$. If an operator $U : L^p \to L^q$ is right stably regular then the operator UU_0 is right stably regular too.*

◁ This property can be obtained from property (b) by duality. ▷

Observe that the conditions $q < \infty$ in (b) and $p > 1$ in (c) are essential.

3.7.3. Lemmas. (a) *If $U : C(K) \to L^2(T, \mu)$ is a stably dominated operator then $U \in N(C(K), L^2(T, \mu))$.*

◁ Let $A = U(B)$, where B stands for the unit ball in $C(K)$. By Corollary 3.4.7, there exists a Hilbert-Schmidt operator $V : H \to L^2(T, \mu)$ such that $A \subset V(B_H)$ where H is a Hilbert space and B_H is the unit ball in H. Without loss of generality, we may assume that the operator V is one-to-one. Let V^{-1} be the operator on $V(H)$ inverse to V. Since $V(H) \supset U(C(K))$, the composition $W = V^{-1}U$ is defined. As is easily verified, the operator W is closed and thus continuous. It is also clear that $U = VW$. By Theorem 3.5.2, we have $W \in \Pi_2(C(K), H)$ and for completing the proof it suffices to appeal to the multiplication theorem 3.3.6. ▷

(b) *Let H be a Hilbert space and let $W \in \mathscr{L}(c_0, H)$. If the operator WR is nuclear for each operator $R : H \to c_0$ then the operator W is nuclear too.*

◁ First of all we note that since the mapping $R \to WR$ from $\mathscr{L}(H, c_0)$ into $N(H, H)$ is closed, it is continuous and there exists a number C such that

$$\nu(WR) \leq C\|R\| \quad (R \in \mathscr{L}(H, c_0)).$$

Let $h_k = W(e_k)$, where $\{e_k\}_{k=1}^{\infty}$ is the canonical basis for c_0. Then

$$W(t) = \sum_{k=1}^{\infty} t_k h_k \quad \left(t = \{t_k\}_{k=1}^{\infty} \in c_0\right).$$

Verify that

$$\sum_{k=1}^{\infty} \|h_k\| < \infty,$$

which yields nuclearity of W. Define the operator $R_n : H \to c_0$ as

$$R_n x = \sum_{k=1}^{n} \left(x, \frac{h_k}{\|h_k\|} \right) e_k.$$

We have

$$\|R_n\| = 1, \quad WR_n(x) = \sum_{k=1}^{n} \left(x, \frac{h_k}{\|h_k\|} \right) h_k, \quad \mathrm{trace}(WR_n) = \sum_{k=1}^{n} \|h_k\|.$$

Since $|\,\mathrm{trace}(WR_n)| \le \nu(WR_n) \le C$ (see 3.3.8), we have

$$\sum_{k=1}^{n} \|h_k\| \le C,$$

which yields convergence of the series $\sum_{k=1}^{\infty} \|h_k\|$ in view of the arbitrariness of n. $\triangleright$

3.7.4. Interrelations between stably regular and 1-absolutely summing operators.

(a) Theorem. *Let $1 \le p \le \infty$ and $U \in \mathscr{L}(L^p, L^2)$. The following assertions are equivalent:*

(1) $U \in \mathscr{L}^{\sim}_{\mathrm{st}}(L^p, L^2)$;

(2) *the operator U is left stably regular;*

(3) $U \in \Pi_1(L^p, L^2)$.

$\triangleleft$ The implication (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (3): If $p = 1$ then $U \in \Pi_1(L^1, L^2)$ by Grothendieck's theorem (see 3.5.4). If $p = \infty$ then $U \in N(L^\infty, L^2)$ by Lemma 3.7.3(a). Let now $1 < p < \infty$. It suffices to verify that $UV \in \Pi_1(c_0, L^2)$ for every operator $V \in \mathscr{L}(c_0, L^p)$ (see 3.1.6). Prove that the more strong assertion is valid: the operator $W = UV$ is nuclear. By Lemma 3.7.3(b), it suffices to check nuclearity of WR for an arbitrary $R \in \mathscr{L}(L^2, c_0)$. Take such an R. By Grothendieck's theorem, the operator R^* as well as the operator $(VR)^*$ belongs to the class Π_1; therefore, the operator VR is dominated (3.4.1(b)). Consequently, the operator VR admits strong factorization $VR = M_g R_0$, where $R_0 \in \mathscr{L}(L^2, L^\infty)$ and M_g is the operator of multiplication by

a function $g \in L^p$. By putting $U_0 = UM_g$, we obtain the following commutative diagram:

$$\begin{tikzcd}
RW : L^2 \arrow[r, "R"] \arrow[drr, "R_0"'] & c_0 \arrow[r, "V"] & L^p \arrow[r, "U"] & L_2 \\
 & & L^\infty \arrow[u, "M_g"] \arrow[ur, "U_0"'] &
\end{tikzcd}$$

Since the operator U_0 satisfies the hypothesis of Lemma 3.7.3(a), it is nuclear as well as the operator RW.

(3) $\Rightarrow$ (1): It suffices to prove that the operator U is regular. Since the operator U^* is dominated (see 3.4.11), it is regular. Therefore, U^{**} is regular and U is regular too. $\triangleright$

(b) Theorem. *Let $1 \leq p \leq \infty$ and $U \in \mathscr{L}(L^2, L^2)$. The following assertions are equivalent:*

(1) $U \in \mathscr{L}_{\mathrm{st}}^{\sim}(L^2, L^2)$;

(2) *U is right stably regular;*

(3) $U^* \in \Pi_1(L^2, L^2)$.

If $1 \leq q \leq 2$ then assertions (1)–(3) are equivalent to the following:

(4) $U \in \Pi_2(L^2, L^q)$.

$\triangleleft$ The equivalence of (1)–(3) can be established by duality and Theorem (a).

(3) $\Rightarrow$ (4): By Theorem 3.4.1(b), the operator U is dominated and consequently $U \in \Pi_q(L^2, L^q)$ (see 1.3.5(c)). It remains to observe that $\Pi_q(L^2, L^q) \subset \Pi_2(L^2, L^q)$ since $q \leq 2$.

(4) $\Rightarrow$ (3): By making use of the canonical factorization of a 2-absolutely summing operator, we obtain the representation $U = WV$, where V is a Hilbert-Schmidt operator. The operator U^* is absolutely summing together with V^*. $\triangleright$

(c) Corollary. *Let $U \in \mathscr{L}(L^2, L^2)$. The following assertions are equivalent:*

(1) *U is left stably regular;*

(2) *U is right stably regular;*

(3) *U is a Hilbert-Schmidt operator.*

3.7.5. (a) Theorem. *Let $1 < p \leq \infty$, $1 \leq q \leq 2$, and $U \in \mathscr{L}(L^p, L^q)$. If U is right stably regular then $U \in \Pi_2(L^p, L^q)$.*

◁ In accord with 3.1.6 it suffices to verify that $UV \in \Pi_2(L^2, L^q)$ for every $V \in \mathscr{L}(L^2, L^p)$. By 3.7.2(b), the operator UV is right stably regular. It remains to appeal to Theorem 3.7.4(b). ▷

(b) Corollary. *If $1 < p \leq \infty$ and $1 \leq q \leq 2$ then $\mathscr{L}_{\mathrm{st}}^{\sim}(L^p, L^q) \subset \Pi_2(L^p, L^q)$.*

(c) REMARK. As will be demonstrated in 3.8.2, the equality

$$\mathscr{L}_{\mathrm{st}}^{\sim}(L^p, L^q) = \Pi_2(L^p, L^q)$$

holds for $1 < p \leq 2$ and $1 \leq q \leq 2$.

(d) REMARK. An example of the identity embedding of the space l^1 into the space l^q shows that Theorem (a) is false for $p = 1$ and $1 \leq q < 2$.

(e) Corollary. *Let $U \in \mathscr{L}(L^p(T, \mu), L^q(T_1, \mu_1))$ with $1 < p \leq 2$, $1 \leq q \leq 2$. If $\dim L^p(T_0, \mu_0) = \infty$ and UV is regular for every $V \in \mathscr{L}(L^p(T_0, \mu_0), L^p(T, \mu))$ then $U \in \Pi_2(L^p(T, \mu), L^q(T, \mu))$.*

◁ It is clear that the operator $UV : L^p(T_0, \mu_0) \to L^q(T_1, \mu_1)$ is right stably regular. By Theorem 3.7.5(a),

$$UV \in \Pi_2(L^p(T_0, \mu_0), L^q(T_1, \mu_1)) \subset \Pi_{p'}(L^p(T, \mu), L^q(T_1, \mu_1)).$$

Since the space l^p is isomorphic to a complemented subspace in $L^p(T_0, \mu_0)$, we have

$$UW \in \Pi_{p'}(l^p, L^q(T_1, \mu_1))$$

for every $W \in \mathscr{L}(l^p, L^p(T, \mu))$. Hence, $U \in \Pi_{p'}(L^p(T, \mu), L^q(T_1, \mu_1))$ (see 3.1.6). Since $L^q(T_1, \mu_1)$ is a 2-cotype space, it remains to appeal to 3.2.4. ▷

3.7.6. Theorem. *Let $1 < p \leq 2 \leq q < \infty$, $1/2 = 1/q + 1/s$, and $U \in \mathscr{L}(L^p, L^q)$. The following assertions are equivalent:*

(1) $U \in \mathscr{L}_{\mathrm{st}}^{\sim}(L^p, L^q)$;

(2) U is left stably regular;

(2′) U is right stably regular;

(3) $M_g U \in \Pi_2(L^p, L^q)$ for every $g \in L^s$, where $M_g : L^q \to L^2$ is the operator of multiplication by the function g;

(4) $VU \in \Pi_2(L^p, L^2)$ for every $V \in \mathscr{L}(L^p, L^2)$.

$\lhd$ The implications (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (2$'$) are trivial. Prove (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (1) and (2$'$) $\Rightarrow$ (1).

(2) $\Rightarrow$ (3): Let $g \in L^s$. By 3.7.2(b), the operator M_gU is left stably regular and thus $M_gU \in \Pi_1(L^p, L^2) \subset \Pi_2(L^p, L^2)$ by Theorem 3.7.4(a).

(3) $\Rightarrow$ (4): Let $V \in \mathscr{L}(L^p, L^2)$. By 3.6.5(b), the operator V^* admits strong factorization $V^* = M_gW$, where $W \in \mathscr{L}(L^2, L^2)$ and $g \in L^s$. Since $M_gU \in \Pi_2(L^p, L^2)$, we have $VU = W^*M_gU \in \Pi_2(L^p, L^2)$.

(4) $\Rightarrow$ (1): It suffices to check regularity of the operator U because an arbitrary operator U_1UU_0 satisfies condition (4) together with U, where $U_1 \in \mathscr{L}(L^q, L^q)$ and $U_0 \in \mathscr{L}(L^p, L^p)$. Regularity of U is equivalent to the fact that the operator UM_h is dominated for every operator $M_h : L^\infty \to L^p$ of multiplication by a function $h \in L^p$. Since $p \le 2$, the operator M_h obviously factors through the space L^2; that is, $M_h = \alpha\beta$, where $\alpha \in \mathscr{L}(L^2, L^p)$ and $\beta \in \mathscr{L}(L^\infty, L^2)$. Thus, it suffices to prove that $U\alpha$ is dominated. To this end, we prove that $(U\alpha)^* \in \Pi_q(L^{q'}, L^2)$. Let $V \in \mathscr{L}(L^q, L^2)$. By hypotheses, we have $VU \in \Pi_2(L^p, L^2)$ and consequently $VU\alpha$ is a Hilbert-Schmidt operator. Therefore,

$$(VU\alpha)^* = (U\alpha)^*V^* \in \Pi_2(L^2, L^2)$$

and

$$(U\alpha)^* \in \Pi_2(L^{q'}, L^2) \subset \Pi_q(L^{q'}, L^2)$$

in view of the arbitrariness of V (see 3.1.6).

(2$'$) $\Rightarrow$ (1): Since the operators U and U^* are stably regular simultaneously, it suffices to observe that U^* is stably regular and to use the equivalence (1) $\Leftrightarrow$ (2) for the operator.

3.8. Certain Operator Lattices

3.8.1. Regularity of 2-absolutely summing operators. *Let $1 \le p \le 2$, $1 \le q \le 2$, $1/p = 1/2 + 1/r$, and $U \in \mathscr{L}(L^q(T, \mu), L^q(T_0, \mu_0))$ and let*

$$M_g : L^2(T, \mu) \to L^p(T, \mu)$$

be the operator of multiplication by a function $g \in L^r(T, \mu)$. The following assertions are equivalent:

(1) $U \in \Pi_2(L^p(T,\mu), L^q(T_0,\mu_0))$;
(2) $UM_g \in \Pi_2(L^2(T,\mu), L^q(T_0,\mu_0))$ *for every function* $g \in L^r(T,\mu)$;
(3) $UM_g \in M(L^2(T,\mu), L^q(T_0,\mu_0))$ *for every function* $g \in L^r(T,\mu)$.
If an operator U *satisfies the conditions* (1)–(3) *then*

$$\pi_2(UM_g) \leq \|UM_g\|_M \leq \sqrt{2}\pi_2(UM_g), \tag{1}$$

$$\pi_2(U) \leq \sqrt{2}\sup\{\pi_2(UM_g) \mid \|g\|_r \leq 1\}. \tag{2}$$

$\lhd$ The implication (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (3): Consider the canonical factorization of the operator UM_g:

$$UM_g : L^2(T,\mu) \xrightarrow{j} C(K) \xrightarrow{i_2} L^2(K,\nu) \xrightarrow{V} L^q(T_0,\mu_0),$$

where $\|V\| = \pi_2(UM_g)$ and $\pi_2(i_2 j) = 1$. Clearly, $V_0 = i_2 j$ and V_0^* are Hilbert-Schmidt operators. Therefore, Theorem 3.4.1(b) implies that the operator UM_g is dominated and

$$\|UM_g\|_M \leq \pi_1((UM_g)^*) \leq \pi_1(V_0^*)\|V\| = \pi_1(V_0^*)\pi_2(UM_g).$$

Since $\pi_1(V_0^*) \leq \sqrt{2}\pi_2(V_0^*)$ by Corollary 3.2.3, we have

$$\|UM_g\|_M \leq \sqrt{2}\pi_2(V_0^*)\pi_2(UM_g) = \sqrt{2}\pi_2(V_0)\pi_2(UM_g) = \sqrt{2}\pi_2(UM_g),$$

which proves the right inequality in (1). The left inequality follows from 3.1.5(c).

(3) $\Rightarrow$ (1): By 3.1.6, it suffices to verify that $UV \in \Pi_2(L^2(T,\mu), L^q(T_0,\mu_0))$ for every operator $V \in \mathscr{L}(L^2(T,\mu), L^p(T,\mu))$. By Remark 3.6.5(c), the operator V admits strong factorization $V = M_g V_0$ with the estimates $\|g\|_r \leq 1$ and $\|V_0\| \leq \sqrt{2}\|V\|$, where $V_0 \in \mathscr{L}(L^2(T,\mu), L^2(T,\mu))$. Consequently, $UV = UM_g V_0 \in \Pi_2(L^2(T,\mu), L^q(T_0,\mu_0))$. Moreover,

$$\begin{aligned}
\pi_2(U) &= \sup\{\pi_2(UV) \mid V \in \mathscr{L}(L^2(T,\mu), L^p(T,\mu)),\ \|V\| \leq 1\} \\
&\leq \sup\{\pi_2(UM_g V_0) \mid \|g\|_r \leq 1,\ V_0 \in \mathscr{L}(L^2(T,\mu), L^2(T,\mu)),\ \|V_0\| \leq \sqrt{2}\} \\
&\leq \sup\{\pi_2(UM_g)\|V_0\| \mid \|g\|_r \leq 1,\ \|V_0\| \leq \sqrt{2}\} \\
&= \sqrt{2}\sup\{\pi_2(UM_g) \mid \|g\|_r \leq 1\}.
\end{aligned}$$

3.8.2. Corollaries. Throughout this subsection we assume that $1 \leq p \leq 2$, $1 \leq q \leq 2$, $L^p = L^p(T,\mu)$, and $L^q = L^q(T_0,\mu_0)$.

(a) *The space* $\Pi_2(L^p, L^q)$ *is an o-ideal in the lattice* $L^\sim(L^p, L^q)$; *more precisely,* $\Pi_2(L^p, L^q) \subset \mathscr{L}_{\mathrm{st}}^\sim(L^p, L^q)$ *and if*

$$U \in \Pi_2(L^p, L^q), \quad V \in L^\sim(L^p, L^q), \quad |V| \le |U|$$

then $V \in \Pi_2(L^p, L^q)$ *and* $\pi_2(V) \le 2\pi_2(U)$.

◁ Let $U \in \Pi_2(L^p, L^q)$. Check that $U \in \mathscr{L}_{\mathrm{st}}^\sim(L^p, L^q)$. Obviously, it suffices to prove that $U \in L^\sim(L^p, L^q)$; i.e., that the image $U(I_h)$ of the interval $I_h = \{x \in L^p \mid |x| \le h\}$ is order bounded for every $h \in L_+^p$. Assign

$$\frac{1}{r} = \frac{1}{p} - \frac{1}{2}, \quad g = h^{p/r}, \quad g_0 = h^{p/2}, \quad I_{g_0} = \{y \in L^2 \mid |y| \le g_0\}.$$

Then $U(I_h) = (UM_g)(I_{g_0})$. Now the order boundedness of $U(I_h)$ follows from the fact that the operator UM_g is dominated and the set I_{g_0} is norm bounded.

If $V \in L^\sim(L^p, L^q)$ and $|V| \le |U|$ then $|VM_g| \le |UM_g|$ for all $g \in L^r = L^r(T, \mu)$; therefore, $VM_g \in M(L^2, L^q)$. By making use of inequalities (1) and (2) in Theorem 3.8.1, we obtain

$$\begin{aligned}
\pi_2(V) &\le \sqrt{2}\sup\{\pi_2(VM_g) \mid \|g\|_r \le 1\} \\
&\le \sqrt{2}\sup\{\|VM_g\|_M \mid \|g\|_r \le 1\} \le \sqrt{2}\sup\{\|UM_g\|_M \mid \|g\|_r \le 1\} \\
&\le 2\sup\{\pi_2(UM_g) \mid \|g\|_r \le 1\} \le 2\pi_2(U). \ \triangleright
\end{aligned}$$

(b) *If* $1 < p \le 2$ *and* $1 \le q \le 2$ *then* $\Pi_2(L^p, L^q) = \mathscr{L}_{\mathrm{st}}^\sim(L^p, L^q)$.

◁ This fact follows from (a) and 3.7.4(e). ▷

(c) *Let* $2 \le p \le \infty$ *and* $2 \le q < \infty$. *If an operator* $U \in \mathscr{L}(L^p, L^q)$ *is left stably regular then* $U \in \mathscr{L}_{\mathrm{st}}^\sim(L^p, L^q)$.

◁ Stable regularity of U is equivalent to stable regularity of U^*. The latter is true in view of right stable regularity of the operator U^*, Theorem 3.7.5(a) and the preceding corollary. ▷

(d) *If* p, q, U, *and* V *satisfy the hypothesis of Corollary* (a) *then* $\pi_1(V) \le 2K_G\pi_1(U)$, *where* K_G *is the Grothendieck constant.*

◁ This follows from (a) and Remark 3.5.4. ▷

(e) If $U \in \Pi_2(L^p, L^q)$ then $U^* \in M(L^{q'}, L^{p'})$ (see 3.4.4 and 3.5.3(b)). Therefore, U^* and U are integral operators. If K is the kernel of U then condition (3) of Theorem 3.8.1 means (see 2.4.13) that

$$\|UM_g\|_M = \left(\int\limits_{T_0} \left(\int\limits_{T} |K(s,t)g(t)|^2\, d\mu(t)\right)^{q/2} d\mu_0\right)^{1/q} < \infty \tag{3}$$

for every $g \in L^r(T, \mu)$. By using (3), we can easily verify that the inclusions

$$M(L^p, L^p) \subset \Pi_2(L^p, L^q) \subset M^{\mathrm{dual}}(L^p, L^q) = \{U \in \mathscr{L}(L^p, L^q) \mid U^* \in M(L^{q'}, L^{p'}\}$$

are proper for $p < 2$.

3.8.3. Two Lemmas.

(a) Lemma. *Let $r_1, r_2, \dots$ be the Rademacher functions. For arbitrary $a_{kj} \in \mathbb{C}$ ($k = 1, \dots, n;\ j = 1, \dots, m$), the following inequality is valid:*

$$\left(\sum_{k,j=1}^{n,m} |a_{kj}|^2\right)^{1/2} \leq 2 \int\limits_0^1 \int\limits_0^1 \left|\sum_{k,j=1}^{n,m} a_{kj} r_k(s) r_j(t)\right| dsdt.$$

◁ To prove the lemma, apply the Khinchin inequality twice (see 3.S.1). Put

$$\Phi(s,t) = \left|\sum_{k,j=1}^{n,m} a_{kj} r_k(s) r_j(t)\right|.$$

Then

$$\left(\sum_{k,j=1}^{n,m}|a_{kj}|^2\right)^{1/2}=\left(\int_0^1\int_0^1\Phi^2(s,t)\,ds\,dt\right)^{1/2}\le\left(\int_0^1 2\left(\int_0^1\Phi(s,t)\,dt\right)^2 ds\right)^{1/2}$$

$$=\sqrt{2}\left\|\int_0^1\Phi(\cdot,t)\,dt\right\|_{L^2(0,1)}\le\sqrt{2}\int_0^1\|\Phi(\cdot,t)\|_{L^2(0,1)}\,dt$$

$$=\sqrt{2}\int_0^1\left(\int_0^1\left|\sum_{k=1}^n\sum_{j=1}^m a_{kj}r_j(t)r_k(s)\right|^2 ds\right)^{1/2}dt$$

$$\le\sqrt{2}\int_0^1\sqrt{2}\int_0^1\left|\sum_{k=1}^n\sum_{j=1}^m a_{kj}r_j(t)\,r_k(s)\right|dsdt$$

$$=2\int_0^1\int_0^1\Phi(s,t)\,dsdt.\ \rhd$$

(b) Lemma. *Let $1\le r\le q\le p\le r'$, $L^p=L^p(T,\mu)$, $L^q=L^q(T_0,\mu_0)$, $L^r=L^r(T_1,\mu_1)$, and $U\in\mathscr{L}(L^p,L^q)$ and let J be an arbitrary isometric embedding of L^q into L^r. The following assertions are equivalent*:

(1) $U\in\Pi_r(L^p,L^q)$;

(2) $U^*\in\Pi_r(L^{q'},L^{p'})$;

(3) $JU\in M(L^p,L^r)$.

If (1)–(3) *are satisfied then the equalities $\pi_r(U)=\pi_r(U^*)=\|JU\|_M$ are valid.*

$\lhd$ (3) $\Rightarrow$ (1) by 3.4.2(c), where the inequality $\pi_r(U)=\pi_r(JU)\le\|JU\|_M$ was established too.

(2) $\Rightarrow$ (3) by 3.4.2(b), whence the estimate $\|JU\|_M\le\pi_r((JU)^*)\le\pi_r(U^*)$ ensues.

The above assertion implies that if $U^*\in\Pi_r(L^{q'},L^{p'})$ then $U\in\Pi_r(L^p,L^q)$; moreover, $\pi_r(U)\le\pi_r(U^*)$.

(1) $\Rightarrow$ (2): By applying the preceding remark to the operator U^* and by using the fact that the operator U^{**} is r-absolutely summing simultaneously with the operator U (see 3.1.11), we infer that $\pi_r(U^*)\le\pi_r(U^{**})=\pi_r(U)$. $\rhd$

3.8.4. Interrelations between the spaces $\Pi_1(L^p, L^q)$ and $M(L^p, L^q)$.

Theorem. *Let $1 \le q \le 2 \le p \le \infty$, $L^p = L^p(T, \mu)$, and $L^q = L^q(T_0, \mu_0)$. Then*

(1) *if $1 \le q < p' \le 2$ then $\Pi_1(L^p, L^q) = M(L^p, L^q)$;*

(2) *if $1 \le p' < q \le 2$ then*

$$\Pi_1(L^p, L^q) = M^{\mathrm{dual}}(L^p, L^q) = \{U \in \mathscr{L}(L^p, L^q) \mid U^* \in M(L^{q'}, L^{p'})\};$$

(3) *if $1 < p' = q < 2$ then $\Pi_1(L^p, L^{p'}) \subsetneq M(L^p, L^{p'}) = \Pi_{p'}(L^p, L^{p'})$;*

(4) *$\Pi_1(L^\infty, L^1) = M(L^\infty, L^1)$; moreover, $\pi_1(U) = \|U\|_M$ for $U \in \Pi_1(L^\infty, L^1)$.*

$\lhd$ (1) By 3.5.8 and 3.1.5(c), we have

$$\Pi_1(L^p, L^q) = \Pi_q(L^p, L^q) \supset M(L^p, L^q).$$

By making use of Lemma 3.8.3(b) for $r = 1$, we find that the relations $U \in \Pi_q(L^p, L^q)$ and $U^* \in \Pi_1(L^{q'}, L^{p'})$ are equivalent. Hence $\Pi_1(L^p, L^q) \subset M(L^p, L^q)$ (see 3.4.1(b)).

(2) It is clear that $\Pi_1(L^p, L^q) \subset M^{\mathrm{dual}}(L^p, L^q)$ (see 3.4.1(a)). On the other hand, if $U \in M^{\mathrm{dual}}(L^p, L^q)$ then $U^* \in \Pi_{p'}(L^{q'}, L^{p'})$ and $U \in \Pi_1(L^p, L^q)$ by 3.5.7 since $p' < q$.

(3) By 3.4.6(b), we have

$$M(L^p, L^{p'}) = \Pi_{p'}(L^p, L^{p'}) \supset \Pi_1(L^p, L^{p'}).$$

Check that the preceding inclusion is proper. Since the spaces l^p and $l^{p'}$ are isomorphic to complemented subspaces of L^p and $L^{p'}$ respectively, it suffices to prove that $\Pi_1(l^p, l^{p'}) \neq \Pi_{p'}(l^p, l^{p'})$.

Take $a = \{a_k\}_{k=1}^\infty \in l^{p'}$, $0 < a_k < 1$. Demonstrate that if

$$\sum_{k=1}^\infty a_k^{p'} \ln \frac{1}{a_k} = \infty$$

then the diagonal operator $M_a : l^p \to l^{p'}$ defined by the equality

$$M_a(x) = \{a_k x_k\}_{k=1}^\infty \quad (x = \{x_k\}_{k=1}^\infty \in l^p)$$

belongs to $\Pi_{p'}(l^p, l^{p'})$ but does not belong to $\Pi_1(l^p, l^{p'})$. The assertion 3.1.5(b) yields $M_a \in \Pi_{p'}(l^p, l^{p'})$. Suppose to the contrary that $M_a \in \Pi_1(l^p, l^{p'})$ and let $\omega_1 : l^{p'} \to L^1(0,1)$ be the isomorphic embedding defined in the proof of Lemma 3.5.6:

$$\omega_1(t) = \sum_{k=1}^{\infty} t_k h_k \quad \left(t = \{t_k\}_{k=1}^{\infty} \in l^{p'}\right),$$

where $\{h_k\}_{k=1}^{\infty}$ is a sequence of symmetric independent random variables on $(0,1)$ distributed by a p'-stable law $F_{p'}$. Since

$$(\omega_1 M_a)^* = M_a \omega_1^* \in \Pi_1(L^\infty(0,1), l^p),$$

the operator $\omega_1 M_a$ is dominated in view of 3.4.1(b) and consequently there exists a summable function f on $(0,1)$ such that

$$|(\omega_1 M_a)(x)| = \left|\sum_{k=1}^{\infty} a_k x_k h_k\right| \le f\|x\|,$$

where $x = \{x_k\}_{k=1}^{\infty} \in l^p$. Thus, the series $\sum_{k=1}^{\infty} |a_k h_k|^{p'}$ converges almost everywhere. The functions $|a_k h_k|^{p'}$ are independent, so, by the three series theorem (see, for example, [14, p. 194]), for the series to converge almost everywhere it is necessary that the series

$$\sum_{k=1}^{\infty} \int_{E_k} |a_k h_k(\tau)|^{p'} \, d\tau$$

converges, where $E_k = \{\tau \in (0,1) \mid |a_k h_k(\tau)| \le 1\}$. As was pointed out in 3.5.6,

$$F'_{p'}(u) \underset{k\to\infty}{\sim} C/u^{1+p'};$$

therefore, there is $a > 0$ such that $F'_{p'}u \ge au^{-1-p'}$ for $u \ge 1$. Consequently,

$$\infty > \sum_{k=1}^{\infty} \int_{E_k} |a_k h_k(\tau)|^{p'} \, d\tau \ge \sum_{k=1}^{\infty} a_k^{p'} \int_1^{a_k^{-1}} u^{p'} \, dF_{p'}(u)$$

$$\ge a \sum_{k=1}^{\infty} a_k^{p'} \int_1^{a_k^{-1}} \frac{du}{u} = a \sum_{k=1}^{\infty} a_k^{p'} \ln \frac{1}{a_k} = \infty$$

which is impossible. Thus, $M_a \notin \Pi_1(l^p, l^{p'})$.

We leave to the reader the proof of assertion (4) grounding on 3.1.5(c) and 3.4.1. ▷

Corollary 1. *Let $1 \le q \le 2 \le p \le \infty$ and let $U \in \Pi_1(L^p, L^q)$. Then U is an integral operator; moreover, its kernel K satisfies the following conditions (see 2.4.13):*

$$\|U\|_M = \left(\int\limits_{T_0}\left(\int\limits_{T} |K(s,t)|^{p'}\,d\mu(t)\right)^{q/p'} d\mu_0(s)\right)^{1/q} < \infty \text{ for } q < p',$$

$$\|U^*\|_M = \left(\int\limits_{T}\left(\int\limits_{T_0} |K(s,t)|^{q}\,d\mu_0(s)\right)^{p'/q} d\mu(t)\right)^{1/p'} < \infty \text{ for } q > p'.$$

Corollary 2. *If $1 \le q \le 2 \le q \le \infty$ and $q \ne p'$ then $\Pi_1(L^p, L^q)$ is an o-ideal in the lattice $\mathscr{L}^\sim(L^p, L^q)$.*

The last corollary leaves the question open whether the set $\Pi_1(L^p, L^{p'})$ is an o-ideal in $L^\sim(L^p, L^{p'})$. Moreover, Theorem 3.8.4 does not contain an estimate for the π_1-norm of V in terms of the π_1-norm of U in the case $|V| \le |U|$. The following assertion fills in the gaps.

Theorem. *Let $1 \le q \le 2 \le p \le \infty$ and $U \in \Pi_1(L^p, L^q)$. Then*

(1) $|U| \in \Pi_1(L^p, L^q)$ and $\pi_1(|U|) \le 2\pi_1(U)$;

(2) if $V \in L^\sim(L^p, L^q)$ and $|V| \le |U|$ then $V \in \Pi_1(L^p, L^q)$ and $\pi_1(V) \le 2\pi_1(|U|)$.

$\lhd$ To begin with, we prove the theorem on assuming that $L^p = l^p_n$ and $L^q = l^q_m$. Moreover, at first we suppose that $p < \infty$ and $q > 1$.

Let the operator U be determined by the matrix $(u_{jk})_{j,k=1}^{m,n}$. Define an isometric embedding J of the space l^q_m into $L^1(0,1)$ by the equality

$$J(t) = C_{q,1}^{-1} \sum_{j=1}^{m} t_j h_j,$$

where $t = \{t_j\}_{j=1}^m \in l^q_m$ (see Lemma 3.5.6). By Lemma 3.8.3(b), we have $\pi_1(U) = \|JU\|_M$ for $r = 1$. Find $\|JU\|_M$. Since

$$JU(x) = C_{q,1}^{-1} \sum_{j=1}^{\infty} \left(\sum_{k=1}^{n} u_{jk} x_k\right) h_j, \quad x = \{x_k\}_{k=1}^n \in l^p_n,$$

we have

$$|JU| = \sup\{|JU(x)| \mid x \in l_n^p,\ \|x\| \le 1\} = C_{q,1}^{-1}\left(\sum_{k=1}^{n}\left|\sum_{j=1}^{m} u_{jk}h_j\right|^{p'}\right)^{1/p'}.$$

Let $\{g_k\}_{k=1}^n$ be a family of independent symmetric random variables on $(0,1)$ distributed by a p'-stable law. By making use of Lemma 3.5.6 again, we obtain

$$\int_0^1 \left|\sum_{k=1}^{n} a_k g_k(\sigma)\right| d\sigma = C_{p',1}\left(\sum_{k=1}^{n}|a_k|^{p'}\right)^{1/p'}.$$

Therefore,

$$|JU|(\tau) = C_{q,1}^{-1}C_{p',1}^{-1}\int_0^1 \left|\sum_{k=1}^{n}\left(\sum_{j=1}^{m} u_{jk}h_j(\tau)\right) g_k(\tau)\right| d\tau$$

and consequently

$$\|JU\|_M = \|\,|JU|\,\|_{L^1(0,1)} = C_{q,1}^{-1}C_{p',1}^{-1}\int_0^1\int_0^1 \left|\sum_{j,k=1}^{m,n} u_{jk}h_j(\tau)\,g_k(\sigma)\right| d\sigma d\tau.$$

Since the random variables g_k and h_j are symmetric, the equality

$$\|JU\|_M = C_{q,1}^{-1}C_{p',1}^{-1}\int_0^1\int_0^1 \left|\sum_{j,k=1}^{m,n} u_{jk}\varepsilon_k\varepsilon_j' h_j(\tau)\,g_k(\sigma)\right| d\sigma d\tau$$

is valid for all $\varepsilon_k = \pm 1$ and $\varepsilon_j' = \pm 1$; therefore,

$$\|JU\|_M = C_{q,1}^{-1}C_{p',1}^{-1}\int_0^1\int_0^1 \left[\int_0^1\int_0^1 \left|\sum_{j,k=1}^{m,n} u_{jk}r_k(s)\,r_j(t)\,h_j(\tau)\,g_k(\sigma)\right| ds\,dt\right] d\sigma d\tau, \quad (4)$$

where $r_1, r_2, \dots$ are the Rademacher functions. By making use of Bunyakovskiĭ's inequality, we obtain

$$\|JU\|_M \le C_{q,1}^{-1}C_{p',1}^{-1}\int_0^1\int_0^1 \left(\sum_{j,k=1}^{m,n} |u_{jk}|^2 h_j^2(\tau)\,g_k^2(\sigma)\right)^{1/2} d\sigma d\tau, \quad (5)$$

and, by Lemma 3.8.3(c),

$$\left(\sum_{j,k=1}^{m,n} |u_{jk}|^2 h_j^2(\tau)\, g_k^2(\sigma)\right)^{1/2} \le 2\int_0^1\int_0^1 \left|\sum_{j,k=1}^{m,n} u_{jk} r_k(s)\, r_j(t)\, h_j(\tau)\, g_k(\sigma)\right| dsdt.$$

Thus, (4) and (5) yield

$$\|JU\|_M \le C_{q,1}^{-1} C_{p',1}^{-1} \int_0^1\int_0^1 \left(\sum_{j,k=1}^{m,n} |u_{jk}|^2 h_j^2(\tau)\, g_k^2(\sigma)\right)^{1/2} d\sigma d\tau \le 2\|JU\|_M. \tag{6}$$

If the matrix $(v_{jk})_{j,k=1}^{m,n}$ corresponds to an operator $V : l_n^p \to l_m^q$ and $|V| \le |U|$ then $|v_{jk}| \le |u_{jk}|$ $(j = 1, \ldots, m;\ k = 1, \ldots, n)$. From (6) it follows that

$$\begin{aligned}\|JV\|_M &\le C_{q,1}^{-1} C_{p',1}^{-1} \int_0^1\int_0^1 \left(\sum_{j,k=1}^{m,n} |v_{jk}|^2 h_j^2(\tau)\, g_k^2(\sigma)\right)^{1/2} d\sigma d\tau \\ &\le C_{q,1}^{-1} C_{p',1}^{-1} \int_0^1\int_0^1 \left(\sum_{j,k=1}^{m,n} |u_{jk}|^2 h_j^2(\tau)\, g_k^2(\sigma)\right)^{1/2} d\sigma d\tau \le 2\|JU\|_M.\end{aligned}$$

By Lemma 3.8.3(b) (with $r = 1$), we obtain

$$\pi_1(V) = \|JV\|_M \le 2\|JU\|_M = 2\pi_1(U). \tag{7}$$

In the case $q = 1$ or $p = \infty$, we have to replace the functions h_j (g_k) with the characteristic functions of pairwise disjoint sets of equal measure and the constant $C_{q,1}$ (respectively $C_{p',1}$) with the measure of these sets.

Now we turn to the case of arbitrary spaces L^p and L^q. By virtue of claim (4) of Theorem 3.8.4, we may except the case of $p = \infty$ and $q = 1$. For all other values p and q, the set of finite rank operators is dense in $\Pi_1(L^p, L^q)$. For $p < \infty$, this fact was established in Theorem 3.3.9 and for $p = \infty$ and $q > 1$, we have $\Pi_1(L^\infty, L^q) = N(L^\infty, L^q)$ by Lemma 3.3.4. Therefore, every operator $U \in \Pi_1(L^p, L^q)$ can be approximated by operators of the form

$$\widetilde{U} = QUP, \tag{8}$$

where P and Q are positive projections ("conditional expectations") onto the subspaces $L \subset L^p$ and $M \subset L^q$ spanned over finite families of characteristic functions of pairwise disjoint sets. The subspaces L and M are isometric (with the preservation of order) to the spaces l_n^p and l_m^q, where $n = \dim L$ and $m = \dim M$. Therefore, inequality (7) remains valid for the operators $\widetilde{U}$ and $\widetilde{V}$ of the form (8) provided that $|\widetilde{V}| \le |\widetilde{U}|$.

Let $U_i = Q_i U P_i$ $(i = 1, 2, \dots)$ be a sequence of operators of the form (8) convergent to U in the norm π_1. Since $|\,|U_k| - |U_j|\,| \le |U_k - U_j|$, we have

$$\pi_1(|U_k| - |U_j|) \le 2\pi_1(U_k - U_j)$$

by (7). Thus, the sequence $\{|U_k|\}_{k=1}^{\infty}$ converges in $\Pi_1(L^p, L^q)$. Prove that

$$\lim_{k\to\infty} |U_k| = |U|.$$

To this end, it suffices to verify that $|U_k| \to |U|$ pointwise. Without loss of generality, we may (and shall) assume that

$$\sum_{k=1}^{\infty} \pi_1(U_{k+1} - U_k) < \infty.$$

Then we have

$$\begin{aligned} |\,|U|(x) - |U_k|(x)\,| \le |U - U_k|(|x|) &= \left|\sum_{j=k}^{\infty}(U_{j+1} - U_j)\right|(|x|) \\ &\le \sum_{j=k}^{\infty} |U_{j+1} - U_j|(|x|), \end{aligned}$$

for all $x \in L^p$. Consequently,

$$\begin{aligned} \|\,|U|(x) - |U_k|(x)\| &\le \sum_{j=k}^{\infty} \|\,|U_{j+1} - U_j|\,\|\,\|x\| \\ &\le \|x\| \sum_{j=k}^{\infty} \pi_1(|U_{j+1} - U_j|) \\ &\le 2\|x\| \sum_{j=k}^{\infty} \pi_1(U_{j+1} - U_j) \xrightarrow[k\to\infty]{} 0. \end{aligned}$$

Thus, $\lim\limits_{k\to\infty} |U_k| = |U|$. Moreover,

$$\pi_1(|U|) = \lim_{k\to\infty} \pi_1(|U_k|) \leq \lim_{k\to\infty} 2\pi_1(U_k) = 2\pi_1(U).$$

If $V \in L^{\sim}(L^p, L^q)$ and $|V| \leq |U|$ then $|QVP| \leq Q|V|P \leq Q|U|P$ and $\pi_1(QVP) \leq 2\pi_1(Q|U|P)$ by (7). By passing to the limit in the preceding inequality, we obtain $\pi_1(V) \leq 2\pi_1(|U|)$. $\rhd$

3.8.5. Structure of a vector lattice in the space $\Pi_r(L^p, L^q)$ for $1 \leq r \leq 2$ and $1 \leq q \leq 2$. If $1 < r \leq 2$ and $1 \leq q \leq 2$ and either (1) $1 \leq p < r'$ or (2) $r = p' < q$ then $\Pi_r(L^p, L^q) = \Pi_1(L^p, L^q)$. For $p \leq 2$, the preceding equality is valid in view of Corollary 3.5.3(b) and for the rest of the cases, by 3.5.8. Thus, the set $\Pi_r(L^p, L^q)$ for these values of p, q, and r is an o-ideal in the lattice $L^{\sim}(L^p, L^q)$ by 3.8.5, that is also true for $1 < r = p' = q$ in view of the equality $\Pi_{p'}(L^p, L^{p'}) = M(L^p, L^{p'})$ (see 3.4.6(b)). In this case, the space $\Pi_1(L^p, L^{p'})$ gives us a nontraditional example of an operator vector lattice. One can also prove that $\Pi_1(L^\infty, L^q)$ is an o-ideal in $\mathscr{L}^\infty(L^\infty, L^q)$ for every q, $1 \leq q \leq \infty$. The question whether the set $\Pi_r(L^{r'}, L^q)$ is an o-ideal in $L^{\sim}(L^{r'}, L^q)$ for $1 \leq q < r < 2$ is left open so far as we know. For the remaining values of p,q, r, $1 \leq q$, $r \leq 2$, the space $\Pi_r(L^p, L^q)$ is not isomorphic to any Banach lattice (see 3.9.7).

In conclusion we observe that the space $\Pi_r(L^p(0,1), L^q(0,1))$ for $1 < r \leq q \leq 2 \leq p \leq r'$ is isometric to a subspace of $L^r((0,1) \times (0,1))$ as follows from Lemma 3.8.3(b).

$\lhd$ Indeed, let U and J be such as in Lemma 3.8.3(b) and let J_0 be an isometric embedding of $L^{p'}(0,1)$ into $L^r(0,1)$. Then $\pi_r(U) = \|JU\|_M = \|JUJ_0^*\|_M$ and since the space $M(L^{r'}(0,1), L^r(0,1))$ is isometric to the space $L^r((0,1) \times (0,1))$ (see 2.4.13), we conclude that the mapping $U \to JUJ_0^*$ is a sought isometry. $\rhd$

3.8.6. The lattice of stably regular operators. As follows from 3.8.2(a) and (b), the set $\mathscr{L}_{\mathrm{st}}^{\sim}(L^p, L^q)$ for $1 \leq p, q \leq 2$ is an o-ideal in the lattice $L^{\sim}(L^p, L^q)$. This fact is also valid for $2 \leq p, q < \infty$ by duality. These results can be supplemented with the following statement.

Theorem. *Suppose that $1 < p \leq 2 < q < \infty$ and let $L^p = L^p(T, \mu)$ and $L^q = L^q(T_0, \mu_0)$. The set $\mathscr{L}_{\mathrm{st}}^{\sim}(L^p, L^q)$ is an o-ideal in $L^{\sim}(L^p, L^q)$. Moreover, $\Pi_t(L^p, L^q) \subset \mathscr{L}_{\mathrm{st}}^{\sim}(L^p, L^q)$ for every $t < \infty$.*

$\lhd$ Let $u \in \mathscr{L}^{\sim}_{\mathrm{st}}(L^p, L^q)$, $V \in L^{\sim}(L^p, L^q)$, and $|V| \leq |U|$. By 3.7.5, $V \in \mathscr{L}^{\sim}_{\mathrm{st}}(L^p, L^q)$ is equivalent to the fact that

$$V_0 = M_h V \in \Pi_2(l^p, L^2(T_0, \mu_0))$$

for every function $h \in L^s(T_0, \mu_0)$, where s is defined by the equality $1/2 = 1/q + 1/s$. By 3.8.1, this is equivalent to the containment $V_0 M_g \in M(L^2(T, \mu), L^2(T_0, \mu_0))$ for arbitrary function $g \in L^r(T, \mu)$, where $1/p = 1/2 + 1/r$. The last containment is valid since $|V_0 M_g| = |M_h V M_g| \leq |M_h U M_g|$ and $M_h U M_g \in M(L^2(T, \mu), L^2(T_0, \mu_0))$.

Finally, observe that if $U \in \Pi_t(L^p, L^q)$ then

$$M_h U \in \Pi_t(L^p, L^2(T_0, \mu_0)) = \Pi_2(L^2, L^2(T_0, \mu_0))$$

and consequently $U \in \mathscr{L}^{\sim}_{\mathrm{st}}(L^p, L^q)$. It is well known (see [42, Chapter XXII]) that the spaces $\Pi_t(L^p, L^q)$ for $1 < p < 2 < q < \infty$ are different for t sufficiently large, which implies that the inclusion $\Pi_t(L^p, L^q) \subset \mathscr{L}^{\sim}_{\mathrm{st}}(L^p, L^q)$ is proper. $\rhd$

Corollary. *Let $1 < p \leq 2 < q < t < \infty$. Then*

$$M(L^p, L^q) \subset \Pi_q(L^p, L^q) \subset \Pi_t(L^p, L^q) \subset \mathscr{L}^{\sim}_{\mathrm{st}}(L^p, L^q).$$

Moreover,

$$M^{\mathrm{dual}}(L^p, L^q) = \{U \in \mathscr{L}(L^p, L^q) \mid U^* \in M(L^{q'}, L^{p'})\} \subset \mathscr{L}^{\sim}_{\mathrm{st}}(L^p, L^q).$$

$\lhd$ The preceding relation is true since $M(L^{q'}, L^{p'}) \subset \mathscr{L}^{\sim}_{\mathrm{st}}(L^{q'}, L^{p'})$. $\rhd$

REMARK. As was established in [58], the set $\mathscr{L}^{\sim}_{\mathrm{st}}(l^p, l^q)$ is included into the set of compact operators for $1 < p \leq 2 \leq q < \infty$ and $1/p - 1/q < 1/2$ and the identity embedding of l^p into l^q is stably regular for $1/p - 1/q \geq 1/2$.

3.9. Operator Spaces and Local Unconditional Structure

3.9.1. DEFINITION. A sequence $\{x_n\}_{n=1}^{\infty}$ of linearly independent vectors in X is called *C-unconditional* if

$$\left\| \sum_{k=1}^{n} \theta_k c_k x_k \right\| \leq C \max_{1 \leq k \leq n} |\theta_k| \left\| \sum_{k=1}^{n} c_k x_k \right\|$$

for all scalar sequences $\{\theta_k\}_{k=1}^{n}$ and $\{c_k\}_{k=1}^{n}$ and for every $n \in \mathbb{N}$.

DEFINITION. We say that a Banach space Z possesses *local unconditional structure* (briefly LUST; denotation: $Z \in \mathrm{LUST}$) if there is a number L such that, for every finite-dimensional subspace $H \subset Z$, there exist a finite-dimensional space U with some 1-unconditional basis and operators $V \in \mathscr{L}(H, U)$ and $W \in \mathscr{L}(U, Z)$ such that $WV = I_H$ and $\|W\|\|V\| \le L$.

The greatest lower bound of L satisfying this condition is called the *local unconditional constant for the space* Z and is denoted by $\chi_u(Z)$.

The following two remarks are obvious:

REMARK 1. If Z_0 is a subspace of Z, P is the projection from Z onto Z_0, and $Z \in \mathrm{LUST}$ then $Z_0 \in \mathrm{LUST}$; moreover, $\chi_u(Z_0) \le \|P\|\chi_u(Z)$.

REMARK 2. If $\dim Z < \infty$ then $\chi_u(Z) = \chi_u(Z^*)$.

3.9.2. In this subsection we prove that each Banach lattice possesses LUST.

Recall that the band projection of an element of a K-space onto a band is called a *fragment* of the element.

Lemma. *Let E be an arbitrary K-space, let $e \in E_+$, let $I(e)$ be the principal ideal generated by e, and let ε be an arbitrary positive number. Given $z \in I(e)$, there exist pairwise disjoint fragments ω_j $(j = 1, \dots, N)$ of e and numbers a_j such that*

$$\left| z - \sum_{j=1}^{N} a_j \omega_j \right| \le \varepsilon e.$$

$\lhd$ First we will assume that the space E is real. Let $|z| \le Ce$. Select scalars $a_0, a_1, \dots, a_N$ so that $a_0 < -C < a_1 < \dots < a_{N-1} < C < a_N$ and $a_j - a_{j-1} < \varepsilon$ for $j = 1, \dots, N$. Let π_x be the band projection onto the band generated by the vector x and let $z_0 = 0$, $z_j = \pi_{(a_j e - z)_+}(e)$, and $\omega_j = z_j - z_{j-1}$. Then

$$z_0 = 0 \le z_1 \le \dots \le z_N = e, \quad \sum_{j=1}^{N} \omega_j = e, \quad \sum_{j=1}^{N} \pi_{\omega_j} = I_E.$$

Moreover, $\pi_{\omega_j}(a_{j-1}e) \le \pi_{\omega_j}(z) \le \pi_{\omega_j}(a_j e)$; i.e., $a_{j-1}\omega_j \le \pi_{\omega_j}(z) \le a_j \omega_j$. Summing the last inequalities, we obtain

$$\sum_{j=1}^{N} a_{j-1}\omega_j \le \sum_{j=1}^{N} \pi_{\omega_j}(z) = z \le \sum_{j=1}^{N} a_j \omega_j.$$

Consequently,

$$\left| z - \sum_{j=1}^{N} a_j \omega_j \right| \leq \sum_{j=1}^{N} (a_j - a_{j-1}) \omega_j \leq \varepsilon \sum_{j=1}^{N} \omega_j = \varepsilon e.$$

Now we turn to the case in which the space E is complex. Represent z as $z = u + iv$ where u and v are real elements of the lattice E. As was proven, there exist real b_j and c_j and fragments ω_j' and ω_j'' of the vector e such that

$$\left| u - \sum_{j=1}^{N} b_j \omega_j' \right| \leq \frac{\varepsilon}{2} e, \quad \omega_j' \wedge \omega_k' = 0 \text{ for } k \neq j,$$

$$\left| v - \sum_{j=1}^{N} c_j \omega_j'' \right| \leq \frac{\varepsilon}{2} e, \quad \omega_j'' \wedge \omega_k'' = 0 \text{ for } k \neq j.$$

Assign $\omega_{jk} = \omega_j' \wedge \omega_k''$ and $a_{jk} = b_j + ic_k$. Then we obviously have

$$\begin{aligned} \left| z - \sum_{j,k=1}^{N} a_{jk} \omega_{jk} \right| &\leq \left| u - \sum_{j,k=1}^{N} b_j \omega_{jk} \right| + \left| v - \sum_{j,k=1}^{N} c_k \omega_{jk} \right| \\ &= \left| u - \sum_{j=1}^{N} b_j \omega_j' \right| + \left| v - \sum_{k=1}^{N} c_k \omega_k'' \right| \leq \frac{\varepsilon}{2} e + \frac{\varepsilon}{2} e = \varepsilon e. \ \rhd \end{aligned}$$

Corollary. *If $\delta > 0$ and $h_1, \dots, h_m \in I(e)$ then there exist pairwise disjoint fragments ω_j of the vector e and numbers $a_j^{(k)}$ such that*

$$\left| h_k - \sum_{k=1}^{N} a_j^{(k)} \omega_j \right| < \delta e$$

for all $k = 1, \dots, m$.

$\lhd$ In the case of two vectors, the proof almost word for word repeats the arguments in the proof of the lemma for the complex case. The general case is settled by induction. $\rhd$

Theorem. *Every Banach lattice E has* LUST; *moreover, $\chi_u(E) = 1$.*

◁ First we suppose that E is a Banach K-space. Fix an arbitrary $\varepsilon > 0$ and consider an arbitrary finite-dimensional subspace $H \subset Z$. Let $h_1, \dots, h_m$ be a basis for H. Put

$$e = \sum_{k=1}^{m} |h_k|$$

and choose a number C so that

$$\sum_{k=1}^{m} |\lambda_k| \le C \left\| \sum_{k=1}^{m} \lambda_k h_k \right\|$$

for all numbers λ_k. Let vectors ω_j and numbers $a_j^{(k)}$ be such as in the corollary to the above-proven lemma, where $\delta > 0$ is chosen so that

$$\delta C \|e\| < 1, \quad m\delta C \|e\| (1 - \delta C \|e\|)^{-1} < \varepsilon.$$

Put

$$U = \operatorname{lin}\big(\{\omega_j\}_{j=1}^{N}\big), \quad \tilde{h}_k = \sum_{j=1}^{N} a_j^{(k)} \omega_j, \quad \widetilde{H} = \operatorname{lin}\big(\{\tilde{h}_k\}_{k=1}^{m}\big).$$

It is clear that $\{\omega_j\}_{j=1}^{N}$ is a 1-unconditional basis for U. Define an operator $V : H \to U$ by the equality

$$V(h) = \sum_{k=1}^{m} \lambda_k \tilde{h}_k, \text{ where } h = \sum_{k=1}^{m} \lambda_k h_k.$$

Then we have

$$\begin{aligned} \|V(h)\| &\le \|h\| + \left\| \sum_{k=1}^{m} \lambda_k (\tilde{h}_k - h_k) \right\| \le \|h\| + \left\| \sum_{k=1}^{m} |\lambda_k| \delta e \right\| \\ &= \|h\| + \delta \|e\| \sum_{k=1}^{m} |\lambda_k| \le (1 + \delta C \|e\|) \|h\| \le (1 + \varepsilon) \|h\|. \end{aligned}$$

On the other hand,

$$\|V(h)\| \ge \|h\| - \left\| \sum_{k=1}^{m} \lambda_k (\tilde{h}_k - h_k) \right\| \ge (1 - \delta C \|e\|) \|h\|.$$

Therefore, $\|V\| \le 1+\varepsilon$ and $\|V^{-1}\| \le (1-\delta C\|e\|)^{-1}$.

Now we define an operator $W : U \to Z$ by the equality

$$Wu = V^{-1}Pu + u - Pu \quad (u \in U),$$

where $P : U \to U$ is the projection carrying U onto $\widetilde{H}$ and such that $\|P\| \le m$. Assuming $P(u) = \sum_{k=1}^{m} \lambda_k \tilde{h}_k$, we have

$$\begin{aligned}\|W(u)\| &\le \|u\| + \left\|\sum_{k=1}^{m} \lambda_k(\tilde{h}_k - h_k)\right\| \le \|u\| + \delta\|e\|C\left\|\sum_{k=1}^{m} \lambda_k h_k\right\| \\ &= \|u\| + \delta C\|e\|\|V^{-1}Pu\| \le (1 + \delta C\|e\|\,\|V^{-1}\|\,\|P\|)\|u\|.\end{aligned}$$

By making use of the estimates for $\|V^{-1}\|$ and $\|P\|$, we obtain

$$\|W(u)\| \le (1 + m\delta C\|e\|\,(1-\delta C\|e\|)^{-1})\|u\| \le (1+\varepsilon)\|u\|.$$

Thus, $WVh = h$ for $h \in H$ and $\|W\|\,\|V\| \le (1+\varepsilon)^2$. In view of the arbitrariness of ε, we have the claim.

Now we turn to the case in which E is an arbitrary Banach lattice. Let again $H \subset E$, $\dim H < \infty$, and $\varepsilon > 0$. Identifying E with a subspace of E^{**} canonically, we see from the above that there is a finite-dimensional subspace U with a 1-unconditional basis and operators $V : H \to U$ and $W_0 : U \to E^{**}$ such that $W_0 V = I_H$ and $\|V\|\,\|W_0\| \le \sqrt{1+\varepsilon}$. Let $M = W_0(U)$. Clearly, $M \supset H$. By the local reflexivity principle (see 3.S.4), there exists an operator $J : M \to E$ such that $\|J\| \le \sqrt{1+\varepsilon}$ and $J(h) = h$ for $h \in H$. To complete the proof, it suffices to put $W = JW_0$. ▷

REMARK. As was proven in [3], there is a space with LUST which is not isomorphic to any Banach lattice. Such is the space of [2] which is predual to l^1 and isomorphic to none of the spaces of continuous functions.

3.9.3. The following theorem is the main tool for studying operator spaces with LUST. The symbol $\mathrm{Av}_{\gamma\in\Gamma}\,\xi_\gamma$ denotes the arithmetic mean of a family $\{\xi_\gamma\}_{\gamma\in\Gamma}$ of finite numbers; δ_{ik} stands for the Kronecker symbol.

Theorem. *Let $Z \in$ LUST and let $\{h_i\}_{i=1}^N$ be a basis for a subspace $H \subset Z$. Suppose that there are operators $R_i \in \mathscr{L}(Z, Z)$ $(i = 1, \dots, N)$, a subset $\mathscr{E}$ of the Cartesian product $\{-1, 1\}^N$, and a constant M such that*

(1) $R_i(h_j) = \delta_{ij} h_j$;

(2) *for every collection $\{a_i\}_{i=1}^N$ of numbers, the following inequality holds:*

$$\left(\sum_{i=1}^N |a_i|^2\right)^{1/2} \le M \operatorname*{Av}_{\{\varepsilon_i\}\in\mathscr{E}} \left|\sum_{i=1}^N \varepsilon_i a_i\right|.$$

Then the basis $\{h_i\}_{i=1}^N$ is $(MK)^2\chi_u(Z)$-unconditional, where

$$K = \max_{\{\varepsilon_i\}\in\mathscr{E}} \left\|\sum_{i=1}^N \varepsilon_i R_i\right\|.$$

$\triangleleft$ Let $x \in H$, $x = \sum\limits_{i=1}^N x_i h_i$, and $|\theta_i| \le 1$ $(i = 1, \dots, N)$. Estimate the quantity

$$\left\|\sum_{i=1}^N \theta_i x_i h_i\right\| = \sup\left\{\left|\sum_{i=1}^N \theta_i x_i f(h_i)\right| \mid f \in Z^*,\ \|f\| \le 1\right\}. \tag{1}$$

Put

$$S = \left|\sum_{i=1}^N \theta_i x_i f(h_i)\right|.$$

It is clear that

$$S \le \sum_{i=1}^N |x_i|\,|f(h_i)| = \sum_{i=1}^N |x_i|\,\big|(R_i^* f)(h_i)\big|. \tag{2}$$

By the definition of $\chi_u(Z)$, for every $L > \chi_u(Z)$, there is a finite-dimensional subspace U with 1-unconditional basis $\{e_n\}_{n=1}^m$, $m = \dim U$, and operators $V \in \mathscr{L}(H, U)$ and $W \in \mathscr{L}(U, Z)$ such that $WV = I_H$ and $\|V\|\,\|W\| \le L$.

Assign

$$V(h_i) = \sum_{n=1}^m a_{in} e_n, \quad W^* R_i^*(f) = \sum_{n=1}^m b_{in} e_n^*, \tag{3}$$

where $\{e_n^*\}_{n=1}^m$ is the basis for U^* dual to $\{e_n\}_{n=1}^m$. Then

$$(R_i^* f)(h_i) = \big(W^* R_i^*(f)\big)(V h_i) = \sum_{n=1}^m a_{in} b_{in}.$$

Substituting the preceding expression in (2), we have

$$S \le \sum_{i=1}^{N} |x_i| \sum_{n=1}^{m} |a_{in}|\,|b_{in}| = \sum_{n=1}^{m}\sum_{i=1}^{N} |x_i|\,|a_{in}|\,|b_{in}|$$
$$\le \sum_{n=1}^{m} \left(\sum_{i=1}^{N} |x_i a_{in}|^2\right)^{1/2} \left(\sum_{i=1}^{N} |b_{in}|^2\right)^{1/2}.$$

Now, by using hypothesis (2) of the theorem, we obtain

$$\begin{aligned} S &\le M^2 \operatorname*{Av}_{\{\varepsilon_i\}\in\mathscr{E}} \operatorname*{Av}_{\{\varepsilon'_i\}\in\mathscr{E}} \sum_{n=1}^{m} \left|\sum_{i=1}^{N} \varepsilon_i x_i a_{in}\right| \left|\sum_{i=1}^{N} \varepsilon'_i b_{in}\right| \\ &\le M^2 \max_{(\varepsilon_i),(\varepsilon'_i)\in\mathscr{E}} \left\langle \sum_{n=1}^{m} \left|\sum_{i=1}^{N} \varepsilon_i x_i a_{in}\right| e'_n, \sum_{n=1}^{m} \left|\sum_{i=1}^{N} \varepsilon'_i b_{in}\right| e^*_n \right\rangle \\ &\le M^2 \max_{(\varepsilon_i),(\varepsilon'_i)\in\mathscr{E}} \left\| \sum_{n=1}^{m} \left|\sum_{i=1}^{N} \varepsilon_i x_i a_{in}\right| e_n \right\|_U \left\| \sum_{n=1}^{m} \left|\sum_{i=1}^{N} \varepsilon'_i b_{in}\right| e^*_n \right\|_{U^*}. \end{aligned} \tag{4}$$

Estimate the norms on the right-hand side of (4). By virtue of (3) and 1-unconditionality of the bases $\{e_n\}_{n=1}^m$ and $\{e^*_n\}_{n=1}^m$, we have

$$\begin{aligned} \left\| \sum_{n=1}^{m} \left|\sum_{i=1}^{N} \varepsilon_i x_i a_{in}\right| e_n \right\|_U &= \left\| \sum_{n=1}^{m}\sum_{i=1}^{N} \varepsilon_i x_i a_{in} e_n \right\|_U = \left\| \sum_{i=1}^{N} \varepsilon_i x_i \sum_{n=1}^{m} a_{in} e_n \right\|_U \\ &= \left\| \sum_{i=1}^{N} \varepsilon_i x_i V(h_i) \right\|_U \le \|V\| \left\| \left(\sum_{i=1}^{N} \varepsilon_i R_i\right)(x) \right\|_Z \\ &\le \|V\|K\|x\|; \end{aligned}$$

analogously,

$$\begin{aligned} \left\| \sum_{n=1}^{m} \left|\sum_{i=1}^{N} \varepsilon'_i b_{in}\right| e^*_n \right\|_{U^*} &= \left\| \sum_{n=1}^{m}\sum_{i=1}^{N} \varepsilon'_i b_{in} e^*_n \right\|_{U^*} = \left\| \sum_{i=1}^{N} \varepsilon'_i \sum_{i=1}^{N} b_{in} e^*_n \right\|_{U^*} \\ &= \left\| \sum_{i=1}^{N} \varepsilon'_i W^* R_i(f) \right\| \le \|W^*\| \left\| \left(\sum_{i=1}^{N} \varepsilon'_i R^*_i\right)(f) \right\|_{Z^*} \\ &\le \|W\|K\|f\|. \end{aligned}$$

Substituting the estimates in (4), we obtain

$$S \leq \|V\| \, \|W\| M^2 K^2 \|x\| \, \|f\| \leq L(MK)^2 \|x\| \, \|f\|.$$

Now (1) implies that

$$\left\| \sum_{i=1}^{N} \theta_i x_i h_i \right\| \leq L(MK)^2 \left\| \sum_{i=1}^{N} x_i h_i \right\|.$$

Since $L > \chi_u(Z)$ was arbitrary, we arrive at the claim.

REMARK. As is easily seen from the proof, we could suppose that the operators R_i acts from Z not into Z but rather into some space Z_1 that contains Z as a subspace. In this case the supremum in (1) should be calculated over all f in the unit ball of the space Z_1^*.

Corollary. *Let $1 \leq p,\ q \leq \infty$, $E = l_n^p$, and $F = l_m^q$, let $\{e_k\}_{k=1}^n$ and $\{f_j\}_{j=1}^m$ be the canonical bases for the spaces l_n^p and l_m^q respectively, let $\{e_k'\}_{k=1}^n$ and $\{f_j'\}_{j=1}^m$ be their dual bases, and let $(\mathfrak{A}, \alpha)$ be an arbitrary Banach operator ideal (see 3.S.3). Then the basis $\{h_{kj}\}_{k,j=1}^{n,m}$ for $\mathfrak{A}(l_n^p, l_m^q)$, where $h_{kj} = e_k' \otimes h_j$, is λ-unconditional, $\lambda := 4\chi_u(\mathfrak{A}(l_n^p, l_m^q))$.*

$\lhd$ Let $H = Z = \mathfrak{A}(l_n^p, l_m^q)$ and let $R_{kj}(U) = Q_j U P_k$, where $P_k = e_k' \otimes e_k$, $Q_j = f_j' \otimes f_j$, and $U \in \mathfrak{A}(l_n^p, l_m^q)$. Put $N = mn$ and represent the set $\{-1, 1\}^N$ as

$$\{\varepsilon_{kj} \mid \varepsilon_{kj} = \pm 1,\ k = 1, \dots, n,\ j = 1, \dots, m\}.$$

Finally, assume that

$$\mathscr{E} = \{\varepsilon_k \varepsilon_j' \mid \varepsilon_k = \pm 1 \text{ for } k = 1, \dots, n; \quad \varepsilon_j' = \pm 1 \text{ for } j = 1, \dots, m\}.$$

Since $R_{kj} h_{il} = \delta_{ki} \delta_{jl} h_{il}$, the reference to the theorem provides the claimed result considering that, obviously, $M = 2$ and by virtue of Lemma 3.8.3(a) $K = 1$. $\rhd$

3.9.4. We need certain auxiliary statements to use Theorem 3.9.3 for finding out conditions under which a band of a Banach ideal (see 3.S.3) is a vector lattice. Lemma 3 is the main of the statements aiming at our goals.

DEFINITION 1. Let E and F be Banach K-spaces. An element $U \in E^* \otimes F$ is referred to as a *quasimatrix operator* if there exist pairwise disjoint functionals $x_1', \dots, x_N' \in E_+^*$, pairwise disjoint vectors $y_1, \dots, y_N \in F$, and coefficients u_{ij} for which the following equality holds:

$$U = \sum_{i,j=1}^{N} u_{ij} x_i' \otimes y_j. \tag{5}$$

Quasimatrix operators possessing representations (5) with the same x_i' and y_j will be called *similar*.

As is easily verified, the quasimatrix operators

$$U = \sum_{i,j=1}^{N} u_{ij} x_i' \otimes y_j$$

and

$$\widetilde{U} = \sum_{i,j=1}^{N} \tilde{u}_{ij} \tilde{x}_i' \otimes \tilde{y}_j$$

are similar provided that, in their representations, the functionals x_i' and $\tilde{x}_i'$ are fragments of the same functional $x_0' \in E^*$ and y_j and $\tilde{y}_j$ are fragments of the same element $y_0 \in F$. To this end, it suffices to represent the operators U and $\widetilde{U}$ as a linear combination of rank-one operators of the form $(x_i' \wedge \tilde{x}_k') \otimes (y_j \wedge \tilde{y}_l)$.

We leave to the reader the proof of the following

Lemma 1. *Let*

$$U = \sum_{i,j=1}^{N} u_{ij} x_i' \otimes y_j$$

be a quasimatrix operator. Then the operator $|U|$ has the form

$$|U| = \sum_{i,j=1}^{N} |u_{ij}| x_i' \otimes y_j.$$

Lemma 2. *Let E and F be Banach K-spaces and let $U_k \in E^* \otimes F$ and $\varepsilon_k > 0$. Then there are pairwise similar quasimatrix operators $\widetilde{U}_k$ such that $\nu(U_k - \widetilde{U}_k) < \varepsilon_k$, $k \in \mathbb{N}$, where ν is the nuclear norm.*

$\triangleleft$ To prove the lemma it suffices to apply the corollary to Lemma 3.9.2, with $\delta_k > 0$ sufficiently small, to the elements x'_{kj} and y_{kj} participating in the representation

$$U_k = \sum_{j=1}^{m_k} x'_{kj} \otimes y_{kj}.$$

As e we have to take

$$x'_0 = \sum_{k=1}^{\infty} \eta_k \sum_{j=1}^{m_k} |x_{kj}|$$

or

$$y_0 = \sum_{k=1}^{\infty} \eta_k \sum_{j=1}^{m_k} |y_{kj}|,$$

where $\eta_k > 0$ are so small that the series containing them converge. $\triangleright$

DEFINITION 2. Let $(\mathfrak{A}, \alpha)$ be a Banach operator ideal. We say that α is a 1-*tensor norm* for spaces X and Y if the inequality

$$\alpha((W \otimes I_Y)A) \leq \|W\|\alpha(A)$$

holds for every operator $W \in \mathscr{L}(X^*, X^*)$ and all $A \in X^* \otimes Y$.

REMARK. If α is the dual norm and if $\alpha(A) = \alpha(A^{**})$ for every finite rank operator then α is a 1-tensor norm for all spaces X and Y. If the space X is reflexive and X^* has the metric approximation property then α is a 1-tensor norm for spaces X and Y for each Y.

The norms π_p and ν_p in the ideals of p-absolutely summing and nuclear operators provide examples of 1-tensor norms (for all spaces X and Y).

Lemma 3. *Let $(\mathfrak{A}, \alpha)$ be a Banach operator ideal; let E and F be Banach K-spaces, and let $\mathfrak{A}(E, F) \supset Z \supset \mathfrak{A}_0(E, F)$, where $\mathfrak{A}_0(E, F)$ is the closure of $E^* \otimes F$ in $\mathfrak{A}(E, F)$. Suppose that $Z \in$ LUST and $U, V \in \mathscr{L}(E, F)$ are similar quasimatrix operators, $|V| \leq |U|$. If the norm in X is order continuous or the operator norm α is 1-tensor for spaces E, F, and $Z = \mathfrak{A}_0(E, F)$ then*

$$\alpha(V) \leq 4\chi_u(Z)\alpha(U).$$

$\triangleleft$ 1. Let the norm in E be order continuous and let

$$U = \sum_{i,j=1}^{N} u_{ij} x'_i \otimes y_j, \quad V = \sum_{i,j=1}^{N} v_{ij} x'_i \otimes y_j,$$

where $x'_i \geq 0$ and $y_j \geq 0$; $x'_i \wedge x'_j = 0$ and $y_i \wedge y_j = 0$ for $j \neq i$. By Lemma 1, we have

$$|U| = \sum_{i,j=1}^{N} |u_{ij}| x'_i \otimes y_j, \quad |V| = \sum_{i,j=1}^{N} |v_{ij}| x'_i \otimes y_j;$$

therefore,

$$|v_{ij}| \leq |u_{ij}|, \quad i,j = 1,\ldots,N. \tag{6}$$

Since the norm in E is order continuous, each functional has the band of essential positivity; moreover, disjoint functionals have disjoints bands of essential positivity (see [18]). Denote by P_i the band projection onto the band of essential positivity of the functional $x'_i \in E^*$ and by Q_j, the band projection onto the band generated by $y_j \in F$. It is clear that

$$\left\| \sum_{i=1}^{N} \varepsilon_i P_i \right\| \leq 1, \quad \left\| \sum_{j=1}^{N} \varepsilon'_j Q_j \right\| \leq 1, \quad \{\varepsilon_i\}_{i=1}^{N}, \ \{\varepsilon'_j\}_{j=1}^{N} \in \{-1,1\}^N, \tag{7}$$

$$Q_j y_l = \delta_{jl} y_l, \quad P_i^* x'_k = \delta_{ik} x_k^*. \tag{8}$$

Define operators R_{ij} with range in $\mathfrak{A}(E,F)$ by the equalities

$$R_{ij}(W) = Q_j W P_i \quad (W \in \mathfrak{A}(E,F); \ i,j = 1,\ldots,N).$$

Then

$$\sum_{i,j=1}^{N} \varepsilon_i \varepsilon'_j R_{ij}(W) = \left(\sum_{j=1}^{N} \varepsilon'_j Q_j \right) W \left(\sum_{i=1}^{N} \varepsilon_i P_i \right)$$

and by (7) we have

$$\left\| \sum_{i,j=1}^{N} \varepsilon_i \varepsilon'_j R_{ij} \right\| \leq 1, \quad \left(\{\varepsilon_i\}_{i=1}^{N}, \ \{\varepsilon'_j\}_{j=1}^{N} \in \{-1,1\}^N \right). \tag{9}$$

Put

$$h_{ij} = x'_i \otimes y_j, \quad H = \mathrm{lin}\left(\{h_{ij}\}_{i,j=1}^{N}\right).$$

Observe that by (8)

$$R_{ij}(h_{kl}) = \delta_{ik}\delta_{jl} h_{kl} \quad (i,j,k,l, = 1,\ldots,N). \tag{10}$$

Lemma 3.8.3(a) and relations (9) and (10) together justify applying Theorem 3.9.3 to the basis $\{h_{kl}\}_{k,l=1}^N$ for the space H; moreover, in the case $Z \neq \mathfrak{A}_0(E,F)$ one has to take into account the remark on Theorem 3.9.3 assuming $Z_1 = \mathfrak{A}(E,F)$. Thus, the basis $\{h_{ij}\}_{i,j=1}^N$ is $4\chi_u(Z)$-unconditional and, considering (6), we obtain the claimed estimate.

2. If the norm α is 1-tensor for the space E then operators $R_{ij} : \mathfrak{A}_0(E,F) \to \mathfrak{A}_0(E,F)$ can be constructed in the following fashion: consider the band projections S_i in E^* onto the principal bands generated by the elements $x_i' \in E^*$ and put $R_{ij}(W) = (S_i \otimes Q_j)(W)$. Arguing as above, we can easily verify that

$$\left\| \sum_{i,j=1}^N \varepsilon_i \varepsilon_j' R_{ij} \right\| \leq \left\| \sum_{i=1}^N \varepsilon_i S_i \right\| \left\| \sum_{j=1}^N \varepsilon_j' Q_j \right\| \leq 1$$

and $R_{ij} h_{kl} = \delta_{ik}\delta_{jl} h_{kl}$ and the proof can be completed as in the preceding case. ▷

3.9.5. Existence of the structure of a vector lattice in an operator space.

Theorem. *Let $(\mathfrak{A},\alpha)$ be a Banach operator ideal; let E and F be Banach K-spaces, and let $\mathfrak{A}_0(E,F)$ be the closure of $E^* \otimes F$ in $\mathfrak{A}(E,F)$. Further, let $\mathfrak{A}_0(E,F) \subset Z \subset \mathfrak{A}(E,F)$ and $Z \in$ LUST. If the norm in E is order continuous or if α is a 1-tensor norm for E, F, and $Z = \mathfrak{A}_0(E,F)$ then $\mathfrak{A}_0(E,F)$ is a sublattice in $L^\sim(E,F)$; moreover, $\alpha(|U|) \leq 4\chi_u(Z)\alpha(U)$ and $\alpha(V) \leq 4\chi_u(Z)\alpha(|U|)$ for U, $V \in \mathfrak{A}_0(E,F)$, $|V| \leq |U|$.*

◁ Let $U \in \mathfrak{A}_0(E,F)$; let U_n be finite rank operators, and let

$$\alpha(U - U_n) \underset{n\to\infty}{\longrightarrow} 0.$$

By Lemma 2 in 3.9.4, we may assume U_n to be pairwise similar quasimatrix operators. Since

$$\big| |U_n| - |U_m| \big| \leq |U_n - U_m|,$$

by Lemmas 1 and 3 from 3.9.4 we have

$$\alpha(|U_n| - |U_m|) \leq 4\chi_u(Z)\alpha(U_n - U_m) \underset{n,m\to\infty}{\longrightarrow} 0.$$

Thus, the sequence $\{|U_n|\}_{n=1}^\infty$ converges in norm α to some operator $W \in \mathfrak{A}_0(E,F)$.

Given $\tilde{x}$, $x \in E$, $|\tilde{x}| \le |x|$, we have

$$|U\tilde{x}| = \lim |U\tilde{x}| \le \lim |U_n|(|\tilde{x}|) \le \lim |U_n|(|x|) = W(|x|).$$

Therefore, the operator U is regular and $|U| \le W$. Show that $|U| = W$. Without loss of generality we may assume that

$$\sum_{n=1}^{\infty} \alpha(U_{n+1} - U_n) < \infty.$$

Moreover,

$$(|U_n| - |U|)(x) \le |U - U_n|(|x|) = \left|\sum_{k=n}^{\infty}(U_{k+1} - U_k)\right|(|x|) \le \sum_{k=n}^{\infty} |U_{k+1} - U_k|(|x|).$$

Consequently,

$$\begin{aligned} \| \, |U_n|(x) - |U|(x)\| &\le \sum_{k=n}^{\infty} \| \, |U_{k+1} - U_k| \, \| \cdot \|x\| \\ &\le \sum_{k=n}^{\infty} \alpha(|U_{k+1} - U_k|)\|x\| \\ &\le 4\chi_u(Z)\|x\| \sum_{k=n}^{\infty} \alpha(U_{k+1} - U_k) \xrightarrow[n\to\infty]{} 0. \end{aligned}$$

Thus, the sequence $\{|U_n|\}_{n=1}^{\infty}$ converges pointwise to $|U|$ and so $W = |U|$. Hence it follows that $\mathfrak{A}_0(E,F)$ is a sublattice in $L^{\sim}(E,F)$. Moreover,

$$\alpha(U) = \lim \alpha(|U_n|) \le 4\chi_u(Z) \lim \alpha_n(U) = 4\chi_u(Z)\alpha(U). \tag{11}$$

Let now $V \in \mathfrak{A}_0(E,F)$ and $|V| \le |U|$. Consider a sequence $\{V_n\}_{n=1}^{\infty}$ of finite-rank operators which converges to V with respect to the norm α. By Lemma 2 from 3.9.4, we may assume that V_n and U_n are similar quasimatrix operators. By the above, we have $|V_n| \to |V|$, $|U_n| \to |U|$, and

$$|\,|U_n| - |V_n|\,| \to |\,|U| - |V|\,| = |U| - |V|$$

in $\mathfrak{A}(E,F)$. Since $|V_n| \le |V_n| + \big||U_n| - |V_n|\big|$, we have

$$\alpha(V_n) \le 4\chi_u(Z)\alpha(|V_n| + \big||U_n| - |V_n|\big|)$$

by Lemma 3 from 3.9.4. By passing to the limit, we obtain

$$\alpha(V) \le 4\chi_u(Z)\alpha(|V| + |U| - |V|) = 4\chi_u(Z)\alpha(|U|). \tag{12}$$

Corollary. *If $\mathfrak{A}(E,F) \in$ LUST and the norm in E is order continuous then $\mathfrak{A}_0(E,F) \in$ LUST; moreover, $\chi_u(\mathfrak{A}_0(E,F)) \le 16[\chi_u(\mathfrak{A}(E,F))]^2$ and so $\mathfrak{A}_0(E,F)$ is a sublattice in $\mathfrak{A}(E,F)$.*

$\lhd$ Indeed, (11) and (12) imply that if $U, V \in \mathfrak{A}(E,F)$ and $|V| \le |U|$ then $\alpha(V) \le \lambda\alpha(U)$, where $\lambda := [4\chi_u(\mathfrak{A}(E,F))]^2$. $\rhd$

REMARK 1. As was proven in 3.9.2, every Banach lattice possesses local unconditional structure. Theorem 3.9.5 in particular implies that the converse implication is valid for a sufficiently large class of operator spaces; moreover, the order of the spaces is the conventional order of the spaces of regular operators.

REMARK 2. In general the lattice $\mathfrak{A}_0(E,F)$ is not an o-ideal in $L^\sim(E,F)$ and is not even a K_σ-space. To see this, consider the next

EXAMPLE. Let $E := l^1$, $F := L^2(0,1)$, and $\mathfrak{A}(l^1, L^2(0,1)) = \mathscr{L}(l^1, L^2(0,1))$. Then $\mathfrak{A}_0(l^1, L^2(0,1))$ is the set $K(l^1, L^2(0,1))$ of compact operators. Since $\mathscr{L}(l^1, L^2(0,1)) = L^\sim(l^1, L^2(0,1))$ (see 2.2.16), the set $K(l^1, L^2(0,1))$ is a sublattice in $L^\sim(l^1, L^2(0,1))$ by the corollary to Theorem 3.9.5. Demonstrate that $K(l^1, L^2(0,1))$ is not a K_σ-space.

$\lhd$ Let $\mathbf{1}$ be the function on $(0,1)$ identically equal to unity and let $\{r_k\}_{k=1}^\infty$ be the sequence of Rademacher functions

$$e_k = \{\delta_{kj}\}_{j=1}^\infty \in l^\infty, \quad h_n = \sum_{k=1}^n e_k, \quad h = \{1, 1, \dots\}.$$

Consider the operators V_n and V in $\mathscr{L}(l^1, L^2(0,1))$:

$$V_n = \sum_{k=1}^n e_k \otimes r_k, \quad V = \sum_{k=1}^\infty e_k \otimes r_k.$$

It is clear that $|V_n| = h_n \otimes 1$ and $|V| = h \otimes 1$. Thus,

$$0 \leq U_n = \frac{V_n + |V_n|}{2} \leq |V|, \quad U_n,\ V \in K(l^1, L^2(0,1)).$$

However,

$$\sup_n U_n = \frac{V + |V|}{2} \notin K(l^1, L^2(0,1))$$

since the operator V is not compact. $\triangleright$

3.9.6. Evaluation of the local unconditional constant for certain operator spaces.

Lemma. *Let $(\mathfrak{A}, \alpha)$ be an arbitrary Banach operator ideal; let $1 \leq p,\ q \leq \infty$, and let $U : l_n^p \to l_m^q$ be an arbitrary operator whose matrix consists of entries with modulus 1. Then*

$$\chi_u(\mathfrak{A}(l_n^p, l_m^q)) \geq \frac{1}{4} \max\left\{\frac{n^{1/p'} m^{1/q}}{\alpha(U)}, \frac{\alpha(U)}{n^{1/p'} m^{1/q}}\right\}.$$

$\triangleleft$ Let $e' = (1, 1, \dots, 1) \in l_n^{p'}$ and $f = (1, 1, \dots, 1) \in l_m^q$. Then, obviously, $|U| = e' \otimes f$ and

$$\alpha(|U|) = \|e'\|\,\|f\| = n^{1/p'} m^{1/q}.$$

From Corollary 3.9.3, we infer that

$$\alpha(|U|) \leq 4\chi_u(\mathfrak{A}(l_n^p, l_m^q))\alpha(U), \quad \alpha(U) \leq 4\chi_u(\mathfrak{A}(l_n^p, l_m^q))\alpha(|U|). \ \triangleright$$

Theorem. *Let $(\mathfrak{A}, \alpha)$ be an arbitrary Banach operator ideal; let $1 \leq p \leq \infty$, $1 \leq q < \infty$, and let I_n be the identity embedding of l_n^p into l_n^2. Then*

$$\chi_u(\mathfrak{A}(l_n^p, l_{2^n}^q)) \geq (4B_q)^{-1} \frac{n^{1/p'}}{\alpha(I_n)},$$

where B_q is the constant in the Khinchin inequality (see 3.S.1).

$\triangleleft$ Let the operator $U : l_n^p \to l_{2^n}^q$ be determined by the matrix (u_{jk}), where

$$u_{jk} = r_k\left(\frac{2j-1}{2^{n+1}}\right), \quad k = 1, \dots, n;\ j = 1, \dots, 2^n,$$

and $r_1, r_2, \dots$ be the Rademacher functions. Then, for $x = \{x_k\}_{k=1}^n \in l_n^p$, we have

$$\|Ux\| = \left(\sum_{j=1}^{2^n}\left|\sum_{k=1}^n x_k r_k\left(\frac{2j-1}{2^{n+1}}\right)\right|^q\right)^{1/q} = 2^{n/q}\left(\int_0^1\left|\sum_{k=1}^n x_k r_k(t)\right|^q dt\right)^{1/q}.$$

By virtue of the Khinchin inequality, we have

$$\|Ux\| \le 2^{n/q} B_q \left(\sum_{k=1}^n |x_k|^2\right)^{1/2}.$$

Therefore, considering the factorization of U as

$$U : l_n^p \xrightarrow{I_n} l_n^2 \xrightarrow{U_0} l_{2^n}^q,$$

where the operator U_0 is defined by the same matrix as U, we obtain the estimate $\|U_0\| \le B_q 2^{n/q}$. Consequently,

$$\alpha(U) \le \alpha(I_n)\|U_0\| \le B_q 2^{n/q}\alpha(I_n).$$

It remains to apply the lemma for $m = 2^n$ and to use the preceding inequality. $\rhd$

3.9.7. Absence of local unconditional structure in some spaces of r-absolutely summing operators. We assume that the spaces L^p and L^q considering below are constructed over arbitrary measure spaces.

(a) *If $1 < r \le 2$, $p > r'$, and $1 \le q \le \infty$ then*

$$\Pi_r(L^p, L^q) \notin \mathrm{LUST}.$$

$\lhd$ Since the space $\Pi_r(l_n^p, l_{2^n}^q)$ is isometric to a 1-complemented subspace in $\Pi_r(L^p, L^q)$, it suffices to prove that

$$\chi_u\big(\Pi_r(l_n^p, l_{2^n}^q)\big) \xrightarrow[n\to\infty]{} \infty.$$

For $q < \infty$, we have

$$\chi_u\big(\Pi_r(l_n^p, l_{2^n}^q)\big) \ge (4B_q)^{-1}\frac{n^{1/p'}}{\pi_r(I_n)}$$

by Theorem 3.9.6; here I_n is the identity embedding of l_n^p into l_n^2. Estimate $\pi_r(I_n)$. It is obvious that

$$\|I_n x\| = \left(\sum_{k=1}^{n} |x_k|^2\right)^{1/2} \leq \left(\sum_{k=1}^{n} |x_k|^r\right)^{1/r} = n^{1/r}\left(\sum_{k=1}^{n} |x_k|^r \frac{1}{n}\right)^{1/r}$$

for $x = (x_1, \ldots, x_n) \in l_n^p$. Hence it follows that $\pi_r(I_n) \leq n^{1/r}$ (see 3.1.4). This fact together with (12) yields

$$\chi_u(\Pi_r(l_n^p, l_{2^n}^q)) \geq (4B_q)^{-1} n^{(1/p')-(1/r)} \xrightarrow[n\to\infty]{} \infty.$$

For $q = \infty$, we use the following inequality that is established in Lemma 3.9.6:

$$\chi_u\big(\Pi_r\big(l_n^p, l_{2^n}^q\big)\big) \geq \frac{1}{4}\frac{\pi_r(U)}{n^{1/p'}} \tag{13}$$

To this end, estimate $\pi_r(U)$ from below. Let $x^{(k)} \in l_n^p$ be the vector coinciding with kth row of the matrix U in Theorem 3.9.6. Then $\|Ux^{(k)}\| = n$ $(k = 1, 2, \ldots, 2^n)$ and since

$$\left(\sum_{k=1}^{2^n} \|Ux^{(k)}\|^r\right)^{1/r} \leq \pi_r(U)\left\{\left(\sum_{k=1}^{2^n} |\langle x^{(k)}, x'\rangle|^r\right)^{1/r} \;\Big|\; \|x'\|_{l_n^{p'}} \leq 1\right\},$$

we have

$$n2^{n/r} \leq \pi_r(U)2^{n/r} \sup\left\{\left(2^{-n}\sum_{k=1}^{2^n} |\langle x^{(k)}, x'\rangle|^r\right)^{1/r} \;\Big|\; \|x'\|_{l_n^{p'}} \leq 1\right\}.$$

By making use of the inequality

$$2^{-n}\sum_{k=1}^{2^n} |\langle x^{(k)}, x'\rangle|^r = \int_0^1 \left|\sum_{k=1}^{n} x'_k r_k(t)\right|^r dt,$$

where $x' := (x'_1, \ldots, x'_n)$, we see that

$$\begin{aligned} n &\leq \pi_r(U) \sup\left\{\left(\int_0^1 \left|\sum_{k=1}^{n} x'_k r_k(t)\right|^r dt\right)^{1/r} \;\Big|\; \|x'\|_{l_n^{p'}} \leq 1\right\} \\ &\leq \pi_r(U) \sup\left\{\left(\sum_{k=1}^{n} |x'_k|^2\right)^{1/2} \;\Big|\; \|x'\|_{l_n^{p'}} \leq 1\right\} \leq \pi_r(U). \end{aligned}$$

This, together with (13), yields

$$\chi_u\big(\mathfrak{A}(l_n^p, l_{2^n}^\infty)\big) \geq \frac{1}{4} n^{1/p} \xrightarrow[n\to\infty]{} \infty. \ \triangleright$$

(b) *If* $1 \leq r \leq 2$, $1 \leq p < \infty$, *and* $2 < q \leq \infty$ *then* $\Pi_r(L^p, L^q) \notin \mathrm{LUST}$.

$\triangleleft$ The proof of the assertion can be made analogously as the proof of assertion (a). $\triangleright$

(c) *If* $1 \leq r \leq 2$, $1 \leq p < \infty$, *and* $2 < q \leq \infty$ *then* $\Pi_r(L^p, L^q) \notin \mathrm{LUST}$.

$\triangleleft$ As in the preceding cases, we may consider only finite-dimensional spaces and prove that

$$\chi_u\big(\Pi_r(l_{2^n}^p, l_n^q)\big) \xrightarrow[n\to\infty]{} \infty.$$

Since

$$\chi_u\big(\Pi_r(l_{2^n}^p, l_n^q)\big) = \chi_u\big(\big[\Pi_r(l_{2^n}^p, l_n^q)\big]^*\big)$$

(see 3.9.1, Remark 2) and

$$\big[\Pi_r(l_{2^n}^p, l_n^q)\big]^* = \mathfrak{A}^*(l_n^p, l_{2^n}^q),$$

where $(\mathfrak{A}^*, \alpha^*)$ is the dual ideal to (Π_r, π_r) (see 3.S.3), by Theorem 3.9.6 the following inequality is valid:

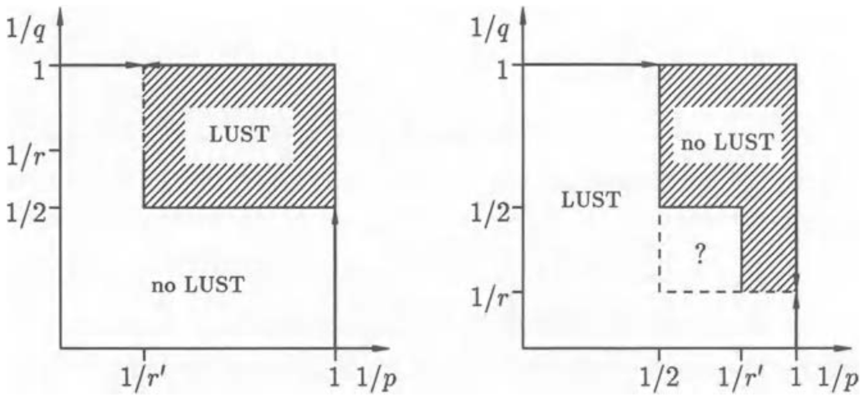
$$\chi_u\big(\Pi_r(l_{2^n}^p, l_n^q)\big) = \chi_u\big(\mathfrak{A}^*(l_n^q, l_{2^n}^p)\big) \geq (4B_q)^{-1} \frac{n^{1/q}}{\alpha^*(I_n)}, \tag{14}$$

where I_n is the identity embedding of l_n^q into l_n^2. Estimate $\alpha^*(I_n)$. Represent I_n as the product $I_n = \gamma\beta$, where γ and β are the identity embeddings of l_n^∞ into l_n^2 and of l_n^q into l_n^∞. Then for every operator $W \in \mathscr{L}(l_n^2, l_n^q)$ we have

$$|\operatorname{trace}(I_n W)| = |\operatorname{trace}(W I_n)| \leq \nu(W I_n) \leq \nu(W\gamma).$$

Considering the canonical basis $\{e_k\}_{k=1}^n$ for l_n^∞, we see that

$$\nu(W\gamma) \leq \sum_{k=1}^n \|W\gamma(e_k)\| \leq \pi_1(W\gamma).$$



(a) (b)

Fig. 1. Presence or absence of LUST in the spaces $\Pi_r(L^p, L^q)$
(a) for $1 < r \leq 2$; (b) for $2 < r < \infty$.

Therefore,

$$\begin{aligned}\alpha^*(I_n) &= \sup\{|\operatorname{trace}(I_n W) \mid W : l_n^2 \to l_n^q,\ \pi_r(W) \leq 1\}\\ &\leq \sup\{\pi_1(W\gamma) \mid W : l_n^2 \to l_n^q,\ \pi_r(W) \leq 1\}\\ &\leq \pi_{r'}(\gamma) \leq \pi_2(\gamma) \leq \sqrt{n}.\end{aligned}$$

This fact together with (14) implies that

$$\chi_u\big(\Pi_r(l_{2^n}^p, l_n^q)\big) \geq (4B_q)^{-1} n^{1/q-1/2} \xrightarrow[n\to\infty]{} \infty. \ \triangleright$$

(d) *If* $2 < r < q < \infty$ *and* $1 \leq p < \infty$ *then* $\Pi_r(L^p, L^q) \notin \mathrm{LUST}$.

$\triangleleft$ The proof of the assertion can be made in analogy with the proof of assertion (c). $\triangleright$

REMARK. Assertions (c) and (d) remain also valid for $p = \infty$; however, we will not expatiate on proving.

The results of the subsection and of the theorems of 3.8 can be visually represented as diagrams (Fig. 1)

On Fig 1,a, the boundary of the shaded rectangle is adjoint to it except the part $p = r'$ and $1 \le q < r$ indicated by the dotted line. The question remains open on presence or absence of LUST in the spaces $\Pi_r(L^{r'}, L^q)$ corresponding to the points of this part.

On Fig 1,b, the boundary of the shaded polygon is included into it except the part $1 \le p < r'$ and $q = r$ indicated by the dotted line. The question remains open on presence or absence of LUST in the spaces $\Pi_r(L^p, L^q)$ corresponding to the points of the part and to the points of the square $r' < p$ and $2 < q \le r$.

3.9.8. Uniqueness of operator ideals with LUST.

Theorem. *Let $1 < p \le 2$ and $1 < q \le 2$ and let $(\mathfrak{A}, \alpha)$ be a Banach operator ideal. If the space $\mathfrak{A}(L^p, L^q)$ has LUST then $\mathfrak{A}(L^p, L^q) = \Pi_2(L^p, L^q)$.*

$\lhd$ Let $\mathfrak{A}_0(L^p, L^q)$ be the closure of the set of finite-rank operators in $\mathfrak{A}(L^p, L^q)$. By Theorem 3.9.5, $\mathfrak{A}_0(L^p, L^q)$ is a sublattice in $L^\sim(L^p, L^q)$. Check that

$$\mathfrak{A}_0(L^p, L^q) = \Pi_2(L^p, L^q).$$

It is obvious that $\mathfrak{A}_0(L^p, L^q) \subset \mathscr{L}^\sim_{\mathrm{st}}(L^p, L^q)$ and since $\mathscr{L}^\sim_{\mathrm{st}}(L^p, L^q) \subset \Pi_2(L^p, L^q)$ by 3.7.2(e), we have

$$\mathfrak{A}_0(L^p, L^q) \subset \Pi_2(L^p, L^q).$$

Let $(\mathfrak{A}^*, \alpha^*)$ be the dual ideal to $(\mathfrak{A}, \alpha)$. Since $(\mathfrak{A}_0(L^p, L^q))^* = \mathfrak{A}^*(L^p, L^q)$ (see 3.S.3), $\mathfrak{A}^*(L^q, L^p)$ is a lattice. Moreover, if an operator $V \in \mathfrak{A}^*(L^p, L^p)$ is positive, $\langle Vy, x' \rangle = \operatorname{trace} VU \ge 0$ for every rank-one positive operator $U = x' \otimes y$, where $x' \in L^{p'}_+$ and $y \in L^q_+$. Thus, the order in $\mathfrak{A}^*(L^q, L^p)$ as in the dual space to $\mathfrak{A}_0(L^p, L^q)$ coincides with the order induced from $L^\sim(L^q, L^p)$ and $\mathfrak{A}^*(L^q, L^p) \subset L^\sim(L^q, L^p)$. Hence

$$\mathfrak{A}^*(L^q, L^p) \subset \mathscr{L}^\sim_{\mathrm{st}}(L^q, L^p) \subset \Pi_2(L^q, L^p). \tag{15}$$

On the other hand, since $\mathfrak{A}_0(L^p, L^q) \subset \Pi_2(L^p, L^q)$ and $[\Pi_2(L^p, L^q)]^* \supset \Pi_2(L^q, L^p)$ (see 3.S.3), we have

$$\mathfrak{A}^*(L^q, L^p) = [\mathfrak{A}_0(L^p, L^q)]^* \supset [\Pi_2(L^p, L^q)]^* \supset \Pi_2(L^q, L^p).$$

This fact, together with (15), yields

$$[\mathfrak{A}_0(L^p, L^q)]^* = \Pi_2(L^q, L^p) = [\Pi_2(L^p, L^q)]^*$$

which implies the equality $\mathfrak{A}_0(L^p, L^q) = \Pi_2(L^p, L^q)$.

With the help of the closed graph theorem, we easily deduce that the norms π_2 and α are equivalent and the inequality $\pi_2(U) \le C\alpha(U)$ holds in particular for some number C and every operator $U \in \mathfrak{A}_0(L^p, L^q)$.

To complete the proof, it now suffices to verify that $\mathfrak{A}(L^p, L^q) \subset \Pi_2(L^p, L^q)$. Let $(P_\theta)_{\theta\in\Theta}$ be a net of finite rank operators in L^p satisfying the following conditions: $x = \lim_\theta P_\theta x$ for every $x \in L^p$ and $\|P_\theta\| \le 1$ for every $\theta \in \Theta$. If $U \in \mathfrak{A}(L^p, L^q)$ and $U_\theta := UP_\theta$ then $U_\theta \in \mathfrak{A}_0(L^p, L^q)$ and $Ux = \lim_\theta U_\theta x$ for each $x \in L^p$. Moreover, $\pi_2(U_\theta) \le C\alpha(UP_\theta) \le C\alpha(U)$. Since the unit ball of the space $\Pi_2(L^p, L^q)$ is closed under strong operator convergence (see 3.1.1), we have $U \in \Pi_2(L^p, L^q)$. $\triangleright$

Remark 1. The following assertions complement Theorem 3.9.8 (the symbol $K(X,Y)$ stands for the set of compact operators):

(1) *If* $\mathfrak{A}(L^2, L^1) \in \mathrm{LUST}$ *then* $\mathfrak{A}_0(L^2, L^1) = N(L^2, L^1)$.

(2) *If* $\mathfrak{A}(L^1, L^2) \in \mathrm{LUST}$ *then* $\mathfrak{A}_0(L^1, L^2) = K(L^1, L^2)$.

(3) *If* $\mathfrak{A}(L^1, L^\infty) \in \mathrm{LUST}$ *then* $\mathfrak{A}_0(L^1, L^\infty) = K(L^1, L^\infty)$.

(4) *If* $\mathfrak{A}(L^\infty, L^1) \in \mathrm{LUST}$ *then* $\mathfrak{A}_0(L^\infty, L^1) = N(L^\infty, L^1)$.

$\triangleleft$ By Theorems 3.9.5 and 3.7.4(d), we have

$$\mathfrak{A}_0(L^2, L^1) \subset \mathscr{L}_{\mathrm{st}}^{\sim}(L^2, L^1) \subset \Pi_2(L^2, L^1),$$

whence $N(L^2, L^1) \subset \mathfrak{A}(L^2, L^1)$ in view of the closure of the unit ball in $\Pi_2(L^2, L^1)$ under pointwise convergence. It remains to observe that $\mathfrak{A}(L^2, L^1) \subset \Pi_2(L^2, L^1)$ and $\Pi_2(L^2, L^1) = N(L^2, L^1)$.

To prove the forth assertion we notice that $\mathfrak{A}_0(L^\infty, L^1) = N(L^\infty, L^1)$ since $\mathfrak{A}_0(L^\infty, L^1) \subset \mathscr{L}_{\mathrm{st}}^{\sim}(L^\infty, L^1)$ by Theorem 3.9.5 (see [49, Chapter IV, § 5]).

Proofs of assertions (2) and (3) can be easily deduced from (1) and (4) by duality. $\triangleright$

Remark 2. The hypotheses $\mathfrak{A}(L^p, L^q) \in \mathrm{LUST}$ in Theorem 3.5.8 and in Remark 1 on it may be replaced with the hypotheses $\mathfrak{A}_0(L^p, L^q) \in \mathrm{LUST}$ which is weaker to some extent as is seen from the corollary to Theorem 3.9.5.

Corollary. *Let* $2 \le p$ *and* $q < \infty$. *If* $\mathfrak{A}(L^p, L^q) \in \mathrm{LUST}$ *then*

$$\mathfrak{A}(L^p, L^q) = \{U \in \mathscr{L}(L^p, L^q) \mid U^* \in \Pi_2(L^{q'}, L^{p'})\} \equiv \Pi_2^{\mathrm{dual}}(L^p, L^q).$$

Moreover, if $\mathfrak{A}(L^\infty, L^2) \in$ LUST *then* $\mathfrak{A}_0(L^\infty, L^2) = N(L^\infty, L^2)$; *if* $\mathfrak{A}(L^2, L^\infty) \in$ *then* $\mathfrak{A}_0(L^2, L^\infty) = K(L^2, L^\infty)$.

The corollary is straightforward from Theorem 3.9.8, Remark 1 on it, and the definition of dual ideal.

The results of Theorem 3.9.8, of the Remark, and of the Corollary to it can be conveniently exhibited as the diagram (see Fig. 2), where at the point $(1/p, 1/q)$, the results are reflected for ideals of operators acting between L^p and L^q and the shaded domain corresponds to uniqueness of an ideal in LUST. It is easy to check that, for boundary points situated between $(1/2, 1)$ and $(1, 1)$ and between $(1, 1/2)$ and $(1, 1)$, the spaces $N(L^p, L^1)$, $\Pi_2(L^p, L^1)$ and $N_2(L^1, L^p)$, $K(L^1, L^p)$ for $1 < p < 2$ and the spaces $N(L^1, L^1)$, $N_2(L^1, L^1)$, and $K(L^1, L^1)$ for $p = 1$ (pairwise disjoint) serve as vector lattices with everywhere dense set of finite rank operators; here $N_2(X, Y)$ stands for the closure of the set of finite rank operators in the space $\Pi_2(X, Y)$. By duality, one can easily obtain the corresponding results for the points $(0, 0)$ and $(0, 1/p)$, $(1/p, 0)$ for $2 < p < \infty$. The uniqueness of operator ideals with vector lattice structure fails at the points $(1/p', 1/p)$ for $1 < p < \infty$, $p \neq 2$ either. For $1 < p < 2$, the ideals $\Pi_p(L^{p'}, L^p)$ and $\Pi_1(L^{p'}, L^p)$ provide the sought examples. For the other points of the diagram, the question of uniqueness seems to remain open.

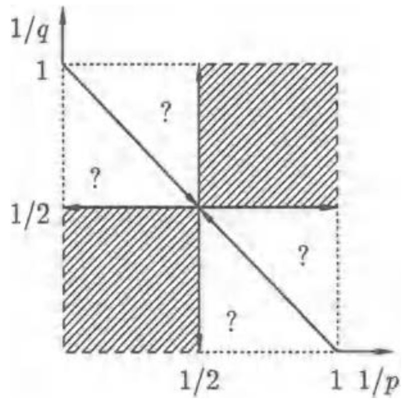


Fig. 2

3.S. Supplement to Chapter 3

3.S.1. The Khinchin-Kahane inequality. The classical Khinchin inequality (see [15, 16]) reads as follows: for every p, $0 < p < \infty$, there exist positive constants A_p and B_p such that

$$A_p \left(\sum_{k=1}^{n} |c_k|^2 \right)^{1/2} \leq \left\| \sum_{k=1}^{n} c_k r_k \right\|_p \leq B_p \left(\sum_{k=1}^{n} |c_k|^2 \right)^{1/2}$$

for all $n \in \mathbb{N}$ and all numbers $c_1, \ldots, c_n$, where $r_1, r_2, \ldots$ are the Rademacher functions. In what follows we assume that the constants A_p and B_p are the best of those possible. It is clear that $A_p = 1$ for $p \geq 2$ and $B_p = 1$ for $p \leq 2$. By direct calculations one can easily establish that $B_4 = \sqrt[4]{3}$ (see, for instance, [16]). In [55], it was proven that $A_1 = 2^{-1/2}$ and, in [50], A_p and B_p were calculated for all p. Observe that $B_p \sim (p/e)^{1/2}$ as $p \to \infty$.

A generalization of the Khinchin inequality for vector-valued functions means that, for every p, $0 < p < \infty$, there are positive constants α_p and β_p such that the inequality

$$\alpha_p \left(\int_0^1 \left\| \sum_{k=1}^n r_k(t) x_k \right\|^2 dt \right)^{1/2} \leq \left(\int_0^1 \left\| \sum_{k=1}^n r_k(t) x_k \right\|^p dt \right)^{1/p}$$
$$\leq \beta_p \left(\int_0^1 \left\| \sum_{k=1}^n r_k(t) x_k \right\|^2 dt \right)^{1/2} \tag{1}$$

holds for all normed space X, all $n \in \mathbb{N}$, and all vectors $x_1, \ldots, x_n \in X$ This inequality we shall call the *Khinchin-Kahane inequality.*

As is easily verified, the left-hand side in inequality (1) is a corollary to the right-hand one. Therefore, it suffices to prove that

$$\left(\int_0^1 \left\| \sum_{k=1}^n r_k(t) x_k \right\|^p dt \right)^{1/p} \leq \sqrt{p-1} \left(\int_0^1 \left\| \sum_{k=1}^n r_k(t) x_k \right\|^2 dt \right)^{1/2} \tag{2}$$

for $p \geq 2$. The result (where $\sqrt{p-1}$ was replaced with p) was essentially obtained in [17]. The proof presented bellow is taken from [43] (see also [23]).

Lemma. *Let $2 \leq p < \infty$ and $\varepsilon = (p-1)^{-1/2}$. Then the inequality*

$$\left(\frac{|1 + \varepsilon z|^p + |1 - \varepsilon z|^p}{2} \right)^{1/p} \leq (1 + |z|^2)^{1/2} \tag{3}$$

holds for every $z \in \mathbb{C}$.

$\lhd$ We propose to the reader to convince himself that it suffices to prove inequality (3) only for $z \in \mathbb{R}$. (If we put $z = |z|e^{i\varphi}$, $0 \leq \varphi \leq 2\pi$, and fix $|z|$ then the left-hand side in inequality (33) attains the greatest value at $\varphi = 0$.) By putting $\varepsilon z = t$, we see that (3) is equivalent to the inequality

$$\left(\frac{|1+t|^p + |1-t|^p}{2}\right)^{1/p} \leq (1+(p-1)t^2)^{1/2}, \quad t \geq 0. \tag{4}$$

First we prove inequality (4) for $0 \leq t \leq 1$. Assign $s = (1-t)/(1+t)$. Then we can rewrite (4) as

$$2\left(\frac{1+s^p}{2}\right)^{1/p} \leq (p+2(p-2)s+ps^2)^{1/2}, \quad 0 \leq s \leq 1. \tag{5}$$

It is clear that (5) is equivalent to the inequality

$$\psi(s) \leq \psi(1) = 2^{1/p-1}, \quad 0 \leq s \leq 1, \tag{6}$$

where $\psi(s) = (1+s^p)^{1/p}(p-2(p-2)s+ps^2)^{-1/2}$. It is easy to verify that the sign of $\psi'(s)$ for $0 < s < 1$ coincides with the sign of the function $h(s) = ps^{p-1} - (p-2)s^p - ps + (p-2)$. Since h is convex on the interval $(0,1)$ and $h(1) = h'(1) = 0$ then $h(s) > 0$ for $0 \leq s < 1$. Thus, ψ increases and inequality (6) holds. Consequently, inequality (4) holds for $0 \leq t \leq 1$.

Let now $t > 1$. Put $\tau = 1/t$. Then $0 < \tau \leq 1$ and

$$\begin{aligned}\left(\frac{|1+t|^p + |1-t|^p}{2}\right)^{1/p} &= t\left(\frac{|1+\tau|^p + |1-\tau|^p}{2}\right)^{1/p} \leq t(1+(p-1)\tau^2)^{1/2} \\ &= (p-1+t^2)^{1/2} \leq (1+(p-1)t^2)^{1/2}. \ \rhd\end{aligned}$$

Corollary. *Let* $p \geq 2$, $\varepsilon = (p-1)^{-1/2}$, *and* $a, b, c, d \in \mathbb{C}$. *Then*

$$\left(\frac{|a+\varepsilon c|^p + |b-\varepsilon d|^p}{2}\right)^{1/p} \leq \left(\frac{A^2+B^2}{2}\right)^{1/2},$$

where $A = \max\{|a+c|, |b+d|\}$ *and* $B = \max\{|a-c|, |b-d|\}$.

◁ Assign $\alpha = a+c$, $\beta = a-c$, $\gamma = b+d$, $\delta = b-d$, and $t = (A-B)/(A+B)$. Then

$$\begin{aligned}|a+\varepsilon c|^p + |b-\varepsilon d|^p &= \left|\frac{\alpha}{2}(1+\varepsilon)+\frac{\beta}{2}(1-\varepsilon)\right|^p + \left|\frac{\gamma}{2}(1-\varepsilon)+\frac{\delta}{2}(1+\varepsilon)\right|^p \\ &\le \left(\frac{A}{2}(1+\varepsilon)+\frac{B}{2}(1-\varepsilon)\right)^p + \left(\frac{A}{2}(1-\varepsilon)+\frac{B}{2}(1+\varepsilon)\right)^p \\ &= \left(\frac{A+B}{2}+\varepsilon\frac{A-B}{2}\right)^p + \left(\frac{A+B}{2}-\varepsilon\frac{A-B}{2}\right)^p \\ &= \left(\frac{A+B}{2}\right)^p [(1-\varepsilon t)^p + (1+\varepsilon t)^p].\end{aligned}$$

By using the lemma, we obtain

$$\begin{aligned}\left(\frac{|a+\varepsilon c|^p + |b-\varepsilon d|^p}{2}\right)^{1/p} &= \frac{A+B}{2}\left[\frac{(1+\varepsilon t)^p + (1-\varepsilon t)^p}{2}\right]^{1/p} \\ &\le \frac{A+B}{2}(1+t^2)^{1/2} = \left[\left(\frac{A+B}{2}\right)^2 + \left(\frac{A-B}{2}\right)^2\right]^{1/2} \\ &= \left(\frac{A^2+B^2}{2}\right)^{1/2}. \ \triangleright\end{aligned}$$

Theorem. *Let $p \ge 2$; let $\varepsilon = (p-1)^{-1/2}$; let X be an arbitrary normed space, and let $x_0, x_1, \dots, x_n \in X$. Then*

$$I_n = \left(\int_0^1 \left\|x_0 + \varepsilon\sum_{k=1}^n r_k(t)x_k\right\|^p dt\right)^{1/p} \le \left(\int_0^1 \left\|x_0 + \sum_{k=1}^n r_k(t)x_k\right\|^2 dt\right)^{1/2}. \quad (7)$$

For $x_0 = 1$, the inequality coincides with (2).

◁ Prove the theorem by induction. Let $n = 1$. Then inequality (7) means that

$$\left(\frac{\|x_0+\varepsilon x_1\|^p + \|x_0-\varepsilon x_1\|^p}{2}\right)^{1/p} \le \left(\frac{\|x_0+x_1\|^2 + \|x_0-x_1\|^2}{2}\right)^{1/2}. \quad (7')$$

Let $f, g \in X^*$ be functionals such that

$$\|f\| = \|g\| = 1, \quad f(x_0+\varepsilon x_1) = \|x_0+\varepsilon x_1\|, \quad g(x_0-\varepsilon x_1) = \|x_0-\varepsilon x_1\|.$$

Then

$$\|x_0 + \varepsilon x_1\|^p + \|x_0 - \varepsilon x_1\|^p = |f(x_0 + \varepsilon f(x_1)|^p + |g(x_0) - \varepsilon g(x_1)|^p.$$

By the corollary to the lemma with

$$\begin{aligned} A &= \max\{|f(x_0) + f(x_1)|, |g(x_0 + g(x_1)|\} \le \|x_0 + x_1\|, \\ B &= \max\{|f(x_0) - f(x_1)|, |g(x_0 - g(x_1)|\} \le \|x_0 - x_1\|, \end{aligned}$$

we convince ourselves that inequality (7′) is true.

Now we assume that inequality (7) holds for all vectors $x_0, x_1, \ldots, x_n \in X$ for $n \le m$ and prove it for $n = m+1$. In view of the equality

$$I_{m+1}^p = \frac{1}{2}\int_0^1 \left[\left\|x_0 + \varepsilon x_{m+1} + \varepsilon \sum_{k=1}^m r_k(t)x_k\right\|^p + \left\|x_0 - \varepsilon x_{m+1} + \varepsilon \sum_{k=1}^m r_k(t)x_k\right\|^p\right] dt,$$

and inductive hypothesis, we obtain

$$\begin{aligned} I_{m+1} &= 2^{-1/p}\left(\int_0^1 \left\|x_0 + \varepsilon x_{m+1} + \varepsilon \sum_{k=1}^m r_k(t)x_k\right\|^p dt \right. \\ &\qquad \left. + \int_0^1 \left\|x_0 + \varepsilon x_{m+1} + \varepsilon \sum_{k=1}^m r_k(t)x_k\right\|^p dt\right)^{1/p} \\ &\le 2^{-1/p}\left(\left(\int_0^1 \left\|x_0 + \varepsilon x_{m+1} + \sum_{k=1}^m r_k(t)x_k\right\|^2 dt\right)^{p/2} \right. \\ &\qquad \left. + \left(\int_0^1 \left\|x_0 + \varepsilon x_{m+1} + \sum_{k=1}^m r_k(t)x_k\right\|^2 dt\right)^{p/2}\right)^{1/p}. \end{aligned}$$

Applying to the right-hand side the inequality

$$\left(\left|\int_0^1 f(t)\,dt\right|^{p/2} + \left|\int_0^1 g(t)\,dt\right|^{p/2}\right)^{2/p} \le \int_0^1 (|f(t)|^{p/2} + |g(t)|^{p/2})^{2/p}\,dt$$

with

$$f = \left\| x_0 + \varepsilon x_{m+1} + \sum_{k=1}^{m} r_k x_k \right\|^2, \quad g = \left\| x_0 - \varepsilon x_{m+1} + \sum_{k=1}^{m} r_k x_k \right\|^2,$$

we see that

$$I_{m+1} \leq \left(\int_0^1 \left(\frac{1}{2} \left(\left\| x_0 + \sum_{k=1}^{m} r_k(t) x_k + \varepsilon x_{m+1} \right\|^p \right.\right.\right.$$
$$\left.\left.\left. + \left\| x_0 + \sum_{k=1}^{m} r_k(t) x_k - \varepsilon x_{m+1} \right\|^p \right) \right)^{2/p} dt \right)^{1/2}.$$

Estimating the integrand on the right-hand side with the help of inequality (7′) (where the vector x_0 is replaced with

$$x_0 + \sum_{k=1}^{m} r_k(t) x_k$$

and x_1, with x_{m+1}), we obtain the claim. ▷

REMARK. Inequality (7) admits the following generalization:

$$\left(\int_0^1 \left\| x_0 + \varepsilon \sum_{k=1}^{m} r_k(t) x_k + \varepsilon^2 \sum_{1 \leq k < l \leq n} r_k(t) r_l(t) x_{kl} \right\|^p dt \right)^{1/p}$$
$$\leq \left(\int_0^1 \left\| x_0 + \sum_{k=1}^{m} r_k(t) x_k + \sum_{1 \leq k < l \leq n} r_k(t) r_l(t) x_{kl} \right\|^2 dt \right)^{1/2}.$$

The proof of the inequality coincides essentially with the proof of the theorem. One can prove some further generalizations of (7) which use the Walsh functions representable as a product of at most three, at most four etc. Rademacher functions. Thus,

$$\left(\int_0^1 \left\| \sum_{1 \leq k_1 < k_2 < \cdots < k_m \leq n} r_{k_1}(t) r_{k_2}(t) \ldots r_{k_m}(t) x_{k_1} x_{k_2} \ldots x_{k_m} \right\|^p dt \right)^{1/p}$$
$$\leq p^{m/2} \left(\int_0^1 \left\| \sum_{1 \leq k_1 < k_2 < \cdots < k_m \leq n} r_{k_1}(t) r_{k_2}(t) \ldots r_{k_m}(t) x_{k_1} x_{k_2} \ldots x_{k_m} \right\|^2 dt \right)^{1/2}$$

for all $m \le n$ and all vectors $x_{k_1 k_2 \dots k_m} \in X$.

As was observed, the left-hand side in inequality (1) is a consequence of the right-hand one. Prove this for $p = 1$ (the general case can be proven similarly).

Let $f \in L^4(0,1)$ and $\|f\|_4 \le a\|f\|_2$. Then

$$\|f\|_2^2 = \int_0^1 |f(t)|^{4/3}|f(t)|^{2/3}\,dt \le \left(\int_0^1 |f(t)|^4\,dt\right)^{1/3}\left(\int_0^1 |f(t)|\,dt\right)^{2/3}$$
$$\le (a\|f\|_2)^{4/3}\|f\|_1^{2/3};$$

i.e., $\|f\|_2 \le a^2\|f\|_1$. By putting

$$f = \left\|\sum_{k=1}^n r_k x_k\right\|, \quad a = \sqrt{3},$$

we obtain the claim with $\alpha_1 = 1/3$.

3.S.2. 2-Cotype spaces.

DEFINITION. We say that a normed space X is of (Rademacher) 2-*cotype* if there is a number C such that the inequality

$$\left(\sum_{k=1}^n \|x_k\|^2\right)^{1/2} \le C\int_0^1 \left\|\sum_{k=1}^n r_k(t)x_k\right\| dt,$$

holds for all $n \in \mathbb{N}$ and $x_1, \dots, x_n \in X$, where $r_1, r_2, \dots$ are the Rademacher functions. The least possible constant C is said to be the 2-*cotype constant* for the space X and is denoted by $C_2(X)$.

By the Khinchin-Kahane inequality, the integral

$$\int_0^1 \left\|\sum_{k=1}^n r_k(t)x_k\right\| dt$$

in the definition of 2-cotype space may be replaced with

$$\left(\int_0^1 \left\|\sum_{k=1}^n r_k(t)x_k\right\|^p dt\right)^{1/p}$$

for every $p > 0$. Recall that in a Hilbert space the following identity holds:

$$\sum_{k=1}^{n} \|x_k\|^2 = \int_0^1 \left\| \sum_{k=1}^{n} r_k(t) x_k \right\|^2 dt.$$

Theorem. *For $1 \leq p \leq 2$, the space $L^p(T,\mu)$ is a 2-cotype space; moreover, $C_2(L^p(T,\mu)) \leq \sqrt{2}$.*

Let $x_1, \ldots, x_n \in L^p(T,\mu)$ and $y_k = |x_k|^2$. Then

$$\sum_{k=1}^{n} \|x_k\|_p^2 = \sum_{k=1}^{n} \|y_k\|_{p/2}.$$

Since $p/2 \leq 1$, we obtain

$$\sum_{k=1}^{n} \|x_k\|_p^2 \leq \left\| \sum_{k=1}^{n} y_k \right\|_{p/2} = \left(\int_T \left(\sum_{k=1}^{n} |x_k(t)|^2 \right)^{p/2} d\mu(t) \right)^{2/p} \tag{8}$$

by the reverse triangle inequality. Estimating the sum $\sum_{k=1}^{n} |x_k(t)|^2$ by the Khinchin inequality, we see that

$$\left(\sum_{k=1}^{n} |x_k(t)|^2 \right)^{1/2} \leq A_1^{-1} \int_0^1 \left| \sum_{k=1}^{n} r_k(u) x_k(t) \right| du,$$

where $A_1^{-1} = \sqrt{2}$. This fact together with (8) yields

$$\left(\sum_{k=1}^{n} \|x_k\|_p^2 \right)^{1/2} \leq \sqrt{2} \left(\int_T \left(\int_0^1 \left| \sum_{k=1}^{n} r_k(u) x_k(t) \right| du \right)^p d\mu(t) \right)^{1/p}. \tag{9}$$

Since

$$\left\| \int_0^1 f(\cdot,u)\, du \right\| \leq \int_0^1 \|f(\cdot,u)\|\, du,$$

from (9) it follows that

$$\left(\sum_{k=1}^{n}\|x_k\|_p^2\right)^{1/2} \le \sqrt{2}\int_0^1\left(\int_T\left|\sum_{k=1}^{n}r_k(u)x_k(t)\right|^p d\mu(t)\right)^{1/p} du$$

$$= \sqrt{2}\int_0^1\left\|\sum_{k=1}^{n}r_k(u)x_k\right\|_p du. \ \triangleright$$

3.S.3. Banach operator ideals. Recall that the symbol $\mathscr{L}(X,Y)$ stands for the set of all continuous linear operators acting from a Banach space X into a Banach space Y.

DEFINITION. A class $\mathfrak{A}$ of operators acting between arbitrary Banach spaces is said to be an *operator ideal* if it satisfies the following conditions:

(1) $\mathfrak{A}(X,Y) = \mathfrak{A}\cap\mathscr{L}(X,Y)$ is a linear set;

(2) $\mathfrak{A}$ contains all finite rank operators;

(3) if $W \in \mathscr{L}(X_0,X)$, $V \in \mathscr{L}(Y,Y_0)$, and $U \in \mathfrak{A}(X,Y)$ then $VUW \in \mathfrak{A}(X_0,Y_0)$.

The set $\mathfrak{A}(X,Y)$ is called a *component of operator ideal* $\mathfrak{A}$.

DEFINITION. Given an operator ideal $\mathfrak{A}$, a nonnegative function α is called a *norm* in the operator ideal if

(1) the restriction of α to $\mathfrak{A}(X,Y)$ is a norm in $\mathfrak{A}(X,Y)$ for arbitrary X and Y;

(2) $\alpha(x'\otimes y) = \|x'\|\,\|y\|$ for every rank-one operator $x'\otimes y : X \to Y$;

(3) if $W \in \mathscr{L}(X_0,X)$, $V \in \mathscr{L}(Y,Y_0)$, and $U \in \mathfrak{A}(X,Y)$ then $\alpha(VUW) \le \|V\|\,\|W\|\,\alpha(U)$.

If the normed space $\mathfrak{A}(X,Y)$ is complete for arbitrary Banach spaces X and Y then the pair $(\mathfrak{A},\alpha)$ is called a *Banach operator ideal.*

The class Π_p $(p \ge 1)$ with the norm π_p and the class of nuclear operators with the nuclear norm provide examples of Banach operator ideals.

A norm α in a Banach operator ideal $(\mathfrak{A},\alpha)$ is called *dual* if there exists a Banach operator ideal $(\mathfrak{B},\beta)$ such that $\alpha(U) = \beta(U^*)$ for every operator $U \in \mathfrak{A}$.

Let $(\mathfrak{A},\alpha)$ be a Banach operator ideal. We say that an operator $U \in \mathscr{L}(X,Y)$ *belongs to the dual operator ideal* $\mathfrak{A}^*$ if there exists a constant C such that

$$|\operatorname{trace}(UAVB)| \le C\|A\|\alpha(V)\|B\|,$$

where $V \in \mathfrak{A}(Y_0, X_0)$, $A \in \mathscr{L}(X_0, X)$, $\operatorname{rank} A < \infty$, $B \in \mathscr{L}(Y, Y_0)$, and $\operatorname{rank} B < \infty$. Put $\alpha^*(U) = \inf\{C\}$.

One can prove that the pair $(\mathfrak{A}^*, \alpha^*)$ is a Banach operator ideal (see [42, Theorem 9.1.3]). It is said to be the *dual ideal to the ideal* $(\mathfrak{A}, \alpha)$. By using the notion of dual ideal, it is possible to describe dual spaces to components of a given ideal in a series of cases. In the cases when the trace of a nuclear operator in a space X is correctly defined, one can try to establish a duality between the spaces $\mathfrak{A}(X, Y)$ and $\mathfrak{A}^*(Y, X)$ by the formula $\langle U, V\rangle = \operatorname{trace} VU$, where $U \in \mathfrak{A}(X, Y)$ and $V \in \mathfrak{A}^*(Y, X)$. Thus, we would like to identify an operator $V \in \mathfrak{A}^*(Y, X)$ with the functional F_V defined on $\mathfrak{A}(X, Y)$ by the equality $F_V = \operatorname{trace} VU$ $(U \in \mathfrak{A}(X, Y))$. This program is implementable in a wide class of cases. The following theorem is a particular case of Theorem 10.3.5 in [42].

Theorem. *Let $1 < p < \infty$ and $1 \le q \le \infty$; let $(\mathfrak{A}, \alpha)$ be an arbitrary Banach operator ideal, and let $\mathfrak{A}_0(L^p, L^q)$ be the closure of the set of finite rank operators in $\mathfrak{A}(L^p, L^q)$.*

(1) *If $V \in \mathfrak{A}^*(L^q, L^p)$ then $VU \in N(L^p, L^p)$ for every operator $U \in \mathfrak{A}_0(L^p, L^q)$ and the equality*

$$F_V = \operatorname{trace} VU \quad (U \in \mathfrak{A}_0(L^p, L^q)) \tag{$*$}$$

defines a continuous linear functional on $\mathfrak{A}_0(L^p, L^q)$; moreover, $\|F_V\| = \alpha^*(V)$.

(2) *For every functional $F \in [\mathfrak{A}_0(L^p, L^q)]^*$, there exists a unique operator $V \in \mathfrak{A}^*(L^p, L^q)$ such that F is representable in the form $(*)$.*

The next theorem is a particular case of the result obtained in [39].

Theorem. *Let $1 < p < \infty$ and $L^p = L^p(T, \mu)$. Then $[\Pi_2(L^p, Y)]^* = \Pi_2(Y, L^p)$.*

$\lhd$ We confine ourselves to a part of the proof of the theorem and establish inclusion $\Pi_2(Y, L^p) \subset [\Pi_2(L^p, Y)]^*$ which is only used in Section 3.9 (as above, we assume that a duality between spaces $\Pi_2(L^p, Y)$ and $\Pi_2(Y, L^p)$ is provided by the trace).

Let $V \in \Pi_2(Y, L^p)$. By Theorem 3.3.6, the operator VU is nuclear for every $U \in \Pi_2(L^p, Y)$. Moreover,

$$|\operatorname{trace}(VU)| \le \nu(VU) \le \pi_2(V)\pi_2(U).$$

Thereby we obtain the sought inclusion.

The reader can find a detailed exposition of the theory of Banach operator ideals in [11, 42].

3.S.4. The local reflexivity principle. A number of statements are consolidated by this title which have their origin in a result obtained by Lindenstrauss and Rosenthal. Its various refinements and generalizations are exposed in [7, 15]. We use the variant of the local reflexivity principle in the formulation presented below. We suppose that the space X is canonically identified with a subspace of X^{**}.

Theorem. *Let X be an arbitrary Banach space; let $L \subset X^{**}$, $\dim L < \infty$; let $M \subset X^{*}$, $\dim M < \infty$, and let $\varepsilon > 0$. There exists an operator $U : L \to X$ such that*

(1) $(1-\varepsilon)\|x''\| \le \|Ux''\| \le (1+\varepsilon)\|x''\|$ *for every* $x'' \in L$;

(2) $\langle Ux'', x'\rangle = \langle x', x''\rangle$ *for all* $x' \in M$ *and* $x'' \in L$;

(3) $Ux'' = x''$ *for* $x'' \in L \cap X$.

3.S.5. The Fan Ky lemma.

Lemma (see [5]). *Let K be a compact convex subset of an Hausdorff topological vector space and let Φ be a convex set of convex functions lower semicontinuous on K. If each of the functions in Φ takes a nonpositive value on K then there is a point $x_0 \in K$ at which the values of all functions in Φ are nonpositive.*

Comments

3.1. The concept of 1-absolutely summing operator (under the name of right semi-integral operator) appeared in the fundamental article [12] by A. Grothendieck. Independently of tensor product technique, 1-absolutely summing operators were studied by A. Pietsch (see [41], where further literature directions are given), who introduced the notion of p-absolutely summing operator in [40]. A rather complete exposition of results obtained in the field in the last 20 years was given in the Pietsch monograph [42].

3.2. Theorem 3.2.3 was obtained in [40], Theorem 3.2.4 for the case in which Y is a Hilbert space, in [36]. In the general form, it is contained in [32].

3.3. Theorem 3.3.6 is a particular case of multiplication theorems obtained in [30].

3.4. Theorems of 3.4.1 are given in [49], where the further bibliography is indicated. As for Theorems 3.4.2–3.4.5 see [6, 34]. Equivalent formulation of these assertions are presented in [25]. Corollary 3.4.6(b) is proven in [38]. As to Theorem 3.4.7 see [25]. Corollary 3.4.7 is obtained in [53]. In 3.4.8 and 3.4.9 we use the idea of the article [52] for generalizing the results of [53, 54]. Theorem 3.4.10 is borrowed from [54]. The result of 3.4.11 was announced in [25].

3.5. Theorem 3.5.2 was proven in [33]. Theorem 3.5.4, due to Grothendieck, is one of the important and deepest results of the theory of p-absolutely summing operators. We expose the proof of this theorem in line with [33, 44]. Other proofs of the theorem may be found in [20, 22, 37].

3.6. Some of the theorems presented in 3.6.1–3.6.3 are valid not only for linear operators but also for convex operators as well as for operators with values in $L^0(T, \mu)$ They were obtained for the first time in [35]. Our exposition adheres to [32]. Theorem 3.6.4 was obtained in [32], and Theorems 3.6.6 and 3.6.7 are given in [35]. The presented proof of Theorem 3.6.7 was proposed by A. V. Bukhvalov.

3.7. The results of this section are borrowed from [27] (see also [26]).

3.8. As to Corollary 3.8.2(a) see [47]. The other results of 3.8.1–3.8.6 are presented in [29]. Theorem 3.8.7 appears for the first time.

3.9. We use the concept of local unconditional structure put forward in [10]. The proof of Theorem 3.9.3 follows that of K. Schütt's theorem [50] which was proposed in [56]. Theorem 3.9.5 was obtained in [28]. Theorem 3.9.6 is given in [46]. As to the results of Subsection 3.9.7 see [46, 50]. The first result on uniqueness of the ideal of operators with local unconditional structure was likely obtained for the first time in [9]. Theorem 3.9.8 was exposed in [27] and its generalization to the case when the spaces L^p and L^q with $1 \leq p$ and $q \leq 2$ are replaced by arbitrary Banach lattices of 2-cotype is given in [48].

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CHAPTER 4

Integral Operators

BY

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†) Sections 4.1–4.5 are written by A. V. Bukhvalov, Section 4.6, by V. B. Korotkov, and Section 4.7, by B. M. Makarov.

In the present chapter, we will consider various problems connected with integral operators. First of all we solve the problem of finding conditions for an operator to be an integral operator. In Section 4.2, a necessary and sufficient condition for integrality of linear operators is given and, in Section 4.3, we apply this criterion to finding sufficient conditions for integrality of some classes of operators (in particular, dominated). In Section 4.4, the criterion for integral representability is used for integral representation of vector measures and, in Section 4.5, of nonlinear operators. The second part of Chapter 4 is mainly devoted to studying operators that remain integral (or belong to some classes of integral operators) after multiplication by an arbitrary operator of some class (for example, a unitary or bounded operator). Moreover, we expose as completely as possible the state of art in regard to the problems raised in the monograph [19] by P. Halmos and V. Sunder.

4.1. Basic Properties of Integral Operators

4.1.1. Throughout this chapter E and F denote ideal spaces over (T, Σ, μ) and (S, Λ, ν) respectively. An operator $U : E \to F$ is called *integral* if there exists a measurable function $K(s,t)$ $(t \in T,\ s \in S)$ such that for every $x \in E$ the value $y = Ux$ is the function

$$y(s) = \int\limits_T K(s,t)\, x(t)\, d\mu(t). \tag{1}$$

The function $K(s,t)$ is referred to as the *kernel* of the integral operator U. The set of all integral operators acting from E into F is denoted by $\mathscr{I}(E,F)$.

4.1.2. Observe that operator (1) must be given on the whole ideal space E and the integral in (1) is understood to be the usual Lebesgue integral. This circumstance excludes the operators that are densely defined in L^p by formula (1) as well as the operators in which the convergence of integrals is understood in the sense of some summation method: singular operators (where the integration is understood in the sense of principal value, pointwise almost everywhere, or in the metric of L^p, $1 < p < \infty$) which in fact fail to be not only integral operators in the sense of (1) but also to be even order bounded operators with values in the widest ideal space L^0. This case formally includes the "integral representation for the resolvent," classical in the theory of ordinary differential operators, which involves passage to the limit in the metric of L^2. Many potentials however admit integral representation for the resolvent. Nevertheless, the question seems to be open for some classes of operators. Note also that passage to the limit in L^2 can lead to loosing the property of integral representability. Indeed, the Fourier transform operator $\mathscr{F} : L^1(-\infty,\infty) \to L^\infty(-\infty,\infty)$ is an integral operator, whereas the Fourier-Plancherel operator obtained by passing to the limit in L^2 is not o-bounded from $L^2(-\infty,\infty)$ into $L^0(-\infty,\infty)$ and, therefore, cannot be an integral operator (see below). Finally, definition (1) excludes the operators whose kernels are distributions, because otherwise every operator could be considered as an integral operator once the class of distributions is defined appropriately.

It was just the discussion between John von Neumann and Paul A. M. Dirac about a possibility of applying the theory of integral operators to studying operators in quantum mechanics that led John von Neumann to the problem of describing the class of integral operators in the sense of (1). Of course, now the discussion is not that challenging due to appearance of the mathematically sound apparatus of distributions and the technique of rigged Hilbert spaces. However historically, it seems to lead to the fundamental article [58] by John von Neumann in which he solved the problem of finding all operators in $L^2(0,1)$ unitarily equivalent to some selfadjoint integral operator (see the monograph [38] by V. B. Korotkov) and posed the problem of finding necessary and sufficient conditions for a given operator in $L^2(0,1)$ to admit integral representation (1) [58, p. 4]. Solving the problem is the topic of Section 4.2.

4.1.3. We turn to considering integral operators. First we present their elementary properties.

The definition of operator (1) presumes the following two conditions to be satisfied:

(1) For every $x \in E$ the integrand in (1) is summable for almost all $s \in S$; i.e.,

$$\int |K(s,t)\,x(t)|\,d\mu(t) < \infty \tag{2}$$

for almost all $s \in S$.

(2) For every $x \in E$ function (1) belongs to the space F; i.e., for instance, in the case of $F = L^2$ we require that

$$\int \left| \int K(s,t)\,x(t)\,d\mu(t) \right|^2 d\nu(s) < \infty. \tag{3}$$

Note that the set of measure zero constituted by the points $s \in S$ at which (2) is violated depends, generally speaking, on the function x. Indeed, suppose that $E = L^2$ and property (2) is satisfied for all $x \in L^2$ and all $s \in S \setminus B$, where $\nu(B) = 0$. Then the function $K(s, \cdot)$ generates a continuous functional on L^2 for all $s \in S \setminus B$; whence,

$$\int |K(s,t)|^2 \, d\mu(t) < \infty \tag{4}$$

for almost all $s \in S$. An integral operator U in L^2 with kernel satisfying (4) is called a *Carleman operator*. The kernel of a Carleman operator possesses some extra summability as compared with (2). There are as many operators in L^2 whose kernels do not satisfy (4) as desired. As a simplest example we take a function $k \in L^1(0,1) \setminus L^2(0,1/2)$ and define the kernel to be $K(s,t) = k(|s-t|)$, i.e.,

$$(Ux)(s) = \int_0^1 k(|s-t|)x(t)\, dt, \quad x \in L^2(0,1).$$

Since $k \in L^1$, the operator U is bounded in L^2, but

$$\int_0^1 |k(|s-t|)|^2 \, dt = \int_0^s |k(t)|^2 \, dt + \int_0^{1-s} |k(t)|^2 \, dt = \infty$$

for every $s \in [0,1]$, i.e., (4) is not valid.

In the case of a finite measure μ, making the substitution $x = 1$, we obtain

$$\int |K(s,t)| \, d\mu(t) < \infty \tag{5}$$

for almost all s. The kernel may fail to meet any summability condition stronger than (5).

4.1.4. Now we turn to condition (2) and, in particular, to (3). Note that in (3) the modulus sign cannot be inserted under the inner integral sign.

Alongside with (1) we consider the integral operator with kernel $|K(s,t)|$:

$$(Wx)(s) = \int |K(s,t)| \, x(t) \, d\mu(t). \tag{6}$$

By (2), the function Wx is defined for all $x \in E$ and is finite almost everywhere, i.e., $Wx \in L^0$. Thus, the operator W always acts from E into L^0. Now what about the action in F? Even in the case $E = F = L^2(0,1)$ the operator W may fail to act from L^2 into L^2 (see, for example, [42, p. 78–81]. In this connection, we give the following definition.

Operator (1) is called a *regular integral operator* from E into F if operator (6) with kernel $|K(s,t)|$ acts from E into F. It is evident that if W acts from E into F, then U acts from E into F. The converse is true only for regular integral operators, and what was said before the definition means that there exist irregular integral operators in L^2. Property (2) shows that every integral operator is a regular integral operator if it is considered as an operator acting from E into L^0, which fact will be of use later.

The set of all regular integral operators is denoted by $\mathscr{I}^{\sim}(E,F)$.

4.1.5. Now we establish a connection between the concept of integral operator with the calculus of o-bounded operators which was discussed in Chapter 2. First of all it is obvious that every regular integral operator from E into F is o-bounded from E into F. In fact, a far less trivial assertion holds.

Proposition. *An integral operator U (see (1)) is a regular integral operator from E into F if and only if it is o-bounded from E into F. Moreover, the modulus $|U|$ in the sense of the calculus of Chapter 2 coincides with the operator W (see (6)), i.e.,*

$$(|U|x)(s) = \int |K(s,t)|\, x(t)\, d\mu(t), \quad x \in E. \tag{7}$$

The proposition has a simple proof for separable measure spaces [8, 88]; we present this proof below. In the general case the proof is very involved. In [50] there is a proof that grounds on approximating the kernel $K(s,t)$ with finite-dimensional kernels. In [25] (see Theorem XI.1.2) there is another proof based on one Yu. I. Gribanov's result [16] (earlier a close result was established by W. Luxemburg [49, 50]). All subtlety of the theorem lies in the fact that, under some conditions on the set of functions in the domain of definition of an integral operator, the supremum of the values of the operator on this set calculated in the K-space L^0 coincides with the pointwise supremum.

We will return to the proof of Proposition 4.1.5 after the next section.

4.1.6. Now consider the question concerning the continuity properties of integral operator (1).

Proposition. *Let $U : E \to L^0$ be an integral operator* (1).

(1) *If $x_n \to 0$ almost everywhere and $|x_n| \le x \in E$ $(n \in \mathbb{N})$ then $Ux_n \to 0$ almost everywhere.*

(2) *If $x_n \to 0(\mu)$ and $|x_n| \le x \in E$ $(n \in \mathbb{N})$ then $Ux_n \to 0$ almost everywhere.*

(3) *If U acts from a Banach ideal space E into a Banach ideal space F, then U is continuous.*

$\lhd$ Assertions (1) and (2) are obvious corollaries to the Lebesgue dominated convergence theorem by virtue of formula (2).

(3): According to the closed graph theorem, it suffices to check that the convergence of $x_n \to x$ in the norm of E and the convergence of $Ux_n \to y$ in the norm of F imply that $y = Ux$. By Proposition 2.1.10, there exists a subsequence $x_{n_k} \xrightarrow{o} x$ in E. Then $Ux_{n_k} \to Ux$ almost everywhere by (1). Using Proposition 2.1.10 again, we infer that $Ux_{n_k} \to y$ in measure. Hence $y = Ux$. $\rhd$

Claim (3) of Proposition 2.2 in the case of a Banach ideal space with some condition (in particular, L^p included) is due to S. Banach; and in the general case, to Yu. I. Gribanov (see the bibliography in [8]).

REMARK. In view of the above presentation, every integral operator from a Banach ideal space E into L^0 is continuous (L^0 is considered as endowed with the topology of convergence in measure) and therefore takes a ball of the space E into a set bounded in measure.

The question of describing in terms of the kernel the situation in which the operator acts from L^p into L^q lies beyond the scope of the work (the sufficient conditions found by L. V. Kantorovich are given in [42, § 7] and [25, § XI.3], and some necessary and sufficient conditions are presented in [79, (7.1.6)]; the idea of the latter conditions is connected with that of the so-called Schur's method).

4.1.7. We prove Proposition of 4.1.5 in the separable case.

$\lhd$ It suffices to establish that if the integral operator U acts from E into L^0 then formula (7) holds for all $x \in E$, $x \ge 0$. Fix such a function x and consider the set $M = \{y \mid |y| \le x\}$ involved in formula (5) of Section 2.2 for calculation of the modulus. We have

$$\int |K(s,t)|\,x(t)\,d\mu(t) = \sup\left\{\int K(s,t)\,y(t)\,d\mu(t) \mid y \in M\right\} \tag{8}$$

for almost all s, where the supremum on the right-hand side of (8) is calculated pointwise. In order to check (8), it is sufficient to observe that, for every s, the function

$$y_s(t) = \operatorname{sign}(K(s,t))\,x(t) \tag{9}$$

satisfies $|y_s| \le x$ and

$$\int |K(s,t)|\,x(t)\,d\mu(t) = \int K(s,t)\,y_s(t)\,d\mu(t). \tag{10}$$

Formula (12) of 2.2 asserts that

$$|U|x = \sup\{Uy \mid y \in M\}, \tag{11}$$

but the supremum on the right-hand side of (11) is that in L^0. The widely-spread fallacy in proving formula (7) consists in the fact that one does not distinguish between the natures of suprema in (8) and (11). In Section 2.1 it was explained that, generally speaking, information about the pointwise supremum tells nothing about the supremum in L^0 even in the case of the Lebesgue measure. Here we ought to use the particularity of our set $U(M)$. By virtue of the separability of L^0, there exists a countable everywhere dense set $\{y_n\}$ in M. Prove that

$$\int |K(s,t)|\,x(t)\,dt = \sup\left\{\int K(s,t)\,y_n(t)\,dt \;\mid\; n \in \mathbb{N}\right\} \quad \text{almost everywhere;} \tag{12}$$

$$\sup\{Uy \;\mid\; y \in M\} = \sup\{Uy_n \;\mid\; n \in \mathbb{N}\} \tag{13}$$

which immediately yields (7). If $y \in M$ is arbitrary, then there is a sequence y_{n_k} such that $y_{n_k} \to y$ in measure. Since the set M is o-bounded in E, we have $Uy_{n_k} \to Uy$ almost everywhere by Proposition of 4.1.6(2). Therefore,

$$Uy \le \sup_k Uy_{n_k} \le \sup Uy_n, \tag{14}$$

which immediately implies (13). Putting $y = y_s$ in (14), where y_s is defined by formula (9), we obtain (12).

Corollary. *An integral operator U is positive if and only if $K(s,t) \geq 0$ μ-almost everywhere. The operator U is identically zero if and only if $K(s,t) = 0$ μ-almost everywhere.*

An elementary proof of the preceding corollary which does not rely upon the more complicated Proposition 4.1.5 is given in [25, p. 393].

4.1.8. Now consider the question about the integrality of the dual of an integral operator.

Let U be an integral operator (1) acting from a Banach ideal space E into a Banach ideal space F. Alongside with (1) we consider the "transpose"

$$(U^{\#}y)(t) = \int K(s,t)\,y(s)\,d\nu(s). \tag{15}$$

If $U \in L^{\sim}(E,F)$ then we have $U \in L_n^{\sim}(E,F)$ by Proposition 4.1.6. We identify E' with $E_n^{\sim}$ and F' with $F_n^{\sim}$. Appealing to Subsections 2.2.25 and 2.2.26, we can assume that the dual operator U^* acts from F' into E' and is regular. Employing Fubini's theorem (see also [25, § XI.1]), we readily derive the following

Proposition. *If $U : E \to F$ is a regular integral operator then $U^* : F' \to E'$ is a (regular) integral operator and $U^* = U^{\#}$.*

The claim of the proposition can be valid without the assumption of the regularity of U (for instance, in the case when the kernel is symmetric or skew-symmetric, see [42, p. 84]), but it does not hold in general [19, Example 7.2; 11, Example II.1.17]. In [63], an example was constructed of a normal integral operator in $L^2(0,1)$ whose dual is not an integral operator (and, hence, the operator U does not act into L^2 [19, Theorem 7.5]), which solves the long-posed problem (see, for example, [19, Problem 11.12]).

4.1.9. Now we state criteria for an operator to belong to some important classes of operators (for proofs see [25, § XI.1]).

Theorem. *Let E be a Banach ideal space. If U is an integral operator (1) then $U \in M(E,F)$ if and only if $K \in F[E']$; moreover, $|U| = |K|$. If F is a Banach ideal space then $\|U\|_M = \|K\|_{F(E')}$.*

4.1.10. *Theorem.* *Let F be a perfect Banach ideal space. If U is an integral operator (1) then $U \in \mathscr{L}(L^1,F)$ if and only if $K \in L^\infty[F]$; moreover, $\|U\| = \|K\|_{L^\infty[F]} = \operatorname{vrai\,sup}_t \|K(\cdot,t)\|_F$.*

These results explain the usefulness of the spaces with mixed norm which were introduced in Chapter 2. Until Section 4.3 we leave open the question about the conditions under which the dominated operators or operators defined on L^1 admit integral representation.

4.2. Integral Representation of Linear Operators

Now we turn to the question of conditions for a linear operator to admit integral representation. This problem was posed by John von Neumann in [58] and solved in [5, 6] (see also the bibliography).

4.2.1. John von Neumann seems to be the first who proved that the identity operator I in $L^2(0,1)$ does not admit integral representation. This example is quite sufficient for illustrating the difference between conditions (1) and (2) of Proposition of 4.1.6 resultant from two perfectly equivalent statements of the Lebesgue dominated convergence theorem. In the case of the operator I condition (1) is obviously satisfied (it is simply a tautology) whereas condition (2) fails definitely, since we can easily indicate a sequence of characteristic functions x_n on $[0,1]$ such that $x_n \to 0$ in measure, but $\{x_n\}$ does not vanish almost everywhere. This argument is the simplest proof of the fact that I is not an integral operator. It turns out that condition (2) of Proposition 4.1.6 characterizes integral operators.

Let E and F be ideal spaces on (T, Σ, μ) and (S, Λ, ν).

Theorem. *Let $U : E \to F$ be a linear operator. The following assertions are equivalent:*

(1) *U is an integral operator, i.e., U admits representation* (1);

(2) *if $0 \le x_n \le x \in E$ $(n \in \mathbb{N})$ and $x_n \to 0$ in measure then $Ux_n \to 0$ almost everywhere;*

(3) *the operator U satisfies the following two conditions:*

(a) *if $\mu(A_n) \to 0$ $(A_n \in \Sigma)$ and $\chi_{A_n} \le x \in E$ $(n \in \mathbb{N})$ then $U(\chi_{A_n}) \to 0$ almost everywhere;*

(b) *if $0 \le x_n \le x \in E$ $(n \in \mathbb{N})$ and $x_n \to 0$ almost everywhere then $Ux_n \to 0$ almost everywhere.*

Remark. If E is a Banach ideal space with condition (A), then, by virtue of Proposition 2.1.10, assertions (1)–(3) are equivalent to the following condition:

(4) *if* $0 \leq x_n \leq x \in E$ $(n \in \mathbb{N})$ *and* $x_n \to 0$ *in norm, then* $Ux_n \to 0$ *almost everywhere.*

4.2.2. The proof of Theorem 4.2.1 will be given in 4.2.11, and now we discuss its statement. The implication (1) $\Rightarrow$ (2) is trivial and is proven in Proposition 4.1.6. The main equivalence of Theorem 4.2.1 is (1) $\Leftrightarrow$ (2) whose nontrivial part happens to be of profound interest because the implication (2) $\Rightarrow$ (1) is a sufficient condition for an operator to be an integral operator. The implication (2) $\Rightarrow$ (3) is obvious. Assertion (3) appears as refinement of the main assertion (2). The main condition in (3) is (a), meaning fulfillment of a condition equivalent to (2) but for characteristic functions, which turns out to be useful in applications (see Sections 4.4 and 4.5). Thus, to prove Theorem 4.2.1, we ought to establish only (3) $\Rightarrow$ (1) which will be done in Subsection 4.2.11.

If the measure μ is discrete, then all operators admit integral representation [25, p. 394–395]; therefore, the discrete case is of no interest. The basic cases are $T = S = [0,1]$ and $T = S = \mathbb{R}^n$ with the Lebesgue measure. Here the reader is free with choosing the desired level of generality, for it in no way simplifies the proof.

Furthermore, even the case $E = F = L^2(0,1)$ reveals all the difficulties. Now we will discuss possible approaches to proving Theorem 4.2.1 just in this case. First of all, looking at condition (2) that presumes some specific continuity of the operator U, one can think that some theorem on integral representation of functionals might be useful. It would be natural to fix $s \in S$ and introduce the functional on L^2 acting by the formula

$$\varphi_s(x) = (Ux)(s).$$

However, the term "value of a measurable function at a point s" (more exactly, "value of a class of equivalent functions") is not defined. There is a quite intricate theorem due to von Neumann and Maharam on existence of a lifting (see Subsection 2.4.5) which enables us to attach rigor to this concept for functions in L^∞, but in the case of the L^p spaces with $p < \infty$ the concept of lifting cannot be defined in principle. Assume, however, that we can overcome this difficulty. Further, checking continuity of φ on L^2 for almost all s, we would face another difficulty; the problem is that the set of measure zero where the convergence $Ux_n \to 0$ of condition (2) may fail to hold depends on $\{x_n\}$. Assume again that we could overcome the difficulty.

Afterward, we can apply Riesz's theorem on the general form of a functional in L^2 and write down

$$\varphi_s(x) = \int y_s(t)\, x(t)\, d\mu(t)$$

or

$$(Ux)(s) = \varphi_s(x) = \int K(s,t)\, x(t)\, d\mu(t),$$

where $K(s,t) = y_s(t)$. Generally speaking, the function $K(s,t)$ is not measurable as a function in two arguments, but in 4.3.6 we will see that it can be amended so as to become measurable. Now observe that Riesz's theorem yields $\int |y_s(t)|^2 d\mu(t) < \infty$ for every s and, hence,

$$\int |K(s,t)|^2 d\mu(t) < \infty$$

for almost all s. Thus, we conclude that the kernel $K(s,t)$ satisfies (4) of Section 4.1 with necessity, which means that U is a Carleman operator. As we know, there are non-Carleman operators; consequently, the chain of our arguments contains essential gaps and such approach to proving Theorem 4.2.1 fails.

Let us try to take another approach. Any theorem on integral representation suggests making use of the Radon-Nikodým theorem. This theorem appears eventually in our proof, but as a simple part of the reasoning, whereas all sophisticated arguments rely upon the calculus of o-bounded operators. We try to explain why an attempt to apply the theorem straightforward is doomed to failure. Assume that the operator U satisfies (2) and suppose for simplicity that $\mu(T), \nu(S) < \infty$. The natural measure whose Radon-Nikodým derivative yields the kernel of U is defined as follows:

$$\lambda(A \times B) = \int\limits_B U(\chi_A)\, d\mu, \quad A \in \Sigma,\ B \in \Lambda. \tag{1}$$

Function (1) can be easily extended to a finite-additive measure (not necessarily positive) over the algebra of subsets of $T \times S$ generated by the product of the σ-algebras Σ and Λ. However, its countable additivity (or, which is the same, absolute continuity with respect to $\mu \otimes \nu$) is difficult to check, because the integration in (1) "obliterates" the difference between condition (2) of Proposition 4.1.6 with convergence in measure and condition (1) with convergence almost everywhere which is in no way sufficient for integral representability. The only known proof of the countable additivity of the extension of (1) relies upon Theorem 4.2.1 itself. We believe that the "operator" version of Theorem 4.2.1 cannot be plainly reduced to measure theory without nontrivial application of operator theory.

Observe that any idea of proving Theorem 4.2.1 must explain where the difference between convergencies in measure and almost everywhere plays its role.

In conclusion, we answer the question why the very fact that an operator is an integral operator draws attention. First of all it is an interesting property in its own right which reveals the structure of an operator; the relevant problems are involved and have long history. Moreover, the fact that an operator is an integral operator yields some information on its spectrum [38, 58] and specific criteria for compactness. Once we have mentioned the concept of spectrum, we close the section with J. von Neumann's classical theorem of the article [58].

4.2.3. Let U be a linear operator in $L^2(0,1)$. A number λ belongs to the *limit spectrum* $\sigma_c(U)$ of the operator U if there exists an orthonormal sequence $\{e_n\}$ such that $\|Ue_n - \lambda e_n\| \to 0$.

The J. von Neumann theorem [58]. *Suppose that U is a bounded selfadjoint operator in $L^2(0,1)$. The following assertions are equivalent:*

(1) *U is unitarily equivalent to an integral operator (in $L^2(0,1)$);*

(2) *U is unitarily equivalent to a Carleman integral operator;*

(3) $0 \in \sigma_c(U)$.

Theorem 4.2.3 extends to the case of nonselfadjoint and unbounded operators, for instance, in [38, 19] (see also the bibliography therein).

Note that, clearly, not every operator unitarily equivalent to an integral one is an integral operator itself; therefore, Theorems 4.2.1 and 4.2.3 are in no way connected—they solve different problems.

4.2.4. Before launching into the proof of Theorem 4.2.1, we present one more criterion for integral representability which has an implicit form in contrast with Theorem 4.2.1; the latter characterizes every individual integral operator, whereas the theorem given below characterizes the whole class of regular integral operators and attaches rigorous meaning to the intuitively clear fact that integral operators must be connected with finite-rank operators. It is evident that the norm in the space of operators in L^2 is not suitable for describing this connection: every compact operator is a limit of finite-rank operators, but not every compact operator is an integral one, since there are integral operators (for example, Volterra operators) not compact in L^2 and, consequently, not approximable in norm by finite-rank operators.

Fix $x' \in E'$ and $y \in F$. Denote by $x' \otimes y$ the one-rank operator

$$(x' \otimes y)(x) = \left(\int x(t)\, x'(t)\, d\mu(t) \right) y, \quad x \in E,$$

that obviously belongs to $L_n^{\sim}(E, F)$ and is a regular integral operator with kernel $K(s,t) = y(s)\, x'(t)$. Denote by $K(E, F)$ the band in the K-space $L_n^{\sim}(E, F)$ which is generated by all operators of the form $x' \otimes y$ $(x' \in E', y \in F)$.

Theorem. $\mathscr{I}^{\sim}(E, F) = K(E, F)$.

In the case when E and F are Kantorovich-Banach spaces on $[0,1]$ the theorem was proven by G. Ya. Lozanovskiĭ [47]. The general form was obtained in [5].

4.2.5. While proving Theorem 4.2.4, we will use the following lemma which looks like the Dunford-Pettis theorem and is well known for a long time at least in the case of the interval $[0,1]$ with the Lebesgue measure. In [5], there is a proof grounded on the Bochner integral (in [67] the proof of Theorem 4.2.4 was reduced to the Radon-Nikodým theorem which is somewhat longer). Here we present a simple proof which is based only on the general form of a linear functional over L^1 and whose idea is similar to that of the proofs of more general facts in [7].

Lemma. *Every continuous linear operator $U : L^1(T, \mu) \to L^\infty(S, \nu)$ is an integral operator.*

$\lhd$ Let M be the set of functions of the form

$$L(s,t) = \sum_{i=1}^{n} \lambda_i \chi_{A_i}(t)\, \chi_{B_i}(s)$$

$(\mu(A_i), \nu(B_i) < \infty)$, where $A_i \cap A_j = \varnothing$ $(i \neq j)$. Put

$$\varphi(L) = \sum_{i=1}^{n} \lambda_i \int\limits_{B_i} U(\chi_{A_i})\, d\nu.$$

It is easy to see that the linear functional φ is defined on M correctly.

Assign $L^1(T \times S) = L^1(T \times S, \mu \otimes \nu)$ and $L^\infty(T \times S) = L^\infty(T \times S, \mu \otimes \nu)$. Then

$$|\varphi(L)| \leq \|U\| \sum_{i=1}^{n} \lambda_i \nu(B_i)\, \mu(A_i) = \|U\|\, \|L\|_{L^1(T\times S)},$$

whence φ is continuous on M endowed with the norm of $L^1(T \times S)$. Since M is dense in $L^1(T \times S)$ in norm [25, Lemma XI.1.2], we can extend φ by continuity to a bounded functional over $L^1(T \times S)$ for which we preserve the previous notation. Then there exists a function $K \in L^\infty(T \times S)$ such that

$$\varphi(L) = \int\limits_{T\times S} L(s,t)\, K(s,t)\, d(\mu \otimes \nu)(s,t), \quad L \in L^1(T \times S).$$

Hence, Ux can be expressed by means of formula (1) of Section 4.1 with kernel $K(s,t)$ for finite-valued functions $x \in L^1(T,\mu)$. In view of $K \in L^\infty(T \times S)$, we infer that the representation can be extended by continuity to the whole $L^1(T,\mu)$. $\triangleright$

4.2.6. Without loss of generality we may assume that $\mathbf{1}_T \in E'$ and $L^\infty(S,\nu) \subset F$ (therefore $\mathbf{1}_S \in F$). The general case easily reduces to the case by decomposition of measure spaces (see [5, 86, 87]). Now we present some technical lemmas of use in proving Theorem 4.2.4.

Lemma. *Denote $V_n = n\mathbf{1}_T \otimes \mathbf{1}_S$ $(n \in \mathbb{N})$. If $0 \leq U \leq \sup V_n$, then U is an integral operator.*

$\triangleleft$ It is obvious that

$$\|Ux\|_{L^\infty} \leq \|V_n(|x|)\|_{L^\infty} = \int |x(t)|\, d\mu(t) = \|x\|_{L^1}$$

for all $x \in L$; therefore, we can extend U to a continuous linear operator from L^1 into L^∞ and apply Lemma 4.2.5. $\triangleright$

4.2.7. Lemma. *For an arbitrary $U \in K(L^2, F)$, $U \geq 0$, we have $U \wedge V_n \uparrow U$.*

Remark. Appealing to the analogy between the theory of functions and the theory of o-bounded operators discussed in Chapter 2, we can attach a simple meaning to Lemma 4.2.7. It consists in the fact that the operator V_1 plays the role of a function identically equal to one. Then the operators $U \wedge V_n$ are cutoffs of the operator U and, therefore, Lemma 4.2.7 means convergence of the cutoffs to the operator itself.

$\lhd$ (1) First we prove the sought relation for $U = x' \otimes y$, where $x' \in E'_+$ and $y \in F_+$. It is clear that

$$(U \wedge V_n)(x)(s) = \int x(t)\,(x'(t)\,y(s) \wedge n\mathbf{1}_{T\times S}(s,t))\,d\mu(t)$$

(this follows, for example, from Proposition 4.1.5, but can be checked straightforwardly). Since

$$x'(t)\,y(s) \wedge n\mathbf{1}_{T\times S}(s,t) \uparrow x'(t)\,y(s),$$

it remains to employ Beppo Levy's theorem and formula (13).

(2) Now we check that $W \in K(E,F)$, $W \geq 0$, and $W \wedge V_1 = 0$ imply $W = 0$.

Take an arbitrary operator $V \geq 0$ of the form $x' \otimes y$. Then $\sup(V \wedge V_n) = V$ by the above. Hence,

$$W \wedge V = W \wedge \sup(V \wedge V_n) = \sup(W \wedge V \wedge V_n) = 0,$$

for $0 \leq W \wedge V_n \leq n(W \wedge V_1) = 0$. By the definition of $K(E,F)$ as the band generated by operators of the form $x' \otimes y$, we have $W = 0$ (we have proven that W is disjoint from $K(E,F)$).

(3) We return to proving the lemma for an arbitrary operator U. By Theorem 2.2.4, there exists an operator $V = \sup(U \wedge V_n) \leq U$ in the K-space $L^\sim(E,F)$. It is clear that

$$\sup(V_n \wedge V) = \sup[(V_n \wedge U) \wedge V] = \sup(V_n \wedge U) \wedge V = V.$$

Put $W = (U - V) \wedge V_1 \geq 0$. Afterward,

$$(V_n \wedge V) + W = (V_n + W) \wedge (V + W) \leq V_{n+1} \wedge U$$

and, therefore,

$$V = \sup(U \wedge V_n) = \sup(U \wedge V_{n+1}) \geq \sup(V \wedge V_n) + W = V + W.$$

Consequently, $W \leq 0$. Thus, $W = 0$, i.e., $(U - V) \wedge V_1 = 0$. The operator $U - V$ belongs to $K(E,F)$; therefore, appealing to Step (2), we have $U = V$, which was required. $\rhd$

4.2.8. Now we pass to proving Theorem 4.2.4.

$\lhd$ (1) Suppose that $U \in K(E, F)$. Since $U = U_+ - U_-$, we can assume that $U \geq 0$. We ought to show that U admits integral representation. By Lemma 4.2.6, for $U_n = U \wedge V_n$ there exists a kernel $K_n(s, t)$, i.e.,

$$(U_n x)(s) = \int K_n(s, t)\, x(t)\, d\mu(t), \quad x \in E.$$

Since $0 \leq U_n \uparrow$, we have $0 \leq K_n \uparrow$ by Proposition 4.1.5. According to Lemma 4.2.7, $U_n \uparrow U$. Put

$$K(s, t) = \sup_n K_n(s, t)$$

(the function can a priori take the value $+\infty$ on a set of positive measure). By formula (13) of Section 2.2, we obtain $U_n x \uparrow Ux$, $x \in E_+$. By Beppo Levy's theorem, we infer that U is an integral operator with kernel $K(s, t)$ (which then is automatically finite almost everywhere).

(2) Let U be a regular integral operator. Show that $U \in K(E, F)$. By the regularity of U, we have $U \in L_n^{\sim}(E, F)$. Without loss of generality we can assume that $U \geq 0$. Assign $K_n(s, t) = K(s, t) \wedge n\mathbf{1}(s, t)$ and let U_n be the integral operator with kernel $K_n(s, t)$. Then for $x \in E_+$ we have

$$(U_n x)(s) \leq \left(n \int x(t)\, d\mu(t)\right) \mathbf{1}_S(s) = (V_n x)(s).$$

Since $V_n \in K(E, F)$ and $0 \leq U_n \leq V_n$, we obtain $U_n \in K(E, F)$, for every band is an ideal. By Beppo Levy's theorem, $U_n \uparrow U$ (here we again use formula (13) of Section 2.2 afterward $U \in K(E, F)$ by the definition of band). $\rhd$

Corollary. *The set of regular integral operators from E into F is a band.*

Despite the fact that the corollary lacks the information on the structure of the band of integral operators, it is very useful. Note that the nontrivial part of the proof consists in checking that regular integral operators form an ideal. In this connection, we state the indicated result as a separate criterion often effective in particular situations (see [8]).

4.2.9. *Theorem.* *An operator $U : E \to L^0$ is an integral operator if and only if there exists an integral operator $V \geq 0$ such that $|U| \leq V$.*

4.2.10. Let us return to proving Theorem 4.2.1. We start with an elementary lemma ensuing from the diagonal sequence theorem.

Lemma. *Let $\{P_m\}$ be a sequence of subsets in $L^0(S, \Lambda, \nu)$ such that $P_m \supset P_{m+1}$ $(m \in \mathbb{N})$ and $\inf\{y \mid y \in P_m\} = 0$ $\forall m$. Then there is a sequence of functions $\{y_m\}$ such that*

$$\lim y_n(s) = 0 \quad \textit{almost everywhere}; \tag{2}$$

$$(\forall m)\,(\exists n(m))\,\{y_n \mid \ n \geq n(m)\} \subset P_m. \tag{3}$$

$\triangleleft$ Since $\inf\{y \mid y \in P_m\} = 0$, for every m there exists a sequence $\{h_{km}\}$ of the form

$$h_{km} = y_1^{(m)} \wedge \dots \wedge y_k^{(m)},$$

where $y_i^{(m)} \in P_m$, such that $h_{km} \downarrow 0$ as $k \to \infty$. By the diagonal sequence theorem [34, Theorem VII.4.5], there exists a sequence $k_1 < k_2 < \dots < k_m < \cdots$ such that $h_{k_m m} \to 0$ almost everywhere as $m \to \infty$. We define the sequence $\{y_n\}$ as follows. First we write down the elements $y_1^{(1)}, \dots, y_{k_1}^{(1)}$ that determine $h_{k_1 1}$; then the $y_1^{(2)}, \dots, y_{k_2}^{(2)}$ that determine $h_{k_2 2}$; etc. For every $l \in \mathbb{N}$ we have

$$\inf\{y_n \mid \ n \geq l\} \leq h_{k_m m}$$

whenever $m \geq m_1$. Letting $m \to 0$, we obtain

$$\inf\{y_n \mid \ n \geq l\} = 0,$$

whence $\lim y_n(s) = 0$ almost everywhere. Property (3) follows by construction from the fact that P_m decreases. $\triangleright$

4.2.11. We prove the implication (3) $\Rightarrow$ (1) in Theorem 4.2.1.

$\triangleleft$ We can assume that $F = L^0(S, \nu)$, since we have to prove only integral representability. Afterward, condition (b) implies that $U \in L_n^\sim(E, L^0)$ by Proposition 2.4.4.

By Theorem 4.2.4, the set $\mathscr{I}(E, L^0) = \mathscr{I}^\sim(E, L^0)$ of integral operators from E into L^0 is the band $K(E, L^0)$ generated by the finite-rank integral operators. Recalling the basic properties of the operator calculus of Chapter 1, we conclude that there exists a projection Pr of the K-space $L_n^\sim(E, L^0)$ onto the band $K(E, L^0)$.

Put $W = U - \Pr U$. The operator $W \in L_n^{\sim}(E, L^0)$ satisfies condition (3a) of Theorem 4.2.1. Since the action of Pr cannot be described constructively, this fact is difficult to be obtained straightforwardly from the fact that the operator U possesses the indicated property. Here we have to make use of Theorem 4.2.4 again. Indeed, by Theorem 4.2.4, the operator $\Pr U$ admits integral representation and, hence, satisfies (a). Now U and $\Pr U$ satisfy (a) and consequently the operator W, their difference, satisfies this condition too.

Assume that U is not an integral operator. Then $W \neq 0$ by Theorem 4.2.4. We will derive a contradiction from it.

By the definition of band projection, the operator W is disjoint from the band $K(E, S)$ and, in particular, from all finite-rank integral operators. Fix an arbitrary set $C \in \Sigma$, $\mu(C) < \infty$, $\chi_C \in E$, and demonstrate that

$$W(\chi_C) = 0. \tag{4}$$

By the above, we have $|W| \wedge (x' \otimes \mathbf{1}_S) = 0$, in particular, for $x' = \chi_C \in E'$. Afterward, using formula (15) of Section 2.2 for the infimum of operators, we obtain

$$\begin{aligned}(|W| \wedge (x' \otimes \mathbf{1}_S))(\chi_C)& \\ = \inf\{\, |W|(\chi_A) + \mu(B)\mathbf{1}_S) \mid C = A \cup B,\ A \cap B = \varnothing\} = 0.&\end{aligned} \tag{5}$$

Assign

$$P_m = \{\, |W|(\chi_A) + \mu(B)\mathbf{1}_S) \mid C = A \cup B,\ A \cap B = \varnothing,\ \mu(B) \le 1/m\}.$$

It is clear that $P_m \supset P_{m+1}$ $(m \in \mathbb{N})$. From (5) we derive

$$\begin{aligned}0 &= \inf P_m \wedge \inf\{\, |W|(\chi_A) + \mu(B)\mathbf{1}_S) \mid C = A \cup B,\ A \cap B = \varnothing,\ \mu(B) > 1/m\} \\ &\ge \inf P_m \wedge ((1/m)\mathbf{1}_S),\end{aligned}$$

whence $\inf P_m = 0$ for all $m \in \mathbb{N}$. By Lemma 4.2.10, there are sequences $\{A_n\}$ and $\{B_n\}$ such that

$$C = A_n \cup B_n, \quad A_n \cap B_n = \varnothing, \tag{6}$$

$$\varliminf_{n \to \infty} (\{\, |W|(\chi_{A_n}) + \mu(B_n)\mathbf{1}_S) = 0 \quad \text{almost everywhere}, \tag{7}$$

$$\mu(B_n) \to 0. \tag{8}$$

Property (7) ensues from (2), and property (8), from (3). Since $|W(\chi_{A_n})| \leq |W|(\chi_{A_n})$, from (7) we obtain

$$\lim_{n\to\infty} (\{\ |W(\chi_{A_n})| + \mu(B_n)\mathbf{1}_S) = 0 \quad \text{almost everywhere.} \tag{9}$$

In view of $\chi_{b_n} \leq \chi_C \in E$ and (8), using the fact that the operator W possesses property (a), we obtain $W(\chi_{B_n}) \to 0$ almost everywhere. Consequently,

$$W(\chi_{A_n}) = W(\chi_C) - W(\chi_{B_n}) \to W(\chi_C) \quad \text{almost everywhere.} \tag{10}$$

Now choose an arbitrary $s \in S$ so that both limit relations (9) and (10) be valid. From (9) we derive

$$\lim_{n\to\infty} |W(\chi_{A_n})(s)| = 0.$$

Comparing it with (10), we see that $W(\chi_C)(s) = 0$ for such s; thus, $W(\chi_C)(s) = 0$ almost everywhere, i.e., (4) holds.

We are about to complete the proof; therefore, it is time to explain where we have used the difference between convergences almost everywhere and in measure (i.e., between conditions (1) and (2) of Proposition 4.1.6). Passing to a subsequence, we could make $\chi_{B_{n_k}} \to 0$ almost everywhere by (8), but then equality (7) would be violated (the problem is that (7) involves the lower limit rather than the usual limit almost everywhere where passage to a subsequence spoils nothing).

Now it remains to complete the proof of the equality $W = 0$ by extending the already-proven equality (4) to all functions in E by continuity, which fact is guaranteed by $W \in L_n^\sim(E, L^0)$. ▷

4.2.12. For the first time Theorem 4.2.1 was published in the articles [5, 6] by A. V. Bukhvalov (about the later articles [67, 68] by A. Schep see [8]). The monograph [86] by A. Zaanen gives a proof very close to the original one.

All above-mentioned works use the idea of the proof of H. Nakano's theorem on bilinear forms [57]. In 1988 Professor R. Nagel from Tübingen University informed A. V. Bukhvalov that a disciple of H. Nakano, a well-known Japanese mathematician T. Andô, presented Theorem 4.2.1 in the course of lectures he delivered at the university in the late 1960s. However, his proof, which relies upon Nakano's theorem as well, was never published.

4.2.13. A new proof of the implication (1) $\Rightarrow$ (2) in Theorem 2.4.1 was found by L. Weis [83]. His proof is remarkable for he managed to use the Dunford-Pettis theorem (Theorem 2.4.11), thereby connecting the theory of representation of operators by means of vector-functions and the theory of integral operators.

Henceforth, we suppose for simplicity that $\mu(T), \nu(S) < \infty$ and $L^\infty(T,\mu) \subset E \subset L^1(T,\mu)$.

In the case of operators acting in L^1, we can reformulate Theorem 2.4.11 on using Proposition 2.3.7 as follows:

Lemma. *A bounded linear operator $U : L^1(T,\mu) \to L^1(S,\nu)$ is an integral operator if and only if for every $\varepsilon > 0$ there exists a set $A \subset T$, $\mu(T \setminus A) < \varepsilon$, such that $U\chi_A$ is a weakly compact operator.*

4.2.14. We state an elementary *exhaustion lemma.*

Lemma. *Let (P) be a property of measurable sets in Σ such that*

(a) $\mu(A) = 0 \Rightarrow A \in (P)$;

(b) $A \in (P) \Rightarrow (B \in \Sigma,\ B \subset A \Rightarrow B \in (P))$;

(c) $A,\ B \in (P) \Rightarrow A \cup B \in (P)$;

(d) *for every $A \in \Sigma$, $\mu(A) > 0$, there is a $B \in (P)$ such that $B \subset A$ and $\mu(B) > 0$.*

Then there exists a sequence $\{A_n\}$ of pairwise disjoint sets such that $\cup A_n = T$ and $A_n \in (P)$ $\forall n \in \mathbb{N}$.

Using Lemma 4.2.13, we can obtain the following result.

4.2.15. Corollary. *If a bounded linear operator $U : L^1(T,\mu) \to L^1(S,\nu)$ is not an integral operator, then there exist $\delta > 0$ and $A \in \Sigma$, $\mu(A) > 0$, such that*

$$\eta(B) = \varlimsup_{\nu(C)\to 0} \|\chi_C U \chi_B\| > \delta$$

for every $B \subset A$, $\mu(B) > 0$.

$\triangleleft$ Suppose to the contrary that, for arbitrary $\varepsilon > 0$ and $A \in \Sigma$, $\mu(A) > 0$, there exists a $B \subset A$, $\mu(B) > 0$, such that $\eta(B) < \varepsilon$. If $B_1 \cap B_2 = \varnothing$, $\eta(B_1) \le \varepsilon$, and $\eta(B_2) \le \varepsilon$ then $\eta(B_1 \cup B_2) \le \varepsilon$. Consequently, the property $\eta(B) \le \varepsilon$ meets the conditions of the exhaustion lemma 4.2.14. Therefore, there is a sequence $\{A_n\}$ such that $\mu(T \setminus A_n) \to 0$ and $\eta(A_n) \le \varepsilon$. Given a $\lambda > 0$ and a numeric sequence

$\{\varepsilon_n\}$ such that $\varepsilon_n \to 0$, find sets B_n satisfying the relations $\mu(T \setminus B_n) \le \lambda/2^n$ and $\eta(B_n) \le \varepsilon_n$. Then for $B = \cap B_n$ we have $\mu(T \setminus B) \le \lambda$ and $\eta(B) = 0$. If $\eta(B) = 0$ then the operator $U\chi_B$ is weakly compact by the criterion for weak compactness in L^1 (see the lemma of Subsection 2.4.11). Since $\lambda > 0$ is arbitrary, U is an integral operator by Lemma 4.2.13, which contradicts the hypothesis. ▷

4.2.16. We will prove the implication (2) $\Rightarrow$ (1) in Theorem 4.2.1 following L. Weis [83].

◁ As above, $U \in L_n^{\sim}(E, L^0)$. Choose a partition $\{B_n\}$ of the set S so as to have $\chi_{B_n}|U|\mathbf{1}_T \in L^\infty(S, \nu)$. Then $\chi_B U \in L_n^{\sim}(L^\infty, L^\infty)$. By 4.1.8, if $(\chi_{B_n}U)^* : L^1 \to L^1$ is an integral operator, then $\chi_{B_n}U$ is an integral operator on L^∞. Recalling that $U \in L_n^{\sim}(E, L^0)$, we can easily infer that U is an integral operator too.

Thus, it suffices to establish that if $U : L^\infty \to L^\infty$ satisfies condition (2), then $U^* : L^1 \to L^1$ is an integral operator.

Suppose to the contrary. By virtue of Corollary 4.2.15, there exist $\delta > 0$ and $C \in \Lambda$, $\nu(C) > 0$, such that

$$\overline{\lim_{\mu(A)\to 0}} \|\chi_D U \chi_A\|_\infty = \overline{\lim_{\mu(A)\to 0}} \|\chi_A U^* \chi_D\|_1 > \delta \tag{11}$$

for every $D \subset C$, $\nu(D) > 0$.

Fix an $\varepsilon > 0$. By (11), for every $D \subset C$, $\nu(D) > 0$, there exists a set $D' \subset D$, $\nu(D') > 0$, satisfying the condition

$$(P_\varepsilon): \text{ there are finitely many functions } x_i \text{ such that}$$
$$0 \le x_i \le \mathbf{1}, \quad \int x_i \, d\mu \le \varepsilon, \text{ and } \sup_i |Ux_i| \ge \delta/2 \text{ almost everywhere on } D'.$$

Property (P_ε) satisfies the conditions of the exhaustion lemma 4.2.14. Consequently, there are sets $D \in (P_\varepsilon)$ such that $\nu(C \setminus D)$ is arbitrarily small. Afterward, for every $m \in \mathbb{N}$, we can successively construct functions $x_{n_m+1}, \dots, x_{n_{m+1}}$ such that $0 \le x_i \le \mathbf{1}$ and $\int x_i \, d\mu \le \frac{1}{m}$ for $i = n_m + 1, \dots, n_{m+1}$; moreover, if

$$B_m = \{s \mid \sup_{i=n_m+1,\dots,n_{m+1}} |(Ux_i)(s)| \ge \delta/2\},$$

then $\nu(C \setminus B_n) \le \nu(C)/2^{m+1}$. Thus the set $B = \cap B_n$ has positive measure and, consequently, $Ux_i \not\to 0$ almost everywhere, though $x_i \to 0$ in measure, which contradicts condition (2) imposed on U. ▷

4.3. Applications of the Criterion for Integral Representability

4.3.1. We start with considering the problem left open in 4.1: When is a dominated operator an integral operator?

Theorem. *Let E be a Banach ideal space and take $U \in M(E, L^0(S, \nu))$. The operator U is an integral operator if and only if the following condition is satisfied:*

$(*)$ if $x_n \to 0$ (in $\sigma(E, E')$) then $Ux_n \to 0$ almost everywhere.

$\lhd$ If U is an integral operator, then its kernel K belongs to $L^0[E']$ by Theorem 4.1.9 and therefore condition $(*)$ is satisfied.

Conversely, if condition $(*)$ is satisfied then condition (2) of Theorem 4.2.1 is obviously satisfied and, consequently, U is an integral operator. $\rhd$

4.3.2. ***Theorem.*** *If E is a Banach ideal space satisfying condition* (4), *then every operator in $M(E, L^0)$ admits integral representation.*

$\lhd$ By virtue of condition (A), we have $E^* = E_n^\sim = E'$; therefore, the convergence in $(*)$ is the usual weak convergence. It remains to use the representation by means of vector-functions given in Theorem 2.4.7.

Hence we immediately obtain the following corollary.

4.3.3. ***Corollary.*** *If E is a Banach ideal space satisfying condition $(*)$ then $L_s^0(E^*) = L_s^0(E', E) = L^0[E']$.*

4.3.4. ***Theorem.*** *If F is a Banach ideal space satisfying condition (A) then every operator $U \in \mathscr{L}(L^1, F')$ is an integral operator.*

$\lhd$ The assertion ensues from Corollary 4.3.3 and Theorem 2.4.9. $\rhd$

Observe that condition (A) in Theorems 4.3.2 and 4.3.4 cannot be omitted, which is demonstrated by the example of the identity operator (in L^∞ and L^1). Recall that the identity operator is not an integral operator which is an immediate consequence of the fact that convergence in measure and convergence almost everywhere do not agree in the case of a continuous measure.

Theorem 4.3.2 involves criteria for an operator to lie in the Hilbert-Schmidt and Carleman classes (see Chapter 2).

4.3.5. Now we indicate simple corollaries to Theorem 4.2.1 which are connected with the integral representability of the superposition of operators.

Let E, F, and G be three ideal spaces and let $U : E \to F$ and $V : F \to G$ be linear operators.

Proposition. (1) *If $U : E \to F$ is a regular integral operator and $V \in L_n^\sim(F, L^0)$ then $W = VU$ is an integral operator.*

(2) *If $U \in L_n^\sim(E, F)$ and $V : F \to G$ is an integral operators then $W = VU$ is an integral operator.*

$\lhd$ Let us check (1), for example. Suppose that $0 \leq x_n \leq x \in E$ and $x_n \to 0$ in measure. Since U is an integral operator, we have $Ux_n \to 0$ almost everywhere; moreover, the regularity of U implies that the sequence $\{Ux_n\}$ is o-bounded in F, i.e., $Ux_n \to 0$ in F. Owing to $V \in L_n^\sim(F, L^0)$, we obtain $VUx_n \to 0$ almost everywhere. $\rhd$

Note that the conditions of the proposition cannot be weakened. In [37] and [62] examples were independently constructed of two integral (compact) operators in $L^2(0,1)$ whose superposition is not an integral operator (the same articles and [36] contain many other interesting results about integrality of the product of operators).

4.3.6. With the help of Theorem 4.2.1, one can easily prove that if an "integral" operator is generated by a nonmeasurable kernel, then the kernel can be replace by a measurable one.

Proposition [5]. *Let E be an ideal space and suppose that a function $\Phi(s,t)$, which may be nonmeasurable in general, is such that the ν-almost everywhere finite ν-measurable function*

$$y(s) = \int \Phi(s,t)\, x(t)\, d\mu(t)$$

is defined for all $x \in E$. Define the operator $Ux = y$ $(x \in E)$. Then there exists a μ-measurable function $K(s,t)$ such that

$$(Uf)(s) = \int \Phi(s,t)\, x(t)\, d\mu(t) = \int K(s,t)\, x(t)\, d\mu(t)$$

for all $x \in E$ and ν-almost all s (the exceptional set depends, generally speaking, on x).

$\lhd$ In the same way as in the proof of Proposition 2.2, using Lebesgue's theorem, we check that U satisfies condition (2) of Theorem 4.2.1. $\rhd$

Corollary [17]. *If, under the conditions of the proposition, (T, Σ, μ) is separable then $K(s,t) = \Phi(s,t)$ for ν-almost all s and μ-almost all t.*

REMARK. The Corollary can be extended to the general measure spaces.

New proofs of Proposition 4.3.6 and corollaries to it were found in [61]. Observe that the Proposition and its Corollary can be interpreted in terms of the theory of stochastic processes. Note also that Proposition 4.3.6 shows a way for obtaining results on integral representability of operators in the sense of (1) from the results on representability of operators by means of vector-functions.

4.4. Linear Operators and Vector Measures

The theory of vector measures and the theory of operators are closely connected with one another. For an arbitrary vector measure we can construct an operator defined on the space $C(K)$ or L^∞ (or some their subspaces). Moreover, it may occur that a certain property of a vector measure can be adequately expressed in terms of the corresponding operator. Condition (3) of Theorem 4.2.1 shows that in the case of the problem on integral representation of a vector measure with values in an ideal space or a Banach ideal space the case of a measure reduces to considering operators. We will clarify this idea in the present section.

4.4.1. A function $\vec{\varphi} : \Sigma \to L^0 = L^0(S, \nu)$ is called *additive* if

$$\vec{\varphi}(A \cup B) = \vec{\varphi}(A) + \vec{\varphi}(B)$$

for all $A, B \in \Sigma$, $A \cap B = \varnothing$. We always suppose that $\mu(A) = 0$ implies $\vec{\varphi}(A) = 0$. Of course, this condition is insufficient for integral representability of $\vec{\varphi}$ (for instance, $\vec{\varphi}(A) = \chi_A$). A function $\vec{\varphi}$ is called *bounded* if there exists

$$g(\vec{\varphi}) = \sup\{|\vec{\varphi}(A)| \mid A \in \Sigma\} \in L^0.$$

Denote by L the set of all measurable finite-valued functions

$$f = \sum_{i=1}^{n} \lambda_i \chi_{A_i} \quad (A_i \cap A_j = \varnothing,\ i \neq j)$$

on T. Every additive function $\vec{\varphi}$ generates a linear operator U on L by the formula

$$U(f) = \sum_{i=1}^{n} \lambda_i \vec{\varphi}(A_i).$$

Conversely, every linear operator U generates the additive function $\vec{\varphi}(A) = U(\chi_A)$, It is well known (and easily verified) that if a function $\vec{\varphi}$ is additive and bounded, then the operator U can be uniquely extended to a regular operator from $L^\infty(T,\mu)$ into $L^0(S,\nu)$ ([26, Theorem VIII.1.17]; the converse is true as well). If $\vec{\varphi}$ is o-continuous, i.e., $\chi_{A_n} \to 0$ almost everywhere implies $\vec{\varphi}(A_n) \to 0$ almost everywhere, then the operator U is o-continuous ([26, Theorem VIII.1.33]; the converse is true as well).

4.4.2. Proposition. *Let $\vec{\varphi} : \Sigma \to L^0$ be an additive function. The following assertions are equivalent:*

(1) *there exists a measurable function $K(s,t)$ such that*

$$\vec{\varphi}(A)(s) = \int\limits_A K(s,t)\,d\mu(t), \quad A \in \Sigma; \tag{1}$$

(2) *$\vec{\varphi}$ is bounded and $\mu(A_n) \to 0$ implies $\vec{\varphi}(A_n) \to 0$ almost everywhere.*

$\triangleleft$ (1) $\Rightarrow$ (2): The boundedness follows from the inequality

$$|\vec{\varphi}(A)(s)| \leq \int\limits_T |K(s,t)|\,d\mu(t) < \infty \quad \text{for almost all } s.$$

Afterward the validity of the second property in (2) is a consequence of the Lebesgue dominated convergence theorem.

(2) $\Rightarrow$ (1): By (2), the measure $\vec{\varphi}$ is o-continuous and then the operator U constructed above is o-continuous from L^∞ into L^0. By Item (3) of Theorem 4.2.1, the operator U and, hence, $\vec{\varphi}$ admit of integral representation. $\triangleright$

4.4.3. We present one immediate corollary to Proposition 4.4.2 which is proven independently in [11].

If X is a Banach space, then $\operatorname{bva}(\Sigma, X)$ denotes the set of all vector measures $\vec{m} : \Sigma \to X$ of finite variation (see [12]).

Theorem. *Let F be a Banach ideal space over (Ω, Λ, ν). The following assertions are equivalent:*

(1) *F is a Kantorovich-Banach space;*

(2) *given a measure space* (T, Σ, ν), *a measure* $m \in \operatorname{bva}(\Sigma, F)$ *admits of the representation*

$$\vec{m}(A) = \int_A \vec{g}\, d\mu, \quad \vec{g} \in L^1(T, \mu; F),$$

if and only if $\vec{m}$ *is o-absolutely continuous with respect to* μ, *i.e.*, $(\mu(A_n) \to 0) \Rightarrow (\vec{m}(A_n) \xrightarrow{o} 0$ *in* $F)$.

$\lhd$ The implication (2) $\Rightarrow$ (1) is not connected with the criterion for integral representability, therefore, we refer the reader to [11].

(1) $\Rightarrow$ (2): By analogy with the proof of Theorem 2.2.16, the condition $\vec{m} \in \operatorname{bva}(\Sigma, F)$ implies that the measure is bounded, afterward, then integral representation (1) holds by virtue of Proposition 4.4.2. Since $\vec{m} \in \operatorname{bva}(\Sigma, F)$ and F is perfect, we easily derive $K \in L^1[F]$; now $F \in (A)$ implies $L^1[F] = L^1(F)$ by 2.3.7. $\rhd$

4.5. Integral Representation of Nonlinear Operators

The main goal of the section is to state a nonlinear analogue of Theorem 4.2.1. However, to make the presentation complete, we start with L. Drewnoski and W. Orlicz's result on representation of nonlinear functionals [13].

4.5.1. Let E be an ideal space over (T, Σ, μ) and let X be a linear space. An operator $U : E \to X$ (in particular, a functional) is called *orthogonally additive* if for arbitrary $x_1, x_2 \in E$ the condition $x_1 \perp x_2$ implies $U(x_1 + x_2) = U(x_1) + U(x_2)$.

We say that a function $K : T \times \mathbb{R} \to \mathbb{R}$ *satisfies the Carathéodory condition* if the function $t \mapsto K(t, r)$ is measurable for every $r \in \mathbb{R}$ and the function $r \to K(t, r)$ is continuous for almost all $t \in T$.

Theorem. *Let* f *be a real functional over* E. *The following assertions are equivalent:*

(1) *there exists a function* $K(t, r)$ *satisfying the Carathéodory condition and such that* $K(t, 0) = 0$ *almost everywhere and*

$$f(x) = \int K(t, x(t))\, d\mu(t)$$

for every $x \in E$;

(2) *the functional* f *is orthogonally additive,* $x_n \to 0$ *almost everywhere, and* $|x_n| \le x \in E$ *implies* $f(x_n) \to 0$.

◁ A proof is given in [13]. It uses, in particular, some ideas of L. V. Kantorovich and A. G. Pinsker's works [27, 28] dating back to the late 1930s. ▷

4.5.2. With the help of Theorem 2.4.1, one can easily obtain the integral representation of orthogonally additive operators on the set of finite-valued functions L on (T, Σ, μ).

Theorem. *Let $W : L \to L^0(S, \nu)$ be an orthogonally additive operator. The following assertions are equivalent:*

(1) *there is a function $K(s,t,r)$ on $S \times T \times \mathbb{R}$ such that the function $(s,t) \mapsto K(s,t,r)$ is measurable for every $r \in \mathbb{R}$, $K(s,t,0) = 0$ for almost all (s,t), and the following representation holds:*

$$(Wx)(s) = \int K(s,t,x(t))\, d\mu(t); \tag{1}$$

(2) *the following two conditions are satisfied:*

(a) $\sup\{|W(r\chi_A)| \mid A \in \Sigma\} \in L^0 \ \forall r \in \mathbb{R}$;

(b) $\mu(A_n) \to 0 \Rightarrow W(r\chi_{A_n}) \to 0 \ \forall r \in \mathbb{R}$ *almost everywhere.*

◁ The implication (1) ⇒ (2) is plain.

(2) ⇒ (1): Introduce the additive function $\vec{\varphi}_r(A) = W(r\chi_A)$. By Proposition 4.4.2, there exists a function $K(s,t,r)$ satisfying the imposed measurability requirements and such that

$$W(r\chi_A)(s) = \vec{\varphi}_r(A)(s) = \int_A K(s,t,r)\, d\mu(t) = \int_A K(s,t,r\chi_A(t))\, d\mu(t). \tag{2}$$

If $r = 0$ then

$$\int_A K(s,t,0)\, d\mu(t) = 0 \quad \forall A \in \Sigma.$$

Therefore, subtracting from $K(s,t,r)$ the value $K(s,t,0)$, we can make the condition $K(s,t,0) = 0$ almost everywhere be valid without violating condition (2). In essence, here we normalize $K(s,t,r)$ so as to guarantee uniqueness of the kernel in representation (1).

Check equality (1). If

$$x = \sum_{i=1}^{n} \lambda_i \chi_{A_i} \quad A_i \cap A_j = \varnothing \ \ (i \neq j),$$

then, by virtue of the orthogonal additivity of W, we have

$$(Wx)(s) = \sum_{i=1}^{n} W(\lambda_i \chi_{A_i})(s) = \sum_{i=1}^{n} \int_A K(s, t, \lambda_i \chi_{A_i}(t))\, d\mu(t)$$
$$= \int_A K\Big(s, t, \sum_{i=1}^{n} \lambda_i \chi_{A_i}(t)\Big)\, d\mu(t) = \int_A K(s, t, x(t))\, d\mu(t). \ \triangleright$$

4.5.3. In the case of linear operators, obtaining integral representation on the set L settles the question of integral representability on the whole ideal space, provided the operator is order continuous. This is guaranteed by the possibility of passing to the limit by Beppo Levy's theorem. If $K(s,t)$ is the kernel of an integral operator then, writing it in the form (1), we obtain $\widetilde{K}(s,t) = K(s,t)r$. Here we easily obtain (and use) monotonicity and continuity with respect to r. In the nonlinear case, we can guarantee neither without additional conditions; therein lies the nontriviality of the assertion. Moreover, in the case of operators we cannot immediately transfer the method for validating the Carathéodory condition developed in [13] for proving Theorem 1.5.1. Furthermore, the fact that the kernel $K(s,t,r)$ has a majorant monotone in r distinguishes a proper subclass of the set of all nonlinear integral operators, the class of regular operators (see [88, 42]).

Here we present a solution to the posed problem obtain by S. Segura de León, a Spanish mathematician and a disciple of J. Mazón [71, 72].

An operator $U : E \to L^0$ is called an *abstract Urysohn operator* if it is orthogonally additive and order bounded (the latter is understood as in the linear case).

An operator $U : E \to L^0$ is called an *Urysohn operator* if representation (1) holds for it with a kernel $K : S \times T \times \mathbb{R} \to \mathbb{R}$ satisfying the following conditions:

(1) $K(s,t,0) = 0$ for almost all $(s,t) \in S \times \mathbb{R}$;

(2) the function $r \mapsto K(s,t,r)$ is continuous for almost all (s,t);

(3) the function $(s,t) \mapsto K(s,t,r)$ is measurable for almost all $r \in \mathbb{R}$.

Theorem. *Let $U : E \to L^0$ be an abstract Urysohn operator. The following conditions are equivalent:*

(1) *U is an Urysohn operator;*

(2) *if $\{x_n\}$ and $\{y_n\}$ are two order bounded sequences in E and $x_n - y_n \to 0$ in measure, then $Ux_n - Uy_n \to 0$ almost everywhere.*

The proof uses Theorem 4.5.2 as the first step. Afterward we have to prove the required properties of the kernel using (2). The order boundedness of the operator guarantees existence of the a monotone dominant.

4.6. Algebraic Properties of Integral Operators

In [19] (see p. 66) P. Halmos and V. Sunder posed the following problems on the algebraic properties of integral operators.

(I) Is the set of all integral operators in L^2 a right ideal?

(II) If $U : L^2(T_2) \to L^2(T_3)$ is an integral operator and $V : L^2(T_1) \to L^2(T_2)$ is a continuous linear operator, does it follow that $UV : L^2(T_1) \to L^2(T_3)$ is an integral operator?

(III) Is UV integral if U is a regular integral operator?

(IV) Even if the set of all integral operators in L^2 is not a right ideal, is it at least an algebra?

In this section we present solutions to the above problems and some other problems from the monograph [19] by Halmos and Sunder. Without loss of generality we may consider operators in $L^2(0,1)$, since the construction of [38] (see pp. 154, 155) enables us to transfer all the results given below to the case of operators in a separable space $L^2(T,\mu)$ with measure μ not purely atomic.

4.6.1. Henceforth in this section $L^2 = L^2(0,1)$; by an operator we mean a continuous linear operator acting from L^2 into L^2; $\|\cdot\|$ and $(\cdot,\cdot)$ stand for the norm and scalar product of L^2; and mes$\,(A)$ denotes the Lebesgue measure of a set $A \subseteq [0,1]$.

Theorem [36, 38]. *If UV is an integral operator for an arbitrary operator V then U is a Carleman integral operator.*

$\lhd$ It is clear that U is an integral operator. Let $K(s,t)$ be its kernel and let $\{r_n\}$ be the Rademacher system, $\{w_n\}$ be the Walsh system [20], and $\{\varphi_n\}$ be an arbitrary orthonormal basis for L^2. Consider the operator

$$V_0 f = \sum_{n=1}^{\infty} (f, r_n)\varphi_n.$$

Let K_0 be the kernel of the integral operator UV_0. Then

$$\int_0^1 |K_0(s,t)|\, dt < \infty, \tag{1}$$

$$\int_0^1 K_0(s,t)\, r_n\, dt = UV_0 r_n(s) = U\varphi_n(s), \quad n = 1,2,\dots, \tag{2}$$

$$\int_0^1 K_0(s,t) r_m^{\perp}\, dt = 0, \quad m = 1,2,3,\dots, \tag{3}$$

for almost all $s \in [0,1]$, where $\{r_m^{\perp}\}$ is the system obtained by excluding the Rademacher system from the Walsh system. From (1)–(3), using Khinchin's inequality (see 3.S.1), we obtain

$$\int_0^1 |K(s,t)|^2\, dt = \sum_{n=1}^{\infty} \left| \int_0^1 K(s,t)\, \varphi_n(t)\, dt \right|^2 = \sum_{n=1}^{\infty} |U\varphi_n(s)|^2 =$$

$$= \sum_{n=1}^{\infty} \left| \int_0^1 K_0(s,t)\, r_n(t)\, dt \right|^2 \leq 64 \left(\int_0^1 |K_0(s,t)|\, dt \right)^2 < \infty$$

for almost all $s \in [0,1]$. $\triangleright$

This proof seems to be simpler than the original of [23, 25].

4.6.2. Theorem 4.6.1 implies that UV_0 is not an integral operator if U is not a Carleman operator. An example of a non-Carleman integral operator is provided by the operator

$$\widetilde{U} f(s) = \int_0^1 K(s-t)\, f(t)\, dt,$$

where $0 \leq K : \mathbb{R} \to \mathbb{R}$ is a periodic measurable function with period 1 whose restriction to $[0,1]$ belongs to $L^1(0,1) \setminus L^2(0,1)$. Thus, the above remark and the example give a negative answer to the first three questions (I)–(III).

Theorem 4.6.1 also gives a negative answer to the following question posed by Halmos and Sunder [35, Problem 17.6]: Is an integral operator U characterized by the conditions that if $\{h_n\}$ is an orthonormal basis and $\alpha = \{\alpha_n\}$ is a sequence in l^2 then there exists a set $A(\alpha) \subset [0,1]$ of measure zero such that

$$\sum_{n=1}^{\infty} |\alpha_n U h_n(s)| < \infty \tag{4}$$

whenever $s \notin A(\alpha)$? Indeed, if the integrality of U is characterized by this condition, then UV is an integral operator itself for every unitary operator V. Since every operator W is a linear combination of four unitary operators, UW is an integral operator. This fact together with Theorem 4.6.1 implies that U must be a Carleman integral operator. Thus, condition (4) characterizes not all integral operators but only Carleman integral operators.

In his article [61, Example 6.6] W. Schachermayer constructed an integral operator U_0 with a nonnegative kernel and a (nonintegral) operator V_0 such that U_0V_0 is an integral operator. This example also gives a negative answer to Halmos and Sunder's questions (I)–(III). Also, in his article [62] W. Schachermayer constructed an example of an integral operator not meeting condition (4).

4.6.3. Theorem 4.6.1 yields the following corollary:

Corollary. *Each right ideal in the set of all operators (in L^2) which is composed of integral operators is contained in the right ideal of this set comprising all Carleman integral operators.*

4.6.4. It is of interest to find the form of the greatest left and right ideals in the set of all operators (in L^2) which are composed of integral operators.

Theorem [39, 40]. *If WU is an integral operator for every operator W then U is an integral Hilbert-Schmidt operator.*

$\lhd$ Show that WU is a regular integral operator for every W. Let E be an arbitrary order bounded absolutely convex set in L^2. Choose $0 \leq h_0 \in L^2$ so that $|z| \leq h_0$ for every function $z \in E$. The set $WU(E)$ is absolutely convex. Assume that the set $WU(E)$ is not order bounded. Then, by V. N. Sudakov's theorem (see 3.4.10), there exist an operator V, a sequence $\{z_i\} \subset E$, and a set $A \subset [0,1]$, $\operatorname{mes}(A) > 0$, such that

$$\sup_i |VWUz_i(s)| = \infty \tag{5}$$

for all $s \in A$. Let $K(s,t)$ be the kernel of the integral operator VWU. Then

$$|VWUz_i(s)| \leq \int_0^1 |K(s,t)|\, h_0(t)\, dt < \infty$$

for almost all $s \in [0,1]$ and all $i = 1, 2, 3, \ldots$, which contradicts (5). Thus, WU is a regular integral operator. Then $(WU)^* = U^*W^*$ is a regular integral operator.

Consider the operator

$$\widetilde{W}f = \sum_{n=1}^{\infty}(f,\varphi_n)r_n,$$

with $\{\varphi_n\}$ an orthonormal basis. Let $\widetilde{K}$ be the kernel of the integral operator $U^*\widetilde{W}^*$. Then

$$\int_0^1 |\widetilde{K}(s,t)|\,dt < \infty,$$

$$\int_0^1 \widetilde{K}(s,t)r_n(t)\,dt = U^*\widetilde{W}^*r_n(s) = U^*\varphi_n(s), \quad n = 1,2,3,\dots,$$

$$\int_0^1 \widetilde{K}(s,t)r_m^{\perp}(t)\,dt = 0, \quad m = 1,2,3,\dots,$$

for almost all $s \in [0,1]$. Hence, as in the proof of Theorem 1, we have

$$\sum_{n=1}^{\infty}|U^*\varphi_n(s)|^2 \le 64\left(\int_0^1 |\widetilde{K}(s,t)|\,dt\right)^2$$

for almost all $s \in [0,1]$ and, consequently,

$$\sum_{n=1}^{\infty}\|U^*\varphi_n\|^2 \le 64\int_0^1\left(\int_0^1 |\widetilde{K}(s,t)|\,dt\right)^2 ds = 2\,\|\,|U^*\widetilde{W}^*|\chi_{[0,1]}\|^2 < \infty.$$

Thus, U^* is an integral Hilbert-Schmidt operator. Therefore, U is an integral Hilbert-Schmidt operator. ▷

The presented proof seems to be simpler than the original proof of the theorem which is given in [39, 40].

4.6.5. Corollary. *Each left or two-sided ideal in the set of all operators (in L^2) which is composed of integral operators is contained in the two-sided ideal of this set constituted of all integral Hilbert-Schmidt operators.*

4.6.6. We now turn to Problem (IV).

Theorem [37, 38]. *The set of all integral operators is not an algebra.*

$\lhd$ To prove the theorem, we construct a compact integral operator M and a compact Carleman integral operator N such that MN is not an integral operator.

Let $\{A_n\}$ be a sequence of pairwise disjoint sets of positive measure in $[0,1]$ and let

$$Nf = \sum_{n=1}^{\infty} \lambda_n (f, r_n) \frac{\chi_{A_n}}{\sqrt{\operatorname{mes}(A_n)}},$$

with the sequence $\{\lambda_n\} \to 0$ to be chosen below. It is clear that N is a compact Carleman integral operator with kernel

$$N(s,t) = \sum_{n=1}^{\infty} \lambda_n \frac{\chi_{A_n}(s)}{\sqrt{\operatorname{mes}(A_n)}} r_n(t).$$

Let $\{\psi_n\}$ be an orthonormal system of absolute convergence for l^2 (i.e., such a sequence that the series

$$\sum_{n=1}^{\infty} |a_n \psi_n(s)|$$

converges almost everywhere for an arbitrary sequence $\{a_n\} \in l^2$ and the set of convergence of the series depends on $\{a_n\}$). Furthermore, assume that $\{\psi_n\}$ satisfies the following condition: there exists a set $A \subset [0,1]$, $\operatorname{mes}(A) > 0$, such that

$$\sum_{n=1}^{\infty} |\psi_n(s)|^2 = \infty$$

for all $s \in A$. Existence of such a system ensues from E. M. Nikishin's theorem [59, Theorem 13]. Then there exist a set $A_0 \subset A$, $\operatorname{mes}(A_0) > 0$, and a sequence $\{\mu_n\} \to 0$ such that

$$\sum_{n=1}^{\infty} |\mu_n \psi_n(s)|^2 = \infty$$

for all $s \in A_0$.

Now choose $\{\lambda_n\} \to 0$ and a set $A_1 \subset A_0$, $\operatorname{mes}(A_1) > 0$, so as to have

$$\sum_{n=1}^{\infty} |\lambda_n \mu_n \psi_n(s)|^2 = \infty \tag{6}$$

for all $s \in A_1$. Let

$$Mf = \sum_{n=1}^{\infty} \mu_n \left(f, \frac{\chi_{A_n}}{\sqrt{\operatorname{mes}(A_n)}} \right) \psi_n,$$

$$M(s,t) = \sum_{n=1}^{\infty} \mu_n \psi_n(s) \frac{\chi_{A_n}(t)}{\sqrt{\operatorname{mes}(A_n)}}.$$

Since

$$\int_0^1 |M(s,t)|\,|f(t)|\,dt = \sum_{n=1}^{\infty} |\mu_n|\,|\psi_n(s)| \left(|f|, \frac{\chi_{A_n}}{\sqrt{\operatorname{mes}(A_n)}} \right) < \infty$$

for every function $f \in L^2$ and almost all $s \in [0,1]$; therefore, M is a compact integral operator with kernel $M(s,t)$. If we assume that $L = MN$ is an integral operator with kernel $L(s,t)$, then

$$\int_0^1 |L(s,t)|\,dt < \infty,$$

$$\int_0^1 L(s,t)\,r_n(t)\,dt = \lambda_n \mu_n \psi_n, \quad n = 1,2,3,\dots,$$

$$\int_0^1 L(s,t)\,r_m^{\perp}(t)\,dt = 0, \quad m = 1,2,3,\dots,$$

for almost all $s \in [0,1]$. Hence, as in the proof of Theorem 4.6.1, we have

$$\sum_{n=1}^{\infty} |\lambda_n \mu_n \psi_n(s)|^2 < \infty$$

for almost all $s \in [0,1]$, which contradicts (6). $\triangleright$

4.6.7. REMARK. Theorem 4.6.6, the Schachermayer-Weis theorem [65, Theorem 5.1], and the lemma on right multiplication [54] imply that there exists a compact regular integral operator and a compact Carleman integral operator such that their product is not an integral operator.

4.6.8. In [61, Example 6.8] Schachermayer constructed an integral operator $N_0 : L^2 \to l^2$ and a compact integral operator $M_0 : l^2 \to L^2$ such that $M_0 N_0$ is not an integral operator.

Henceforth, we need the so-called generalized Rademacher functions. Let $A \subset [0,1]$ be a measurable set of positive measure. Put $r_{1,A} = \chi_A / \sqrt{\operatorname{mes}(A)}$. Split A into two disjoint sets A_1 and A_2 of equal measure and put

$$r_{2,A} = \frac{1}{\sqrt{\operatorname{mes}(A)}} (\chi_{A_1} - \chi_{A_2}).$$

Further, split each set A_i, $i = 1, 2$, into two disjoint sets $A_{i,k}$, $k = 1, 2$, of equal measure and put

$$r_{3,A} = \frac{1}{\sqrt{\operatorname{mes}(A)}} (\chi_{A_{1,1}} - \chi_{A_{1,2}} + \chi_{A_{2,1}} - \chi_{A_{2,2}}).$$

Repeating the process, we obtain the system of the generalized Rademacher functions $\{r_{n,A}\}$. It is clear that $\{r_{n,A}\}$ is an orthonormal system.

4.6.9. Also, we need the following lemma:

The Riemann-Lebesgue lemma. *Let $f \in L^1(0,1)$ and let $\{u_n\}$ be an orthonormal system such that $|u_n(s)| \le C$ for all $n \in \mathbb{N}$ and almost all $s \in [0,1]$. Then*

$$\lim_{n\to\infty} (f, u_n) = 0.$$

$\lhd$ Given an arbitrary $\varepsilon > 0$, we can find $f_\varepsilon \in L^2$ so that

$$\int_0^1 |f - f_\varepsilon| \, dt < \varepsilon.$$

Afterward, we have

$$|(f, u_n)| \le C \int_0^1 |f - f_\varepsilon| \, dt + |(f_\varepsilon, u_n)| < C\varepsilon + |(f_\varepsilon, u_n)|.$$

Since $f_\varepsilon \in L^2$, Bessel's inequality implies that $\lim_{n\to\infty}(f_\varepsilon, u_n) = 0$, which was required. $\rhd$

4.6.10. Lemma. *Let U be an integral operator with kernel $K(s,t)$. Then for every $\varepsilon > 0$ there exists a set $A = A_\varepsilon \subset [0,1]$ such that* $\operatorname{mes}([0,1] \setminus A) < \varepsilon$ *and*

$$\lim_{n\to\infty} \|U^* r_{n,A}\| = 0.$$

$\lhd$ By E. M. Nikishin's theorem [59], for every $\varepsilon > 0$ there exists a set $A = A_\varepsilon \subset [0,1]$ such that $\operatorname{mes}([0,1] \setminus A) < \varepsilon$ and the integral operator $U_+ = P_A|U|$ with kernel $\chi_A(s)|K(s,t)|$ acts continuously from L^2 into L^1. Here $P_A f = \chi_A f$. Then the adjoint operator $U_+^* : L^\infty \to L^2$ is an integral operator; namely,

$$U_+^* h(t) = \int_0^1 \chi_A(s)\, |K(s,t)|\, h(s)\, ds.$$

Consequently, $U^* P_A : L^\infty \to L^2$ is an integral operator too and

$$U^* P_A h(t) = \int_0^1 \chi_A(s)\, \overline{K(s,t)}\, h(s)\, ds.$$

Since

$$\int_0^1 \chi_A(s)|K(s,t)|\, ds < \infty$$

for almost all $t \in [0,1]$ and $|r_{n,A}| \leq 1/\sqrt{\operatorname{mes}(A)}$, $n \in \mathbb{N}$; on appealing to the Riemann-Lebesgue lemma, we obtain

$$U^* r_{n,A}(t) = U^* P_A r_{n,A}(t) = \int_0^1 \chi_A(s)\, \overline{K(s,t)}\, r_{n,A}(s)\, ds \to 0$$

as $n \to \infty$ for almost all $t \in [0,1]$. Moreover,

$$|U^* r_{n,A}| = |U^* P_A r_{n,A}| \leq U_+^* \frac{\chi_A}{\sqrt{\operatorname{mes}(A)}} \in L^2.$$

Hence, $\lim_{n\to\infty} \|U^* r_{n,A}\| = 0$ by the Lebesgue dominated convergence theorem. $\rhd$

4.6.11. Now consider the following question: Is the adjoint of an integral operator an integral operator? As the following example shows [22], the answer to the question is negative in general.

Put

$$U_0 f = \sum_{n=1}^{\infty} (f, \varphi_n) \frac{\chi_{A_n}}{\sqrt{\operatorname{mes}(A_n)}},$$

where $\{A_n\}$ is some sequence of pairwise disjoint sets of positive measure in $[0,1]$ and $\{\varphi_n\}$ is some orthonormal basis. It is evident that U_0 is a Carleman integral operator with kernel

$$U_0(s,t) = \sum_{n=1}^{\infty} \frac{\chi_{A_n}(s)}{\sqrt{\operatorname{mes}(A_n)}} \overline{\varphi_n(t)}.$$

We will show that not only U_0^* fails to be an integral operator but also it fails to be similar to any integral operator. Indeed, if there is a linear homeomorphism V such that $\widetilde{U} = VU_0^*V^{-1}$ is an integral operator then, by Lemma 4.6.10, there exists $A \subset [0,1]$, $\operatorname{mes}(A) > 0$, such that $\|\widetilde{U}^* r_{n,A}\| \to 0$ as $n \to \infty$. Put $h_n = V^* r_{n,A}$, $n \in \mathbb{N}$. Therefore,

$$\|U_0 h_n\| = \|V^* \widetilde{U}^* r_{n,A}\| \to 0 \text{ as } n \to \infty.$$

But $\|U_0 h_n\| = \|h_n\|$, $n \in \mathbb{N}$, and we arrive at a contradiction.

The situation is different when U is a normal Carleman integral operator, thus, satisfying the identity $U^*U = UU^*$. In this case, as was established by B. Misra, D. Speiser, and G. Targonski [54], the adjoint operator is an integral operator too. Indeed, since U is a normal operator, applying the spectral theorem, we have $U^* = WU$, where W is a unitary operator. Consequently, by the lemma on right multiplication [54], U^* is a Carleman integral operator.

In this connection, in [35, Problem 11.12] Halmos and Sunder posed the following question: Is the adjoint of a normal integral operator an integral operator?

A negative answer to this question was given by Schachermayer [63, Proposition 1.1].

4.6.12. The integral operators considered above admit integral representation on the whole L^2. However, we can study operators that admit integral representation only on linear manifolds of L^2. Here we confine ourselves to considering the operators that admit integral representation on L^∞. Such operators were introduced in [34] and called *partially integral* operators.

The class of partially integral operators is much broader than the class of integral operators. Indeed, consider the operator

$$U_0 f = \sum_{n=1}^{\infty}(f,\varphi_n)\frac{\chi_{A_n}}{\sqrt{\operatorname{mes}(A_n)}},$$

where $\{A_n\}$ is any sequence of pairwise disjoint sets of positive measure in $[0,1]$ satisfying the condition

$$\sum_{n=1}^{\infty}\sqrt{\operatorname{mes}(A_n)} < \infty$$

and $\{\varphi_n\}$ is any orthonormal basis. As was shown above, the operator

$$U_0^* f = \sum_{n=1}^{\infty}\left(f, \frac{\chi_{A_n}}{\sqrt{\operatorname{mes}(A_n)}}\right)\varphi_n$$

is not similar to any integral operator. Show that U_0^* is a partially integral operator with kernel

$$K(s,t) = \sum_{n=1}^{\infty}\varphi_n(s)\,\frac{\chi_{A_n}(t)}{\sqrt{\operatorname{mes}(A_n)}}.$$

Take $f \in L^\infty$ and let $\|f\|_\infty$ denote the norm of f in L^∞. Then

$$\int_0^1 |K(s,t)|\,|f(t)|\,dt \le \|f\|_\infty \sum_{n=1}^{\infty}\sqrt{\operatorname{mes}(A_n)}|\varphi_n(s)|$$

$$\le \|f\|_\infty\left(\sum_{n=1}^{\infty}\sqrt{\operatorname{mes}(A_n)}\right)^{1/2}\left(\sum_{n=1}^{\infty}\sqrt{\operatorname{mes}(A_n)}|\varphi_n(s)|^2\right)^{1/2} < \infty$$

for almost all $s \in [0,1]$, because

$$\int_0^1 \sum_{n=1}^{\infty}\sqrt{\operatorname{mes}(A_n)}|\varphi_n(s)|^2\,ds = \sum_{n=1}^{\infty}\sqrt{\operatorname{mes}(A_n)} < \infty.$$

Note that U_0 is a Carleman integral operator and $U_0^*U_0 = I$, where I is the identity operator.

4.6.13. The following theorem essentially strengthens the above result.

Theorem [41]. *Every operator U (in L^2) can be represented as $U = VW$, where V is a partially integral operator and W is a Carleman integral operator.*

$\lhd$ Alongside the partially integral operator U_0^* acting by the rule

$$U_0^* f = \sum_{n=1}^{\infty} \left(f, \frac{\chi_{A_n}}{\sqrt{\operatorname{mes}(A_n)}} \right) \varphi_n, \quad \sum_{n=1}^{\infty} \sqrt{\operatorname{mes}(A_n)} < \infty,$$

we will consider the operators

$$Vf = UU_0^* f = \sum_{n=1}^{\infty} \left(f, \frac{\chi_{A_n}}{\sqrt{\operatorname{mes}(A_n)}} \right) U\varphi_n,$$

$$Wf = \sum_{n=1}^{\infty} (f, \varphi_n) \frac{\chi_{A_n}}{\sqrt{\operatorname{mes}(A_n)}};$$

the so-defined operator W is a Carleman integral operator with kernel

$$W(s,t) = \sum_{n=1}^{\infty} \frac{\chi_{A_n}(s)}{\sqrt{\operatorname{mes}(A_n)}} \overline{\varphi_n(t)}.$$

Consider the kernel

$$V(s,t) = \sum_{n=1}^{\infty} U\varphi_n(s) \frac{\chi_{A_n}(t)}{\sqrt{\operatorname{mes}(A_n)}}.$$

Let $f \in L^\infty$. Then

$$\int_0^1 |V(s,t)|\,|f(t)|\,dt \le \|f\|_\infty \sum_{n=1}^{\infty} \sqrt{\operatorname{mes}(A_n)}\,|U\varphi_n(s)|$$

$$\le \|f\|_\infty \left(\sum_{n=1}^{\infty} \sqrt{\operatorname{mes}(A_n)} \right)^{1/2} \left(\sum_{n=1}^{\infty} \sqrt{\operatorname{mes}(A_n)}\,|U\varphi_n(s)|^2 \right)^{1/2} < \infty$$

for almost all $s \in [0,1]$, because

$$\int_0^1 \sum_{n=1}^{\infty} \sqrt{\operatorname{mes}(A_n)}\,|U\varphi_n(s)|^2\,ds \le \|U\|^2 \sum_{n=1}^{\infty} \sqrt{\operatorname{mes}(A_n)} < \infty.$$

Therefore, V is a partially integral operator. Since $U\varphi_n = VW\varphi_n$, $n \in \mathbb{N}$, we conclude that $U = VW$. $\rhd$

The following theorem demonstrates why the partially integral operator V in the preceding theorem may fail to be an integral operator.

DEFINITION. We say that 0 belongs to the *limit spectrum* of an operator U if there exists an orthonormal system $\{f_n\}$ such that $\lim_{n\to\infty} \|Uf_n\| = 0$.

4.6.14. Theorem [41]. *An operator U with 0 not in the limit spectrum of U^* cannot be approximated in operator norm by linear combinations of finitely many operators of the form KL, where K is an arbitrary integral operator and L is an arbitrary operator.*

◁ Suppose to the contrary that there is a sequence

$$U_i = \sum_{j=1}^{N(i)} K_{i,j} L_{i,j}$$

converging to U in the operator norm, where $K_{i,j}$ is an integral operator and $L_{i,j}$ is an operator (presumed to be neither integral nor partially integral), $j = 1, \dots, N(i)$, $i = 1, 2, 3 \dots$. By Lemma 4.6.10, there exists a set $A \subset [0,1]$, $\operatorname{mes}(A) > 0$, such that

$$\lim_{n\to\infty} \|K_{i,j} r_{n,A}\| = 0, \quad j = 1, \dots, N(i), \; i = 1, 2, 3, \dots .$$

Hence,

$$\lim_{n\to\infty} \|L_{i,j}^* K_{i,j}^* r_{n,A}\| = 0.$$

But then

$$\lim_{n\to\infty} \|U^* r_{n,A}\| = 0,$$

i.e., 0 belongs to the limit spectrum of U^*. This contradiction proves the theorem. ▷

4.7. Universal Integral Operators and Operators with Integral Commutators

4.7.1. In his article [33] (see also [38]) V. B. Korotkov described the operators in $L^2(0,1)$ whose unitary orbits consist of integral operators. In [19], such an operator is called *universal integral* which accounts for the terminology used below.

Our goal is, first, to generalize the V. B. Korotkov result and the result of [75] to the case of operators acting in the spaces L^p with $p \neq 2$ and, second, to modify the result by considering the set of all commutators of a given operator instead of the orbit. Throughout the section all spaces are assumed complex.

DEFINITION. Let $1 < p < \infty$. An operator $A : L^p(T,\mu) \to L^p(T,\mu)$ is called a *universal regular operator* if the operator UAU^{-1} is regular for every isomorphism $U : L^p(T,\mu) \to L^p(T,\mu)$. Recall the definition of essential spectrum (see [31]) which will be used below.

DEFINITION. Let X be a (complex) Banach space and let $A \in \mathscr{L}(X,X)$. A number α *belongs to the essential spectrum* of the operator A if at least one of the following conditions is satisfied:

(a) the range $(A-\alpha I)(X)$ of the operator $A - \alpha I$ is not closed;

(b) $\dim\ker(A-\alpha I) = \infty$ and $\dim\ker(A-\alpha I)^* = \infty$.

It is well known that if X is an infinite-dimensional space then the essential spectrum of an operator T in $B(X)$ is nonempty and is preserved under compact perturbations of T (see [31, p. 306]).

In the sequel, we will need the following lemma whose proof is left to the reader.

Lemma. *Suppose that X and Y are Banach spaces, with X reflexive, and $U \in \mathscr{L}(X,Y)$. If the set $U(X)$ is not closed then there exists a sequence $\{x_n\}_{n=1}^{\infty} \subset X$ possessing the following properties: $\|x_n\| = 1$ $(n \in \mathbb{N})$, $x_n \underset{n\to\infty}{\longrightarrow} 0$ weakly, and $\|Ux_n\| \underset{n\to\infty}{\longrightarrow} 0$.*

4.7.2. Theorem. *Let $1 < p < \infty$; $(T, \mathfrak{A}, \mu)$ be a space with σ-finite measure; $L^p = L^p(T,\mu)$; $A \in \mathscr{L}(L^p, L^p)$; and $0 < \varepsilon < 1$. The following assertions are equivalent:*

(1) *A is a universal regular operator;*

(2) *UAU^{-1} is a regular operator for every isomorphism U in $\mathscr{L}(L^p, L^p)$ such that $\|U - I\| < \varepsilon$, where I is the identity operator in L^p;*

(3) *the operator A can be represented as $A = \alpha I + B$, where $\alpha \in \mathbb{C}$ and $B \in \mathscr{L}^{\sim}_{\mathrm{st}}(L^p, L^p)$.*

$\lhd$ The implications (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1) are plain. Prove (2) $\Rightarrow$ (3). First of all note that we can assume that zero belongs to the essential spectrum of A, for otherwise we can replace A by $A - \alpha I$, where α belongs to the essential spectrum of A. Prove that A is stably regular. Obviously, it suffices to prove the assertion for $p \le 2$, since the case of $p > 2$ will be settled by replacing A with A^*.

Also, observe that we can evidently exclude the case when the measure μ is continuous and has finitely many point masses. Afterward, since the measure μ is

σ-finite, we can decompose the set T into some parts e_n so that (see [81, 51] and also [32, 73])

$$\begin{gathered}\bigcup_{n=1}^{\infty} e_n = T, \quad e_n \cap e_m = \varnothing \text{ for } n \neq m \ (n, m \in \mathbb{N}), \\ \text{and every space } L^p(e_n, \mu) \text{ is isomorphic,} \\ \text{as a Banach lattice, to } L^p. \end{gathered} \tag{1}$$

In this event, it is obvious that $L^p(e'_n, \mu)$ with $e'_n = T \setminus e_n$ is isomorphic to L^p, since

$$L^p = \left(\sum_{k=1}^{\infty} L^p(e_k, \mu) \right)_{l^p}$$

and

$$L^p(e'_n, \mu) = \left(\sum_{k \neq n} L^p(e_k, \mu) \right)_{l^p}.$$

We split the further proof into several steps. If $e \subset T$ then we use the notation $L^p(e, \mu)$ rather than $L^p(e)$.

I. Decompose the set T into two parts e and e' so that $L^p(e)$ and $L^p(e')$ be isomorphic to L^p. We naturally identify $L^p(e)$ and $L^p(e')$ with subspaces (bands) of L^p and let P and P' be the band projections in L^p onto $L^p(e)$ and $L^p(e')$ respectively (it is evident that P and P' are merely the operators of multiplication by χ_e and $\chi_{e'}$). Check that the operators $A_{12} = PAP'$ and $A_{21} = P'AP$ are 2-absolutely summing (and consequently compact and stably regular (see 3.7.5(a)). Let

$$W \in \mathscr{L}(L^p, L^p), \quad U = I - \delta PWP', \tag{2}$$

where $0 < \delta < \varepsilon \|W\|^{-1}$. Then $U^{-1} = I + \delta PWP'$ and $\|U - I\| < \varepsilon$. By hypothesis, the operator UAU^{-1} is regular and consequently

$$P'UAU^{-1} = P'A - P'APWP' = P'A - A_{21}WP'$$

is a regular operator as well. Hence, in view of the arbitrariness of W and Corollary 3.7.5(d), we conclude that $A_{21} \in \Pi_2(L^p, L^p)$. Interchanging e and e', we find that $A_{12} \in \Pi_2(L^p, L^p)$.

II. Consider $A_{11} = PAP$ and $A_{22} = P'AP'$ as operators in $L^p(e)$ and $L^p(e')$ and prove that zero belongs to their essential spectra. Let $\omega : L^p(e') \to L^p(e)$ be an arbitrary isomorphism. "Transfer" the operator A_{22} into $L^p(e)$ by replacing it by the operator $\omega A_{22}\omega^{-1}$ which has obviously the same spectrum and the same essential spectrum as the operator A_{22}. We prove that the difference $A_{11} - \omega A \omega^{-1}$ is a compact operator. Take an operator U defined by equality (2). Then the operator UAU^{-1} satisfies condition (2) of the theorem with ε replaced by a sufficiently small number $\varepsilon_1 > 0$ (it is sufficient that ε_1 satisfy the inequality

$$(1+\varepsilon)\varepsilon_1 + \|I + U\| < \varepsilon).$$

Appealing to the first step of the proof, we conclude that the operator $PUAU^{-1}P'$ belongs to the class Π_2 and consequently is compact. Simple calculations show that

$$P(UAU^{-1} - A)P' = \delta(A_{11}WP' - PWA_{22}P') = \delta\bar{A}.$$

Thus,

$$\bar{A} = A_{11}WP' - PWA_{22}P' \in K(L^p, L^p), \tag{3}$$

where $K(L^p, L^p)$ is the set of compact operators. Put $W = j\omega P'$, where j is the identical embedding of $L^p(e)$ into L^p. Then (3) implies the compactness of the operator $\bar{A}|_{L^p(e')} = A_{11}\omega - \omega A_{22}$. Consequently, the operator $A_{11} - \omega A \omega^{-1}$ is also compact. Thus, the operators A_{11} and A_{22} have the same essential spectrum; moreover, the kernels of these operators are or are not finite-dimensional simultaneously. Now verify that 0 belongs to the essential spectrum of these operators. Suppose to the contrary. Then the ranges of the operators are closed, which implies the closure property for the ranges of the operators $PA_{11}P + P'A_{22}P'$ and A. If $\dim\ker(A_{11}) < \infty$ (or $\dim\ker(A_{11}^*) < \infty$) then $\dim\ker(A_{22}) < \infty$ (or $\dim\ker(A_{22}^*) < \infty$); hence, $\dim\ker(A) < \infty$ ($\dim\ker(A^*) < \infty$). Thus, 0 does not belong to the essential spectrum of A, which is impossible.

III. Now prove that, using decomposition (1), we can construct a projection $Q : L^p \to L^p$ such that the subspace $Q(L^p)$ is isomorphic to l^p and QA is a nuclear operator. Let P_j be the band projection onto $L^p(e_j)$ and let $A_j = P_jAP_j$. As was established at the preceding step of the proof, 0 belongs to the essential spectrum of A_j (with A considered as an operator in $L^p(e_j)$). Therefore, there exists a sequence $\{x'_{jn}\}_{n=1}^{\infty} \subset L^{p'}(e_j)$ such that

$$\|x'_{jn}\|_{p'} = 1, \quad x'_{jn} \underset{n\to\infty}{\longrightarrow} 0 \text{ weakly}, \quad \|A_j^* x'_{jn}\|_{p'} \underset{n\to\infty}{\longrightarrow} 0. \tag{4}$$

Indeed, if $\dim\ker(A_j^*) = \infty$ it is obvious. Otherwise, it follows from the fact that the range of the operator A_j and consequently that of A_j^* are not closed (see Lemma 4.7.1). Set $Q_j = I - P_j$ and $B_j = P_j A Q_j$. As was proven at the first step of the proof, the operators B_j are compact. Therefore, if the sequence $\{x'_{jn}\}_{n=1}^{\infty}$ satisfies condition (4), then

$$\left\|B_j^* x'_{jn}\right\|_{p'} \underset{n\to\infty}{\longrightarrow} 0. \tag{4'}$$

From (4) and (4′) it follows that for every $j = 1, 2, \ldots$ there is a function $x'_j \in L^{p'}(e_j)$ such that $\|x'_j\| = 1$ and $\|A_j^* x'_j\| + \|B_j^* x'_j\| < 2^{-j}$. Afterward

$$\|A^* x'_j\| = \|A^* P_j^* x'_j\| \le \|A_j^* x'_j\| + \|B_j^* x'_j\| < 2^{-j}$$

and, therefore,

$$\sum_{j=1}^{\infty} \|A^* x'_j\| < \infty. \tag{5}$$

Let $x_j \in L^p(e_j)$ be functions such that $\|x_j\|_p = 1 = \langle x_j, x'_j\rangle$. We define the projection $Q : L^p \to L^p$ by the equality

$$Qx = \sum_{j=1}^{\infty} \langle x, x'_j\rangle x_j \quad (x \in L^p).$$

It is obvious that

$$QAx = \sum_{j=1}^{\infty} \langle x, A^* x'_j\rangle x_j$$

and, by (5), we conclude that

$$\sum_{j=1}^{\infty} A^* x'_j \otimes x_j$$

is a nuclear representation of the operator QA. Since QA is a nuclear operator, it is a stably regular operator.

IV. Let $P_0 = I - Q$. It is evident that the operator $P_0 A = A - QA$ satisfies condition (2) of the theorem. Demonstrate that the operator $P_0 AWQ$ is regular for every operator W in $\mathscr{L}(L^p, L^p)$. By analogy with (2), we put $U = I - \delta P_0 WQ$, where $\delta > 0$ is a sufficiently small number. Then

$$L^{\sim}(L^p, L^p) \ni UP_0AU^{-1} = P_0A - \delta P_0 A P_0 WQ. \tag{6}$$

Afterward we assign $V = I - \delta QWQ$, where the number $\delta > 0$ is chosen so small that $\delta\|QWQ\| < 1$ and $\|I - V^{-1}\| < \varepsilon$. Note that the operator V^{-1} has the form

$$V^{-1} = I + R = I + RQ,$$

where

$$R = \sum_{k=1}^{\infty} (\delta QWQ)^k$$

and $\|R\| < \varepsilon$. Therefore,

$$L^{\sim}(L^p, L^p) \ni V^{-1} P_0 AV = P_0 A - \delta P_0 AQWQ.$$

Subtracting the last equality from (6), we obtain what was required.

V. By Corollaries 3.1.5(d) and 3.8.2, in order to prove the stable regularity of the operator P_0A, it suffices to check that P_0AS is a regular operator for every operator $S : l^p \to L^p$. Define the mapping $\varphi : l^p \to L^p$ by the equality

$$\varphi(t) = \sum_{k=1}^{\infty} t_k x_k \quad \big(t = \{t_k\}_{k=1}^{\infty} \in l^p\big).$$

It is obvious that φ is a regular isometric mapping of l^p onto $\overline{\operatorname{lin}}\big(\{x_k\}_{k=1}^{\infty}\big)$. Put $W = S\varphi^{-1}Q$. Then $S = W\varphi$ and $W = WQ$, and it follows from what was proven above that the operator $P_0AW = P_0AWQ$ is regular and thereby such is the operator $P_0AS = P_0AW\varphi$. $\triangleright$

4.7.3. Theorem 4.7.2 claims that if the orbit of an operator is "small" in the sense that it is contained in $L^{\sim}(L^p, L^p)$, then the operator differs from a scalar operator by a stably regular summand. Another classical quantity, alongside the orbit, showing to what extent an operator differs from a scalar one is the set of commutators of the operator. It turns out that, replacing the orbit by this set, we obtain the following theorem which is similar to Theorem 4.7.2.

Theorem. *Let $1 < p < \infty$, $(T, \mathfrak{A}, \mu)$ be a space with σ-finite measure, $L^p = L^p(T, \mu)$, $A \in \mathscr{L}(L^p, L^p)$, and $0 < \varepsilon < 1$. The following assertions are equivalent:*

(1) *$UA - AU \in L^{\sim}(L^p, L^p)$ for every operator U in $\mathscr{L}(L^p, L^p)$;*

(2) *$UA - AU \in L^{\sim}(L^p, L^p)$ for every isomorphism $U : L^p \to L^p$ such that $\|U - I\| < \varepsilon$, where I is the identity operator in L^p;*

(3) *the operator A can be represented as $\alpha I + B$, where $\alpha \in \mathbb{C}$ and $B \in \mathscr{L}^{\sim}_{\mathrm{st}}(L^p, L^p)$.*

$\lhd$ The implication (3) $\Rightarrow$ (2) is trivial. The implication (2) $\Rightarrow$ (1) ensues from the identity

$$(I + \delta U)A - A(I + \delta U) = \delta(UA - AU),$$

where $0 < \delta < \varepsilon \|U\|^{-1}$ and U is an arbitrary operator in $\mathscr{L}(L^p, L^p)$.

(1) $\Rightarrow$ (3): The proof of this implication represents a modification of the arguments proving Theorem 4.7.2. As in that case, we can assume that $1 < p \leq 2$ and 0 belongs to the essential spectrum of A. Split the further proof into several steps.

I. We decompose the set T into two parts e and e' so that the spaces $L^p(e) = L^p(e, \mu)$ and $L^p(e') = L^p(e', \mu)$, as Banach lattices, be isomorphic to L^p (as in the proof of Theorem 4.9.2 we may assume that the measure μ is continuous or has infinitely many atoms). Identify $L^p(e)$ and $L^p(e')$ with subspaces (bands) of L^p, and let P and P' be the band projections onto these subspaces in L^p. Let $W \in \mathscr{L}(L^p, L^p)$. By hypothesis, the operator $PWA - APW$ is regular. Multiplying it by P' from the left, we see that $P'APW \in L^\sim(L^p, L^p)$. Thus, the operator $P'AP$ is right stably regular and consequently (see 3.7.5(e) and 3.8.2) we obtain

$$P'AP \in \Pi_2(L^p, L^p) = \mathscr{L}^\sim_{\mathrm{st}}(L^p, L^p) \subset K(L^p, L^p).$$

In a similar way,

$$PAP' \in \Pi_2(L^p, L^p) = \mathscr{L}^\sim_{\mathrm{st}}(L^p, L^p) \subset K(L^p, L^p).$$

II. Consider the operators $A_{11} = PAP$ and $A_{22} = P'AP'$ as operators in $L^p(e)$ and $L^p(e')$ respectively and prove that 0 is a point of their essential spectra. Putting $U = PWP'$, we find that

$$PWP'A - APWP' \in L^\sim(L^p, L^p).$$

Afterward multiplying the left-hand side of the containment by P from the left and by P' from the right, we obtain

$$PWA_{22}P' - A_{11}WP' \in L^\sim(L^p, L^p). \tag{7}$$

Fix an arbitrary order isomorphism $\omega : L^p(e') \to L^p(e)$ and assign $W = \omega P'V$, where V is an arbitrary operator in $\mathscr{L}(L^p, L^p)$. Then (7) implies that

$$\omega P'VA_{22}P' - A_{11}\omega P'VP' \in L^\sim(L^p, L^p)$$

and consequently

$$P'VA_{22}P' - \omega^{-1}A_{11}\omega P'VP' \in L^{\sim}(L^p, L^p). \tag{8}$$

Furthermore,

$$P'VP'A - AP'VP' \in L^{\sim}(L^p, L^p).$$

Multiplying the left-hand side of the containment by P' from the left and the right, we obtain

$$P'VA_{22}P' - A_{22}VP' \in L^{\sim}(L^p, L^p).$$

Subtracting the left-hand side of the preceding containment from the left-hand side of (8), we infer that

$$(A_{22} - \omega^{-1}A_{11}\omega)P'VP' \in L^{\sim}(L^p, L^p).$$

Since the operator V is arbitrary, the latter containment shows that the operator $A_{22} - \omega^{-1}A_{11}\omega$ is right stably regular and consequently (see 3.7.5(c))

$$A_{22} - \omega^{-1}A_{11}\omega \in \Pi_2(L^p(e'), L^p(e')) \subset K(L^p(e'), L^p(e')).$$

Thus, the operators A_{22} and $\omega^{-1}A_{11}\omega$ and, hence, A_{22} and A_{11} have the same essential spectrum and the kernels of the operators are finite-dimensional or not simultaneously. Now repeating verbatim the closing part of the second step of the proof of Theorem 4.7.2, we infer that 0 belongs to the essential spectrum of the operators.

III. Repeating the arguments of the third step of the proof of Theorem 4.7.2, we can construct an operator $Q : L^p \to L^p$ such that QA is a nuclear operator and the subspace $Q(L^p)$ coincides with $\overline{\mathrm{lin}}(\{x_k\}_{k=1}^{\infty})$, where the functions x_k have pairwise disjoint supports.

IV. Let $P_0 = I - Q$. It is obvious that the operator $P_0A = A - QA$ satisfies condition (1) of the theorem. Prove that $P_0AWQ \in L^{\sim}(L^p, L^p)$ for every operator W in $\mathscr{L}(L^p, L^p)$. Indeed, $UP_0A - P_0AU \in L^{\sim}(L^p, L^p)$ and, putting $U = WQ$, we obtain $P_0AWQ \in L^{\sim}(L^p, L^p)$.

V. Now, to prove the stable regularity of the operator P_0A, we repeat the arguments of the last (fifth) step of the proof of Theorem 4.7.2. $\rhd$

4.7.4. In the case $p = 2$ arbitrary isomorphisms in Theorems 4.7.2 and 4.7.3 may be replaced by unitary operators.

Theorem. *Let $(T, \mathfrak{A}, \mu)$ be a space with σ-finite measure, $L^2 = L^2(T, \mu)$, and $A \in \mathscr{L}(L^2, L^2)$. The following assertions are equivalent:*

(1) *the operator UAU^{-1} is regular for every unitary operator $U : L^2 \to L^2$;*

(2) *the operator $UA - AU$ is regular for every unitary operator $U : L^2 \to L^2$;*

(3) *the operator A can be represented as $A = \alpha I + B$, where $\alpha \in \mathbb{C}$ and B is a Hilbert-Schmidt operator.*

$\lhd$ The implications (3) $\Rightarrow$ (1) and (3) $\Rightarrow$ (2) are trivial. The proof of the implication (1) $\Rightarrow$ (3) is similar to that of Theorem 3.9.2 (see also the proofs of close assertions in [19, Theorem 16.5] and in [75]). To prove the implication (2) $\Rightarrow$ (3), it suffices to recall that every operator is a linear combination of unitary operators (see the Remark next to Corollary 3.4.10). $\rhd$

4.7.5. L^0-regular operators.

DEFINITION. Let $1 \leq p, q \leq \infty$. An operator $U : L^p(T, \mu) \to L^q(T_1, \mu_1)$ is said to be *L^0-regular* if it carries each $L^p(T, \mu)$-order bounded set to an $L^0(T_1, \mu_1)$-order bounded set; i.e., if it is regular as an operator from $L^p(T, \mu)$ to $L^0(T_1, \mu_1)$.

If an operator UV (WU) is L^0-regular for every $V \in \mathscr{L}(L^p(T, \mu), L^p(T, \mu))$ ($W \in \mathscr{L}(L^q(T_1, \mu_1), L^q(T_1, \mu_1))$) then the operator U is called *right* (*left*) *stably L^0-regular.*

It is clear that if the measure μ_1 is purely atomic then every operator with values in $L^q(T_1, \mu_1)$ is L^0-regular. For this reason, we henceforth confine ourselves to the case when $T_1 = (0, 1)$ and μ_1 is the Lebesgue measure.

(a) Lemma. *If $0 < \varepsilon < 1$, $A \in \mathscr{L}(L^p(T, \mu), L^q(0, 1))$, and the operator UA is L^0-regular for every isomorphism $U : L^q(0, 1) \to L^q(0, 1)$ such that $\|U - I\| < \varepsilon$ (in particular, if the operator A is left stably L^0-regular), then the operator A is left stably regular.*

$\lhd$ The assertion of the lemma is a restatement of Corollary 3.4.11. $\rhd$

(b) Lemma. *Let $0 < r < \infty$ and $U \in \mathscr{L}(X, L^p(T, \mu))$. If $VU \in \Pi_r(X, l^p)$ for every operator V in $\mathscr{L}(L^p(T, \mu), l^p)$ then $U \in \Pi_r(X, L^p(T, \mu))$.*

$\lhd$ The closed graph theorem implies that the inequality $\pi_r(VU) \leq C\|V\|$ holds for some number C and an arbitrary operator V in $\mathscr{L}(L^p(T, \mu), l^p)$. Consider an arbitrary collection $x_1, \dots, x_n$ of vectors in the space X and put $f_k = Ux_k$. Fix an arbitrary number ε, $0 < \varepsilon < 1$, and for every function f_k find a step function g_k such that

$$\|f_k - g_k\| \le \varepsilon \|f_k\| \quad (k = 1, 2, \dots, n).$$

Let $L = \operatorname{lin}(\chi_{e_1}, \dots, \chi_{e_N})$, where χ_{e_j} are the characteristic functions of pairwise disjoint sets $e_j \subset T$, so that the subspace L contains all functions g_k; and let P be a projection of $L^p(T, \mu)$ onto L, $\|P\| = 1$. Finally, let ω be the isometric embedding of L into l^p and let $V = \omega P$. Since $\|f_k\| \le \frac{1}{1-\varepsilon} \|g_k\|$, we have

$$\begin{aligned}
\left(\sum_{k=1}^{n} \|Ux_k\|^r\right)^{1/r} &\le \frac{1}{1-\varepsilon} \left(\sum_{k=1}^{n} \|g_k\|^r\right)^{1/r} \\
&= \frac{1}{1-\varepsilon} \left(\sum_{k=1}^{n} \|VUx_k - V(Ux_k - g_k)\|^r\right)^{1/r} \\
&\le \frac{1}{1-\varepsilon} \left[\left(\sum_{k=1}^{n} \|VUx_k\|^r\right)^{1/r} + \left(\sum_{k=1}^{n} \|f_k - g_k\|^r\right)^{1/r}\right] \\
&\le \frac{1}{1-\varepsilon} \pi_r(VU) \sup\left\{\left(\sum_{k=1}^{n} |\langle x_k, x'\rangle|^r\right)^{1/r} \mid x' \in X^*,\ \|x'\| \le 1\right\} + \frac{\varepsilon}{1-\varepsilon}.
\end{aligned}$$

It follows from the arbitrariness of ε that

$$\begin{aligned}
\left(\sum_{k=1}^{n} \|Ux_k\|^r\right)^{1/r} &\le \pi_r(VU) \sup\left\{\left(\sum_{k=1}^{n} |\langle x_k, x'\rangle|^r\right)^{1/r} \mid \|x'\| \le 1\right\} \\
&\le C \sup\left\{\left(\sum_{k=1}^{n} |\langle x_k, x'\rangle|^r\right)^{1/r} \mid \|x'\| \le 1\right\}. \ \triangleright
\end{aligned}$$

(c) Lemma. *Let $S \in \mathscr{L}(X, L^p(0,1))$. If $\infty = \sup\{|Sx| \mid \|x\| \le 1\}$ on some set $E_0 \subset [0,1]$, $\operatorname{mes}(E_0) > 0$, then there exist a sequence $(x_k)_{k=1}^{\infty} \subset X$ and a set $E \subset E_0$ such that*

$$\operatorname{mes}(E) > 0, \quad \lim_{n\to\infty} \|x_n\| = 0, \quad \sup_n\{\,|Sx_n|\,\} = \infty \text{ on } E.$$

$\triangleleft$ By Lemma 3.4.11(a), for every $n \in \mathbb{N}$ there are vectors $x_1^{(n)}, \dots, x_{m_n}^{(n)}$ such that

(1) $\|x_j^{(n)}\| \leq 1$ for $n \in \mathbb{N}$, $j = 1, 2, \dots, m_n$;

(2) $\sup_{1 \leq j \leq m_n} |Sx_j^{(n)}| > 4^n$ on the set $E_0 \setminus e_n$ and $\operatorname{mes}(e_n) < 2^{-n-1} \operatorname{mes}(E_0)$.

Putting $E := E_0 \setminus \bigcup_{n=1}^{\infty} e_n$ and enumerating the family $\{2^{-u} x_j^{(n)}\}$, where $n \in \mathbb{N}$ and $1 \leq j \leq m_n$, in an arbitrary manner, we obtain the required assertion. $\triangleright$

4.7.6. There is an assertion strengthening Theorem 4.9.2 for L^0-regular operators in $L^p(0,1)$. As a preliminary, we give the following definition:

DEFINITION. An operator $A \in \mathscr{L}(L^p(0,1), L^p(0,1))$ is a *universal L^0-regular* (*universal regular*) *operator* if UAU^{-1} is a L^0-regular (integral) operator for every isomorphism $U : L^p(0,1) \to L^p(0,1)$.

Theorem. *Let $1 < p < \infty$, $L^p := L^p(0,1)$, and $A = \mathscr{L}(L^p, L^p)$. The following assertions are equivalent:*

(1) *the operator A is universal L^0-regular;*

(2) *the operator UAU^{-1} is L^0-regular for every isomorphism $U : L^p \to L^p$ such that $\|U - I\| < \varepsilon$;*

(3) *the operator A can be represented as $A = \alpha I + B$, where $\alpha \in \mathbb{C}$ and $B \in \mathscr{L}_{\mathrm{st}}^{\sim}(L^p, L^p)$.*

$\triangleleft$ The implications (3) $\Rightarrow$ (1) and (1) $\Rightarrow$ (2) are trivial. Prove (2) $\Rightarrow$ (3). As in the proof of Theorem 4.7.2, we may assume that 0 belongs to the essential spectrum of the operator A. However, we cannot confine ourselves to the case $1 < p \leq 2$, since passing to the adjoint operator, we, generally speaking, loose the property of L^0-regularity. Therefore, the cases $p \leq 2$ and $p > 2$ should be considered separately, despite they have much in common, as we will see. First we suppose that $1 < p \leq 2$ and split the proof into several steps that are modifications of the corresponding steps of the proof of Theorem 4.7.2. The symbol $\operatorname{mes}(e)$ stands for the Lebesgue measure of a set e.

I. Let $e \subset (0,1)$, $0 < \operatorname{mes}(e) < 1$, and $e' := (0,1) \setminus e$. We identify the spaces $L^p(e)$ and $L^p(e')$ with subspaces of L^p. Let V and W be automorphisms of $L^p(e)$ and $L^p(e')$ such that

$$\|V - I_{L^p(e)}\| < \varepsilon, \quad \|W - I_{L^p(e')}\| \leq \varepsilon,$$

and $U = V \oplus W$; and let P and P' be the band projections onto $L^p(e)$ and $L^p(e')$ in L^p. Note that $PU = UP$ and $P'U^{-1} = U^{-1}P'$. The operator UAU^{-1} and, hence, the operator $PUAU^{-1}P'$ are L^0-regular. Putting $A_{12} := PAP'$, we see that the operator $VA_{12}W^{-1}$ is L^0-regular too. It follows from the arbitrariness of the operator V that the operator $A_{12}W^{-1}$ is regular (moreover, it is stably regular from the right). Taking $W^{-1} = \delta R + I_{L^p(e')}$, where R is an arbitrary operator in $\mathscr{L}(L^p(e'), L^p(e'))$ and δ is a sufficiently small positive number, we find that the operator A_{12} is stably regular from the right. Consequently (see 3.7.5(a)), we have

$$A_{12} \in \Pi_2(L^p, L^p) = \mathscr{L}^{\sim}_{\mathrm{st}}(L^p, L^p) \subset K(L^p, L^p).$$

Similar inclusions are obviously valid for the operator $A_{21} := P'AP$.

II. The arguments of this step of the proof repeat verbatim those of the second and third step of the proof of Theorem 4.7.2 with the only difference that the operator Q is defined by the equality

$$Q := \sum_{k=2}^{\infty} x'_k \otimes x_k,$$

where the summation is carried out over $k \geq 2$ rather than over $k \geq 1$. In this regard, the subspace $(I - Q)(L^p)$ contains $L^p(e_1)$.

III. Put $P_0 := I - Q$ and check that the operator P_0A is stably regular. Indeed, the operator $P_0A = A - QA$ obviously satisfies condition (2) of the theorem. Therefore, putting $U := I - \delta P_0VQ$ for an arbitrary $V \in \mathscr{L}(L^p, L^p)$ and a sufficiently small $\delta > 0$, we see that the operator

$$UP_0AU^{-1} = P_0A + \delta A_0VQ,$$

where $A_0 := P_0AP_0$, and thereby A_0VQ satisfy condition (2) of the theorem with ε replaced by $\varepsilon_1 > 0$. Now take an arbitrary operator W in $\mathscr{L}(L^p, L^p)$ and assign

$$U_1 := I - \delta_1 P_0WP_0,$$

where δ_1 is a sufficiently small positive number. It is clear that

$$U_1^{-1} = I + \sum_{k=1}^{\infty}(\delta_1 P_0WP_0)^k = I + R = I + P_0R,$$

where $R := U_1^{-1} - I$ and $\|R\| < \varepsilon_1$. Simple calculations show that

$$U_1 A_0 V Q U_1^{-1} = A_0 V Q - \delta_1 P_0 W A_0 V Q. \tag{9}$$

As follows from the definition of the operator Q at the end of the preceding step of the proof, the band projection P_1 onto $L^p(e_1)$ satisfies the equalities $P_1 P_0 = P_0 P_1 = P_1$. Multiplying the equality by P_1 we find that

$$\text{the operator } P_1 W A_0 V Q \text{ is } L^0\text{-regular.} \tag{10}$$

Since W is arbitrary (and the spaces L^p and $L^p(e)$ are order isomorphic), we conclude that the operator $A_0 V Q$ is regular by Lemma 4.9.5. Arguing as in part V of the proof of Theorem 4.7.2 (with A_0 instead of $P_0 A$), we infer that the operator A_0 is stably regular.

Now put $A_0' := P_0 A Q$ and prove that the operator $A_0' V Q$ is regular. Like $P_0 A$ the operator $A_0' = P_0 A - A_0$ satisfies condition (2) of the theorem too. Assign $U := I - \delta P_0 W P_0$, where W is an arbitrary operator in $\mathscr{L}(L^p, L^p)$ and δ is a sufficiently small positive number. Then

$$U A_0' U^{-1} = U A_0' = A_0' - \delta P_0 W A_0'.$$

It shows that the operator $P_0 W A_0'$ meets condition (2) of the theorem with ε replaced by a sufficiently small positive number. Now put $U_1^{-1} = I - \delta_1 Q V Q$, where V is an arbitrary operator in $\mathscr{L}(L^p, L^p)$ and δ_1 is a sufficiently small positive number. Then

$$U_1 = I + \sum_{k=1}^{\infty} (\delta_1 Q V Q)^k = I + R = I + RQ.$$

Therefore

$$U_1 P_0 A_0' U_1^{-1} = P_0 W A_0' U_1^{-1} = P_0 W A_0' - \delta P_0 W A_0' V Q.$$

Consequently, the operator $P_0 W A_0' V Q$ is L^0-regular and thereby such is the operator $P_1 W A_0' V Q$. Afterward the arguments can be completed as for the operator $P_1 W A_0 V Q$.

Thus, $P_0 A = A_0 + A_0' \in \mathscr{L}_{\mathrm{st}}^{\sim}(L^p, L^p)$ and the theorem is proven in the case $1 < p \leq 2$.

IV. Now we turn to the case $2 \leq p < \infty$. At the first step of the proof, as in the case $p \leq 2$, we conclude that the operator A_{12} is left stably regular. Consequently, the operator A_{12}^* is right stably regular in $L^{p'}$. Since $1 < p' \leq 2$, by Theorem 3.7.5(a), this means that

$$A_{12}^* \in \mathscr{L}_{\mathrm{st}}^{\sim}(L^{p'}, L^{p'}) \subset K(L^{p'}, L^{p'}).$$

It is evident that a similar inclusion holds for the operator A_{12} itself.

The further argument remains the same up to relation (10) inclusively.

Since W is arbitrary; therefore, (10) implies that A_0VQ is left stably regular by Lemma 4.7.5(a). Consequently, the operator $(A_0VQ)^* = Q^*V^*A_0^*$, which is right stably regular, is 2-absolutely summing by Theorem 3.7.5(a). Since the space $Q^*(L^{p'})$ is isomorphic to $l^{p'}$ and V is arbitrary, we obtain

$$A_0^* \in \Pi_2(L^{p'}, L^{p'}) = \mathscr{L}_{\mathrm{st}}^{\sim}(L^{p'}, L^{p'})$$

by Lemma 7.4.5(b). Consequently, $A_0 \in \mathscr{L}_{\mathrm{st}}^{\sim}(L^{p'}, L^{p'})$.

Passing to the operator $A_0' := P_0AQ$, as in the case $p \leq 2$, we find that the operator P_0WA_0' is L^0-regular for every $W \in \mathscr{L}(L^p, L^p)$. The operator P_0WA_0' is L^0-regular along with it. It follows from Lemma 4.7.5(a) that the operator A_0' is left stably regular. Consequently, $A_0' \in \mathscr{L}_{\mathrm{st}}^{\sim}(L^p, L^p)$ by Corollary 3.7.5(f). ▷

Corollary. *Let $1 < p < \infty$, $L^p = L^p(0,1)$, and $A \in \mathscr{L}(L^p, L^p)$. If UAU^{-1} is an integral operator for every $U : L^p \to L^p$ then $A \in \mathscr{L}_{\mathrm{st}}^{\sim}(L^p, L^p)$.*

◁ By Theorem 4.7.6, $A = \alpha I + B$, where $B \in \mathscr{L}_{\mathrm{st}}^{\sim}(L^p, L^p)$. As was mentioned in 3.8.2(d), B is an integral operator. Consequently, $\alpha I = A - B$ is an integral operator, which is possible only for $\alpha = 0$. ▷

REMARK. For $p = 2$ assertions (1) and (3) of Theorem 4.7.6 remain equivalent if we consider in (1) only unitary operators (see [75]).

4.7.7. In the case of the space $L^p(0,1)$ Theorem 4.7.3 can be strengthened as follows:

Theorem. *Let $2 \leq p < \infty$, $L^p = L^p(0,1)$, $A \in \mathscr{L}(L^p, L^p)$, and $0 < \varepsilon < 1$. The following assertions are equivalent:*

(1) *the operator $UA - AU$ is L^0-regular for every operator $U \in \mathscr{L}(L^p, L^p)$;*

(2) *the operator* $UA - AU$ *is* L^0*-regular for every isomorphism* $U : L^p \to L^p$ *such that* $\|U - I\| < \varepsilon$;

(3) *the operator* A *may be represented as* $A = \alpha I + B$, *where* $\alpha \in \mathbb{C}$ *and* $B \in \mathscr{L}_{\mathrm{st}}^{\sim}(L^p, L^p)$.

To prove the theorem, we need the following lemma (see, for example, [60, Theorem 49]).

Lemma. *If* $g \in L^p(T,\mu)$, $M_g : L^\infty(T,\mu) \to L^p(T,\mu)$ *denotes the operator of multiplication by the function* g, *and* $U : X \to L^\infty(T,\mu)$ *is a compact operator; then the operator* $M_g U$ *admits the following factorization:*

$$M_g U : X \xrightarrow{\beta} l^\infty \xrightarrow{M_\delta} l^p \xrightarrow{\alpha} L^p(T,\mu),$$

where $\alpha \in \mathscr{L}(l^p, L^p(T,\mu))$, $\beta \in \mathscr{L}(X, l^\infty)$, *and* M_δ *is the operator of multiplication by a sequence* $\delta \in l^p$.

Let us turn to proving the theorem.

$\lhd$ The implication (3) $\Rightarrow$ (2) is trivial, and the implication (2) $\Rightarrow$ (1) is proven as in Theorem 4.7.3. We will prove (1) $\Rightarrow$ (2).

As in Theorem 4.7.3, we can assume that 0 belongs to the essential spectrum of the operator A. Split the further arguments into several steps, following the general scheme of the proofs of Theorems 4.7.2, 4.7.3, and 4.7.6. As above, we identify the space $L^p(e)$, where $e \subset (0,1)$, with a subspace of L^p.

I. Let e be an arbitrary measurable subset of the interval $(0,1)$, $0 < \mathrm{mes}\,(e) < 1$, $e' := (0,1) \setminus e$, and P and P' be the band projections onto $L^p(e)$ and $L^p(e')$ in the space L^p. The operator $PVPA - APVP$ is L^0-regular for every $V \in \mathscr{L}(L^p, L^p)$ by hypothesis; therefore, multiplying it by P' from the right, we see that $PVPAP'$ is an L^0-regular operator. Since the operator V is arbitrary and the spaces L^p and $L^p(e)$ are order isomorphic, owing to Lemma 4.7.5(a), we infer that the operator PAP' is left stably regular and, consequently, bilaterally stably regular by Corollary 3.7.5(f). It is clear that the operator $P'AP$ is stably regular as well.

II. Now consider the operators $A_{11} := PAP$ and $A_{22} := P'AP'$ as operators in $L^p(e)$ and $L^p(e')$ and check that 0 belongs to their essential spectra. Let $W \in \mathscr{L}(L^p, L^p)$ and $V' := PWP'$. Then it is easy to establish that

$$V'A - AV' = PWA_{22}P' - jA_{11}WP' + \widetilde{U},$$

where j is the identical embedding of $L^p(e)$ into L^p and $\widetilde{U} \in \mathscr{L}^{\sim}_{\mathrm{st}}(L^p, L^p)$. Therefore, the operator $PWA_{22}P' - jA_{11}WP'$ is L^0-regular. Fix an arbitrary order isomorphism $\omega : L^p(e) \to L^p(e')$ and put $W := jV\omega$, where $V \in \mathscr{L}(L^p(e), L^p(e))$. Afterward, the L^0-regularity of the operator $PWA_{22}P' - jA_{11}WP'$ implies the L^0-regularity of the operator

$$V\omega A_{22} - A_{11}V\omega : L^p(e') \to L^p(e)$$

and, consequently, the L^0-regularity of the operator

$$V\omega A_{22}\omega^{-1} - A_{11}V : L^p(e) \to L^p(e). \tag{11}$$

Moreover, since the commutator $V_1A - AV_1$ is L^0-regular for $V_1 := PjVP$, multiplying it by P from both sides, we infer that the operator $PV_1AP - PAV_1P$ is L^0-regular and, hence, so is the operator

$$VA_{11} - A_{11}V : L^p(e) \to L^p(e).$$

Using (11), we see that the operator

$$V(\omega A_{22}\omega^{-1} - A_{11}) : L^p(e) \to L^p(e)$$

is L^0-regular. This fact, together with Lemma 4.7.5 and Corollary 3.7.5(f), yields $\omega A_{22}\omega^{-1} - A_{11} \in \mathscr{L}^{\sim}_{\mathrm{st}}(L^p, L^p)$, which implies that $\omega A_{22}\omega^{-1} - A_{11}$ is a compact operator.

Repeating the arguments of steps II and III of the proof of Theorem 4.7.2, we see that, in fact, zero belongs to the essential spectrum of the operators A_{11} and A_{22}; hence, there are sequences $\{x_n\}_{n=1}^{\infty} \subset L^p$ and $\{x'_n\}_{n=1}^{\infty} \subset L^p$ such that

(1) the supports e_n of the function x_n are pairwise disjoint;

(2) $\|x_n\|_p = \|x'_n\|_{p'} = \langle x_n, x'_n \rangle = 1 \quad (n := 1, 2, \dots)$;

(3) $\sum_{n=1}^{\infty} \|A^*x'_n\| < \infty$.

We can obviously assume that $\operatorname{mes}\left(\bigcup_{k=1}^{\infty} e_k\right) < 1$. Put

$$e_0 := (0,1) \setminus \bigcup_{k=1}^{\infty} e_k$$

and let P_0 be the band projection onto $L^p(e_0)$ in L^p. It is clear that the operator $Q : L^p \to L^p$ defined by the equality

$$Qx = \sum_{k=1}^{\infty} \langle x_k, x'_k \rangle x_k \quad (x \in L^p)$$

is a projection onto the subspace $L = \overline{\mathrm{lin}}(\{x_k\}_{k=1}^{\infty})$ which is isomorphic to the space l^p (as a Banach space). Furthermore, it is clear that $P_0\,(I-Q) = P_0$ and QA is a nuclear operator. Since $QA \in \mathscr{L}^{\sim}_{\mathrm{st}}(L^p, L^p)$, all commutators of the operator $A_0 = (I-Q)\,A$ are L^0-regular. Therefore, the operator

$$WQA_0 - A_0WQ = -A_0WQ \tag{12}$$

is L^0-regular for every operator $W \in \mathscr{L}(L^p, L^p)$. Check that the operator AW is L^0-regular too. Suppose the contrary. Then there exist an operator $W_0 : L^p \to L^p$ and a function $g_0 \in L^p_+$ such that the set

$$C := \{|A_0W_0x| \mid x \in I_{g_0}\},$$

where $I_{g_0} := \{x \in L^p \mid |x| \le g_0\}$, is unbounded in $L^0(0,1)$. Assign $E_0 := \{t \in (0,1) \mid (\sup C)(t) = \infty\}$. Put $S = A_0W_0M_{g_0}$, where $M_{g_0} : L^\infty(0,1) \to L^p$ is the operator of multiplication by the function g_0. By Lemma 4.7.5(c), there exist a compact set K contained in the ball $B_{L^\infty(0,1)}$ and a set $E \subset E_0$, $\mathrm{mes}\,(E) < 0$, such that $\sup\{|Sx| \mid x \in K\} = \infty$ on E. Obviously, we may suppose that the set K is absolutely convex. Put $X = \mathrm{lin}(K)$ and make the set X into a Banach space by furnishing it with the norm defined as the Minkowski functional of K. Appealing to the lemma stated before the proof of the theorem, we see that the operator $M_g i$, where i is the identical embedding of X into $L^\infty(0,1)$, meets the following commutative diagram:

$$\begin{array}{ccccc}
X & \xrightarrow{\ i\ } & L^\infty(0,1) & \xrightarrow{M_{g_0}} & L^p \\
\beta\big\downarrow & & & & \big\uparrow\alpha \\
l^\infty & & \xrightarrow{\qquad M_g \qquad} & & l^p,
\end{array}$$

where $\alpha \in \mathscr{L}(l^p, L^p)$, $\beta \in \mathscr{L}(X, l^\infty)$, $\|\beta\| \leq 1$, and M_δ is the operator of multiplication by a sequence $\delta = \{\delta_k\}_{k=1}^\infty$ in l^p. Since the subspace L is isomorphic to l^p, the operator α admits the following factorization:

$$\alpha : l^p \xrightarrow{\sigma} L \xrightarrow{j_0} L^p \xrightarrow{\alpha_1} L^p,$$

where σ is the isomorphism between l^p and L sending the standard basis of l^p into the functions x_k, $\alpha = \alpha_1 Q$, and j_0 is the identical embedding of L into L^p. Let

$$g_1 = \sum_{k=1}^{\infty} |\delta_k| \cdot |x_k|, \quad I_{g_1} = \{x \in L^p \mid |x| \leq g_1\},$$

and let $C_1 = \sigma M_\delta \beta(K)$. It is clear that $C_1 \subset I_{g_1}$ and $C_1 = Q(C_1) \subset Q(I_{g_1})$. Therefore, $M_{g_0}(K) = \alpha_1(C_1) \subset \alpha_1 Q(I_{g_1})$. Consequently,

$$A_0 W_0 M_{g_0}(K) \subset A_0 W_0 \alpha_1 Q(I_{g_1})$$

and

$$\sup |(A_0 W_0 \alpha_1 Q)(I_{g_1})| \geq \sup |(A_0 W_0 M_{g_0})(K)| = \infty \quad \text{on} \quad E.$$

Thus, the operator $A_0 W_0 \alpha_1 Q$ is not L^0-regular, which contradicts the above-proven L^0-regularity of an operator of the form (12).

Thus, we have checked that the operator $A_0 W$ is L^0-regular for every $W \in \mathscr{L}(L^p, L^p)$. Since the commutator $WA_0 - A_0 W$ is L^0-regular as well, the operator WA is L^0-regular too. Hence, Lemma 4.7.5(a) implies that the operator A_0 is left stably regular. By Corollary 3.7.5(f), the latter is equivalent to the stable regularity of the operator A_0. $\triangleright$

Corollary. *Let $2 \leq p < \infty$, $L^p = L^p(0,1)$, and $A \in \mathscr{L}(L^p, L^p)$. If $UA - AU$ is an integral operator for every $U \in \mathscr{L}(L^p, L^p)$, then the operator A can be represented as $A = \alpha I + B$, where $\alpha \in \mathbb{C}$ and $B \in \mathscr{L}_{\mathrm{st}}^{\sim}(L^p, L^p)$.*

4.7.8. In the case of Hilbert space, Theorems 4.7.6 and 4.7.7 may be restated as follows:

Theorem. *Let $L^2 = L^2(0,1)$ and $A \in \mathscr{L}(L^2, L^2)$. The following assertions are equivalent:*

(1) *the operator UAU is L^0-regular for every unitary operator $U : L^2 \to L^2$;*

(2) *the operator* $UA - AU$ *is* L^0*-regular for every unitary operator* $U : L^2 \to L^2$;

(3) *the operator* A *is representable as* $A = \alpha I + B$, *where* $\alpha \in \mathbb{C}$ *and* B *is a Hilbert-Schmidt operator.*

$\lhd$ The implications (3) $\Rightarrow$ (1) and (3) $\Rightarrow$ (2) are trivial. As was mentioned in Corollary 4.7.6, the implication (1) $\Rightarrow$ (3) was actually established in [75]. The implication (2) $\Rightarrow$ (3) can be proven in the same way as the corresponding implication in Theorem 4.7.4. $\rhd$

4.7.9. Theorems 4.7.6 and 4.7.7 generalize Theorems 4.7.2 and 4.7.3. However, unlike Theorem 4.7.6, which is valid similarly as Theorems 4.7.2 and 4.7.3 for every p in the interval $(1, \infty)$, Theorem 4.7.7 covers only the case $2 \le p < \infty$. It turns out that this restriction is essential and Theorem 4.7.7 fails for $1 < p < 2$. The aim of this section is to construct a corresponding counterexample:

Theorem. *Let* $1 < p < 2$ *and* $L^p = L^p(0,1)$. *There exists a positive compact operator* $A : L^p \to L^p$ *with the following properties:*

(1) UA *and* AU *are integral operators for every operator* $U : L^p \to L^p$;

(2) *the operator* A^* *is dominated;*

(3) *there is a function* g *in* $L^0(0,1)$ *such that* $\|Ax\| \le g\,\|x\|$ *for every* $X \in L^p$;

(4) *the operator* A *is not a stably regular but is left stably regular.*

Remark. Property (1) implies that all commutators of the operator A are integral operators. Comparing this fact with property (4), we see that Theorem 4.7.7 fails for $1 < p < 2$.

$\lhd$ Let $\{e_k\}_{k=1}^{\infty}$ be a sequence of pairwise disjoint subsets of the interval $(0,1)$ such that

$$\alpha_k = \operatorname{mes}(e_k) > 0, \quad \sum_{k=1}^{\infty} \alpha_k^{p/2} = \infty.$$

Put

$$K(s,t) = \sum_{k=1}^{\infty} \alpha_k^{-1/p} \chi_{e_k}(s)\, \chi_{e_k}(t) \quad (s,t \in (0,1)) \tag{13}$$

and define

$$(Ax)(s) = \int_0^1 K(s,t)\, x(t)\, dt \quad (x \in L^p,\ s \in (0,1)).$$

It is easy to check that $A \in \mathscr{L}(L^p, L^p)$, $\|A\| \le 1$, and $A \ge 0$. The operator A is a compact operator, since it is approximated by the operators A_n generated by the partial sums of series (13):

$$\|A - A_n\| \le \max_{k>n} \alpha_k^{1/p'}.$$

Verify that the operator A possesses all other required properties. We have

$$(A^*y)(t) = \int_0^1 K(s,t)\, y(s)\, ds \quad (y \in L^{p'},\ t \in (0,1));$$

therefore,

$$\begin{aligned} |(A^*y)(t)| &\le \sum_{k=1}^{\infty} \alpha_k^{-1/p} \chi_{e_k}(t) \int_{e_k} |y(s)|\, ds \\ &\le \sum_{k=1}^{\infty} \chi_{e_k}(t) \Big(\int_{e_k} |y(s)|^{p'} ds \Big)^{1/p'} = \Big(\sum_{k=1}^{\infty} \chi_{e_k}(t) \int_{e_k} |y(s)|^{p'} ds \Big)^{1/p'} \le \|y\|_{p'} \end{aligned}$$

for $t \in (0,1)$. Thus, the operator A^* is dominated. It follows that all possible operators A^*U^*, where $U \in \mathscr{L}(L^p, L^p)$, are dominated and consequently integral and regular operators. Hence, UA are regular and integral operators as well. In particular, this guarantees left stable regularity for the operator A.

Now consider the operators AU. From the equality

$$(AUx)(s) = \sum_{k=1}^{\infty} \alpha_k^{-1/p} \chi_{e_k}(s) \langle Ux, \chi_{e_k} \rangle = \sum_{k=1}^{\infty} \alpha_k^{-1/p} \chi_{e_k}(s) \langle x, U^* \chi_{e_k} \rangle$$

it follows that the operator AU is generated by the kernel

$$K_U(s,t) = \sum_{k=1}^{\infty} \alpha_k^{-1/p} \chi_{e_k}(s) (U^* \chi_{e_k})(t) \quad (s, t \in (0,1)).$$

Moreover,

$$|(AUx)(t)| \le \sum_{k=1}^{\infty} \alpha_k^{-1/p} \chi_{e_k}(s) \|x\| \, \|U^* \chi_{e_k}\| \le \|U\| g(s) \|x\|,$$

where

$$g(s) = \sum_{k=1}^{\infty} \alpha_k^{-1/p'-1/p} \chi_{e_k}(s).$$

In order to check that $U \notin \mathscr{L}_{\mathrm{st}}^{\sim}(L^p, L^p)$, we use Remark 3.8.2(d) which demonstrates that the inclusion $U \in \mathscr{L}_{\mathrm{st}}^{\sim}(L^p, L^p) = \Pi_2(L^p, L^p)$ implies that the integral

$$\int_0^1 \left(\int_0^1 K(s,t)\, dt \right)^{p/2} ds$$

is finite. However,

$$K^2(s,t) = \sum_{k=1}^{\infty} \alpha_k^{-2/p} \chi_{e_k}(s)\, \chi_{e_k}(t), \quad \left(\int_0^1 K^2(s,t)\, dt \right)^{p/2} = \sum_{k=1}^{\infty} \alpha_k^{p/2-1} \chi_{e_k}(s)$$

and, therefore,

$$\int_0^1 \left(\int_0^1 K^2(s,t)\, dt \right)^{p/2} ds = \sum_{k=1}^{\infty} \alpha_k^{p/2} = \infty.$$

Comments

The monographs [38, 42, 19] are the main sources as regards the questions of the theory of integral operators which are discussed in this chapter; see also [8, 10, 25].

Some relevant references to the results of the introductory Section 4.1 are given in the text. The results of 4.19 (in a more general statement) were established in [7] and repeated in [67, 68].

As was mentioned in the main text, Theorem 4.2.1 was obtained in [5, 6]; for a thorough priority analysis see [8] and 4.2.11.

Theorem 4.2.1. was generalized to various fields: to spaces of measurable vector-functions [43, 44], noncommutative spaces of measurable operators, nonlinear operators (see 4.5). As to the material of Section 4.2 see also [45, 46, 55–57, 61, 67–70, 86, 87]. Applications to the theory of semigroups of linear operators are exposed in the article [3] by W. Arendt and A. V. Bukhvalov.

The results of Section 4.3 were mainly published in [5–7].

Proposition 4.4.2 was proven in [9]. The presented proof of Theorem 4.4.3 is given in more detail in [10].

Theorem 4.5.1 was proven in [9]. Certain results on representation of nonlinear operators were obtained in [89]. As to the material of Section 4.6 see also [1, 2, 74–76, 78].

Let us briefly discuss the problems raised in the monograph [19] by P. Halmos and V. Sunder. The problem of finding a criterion for integrability posed in the introduction to the book was solved earlier in [5, 6] and Problem 8.4, in [5, 17] and later in [61].

The main contribution to solving many other problems belongs to V. B. Korotkov (see Section 4.6) and W. Schachermayer (see Section 4.6 and [61–65]). This series of articles by W. Schachermayer comprises many other remarkable results. In particular, there is an example of a positive nonintegral operator $U \in \mathscr{L}(L^2)$ such that $U \in \mathfrak{S}_p$ for all $p > 2$. It is clear that for $p = 2$ we obtain the Hilbert-Schmidt class of integral operators; i.e., the example cannot be refined in the power scale.

Section 4.7 is devoted to generalization of the results of the articles [33, 75] to the case of operators in $\mathscr{L}(L^p)$ with an arbitrary p, $1 < p < \infty$. Theorems 4.7.2 and 4.7.3 in a weaker form were announced in [53]. Theorems 4.7.6, 4.7.7, and 4.7.9 are published here for the first time.

In [52], D. Maharam surveyed the results of her original approach to the problem of representing positive operators.

Since the late 1970s, the theory has been intensively developed of pseudointegral operators, i.e., those of the form

$$(Ux)(s) = \int\limits_A x(t)\, d\mu_s(t),$$

where $\{\mu_s\}$ is a "random" measure, i.e., a family of measures satisfying certain measurability conditions. Formula (1) yields a general form of an o-continuous operator in the space of measurable functions [77]. Interest to representation (1) is stirred up not by its complexity but rather by various applications in the geometry of Banach and quasi-Banach spaces, in the spectral theory of operators, in studying the particle transport equation, etc. (see, for instance, [4, 15, 21–24, 83–85]).

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SUPPLEMENT TO CHAPTER 4

Integral Operators of Convolution

BY

V. D. Stepanov

The goal of the Supplement is to expose a solution to Problem 11.8 from the P. R. Halmos and V. S. Sunder book [4] in a strengthened form; namely, a counterexample of nonintegrability of a superposition is constructed in the class of integral operators of convolution which is essentially narrower than the class of all integral operators. This result was obtained by V. D. Stepanov [16, 18]; as was mentioned, it refines the results of V. B. Korotkov which are exposed in Chapter 4. The complete proof is presented here in the case $p = 2$ which is simpler. As regards the method of proof, the result relates to harmonic analysis and stands aside of the tools used in this book. It is for that reason that this material appears in a separate supplement.

We emphasize that integral operators of convolution are as before understood to be integral operators in the sense of the definitions of Chapter 4 (i.e., singular operators are not considered still). Thus, an *integral operator of convolution* with a measurable function k given on the axis $\mathbb{R}$ is defined for functions $f \in L^p(\mathbb{R})$ as

$$(k * f)(t) = \int_{-\infty}^{\infty} k(s-t)f(t)\,dt;$$

moreover, the condition

$$\int_{-\infty}^{\infty} |k(s-t)f(t)|\,dt < \infty$$

is supposed to be valid for all $f \in L^p(\mathbb{R})$. It is clear that we obtain a particular case of the conventional definition of Chapter 4 of integral operator with kernel $K(s,t) = k(s-t)$.

0.1. Let $0 < p, q \leq \infty$. The *amalgam* (L^p, l^q) is the space of measurable functions f on the real axis for which

$$\|f\|_{p,q} = \left\{ \sum_{n=-\infty}^{\infty} \left(\int_{n}^{n+2} |f(x)|^p\,dx \right)^{q/p} \right\}^{1/q} < \infty.$$

0.2. It is easy to see that $(L^p, l^p) = L^p(\mathbb{R})$ and amalgams form a scale in each of the exponents with the other fixed. For amalgams, valid are some analogs of many classical results such as the Hölder inequality, the Young inequality for convolutions, the Young-Hausdorff inequality for the Fourier operator, the interpolation theorems, etc. The survey of J. J. F. Fourier and J. Stewart can help to acquire more detailed knowledge of the subject.

1.1. Consider the convolution operators $f \mapsto k * f$. In the theory of integral operators, the problem is well known of finding criteria on the kernel of operator for its natural domain to include some ideal space of measurable functions. The case in which the latter is the Lebesgue space $L^p(\mathbb{R})$ was considered in [7]. For convolutions, this case is solvable in terms of amalgams.

1.2. *Let $1 \le p \le \infty$ and $1/p + 1/p' = 1$. Then*

$$\forall f \in L^p(\mathbb{R}) \quad \int_{-\infty}^{\infty} |k(s-t)f(t)|\,dt < \infty \text{ for almost all } s \in \mathbb{R} \tag{1}$$

if and only if $k \in (L^1, l^{p'})$.

The result was obtained by R. S. Busby and H. A. Smith [2] and independently in [11, 14].

◁ First we consider the case $p = 1$. We need to show that $k \in (L^1, l^\infty)$. Suppose to the contrary. Then there exists a sequence $\{n_m\}$ of integers such that $|n_m| \to \infty$ and

$$\int_{n_m}^{n_m+1} |k(t)|\,dt = \lambda_m \to \infty.$$

By passing to a subsequence, if necessary, we may assume that $n_m \ge 0$, $n_{m+1} - n_m \ge 2$, and $\lambda_m = m^2$. Let $E_m = [-n_m - 1, -n_m + 1/2]$. Put

$$f(t) = \sum_{m=1}^{\infty} (1/m \log^2(m+1)) \chi_{E_m}(t).$$

It is obvious that $f \in L^1(\mathbb{R})$; therefore, (1) implies that

$$\infty > \int_{-\infty}^{\infty} |k(s-t)f(t)|\,dt = \sum_{m=1}^{\infty} (1/m \log^2(m+1)) \int_{E_m} |k(s-t)|\,dt$$

$$= \sum_{m=1}^{\infty} (1/m \log^2(m+1)) \int_{n_m-1/2}^{n_m+1} |k(s+t)|\,dt$$

for almost all $s \in (0,1/2)$. Assigning a suitable value $s_0 \in (0,1/2)$ to s, we find out

$$\infty > \sum_{m=1}^{\infty} (1/m\log^2(m+1)) \int_{n_m-1/2}^{n_m+1} |k(s_0+t)|\,dt$$
$$\geq \sum_{m=1}^{\infty} (1/m\log^2(m+1)) \int_{n_m}^{n_m+1} |k(t)|\,dt = \infty.$$

The contradiction shows that $k \in (L^1, l^\infty)$.

The case $p = \infty$ is trivial since $k \in L^1(\mathbb{R}) = (L^1, l^1)$ in view of (1) with $f(x) \equiv 1$.

Consider the case $1 < p < \infty$. Let us employ the E. Landau theorem [6, § 1.1.3] which states that if $p > 1$ and, for every sequence $\{a_n\} \in l^{p'}$, the series $\sum_n a_n b_n$ is convergent then $\{b_n\} \in l^{p'}$. Let $\{a_n\} \in l^p$ and $f(t) = \sum_n a_n \chi_{[n,n+1]}(t)$. By (1), we have

$$\infty > \int_{-\infty}^{\infty} |k(s-t)f(t)|\,dt = \sum_n |a_n| \int_{n-1}^{n} |k(s+t)|\,dt$$

for almost all $s \in (-1/2, 1/2)$. Assigning suitable values to $s_1 \in (0,1/2)$ and $s_2 \in (-1/2, 0)$, we find out

$$\infty > \sum_n |a_n| \int_{n-1/2}^{n} |k(t)|\,dt, \quad \infty > \sum_n |a_n| \int_{n-1}^{n-1/2} |k(t)|\,dt.$$

Hence it follows that $k \in (L^1, l^{p'})$ by the E. Landau theorem. $\rhd$

This proof is simpler than the initial one in [14].

1.2.1. **(a)** Conditions (1) in statement 1.2 can be relaxed by replacing the requirement $f \in L^p(\mathbb{R})$ with $f \in (L^\infty, l^p)$.

(b) A particular case of the Young inequality for convolutions [3] has the form

$$\|k * f\|_{p,r} \leq C_1 \|k\|_{1,q} \|f\|_{p,p},$$

where $1 \leq p \leq \infty$, $1 \leq q \leq p'$, $1/r = 1/q + 1/p - 1$, and C_1 is an absolute constant. For $r = p, \infty$, the inequality admits a generalization for $k(t) \geq 0$ in the following sense. If $k(t) \geq 0$, $r = p$ or $r = \infty$, and

$$\|k * f\|_{p,r} \leq C \|f\|_{p,p}$$

for all $f \in L^p(\mathbb{R})$ then $k \in (L^1, l^q)$, where $q = 1$ or $q = p'$ respectively; moreover, $\|k\|_{1,q} \le C_2 C$, where $C_2 > 0$ is an absolute constant ($C_2 = 1$ for $r = p$). The case $r = p$ was considered, for instance, in [10]; the case $r = \infty$ follows from 1.2. For $p < r < \infty$, a similar result is false by the Hardy-Littlewood-Polya theorem [5, Theorem 381].

2.1. We illustrate the application of Theorem 1.2 to two problems on bounded convolution operators in L^p. The first problem consists in studying asymptotic behaviour of the symbol of the kernel of such an operator; the second is an analog of the Halmos and Sunder problem [4, Problem 11.8] for convolutions asking whether such operators form an algebra.

2.2. Let $1 \le p \le \infty$. Following [4], we denote

$$\mathrm{Int}(p) = \{k \in (L^1, l^{p'}) \mid \|k * f\|_p \le C\|f\|_p \ \forall f \in L^p(\mathbb{R})\}.$$

The elements of $\mathrm{Int}(p)$ will be called *kernels*; they are in one-to-one correspondence with integral operators of convolution acting boundedly in $L^p(\mathbb{R})$.

2.3. In the asymptotic behaviour of the symbols (= the Fourier transforms in the distribution sense) of kernels in $\mathrm{Int}(p)$, the "Riemann-Lebesgue effect" is observed; i.e., the symbol vanishes at infinity in some sense. This contrasts the asymptotic behaviour of kernels of singular convolutions, which can be attracted at infinity to a constant as, for instance, the symbol of the kernel of the Hilbert transform [9, Chapter 6, § 1].

2.3.1. *Let* $1 \le p \le \infty$ *and let* p *be fixed. If* $k \in \mathrm{Int}(p)$ *and*

$$\Delta(n, \lambda) = \mathrm{mes}\{\xi \in (n - 1/2, n + 1/2) \mid |k^\wedge(\xi)| > \lambda\} \tag{2}$$

then $\lim\limits_{|n| \to \infty} \Delta(n, \lambda) = 0$ *for every* $\lambda > 0$.

The proposition was obtained for $p = 2$ by V. B. Korotkov and in the general case, in [15, 16].

$\lhd$ Let $\varphi_n^\wedge(\xi) = \chi_{[-1,1]} \times \chi_{[-1,1]}(\xi - n)$. Then $\varphi_n(y) = \sin^2 2\pi y e^{2\pi i n y}/(\pi y)^2$ and

$$\|\varphi_n\|_p \le A, \quad \|\varphi_0\|_{p',1} \le A,$$

where A is an absolute constant. For $|\xi - n| \le 1/2$, we have $\varphi_n^\wedge(\xi) \ge 1$; therefore, we obtain

$$\begin{aligned}\Delta(n,\lambda) &\le \operatorname{mes}\{\xi \in (n-1/2, n+1/2) \mid |k^\wedge(\xi)\varphi_n^\wedge(\xi)| > \lambda\} \\ &\le \frac{1}{\lambda^2}\|k^\wedge\varphi_n^\wedge\|_2^2 = \frac{1}{\lambda^2}\|k * \varphi_n\|_2^2 \le \frac{1}{\lambda^2}\|k*\varphi_n\|_p\|k*\varphi_n\|_{p'} \\ &\le \frac{C\|\varphi_n\|_p}{\lambda^2}\|k*\varphi_n\|_{p'} \le \frac{CA}{\lambda^2}\|k*\varphi_n\|_{p'}.\end{aligned}$$

by the Parseval inequality and the Hölder inequality. Further,

$$k * \varphi_n(x) = \int_{-\infty}^{\infty} k(x-y)\frac{\sin^2 2\pi y}{(\pi y)^2}e^{2\pi i n y}\,dy.$$

Since $k \in (L^1, l^{p'})$, we have $k(x-y)\sin^2 2\pi y/(\pi y)^2 \in L^1(\mathbb{R})$ for each $x \in \mathbb{R}$ fixed. By the Riemann-Lebesgue lemma, we obtain

$$\lim_{|n|\to\infty} k * \varphi_n(x) = 0 \quad \forall x \in \mathbb{R}.$$

In virtue of the Young inequality for convolutions [19]

$$\|\,|k| * |\varphi_n|\,\|_{p',p'} \le \|k\|_{(1,p')}\|\varphi_0\|_{p',1},$$

we find that

$$|k * \varphi_n(x)| \le \Phi(x) = |k| * |\varphi_0| \in L^{p'}(\mathbb{R}).$$

Hence

$$\lim_{|n|\to\infty} \|k*\varphi_n\|_{p'} = 0$$

by the Lebesgue dominated convergence theorem, and Proposition 2.3.1 is proven. ⊳

2.3.2. Generally speaking, condition (2) is not sufficient for $k \in \operatorname{Int}(p)$ even if k determines a bounded operator in $L^p(\mathbb{R})$ by the formula $(T_k f)^\wedge = k^\wedge f^\wedge$. The function

$$k^\wedge(\xi) = \sum_{|k|\ge 2} \operatorname{sign} k\,\chi_{[-1/2\log|k|, 1/2\log|k|]}(\xi - k)$$

provides an instance of this. In virtue of the Parseval equality, it is obvious that $T_k : L^2(\mathbb{R}) \to L^2(\mathbb{R})$; however,

$$k(x) = \frac{2i}{\pi x} \sum_{k=2}^{\infty} \sin \frac{\pi x}{\log k} \sin 2\pi k x \notin (L^1, l^2),$$

since k has a nonsummable singularity at zero. The last is derived from the Salem theorem [1, Chapter 10, § 7].

2.3.3. Let $k \in \mathrm{Int}(2)$. What is the rate of decrease of the portions $\Delta(n, \lambda)$ as $|n| \to \infty$ with $\lambda > 0$ fixed in relation (2)? The question arises from a natural attempt at finding a metric criterion for the containment $k \in \mathrm{Int}(2)$ in terms of the portions $\Delta(n, \lambda)$, since the space $\mathrm{Int}(2)$ has the simplest structure and it can be easily made into a Banach space by introduction of the norm $\|k\|_{\mathrm{Int}(2)} = \|k\|_{1,2} + \|k^\wedge\|_\infty$. It turns out that the rate of decrease of the portions $\Delta(n, \lambda)$ can be arbitrarily small in some sense. For instance, there is no finite constant $b \geq 1$ universal for the whole space $\mathrm{Int}(2)$ and such that $\sum_n |\Delta(n, \lambda)|^b < \infty$ for all $\lambda > 0$. An example here is provided by the kernel

$$K_a^\wedge(\xi) = \sum_{k \neq 0} \chi_{[-1/2|k|^a, 1/2|k|^a]}(\xi - k).$$

with $a \in (0, \infty)$ fixed. It was demonstrated in [17, 19] that $K_a \in (L^1, l^2)$ for every $a > 0$ and the operator of convolution with kernel K_a acts boundedly from $L^p(\mathbb{R})$ into $L^p(\mathbb{R})$ only for $p = 2$.

2.3.4. *Let $1 \leq p \leq \infty$. The operators of convolution $f \mapsto k * f$ with kernels $k \in \mathrm{Int}(p)$ form an algebra only for $p = 1, \infty$.*

$\lhd$ The positive part of the assertion is trivial since $\mathrm{Int}(1) = \mathrm{Int}(\infty) = L^1(\mathbb{R})$.

Let $1 < p < \infty$. To prove the result it suffices to exhibit an example of a kernel $k \in \mathrm{Int}(p)$ such that $k * k \notin \mathrm{Int}(p)$. Consider a family $k_{\varepsilon,\delta}$ of functions given by the formula

$$k_{\varepsilon,\delta}(x) = \sum_{n=1}^{\infty} e^{2\pi i n^{1-\varepsilon+\delta}} \chi_{[-1/2n^{(1+\varepsilon)/2}, 1/2n^{(1+\varepsilon)/2}]}(x - n),$$

where $1/2 < \varepsilon < 1$ and $0 < \delta < \varepsilon$.

The following properties hold:

$$k_{\varepsilon,\delta} \in \mathrm{Int}(p), \quad 2\left(1 - \frac{\delta}{1 - \varepsilon + 2\delta}\right) \leq p \leq 2\left(1 + \frac{\delta}{1 - \varepsilon}\right), \tag{3}$$

$$k_{\varepsilon,\delta} * k_{\varepsilon,\delta} \notin (L^1, l^{p'}), \quad p \geq 2/(2-\delta). \tag{4}$$

By letting $2/(2-\delta) = 2(1-\delta/(1-\varepsilon+2\delta))$, we find the relation

$$\delta = \sqrt{((1-\varepsilon)/2)^2 + 1 - \varepsilon} - (1-\varepsilon)/2$$

under which the kernel $k_{\varepsilon,\delta} \in \mathrm{Int}(p)$ determines an integral operator of convolution, which acts boundedly in $L^p(\mathbb{R})$ and has a nonsummable square since $k_{\varepsilon,\delta} * k_{\varepsilon,\delta} \notin (L^1, l^{p'})$; moreover, the interval in which p changes in (3) can be arbitrarily full.

A complete proof of assertion (3) is sufficiently cumbersome. It is grounded on the A. Miyachi theorem on multiplicators of the Fourier transform with a narrow interval of action [8] and the methods of the articles [12, 13]. However, it is not difficult to verify that $k_{\varepsilon,\delta} \in (L^1, l^2)$ and $k^\wedge_{\varepsilon,\delta} \in L^\infty(\mathbb{R})$. The first inclusion is elementary; the second can be deduced by applying a Van der Corput theorem [20, Chapter 5, § 4, 5]. Therefore, $k_{\varepsilon,\delta} \in \mathrm{Int}(2)$ which together with (2) provides the claimed for $p = 2$.

Prove (4). Let $k(x) = k_{\varepsilon,\delta} * k_{\varepsilon,\delta}(x)$. For $x = m + t$, where m is an integer and $|t| \leq 1/2$, we write down

$$k(m+t) = \sum_{n=1}^{m-1} e^{2\pi i(n^{1-\varepsilon+\delta} + (m-n)^{1-\varepsilon+\delta})} \chi_n * \chi_{m-n}(t),$$

where $\chi_n \equiv \chi_{[-1/2n^{(1+\varepsilon)/2}, 1/2n^{(1+\varepsilon)/2}]}$. Let $m = 2(l+1)$ and $n = k + l + 1$. Then

$$k(2(l+1)+t) = \sum_{k=-l}^{l} J_k(t) \equiv J_0(t) + V_l(t).$$

We have

$$J_0(t) = e^{4\pi i (l+1)^{1-\omega}} \chi_{l+1} * \chi_{l+1}(t),$$

where $\omega = \varepsilon - \delta$. As easily seen,

$$\int_{-1/2}^{1/2} |J_0(t)|\, dt \leq \|\chi_{l+1} * \chi_{l+1}\| \leq \frac{1}{l^{1+\varepsilon}}.$$

Further,

$$V_l(t) \equiv \sum_{k=1}^{l} e^{2\pi i((k+l+1)^{1-\omega} + (l+1-k)^{1-\omega})} \chi_{k+l+1} * \chi_{l+1-k}(t).$$

Assertion (4) follows from the equality

$$\sum_{l=1}^{\infty}\left(\int_{-1/2}^{1/2}|k(2(l+1)+t)|\,dt\right)^{2/\delta}=\infty. \tag{5}$$

Put

$$v_k=\chi_{l+1+k}*\chi_{l+1-k},\quad \varphi(t)=(l+1+t)^{1-\omega}+(l+1-t)^{1-\omega},\quad U_k=\sum_{m=1}^{k}e^{2\pi i\varphi(m)}.$$

By applying the Abel transform for sums, we obtain

$$V_l(t)=\sum_{k=1}^{l-1}U_k(\Delta v)_k+U_l v_l.$$

We have

$$\varphi'(t)=(1-\omega)\left(\frac{1}{(l+1+t)^{\omega}}-\frac{1}{(l+1-t)^{\omega}}\right)<0,\quad t\in[1,l],$$

$$|\varphi'(t)|\le(1-\omega)[1-1/(2l+1)^{\omega}]\le q_\omega<1,\quad l>l_\omega.$$

Hence

$$U_l=\int_1^l e^{2\pi i\varphi(t)}\,dt+C_l,\quad |C_l|\le A_\omega.$$

by the Van der Corput theorem. Here and henceforth A_ω stands for a finite constant depending only on ω.

Let

$$J(l)=\int_1^l \exp(2\pi i\varphi(t))\,dt.$$

As $l\to\infty$, the following asymptotic formula holds:

$$J(l)=\frac{e^{i\pi/4}}{2\sqrt{2\omega(1-\omega)}}e^{2\pi i(l+1)^{1-\omega}}l^{(1+\omega)/2}\big(1+O\big(l^{-(1-\omega)/6}\big)\big),\quad l\to\infty;$$

this can be obtained by the method of stationary phase. Hence

$$\int_{-1/2}^{1/2} |U_l(t)v_l(t)|\, dt \geq A_\omega l^{-\delta/2}, \quad l > l_\omega.$$

By applying the Van der Corput theorem again, we obtain

$$\int_{-1/2}^{1/2} \left| \sum_{k=1}^{l-1} U_k(\Delta v)_k \right| dt \leq A_\omega l^{-(1+\varepsilon+\delta)/2}.$$

The last two estimates yield (5). ▷

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CHAPTER 5

Disjointness Preserving Operators

BY

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Disjointness preserving operators have its own theory which is very rich in results and includes such questions as boundedness, continuity, spectral and geometric properties, multiplicativity, compactness, etc. The list of publications devoted to studying disjointness preserving operators is so extensive that it could serve as a reason for a separate review. Leaving aside many rather interesting directions, we will only concentrate our attention on analytic representation and decomposition of disjointness preserving operators. B. Z. Vulikh [7–9] was one of the first who considered these questions. Later, disjointness preserving operators were studied by Yu. A. Abramovich, E. L. Arenson, D. R. Hart, A. K. Kitover, A. V. Koldunov, P. T. N. MacPolin, A. I. Veksler, A. W. Wickstead, A. C. Zaanen, and many others (see, for instance, [1–3, 19, 32, 37, 41, 42]). We also observe that the question of analytic representation of disjointness preserving operators includes such a powerful direction as descriptions of isometries of vector-valued L^p-spaces (the so-called Banach-Stone theorems).

In the current chapter, we study disjointness preserving operators in K-spaces and lattice-normed spaces. In particular, we find their analytic representations and decompositions into simpler components. We begin with studying general properties of disjointness preserving operators; then we consider orthomorphisms, shift operators, weighted shift operators, and, finally, return to arbitrary operators and apply the accumulated experience.

5.1. Prerequisites

This section contains some preliminary information about Boolean homomorphisms, vector lattices, and lattice-normed spaces which was not exposed in the previous chapters.

5.1.1. In the sequel, we deal with various convergences (such as o- and r-convergences) and related notions (such as o- and r-closures, o- and r-continuity, etc.). For the sake of convenience and in order to avoid duplication, we present some general definitions now.

Let X be an arbitrary set and let c be some convergence in X. The totality of the c-limits of all c-convergent nets in X constituted by elements of some subset $X_0 \subset X$ is called the *c-closure* of X_0. A set is called *c-closed* if it coincides with the c-closure of itself. The set X_0 is said to be *c-dense* in X if X is the c-closure of X_0. Suppose now that X_1 and X_2 are some sets with convergences c_1 and c_2, respectively. A mapping $f: X_1 \to X_2$ is called *c_1-c_2-continuous* if c_1-convergence $x_\alpha \to x$ implies c_2-convergence $f(x_\alpha) \to f(x)$ for every net $(x_\alpha)_{\alpha \in \mathrm{A}}$ in X_1 and every element $x \in X_1$. If the convergences c_1 and c_2 have the same notation c, then any c_1-c_2-continuous mapping is called *c-continuous*.

Considering only countable nets in the above definitions results in the notions of *countable c-closure, countable c-closedness, countable c-density,* and *countable c_1-c_2-continuity.* By replacing nets with sequences, we obtain the notions of *sequential c-closure, sequential c-closedness, sequential c-density,* and *sequential c_1-c_2-continuity.*

5.1.2. Ring and Boolean homomorphisms are often met in our further consideration. We recall the relevant definitions.

Let A and B be Boolean algebras. A mapping $h: A \to B$ is called a *ring homomorphism* if the following equalities hold for all $a_1, a_2 \in A$:

(a) $h(a_1 \vee a_2) = h(a_1) \vee h(a_2)$;

(b) $h(a_1 \wedge a_2) = h(a_1) \wedge h(a_2)$;

(c) $h(a_1 \backslash a_2) = h(a_1) \backslash h(a_2)$,

where $x \backslash y$ stands for $x \wedge y^\perp$ and $x^\perp$ stands for the complement of x in a Boolean algebra. We observe that (a) is a consequence of (b) and (c), as well as (b) is a consequence of (a) and (c). Every ring homomorphism $h: A \to B$ preserves order, i.e., $a_1 \leqslant a_2$ implies $h(a_1) \leqslant h(a_2)$ for all $a_1, a_2 \in A$.

A ring homomorphism $h: A \to B$ is called a *Boolean homomorphism* in case $h(1) = 1$. Obviously, a mapping $h: A \to B$ is a Boolean homomorphism if and only if it satisfies one of the conditions (a) or (b) and, in addition, $h(a^\perp) = h(a)^\perp$ for all $a \in A$. Every ring homomorphism $h: A \to B$ is a Boolean homomorphism into the Boolean algebra $B_{h(1)} = \{b \in B : b \leqslant h(1)\}$. The image $h[A]$ of the homomorphism h is a Boolean subalgebra of $B_{h(1)}$. A bijective Boolean homomorphism is called a *Boolean isomorphism.*

The following description of Boolean homomorphisms is convenient in studying disjointness preserving operators.

Proposition. *Let A and B be Boolean algebras. A mapping $h: A \to B$ is a Boolean homomorphism if and only if, for every partition (a_1, a_2, a_3) of unity in A, the triple $\big(h(a_1), h(a_2), h(a_3)\big)$ is a partition of unity in B.*

$\triangleleft$ Necessity is obvious; thus, we only prove sufficiency. Suppose that the mapping h preserves triple partitions. Applying this property of h to the triple $(0, 0, 1)$, we obtain the equality $h(0) = 0$. Considering the triple $(a, a^\perp, 0)$, we conclude that $h(a^\perp) = h(a)^\perp$ for every $a \in A$. It remains to establish the relation $h(a_1 \vee a_2) = h(a_1) \vee h(a_2)$. First, we prove this equality for disjoint a_1 and a_2. To this end, it is sufficient to apply the partition preservation property of h to the triples $\big(a_1, a_2, (a_1 \vee a_2)^\perp\big)$ and $\big(a_1 \vee a_2, (a_1 \vee a_2)^\perp, 0\big)$. Finally, taking arbitrary elements $a_1, a_2 \in A$ and using the above-established facts, we obtain

$$\begin{aligned} h(a_1 \vee a_2) &= h\big((a_1 \backslash a_2) \vee (a_1 \wedge a_2) \vee (a_2 \backslash a_1)\big) \\ &= h(a_1 \backslash a_2) \vee h(a_1 \wedge a_2) \vee h(a_2 \backslash a_1) \\ &= \big(h(a_1 \backslash a_2) \vee h(a_1 \wedge a_2)\big) \vee \big(h(a_1 \wedge a_2) \vee h(a_2 \backslash a_1)\big) \\ &= h(a_1) \vee h(a_2). \quad \triangleright \end{aligned}$$

5.1.3. *Proposition.* *Let A and B be Boolean algebras.*

(a) *The following properties of a Boolean homomorphism $h\colon A \to B$ are equivalent:*
   (1) *h is o-continuous;*
   (2) *if a subset $C \subset A$ has a supremum then $h(\sup C) = \sup h[C]$;*
   (3) *if a subset $C \subset A$ has an infimum then $h(\inf C) = \inf h[C]$;*
   (4) *if $(a_\lambda)_{\lambda \in \Lambda}$ is a net in A and $a_\lambda \uparrow 1$ then $\sup_{\lambda \in \Lambda} h(a_\lambda) = 1$;*
   (5) *if $(a_\lambda)_{\lambda \in \Lambda}$ is a net in A and $a_\lambda \downarrow 0$ then $\inf_{\lambda \in \Lambda} h(a_\lambda) = 0$.*

(b) *The following properties of a Boolean homomorphism $h\colon A \to B$ are equivalent:*
   (1) *h is countably o-continuous;*
   (2) *if a countable subset $C \subset A$ has a supremum then $h(\sup C) = \sup h[C]$;*
   (3) *if a countable subset $C \subset A$ has an infimum then $h(\inf C) = \inf h[C]$;*
   (4) *if $(a_n)_{n \in \mathbb{N}}$ is a sequence in A and $a_n \uparrow 1$ then $\sup_{n \in \mathbb{N}} h(a_n) = 1$;*
   (5) *if $(a_n)_{n \in \mathbb{N}}$ is a sequence in A and $a_n \downarrow 0$ then $\inf_{n \in \mathbb{N}} h(a_n) = 0$.*

If the Boolean algebra A is complete (σ-complete) then each of the five conditions (a) *(respectively,* (b)*) is equivalent to the following one:* $\sup h[D] = 1$ *for every (countable) partition D of unity in A.*

In view of the equivalence of conditions (a)(1)–(a)(3), o-continuous homomorphisms are often called *full* or *complete*. Observe that the implication (b)(5)$\Rightarrow$(b)(1) implies equivalence of countable and sequential o-continuities of a Boolean homomorphism.

5.1.4. Let A and B be Boolean algebras. We say that a ring homomorphism $h\colon A \to B$ *dominates* a function $h_0\colon A \to B$ (and write $h_0 \leqslant h$), if $h_0(a) \leqslant h(a)$ for all $a \in A$.

Proposition. *Let A and B be Boolean algebras. A ring homomorphism $h\colon A \to B$ dominates a ring homomorphism $h_0\colon A \to B$ if and only if $h_0(a) = h_0(1) \wedge h(a)$ for all $a \in A$.*

$\lhd$ The equality $h_0(a) = h_0(1) \wedge h(a)$ ensues from the relations $h_0(a) \leqslant h_0(1) \wedge h(a)$, $h_0(a^\perp) \leqslant h_0(1) \wedge h(a^\perp)$, and $h_0(a) \vee h_0(a^\perp) = h_0(1)$. $\rhd$

5.1.5. Let E be a vector lattice. Given an element $e \in E$, the symbol $\langle e\rangle$ denotes the band projection onto the principal band $\{e\}^{\perp\perp}$ (if such a projection exists). For $e, f \in E$, we define $\langle e < f\rangle := \langle (f-e)^+\rangle$, $\langle e \leqslant f\rangle := \langle f < e\rangle^\perp$, $\langle e > f\rangle := \langle f < e\rangle$, and $\langle e \geqslant f\rangle := \langle f \geqslant e\rangle$. It is clear that $\langle e \leqslant f\rangle = \max\{\pi \in \mathrm{Pr}(E) : \pi e \leqslant \pi f\}$.

5.1.6. Let f be an arbitrary positive element of a vector lattice E. An element $s \in E$ is called an *f-step element*, if $s = \sum_{i=1}^n \lambda_i \pi_i f$ for some $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ and $\pi_1, \dots, \pi_n \in \mathrm{Pr}(E)$.

Proposition. *Suppose that a vector lattice E possesses the principal projection property (for instance, E is a K_σ-space). Let E_f be the ideal of E generated by a positive element $f \in E$. Then, for every element $e \in E_f$ and every number $\varepsilon > 0$, there is an f-step element $s \in E_f$ such that $|s| \leqslant |e|$ and $|e - s| \leqslant \varepsilon f$. In particular, the set of all f-step elements is r-dense in E_f.*

$\lhd$ Assume all the hypotheses of the proposition to be satisfied and consider an arbitrary element $e \in E_f$ and a number $\varepsilon > 0$. Let numbers $m, n \in \mathbb{N}$ be such that $|e| \leqslant mf$ and $1/n \leqslant \varepsilon$. Then the sum

$$\sum_{i=-mn}^{-1} \frac{i}{n}\left\langle \frac{i-1}{n} f < e \leqslant \frac{i}{n} f\right\rangle f + \sum_{i=1}^{mn} \frac{i}{n}\left\langle \frac{i}{n} f \leqslant e < \frac{i+1}{n} f\right\rangle f$$

is a desired f-step element. $\rhd$

5.1.7. We use the abbreviation LNS for "lattice-normed space" (see 1.6.1) and BKS for "Banach-Kantorovich space" (see 1.6.3). Each LNS considered in the current chapter is assumed to be normed by a K-space (if the opposite is not stated explicitly). The lattice-valued norm in an LNS is denoted by $|\cdot|$ by default. The phrase "$\mathscr{U}$ is an LNS over E" means that E is a K-space and $(\mathscr{U}, |\cdot|, E)$ is an LNS in the sense of 1.6.1. We assume that the equality $\{|u| : u \in \mathscr{U}\}^{\perp\perp} = E$ holds for every LNS $\mathscr{U}$ over E that we consider. We also assume by default that all the LNSs are d-decomposable (see 1.6.2). The Boolean algebra of band projections in an LNS $\mathscr{U}$ over E is denoted by $\mathrm{Pr}(\mathscr{U})$ and conventionally identified with the Boolean algebra $\mathrm{Pr}(E)$ of band projections in E (see 1.6.2).

5.1.8. One of useful properties of d-decomposable LNSs is realized in the possibility of evaluating band projections on their elements and mixing them to obtain new elements. Here, we will discuss such operations.

If $(u_\xi)_{\xi\in\Xi}$ is an arbitrary family in an LNS $\mathscr{U}$ and $(\pi_\xi)_{\xi\in\Xi}$ is a partition of unity in the Boolean algebra $\mathrm{Pr}(\mathscr{U})$, then the sum $o\text{-}\sum_{\xi\in\Xi}\pi_\xi u_\xi$ (if the latter exists) is called the *mixing* of the family $(u_\xi)_{\xi\in\Xi}$ with respect to $(\pi_\xi)_{\xi\in\Xi}$. Let $\mathscr{V}$ be a subset of $\mathscr{U}$. The totality of all mixings of arbitrary (finite) families in $\mathscr{V}$ is called the *cyclic hull* (the *finitely cyclic hull*) of $\mathscr{V}$ and denoted by $\operatorname{mix}\mathscr{V}$ (by $\operatorname{mix}_{\mathrm{fin}}\mathscr{V}$). The cyclic hull of the union $\mathscr{V}\cup\{0\}$ is called the *d-closure* of $\mathscr{V}$ and denoted by $d\mathscr{V}$. Similarly, the symbol $d_{\mathrm{fin}}\mathscr{V}$ is used to denote the finitely cyclic hull of $\mathscr{V}\cup\{0\}$. The set $\mathscr{V}$ is called *cyclic* (*finitely cyclic*) if $\operatorname{mix}\mathscr{V} = \mathscr{V}$ ($\operatorname{mix}_{\mathrm{fin}}\mathscr{V} = \mathscr{V}$). It easy to verify that the (finitely) cyclic hull of a set $\mathscr{V}$ is the smallest (finitely) cyclic set that includes $\mathscr{V}$. Obviously, for a set $\mathscr{V}$ to be finitely cyclic, it is sufficient that it contain the sums $\pi v + \pi^\perp w$ for all $v, w \in \mathscr{V}$ and $\pi \in \mathrm{Pr}(\mathscr{U})$.

5.1.9. Let $\mathscr{U}$ be an arbitrary, not necessarily d-decomposable, LNS over an arbitrary vector lattice E. Suppose that a d-decomposable LNS $\overline{\mathscr{U}}$ over E contains $\mathscr{U}$ as a subspace with the induced norm. We say that the LNS $\overline{\mathscr{U}}$ is a *d-decomposable hull* of $\mathscr{U}$, if $d_{\mathrm{fin}}\mathscr{U} = \overline{\mathscr{U}}$, i.e., $\overline{\mathscr{U}}$ is a minimal d-decomposable LNS that contains $\mathscr{U}$ as a subspace with the induced norm.

Proposition. *Suppose that a vector lattice E possesses the principal projection property. Then every (not necessarily d-decomposable) LNS over E has a d-decomposable hull which is unique to within an isometry.*

◁ In order to construct a d-decomposable hull of an LNS $\mathscr{U}$ over E, we employ the schema of formal mixing, which is traditionally used in similar situations (cf. [5, 6, 16]). Denote by $\widetilde{\mathscr{U}}$ the totality of all finite families $\big((\pi_i, u_i)\big)_{i\in I}$ of elements in $\Pr(E)\times\mathscr{U}$ such that $(\pi_i)_{i\in I}$ is a partition of unity in the Boolean algebra $\Pr(E)$. Introduce in $\widetilde{\mathscr{U}}$ the equivalence relation by letting $\big((\pi_i, u_i)\big)_{i\in I} \sim \big((\rho_j, v_j)\big)_{j\in J}$ if and only if $\pi_i\rho_j\big|u_i - v_j\big| = 0$ for all $i \in I$ and $j \in J$. Define $\overline{\mathscr{U}}$ to be the quotient set $\widetilde{\mathscr{U}}/\!\sim$ and agree to denote the coset of a family $\big((\pi_i, u_i)\big)_{i\in I}$ by $\sum_{i\in I}\pi_i u_i$. By identifying the elements $u \in \mathscr{U}$ with "monomials" $1u \in \overline{\mathscr{U}}$, we assume that $\mathscr{U} \subset \overline{\mathscr{U}}$. It is easy to become convinced that $\overline{\mathscr{U}}$ is an LNS over E under the operations

$$\sum_{i\in I}\pi_i u_i + \sum_{j\in J}\rho_j v_j := \sum_{i\in I, j\in J}\pi_i\rho_j(u_i + v_j),$$

$$\lambda\sum_{i\in I}\pi_i u_i := \sum_{i\in I}\pi_i\lambda u_i,$$

$$\Big|\sum_{i\in I}\pi_i u_i\Big| := \sum_{i\in I}\pi_i|u_i|$$

and is a d-decomposable hull of $\mathscr{U}$. Uniqueness of a d-decomposable hull is obvious. ▷

5.1.10. Let E be a universally complete K-space and let $(E_\xi)_{\xi\in\Xi}$ be a family of pairwise disjoint ideals of E. The symbol $\bigoplus_{\xi\in\Xi} E_\xi$ denotes the ideal of the K-space E constituted by all elements $e \in E$ that satisfy the relation $\langle E_\xi\rangle e \in E_\xi$ for each $\xi \in \Xi$. Obviously,

$$\bigoplus_{\xi\in\Xi} E_\xi = \Big\{ o\text{-}\sum_{\xi\in\Xi} e_\xi \ :\ (e_\xi)_{\xi\in\Xi} \in \prod_{\xi\in\Xi} E_\xi \Big\}.$$

Suppose that, for every $\xi \in \Xi$, we are given an LNS $\mathscr{U}_\xi$ over E_ξ. It is not difficult to become convinced that the vector space $\prod_{\xi\in\Xi}\mathscr{U}_\xi$ is an LNS over $\bigoplus_{\xi\in\Xi} E_\xi$ with respect to the norm $\big|(u_\xi)_{\xi\in\Xi}\big| = o\text{-}\sum_{\xi\in\Xi}\big|u_\xi\big|$. This LNS is denoted by $\bigoplus_{\xi\in\Xi}\mathscr{U}_\xi$ and called the *disjoint sum of the family of* LNSs $(\mathscr{U}_\xi)_{\xi\in\Xi}$.

5.1.11. Let E and F be K-spaces and let $\mathscr{U}$ be an LNS over E. Suppose that a function $S\colon E \to F$ satisfies the following conditions:

(a) $S(e_1 + e_2) \leqslant Se_1 + Se_2$ for all positive $e_1, e_2 \in E$;

(b) $S(\lambda e) = \lambda Se$ for all positive $e \in E$ and $\lambda \in \mathbb{R}$;

(c) if $0 \leqslant e_1 \leqslant e_2$ then $Se_1 \leqslant Se_2$.

Consider the vector subspace $\mathcal{U}_0 := \{u \in \mathcal{U} : S|u| = 0\}$ and agree to denote by $S_{\mathcal{U}}u$ the coset in $\mathcal{U}/\mathcal{U}_0$ containing an $u \in \mathcal{U}$. It is easy to become convinced that the space $\mathcal{U}/\mathcal{U}_0$ is an LNS over F with respect to the norm $|S_{\mathcal{U}}u| := S|u|$. Observe that the LNS $\mathcal{U}/\mathcal{U}_0$ need not be d-decomposable (for instance, in case $\mathcal{U} = E = F = \mathbb{R}^2$ and $S(x,y) = (x,x)$). Slightly abusing the language, we call a d-decomposable hull of the LNS $\mathcal{U}/\mathcal{U}_0$ the *norm transformation of $\mathcal{U}$ by means of S* and denote it by $S\mathcal{U}$. The linear operator $S_{\mathcal{U}} : \mathcal{U} \to S\mathcal{U}$ is called the *operator of norm transformation of $\mathcal{U}$ by means of S.*

5.1.12. As is known (see 1.3.7 (8)), every universally complete K-space E can be endowed with multiplication so that E becomes a commutative ordered algebra. If we additionally fix an order unity in E and require it to be a multiplication unity then the way of introducing multiplication in E becomes unique. Furthermore, for every $f \in E$, there exists a unique element $g \in E$ such that $fg = \langle f \rangle 1$, where $1 \in E$ is the multiplication unity. We denote such an element g by $1/f$. The product $e(1/f)$ is denoted by e/f for brevity.

As is known (see 1.6.5), every BKS $\mathcal{U}$ over a universally complete K-space E with a fixed order unity 1_E can be endowed with the structure of a module over E so that $1_E u = u$ and $|eu| = |e||u|$ for all $e \in E$ and $u \in \mathcal{U}$. Below (see 5.5.17) we will see that the relation $|eu| = |e||u|$ uniquely determines the structure of a module in $\mathcal{U}$.

Let $\mathcal{U}$ be an arbitrary BKS over an order-dense ideal E of a universally complete K-space $\mathcal{E}$ with a fixed order unity. Given arbitrary $e \in \mathcal{E}$ and $u \in \mathcal{U}$, we say that *the product eu is defined in $\mathcal{U}$* (and write $eu \in \mathcal{U}$), if the product eu calculated in the universal completion of $\mathcal{U}$ belongs to $\mathcal{U}$. Obviously, the latter is true if and only if $|e||u| \in E$.

5.1.13. The module structure of a BKS is often used for finding elements that satisfy certain conditions imposed on their norm. Here is one of typical examples.

Lemma. *Let $\mathcal{U}$ be a* BKS *over E. For all $u \in \mathcal{U}$ and $e \in E^+$, there exists an element $u_e \in \mathcal{U}$ such that $|u_e| = e$ and $|u - u_e| = ||u| - e|$.*

$\lhd$ Fix an order unity 1 in the universal completion $\overline{E}$ of the K-space E, endow $\overline{E}$ with the corresponding multiplication and introduce in the universal completion $\overline{\mathscr{U}}$ of $\mathscr{U}$ the structure of a module over $\overline{E}$. Let $\bar{u}$ be an element of $\overline{\mathscr{U}}$ such that $|\bar{u}| = 1$ and $u = |u|\bar{u}$. Obviously, $u_e := e\bar{u}$ is the desired element. $\triangleright$

5.2. Order Approximating Sets

In this section, we introduce the notions of order approximating and h-approximating subsets of an LNS. We also present equivalent descriptions of the notions in terms of convergences of various types. The notion of order approximation seems to be useful in the general theory of LNSs. As for h-approximation, it will play its role in studying disjointness preserving operators.

5.2.1. Lemma. *Let $\mathscr{U}$ be an LNS over a K-space E and let $\mathscr{V}$ be a finitely cyclic subset of $\mathscr{U}$. Then, for every $u \in \mathscr{U}$, there exists a net $(v_\alpha)_{\alpha\in\mathrm{A}}$ in $\mathscr{V}$ such that the net $\big(|u - v_\alpha|\big)_{\alpha\in\mathrm{A}}$ decreases and $\{|u - v_\alpha| : \alpha \in \mathrm{A}\} = \{|u - v| : v \in \mathscr{V}\}$. In particular, $|u - v_\alpha| \downarrow \inf_{v\in\mathscr{V}} |u - v|$.*

$\triangleleft$ Suppose that a set $\mathscr{V} \subset \mathscr{U}$ meets the hypothesis of the lemma and fix an arbitrary element $u \in \mathscr{U}$. We introduce in $\mathscr{V}$ relations of equivalence and preorder as follows:

$$v \sim w \;\Leftrightarrow\; |u - v| = |u - w|,$$
$$v \preccurlyeq w \;\Leftrightarrow\; |u - v| \geqslant |u - w|.$$

For any two elements $v, w \in \mathscr{V}$ we can find a projection $\pi \in \mathrm{Pr}(E)$ such that $\big|u - (\pi v + \pi^\perp w)\big| = |u - v| \wedge |u - w|$. Since $\mathscr{V}$ is finitely cyclic, the latter means that the set $(\mathscr{V}, \preccurlyeq)$ is directed. Therefore, the quotient set $\mathrm{A} := \mathscr{V}/{\sim}$ (endowed with the quotient order) is a directed ordered set. Taking an element $v_\alpha \in \alpha$ in every coset $\alpha \in \mathrm{A}$, we obtain the desired net $(v_\alpha)_{\alpha\in\mathrm{A}}$. $\triangleright$

5.2.2. Let $\mathscr{V}$ be a subset of an LNS $\mathscr{U}$. We say that $\mathscr{V}$ *(orderly) approximates an element* $u \in \mathscr{U}$ if $\inf_{v\in\mathscr{V}} |u - v| = 0$. We say that $\mathscr{V}$ *(orderly) approximates a subset* $\mathscr{W} \subset \mathscr{U}$ if $\mathscr{V}$ approximates every element of $\mathscr{W}$. A subset of $\mathscr{U}$ is called *(order) approximating* if it approximates $\mathscr{U}$. Any order dense ideal of an LNS is an example of an approximating set.

Proposition. *Let X, Y, and Z be subsets of an LNS. If X approximates Y and Y approximates Z, then X approximates Z.*

$\triangleleft$ For an arbitrary element $z \in Z$, denote $\inf_{x \in X} |x - z|$ by e and assume by way of contradiction that $e \neq 0$. Since $\inf_{y \in Y} |y - z| = 0$, there is an element $y \in Y$ and band projection ρ such that $\rho|y - z| < \rho e/2$. Similarly, in view of the equality $\inf_{x \in X} |x - y| = 0$, there is an element $x \in X$ and an band projection π such that $\pi|x - y| < \pi\rho e/2$. The following contradictory relations complete the proof:

$$\pi\rho e \leqslant \pi\rho|x - z| \leqslant \pi\rho|x - y| + \pi\rho|y - z| < \pi\rho e/2 + \pi\rho e/2 = \pi\rho e. \quad \triangleright$$

5.2.3. Proposition. *Let $\mathscr{V}$ be a subset and let u be an element of an LNS. The set $\mathscr{V}$ approximates u if and only if u is the o-limit of some net in* $\mathrm{mix}_{\mathrm{fin}}\, \mathscr{V}$.

$\triangleleft$ If $\mathscr{V}$ approximates u then $\inf\{|u - w| : w \in \mathrm{mix}_{\mathrm{fin}}\, \mathscr{V}\} = 0$. Therefore, in view of 5.2.1, there exists a net $(w_\alpha)_{\alpha \in \mathrm{A}}$ in $\mathrm{mix}_{\mathrm{fin}}\, \mathscr{V}$ such that $|u - w_\alpha| \downarrow 0$.

Conversely, if u is the o-limit of a net in $\mathrm{mix}_{\mathrm{fin}}\, \mathscr{V}$ then $\mathrm{mix}_{\mathrm{fin}}\, \mathscr{V}$ approximates u. It remains to use Proposition 5.2.2 on observing that $\mathscr{V}$ approximates $\mathrm{mix}_{\mathrm{fin}}\, \mathscr{V}$. $\triangleright$

Corollary. *If a subset $\mathscr{V}$ of an LNS $\mathscr{U}$ is finitely cyclic, then its o-closure consists of all elements $u \in \mathscr{U}$ approximated by $\mathscr{V}$.*

Corollary. *If a subset $\mathscr{V}$ of an LNS $\mathscr{U}$ is finitely cyclic, then its o-closure is o-closed and, hence, is the least o-closed subset of $\mathscr{U}$ that includes $\mathscr{V}$.*

$\triangleleft$ The claim follows from the previous corollary and Proposition 5.2.2. $\triangleright$

5.2.4. Proposition. *The following properties of a subset $\mathscr{V}$ of an LNS $\mathscr{U}$ are equivalent:*

(1) *$\mathscr{V}$ is an approximating subset of $\mathscr{U}$;*
(2) *for every ideal $\mathscr{U}_0 \subset \mathscr{U}$, the set $d_{\mathrm{fin}}\mathscr{V} \cap \mathscr{U}_0$ is o-dense in $\mathscr{U}_0$;*
(3) *the set $d_{\mathrm{fin}}\mathscr{V}$ is o-dense in $\mathscr{U}$;*
(4) *$d_{\mathrm{fin}}\mathscr{V}$ is an approximating subset of $\mathscr{U}$.*

$\triangleleft$ The implications (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious. It remains to prove that (1) $\Rightarrow$ (2) and (4) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2): Suppose that the set $\mathscr{V} \subset \mathscr{U}$ satisfies condition (1), fix an arbitrary ideal $\mathscr{U}_0 \subset \mathscr{U}$ and its element $u \in \mathscr{U}_0$, denote the set $d_{\mathrm{fin}}\mathscr{V} \cap \mathscr{U}_0$ by $\mathscr{W}$, and assign $e := \inf_{w \in \mathscr{W}} |u - w|$. Obviously, $e \leqslant |u|$. According to 5.2.1, there exists a net $(w_\alpha)_{\alpha \in \mathrm{A}}$ in $\mathscr{W}$ such that $|u - w_\alpha| \downarrow e$. It remains to show that $e = 0$. If $e \neq 0$ then, in view of 5.2.3, there are $w \in \mathrm{mix}_{\mathrm{fin}}\, \mathscr{V}$ and $\pi \in \mathrm{Pr}(E)$ such that

$\pi|u-w| < \pi e$. The inequalities $|\pi w| \leqslant |\pi w - \pi u| + |\pi u| \leqslant e + |u| \leqslant 2|u|$ ensure the containment $\pi w \in \mathscr{W}$ and, thus, we have the following contradictory relations: $\pi e \leqslant \pi|u - \pi w| < \pi e$.

(4) $\Rightarrow$ (1): Denote the set $d_{\mathrm{fin}}\mathscr{V}$ by $\mathscr{W}$ and suppose that it is an approximating subset of $\mathscr{U}$.

Denote $\inf_{v\in\mathscr{V}} |v|$ by e and prove that $e = 0$. If it is not so, then there is an element $u \in \mathscr{U}$ that satisfies the inequalities $0 < |u| \leqslant e/2$. Since $\inf_{w\in\mathscr{W}} |u-w| = 0$, there is an band projection $\pi \neq 0$ and an element $w = \pi_1 v_1 + \dots + \pi_n v_n \in \mathscr{W}$ ($v_i \in \mathscr{V}$) such that $\pi_0|u-w| < \pi_0|u|$ for all $0 \neq \pi_0 \leqslant \pi$. It is clear that $\pi w \neq 0$ and, hence, $\rho := \pi_i \wedge \pi \neq 0$ for some i. Now, the inequalities $\rho|u - v_i| < \rho|u| \leqslant \rho e/2$ lead to a contradiction: $\rho e \leqslant \rho|v_i| \leqslant \rho|u - v_i| + \rho|u| < \rho e/2 + \rho e/2 = \rho e$.

Thus, $\inf_{v\in\mathscr{V}} |v| = 0$, which implies that $\mathscr{V}$ approximates $\mathscr{V} \cup \{0\}$. However, it is obvious that the set $\mathscr{V} \cup \{0\}$ approximates $d_{\mathrm{fin}}\mathscr{V}$ and the latter approximates $\mathscr{U}$. It remains to apply Proposition 5.2.2. $\rhd$

REMARK. Replacing $d_{\mathrm{fin}}\mathscr{V}$ by $\mathrm{mix}_{\mathrm{fin}}\,\mathscr{V}$ in condition (2) of the last proposition can lead to a nonequivalent assertion even if $\mathscr{U} = E$. Indeed, the totality $\mathscr{V}$ of all numeric sequences convergent to 1 is an approximating subset of the K-space $\mathscr{U}$ of all sequences; however, the set $\mathrm{mix}_{\mathrm{fin}}\,\mathscr{V}$ coincides with $\mathscr{V}$ and has empty intersection with the order-dense ideal $\mathscr{U}_0 \subset \mathscr{U}$ of all vanishing sequences.

5.2.5. Lemma. *If $\mathscr{V}$ is an approximating subset of a d-complete LNS $\mathscr{U}$ over E then, for every $u \in \mathscr{U}$, $e \in E$, and $n \in \mathbb{N}$, there exists an element $w \in \mathrm{mix}\,\mathscr{V}$ satisfying the inequality $\langle e\rangle|u - w| \leqslant e/n$.*

$\lhd$ Suppose that $\mathscr{U}$ and $\mathscr{V}$ meet the hypotheses of the lemma and consider arbitrary elements $u \in \mathscr{U}$, $e \in E$, and $n \in \mathbb{N}$. According to 5.2.4, there is a net $(v_\alpha)_{\alpha\in\mathrm{A}}$ in $\mathrm{mix}_{\mathrm{fin}}\,\mathscr{V}$ o-convergent to u. We may assume that this net is order bounded. In view of 1.3.9, there is a partition of unity $(\pi_\alpha^n)_{\alpha\in\mathrm{A}}$ in the Boolean algebra $\mathrm{Pr}(E)$ such that $\pi_\alpha^n\langle e\rangle|v_\alpha - u| \leqslant e/n$ for all $\alpha \in \mathrm{A}$. It is clear that the sum $w := o\text{-}\sum_{\alpha\in\mathrm{A}} \pi_\alpha^n v_\alpha$ is the desired element of $\mathrm{mix}\,\mathscr{V}$. $\rhd$

5.2.6. Suppose that an order unity $\mathbb{1}$ is fixed in the norming K-space of an LNS. Then the r-convergence with regulator $\mathbb{1}$ is called the *uniform convergence* in the LNS. The notions of *uniformly dense subset* and *uniform closure* are introduced in such an LNS similarly.

Proposition. *Let $\mathscr{V}$ be a subset and let u be an element of an LNS over a K-space with a fixed order unity. The set $\mathscr{V}$ approximates u if and only if u is a uniform limit of some sequence in* $\operatorname{mix}\mathscr{V}$.

$\lhd$ Necessity is a straightforward consequence of Lemma 5.2.5; sufficiency is established as in the proof of Proposition 5.2.3. $\rhd$

5.2.7. Proposition. *Let $\mathscr{U}$ be a d-complete LNS over a K-space with a fixed order unity. The following properties of a subset $\mathscr{V} \subset \mathscr{U}$ are equivalent:*

(1) *$\mathscr{V}$ is an approximating subset of $\mathscr{U}$;*
(2) *for every ideal $\mathscr{U}_0 \subset \mathscr{U}$, the set $d\mathscr{V} \cap \mathscr{U}_0$ is uniformly dense in $\mathscr{U}_0$;*
(3) *$d\mathscr{V}$ is uniformly dense in $\mathscr{U}$;*
(4) *$d\mathscr{V}$ is an approximating subset of $\mathscr{U}$.*

$\lhd$ Suppose that an LNS $\mathscr{U}$ over E meets the hypotheses of the proposition and 1 is an order unity in E. The implications (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious and the implication (4) $\Rightarrow$ (1) is established as in the proof of Proposition 5.2.4. It remains to show that (1) $\Rightarrow$ (2).

Suppose that a subset $\mathscr{V} \subset \mathscr{U}$ satisfies condition (1), fix an arbitrary ideal $\mathscr{U}_0 \subset \mathscr{U}$, and denote the set $d\mathscr{V} \cap \mathscr{U}_0$ by $\mathscr{W}$.

Show that $\mathscr{W}$ approximates $\mathscr{U}_0$. For this purpose, we fix an arbitrary element $u \in \mathscr{U}_0$, assign $e := \inf_{w \in \mathscr{W}} |u - w|$, and establish the equality $e = 0$. If $e \neq 0$ then, in view of 5.2.6, there are $w \in \operatorname{mix}\mathscr{V}$ and $\pi \in \operatorname{Pr}(E)$ such that $\pi|u - w| < \pi e$. Obviously, $e \leqslant |u|$. The inequalities $|\pi w| \leqslant |\pi w - \pi u| + |\pi u| \leqslant e + |u| \leqslant 2|u|$ ensure the containment $\pi w \in \mathscr{W}$ and, thus, we have the following contradictory relations: $\pi e \leqslant \pi|u - \pi w| < \pi e$.

Since $\mathscr{W}$ approximates $\mathscr{U}_0$, in view of 5.2.5 there exists a sequence $(w_n)_{n\in\mathbb{N}}$ in $\operatorname{mix}\mathscr{W}$ such that $\langle u\rangle|u - w_n| \leqslant (|u| \wedge 1)/n$ for all $n \in \mathbb{N}$. It is clear that the sequence $\big(\langle u\rangle w_n\big)_{n\in\mathbb{N}}$ is constituted by elements of $\mathscr{W}$ and r-converges to u with regulator 1. $\rhd$

Remark. Replacing $d\mathscr{V}$ by $\operatorname{mix}\mathscr{V}$ in condition (2) of the last proposition can lead to a nonequivalent assertion even if $\mathscr{U} = E$. Indeed, the totality $\mathscr{V}$ of all numeric sequences with every member nonzero is an approximating subset of the K-space $\mathscr{U}$ of all sequences; however, the set $\operatorname{mix}\mathscr{V}$ coincides with $\mathscr{V}$ and has empty intersection with the order dense ideal $\mathscr{U}_0 \subset \mathscr{U}$ of all finitary (= terminating) sequences.

5.2.8. Proposition. *Let $\mathscr{V}$ be a subset and let u be an element of an LNS. The set $\mathscr{V}$ approximates u if and only if u is the r-limit of some sequence in* $\operatorname{mix}\mathscr{V}$.

$\lhd$ SUFFICIENCY: Suppose that $\mathscr{V}$ approximates u. Consider an arbitrary element $v \in \mathscr{V}$ and assign $e := |u| \vee |v|$. It is sufficient to fix an $n \in \mathbb{N}$ and find an element $w \in \operatorname{mix}\mathscr{V}$ that satisfies the inequality $|u - w| \leqslant e/n$. According to Lemma 5.2.5, there exists an element $w_0 \in \operatorname{mix}\mathscr{V}$ that satisfies the inequality $\langle e\rangle |u - w_0| \leqslant e/n$. It is clear that the sum $\langle e\rangle w_0 + \langle e\rangle^{\perp} v$ belongs to $\operatorname{mix}\mathscr{V}$, coincides with $\langle e\rangle w_0$, and, thus, is the desired element w.

Necessity is established in the same way as in Proposition 5.2.3. $\rhd$

5.2.9. Proposition. *Let $\mathscr{U}$ be a d-complete LNS. The following properties of a subset $\mathscr{V} \subset \mathscr{U}$ are equivalent:*

(1) *$\mathscr{V}$ is an approximating subset of $\mathscr{U}$;*
(2) *for every ideal $\mathscr{U}_0 \subset \mathscr{U}$, the set $d\mathscr{V} \cap \mathscr{U}_0$ is r-dense in $\mathscr{U}_0$;*
(3) *$d\mathscr{V}$ is r-dense in $\mathscr{U}$;*
(4) *$d\mathscr{V}$ is an approximating subset of $\mathscr{U}$.*

$\lhd$ The implications (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious, the equivalence (4) $\Leftrightarrow$ (1) is established in Proposition 5.2.7, and the proof of the implication (1) $\Rightarrow$ (2) word for word repeats that of the analogous implication in Proposition 5.2.7, with the only difference that 1 is replaced by $|u|$. $\rhd$

REMARK. Replacing $d\mathscr{V}$ by $\operatorname{mix}\mathscr{V}$ in condition (2) of the last proposition can lead to a nonequivalent assertion. There is an appropriate example in the previous remark (see 5.2.7).

5.2.10. A net $(e_\alpha)_{\alpha\in A}$ in a vector lattice E is said to be *asymptotically bounded* if there exists an index $\bar{\alpha} \in A$ such that the set $\{e_\alpha : \alpha \geqslant \bar{\alpha}\}$ is order bounded. Obviously, every o-convergent net is asymptotically bounded.

In the sequel, we need some modification of Theorem 1.3.9.

Lemma. *Let $(e_\alpha)_{\alpha\in \mathrm{A}}$ be a net in a K-space E and let $e \in E$.*

(1) *The net $(e_\alpha)_{\alpha\in\mathrm{A}}$ o-converges to e if and only if it is asymptotically*

bounded and the relation $o\text{-}\lim_{\alpha\in\mathrm{A}}\langle d\rangle\big\langle|e_\alpha - e| > d\big\rangle = 0$ *holds in the Boolean algebra* $\mathrm{Pr}(E)$ *for all positive* $d \in E$.

(2) *Let* D *be a set of positive elements in* E *such that the band* $D^{\perp\perp}$ *contains* e *and all the members of the net* $(e_\alpha)_{\alpha\in\mathrm{A}}$. *If the net* $(e_\alpha)_{\alpha\in\mathrm{A}}$ *is asymptotically bounded and* $o\text{-}\lim_{\alpha\in\mathrm{A}}\langle d\rangle\big\langle|e_\alpha - e| > d/n\big\rangle = 0$ *for all* $d \in D$ *and* $n \in \mathbb{N}$, *then* $o\text{-}\lim_{\alpha\in\mathrm{A}} e_\alpha = e$.

$\lhd$ (1) It is easy to verify the necessity of the criterion formulated, and its sufficiency follows from (2).

(2) Let an index $\bar\alpha \in \mathrm{A}$ be such that the set $\{e_\alpha : \alpha \geqslant \bar\alpha\}$ is bounded. Assign $e_0 := \inf_{\alpha\geqslant\bar\alpha}\sup_{\beta\geqslant\alpha}|e_\beta - e|$. If the net $(e_\alpha)_{\alpha\in\mathrm{A}}$ does not converge to e then $e_0 > 0$ and, thus, there are $\pi \in \mathrm{Pr}(E)$, $d \in D$, and $n \in \mathbb{N}$ such that $0 < \pi d/n < e_0$. Therefore, for each index $\alpha \geqslant \bar\alpha$, we have

$$\sup_{\beta\geqslant\alpha}\langle d\rangle\big\langle|e_\beta - e| > d/n\big\rangle = \langle d\rangle\Big\langle\sup_{\beta\geqslant\alpha}|e_\beta - e| > d/n\Big\rangle \geqslant \pi,$$

which contradicts the convergence of $\langle d\rangle\big\langle|e_\alpha - e| > d/n\big\rangle$ to zero. $\rhd$

Corollary. *Suppose that a* K*-space* E *has an order unity* $\mathbb{1}$, $(e_\alpha)_{\alpha\in\mathrm{A}}$ *is an asymptotically bounded net in* E, *and* $e \in E$. *Then* $o\text{-}\lim_{\alpha\in\mathrm{A}} e_\alpha = e$ *if and only if the relation* $o\text{-}\lim_{\alpha\in\mathrm{A}}\big\langle|e_\alpha - e| > \mathbb{1}/n\big\rangle = 0$ *holds in the Boolean algebra* $\mathrm{Pr}(E)$ *for all* $n \in \mathbb{N}$.

The condition of asymptotic boundedness presented in the above assertions is essential. Indeed, let a net $(\pi_\alpha)_{\alpha\in\mathrm{A}}$ of band projections and an element $e \in E$ be such that $o\text{-}\lim_{\alpha\in\mathrm{A}}\pi_\alpha = 0$ and $\pi_\alpha e \neq 0$ for all $\alpha \in \mathrm{A}$. Endow the Cartesian product $\mathrm{A} \times \mathbb{N}$ with the lexicographic order:

$$(\alpha, m) < (\beta, n) \quad \Leftrightarrow \quad \alpha < \beta \ \text{ or } \ (\alpha = \beta \ \text{ and } \ m < n).$$

Then $o\text{-}\lim_{(\alpha,n)\in\mathrm{A}\times\mathbb{N}}\langle d\rangle\big\langle|n\pi_\alpha e| > d\big\rangle = 0$ for all positive $d \in E$; however, the net $(n\pi_\alpha e)_{(\alpha,n)\in\mathrm{A}\times\mathbb{N}}$ is not asymptotically bounded and, hence, has no order limit.

5.2.11. By simplifying the proof of Lemma 5.2.10, we can obtain the following assertion.

Lemma. *Let $(e_\xi)_{\xi\in\Xi}$ be a family of positive elements of a K-space E.*

(1) *The equality $\inf_{\xi\in\Xi} e_\xi = 0$ is valid in the K-space E if and only if the relation $\inf_{\xi\in\Xi}\langle d\rangle\big\langle e_\xi > d\big\rangle = 0$ holds in the Boolean algebra $\mathrm{Pr}(E)$ for all positive $d \in E$.*

(2) *Let D be a set of positive elements of E such that $e_\xi \in D^{\perp\perp}$ for all $\xi \in \Xi$. If $\inf_{\xi\in\Xi}\langle d\rangle\big\langle e_\xi > d/n\big\rangle = 0$ for all $d \in D$ and $n \in \mathbb{N}$, then $\inf_{\xi\in\Xi} e_\xi = 0$.*

Corollary. *Suppose that a K-space E has an order unity 1 and $(e_\xi)_{\xi\in\Xi}$ is a family of positive elements of E. Then $\inf_{\xi\in\Xi} e_\xi = 0$ if and only if the relation $\inf_{\xi\in\Xi}\big\langle e_\xi > 1/n\big\rangle = 0$ holds in the Boolean algebra $\mathrm{Pr}(E)$ for all $n \in \mathbb{N}$.*

5.2.12. Throughout the remainder of the current section, we assume that E is a K-space, B is a complete Boolean algebra, and $h : \mathrm{Pr}(E) \to B$ is a ring homomorphism. Say that a net $(\pi_\alpha)_{\alpha\in\mathrm{A}}$ in $\mathrm{Pr}(E)$ ***h-converges to zero*** and write $h\text{-}\lim_{\alpha\in\mathrm{A}} \pi_\alpha = 0$ whenever $o\text{-}\lim_{\alpha\in\mathrm{A}} \pi_\alpha = 0$ in the Boolean algebra $\mathrm{Pr}(E)$ and $o\text{-}\lim_{\alpha\in\mathrm{A}} h(\pi_\alpha) = 0$ in the Boolean algebra B. In case $h\text{-}\lim_{\alpha\in\mathrm{A}} \pi_\alpha^\perp = 0$, i.e., if $o\text{-}\lim_{\alpha\in\mathrm{A}} \pi_\alpha = 1$ and $o\text{-}\lim_{\alpha\in\mathrm{A}} h(\pi_\alpha) = h(1)$, we say that the net $(\pi_\alpha)_{\alpha\in\mathrm{A}}$ ***h-converges to unity*** and write $h\text{-}\lim_{\alpha\in\mathrm{A}} \pi_\alpha = 1$. We say that a net $(e_\alpha)_{\alpha\in\mathrm{A}}$ in E ***h-converges to*** $e \in E$ and write $h\text{-}\lim_{\alpha\in\mathrm{A}} e_\alpha = e$ if the net $(e_\alpha)_{\alpha\in\mathrm{A}}$ is asymptotically bounded and $h\text{-}\lim_{\alpha\in\mathrm{A}}\langle d\rangle\big\langle |e_\alpha - e| > d\big\rangle = 0$ for all positive $d \in E$. In this case, we call the element e the ***h-limit*** of the net $(e_\alpha)_{\alpha\in\mathrm{A}}$. We say that a net $(u_\alpha)_{\alpha\in\mathrm{A}}$ in $\mathscr{U}$ ***h-converges to*** $u \in \mathscr{U}$ and write $h\text{-}\lim_{\alpha\in\mathrm{A}} u_\alpha = u$ if $h\text{-}\lim_{\alpha\in\mathrm{A}} |u_\alpha - u| = 0$. In this case, we call the element u the ***h-limit*** of the net $(u_\alpha)_{\alpha\in\mathrm{A}}$. The totality of the h-limits of all h-convergent nets in a subset $\mathscr{V} \subset \mathscr{U}$ is called the ***h-closure*** of $\mathscr{V}$. We call a set ***h-closed*** if it coincides with the h-closure of itself. We say that a set is ***h-dense in*** $\mathscr{U}$ if its h-closure coincides with $\mathscr{U}$.

If a family $(\pi_\xi)_{\xi\in\Xi}$ in $\mathrm{Pr}(E)$ is such that $\inf_{\xi\in\Xi}\pi_\xi = 0$ in the Boolean algebra $\mathrm{Pr}(E)$ and $\inf_{\xi\in\Xi} h(\pi_\xi) = 0$ in the Boolean algebra B, then we write $h\text{-}\inf_{\xi\in\Xi}\pi_\xi = 0$. In case $h\text{-}\inf_{\xi\in\Xi}\pi_\xi^\perp = 0$, i.e., if $\sup_{\xi\in\Xi}\pi_\xi = 1$ and $\sup_{\xi\in\Xi} h(\pi_\xi) = h(1)$, we write $h\text{-}\sup_{\xi\in\Xi}\pi_\xi = 1$. For an arbitrary family $(e_\xi)_{\xi\in\Xi}$ of positive elements of a K-space E, the notation $h\text{-}\inf_{\xi\in\Xi} e_\xi = 0$ means that $h\text{-}\inf_{\xi\in\Xi}\langle d\rangle\big\langle e_\xi > d\big\rangle = 0$ for all positive $d \in E$.

REMARK. The criterion for o-convergence which is formulated in Corollary 5.2.10 has no analog for h-convergence. The same is true of Corollary 5.2.11. Indeed, consider as E the K-space of all numeric sequences. Let the Boolean homomorphism $h : \mathrm{Pr}(E) \to \{0,1\}$ be the characteristic function of some nonprincipal ultrafilter in the Boolean algebra $\mathrm{Pr}(E)$. Denote by F the set of all positive sequences convergent to 1. Obviously, the sequence $e = (m)_{m\in\mathbb{N}}$ is an order unity in E and the relation $h\text{-}\inf_{f\in F}\langle f > e/n\rangle = 0$ holds for all $n \in \mathbb{N}$. Moreover, indexing each element of F by itself and endowing the index set with the reverse pointwise order, we obtain a set $(f)_{f\in F}$ that satisfies the relation $h\text{-}\lim_{f\in F}\langle f > e/n\rangle = 0$. Nevertheless, $h\langle f > 1/2\rangle = 1$ for all $f \in F$.

The following assertion follows from Lemmas 5.2.10 and 5.2.11.

Proposition. (a) *For every net $(e_\alpha)_{\alpha\in\mathrm{A}}$ in E and arbitrary element $e \in E$, from $h\text{-}\lim_{\alpha\in\mathrm{A}} e_\alpha = e$ it follows that $o\text{-}\lim_{\alpha\in\mathrm{A}} e_\alpha = e$. If the homomorphism h is o-continuous, then the relations $h\text{-}\lim_{\alpha\in\mathrm{A}} e_\alpha = e$ and $o\text{-}\lim_{\alpha\in\mathrm{A}} e_\alpha = e$ are equivalent.*

(b) *For every net $(e_\xi)_{\xi\in\Xi}$ of positive elements of E, from $h\text{-}\inf_{\xi\in\Xi} e_\xi = 0$ it follows that $\inf_{\xi\in\Xi} e_\xi = 0$. If the homomorphism h is o-continuous, then the relations $h\text{-}\inf_{\xi\in\Xi} e_\xi = 0$ and $\inf_{\xi\in\Xi} e_\xi = 0$ are equivalent.*

5.2.13. REMARK. In the sequel of the current chapter, while establishing equalities of the form $\lim_{\alpha\in\mathrm{A}} h\big(\langle d\rangle\langle e_\alpha > d\rangle\big) = 0$ or $\inf_{\xi\in\Xi} h\big(\langle d\rangle\langle e_\xi > d\rangle\big) = 0$, we often assume that $h\langle d\rangle = 1$. This assumption does not restrict generality. Indeed, leaving aside the trivial case $h\langle d\rangle = 0$ and replacing B by the Boolean algebra $\{b \in B : b \leqslant h\langle d\rangle\}$, we arrive at the situation $h\langle d\rangle = 1$.

5.2.14. Let $\mathscr{V}$ be a subset of an LNS $\mathscr{U}$. We say that $\mathscr{V}$ ***h-approximates an element*** $u \in \mathscr{U}$ if $h\text{-}\inf_{v\in\mathscr{V}} |u - v| = 0$. We say that $\mathscr{V}$ ***h-approximates a set*** $\mathscr{W} \subset \mathscr{U}$ if $\mathscr{V}$ h-approximates every element of $\mathscr{W}$. A subset of an LNS $\mathscr{U}$ is called ***h-approximating*** if it h-approximates $\mathscr{U}$. From Proposition 5.2.12 it follows that every h-approximating set is approximating and, in case the homomorphism h is o-continuous, the notions of approximating and h-approximating set coincide.

Proposition. *Let X, Y, and Z be subsets of an LNS. If X h-approximates Y and Y h-approximates Z, then X h-approximates Z.*

$\lhd$ Consider an arbitrary element $z \in Z$, fix a positive element d of the norming lattice, and assign $b := \inf_{x\in X} h\big(\langle d\rangle\langle |x - z| > d\rangle\big)$. Due to 5.2.2, it is sufficient to establish the equality $b = 0$. For simplicity, we assume that $h\langle d\rangle = 1$ (see 5.2.13). Suppose to the contrary that $b \neq 0$. Then, in view of $\inf_{y\in Y} h\langle |y - z| > d/2\rangle = 0$, there is an element $y \in Y$ such that $b_0 := b \wedge h\langle |y - z| > d/2\rangle < b$. Similarly, in view of the equality $\inf_{x \in X} h\langle |x - y| > d/2\rangle = 0$, there is an element $x \in X$ such that $(b \backslash b_0) \wedge h\langle |x - y| > d/2\rangle < (b \backslash b_0)$. It is easy to verify that x satisfies the inequality $b \wedge h\langle |x - z| > d\rangle < b$, which contradicts the definition of b. $\triangleright$

5.2.15. *Proposition.* *Let $\mathscr{V}$ be a subset and let u be an element of an LNS. The set $\mathscr{V}$ h-approximates u if and only if u is the h-limit of some net in* $\operatorname{mix_{fin}} \mathscr{V}$.

$\triangleleft$ NECESSITY: If $\mathscr{V}$ h-approximates u then, in view of 5.2.1, there exists a net $(w_\alpha)_{\alpha \in \mathrm{A}}$ in $\operatorname{mix_{fin}} \mathscr{V}$ such that the net $\big(|u - w_\alpha|\big)_{\alpha \in \mathrm{A}}$ decreases and

$$\{|u - w_\alpha| : \alpha \in \mathrm{A}\} = \{|u - w| : w \in \operatorname{mix_{fin}} \mathscr{V}\}.$$

It remains to observe that $h\text{-}\lim_{\alpha \in \mathrm{A}} |u - w_\alpha| = 0$.

SUFFICIENCY: If u is the h-limit of a net in $\operatorname{mix_{fin}} \mathscr{V}$, then $\operatorname{mix_{fin}} \mathscr{V}$ h-approximates u. It remains to observe that $\mathscr{V}$ h-approximates $\operatorname{mix_{fin}} \mathscr{V}$ and to use Proposition 5.2.14. $\triangleright$

Corollary. *If a subset $\mathscr{V}$ of an LNS $\mathscr{U}$ is finitely cyclic, then its h-closure consists of all elements $u \in \mathscr{U}$ h-approximated by $\mathscr{V}$.*

Corollary. *If a subset $\mathscr{V}$ of an LNS $\mathscr{U}$ is finitely cyclic, then its h-closure is h-closed and, hence, is the least h-closed subset of $\mathscr{U}$ that includes $\mathscr{V}$.*

$\triangleleft$ The claim follows from the previous corollary and Proposition 5.2.14. $\triangleright$

5.2.16. *Proposition.* *Let $\mathscr{V}$ be a subset of an LNS $\mathscr{U}$ and satisfy the relation $h\text{-}\inf_{v \in \mathscr{V}} |v| = 0$. Then the following assertions are equivalent:*

(1) *$\mathscr{V}$ is an h-approximating subset of $\mathscr{U}$;*

(2) *for every ideal $\mathscr{U}_0 \subset \mathscr{U}$, the set $d_{\mathrm{fin}}\mathscr{V} \cap \mathscr{U}_0$ is h-dense in $\mathscr{U}_0$;*

(3) *the set $d_{\mathrm{fin}}\mathscr{V}$ is h-dense in $\mathscr{U}$;*

(4) *$d_{\mathrm{fin}}\mathscr{V}$ is an h-approximating subset of $\mathscr{U}$.*

$\triangleleft$ The implications (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious. It remains to prove that (1) $\Rightarrow$ (2) and (4) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2): Suppose that a subset $\mathscr{V} \subset \mathscr{U}$ satisfies condition (1). Fix an arbitrary ideal $\mathscr{U}_0 \subset \mathscr{U}$ and denote the set $d_{\mathrm{fin}}\mathscr{V} \cap \mathscr{U}_0$ by $\mathscr{W}$. Consider an arbitrary

element $u \in \mathscr{U}_0$. According to 5.2.15, there exists a net $(w_\alpha)_{\alpha\in\mathrm{A}}$ in $\mathrm{mix}_{\mathrm{fin}}\mathscr{V}$ that h-converges to u. For each $\alpha \in \mathrm{A}$, we assign $\pi_\alpha := \langle |u - w_\alpha| \leqslant |u| \rangle$. The relations

$$\begin{aligned} |\pi_\alpha w_\alpha| &\leqslant |u| + |u - \pi_\alpha w_\alpha| = |u| + (\pi_\alpha |u - w_\alpha| + \pi_\alpha^\perp |u|) \\ &\leqslant |u| + (\pi_\alpha |u| + \pi_\alpha^\perp |u|) = 2|u| \end{aligned}$$

ensure that the net $(\pi_\alpha w_\alpha)_{\alpha\in\mathrm{A}}$ is constituted by elements of $\mathscr{W}$ and the relations

$$|u - \pi_\alpha w_\alpha| = \pi_\alpha |u - w_\alpha| + \pi_\alpha^\perp |u| \leqslant \pi_\alpha |u - w_\alpha| + \pi_\alpha^\perp |u - w_\alpha| = |u - w_\alpha|$$

together with $h\text{-}\lim_{\alpha\in\mathrm{A}} |u - w_\alpha| = 0$ give $h\text{-}\lim_{\alpha\in\mathrm{A}} |u - \pi_\alpha w_\alpha| = 0$.

(4) $\Rightarrow$ (1): From the relation $h\text{-}\inf_{v\in\mathscr{V}} |v| = 0$ it follows that $\mathscr{V}$ h-approximates $\mathscr{V} \cup \{0\}$. On the other hand, the set $\mathscr{V} \cup \{0\}$ obviously h-approximates $d_{\mathrm{fin}}\mathscr{V}$, the latter in turn h-approximating $\mathscr{U}$. It remains to apply Proposition 5.2.14. $\triangleright$

5.2.17. The difference between the statements of Propositions 5.2.4 and 5.2.16 is essential: the condition $h\text{-}\inf_{v\in\mathscr{V}} |v| = 0$ in the latter proposition cannot be omitted. Indeed, consider the K-space E of all numeric sequences and assign $\mathscr{U} := \{u \in E : \inf(\mathrm{Lim}\,|u| \backslash \{0\}) > 0\}$, where $\mathrm{Lim}\,|u|$ is the set of all partial limits of the sequence $|u|$. We make $\mathscr{U}$ an LNS over E by defining $|u| := |u|$ for all $u \in \mathscr{U}$. As in Remark 5.2.12, let the Boolean homomorphism $h : \mathrm{Pr}(E) \to \{0, 1\}$ be the characteristic function of some nonprincipal ultrafilter in the Boolean algebra $\mathrm{Pr}(E)$. Consider as $\mathscr{V}$ the set $\{u \in E : \inf \mathrm{Lim}\,|u| > 0\}$ and assign $d := (1/n)_{n\in\mathbb{N}}$. It is clear that $d_{\mathrm{fin}}\mathscr{V} = \mathscr{U}$; however, $h\langle |v| > d \rangle = 1$ for all $v \in \mathscr{V}$.

Proposition. *Let $\mathscr{U}$ be an LNS over E. Suppose that, for every positive $e \in E$, there is an element $u \in \mathscr{U}$ satisfying the inequalities $e \leqslant |u| \leqslant 2e$ (this is true, for instance, in case $\mathscr{U}$ is o-complete, see* 1.6.3*). Then the condition $h\text{-}\inf_{v\in\mathscr{V}} |v| = 0$ in the statement of Proposition* 5.2.16 *can be omitted.*

$\triangleleft$ Consider an arbitrary subset $\mathscr{V} \subset \mathscr{U}$, denote $d_{\mathrm{fin}}\mathscr{V}$ by $\mathscr{W}$, suppose that $\mathscr{W}$ h-approximates $\mathscr{U}$, and establish the relation $h\text{-}\inf_{v\in\mathscr{V}} |v| = 0$. Due to 5.2.4 (we now use the implication (4) $\Rightarrow$ (1)), it is sufficient to fix an arbitrary positive element $d \in E$ and to show that

$$\inf_{v\in\mathscr{V}} h\big(\langle d\rangle \langle |v| > d\rangle\big) = 0.$$

For the sake of simplicity, we assume that $h\langle d\rangle = 1$ (see 5.2.13). Denote the element $\inf_{v\in\mathscr{V}} h\langle |v| > d\rangle$ by b and assume to the contrary that $b \neq 0$. Consider an arbitrary element $u \in \mathscr{U}$ satisfying the inequalities $d/4 \leqslant |u| \leqslant d/2$. In view of the equality $\inf_{w\in\mathscr{W}} h\langle |u-w| > d/5\rangle = 0$, there exists an element $w = \pi_1 v_1 + \dots + \pi_n v_n \in \mathscr{W}$ ($v_i \in \mathscr{V}$) such that $b \wedge h\langle |u-w| > d/5\rangle < b$. Using the equality

$$\begin{aligned}&\langle |u-w| > d/5\rangle \\ &\quad = \pi_1\langle |u-v_1| > d/5\rangle \vee \dots \vee \pi_n\langle |u-v_n| > d/5\rangle \vee (\pi_1 \vee \dots \vee \pi_n)^{\perp}\langle d\rangle,\end{aligned}$$

it is easy to verify that $b\wedge h\langle |u-v_i| > d/5\rangle < b$ for at least one index $i \in \{1,\dots,n\}$. Then, applying the relations

$$\langle |v_i| > d\rangle \leqslant \langle |u-v_i| + |u| > d\rangle \leqslant \langle |u-v_i| > d/2\rangle \leqslant \langle |u-v_i| > d/5\rangle,$$

we arrive at the equality $b \wedge h\langle |v_i| > d\rangle < b$, which contradicts the definition of b. $\triangleright$

5.2.18. A disjoint family $(\pi_\xi)_{\xi\in\Xi}$ in the Boolean algebra $\mathrm{Pr}(E)$ is called an *h-partition of unity* if $h\text{-}\sup_{\xi\in\Xi}\pi_\xi = 1$. If $(u_\xi)_{\xi\in\Xi}$ is an arbitrary family in an LNS $\mathscr{U}$ over E and $(\pi_\xi)_{\xi\in\Xi}$ is an h-partition of unity in $\mathrm{Pr}(E)$, then we call the sum $o\text{-}\sum_{\xi\in\Xi}\pi_\xi u_\xi$ (if it exists) the *h-mixing* of the family $(u_\xi)_{\xi\in\Xi}$ with respect to $(\pi_\xi)_{\xi\in\Xi}$. For an arbitrary subset $\mathscr{V} \subset \mathscr{U}$, the totality of various h-mixings of all (all countable) families in $\mathscr{V}$ is called the *h-cyclic hull* (the *countably h-cyclic hull*) of the set $\mathscr{V}$ and denoted by $h\text{-mix}\,\mathscr{V}$ (by $h\text{-mix}_\sigma\mathscr{V}$, respectively). A set $\mathscr{V} \subset \mathscr{U}$ is called *h-cyclic* if it coincides with the h-cyclic hull of itself. It is easy to verify that the h-cyclic hull of $\mathscr{V}$ is the least h-cyclic set that includes $\mathscr{V}$.

5.2.19. REMARK. We confine ourselves to the criteria for h-approximation given in Propositions 5.2.15 and 5.2.16. We did not succeed in using the notion of h-cyclic hull to obtain efficient descriptions for h-approximation analogous to those presented in 5.2.6–5.2.9.

5.3. Order Bounded Operators

In this section, we depart from the general conventions made in 5.1.7 and consider not only decomposable LNSs over K-spaces but also arbitrary LNSs over arbitrary vector lattices. We do it not for the sake of generality but rather for avoiding duplication of formulations both for LNSs and vector lattices. Indeed, every vector lattice together with the modulus function $|\cdot|$ is an LNS over itself. Thus, a definition or an assertion formulated for LNSs can be formally extended to the case of vector lattices. Observe that a vector lattice is o-complete as an LNS (i.e., is a BKS) if and only if it is a K-space.

5.3.1. Let $\mathscr{U}$ be an LNS over a vector lattice E. A net $(u_\alpha)_{\alpha\in\mathrm{A}}$ in $\mathscr{U}$ is called ***asymptotically bounded*** if the net $(|u_\alpha|)_{\alpha\in\mathrm{A}}$ possesses this property; i.e., if there exists an index $\bar\alpha\in\mathrm{A}$ such that the set $\{|u_\alpha| : \alpha \geqslant \bar\alpha\}$ is order bounded in E.

(a) We say that a subset $\mathscr{W}\subset\mathscr{U}$ is ***r-annullable*** (***o-annullable, boundable***) if, for every net $(w_\alpha)_{\alpha\in\mathrm{A}}$ in $\mathscr{W}$ and every vanishing numeric net $(\varepsilon_\alpha)_{\alpha\in\mathrm{A}}$, the net $(\varepsilon_\alpha w_\alpha)_{\alpha\in\mathrm{A}}$ is r-convergent to zero (o-convergent to zero, asymptotically bounded).

(b) We say that a subset $\mathscr{W}\subset\mathscr{U}$ is ***countably r-annullable*** (***countably o-annullable, countably boundable***) if, for every countable net $(w_\alpha)_{\alpha\in\mathrm{A}}$ in $\mathscr{W}$ and every vanishing numeric net $(\varepsilon_\alpha)_{\alpha\in\mathrm{A}}$, the net $(\varepsilon_\alpha w_\alpha)_{\alpha\in\mathrm{A}}$ is r-convergent to zero (o-convergent to zero, asymptotically bounded).

(c) We say that a subset $\mathscr{W}\subset\mathscr{U}$ is ***sequentially r-annullable*** (***sequentially o-annullable, sequentially boundable***) if, for every sequence $(w_n)_{n\in\mathrm{N}}$ in $\mathscr{W}$ and every vanishing numeric sequence $(\varepsilon_n)_{n\in\mathrm{N}}$, the sequence $(\varepsilon_n w_n)_{n\in\mathrm{N}}$ is r-convergent to zero (o-convergent to zero, bounded).

(d) We say that a subset $\mathscr{W}\subset\mathscr{U}$ is ***semibounded*** (***countably semibounded, sequentially semibounded***) if, for every net (countable net, sequence) $(w_\alpha)_{\alpha\in\mathrm{A}}$ in $\mathscr{W}$ and every vanishing numeric net $(\varepsilon_\alpha)_{\alpha\in\mathrm{A}}$, the relation $\inf_{\alpha\in\mathrm{A}}|\varepsilon_\alpha w_\alpha| = 0$ holds in the vector lattice E.

Theorem. *Let $\mathscr{U}$ be an LNS over a vector lattice E and let $\mathscr{W}\subset\mathscr{U}$.*

(a) *The following assertions are equivalent:*

(1) *the set $\mathscr{W}$ is r-annullable;*

(2) *the set $\mathscr{W}$ is o-annullable;*

(3) *the set $\mathscr{W}$ is boundable;*

(4) *the set $\{|w| : w\in\mathscr{W}\}$ is order bounded in E.*

(b) *The following assertions are equivalent:*

(1) *the set $\mathscr{W}$ is countably r-annullable;*

(2) *the set $\mathscr{W}$ is countably o-annullable;*

(3) *the set $\mathscr{W}$ is countably boundable;*

(4) *for every countable subset* $\mathscr{W}_0 \subset \mathscr{W}$, *the set* $\{|w| : w \in \mathscr{W}_0\}$ *is order bounded in* E.

(c) *The following assertions are equivalent:*

(1) *the set* $\mathscr{W}$ *is sequentially* r*-annullable;*

(2) *the set* $\mathscr{W}$ *is sequentially* o*-annullable;*

(3) *the set* $\mathscr{W}$ *is sequentially boundable.*

(d) *The following assertions are equivalent:*

(1) *the set* $\mathscr{W}$ *is semibounded;*

(2) *the set* $\mathscr{W}$ *is countably semibounded;*

(3) *the set* $\mathscr{W}$ *is sequentially semibounded;*

(4) $\inf_{n\in\mathbb{N}} |w_n|/n = 0$ *for every sequence* $(w_n)_{n\in\mathbb{N}}$ *in* $\mathscr{W}$.

$\lhd$ (a) The implications (4) $\Rightarrow$ (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious. We will show that (3) $\Rightarrow$ (4). Order the Cartesian product $\mathscr{W} \times \mathbb{N}$ by comparing the second component: $(w_1, n_1) < (w_2, n_2) \Leftrightarrow n_1 < n_2$. Applying assertion (3) to the nets $(w)_{(w,n)\in\mathscr{W}\times\mathbb{N}}$ and $(1/n)_{(w,n)\in\mathscr{W}\times\mathbb{N}}$, we obtain a pair $(\bar{w}, \bar{n}) \in \mathscr{W} \times \mathbb{N}$ and an element $e \in E$ such that $\big|w/n\big| \leqslant e$ for all $(w,n) \geqslant (\bar{w}, \bar{n})$. In particular, $\big|w/(\bar{n}+1)\big| \leqslant e$ for all $w \in \mathscr{W}$, which implies that the set $\{|w| : w \in \mathscr{W}\}$ is bounded from above by $(\bar{n}+1)e$.

(b) This is established in the same way as (a).

(c) The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious. We will show that (3) $\Rightarrow$ (1). Fix an arbitrary sequence $(w_n)_{n\in\mathbb{N}}$ in $\mathscr{W}$ and a vanishing numeric sequence $(\varepsilon_n)_{n\in\mathbb{N}}$. According to (3), the set $\big\{\big||\varepsilon_n|^{1/2} w_n\big| : n \in \mathbb{N}\big\}$ has some upper bound $e \in E$. In order to prove assertion (1), it remains to observe that $|\varepsilon_n w_n| \leqslant |\varepsilon_n|^{1/2} e$ for all $n \in \mathbb{N}$.

(d) The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious. We show that (4) $\Rightarrow$ (1). Fix an arbitrary net $(w_\alpha)_{\alpha\in\mathrm{A}}$ in $\mathscr{W}$ and a vanishing numeric net $(\varepsilon_\alpha)_{\alpha\in\mathrm{A}}$. For each natural $n \in \mathbb{N}$, choose an index $\alpha(n) \in \mathrm{A}$ so that $\varepsilon_{\alpha(n)} \leqslant 1/n$. Then, using (4), we obtain the relations $\inf_{\alpha\in\mathrm{A}} |\varepsilon_\alpha w_\alpha| \leqslant \inf_{n\in\mathbb{N}} \big|\varepsilon_{\alpha(n)} w_{\alpha(n)}\big| = 0$. $\rhd$

A subset $\mathscr{W} \subset \mathscr{U}$ satisfying the conditions listed in items (a), (b), and (c) of the last theorem is called ***bounded***, ***countably bounded***, and ***sequentially bounded***, respectively.

5.3.2. Obviously, every bounded set is countably bounded, every countably bounded set is sequentially bounded, and every sequentially bounded set is semibounded. We observe that the four types of boundedness differ pairwise even if $\mathcal{U} = E$. Indeed, in the K-space of all functions $e : \mathbb{R} \to \mathbb{R}$ with countable supports $e^{-1}[\mathbb{R}\backslash\{0\}]$, the set $\{e_t : t \in \mathbb{R}\}$ of the characteristic functions of all singletons $\{t\} \subset \mathbb{R}$ is countably bounded but not bounded. The set $\{e_n : n \in \mathbb{N}\}$ of the characteristic functions of all singletons $\{n\} \subset \mathbb{N}$ is a sequentially bounded but not a countably bounded subset of the K-space of vanishing numeric sequences.

We will give an example of a semibounded but not sequentially bounded subset in the K-space $M([0,1])$ of cosets of real-valued Lebesgue-measurable functions on the interval $[0,1]$. For this purpose, we construct the family of intervals I_m^n $(n \in \mathbb{N}$, $m \in \{1,2,\dots,2^n\})$ as follows:

$$I_1^1 := \left[\tfrac{0}{2}, \tfrac{1}{2}\right], \quad I_2^1 := \left[\tfrac{1}{2}, \tfrac{2}{2}\right];$$

$$I_1^2 := \left[\tfrac{0}{4}, \tfrac{1}{4}\right], \quad I_2^2 := \left[\tfrac{1}{4}, \tfrac{2}{4}\right], \quad I_3^2 := \left[\tfrac{2}{4}, \tfrac{3}{4}\right], \quad I_4^2 := \left[\tfrac{3}{4}, \tfrac{4}{4}\right];$$

$$\dots$$

$$I_1^n := \left[\tfrac{0}{2^n}, \tfrac{1}{2^n}\right], \quad I_2^n := \left[\tfrac{1}{2^n}, \tfrac{2}{2^n}\right], \quad \dots, \quad I_{2^n}^n := \left[\tfrac{2^n-1}{2^n}, \tfrac{2^n}{2^n}\right];$$

$$\dots$$

and denote by $\mathbf{f}_m^n$ the coset in $M([0,1])$ containing the characteristic function of the interval I_m^n. Then the set $\{2^n\mathbf{f}_m^n : n \in \mathbb{N},\ m \in \{1,2,\dots,2^n\}\}$ is the desired one.

5.3.3. Theorem. *Let $\mathcal{U}$ and $\mathcal{V}$ be LNSs over respective vector lattices E and F and let T be a linear operator from $\mathcal{U}$ into $\mathcal{V}$.*

(a) *The following assertions are equivalent:*

 (1) *T is r-continuous;*

 (2) *T is r-o-continuous;*

 (3) *if $r\text{-}\lim_{\alpha\in\mathrm{A}} u_\alpha = 0$ in $\mathcal{U}$ then the net $(Tu_\alpha)_{\alpha\in\mathrm{A}}$ is asymptotically bounded;*

 (4) *T takes bounded subsets of $\mathcal{U}$ into bounded subsets of $\mathcal{V}$;*

 (5) *for every $e \in E^+$, the set $\{|Tu| : |u| \leqslant e\}$ is bounded in F.*

(b) *The following assertions are equivalent:*

 (1) *T is countably r-continuous;*

 (2) *T is countably r-o-continuous;*

 (3) *if $r\text{-}\lim_{\alpha\in\mathrm{A}} u_\alpha = 0$ in $\mathcal{U}$ and the index set A is countable, then the net $(Tu_\alpha)_{\alpha\in\mathrm{A}}$ is asymptotically bounded;*

(4) *T takes countably bounded subsets of $\mathscr{U}$ into countably bounded subsets of $\mathscr{V}$;*

(5) *T takes bounded subsets of $\mathscr{U}$ into countably bounded subsets of $\mathscr{V}$;*

(6) *T takes countable bounded subsets of $\mathscr{U}$ into bounded subsets of $\mathscr{V}$.*

(c) *The following assertions are equivalent:*

(1) *T is sequentially r-continuous;*

(2) *T is sequentially r-o-continuous;*

(3) *if $r\text{-}\lim_{n\in\mathbb{N}} u_n = 0$ in $\mathscr{U}$ then the sequence $(Tu_n)_{n\in\mathbb{N}}$ is bounded;*

(4) *T takes sequentially bounded subsets of $\mathscr{U}$ into sequentially bounded subsets of $\mathscr{V}$;*

(5) *T takes bounded subsets of $\mathscr{U}$ into sequentially bounded subsets of $\mathscr{V}$.*

(d) *The following assertions are equivalent:*

(1) *if $r\text{-}\lim_{\alpha\in\mathrm{A}} u_\alpha = 0$ in $\mathscr{U}$ then $\inf_{\alpha\in\mathrm{A}} |Tu_\alpha| = 0$;*

(2) *if $r\text{-}\lim_{\alpha\in\mathrm{A}} u_\alpha = 0$ in $\mathscr{U}$ and the index set A is countable, then $\inf_{\alpha\in\mathrm{A}} |Tu_\alpha| = 0$;*

(3) *if $r\text{-}\lim_{n\in\mathbb{N}} u_n$ in $\mathscr{U}$ then $\inf_{n\in\mathbb{N}} |Tu_n| = 0$;*

(4) *T takes semibounded subsets of $\mathscr{U}$ into semibounded subsets of $\mathscr{V}$;*

(5) *T takes bounded subsets of $\mathscr{U}$ into semibounded subsets of $\mathscr{V}$.*

$\triangleleft$ (a) The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (5) are obvious. Using boundability as a criterion for boundedness (see Theorem 5.3.1 (a)), it is easy to deduce (4) from (3). It remains to show that (5) $\Rightarrow$ (1). Suppose that the operator T satisfies condition (5) and, for every positive element $e \in E$, denote by f_e some upper bound of the set $\{|Tu| : |u| \leqslant e\}$ in the lattice F. Let $(u_\alpha)_{\alpha\in\mathrm{A}}$ be an arbitrary net in $\mathscr{U}$ r-convergent to zero with regulator $e \in E$. Fix an arbitrary number $\varepsilon > 0$ and choose an index $\bar{\alpha} \in A$ so that $|u_\alpha| \leqslant \varepsilon e$ for all $\alpha \geqslant \bar{\alpha}$. Then, for all $\alpha \geqslant \bar{\alpha}$, we have: $|Tu_\alpha| = \varepsilon \big|Tu_\alpha/\varepsilon\big| \leqslant \varepsilon f_e$.

(b) The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (5) $\Rightarrow$ (6) are obvious. Using countable boundability as a criterion for countable boundedness (see Theorem 5.3.1 (b)), it is easy to deduce (4) from (3). It remains to show that (6) $\Rightarrow$ (1). Suppose that the operator T satisfies condition (6). Let $(u_\alpha)_{\alpha\in\mathrm{A}}$ be an arbitrary countable net in $\mathscr{U}$ r-convergent to zero with regulator $e \in E$. For every natural n, denote by α_n an element of A such that $|u_\alpha| \leqslant e/n$ for all $\alpha \geqslant \alpha_n$.

The set $\mathscr{U}_0 := \{nu_\alpha : n \in \mathbb{N},\ \alpha \in \mathrm{A},\ \alpha \geqslant \alpha_n\}$ is countable and bounded; hence, there is an element $f \in F$ such that $|Tu| \leqslant f$ for all $u \in \mathscr{U}_0$. Then $|Tu_\alpha| = |Tnu_\alpha|/n \leqslant f/n$ for all $\alpha \geqslant \alpha_n$.

(c) The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (5) are obvious. Using sequential boundability as a criterion for sequential boundedness (see Theorem 5.3.1 (c)), it is easy to deduce (4) from (3). It remains to show that (5) $\Rightarrow$ (1). Let $(u_n)_{n\in\mathbb{N}}$ be an arbitrary sequence in $\mathscr{U}$ r-convergent to zero with regulator $e \in E$. Then there exists a vanishing numeric sequence $\varepsilon_n > 0$ such that $|u_n| \leqslant \varepsilon_n e$ for all $n \in \mathbb{N}$. Boundedness of the set $\{u_n/\varepsilon_n : n \in \mathbb{N}\}$ and condition (5) allow us to conclude that the set $\{Tu_n/\varepsilon_n : n \in \mathbb{N}\}$ is sequentially r-annullable and, hence, the sequence $(Tu_n)_{n\in\mathbb{N}}$ r-converges to zero.

(d) The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious. We will show that (4) $\Rightarrow$ (1). Let $(u_\alpha)_{\alpha\in\mathrm{A}}$ be an arbitrary net in $\mathscr{U}$ r-convergent to zero with regulator $e \in E$. Then, for every natural $n \in \mathbb{N}$, there exists an index $\alpha(n) \in \mathrm{A}$ such that $\left|u_{\alpha(n)}\right| \leqslant e/n$. Boundedness of the set $\{nu_{\alpha(n)} : n \in \mathbb{N}\}$ and condition (4) allow us to conclude that the set $\{Tnu_{\alpha(n)} : n \in \mathbb{N}\}$ is semibounded, hence, $\inf_{\alpha\in\mathrm{A}} |Tu_\alpha| \leqslant \inf_{n\in\mathbb{N}} \left|Tu_{\alpha(n)}\right| = \inf_{n\in\mathbb{N}} \left|Tnu_{\alpha(n)}\right|/n = 0$ (see Theorem 5.3.1 (d)). $\triangleright$

An operator $T : \mathscr{U} \to \mathscr{V}$ satisfying the conditions listed in items (a), (b), (c), and (d) of the last theorem is called ***bounded***, ***countably bounded***, ***sequentially bounded***, and ***semibounded***, respectively. Obviously, every bounded operator is countably bounded, every countably bounded operator is sequentially bounded, and every sequentially bounded operator is semibounded. We devote a large part of this section to presenting examples which show that the four types of boundedness of operators differ pairwise. Operators arising in each of the examples below act from Banach spaces into K-spaces.

5.3.4. EXAMPLE. There exist a Banach space X, a universally complete K-space F, and an operator $T : X \to F$ that is countably bounded but not bounded.

We call a sequence $(\alpha_1, \alpha_2, \dots)$ of countable ordinals α_n *finitary* if there is an index $n \in \mathbb{N}$ such that $\alpha_n \neq 0$ and $\alpha_m = 0$ for all $m > n$. In this case, the number n is called the ***dimension*** of the sequence α and denoted by $\dim(\alpha)$. Denote the set of all finitary sequences of countable ordinals by A and endow it with the lexicographic order by defining $\alpha < \beta$ if and only if, for some $n \in \mathbb{N}$, we have $\alpha_1 = \beta_1, \ \ldots, \ \alpha_{n-1} = \beta_{n-1}$, and $\alpha_n < \beta_n$. For all $\alpha, \beta \in A$, we denote by $]\alpha, \beta[$ the open interval $\{\gamma \in A : \alpha < \gamma < \beta\}$.

For every sequence $\alpha \in A$, assign

$$\alpha + 1 := \big(\alpha_1, \ldots, \alpha_{\dim(\alpha)-1}, \alpha_{\dim(\alpha)} + 1, 0, 0, \ldots\big).$$

Consider $\alpha, \beta \in A$. We say that α is a *fragment* of β and write $\alpha \sqsubset \beta$ if $\alpha = (\beta_1, \beta_2, \ldots, \beta_{\dim(\alpha)}, 0, 0, \ldots)$.

Lemma 1. *For all $\alpha, \beta \in A$, the following relations are equivalent:*

(1) $]\alpha, \alpha+1[\,\cap\,]\beta, \beta+1[\,\neq \varnothing$;

(2) $]\alpha, \alpha+1[\,\subset\,]\beta, \beta+1[$ or $]\alpha, \alpha+1[\,\supset\,]\beta, \beta+1[$;

(3) $\alpha \sqsubset \beta$ or $\beta \sqsubset \alpha$.

$\lhd$ If $\dim(\alpha) = \dim(\beta)$ then the claim is obvious. For definiteness, assume that $\dim(\alpha) < \dim(\beta)$. Therefore, if $\alpha < (\beta_1, \ldots, \beta_{\dim(\alpha)}, 0, 0, \ldots)$ then $\alpha + 1 < \beta$, and if $\alpha > (\beta_1, \ldots, \beta_{\dim(\alpha)}, 0, 0, \ldots)$ then $\alpha > \beta + 1$. In both cases, the intervals $]\alpha, \alpha+1[$ and $]\beta, \beta+1[$ are disjoint. The lemma is proven. $\rhd$

Endow the set A with the order topology, for which $\big\{]\alpha, \beta[\ : \ \alpha, \beta \in A\big\}$ is a base of open sets. Denote by Q the Stone compactum of the Boolean algebra $\mathrm{Rop}(A)$ of regular open subsets of A. Let $U \mapsto \hat{U}$ be an isomorphism of $\mathrm{Rop}(A)$ onto the Boolean algebra $\mathrm{Clop}(Q)$ of clopen subsets of Q. Observe that $\mathrm{Rop}(A)$ contains all intervals $]\alpha, \beta[$ $(\alpha, \beta \in A)$. For every sequence $\alpha \in A$, assign $Q_\alpha :=]\alpha, \alpha+1[^\wedge \in \mathrm{Clop}(Q)$ and denote the characteristic function of the subset $Q_\alpha \subset Q$ by χ_α. Thus, $\chi_\alpha \in C(Q)$.

Lemma 2. *For every nonempty open set $U \subset A$ and every $n \in \mathbb{N}$, there is a sequence $\alpha \in A$ such that $\dim(\alpha) > n$ and $]\alpha, \alpha+1[\,\subset U$.*

$\lhd$ By the definition of order topology, the set U includes some interval $]\alpha, \beta[$, $\alpha < \beta$. Assign $m := \min\{i \in \mathbb{N} : \alpha_i < \beta_i\}$ and $k := \max\{m, n\}$. The sequence $(\alpha_1, \ldots, \alpha_k, \alpha_{k+1} + 1, 0, 0, \ldots)$ is the desired one. $\rhd$

Lemma 3. *For every $n \in \mathbb{N}$, the relation*

$$\sup\big\{]\alpha, \alpha+1[\ : \ \alpha \in A, \ \dim(\alpha) \geqslant n\big\} \ = \ 1$$

holds in the Boolean algebra $\mathrm{Rop}(A)$.

$\lhd$ The claim is immediate from Lemma 2. $\rhd$

Lemma 4. *In the K-space $C_\infty(Q)$ the sum $f_S := o\text{-}\sum_{\alpha\in S}\dim(\alpha)\chi_\alpha$ exists for every countable subset $S \subset A$.*

$\lhd$ The formula $f(q) := \sum_{\alpha\in S}\dim(\alpha)\chi_\alpha(q)$ defines a function $f : Q \to \overline{\mathbb{R}}$. According to [12: Chapter XIII, Theorem 2.32], in order to prove the lemma, it is sufficient to establish that $f^{-1}(\infty)$ is a meager subset of Q. Taking account of Lemma 1, we conclude the following: if a point $q \in Q$ satisfies $f(q) = \infty$, then there is a chain $\alpha^{(1)} \sqsubset \alpha^{(2)} \sqsubset \cdots \sqsubset \alpha^{(n)} \sqsubset \cdots$ of pairwise different elements in S such that $q \in \bigcap_{n\in\mathbb{N}} Q_{\alpha^{(n)}}$. Thus, $f^{-1}(\infty) \subset \bigcap_{n\in\mathbb{N}}\bigcup_{\alpha\in S(n)} Q_\alpha$, where $S(n) = \{\alpha \in S : \dim(\alpha) \geqslant n\}$. Consequently, the lemma will be proven if we establish that $\operatorname{int}\bigcap_{n\in\mathbb{N}}\operatorname{cl}\bigcup_{\alpha\in S(n)} Q_\alpha = \varnothing$, i.e., $\inf_{n\in\mathbb{N}}\sup_{\alpha\in S(n)} Q_\alpha = 0$ in the Boolean algebra $\operatorname{Clop}(Q)$ or, equivalently, $\inf_{n\in\mathbb{N}}\sup_{\alpha\in S(n)}]\alpha,\alpha+1[= 0$ in the Boolean algebra $\operatorname{Rop}(A)$.

Assume that the last equality does not hold. Then, according to Lemma 2, there exists a sequence $\beta \in A$ such that the interval $]\beta,\beta+1[$ is included in $\sup_{\alpha\in S(n)}]\alpha,\alpha+1[$ for every $n \in \mathbb{N}$ and, in particular, for $n = \dim(\beta)+1$. Denote the set $\{\gamma \in S(\dim(\beta)+1) :]\beta,\beta+1[\cap]\gamma,\gamma+1[\neq \varnothing\}$ by Γ. Obviously, $]\beta,\beta+1[\subset \sup_{\gamma\in\Gamma}]\gamma,\gamma+1[$ and, consequently, for every sequence $\alpha < \beta+1$, there exists an element $\gamma \in \Gamma$ such that $\gamma + 1 \geqslant \alpha$. However, Lemma 1 implies that β is a fragment of every element of Γ; therefore, for all $\gamma \in \Gamma$, we have

$$\begin{aligned}\gamma + 1 &= \big(\beta_1,\dots,\beta_{\dim(\beta)},\gamma_{\dim(\beta)+1},\dots,\gamma_{\dim(\gamma)}+1,0,0,\dots\big)\\ &\leqslant \big(\beta_1,\dots,\beta_{\dim(\beta)},\gamma_{\dim(\beta)+1}+1,0,0,\dots\big)\\ &\leqslant \Big(\beta_1,\dots,\beta_{\dim(\beta)},\sup_{\gamma'\in\Gamma}\big(\gamma'_{\dim(\beta)+1}+1\big),0,0,\dots\Big) < \beta+1,\end{aligned}$$

which easily yields a contradiction. $\rhd$

Let $\mathscr{X}$ be the vector space of all bounded functions $x : A \to \mathbb{R}$ with countable support $\{\alpha \in A : x(\alpha) \neq 0\}$. Obviously, $\mathscr{X}$ is a Banach space with respect to the uniform norm $\|\cdot\|_\infty$ and a K-space with respect to the pointwise order.

Lemma 5. *For every function $x \in \mathscr{X}$, the sum $o\text{-}\sum_{\alpha\in A}\dim(\alpha)x(\alpha)\chi_\alpha$ exists in the K-space $C_\infty(Q)$.*

$\lhd$ Denote by S the support of the function $x \in \mathscr{X}$. Applying Lemma 4, we have $\sum_{\alpha\in A}\dim(\alpha)x^+(\alpha)\chi_\alpha(q) \leqslant \|x\|_\infty f_S(q)$ at every point $q \in Q$, which implies the existence of the sum $o\text{-}\sum_{\alpha\in A}\dim(\alpha)x^{+}(\alpha)\chi_\alpha$ in $C_\infty(Q)$. Similar arguments for the function x^- complete the proof of the lemma. $\triangleright$

We now begin defining the spaces X and F and the operator T. The Banach space X is defined as the closure of the subspace of $\mathscr{X}$ constituted by all functions with finite supports. As the K-space F, we take $C_\infty(Q)$. Finally, the operator $T : X \to F$ is defined by the formula

$$Tx = o\text{-}\sum_{\alpha\in A}\dim(\alpha)x(\alpha)\chi_\alpha,$$

in which the existence of the o-sum is guaranteed by Lemma 5.

The operator T is countably bounded. Indeed, if the norms of all the elements of a countable subset $X_0 \subset X$ are bounded from above by a number λ and S is the union of the supports of all the functions in X_0, then, in view of Lemma 4, we have $|Tx| \leqslant \lambda f_S$ for all $x \in X_0$. Thus, the operator T satisfies condition (b)(6) of Theorem 5.3.3, i.e., it is countably bounded.

We show that the operator T is not bounded. For every sequence $\alpha \in A$, denote the characteristic function of the singleton $\{\alpha\} \subset A$ by x_α. If the set $\{Tx : x \in X, \|x\|_\infty \leqslant 1\}$ had an upper bound in the K-space F, then, according to Lemma 3, for every $n \in \mathbb{N}$ we should have

$$\begin{aligned}\sup\{Tx : x \in X,\ \|x\|_\infty \leqslant 1\} &\geqslant \sup\{Tx_\alpha : \alpha \in A,\ \dim(\alpha) \geqslant n\}\\ &\geqslant \sup\{n\chi_\alpha : \alpha \in A,\ \dim(\alpha) \geqslant n\} = n1_F,\end{aligned}$$

where 1_F is the identical unity. Thus, the operator T does not satisfy condition (a)(5) of Theorem 5.3.3, i.e., it is not bounded.

5.3.5. EXAMPLE. There exist a Banach space X, a K-space F, and an operator $T : X \to F$ that is sequentially bounded but not countably bounded.

$\triangleleft$ Endowing the vector space c_0 of vanishing numeric sequences with the uniform norm $\|\cdot\|$, we obtain a Banach space to be denoted by X. On the other hand, endowing the space c_0 with pointwise order, we obtain a K-space which we denote by F. Consider the identity mapping $T : c_0 \to c_0$ as an operator from X into F. For every natural $n \in \mathbb{N}$, denote by e_n the characteristic function of the subset $\{n\} \subset \mathbb{N}$. The operator T is not countably bounded, since it takes the bounded countable subset $\{e_n : n \in \mathbb{N}\}$ of the Banach space X into an unbounded subset of the K-space F (see (b)(6) of Theorem 5.3.3).

We will show that the operator T is sequentially bounded by using criterion (c)(3) of Theorem 5.3.3. Consider an arbitrary sequence $(x_n)_{n\in\mathbb{N}}$ in X convergent in norm to zero and define a sequence x by the formula $x(m) = \sup_{n\in\mathbb{N}} |x_n(m)|$ $(m \in \mathbb{N})$. It is sufficient to show that $x(m) \to 0$ as $m \to \infty$. Fix an arbitrary number $\varepsilon > 0$. Let a number $\bar{n} \in \mathbb{N}$ be such that $\|x_n\| \leqslant \varepsilon$ for all $n > \bar{n}$ and let $\bar{m} \in \mathbb{N}$ be such that $\big(|x_1| \vee |x_2| \vee \dots \vee |x_{\bar{n}}|\big)(m) \leqslant \varepsilon$ for all $m > \bar{m}$. Then $x(m) \leqslant \varepsilon$ for all $m > \bar{m}$. $\rhd$

5.3.6. EXAMPLE. There exist a Banach space X, a universally complete K-space F, and an operator $T : X \to F$ that is semibounded but not sequentially bounded.

$\lhd$ Denote by Δ the set of all finite sequences of unities and zeroes: $\Delta := \big\{\big(\delta(1), \dots, \delta(n)\big) : n \in \mathbb{N},\ \delta(i) \in \{0, 1\}\big\}$. Enumerate the elements of the set Δ, listing first all the sequences of length 1, then of length 2, etc.:

$$\delta_1 := (0), \quad \delta_2 := (1);$$

$$\delta_3 := (0,0), \quad \delta_4 := (0,1), \quad \delta_5 := (1,0), \ \delta_6 := (1,1);$$

$$\dots$$

$$\delta_{2^n-1} := (0,0,\dots,0), \quad \delta_{2^n} := (0,0,\dots,1), \quad \dots, \quad \delta_{2^{n+1}-2} := (1,1,\dots,1);$$

$$\dots$$

For every element $\delta = \big(\delta(1), \dots, \delta(n)\big) \in \Delta$, denote by I_δ the following interval of the real line:

$$\Big[\tfrac{\delta(1)}{2^1} + \tfrac{\delta(2)}{2^2} + \dots + \tfrac{\delta(n)}{2^n},\ \tfrac{\delta(1)}{2^1} + \tfrac{\delta(2)}{2^2} + \dots + \tfrac{\delta(n)}{2^n} + \tfrac{1}{2^n}\Big].$$

By way of explication, we observe that

$$I_{\delta_1} = I_1^1, \quad I_{\delta_2} = I_2^1, \quad I_{\delta_3} = I_1^2, \quad I_{\delta_4} = I_2^2, \quad I_{\delta_5} = I_3^2, \quad I_{\delta_6} = I_4^2, \quad \dots,$$

where I_m^n are the intervals considered in 5.3.2.

Denote by X the Banach space $\ell^1(\Delta)$ of summable functions $x : \Delta \to \mathbb{R}$ with the norm $\|x\| = \sum_{\delta\in\Delta} |x(\delta)|$ and define F to be the K-space $M\big([0,1]\big)$ of cosets of real-valued Lebesgue-measurable functions on the interval $[0,1]$. For every element $\delta \in \Delta$, denote by f_δ the characteristic function of the interval I_δ and by $\mathbf{f}_\delta$ the coset in $M([0,1])$ that contains the function f_δ. Define the operator $T : X \to F$ by the formula $Tx = o\text{-}\sum_{\delta\in\Delta} 2^{\dim\delta} x(\delta)\,\mathbf{f}_\delta$, where $\dim\delta$ is the length of a sequence δ. The last o-sum exists, since the corresponding pointwise sum $\sum_{\delta\in\Delta} 2^{\dim\delta} x(\delta) f_\delta$ is, obviously, measurable and the integral of its modulus is equal to $\sum_{\delta\in\Delta} 2^{\dim\delta} |x(\delta)|\, \mu(I_\delta) = \sum_{\delta\in\Delta} |x(\delta)| = \|x\|$ and, hence, is finite. Thus, $\int |Tx| = \|x\|$, which immediately implies semiboundedness of the operator T.

We show that the operator T constructed is not sequentially bounded. For every element $\delta \in \Delta$, denote by e_δ the characteristic function of the singleton $\{\delta\} \subset \Delta$. Then the sequence $\left(2^{-\dim\delta_n} e_{\delta_n}\right)_{n\in\mathbb{N}}$ converges in norm to zero; however, its image $\left(\mathbf{f}_{\delta_n}\right)_{n\in\mathbb{N}}$ with respect to the operator T does not r-converge to zero. ▷

5.3.7. Let $\mathscr{U}$ and $\mathscr{V}$ be LNSs over respective vector lattices E and F. A positive operator $S\colon E \to F$ is said to be a *dominant* of an operator $T\colon \mathscr{U} \to \mathscr{V}$ if $|Tu| \leqslant S|u|$ for all $u \in \mathscr{U}$. An operator possessing a dominant is called *dominated.* The totality of all dominated operators from $\mathscr{U}$ into $\mathscr{V}$ is denoted by $M(\mathscr{U},\mathscr{V})$. Obviously, $M(\mathscr{U},\mathscr{V})$ is a vector subspace of the space of all linear operators from $\mathscr{U}$ into $\mathscr{V}$.

Proposition. *Let E and F be vector lattices and let $\mathscr{U}$ and $\mathscr{V}$ be* LNSs.

(1) *An operator $T\colon E \to F$ is regular if and only if it is dominated.*

(2) *If an operator $T\colon \mathscr{U} \to \mathscr{V}$ is dominated then it is bounded.*

(3) *If F is a K-space and an operator $T\colon E \to F$ is bounded then it is dominated (= regular).*

◁ Assertions (1) and (2) are obvious. A proof of (3) is presented in [12: VII.1.27; 10: Theorem VIII.2.2]. ▷

REMARK. A bounded operator need not be dominated. Indeed, by endowing the vector space ℓ^∞ of bounded numeric sequences with coordinatewise order, we obtain a K-space (hence, a BKS) which is denoted by $\mathscr{U}$. On the other hand, by endowing ℓ^∞ with the uniform norm, we obtain a Banach space (hence, a BKS) which is denoted by $\mathscr{V}$. Then the identity mapping of ℓ^∞ onto itself, as an operator from $\mathscr{U}$ into $\mathscr{V}$, is bounded but not dominated.

5.3.8. Theorem [16]. *Let $\mathscr{U}$ be an LNS over E and let $\mathscr{V}$ be an LNS over F.*

(1) *Every dominated operator $T\colon \mathscr{U} \to \mathscr{V}$ possesses a least dominant (with respect to the order of the vector lattice $M(E,F)$ of regular operators), denoted by $|T|$ and called the exact dominant of T.*

(2) *If $\mathscr{V}$ is a BKS then the mapping $|\cdot|\colon T \mapsto |T|$ is a decomposable $M(E,F)$-valued norm on $M(\mathscr{U},\mathscr{V})$ under which $M(\mathscr{U},\mathscr{V})$ is a BKS.*

5.3.9. Theorem [16]. *Consider a BKS $\mathscr{U}$ over E, an LNS $\mathscr{V}$ over F, and a linear operator $T\colon \mathscr{U} \to \mathscr{V}$. For each positive element $e \in E$, assign*

$$
\begin{aligned}
T_{\leqslant}(e) &:= \{|Tu_1| + \dots + |Tu_n| \,:\, u_i \in \mathscr{U},\ |u_1| + \dots + |u_n| \leqslant e\},\\
T_{=}(e) &:= \{|Tu_1| + \dots + |Tu_n| \,:\, u_i \in \mathscr{U},\ |u_1| + \dots + |u_n| = e\},\\
T_{\perp}(e) &:= \{|Tu_1| + \dots + |Tu_n| \,:\, u_i \in \mathscr{U} \text{ are pairwise disjoint},\\
&\qquad\qquad\qquad\qquad\qquad\qquad |u_1| + \dots + |u_n| \leqslant e\}.
\end{aligned}
$$

The operator T is dominated if and only if, for every positive element $e \in E$, one (hence, each) of the sets $T_{\leqslant}(e)$, $T_{=}(e)$, or $T_{\perp}(e)$ is bounded. In this case, $|T|e = \sup T_{\leqslant}(e) = \sup T_{=}(e) = \sup T_{\perp}(e)$ for all $e \geqslant 0$.

5.4. The Shadow of an Operator

Our main tool for studying disjointness preserving operators is the so-called shadow, a ring homomorphism in Boolean algebras which is generated by the action of the operator on bands. Many properties of an operator are expressible in terms of its shadow. In particular, this is true of certain questions of continuity.

5.4.1. Let $\mathscr{U}$ and $\mathscr{V}$ be LNSs. An operator $T\colon \mathscr{U} \to \mathscr{V}$ is said to be *disjointness preserving* whenever $u_1 \perp u_2$ implies $Tu_1 \perp Tu_2$ for all $u_1, u_2 \in \mathscr{U}$. It is not difficult to become convinced that every disjointness preserving positive operator in K-spaces is a lattice homomorphism. The following assertion shows that all disjointness preserving operators, not only positive, are closely related to lattice homomorphisms.

Theorem. *Let E be a vector lattice, let F be a K-space, and let $T\colon E \to F$ be a regular disjointness preserving operator. Assign $\rho := \langle T^+[E^+]\rangle$, where $E^+ = \{e \in E : e \geqslant 0\}$. Then the operators $\rho \circ T$ and $-\rho^\perp \circ T$ are lattice homomorphisms. In particular, $T = (\rho - \rho^\perp)|T|$.*

◁ The claim follows directly from [21: Theorem 3.3]. ▷

In the sequel, we repeatedly use the last theorem in order to reduce consideration of an arbitrary regular disjointness preserving operator to the case of a positive operator.

5.4.2. The *shadow* of an operator $T\colon \mathscr{U} \to \mathscr{V}$ is the mapping $h\colon \Pr(\mathscr{U}) \to \Pr(\mathscr{V})$ defined by the formula $h(\pi) = \sup_{u\in\mathscr{U}}\langle T\pi u\rangle$. In other words, $h(\pi) = \big\langle T[\pi\mathscr{U}]\big\rangle$.

Proposition. *A linear operator in LNSs is disjointness preserving if and only if its shadow is a ring homomorphism.*

◁ Only necessity requires proving. Assume that a linear operator $T: \mathscr{U} \to \mathscr{V}$ in LNSs $\mathscr{U}$ and $\mathscr{V}$ is disjointness preserving. Without loss of generality, we may assume that $(\operatorname{im} T)^{\perp\perp} = \mathscr{V}$. Prove that the shadow $h\colon \Pr(\mathscr{U}) \to \Pr(\mathscr{V})$ of T is a Boolean homomorphism. To this end, use Proposition 5.1.2. Let (π_1, π_2, π_3) be a partition of unity in the algebra $\Pr(\mathscr{U})$. Then

$$h(\pi_1) \wedge h(\pi_2) = \sup_{u_1\in\mathscr{U}} \langle T\pi_1 u_1\rangle \wedge \sup_{u_2\in\mathscr{U}} \langle T\pi_2 u_2\rangle = \sup_{u_1,u_2\in\mathscr{U}} \langle T\pi_1 u_1\rangle \wedge \langle T\pi_2 u_2\rangle = 0,$$

i.e., $h(\pi_1) \perp h(\pi_2)$. The relations $h(\pi_1) \perp h(\pi_3)$ and $h(\pi_2) \perp h(\pi_3)$ can be established similarly. Moreover,

$$\begin{aligned} h(\pi_1) \vee h(\pi_2) \vee h(\pi_3) &= \sup_{u_1,u_2,u_3\in\mathscr{U}} \langle T\pi_1 u_1\rangle \vee \langle T\pi_2 u_2\rangle \vee \langle T\pi_3 u_3\rangle \\ &= \sup_{u_1,u_2,u_3\in\mathscr{U}} \big\langle T(\pi_1 u_1 + \pi_2 u_2 + \pi_3 u_3)\big\rangle = \sup_{u\in\mathscr{U}} \langle Tu\rangle = 1, \end{aligned}$$

whence it follows that $\big(h(\pi_1), h(\pi_2), h(\pi_3)\big)$ is a partition of unity in the algebra $\Pr(\mathscr{V})$. ▷

5.4.3. ***Proposition.*** *Consider LNSs $\mathscr{U}$ and $\mathscr{V}$, a linear operator $T: \mathscr{U} \to \mathscr{V}$ and a ring homomorphism $h\colon \Pr(\mathscr{U}) \to \Pr(\mathscr{V})$. The following assertions are equivalent:*

(1) *h dominates the shadow of T* (see 5.1.4);

(2) *$\langle Tu\rangle \leqslant h\langle u\rangle$ for all $u \in \mathscr{U}$;*

(3) *$T\pi u = h(\pi)Tu$ for all $u \in \mathscr{U}$ and $\pi \in \Pr(\mathscr{U})$.*

If, in addition, $h(1) = \langle \operatorname{im} T\rangle$ then each of conditions (1)–(3) is equivalent to coincidence of the shadow of T with h.

◁ The implications (3)⇒(1)⇒(2) are obvious. Assume (2) to be satisfied and prove (3). Fix arbitrary elements $u \in \mathscr{U}$ and $\pi \in \Pr(\mathscr{U})$. From (2) it follows that $T\pi u$ and $T\pi^{\perp}u$ are disjoint. Consequently, there exist a projection $\rho \in \Pr(\mathscr{V})$ such that $T\pi u = \rho Tu$ and $T\pi^{\perp}u = \rho^{\perp}Tu$. In order to ensure the equality $\rho Tu = h(\pi)Tu$, it is sufficient to show that $\rho\langle Tu\rangle = h(\pi)\langle Tu\rangle$. The relations $\rho\langle Tu\rangle = \langle T\pi u\rangle \leqslant h(\pi)$ imply the inequality $\rho\langle Tu\rangle \leqslant h(\pi)\langle Tu\rangle$. One can establish similarly that $\rho^{\perp}\langle Tu\rangle \leqslant h(\pi^{\perp})\langle Tu\rangle$. The two last inequalities directly imply the equality $\rho\langle Tu\rangle = h(\pi)\langle Tu\rangle$.

According to Proposition 5.1.4, condition (1) and the equality $h(1) = \langle \operatorname{im} T\rangle$ imply that the shadow of T coincides with h. ▷

5.4.4. Proposition. *Let T be a dominated operator acting from a BKS into an LNS. Then the shadows of T and $|T|$ coincide.*

◁ Let an operator T acts from a BKS $\mathscr{U}$ over E into an LNS $\mathscr{V}$ over F. Denote the shadow of T by h_T and the shadow of $|T|$ by $h_{|T|}$. Of course, coincidence of the functions $h_T\colon \Pr(\mathscr{U}) \to \Pr(\mathscr{V})$ and $h_{|T|}\colon \Pr(E) \to \Pr(F)$ is understood with the identifications $\Pr(\mathscr{U}) = \Pr(E)$ and $\Pr(\mathscr{V}) = \Pr(F)$ taken into account (see 1.6.3). The inequality $h_T(\pi) \leqslant h_{|T|}(\pi)$ $\big(\pi \in \Pr(E)\big)$ is obvious. To prove the reverse inequality, it is sufficient to observe, that the conditions

$$e \in E, \quad \pi \in \Pr(E), \quad u_1, \dots, u_n \in \mathscr{U}, \quad |u_1| + \dots + |u_n| \leqslant \pi e$$

imply

$$\langle |Tu_1| + \dots + |Tu_n|\rangle = \langle |T\pi u_1| + \dots + |T\pi u_n|\rangle = \langle T\pi u_1\rangle \vee \dots \vee \langle T\pi u_n\rangle \leqslant h_T(\pi),$$

and to use the formula $|T|\pi e = \sup T_{\leqslant}(\pi e)$ (see 5.3.9). ▷

Corollary. *A dominated operator T from a BKS into an LNS is disjointness preserving if and only if its exact dominant $|T|$ is disjointness preserving.*

5.4.5. Let $\mathscr{U}$ and $\mathscr{V}$ be LNSs and let $h\colon \Pr(\mathscr{U}) \to \Pr(\mathscr{V})$ be a ring homomorphism. Following the general rule (see 5.1.1), we say that the mapping $T\colon \mathscr{U} \to \mathscr{V}$ is *h-o-continuous* whenever $h\text{-}\lim_{\alpha\in\mathrm{A}} u_\alpha = u$ (see 5.2.12) implies $o\text{-}\lim_{\alpha\in\mathrm{A}} Tu_\alpha = Tu$ for every net $(u_\alpha)_{\alpha\in\mathrm{A}}$ in $\mathscr{U}$ and every $u \in \mathscr{U}$.

Theorem. *Let E and F be K-spaces. Every disjointness preserving operator $T\colon E \to F$ is h-o-continuous, where h is the shadow of T.*

$\lhd$ Since the shadow of $|T|$ coincides with the shadow of T (see Proposition 5.4.4), we may assume that the operator T is positive. To prove h-o-continuity of T, it is sufficient to consider a net $(e_\alpha)_{\alpha \in \mathrm{A}}$ in E, which is h-convergent to zero, and to show that $o\text{-}\lim\limits_{\alpha \in \mathrm{A}} Te_\alpha = 0$. Asymptotic boundedness of the net $(Te_\alpha)_{\alpha \in \mathrm{A}}$ follows from that of $(e_\alpha)_{\alpha \in \mathrm{A}}$ and from boundedness of T. According to Lemma 5.2.10 (2), o-convergence of Te_α to zero will be established if we prove that $o\text{-}\lim\limits_{\alpha \in \mathrm{A}} \langle Te \rangle \langle Te_\alpha > Te/n \rangle = 0$ for all $e \in E$ and $n \in \mathbb{N}$. The latter relation can be obtained as follows:

$$\langle Te \rangle \langle Te_\alpha > Te/n \rangle = \langle Te \rangle \big\langle \big(T(e_\alpha - e/n)\big)^+ \big\rangle = \langle Te \rangle \big\langle T\big((e_\alpha - e/n)^+\big)\big\rangle$$
$$\leqslant h\big(\langle e \rangle\big) h\big(\langle (e_\alpha - e/n)^+ \rangle\big) = h\big(\langle e \rangle \langle e_\alpha > e/n \rangle\big) \xrightarrow{o} 0. \quad \rhd$$

Corollary. *Every disjointness preserving dominated operator from a BKS into an LNS is h-o-continuous, where h is its shadow.*

$\lhd$ The claim follows from Proposition 5.4.4 and the last theorem. $\rhd$

REMARK. It is sometimes useful to take the following fact into account (the fact follows directly from the last assertion): if $\mathcal{U}$ is a BKS, $\mathcal{V}$ is an LNS, and a ring homomorphism $h\colon \mathrm{Pr}(\mathcal{U}) \to \mathrm{Pr}(\mathcal{V})$ dominates the shadow of an operator $T\colon \mathcal{U} \to \mathcal{V}$, then the latter is h-o-continuous.

5.4.6. *Corollary.* *The following properties of a disjointness preserving dominated operator T from a BKS into an LNS are equivalent:*

(1) *T is (sequentially) o-continuous;*

(2) *$|T|$ is (sequentially) o-continuous;*

(3) *the shadow of T is (sequentially) o-continuous.*

Countable and sequential o-continuity of the operator T are equivalent.

$\lhd$ It is sufficient to combine 5.4.4, 5.1.3, 5.2.12, and 5.4.5. $\rhd$

5.4.7. *Corollary.* *Consider a BKS $\mathcal{U}$ and an LNS $\mathcal{V}$ and assume that the shadows of two dominated operators $S, T\colon \mathcal{U} \to \mathcal{V}$ are dominated by the same ring homomorphism $h\colon \mathrm{Pr}(\mathcal{U}) \to \mathrm{Pr}(\mathcal{V})$. If S and T coincide on some h-approximating subset of $\mathcal{U}$* (see 5.2.14) *then they coincide on the entire $\mathcal{U}$.*

$\triangleleft$ The claim follows from Remark 5.4.5 and Propositions 5.2.16 and 5.2.17. $\triangleright$

5.4.8. Proposition. *Let $\mathscr{U}$ be an LNS over E, let $\mathscr{V}$ be a vector subspace of F, let $\mathscr{U}_0$ let $\mathscr{U}$, $T_0 : \mathscr{U}_0 \to \mathscr{V}$ be a linear operator, let $S\colon E \to F$ be a disjointness preserving positive operator, and let $h\colon \mathrm{Pr}(E) \to \mathrm{Pr}(F)$ be the shadow of S. Denote by $h\mathscr{U}_0$ the LNS of all elements of $\mathscr{U}$ that are h-approximated by $\mathscr{U}_0$* (see 5.2.14, 5.2.15). *Assume that $|T_0u_0| \leqslant S|u_0|$ (respectively, $|T_0u_0| = S|u_0|$) for all $u_0 \in \mathscr{U}_0$. Then there exists a unique linear extension $T\colon h\mathscr{U}_0 \to \mathscr{V}$ of the operator T_0 such that $|Tu| \leqslant S|u|$ (respectively, $|Tu| = S|u|$) for all $u \in h\mathscr{U}_0$.*

$\triangleleft$ First, we prove the assertion about extension with the inequality preserved. If $\pi \in \mathrm{Pr}(\mathscr{U})$ and $u_0 \in \mathscr{U}_0$ are such that $\pi u_0 = 0$, then $h(\pi)T_0u_0 = 0$, since $h(\pi)|T_0u_0| \leqslant h(\pi)S|u_0| = S\pi|u_0| = 0$. This fact implies that the following definition of an operator $\overline{T}_0$ is sound:

$$\overline{T}_0\left(\sum_{i=1}^{n} \pi_i u_i\right) := \sum_{i=1}^{n} h(\pi_i)T_0u_i$$

$$\big(\pi_i \in \mathrm{Pr}(\mathscr{U}) \text{ are pairwise disjoint, } u_i \in \mathscr{U}_0\big),$$

which extends T_0 onto $d_{\mathrm{fin}}\mathscr{U}_0$ and satisfies the inequality $\left|\overline{T}_0u\right| \leqslant S|u|$ for all $u \in d_{\mathrm{fin}}\mathscr{U}_0$. In view of Proposition 5.2.15, for every $u \in h\mathscr{U}_0$, there exists a net $(u_\alpha)_{\alpha\in\mathrm{A}}$ in $d_{\mathrm{fin}}\mathscr{U}_0$ that is h-convergent to u. From the inequality $\left|\overline{T}_0u_\alpha - \overline{T}_0u_\beta\right| \leqslant S\left|u_\alpha - u_\beta\right|$ and h-o-continuity of S (see 5.4.5) it follows that the net $(\overline{T}_0u_\alpha)_{\alpha\in\mathrm{A}}$ is o-fundamental. Since the LNS $\mathscr{V}$, is o-complete, it contains an o-limit of the net. Obviously, the limit depends only on u and, therefore, can be denoted by Tu. It is not difficult to become convinced that the operator $T : h\mathscr{U}_0 \to \mathscr{V}$ thus obtained is the desired one. Uniqueness of the extension constructed is ensured by its h-o-continuity inherited from S.

Assume now that $|T_0u_0| = S|u_0|$ for all $u_0 \in \mathscr{U}_0$. In view of what was proven above, there exists an extension $T\colon h\mathscr{U}_0 \to \mathscr{V}$ of the operator T_0 such that $|Tu| \leqslant S|u|$ for all $u \in h\mathscr{U}_0$. For every $u_0 \in \mathscr{U}_0$ and $\pi \in \mathrm{Pr}(\mathscr{U})$, the relations

$$S|u_0| = |Tu_0| = |T\pi u_0| + \left|T\pi^\perp u_0\right| \leqslant S|\pi u_0| + S\left|\pi^\perp u_0\right| = S|u_0|$$

and the inequalities $|T\pi u_0| \leqslant S|\pi u_0|$ and $\left|T\pi^\perp u_0\right| \leqslant S\left|\pi^\perp u_0\right|$ imply $|T\pi u_0| = S|\pi u_0|$. Since $u_0 \in \mathscr{U}_0$ and $\pi \in \mathrm{Pr}(\mathscr{U})$ were chosen arbitrarily, we have $|Tu| = S|u|$ for all $u \in d_{\mathrm{fin}}\mathscr{U}_0$. The equality $|Tu| = S|u|$ for all $u \in h\mathscr{U}_0$ is now deduced from what was proven with the help of Proposition 5.2.16. $\triangleright$

Corollary. *Let $\mathscr{U}$ be an LNS over E, let $\mathscr{V}$ be a BKS over F, let $\mathscr{U}_0$ be an approximating vector subspace of $\mathscr{U}$, let $T_0 \colon \mathscr{U}_0 \to \mathscr{V}$ be a linear operator, and let $S \colon E \to F$ be a disjointness preserving o-continuous positive operator. Assume that $|T_0 u_0| \leqslant S|u_0|$ (respectively, $|T_0 u_0| = S|u_0|$) for all $u_0 \in \mathscr{U}_0$. Then there exists a unique linear extension $T \colon \mathscr{U} \to \mathscr{V}$ of T_0 such that $|Tu| \leqslant S|u|$ (respectively, $|Tu| = S|u|$) for all $u \in \mathscr{U}$.*

5.4.9. If D is a subset of a K-space E then $|D|$ denotes the set $\{|d| : d \in D\}$, and $\operatorname{lin}|D|$ stands for the linear span of $|D|$. The smallest ideal of E that contains D is conventionally denoted by E_D.

Lemma. *Let E be a K-space, let D be a subset of E, let $\mathscr{V}$ and $\mathscr{W}$ be arbitrary LNSs over the same K-space F, and let $S \colon E \to \mathscr{V}$ and $T \colon E \to \mathscr{W}$ be dominated operators. Assume that the shadows of S and T are dominated by the same ring homomorphism $h \colon \mathrm{Pr}(E) \to \mathrm{Pr}(F)$ and denote the h-closure of the ideal E_D by hE_D.*

(1) If $\mathscr{V} = \mathscr{W}$ and the operators S and T coincide on D, then they coincide on hE_D.

(2) If $|Se| = |Te|$ for all $e \in \operatorname{lin}|D|$ then $|Se| = |Te|$ for all $e \in hE_D$.

$\lhd$ We only prove assertion (1), since (2) can be proven similarly and even easier. Assume that the operators S and T meet all the hypotheses of the lemma and coincide on D. We will prove coincidence of S and T on hE_D in several steps.

(a) Suppose that $e \in |D|$, i.e., $e = |d|$ for some $d \in D$. Then

$$
\begin{aligned}
Se &= S\langle d^+\rangle d + S\langle d^-\rangle d = h(\langle d^+\rangle)Sd + h(\langle d^-\rangle)Sd \\
&= h(\langle d^+\rangle)Td + h(\langle d^-\rangle)Td = T\langle d^+\rangle d + T\langle d^-\rangle d = Te.
\end{aligned}
$$

(b) From (a) it follows that the operators S and T coincide on the set $\operatorname{lin}|D|$.

(c) Let e be a d-step element of E with $d \in \operatorname{lin}|D|$, i.e., $e = \sum_{i=1}^{n} \pi_i \lambda_i d$ for some numbers λ_i and pairwise disjoint projections $\pi_i \in \mathrm{Pr}(E)$. Then, in view of (b), we have

$$
Se = \sum_{i=1}^{n} S(\pi_i \lambda_i d) = \sum_{i=1}^{n} \lambda_i h(\pi_i) Sd = \sum_{i=1}^{n} \lambda_i h(\pi_i) Td = \sum_{i=1}^{n} T(\pi_i \lambda_i d) = Te.
$$

(d) Suppose now that $e \in E_D$. Then $|e| \leqslant d$ for some $d \in \mathrm{lin}|D|$. In view of 5.1.6, there exists a sequence $(e_n)_{n\in\mathbb{N}}$ of d-step elements of E that is r-convergent to e. According to (c), the operators S and T coincide on the elements e_n. Therefore, using r-continuity of S and T, we arrive at the equality $Se = Te$.

(e) Finally, if e is an arbitrary element of hE_D then the equality $Se = Te$ follows from (d) and h-o-continuity of S and T. $\triangleright$

Corollary. *Let $\mathcal{U}$ be a BKS over E, let D be a set of positive elements in E, let $\mathcal{V}$ and $\mathcal{W}$ be arbitrary LNSs over the same K-space F, and let $S\colon \mathcal{U} \to \mathcal{V}$ and $T\colon \mathcal{U} \to \mathcal{W}$ be dominated operators. Assume that the shadows of S and T are dominated by the same ring homomorphism $h\colon \mathrm{Pr}(E) \to \mathrm{Pr}(F)$ and denote by hE_D the h-closure of the ideal E_D.*

(1) If $\mathcal{V} = \mathcal{W}$ and the operators S and T coincide on the set $\{u \in \mathcal{U} : |u| \in D\}$ then they coincide on the set $\{u \in \mathcal{U} : |u| \in hE_D\}$.

(2) If $|Su| = |Tu|$ for all $u \in \mathcal{U}$ with norm $|u| \in \mathrm{lin}D$ then $|Su| = |Tu|$ for all $u \in \mathcal{U}$ with norm $|u| \in hE_D$.

$\triangleleft$ Prove assertion (1) (assertion (2) can be proven similarly). Assume that the operators S and T meet all the hypotheses of the corollary and coincide on the set $\{u \in \mathcal{U} : |u| \in D\}$. Consider an arbitrary element $u \in \mathcal{U}$ with norm $|u| \in hE_D$ and establish the equality $Su = Tu$.

Fix an order unit 1 in the universal completion $\overline{E}$ of the K-space E, introduce the corresponding multiplication in $\overline{E}$ and endow the universal completion $\overline{\mathcal{U}}$ of $\mathcal{U}$ the structure of a module over $\overline{E}$ (see Corollary 5.1.12). Let $\bar{u}$ be an element of $\overline{\mathcal{U}}$ such that $|\bar{u}| = 1$ and $u = |u|\bar{u}$. Consider operators $S_u, T_u\colon E \to \mathcal{V}$ acting by the rules $S_u e = S(e\bar{u})$ and $T_u e = T(e\bar{u})$. It is clear that the shadows of S_u and T_u are dominated by the homomorphism h and the operators themselves coincide on D. Therefore, according to assertion (1) of the last lemma, the operators S_u and T_u coincide on hE_D. In particular, $Su = S_u|u| = T_u|u| = Tu$. $\triangleright$

5.4.10. As is seen from the following theorem, all the four types of boundedness introduced in 5.3.3 coincide for each disjointness preserving operator defined on a vector lattice.

Theorem. *Let E be a vector lattice and let $\mathcal{V}$ be an LNS. The following properties of a disjointness preserving operator $T\colon E \to \mathcal{V}$ are equivalent:*

(1) *T is bounded;*

(2) *T is countably bounded;*

(3) *T is sequentially bounded;*

(4) *T is semibounded;*

(5) *if $e_1, e_2 \in E$ and $|e_1| \leqslant |e_2|$ then $|Te_1| \leqslant |Te_2|$.*

$\lhd$ The implications (5)$\Rightarrow$(1)$\Rightarrow$(2)$\Rightarrow$(3)$\Rightarrow$(4) are obvious. The proof of Theorem 2.1 in [37] that establishes the implication (4)$\Rightarrow$(5) is presented for the case $\mathscr{V} = E$; however, it remains valid for an operator with values in an arbitrary LNS. $\rhd$

The proof of the implication (4)$\Rightarrow$(5) becomes particularly simple and clear in the case when E possesses the principal projection property (for instance, when E is a K_σ-space). Indeed, assume that an operator T meets condition (4), fix arbitrary elements $e_1, e_2 \in E$ satisfying the inequality $|e_1| \leqslant |e_2|$, and denote by S the set $\left\{\sum_{i=1}^n \pi_i\lambda_i|e_2| : \pi_i \in \mathrm{Pr}(E),\ |\lambda_i| \leqslant 1\right\}$. It is not difficult to become convinced that $|Ts| \leqslant |Te_2|$ for all $s \in S$. Moreover, in view of 5.1.6, there exists a sequence $(s_n)_{n\in\mathbb{N}}$ of elements in S that is r-convergent to e_1 with regulator $|e_2|$. Condition (4) together with the relations $|Te_1| \leqslant |Te_1 - Ts_n| + |Te_2|$ $(n \in \mathbb{N})$ now yields the desired inequality $|Te_1| \leqslant |Te_2|$.

5.4.11. The analog of Theorem 5.4.10 for operators defined on LNSs is not true. Moreover, all the four types of boundedness are pairwise different for this class of operators. Indeed, every normed space is an LNS over $\mathbb{R}$ and every linear operator from a normed space into an arbitrary LNS is disjointness preserving. Consequently, operators considered in Examples 5.3.4–5.3.6 act from BKSs into BKSs and are disjointness preserving.

5.4.12. *Lemma.* *Let $\mathscr{U}$ be a BKS over E, let $\mathscr{V}$ be an LNS, let $T\colon \mathscr{U} \to V$ be a disjointness preserving semibounded operator, and let e be a positive element of E. For each $u \in \mathscr{U}$ satisfying the inequality $|u| \leqslant e$, there is an element $\bar{u} \in \mathscr{U}$ such that $|\bar{u}| = e$ and $|Tu| \leqslant |T\bar{u}|$.*

$\lhd$ Suppose that $|u| \leqslant e$. Due to the equality $\{|u| : u \in \mathscr{U}\} = \{e \in E : e \geqslant 0\}$, we do not restrict generality by assuming that $\langle u\rangle = \langle e\rangle$. Obviously, the product $(e'/|u|)u$ is defined in $\mathscr{U}$ for all $e' \in E$ (see 5.1.12). Define an operator $S\colon E \to \mathscr{V}$ by the formula $S(e') = T\big((e'/|u|)u\big)$ and assign $\bar{u} := (e/|u|)u$. It is easy to see that the operator S is disjointness preserving and semibounded. According to Theorem 5.4.10, the operator S meets condition 5.4.10 (5). This allows us to conclude that $|Tu| = S|u| \leqslant Se = |T\bar{u}|$. It remains to observe that $|\bar{u}| = e$. $\rhd$

Proposition. *Let $\mathscr{U}$ be a BKS over E and let $\mathscr{V}$ be an LNS. A disjointness preserving operator $T\colon \mathscr{U} \to V$ is dominated if and only if it is bounded. Furthermore, $|T|e = \sup\{|Tu| : u \in \mathscr{U},\ |u| \leqslant e\} = \sup\{|Tu| : u \in \mathscr{U},\ |u| = e\}$ for all positive $e \in E$.*

$\lhd$ For an arbitrary positive element $e \in E$, the equality $|T|e = \sup\{|Tu| : u \in \mathscr{U},\ |u| \leqslant e\}$ is easily deduced from the criterion 5.3.9 involving the set $T_{\perp}(e)$. It remains to employ the lemma proven above. $\rhd$

The last result does not provide any new information about operators in vector lattices, since dominatedness and boundedness are always equivalent for operators with values in a K-space (see Proposition 5.3.7 (3)). However, an analog of the last proposition is true in the case of vector lattices:

Theorem [38]. *Let E and F be arbitrary vector lattices. A disjointness preserving operator $T\colon E \to F$ is regular (= dominated) if and only if it is bounded.*

5.4.13. As was noted in 5.4.11, countable boundedness is not sufficient for boundedness of a disjointness preserving operator. It is interesting to clarify which (easily verified) additional assumptions yield boundedness of operators bounded in a weaker sense. Leaving this question open, we only formulate one corollary to Lemma 5.4.12 which is a small step in the indicated direction.

Proposition. *Let $\mathscr{U}$ be a BKS over E and let $\mathscr{V}$ be an LNS over F. A disjointness preserving operator $T\colon \mathscr{U} \to \mathscr{V}$ is bounded if and only if it is semibounded and, for every positive element $e \in E$, the set $\{|Tu| : u \in \mathscr{U},\ |u| = e\}$ is order-bounded in F.*

Note that any semibounded disjointness preserving operator defined on a vector lattice obviously meets the hypotheses of the last proposition. This allows us to consider Proposition 5.4.13 as a generalization of Theorem 5.4.10.

5.4.14. One of the main results concerning disjointness preserving operators provides their representation as sums of certain special operators taking pairwise disjoint values (see Section 5.7). Here we pay attention to such sums.

Lemma. *Let $\mathscr{U}$ and $\mathscr{V}$ be LNSs and let $S, T\colon \mathscr{U} \to \mathscr{V}$ be linear operators. The following assertions are equivalent:*

(1) *$Su \perp Tu$ for all $u \in \mathscr{U}$;*

(2) *$Su_1 \perp Tu_2$ for all $u_1, u_2 \in \mathscr{U}$, i.e.,* $\operatorname{im} S \perp \operatorname{im} T$.

$\lhd$ Only the implication (1)$\Rightarrow$(2) requires proving. Let u_1 and u_2 be arbitrary elements of $\mathscr{U}$. The relations $Su_1 \perp Tu_1$ and $Su_2 \perp Tu_2$ imply:

$$|Su_1| \wedge |Tu_2| = |Su_1| \wedge |Tu_1 + Tu_2| \leqslant \big|T(u_1 + u_2)\big|,$$
$$|Su_1| \wedge |Tu_2| = |Su_1 + Su_2| \wedge |Tu_2| \leqslant \big|S(u_1 + u_2)\big|.$$

It remains to observe that $S(u_1 + u_2) \perp T(u_1 + u_2)$. $\rhd$

Operators S and T that meet each of the equivalent conditions (1) or (2) are called ***strongly disjoint***. Let $\mathscr{U}$ and $\mathscr{V}$ be LNSs and let $(T_\xi)_{\xi \in \Xi}$ be a family of linear operators from $\mathscr{U}$ into $\mathscr{V}$. We say that an operator $T\colon \mathscr{U} \to \mathscr{V}$ is decomposable into the ***strongly disjoint sum*** of operators T_ξ $\big($and write $T = \bigoplus_{\xi \in \Xi} T_\xi\big)$, whenever the operators T_ξ are strongly disjoint and, for every $u \in \mathscr{U}$, the relation $Tu = o\text{-}\sum_{\xi \in \Xi} T_\xi u$ holds.

Assume that $T = \bigoplus_{\xi \in \Xi} T_\xi$ and assign $\rho_\xi := \langle \operatorname{im} T_\xi \rangle$ for each $\xi \in \Xi$. According to the lemma, the projections ρ_ξ are pairwise disjoint; therefore, for all $\xi \in \Xi$ the equality $T_\xi = \rho_\xi \circ T$ holds. In particular, this implies that the strongly disjoint sum $\bigoplus_{\xi \in \Xi} T_\xi$ is disjointness preserving if and only if so is each summand T_ξ.

5.5. Orthomorphisms

This section is devoted to one of the simplest classes of disjointness preserving operators, the class of band preserving operators. Simplicity of such operators notwithstanding, the question about their regularity (= order boundedness) is far from trivial. It is known that all band preserving operators in a universally complete K-space are regular if and only if the K-space is locally one-dimensional. However, it seems to have been unknown so far whether there exist nondiscrete locally one-dimensional K-spaces. In the present section we give a positive answer to the question. As an auxiliary result, we establish that a K-space is locally one-dimensional if and only if its base is σ-distributive.

Throughout the section, G is a universally complete K-space with a fixed order unity 1_G, Q is the Stone compact space of the Boolean algebra $\Pr(G)$ (recall that this algebra is the base of G), E and F are order-dense ideals of G, and $\mathscr{U}$ and $\mathscr{V}$ are LNSs over E and F, respectively. We introduce a multiplication in the K-space G which makes it a commutative ordered algebra with unity 1_G (see 5.1.12). Recall also that we identify the Boolean algebras $\mathrm{Pr}(G)$, $\mathrm{Pr}(E)$, $\mathrm{Pr}(F)$, $\mathrm{Pr}(\mathscr{U})$ and $\mathrm{Pr}(\mathscr{V})$. A subset of a Boolean algebra with supremum unity is called a *cover of the algebra*. A cover constituted by pairwise disjoint elements (i.e. a partition of unity) is referred to as a *partition of the algebra* for brevity.

5.5.1. An element $g \in G^+$ is called *locally constant* with respect to an $f \in G^+$, if $g = \bigvee_{\xi\in\Xi} \lambda_\xi \pi_\xi f$ for some numeric family $(\lambda_\xi)_{\xi\in\Xi}$ and a family $(\pi_\xi)_{\xi\in\Xi}$ of pairwise disjoint band projections. A universally complete K-space G is called *locally one-dimensional* if it satisfies one of the following equivalent conditions (see [37: Theorem 3.1]):

(1) all elements of G^+ are locally constant with respect to some order unity of G;

(2) all elements of G^+ are locally constant with respect to every order unity of G;

(3) for every function $g \in C_\infty(Q)$, there exists a partition $(U_\xi)_{\xi\in\Xi}$ of the algebra $\mathrm{Clop}(Q)$ such that the function g is constant on each of the sets U_ξ.

5.5.2. A linear operator $T\colon G \to G$ is said to be *band preserving* if, for all $f, g \in G$, from $f \perp g$ it follows that $Tf \perp g$.

The following statement combines a result of Yu. A. Abramovich, A. I. Veksler, and A. V. Koldunov ([3: Theorem 2.1]) and that of P. T. N. McPolin and A. W. Wickstead ([37: Theorem 3.2]).

Theorem. *Let G be a universally complete K-space. Every band preserving operator $T\colon G \to G$ is regular if and only if G is locally one-dimensional.*

In order to avoid misunderstanding, while reading the articles [3] and [37], one should be aware of the following two circumstances. First, despite of the fact that an arbitrary nondiscrete K-space is mentioned in the statement of Theorem 2.1 of [3], the proof of the theorem is given only for locally one-dimensional K-spaces. Second, the example of a nondiscrete locally one-dimensional K-space presented in [37] contains an error, which was recently reported by A. W. Wickstead in the article [22]. Thus, the question whether every locally one-dimensional K-space must be discrete (i.e. have an atomic base) seems to have been open so far.

5.5.3. The notion of locally one-dimensional K-space admits the following Boolean-valued interpretation. (For an explanation of the main notions of Boolean-valued analysis, we refer the reader to Chapter 1 of the present book and to the second part of the monograph [15].) Let B be a complete Boolean algebra, let $\mathscr{R}$ be the field of reals inside $\mathbb{V}^{(B)}$, and let $\mathbb{R}^\wedge$ be the canonical embedding of $\mathbb{R}$ into $\mathbb{V}^{(B)}$.

Theorem. *The equality $\mathbb{R}^\wedge = \mathscr{R}$ holds if and only if the descent of $\mathscr{R}$ is a locally one-dimensional K-space.*

$\lhd$ Knowing the general structure of the descents of objects that have the form $X^\wedge$, it is easy to deduce the stated assertion from E. I. Gordon's theorem 1.3.2 (see also [15: 3.1.1 (1), 5.2.1, and 5.2.2]). $\rhd$

From private conversations with colleagues, the author of the present chapter is aware that, among the specialists in the domain of Boolean-valued analysis, the superstition is rather popular of atomicity of all Boolean algebras B that provide the equality $\mathbb{R}^\wedge = \mathscr{R}$ in $\mathbb{V}^{(B)}$. Thus, the question about the connection between discrete and locally one-dimensional K-spaces has a rather wide domain of applications, at least including vector lattices, positive operators, and Boolean-valued analysis.

After a certain preliminary discussion of the main notions, we give an example of a purely nonatomic locally one-dimensional K-space. Due to Theorem 5.5.2, we shall thus obtain a purely nonatomic universally complete K-space G, for which all band preserving operators $T\colon G \to G$ are regular. Due to Theorem 5.5.3, we shall have a purely nonatomic complete Boolean algebra B, for which $\mathbb{R}^\wedge = \mathscr{R}$ in $\mathbb{V}^{(B)}$.

5.5.4. A σ-complete Boolean algebra B is called *σ-distributive* if it satisfies one of the following equivalent conditions (see [18: 19.1]):

(1) $\bigwedge_{n\in\mathbb{N}} \bigvee_{m\in\mathbb{N}} b^n_m = \bigvee_{m\in\mathbb{N}^\mathbb{N}} \bigwedge_{n\in\mathbb{N}} b^n_{m(n)}$ for all $b^n_m \in B$ $(n, m \in \mathbb{N})$;
(2) $\bigvee_{n\in\mathbb{N}} \bigwedge_{m\in\mathbb{N}} b^n_m = \bigwedge_{m\in\mathbb{N}^\mathbb{N}} \bigvee_{n\in\mathbb{N}} b^n_{m(n)}$ for all $b^n_m \in B$ $(n, m \in \mathbb{N})$;
(3) $\bigvee_{\varepsilon\in\{1,-1\}^\mathbb{N}} \bigwedge_{n\in\mathbb{N}} \varepsilon(n) b_n = 1$ for all $b_n \in B$ $(n \in \mathbb{N})$, where $1b_n = b_n$ and $(-1)b_n$ is the complement of b_n.

5.5.5. Let B be an arbitrary Boolean algebra and let C be a cover of B. A subset C_0 of the algebra B is said to be *refined* from C if, for each $c_0 \in C_0$, there exists a $c \in C$ such that $c_0 \leqslant c$. An element $b \in B$ is called refined from C if the set $\{b\}$ is refined from C, i.e., $b \leqslant c$ for some element $c \in C$. If $(C_n)_{n\in\mathbb{N}}$ is a sequence of covers of the algebra B and an element $b \in B$ is refined from each of the covers C_n ($n \in \mathbb{N}$), then we say that b is refined from the sequence $(C_n)_{n\in\mathbb{N}}$. We also refer to a cover, all elements of which are refined from the sequence $(C_n)_{n\in\mathbb{N}}$, as refined from the sequence.

Proposition. *Let B be a σ-complete Boolean algebra. The following assertions are equivalent:*

(1) *the algebra B is σ-distributive;*
(2) *from every sequence of countable covers of B, one can refine a (possibly, uncountable) cover;*
(3) *from every sequence of finite covers of B, one can refine a (possibly, infinite) cover;*
(4) *from every sequence of two-element partitions of B, one can refine a cover.*

$\lhd$ A proof of the equivalence (1)$\Leftrightarrow$(2) can be found in [18: 19.3]). Assertion (4) is a reformulation of condition (3) in the definition of σ-distributivity. The implications (2)$\Rightarrow$(3)$\Rightarrow$(4) are obvious. $\rhd$

Corollary. *Let B be a complete Boolean algebra. The following assertions are equivalent:*

(1) *the algebra B is σ-distributive;*
(2) *from every sequence of countable partitions of B, one can refine a (possibly, uncountable) partition;*
(3) *from every sequence of finite partitions of B, one can refine a (possibly, infinite) partition;*
(4) *from every sequence of two-element partitions of B, one can refine a partition.*

$\lhd$ The claim follows from the last proposition in view of the exhaustion principle. $\rhd$

5.5.6. We say that a function $g \in C_\infty(Q)$ is *refined from a cover C* of the Boolean algebra $\mathrm{Clop}(Q)$ if, for every two points $q', q'' \in Q$ satisfying the equality $g(q') = g(q'')$, there exists an element $U \in C$ such that $q', q'' \in U$. If $(C_n)_{n\in\mathbb{N}}$ is a sequence of covers of the algebra $\mathrm{Clop}(Q)$ and a function g is refined from each of the covers C_n ($n \in \mathbb{N}$), then we say that g is *refined from the sequence* $(C_n)_{n\in\mathbb{N}}$.

Lemma. *From every sequence of finite covers of the algebra* Clop(Q), *one can refine a function of* $C(Q)$.

$\lhd$ Let $(C_n)_{n\in\mathbb{N}}$ be a sequence of finite covers of the algebra Clop(Q). With the help of induction, it is not difficult to construct a sequence of partitions $P_m = \{U_1^m, U_2^m, \dots, U_{2^m}^m\}$ of the algebra Clop(Q) possessing the following properties:

(1) for every $n \in \mathbb{N}$ there is a number $m \in \mathbb{N}$ such that the partition P_m is refined from the cover C_n;

(2) $U_n^m = U_{2j-1}^{m+1} \vee U_{2j}^{m+1}$ for all $m \in \mathbb{N}$ and $j \in \{1, 2, \dots, 2^m\}$.

For each number $m \in \mathbb{N}$, define a two-valued function $\chi_m \in C(Q)$ as follows:

$$\chi_m := \sum_{i=1}^{2^{m-1}} \chi(U_{2i}^m),$$

where $\chi(U)$ is the characteristic function of a subset $U \subset Q$. Since the series $\sum_{m=1}^{\infty} \frac{1}{3^m}\chi_m$ is uniformly convergent, its sum g belongs to $C(Q)$. We will show that the function g is refined from $(C_n)_{n\in\mathbb{N}}$. Due to property (1) of the sequence $(P_m)_{m\in\mathbb{N}}$, it is sufficient for this to establish that the function g is refined from $(P_m)_{m\in\mathbb{N}}$.

Assume the contrary and consider the smallest number $m \in \mathbb{N}$, for which the function g is not refined from the partition P_m. In this case, there are two points $q', q'' \in Q$ that satisfy the equality $g(q') = g(q'')$ and belong to distinct elements of P_m. Since the function g is refined from the partition P_{m-1} (for $m > 1$), from property (2) of the sequence $(P_m)_{m\in\mathbb{N}}$ it follows that the points q' and q'' belong to adjacent elements of P_m, i.e. elements of the form U_j^m and U_{j+1}^m, where $j \in \{1, \dots, 2^m - 1\}$. For definiteness, suppose that q' belongs to an element with even subscript and q'' with odd one, i.e., $\chi_m(q') = 1$ and $\chi_m(q'') = 0$. Therefore, taking into account the fact that $\chi_i(q') = \chi_i(q'')$ for all $i \in \{1, \dots, m-1\}$, we have:

$$g(q') - g(q'') = \frac{1}{3^m} + \sum_{i=m+1}^{\infty} \frac{1}{3^i}(\chi_i(q') - \chi_i(q'')) \geqslant \frac{1}{3^m} - \sum_{i=m+1}^{\infty} \frac{1}{3^i} = \frac{1}{2\cdot 3^m} > 0,$$

which contradicts the equality $g(q') = g(q'')$. $\rhd$

5.5.7. *Theorem.* *A universally complete K-space is locally one-dimensional if and only if its base is σ-distributive.*

◁ Let G be a universally complete K-space and let Q be the Stone compact space of its base. Suppose that G is locally one-dimensional and consider an arbitrary sequence $(P_n)_{n\in\mathbb{N}}$ of finite partitions of the Boolean algebra $\mathrm{Clop}(Q)$. According to Corollary 5.5.5, in order to prove σ-distributivity of the base of G, it is sufficient to refine a cover of $\mathrm{Clop}(Q)$ from $(P_n)_{n\in\mathbb{N}}$. In view of Lemma 5.5.6, one can refine a function $g \in C_\infty(Q)$ from the sequence $(P_n)_{n\in\mathbb{N}}$. Since G is locally one-dimensional, there exists a partition $(U_\xi)_{\xi\in\Xi}$ of the algebra $\mathrm{Clop}(Q)$ such that the function g is constant on each of the sets U_ξ. Show that the partition $(U_\xi)_{\xi\in\Xi}$ is refined from the sequence $(P_n)_{n\in\mathbb{N}}$. To this end, we fix arbitrary indices $\xi \in \Xi$ and $n \in \mathbb{N}$ and establish that the set U_ξ is refined from the partition P_n. We may assume that $U_\xi \neq \varnothing$. Let q_0 be an element of U_ξ. Finiteness of the partition P_n allows us to find an element U of it such that $q_0 \in U$. It remains to observe that $U_\xi \subset U$. Indeed, if $q \in U_\xi$ then $g(q) = g(q_0)$ and, since the function g is refined from P_n, the points q and q_0 belong to the same element of the partition P_n, i.e., $q \in U$.

Now, assume that the base of G is σ-distributive and consider an arbitrary function $g \in C_\infty(Q)$. According to condition (3) of the definition of a locally one-dimensional K-space, it is sufficient to construct a partition $(U_\xi)_{\xi\in\Xi}$ of the algebra $\mathrm{Clop}(Q)$ such that the function g is constant on each of the sets U_ξ. For every natural n and every integer m, denote by U_m^n the interior of the closure of the set of all points $q \in Q$ for which $\frac{m}{n} \leqslant g(q) < \frac{m+1}{n}$ and define $P_n := \{U_m^n : m \in \mathbb{Z}\}$. Due to Corollary 5.5.5, from the sequence $(P_n)_{n\in\mathbb{N}}$ of countable partitions of the algebra $\mathrm{Clop}(Q)$, one can refine some partition $(U_\xi)_{\xi\in\Xi}$. It is not difficult to become convinced that the partition constructed is the desired one. ▷

Thus, the question about existence of a purely nonatomic locally one-dimensional K-space is reduced to existence of a purely nonatomic σ-distributive complete Boolean algebra. The remainder of the note is devoted to constructing such an algebra.

5.5.8. A Boolean algebra B is called *σ-inductive* if every decreasing sequence of nonzero elements of B admits a nonzero lower bound. A subalgebra B_0 of a Boolean algebra B is said to be *dense* if, for every nonzero element $b \in B$, there exists a nonzero element $b_0 \in B_0$ such that $b_0 \leqslant b$.

Lemma. *If a σ-complete Boolean algebra contains a σ-inductive dense sub-*

algebra then it is σ-distributive.

$\lhd$ Let B be a σ-complete Boolean algebra and let B_0 be a σ-inductive dense subalgebra of B. Consider an arbitrary sequence $(C_n)_{n\in\mathbb{N}}$ of countable covers of B, denote by C the set of all elements in B that are refined from $(C_n)_{n\in\mathbb{N}}$, and assume by way of contradiction that C is not a cover of B. Then there exists a nonzero element $b \in B$ that is disjoint with all elements of C.

By induction, we construct sequences $(b_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$ as follows. Let c_1 be an element of C_1 such that $b \wedge c_1 \neq 0$. Since B_0 is dense, there is an element $b_1 \in B_0$ such that $0 < b_1 \leqslant b \wedge c_1$. Suppose that the elements b_n and c_n are already constructed. Let c_{n+1} be an element of C_{n+1} such that $b_n \wedge c_{n+1} \neq 0$. As b_{n+1} we take an arbitrary element of B_0 that satisfies the inequalities $0 < b_{n+1} \leqslant b_n \wedge c_{n+1}$.

Thus, we have constructed sequences $(b_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$ such that $b_n \in B_0$, $b_n \leqslant c_n \in C_n$ and $0 < b_{n+1} \leqslant b_n \leqslant b$ for all $n \in \mathbb{N}$. Due to the fact that B_0 is σ-inductive, it contains an element b_0 which satisfies $b_0 \leqslant b_n$ for all $n \in \mathbb{N}$. In view of the inequalities $b_0 \leqslant c_n$, the element b_0 is refined from $(C_n)_{n\in\mathbb{N}}$, i.e., belongs to C. On the other hand, $b_0 \leqslant b$, which contradicts disjointness of b with all elements of C. $\rhd$

5.5.9. As is known, for every Boolean algebra B, there exists a complete Boolean algebra $\overline{B}$ that contains B as a dense subalgebra (see [18: Section 35]). Such an algebra $\overline{B}$ is unique to within an isomorphism and called a *completion* of B. Obviously, a completion of a purely nonatomic Boolean algebra is purely nonatomic. In addition, due to Lemma 5.5.8, a completion of a σ-inductive algebra is σ-distributive. Therefore, in order to prove existence of a purely nonatomic σ-distributive complete Boolean algebra, it is sufficient to present an arbitrary purely nonatomic σ-inductive Boolean algebra. Examples of such algebras are readily available. For the sake of completeness, we present here one of the simplest constructions.

EXAMPLE. Let B be the Boolean algebra of all subsets of $\mathbb{N}$ and let I be the ideal of B consisting of all finite subsets of $\mathbb{N}$. Then the quotient algebra B/I (see [18: Section 10]) is purely nonatomic and σ-inductive.

$\lhd$ Pure nonatomicity of the algebra B/I is obvious. In order to prove that the algebra is σ-inductive, it is sufficient to consider an arbitrary decreasing sequence $(b_n)_{n\in\mathbb{N}}$ of infinite subsets of $\mathbb{N}$ and construct an infinite subset $b \subset \mathbb{N}$ such that the difference $b \setminus b_n$ is finite for each $n \in \mathbb{N}$. We can easily obtain the desired set $b = \{m_n : n \in \mathbb{N}\}$ with the help of induction by letting $m_1 := \min b_1$ and $m_{n+1} := \min\{m \in b_{n+1} : m > m_n\}$. $\triangleright$

5.5.10. A linear operator $T : \mathcal{U} \to \mathcal{V}$ is said to be *band preserving* if it satisfies one of the following equivalent conditions:

(1) $\langle Tu \rangle \leqslant \langle u \rangle$ for all $u \in \mathcal{U}$;

(2) $T\pi u = \pi Tu$ for all $u \in \mathcal{U}$ and $\pi \in \mathrm{Pr}(G)$;

(3) $\pi u = 0$ implies $\pi Tu = 0$ for all $u \in \mathcal{U}$ and $\pi \in \mathrm{Pr}(G)$;

(4) $|u| \perp g$ implies $|Tu| \perp g$ for all $u \in \mathcal{U}$ and $g \in G$;

(5) $|u| \perp g$ implies $|Tu| \perp g$ for all $u \in \mathcal{U}$ and all elements g of some order-dense ideal of the K-space G.

Obviously, the last definition generates the known notion of band preserving operator acting in vector lattices (see 5.5.2 and [2, 3, 37, 41, 42]).

5.5.11. Bounded band preserving operators are called *orthomorphisms*. The totality of all orthomorphisms from $\mathcal{U}$ into $\mathcal{V}$ is denoted by $\mathrm{Orth}(\mathcal{U}, \mathcal{V})$. We write $\mathrm{Orth}(\mathcal{U})$ instead of $\mathrm{Orth}(\mathcal{U}, \mathcal{U})$.

In accordance with Theorem 5.5.2, it seems interesting to clarify, which additional requirements imposed on band preserving operators yield their boundedness. Of course, band preserving operators are disjointness preserving and, therefore, they are subject for such boundedness criteria as 5.4.10 and 5.4.13. It is known (see 5.3.4–5.3.6), that semiboundedness, sequential boundedness, and even countable boundedness of a disjointness preserving operator do not yield its boundedness. In the case of band preserving operators, the situation is different:

Theorem. *The following properties of a band preserving operator T from a BKS into an LNS are equivalent:*

(1) *T is bounded;*

(2) *T is countably bounded;*

(3) *T is sequentially bounded;*

(4) *T is semibounded.*

$\triangleleft$ The implications (1)$\Rightarrow$(2)$\Rightarrow$(3)$\Rightarrow$(4) are obvious. It remains to show that (4)$\Rightarrow$(1). Assume that an LNS $\mathcal{U}$ is order-complete and an operator $T : \mathcal{U} \to \mathcal{V}$ is band preserving and semibounded. Fix an arbitrary positive element $e \in G$ and prove that the set $\{|Tu| : |u| \leqslant e\}$ is order-bounded in F. We divide the proof into two steps.

(a) Show first that the set $\{|Tu| : |u| \leqslant e\}$ is order-bounded in the universally complete K-space G. Without loss of generality, we may assume that $G = C_\infty(Q)$, where Q is an extremally disconnected compact space (see Theorem 1.4.6 (3)). Denote by D the totality of those points $q \in Q$, for which $\sup\{|Tu|(q) : |u| \leqslant e\} = \infty$. Assume that the set $\{|Tu| : |u| \leqslant e\}$ is not bounded in $C_\infty(Q)$. Then, according to [12: Chapter XIII, Theorem 2.32], the clopen set $U := \operatorname{int}\operatorname{cl} D$ is nonempty. For each natural n and each point $q \in U \cap D$, consider an element $u_n^q \in \mathscr{U}$ satisfying the conditions $|u_n^q| \leqslant e$ and $|Tu_n^q|(q) > n$. Denote by U_n^q a clopen subset of Q such that $q \in U_n^q \subset U$ and $|Tu_n^q|(p) \geqslant n$ for all $p \in U_n^q$. It is clear that, for each $n \in \mathbb{N}$ the relation $\sup_{q \in U \cap D} U_n^q = U$ holds in the Boolean algebra $\mathrm{Clop}(Q)$. In view of the exhaustion principle, there exists a family $(V_n^q)_{q \in U \cap D}$ of pairwise disjoint elements of $\mathrm{Clop}(Q)$ such that $V_n^q \subset U_n^q$ for all $q \in U \cap D$, and $\sup_{q \in U \cap D} V_n^q = U$. According to 1.6.7 (5), the sum $o\text{-}\sum_{q \in U \cap D} \langle V_n^q \rangle u_n^q$ exists in the BKS $\mathscr{U}$. Denote the sum by u_n. For all $n \in \mathbb{N}$ and $q \in U \cap D$, we have

$$\langle V_n^q \rangle |Tu_n| = |T \langle V_n^q \rangle u_n| = |T \langle V_n^q \rangle u_n^q| = \langle V_n^q \rangle |Tu_n^q| \geqslant n \chi_{V_n^q}.$$

After passing to the supremum over $q \in U \cap D$, we obtain $|Tu_n| \geqslant n\chi_U$ for all $n \in \mathbb{N}$; which, together with the inequalities $|u_n| \leqslant e$, yields a contradiction with semiboundedness of T.

(b) Denote by f the upper envelope of the set $\{|Tu| : |u| \leqslant e\}$ in the K-space G and show that $f \in F$. Without loss of generality, we may assume that $f > 0$ on some comeager subset of Q. Then, according to [12: Chapter XIII, Theorem 2.32], the set of all points $q \in Q$, for which $0 < \sup\{|Tu|(q) : |u| \leqslant e\} = f(q) < \infty$, is comeager in Q. For any such point q, consider an element $u_q \in \mathscr{U}$ satisfying the conditions $|u_q| \leqslant e$ and $|Tu_q|(q) > f(q)/2$. By repeating the idea of step (a) and "mixing up" the elements u_q in an appropriate way, we can construct an element $u \in \mathscr{U}$ such that $|Tu| \geqslant f/2$; whence the containment $f \in F$ follows directly. $\triangleright$

Additional requirements, yielding boundedness of band preserving operators, can be imposed on the spaces rather than on operators acting in them. In the present article, we are not going to develop this idea. We only observe that many results in the indicated direction are presented in [2: Theorem 2; 3: Theorem 3.2 and 3.3; 37: Corollaries 2.3 and 2.4].

5.5.12. It is easy to become convinced that $\operatorname{Orth}(E, F)$ is an ideal of the K-space $M(E, F)$ and, therefore, is also a K-space.

If an element $g \in G$ is such that $g \cdot e \in F$ for all $e \in E$ then the operator of multiplication by g is obviously an orthomorphism from E into F. Many papers about disjointness preserving operators contain results in this direction (see, for instance, [1–3, 23, 24, 41, 42]). The following statement generalizes, in a sense, the experience from finding multiplication representation of orthomorphisms acting in K-spaces.

Theorem. *For every orthomorphism $T\colon E \to F$, there exists a unique element $g_T \in G$ such that $Te = g_T \cdot e$ for all $e \in E$. The mapping $T \mapsto g_T$ performs a linear and order isomorphism of the K-space* $\operatorname{Orth}(E, F)$ *onto the ideal $\{g \in G : g \cdot e \in F$ for all $e \in E\}$ of the K-space G.*

Identifying an orthomorphism T with the element $g_T \in G$, we assume in the sequel that $\operatorname{Orth}(E, F) \subset G$. Obviously, $\operatorname{Orth}(E)$ contains 1_G and is a subalgebra of G. In particular, $\operatorname{Orth}(E)$ is an f-algebra (see [32, 42]). The last theorem justifies the term *weight operator* which is sometimes used instead of "orthomorphism."

5.5.13. Proposition. *Let an LNS $\mathscr{U}$ be order-complete. A linear operator $T\colon \mathscr{U} \to \mathscr{V}$ is an orthomorphism if and only if it is dominated and its exact dominant $|T|\colon E \to F$ is an orthomorphism. In particular, the space* $\operatorname{Orth}(\mathscr{U}, \mathscr{V})$ *endowed with the dominant-norm is a BKS over the K-space* $\operatorname{Orth}(E, F)$.

$\triangleleft$ The claim follows directly from Propositions 5.4.12 and 5.4.4. $\triangleright$

5.5.14. Corollary. *Every orthomorphism from a BKS into an LNS is o-continuous.*

5.5.15. Corollary. *If two orthomorphisms from a BKS $\mathscr{U}$ into an LNS $\mathscr{V}$ coincide on some order-approximating subset of $\mathscr{U}$* (see 5.2.2), *then they coincide on the entire $\mathscr{U}$.*

$\triangleleft$ The claim follows from 5.5.14 and Proposition 5.2.4. $\triangleright$

5.5.16. Corollary. *If two orthomorphisms $S, T \in$* $\operatorname{Orth}(E, \mathscr{V})$ *inequality on a subset $E_0 \subset E$ then they coincide on $E_0^{\perp\perp}$. In particular, if the K-space E has an order unity 1 and $S(1) = T(1)$ then $S = T$.*

5.5.17. Proposition. *For every BKS $\mathscr{U}$ over E there exists a unique operation* $\mathrm{Orth}(E)\times\mathscr{U}\to\mathscr{U}$ *making* $\mathscr{U}$ *a module over* $\mathrm{Orth}(E)$ *such that* $|gu| = |g||u|$ *for all* $g\in\mathrm{Orth}(E)$ *and* $u\in\mathscr{U}$. *Furthermore,* $\mathscr{U}$ *is a unital module, i.e.,* $1_G u = u$ *for all* $u\in\mathscr{U}$. *For every* $g\in\mathrm{Orth}(E)$ *and* $u\in\mathscr{U}$, *the element* gu *coincides with the product of* g *and* u *calculated in the universal completion of* $\mathscr{U}$ (see 5.1.12).

$\lhd$ Let a BKS $m\mathscr{U}$ over G be a universal completion of $\mathscr{U}$. Then $\mathscr{U} = \{u\in m\mathscr{U} : |u|\in E\}$. In view of 5.1.12, the space $m\mathscr{U}$ can be endowed with the structure of a module over the ring G so that $1_G u = u$ and $|gu| = |g||u|$ for all $g\in G$ and $u\in m\mathscr{U}$. In order to prove existence of a desired module structure in the BKS $\mathscr{U}$, it is sufficient to observe that, for all $g\in\mathrm{Orth}(E)$ and $u\in\mathscr{U}$, we have $|g||u|\in E$ and, consequently, $gu\in\mathscr{U}$.

Now prove uniqueness. Assume that, together with the operation $(g,u)\mapsto gu$ introduced above, there is another one, $(g,u)\mapsto g*u$, also making $\mathscr{U}$ a module over $\mathrm{Orth}(E)$ and satisfying the condition $|g*u| = |g||u|$ for all $g\in\mathrm{Orth}(E)$ and $u\in\mathscr{U}$. Fix an element $u\in\mathscr{U}$ and define the mappings $S,T\colon \mathrm{Orth}(E)\to\mathscr{V}$ by the formulas $S(g) = gu$ and $T(g) = g*u$. Obviously, S and T are orthomorphisms. Observe that $T(1_G) = S(1_G)$, i.e., $1_G * u = u$. Indeed,

$$
\begin{aligned}
|1_G*u-u| &= 1_G\cdot|1_G*u-u| = \big|1_G*(1_G*u-u)\big|\\
&= \big|(1_G\cdot 1_G)*u - 1_G*u\big| = 0.
\end{aligned}
$$

For proving the equality $S=T$, it remains to employ 5.5.16. $\rhd$

The fact that any BKS over G can be endowed with the structure of a module over G allows us to define a simple class of orthomorphisms. If a BKS $\mathscr{U}$ over E and a BKS $\mathscr{V}$ over F are order-dense ideals of the same BKS over G and $g\in\mathrm{Orth}(E,F)$, then the operator $u\mapsto gu$ is an orthomorphism from $\mathscr{U}$ into $\mathscr{V}$. We call such operators *scalar orthomorphisms*.

5.5.18. Proposition. *Let* $\mathscr{U}$ *be an order-complete LNS,* $T\in\mathrm{Orth}(\mathscr{U},\mathscr{V})$, $g\in G$, *and* $u\in\mathscr{U}$. *If the product* gu *is defined in* $\mathscr{U}$ (see 5.1.12) *then the product* $gT(u)$ *is defined in* $\mathscr{V}$ *and the equality* $T(gu) = gT(u)$ *holds. In particular,* $T\circ g = g\circ T$ *for every orthomorphism* $g\in\mathrm{Orth}(E)$.

$\lhd$ Fix an arbitrary element $u\in\mathscr{U}$ and denote by G_u the order-dense ideal $\{g\in G : gu\in\mathscr{U}\}$ of the K-space G. Let $m\mathscr{V}$ be the universal completion of $\mathscr{V}$.

Consider the mappings $L, R\colon G_u \to m\mathscr{V}$ defined by the formulas $L(g) = T(gu)$ and $R(g) = gT(u)$. Obviously, L and R are orthomorphisms and $L(1_G) = R(1_G)$. From 5.5.16 it follows that $L = R$. $\rhd$

5.5.19. We conclude this section by a useful fact, which will be repeatedly employed in the sequel.

Theorem [17]. *Let E be a vector lattice and let F be a K-space. A positive operator $T\colon E \to F$ is disjointness preserving if and only if, for every operator $S\colon E \to F$ satisfying the inequalities $0 \leqslant S \leqslant T$, there is an orthomorphism $g \in \mathrm{Orth}(F)$ such that $0 \leqslant g \leqslant \mathrm{id}_F$ and $S = g \circ T$, where $\mathrm{id}_F\colon F \to F$ is the identity operator.*

Combining the last theorem with Theorem 5.4.1, we obtain the following result.

Corollary. *Let E be a vector lattice and let F be a K-space. A regular operator $T\colon E \to F$ is disjointness preserving if and only if, for every regular operator $S\colon E \to F$ satisfying the inequality $|S| \leqslant |T|$, there is an orthomorphism $g \in \mathrm{Orth}(F)$ such that $|g| \leqslant \mathrm{id}_F$ and $S = g \circ T$, where $\mathrm{id}_F\colon F \to F$ is the identity operator.*

5.6. Shift Operators

Another class of disjointness preserving operators is considered in this section. Here, we introduce and study so-called shift operators, which are abstract analogs of the composition mappings $f \mapsto f \circ s$. This class of operators is closely related to another notion discussed here, the notion of operator "wide on a set." While studying shift operators, we suggest their equivalent characterizations, describe the maximal domain of definition on which they can be extended, and show that the notions of shift operator and that of a multiplicative operator coincide. We also introduce here the notion of the shift of a disjointness preserving operator, which concentrates, in a sense, multiplicative properties of the operator.

Throughout the section, $\mathscr{E}$ and $\mathscr{F}$ are universally complete K-spaces. In case order unities $1_{\mathscr{E}}$ and $1_{\mathscr{F}}$ are fixed in $\mathscr{E}$ and $\mathscr{F}$, we regard the K-spaces as ordered algebras with unities $1_{\mathscr{E}}$ and $1_{\mathscr{F}}$ (see 5.1.12). The ideal of the K-space $\mathscr{E}$ generated by $d \in \mathscr{E}$ is denoted by $\mathscr{E}_d$. In particular, $\mathscr{E}_1$ stands for the ideal of $\mathscr{E}$ generated by $1_{\mathscr{E}}$. We point out that some notions introduced in this section depend of a concrete choice of $1_{\mathscr{E}}$ and $1_{\mathscr{F}}$.

5.6.1. Let E be a K-space, let D be a subset of E, and let $\mathscr{Y}$ be an LNS. We say that an operator $T\colon E \to \mathscr{Y}$ is *wide on the set* D whenever $T[D]^{\perp\perp} = T[E]^{\perp\perp}$.

Proposition. *Suppose that E is a K-space, D is a subset of E, $\mathscr{Y}$ is an LNS, $T\colon E \to \mathscr{Y}$ is a disjointness preserving operator, and $h\colon \mathrm{Pr}(E) \to \mathrm{Pr}(\mathscr{Y})$ is its shadow. The following assertions are equivalent:*

(1) *T is wide on the set D;*
(2) *T is wide on the ideal E_D;*
(3) *the shadow of the restriction of T onto E_D coincides with the shadow of T;*
(4) *the set $T[E_D]$ is o-dense in $T[E]$;*
(5) *the ideal E_D h-approximates the space E.*

$\lhd$ The implications (1)$\Rightarrow$(2)$\Leftarrow$(4) are obvious. Since the shadow of T dominates that of the restriction of T onto E_D, the equivalence (2)$\Leftrightarrow$(3) readily follows from Proposition 5.1.4. We show that (1)$\Leftarrow$(2)$\Rightarrow$(5)$\Rightarrow$(4).

(2)$\Rightarrow$(5): Assume condition (2) to be satisfied, consider an arbitrary element $e \in E$, and show that $h\text{-}\inf_{\pi\in\Pi} \pi e = e$, where $\Pi = \{\pi \in \mathrm{Pr}(E) : \pi e \in E_D\}$. For every $n \in \mathbb{N}$ and $d \in E_D$, assign $\pi_n^d := \big\langle |e| \leqslant n|d| \big\rangle$. Obviously, $\pi_n^d \in \Pi$. Since

$$\left|d - \pi_n^d d\right| = \left(\pi_n^d\right)^{\perp}|d| \leqslant \left(\pi_n^d\right)^{\perp}|e|/n \leqslant |e|/n$$

for all $n \in \mathbb{N}$, we have $r\text{-}\lim\limits_{n\to\infty} \pi_n^d d = d$. Using r-continuity of the operator T and taking account of the equality $T\left(\pi_n^d d\right) = h\left(\pi_n^d\right)Td$, we arrive at the relation $\sup_{n\in\mathbb{N}} h\left(\pi_n^d\right) \geqslant \langle Td\rangle$. Since the element $d \in E_D$ was chosen arbitrarily, we conclude by (2) that $\sup_{\pi\in\Pi} h(\pi) = h(1)$ and, consequently, $h\text{-}\inf_{\pi\in\Pi} \pi e = e$.

(5)$\Rightarrow$(4): Consider an arbitrary element $e \in E$. From (5) and Proposition 5.2.3 it follows that e is the h-limit of some net $(e_\alpha)_{\alpha\in\mathrm{A}}$ of elements in E_D. In view of Corollary 5.4.5, we have $o\text{-}\lim\limits_{\alpha\in\mathrm{A}} T(e_\alpha) = Te$.

(2)$\Rightarrow$(1): For every element $e \in E_D$, there exist $d_1, \dots, d_n \in D$ such that $|e| \leqslant |d_1| + \dots + |d_n|$. In view of Theorem 5.4.10, we conclude that $\langle Te\rangle \leqslant \langle Td_1\rangle \vee \dots \vee \langle Td_n\rangle$. It remains to employ condition (2). $\rhd$

REMARK. As is seen from the last proposition, the fact that an operator T is wide on a set D reflects connection of D with the domain of definition and with the shadow of T rather then with the operator T itself.

5.6.2. Let $\mathscr{U}$ and $\mathscr{V}$ be LNSs and let D be a subset of the norming lattice of $\mathscr{U}$. We say that an operator $T\colon \mathscr{U} \to \mathscr{V}$ is *wide on the set* D, whenever $\{Tu : |u| \in D\}^{\perp\perp} = (\operatorname{im} T)^{\perp\perp}$. If $\mathscr{U}$ and $\mathscr{V}$ are K-spaces then the last definition is equivalent to that given in 5.6.1, which justifies preservation of terminology.

Lemma. *Let $\mathscr{U}$ be a BKS over a K-space E, let $\mathscr{V}$ be an arbitrary LNS, and let D be a subset of positive elements in E. A disjointness preserving operator $T\colon \mathscr{U} \to \mathscr{V}$ is wide on D if and only if its exact dominant $|T|$ is wide on D.*

$\lhd$ A proof can be easily obtained with the help of Proposition 5.4.12. Indeed, the relations

$$\big\langle |T|e \big\rangle = \sup_{|u|=e} \langle Tu \rangle \leqslant \sup_{|u|\in D} \langle Tu \rangle = \sup_{d\in D} \sup_{|u|=d} \langle Tu \rangle = \sup_{d\in D} \big\langle |T|d \big\rangle,$$

which hold for every positive element $e \in E$, prove necessity; whereas the relations

$$\langle Tu \rangle \leqslant \big\langle |T||u| \big\rangle \leqslant \sup_{d\in D} \big\langle |T|d \big\rangle = \sup_{d\in D} \sup_{|u|=d} \langle Tu \rangle = \sup_{|u|\in D} \langle Tu \rangle,$$

that are valid for each element $u \in \mathscr{U}$, establish sufficiency. $\rhd$

Proposition. *Suppose that $\mathscr{U}$ is a BKS over a K-space E, D is a subset of positive elements in E, $\mathscr{V}$ is an arbitrary LNS, $T\colon \mathscr{U} \to \mathscr{V}$ is a disjointness preserving bounded operator, and $h\colon \mathrm{Pr}(\mathscr{U}) \to \mathrm{Pr}(\mathscr{V})$ is its shadow. The following assertions are equivalent:*

(1) *T is wide on the set D;*
(2) *T is wide on the ideal E_D;*
(3) *the shadow of the restriction of T onto the set $\{u \in \mathscr{U} : |u| \in E_D\}$ coincides with the shadow of T;*
(4) *the set $\{Tu : |u| \in E_D\}$ is o-dense in $\operatorname{im} T$;*
(5) *the ideal E_D h-approximates the space E.*

$\lhd$ The equivalence (2)$\Leftrightarrow$(3) is established in the same way as in 5.6.1. Equivalence of assertions (1), (2), and (5) ensues from Propositions 5.4.4 and 5.6.1 and the last lemma. The implication (4)$\Rightarrow$(2) is obvious. It remains to show that (5)$\Rightarrow$(4).

Let u be an arbitrary element of $\mathscr{U}$. From (5) and Proposition 5.2.3 it follows that $|u|$ is the h-limit of some net $(e_\alpha)_{\alpha\in\mathrm{A}}$ of positive elements in E_D.

In view of Lemma 5.1.13, there exists a net $(u_\alpha)_{\alpha\in\mathrm{A}}$ in $\mathscr{U}$ such that $|u_\alpha| = e_\alpha$ and $|u - u_\alpha| = ||u| - e_\alpha|$. Then $h\text{-}\lim\limits_{\alpha\in\mathrm{A}} u_\alpha = u$ and, according to Corollary 5.4.5, we have $o\text{-}\lim_{\alpha\in\mathrm{A}} Tu_\alpha = Tu$. ▷

5.6.3. Proposition. *Let E be an ideal of $\mathscr{E}$ generated by a positive element $d \in \mathscr{E}$. For every ring homomorphism $h\colon \mathrm{Pr}(\mathscr{E}) \to \mathrm{Pr}(\mathscr{F})$, the following sets coincide:*

(1) *the h-closure of E;*
(2) *the h-cyclic hull of E;*
(3) *the countably h-cyclic hull of E;*
(4) *the set of such $e \in \mathscr{E}$ that $\inf_{n\in\mathbb{N}} h\langle |e| > nd\rangle = 0$.*

◁ The relations (4)⊂(3)⊂(2)⊂(4) are obvious. The inclusion (4)⊂(1) can be easily established with the help of the first corollary in 5.2.18. It remains to show that (1)⊂(4). Suppose that a net $(e_\alpha)_{\alpha\in\mathrm{A}}$ of elements in E h-converges to $e \in \mathscr{E}$. For each $\alpha \in \mathrm{A}$, denote by n_α the natural number satisfying the inequality $|e_\alpha| \leqslant n_\alpha d$. By using the relations $h\text{-}\inf_{\alpha\in\mathrm{A}} |e - e_\alpha| = 0$ and

$$\begin{aligned} h\langle |e| > 2n_\alpha d\rangle &\leqslant h\langle |e| > 2|e_\alpha|\rangle \\ = h\big(\langle e\rangle \langle |e| - |e_\alpha| > |e|/2\rangle\big) &\leqslant h\big(\langle e\rangle\langle |e - e_\alpha| > |e|/2\rangle\big), \end{aligned}$$

we obtain the desired equality $\inf_{n\in\mathbb{N}} h\langle |e| > nd\rangle = 0$. ▷

The coincident sets (1)–(4) described in the last proposition are denoted by hE.

5.6.4. Proposition. *Fix an order unity $1_{\mathscr{E}}$ in the K-space $\mathscr{E}$. Then the set $h\mathscr{E}_1$ is a subalgebra of $\mathscr{E}$.*

◁ This fact ensues from 5.6.3 (we mean the equality $h\mathscr{E}_1 = (4)$ for $d = 1_{\mathscr{E}}$) and from the following relations:

$$\begin{aligned} \inf_{n\in\mathbb{N}} h\langle |ef| > n1_{\mathscr{E}}\rangle &= \inf_{m,n\in\mathbb{N}} h\langle |ef| > mn1_{\mathscr{E}}\rangle \\ &\leqslant \inf_{m,n\in\mathbb{N}} h\big(\langle |e| > m1_{\mathscr{E}}\rangle \vee \langle |f| > n1_{\mathscr{E}}\rangle\big) \\ &= \inf_{m,n\in\mathbb{N}} \big(h\langle |e| > m1_{\mathscr{E}}\rangle \vee h\langle |f| > n1_{\mathscr{E}}\rangle\big) \\ &= \inf_{m\in\mathbb{N}} h\langle |e| > m1_{\mathscr{E}}\rangle \vee \inf_{n\in\mathbb{N}} h\langle |f| > n1_{\mathscr{E}}\rangle. \quad ▷ \end{aligned}$$

5.6.5. Lemma. *Let d be an arbitrary order unity in $\mathscr{E}$. For every sequence $(\pi_n)_{n\in\mathbb{N}}$ of projections in $\mathrm{Pr}(\mathscr{E})$ that decreases to zero, there is an element $e \in \mathscr{E}$ such that $\pi_n = \langle |e| > nd\rangle$ for all $n \in \mathbb{N}$.*

$\lhd$ Since the K-space $\mathscr{E}$ is universally complete, the series $\sum_{n=1}^{\infty} \pi_n d$ has an o-sum in it. Denote the sum by s. It is clear that $\langle s > nd\rangle = \pi_{n+1}$ for all $n \in \mathbb{N}$ and, consequently, we can take $s + d$ as the desired element e. $\rhd$

Corollary. *Let $h\colon \mathrm{Pr}(\mathscr{E}) \to \mathrm{Pr}(\mathscr{F})$ be a ring homomorphism and let d be an arbitrary order unity in $\mathscr{E}$. The equality $h\mathscr{E}_d = \mathscr{E}$ holds if and only if the homomorphism $h\colon \mathrm{Pr}(\mathscr{E}) \to \mathrm{Pr}(\mathscr{F})$ is sequentially o-continuous.*

5.6.6. Let $\mathscr{U}$ be an LNS over an order-dense ideal E of the universally complete K-space $\mathscr{E}$, let d be a positive element of $\mathscr{E}$, and let $\mathscr{V}$ be an arbitrary LNS. We say that an operator $T\colon \mathscr{U} \to \mathscr{V}$ is *wide at the element* d whenever it is wide on the set $\{e \in E : e \text{ is a fragment of } d\}$.

Lemma. *Suppose that E is an order-dense ideal of $\mathscr{E}$, d is a positive element of $\mathscr{E}$, $\mathscr{V}$ is an LNS, $T\colon E \to \mathscr{V}$ is a disjointness preserving bounded operator, and h is its shadow. Assign $\Pi := \{\pi \in \mathrm{Pr}(\mathscr{E}) : \pi d \in E\}$. The following assertions are equivalent:*

(1) *the operator T is wide at the element d;*
(2) $\sup_{\pi\in\Pi} h(\pi) = h(1)$ *and, for all $\pi \in \Pi$ the equality $\langle T\pi d\rangle = h(\pi)$ holds;*
(3) $E \subset h\mathscr{E}_d$.

$\lhd$ The equivalence of (1) and (3) is contained in Proposition 5.6.1, the implication (2)$\Rightarrow$(1) is obvious. It remains to show that (1)$\Rightarrow$(2). If (1) is valid then, for every projection $\pi_0 \in \Pi$, we have

$$h(\pi_0) = h(\pi_0)\sup_{e\in E}\langle Te\rangle = h(\pi_0)\sup_{\pi\in\Pi}\langle T\pi d\rangle = \sup_{\pi\in\Pi}\langle T\pi_0\pi d\rangle = \langle T\pi_0 d\rangle. \quad \rhd$$

5.6.7. Proposition. *Fix arbitrary order unities $1_{\mathscr{E}}$ and $1_{\mathscr{F}}$ in the K-spaces $\mathscr{E}$ and $\mathscr{F}$. For every ring homomorphism $h\colon \mathrm{Pr}(\mathscr{E}) \to \mathrm{Pr}(\mathscr{F})$, there exists a unique regular operator $S\colon h\mathscr{E}_1 \to \mathscr{F}$ such that the shadow of S is equal to h and $S(1_{\mathscr{E}}) = h(1)1_{\mathscr{F}}$. Furthermore, the operator S is positive.*

$\lhd$ For the sake of convenience, assume that $h(1) = 1$. We divide the construction of the operator S into three steps.

1. Define the operator S on the set of step-elements of $\mathscr{E}$ by letting

$$S\Big(\sum_{i=1}^{n}\lambda_i\pi_i 1_{\mathscr{E}}\Big) := \sum_{i=1}^{n}\lambda_i h(\pi_i)1_{\mathscr{F}}$$

for arbitrary $\lambda_1,\dots,\lambda_n \in \mathbb{R}$ and $\pi_1,\dots,\pi_n \in \mathrm{Pr}(\mathscr{E})$.

2. Extend the operator S onto $\mathscr{E}_1$. To this end, fix an arbitrary element $e \in \mathscr{E}_1$ and choose a sequence $(e_n)_{n\in\mathbb{N}}$ of step-elements in $\mathscr{E}$ so that it r-converges to e with regulator $1_{\mathscr{E}}$. It is easy to verify that the sequence $(Se_n)_{n\in\mathbb{N}}$ is r-fundamental (with regulator $1_{\mathscr{F}}$). Assign $Se := r\text{-}\lim\limits_{n\to\infty} Se_n$.

3. Finally, extend S onto the entire set $h\mathscr{E}_1$. Every element $e \in h\mathscr{E}_1$ can be represented as the mixing $o\text{-}\sum_{n\in\mathbb{N}}\pi_n e_n$ of elements $e_n \in \mathscr{E}_1$ by means of an h-partition $(\pi_n)_{n\in\mathbb{N}}$. Assign $Se := o\text{-}\sum_{n\in\mathbb{N}} h(\pi_n)Se_n$.

It is easy to verify that the definition of S is sound at each of the steps. Obvious positiveness of S ensures its regularity. In order to prove uniqueness of S, it is sufficient to observe that, at step 3, the sequence $\big(\sum_{n=1}^{m}\pi_n e_n\big)_{m\in\mathbb{N}}$ is r-convergent to e with regulator $o\text{-}\sum_{n\in\mathbb{N}} n\pi_n|e_n| \in h\mathscr{E}_1$. $\triangleright$

The operator S, whose existence is asserted in the last proposition, is called the ***shift by*** h and denoted by S_h. Let E be an order-dense ideal of $\mathscr{E}$ and F be an order-dense ideal of $\mathscr{F}$. We say that an operator $S\colon E \to F$ is a ***shift operator***, if there exists a ring homomorphism $h\colon \mathrm{Pr}(\mathscr{E}) \to \mathrm{Pr}(\mathscr{F})$ such that $E \subset h\mathscr{E}_1$ and $S = S_h$ on E. It is clear that, in this case, the homomorphism h is the shadow of S. Observe that the notion of the shift and that of shift operator depend on the choice of unities $1_{\mathscr{E}}$ and $1_{\mathscr{F}}$ in the K-spaces $\mathscr{E}$ and $\mathscr{F}$.

5.6.8. *Proposition.* *Fix order unities in the universally complete K-spaces $\mathscr{E}$ and $\mathscr{F}$. Let E be an order-dense ideal of $\mathscr{E}$, let F be an order-dense ideal of $\mathscr{F}$, and let $S, \overline{S}\colon E \to F$ be shift operators. If $S \leqslant \overline{S}$ then $S = \rho\circ\overline{S}$ for some projection $\rho \in \mathrm{Pr}(F)$.*

$\triangleleft$ The claim ensues from Propositions 5.1.4 and 5.6.7. $\triangleright$

Let $\rho \in \mathrm{Pr}(\mathscr{F})$, let $h\colon \mathrm{Pr}(\mathscr{E}) \to \mathrm{Pr}(\mathscr{F})$ be a ring homomorphism, and let S be the shift by h. Then the shift by the homomorphism $\rho\circ h$ is denoted by ρS. Observe that, in general, $\mathrm{dom}\,\rho S$ is wider than $\mathrm{dom}\,S$; therefore, ρS differs from the composition $\rho\circ S$. However, in view of the last proposition, the operators ρS and $\rho\circ S$ coincide on $\mathrm{dom}\,S$ and, thus, ρS extends $\rho\circ S$.

5.6.9. Theorem. *Fix order unities $1_{\mathscr{E}}$ and $1_{\mathscr{F}}$ in the K-spaces $\mathscr{E}$ and $\mathscr{F}$. Let E be an order-dense ideal of $\mathscr{E}$ and let F be an order-dense ideal of $\mathscr{F}$. A linear operator $S\colon E \to F$ is a shift operator if and only if it satisfies the following conditions:*

(a) *S is disjointness preserving;*
(b) *S is regular;*
(c) *S takes fragments of $1_{\mathscr{E}}$ into fragments of $1_{\mathscr{F}}$;*
(d) *S is wide at $1_{\mathscr{E}}$.*

$\lhd$ Necessity of conditions (a)–(c) is obvious and necessity of (d) follows from 5.6.6. Let us show sufficiency. Suppose that the operator S satisfies conditions (a)–(d), denote the shadow of S by h and assign $\Pi := \{\pi \in \mathrm{Pr}(\mathscr{E}) : \pi 1_{\mathscr{E}} \in E\}$. Lemma 5.6.6 implies the equality $\langle S(\pi 1_{\mathscr{E}})\rangle = h(\pi)$ for each $\pi \in \Pi$, which, together with condition (c), yields $S(\pi 1_{\mathscr{E}}) = S_h(\pi 1_{\mathscr{E}})$. The same lemma ensures the inclusion $E \subset h\mathscr{E}_1$. In view of Lemma 5.4.9, we now conclude that $S = S_h$ on E. $\rhd$

Corollary. *Fix order unities $1_{\mathscr{E}}$ and $1_{\mathscr{F}}$ in the K-spaces $\mathscr{E}$ amd $\mathscr{F}$. Let E be an order-dense ideal of $\mathscr{E}$ containing $1_{\mathscr{E}}$ and let F be an arbitrary order-dense ideal of $\mathscr{F}$. A linear operator $S\colon E \to F$ is a shift operator if and only if it satisfies the following conditions:*

(a) *S is disjointness preserving;*
(b) *S is regular;*
(c) *$S(1_{\mathscr{E}})$ is a fragment of $1_{\mathscr{F}}$;*
(d) $\{S(1_{\mathscr{E}})\}^{\perp\perp} = (\operatorname{im} S)^{\perp\perp}$.

REMARK. Conditions (d) in the statements of the theorem and the corollary may not be omitted. Indeed, let $\mathscr{F} = \mathbb{R}$, let $\mathscr{E}$ be the space of all sequences, and let E be the ideal of $\mathscr{E}$ generated by the sequence $e_0(n) = n$ $(n \in \mathbb{N})$. Denote by Q the Stone-Čech compactification of the discrete topological space $\mathbb{N}$ and fix an arbitrary point $q \in Q \backslash \mathbb{N}$. Naturally identifying the spaces $\mathscr{E}$ and $C_\infty(Q)$, define an operator $S\colon E \to \mathscr{F}$ by the formula $Se = (e/e_0)(q)$. Letting $1_{\mathscr{E}}(n) = 1$ $(n \in \mathbb{N})$ and $1_{\mathscr{F}} = 1$, we see that the operator S satisfies conditions (a)–(c) of the last lemma, but $S(1_{\mathscr{E}}) = 0$.

REMARK. In particular, from the last corollary it is clear that the domain of definition $h\mathscr{E}_1$ of the shift by h is maximally wide. More precisely, $h\mathscr{E}_1$ contains the domain of definition of every regular operator S acting from an order-dense ideal of $\mathscr{E}$ into $\mathscr{F}$, having shadow h, and satisfying the equality $S(1_{\mathscr{E}}) = h(1)1_{\mathscr{F}}$.

5.6.10. Fix order unities $1_{\mathscr{E}}$ and $1_{\mathscr{F}}$ in the K-spaces $\mathscr{E}$ and $\mathscr{F}$. A linear operator $S\colon E \to \mathscr{F}$ defined on an order-dense ideal $E \subset \mathscr{E}$ is called *multiplicative* if $Se_1Se_2 = S(e_1e_2)$ for any two elements $e_1, e_2 \in E$, whose product belongs to E. Observe that the notion of multiplicative operator depends on the choice of unities $1_{\mathscr{E}}$ and $1_{\mathscr{F}}$.

Theorem. *Let E be an order-dense ideal of $\mathscr{E}$. A linear operator $S\colon E \to \mathscr{F}$ is a shift operator if and only if it is multiplicative.*

$\lhd$ The fact that every shift operator is multiplicative is easily established by checking all the steps of its construction in 5.6.7. We will show that any multiplicative operator $S\colon E \to \mathscr{F}$ is a shift operator by verifying conditions (a)–(d) of Theorem 5.6.9.

(a) Disjointness of elements $e_1, e_2 \in E$ is equivalent to the equality $e_1e_2 = 0$. The same is true for elements of $\mathscr{F}$. Consequently, S is disjointness preserving.

(b) Show that the operator S is positive. We divide the proof into three steps.

(b_1) If $e \in E$ and $0 \leqslant e \leqslant 1_{\mathscr{E}}$ then $Se \geqslant 0$. Indeed, in this case, e^3 and $e\sqrt{e}$ belong to E in view of the inequalities $e^3 \leqslant e$ and $e\sqrt{e} \leqslant e$; consequently, $(Se)^3 = S(e^3) = S\big((e\sqrt{e})^2\big) = S\big(e\sqrt{e}\big)^2 \geqslant 0$.

(b_2) If $e \in E$ and $e \geqslant 1_{\mathscr{E}}$ then $Se \geqslant 0$. Indeed, in this case, $\sqrt{e} \in E$ in view of the inequality $\sqrt{e} \leqslant e$; consequently, $Se = S\big((\sqrt{e})^2\big) = S\big(\sqrt{e}\big)^2 \geqslant 0$.

(b_3) If $e \in E$ and $e \geqslant 0$ then $Se \geqslant 0$. Indeed, $Se = S\langle e \leqslant 1_{\mathscr{E}}\rangle e + S\langle e > 1_{\mathscr{E}}\rangle e \geqslant 0$ in view of (b_1) and (b_2).

(c) The fact that an element $e \in E$ is a fragment of $1_{\mathscr{E}}$ is equivalent to the equality $e^2 = e$. The same is true for fragments of $1_{\mathscr{F}}$. Consequently, S takes fragments of $1_{\mathscr{E}}$ into fragments of $1_{\mathscr{F}}$.

(d) Show that $\{Se : |e| \leqslant 1_{\mathscr{E}}\}^{\perp\perp} = (\operatorname{im} S)^{\perp\perp}$. Consider the projection $\rho \in \Pr(\mathscr{F})$ onto the band $\{Se : |e| \leqslant 1_{\mathscr{E}}\}^{\perp}$ and define an operator $T\colon E \to \mathscr{F}$ by the formula $Te = \rho Se$. The proof will be completed if we establish that $T = 0$. Obviously, the operator T is multiplicative and $Te = 0$ whenever $|e| \leqslant 1_{\mathscr{E}}$. We also observe that, in view of (b), the operator T is positive. Let e be an arbitrary positive element of E. For each $n \in \mathbb{N}$, the equality $T(e/n) = Te_n$ holds, where $e_n = \langle e/n > 1_{\mathscr{E}}\rangle e/n$. Since $\sqrt{e_n} \leqslant e_n \leqslant e/n$, we have the inclusions $\sqrt{e_n}, e_n \in E$

and the inequality $T\sqrt{e_n} \leqslant Te_n$. Consequently,

$$Te = nTe_n = nT\big(\sqrt{e_n}^2\big) = n\big(T\sqrt{e_n}\big)^2 \leqslant n(Te_n)^2 = \; n(Te/n)^2 = (Te)^2/n$$

for all $n \in \mathbb{N}$, which is possible only in case $Te = 0$. $\triangleright$

5.6.11. REMARK. There is a number of results describing multiplicative operators (= shift operators) as extreme points of certain sets of operators (see [25, 26, 39]).

5.6.12. REMARK. It is known (see [10: Theorem VIII.10.1]) that every regular operator $T : \mathscr{E}_1 \to \mathscr{F}$ admits an integral representation

$$Te = \int\limits_{-\infty}^{\infty} \lambda \, d\varphi\big(\langle e \leqslant \lambda 1_{\mathscr{E}}\rangle\big) \quad (e \in \mathscr{E}_1),$$

where φ is an arbitrary order-bounded additive function from $\operatorname{Pr}(\mathscr{E})$ into $\mathscr{F}$. It is not difficult to become convinced that T is a shift operator if and only if the values of the function φ are fragments of $1_{\mathscr{F}}$. Furthermore, the shadow h of T is defined by the formula $h(\pi) = \langle \varphi(\pi)\rangle$. Some classes of multiplicative operators (= shift operators) are described form the viewpoint of the integral representation in the papers by B. Z. Vulikh [7, 9].

5.6.13. Fix order unities $1_{\mathscr{E}}$ and $1_{\mathscr{F}}$ in the K-spaces $\mathscr{E}$ and $\mathscr{F}$. Let $\mathscr{U}$ and $\mathscr{V}$ be LNSs over order-dense ideals $E \subset \mathscr{E}$ and $F \subset \mathscr{F}$, let $T : \mathscr{U} \to \mathscr{V}$ be a disjointness preserving operator, and let $h : \operatorname{Pr}(E) \to \operatorname{Pr}(F)$ be its shadow. Then the shift $S_h : h\mathscr{E}_1 \to \mathscr{F}$ by h is called the *shift of the operator* T.

Proposition. *Let $\mathscr{U}$ and $\mathscr{V}$ be LNSs over order-dense ideals $E \subset \mathscr{E}$ and $F \subset \mathscr{F}$ and suppose that the LNS $\mathscr{U}$ is order-complete. Assume that $T : \mathscr{U} \to \mathscr{V}$ is a disjointness preserving bounded operator with shift S. If elements $e \in \operatorname{dom} S$ and $u \in \mathscr{U}$ are such that the product eu is defined in $\mathscr{U}$, then the product $S(e)T(u)$ is defined in $\mathscr{V}$ and the equality $T(eu) = S(e)T(u)$ holds. In particular, $T \circ g = S(g) \circ T$ for every orthomorphism $g \in \operatorname{Orth}(E) \cap \operatorname{dom} S$.*

$\triangleleft$ Fix an arbitrary element $u \in \mathscr{U}$ and denote by $\mathscr{E}_u$ the order-dense ideal $\{e \in \operatorname{dom} S : eu \in \mathscr{U}\}$ of the K-space $\mathscr{E}$. Let $m\mathscr{V}$ be the universal completion of $\mathscr{V}$. Consider the mappings $L, R : \mathscr{E}_u \to m\mathscr{V}$ defined by the formulas $L(e) = T(eu)$ and

$R(e) = S(e)T(u)$. Obviously, the operators L and R are bounded (= dominated) and disjointness preserving; moreover, their shadows are dominated by the shadow of T. Since $L(1_{\mathscr{E}}) = R(1_{\mathscr{E}})$ and $\mathscr{E}_u \subset \operatorname{dom} S$, Lemma 5.4.9 implies the equality $L = R$. $\triangleright$

5.6.14. Fix order unities $1_{\mathscr{E}}$ and $1_{\mathscr{F}}$ in the K-spaces $\mathscr{E}$ and $\mathscr{F}$. Let $\mathscr{U}$ be an LNS over an order-dense ideal $E \subset \mathscr{E}$ and let $\mathscr{V}$ be an LNS over an order-dense ideal $F \subset \mathscr{F}$. An operator $S : \mathscr{U} \to \mathscr{V}$ is called a ***shift operator*** if there exists a shift operator $s : E \to F$ such that $|Su| = s|u|$ for all $u \in \mathscr{U}$. Obviously, $s = |S|$, i.e., the operator s is the exact dominant of S (see 5.3.8).

REMARK. Thus, if $S : \mathscr{U} \to \mathscr{V}$ is a shift operator then it is dominated and its exact dominant $|S| : E \to F$ if a shift operator. The converse is false in general. Indeed, if $\mathscr{U}$ and $\mathscr{V}$ are Banach spaces and the norm of an operator $S : \mathscr{U} \to \mathscr{V}$ is equal to unity then its exact dominant $|S| : \mathbb{R} \to \mathbb{R}$ is the identity operator (and, hence, a shift operator), while the operator S itself is a shift operator only if it is an isometric embedding.

Proposition. *Let $\mathscr{U}$ be an LNS over an order-dense ideal $E \subset \mathscr{E}$ and let $\mathscr{V}$ be an LNS over an order-dense ideal $F \subset \mathscr{F}$. An operator $S : \mathscr{U} \to \mathscr{V}$ is a shift operator if and only if there exist a shift operator $s : E \to F$ and an F-isometric embedding $\iota : s\mathscr{U} \to \mathscr{V}$ such that $S = \iota \circ s_{\mathscr{U}}$, where $s_{\mathscr{U}} : \mathscr{U} \to s\mathscr{U}$ is the norm transformation of $\mathscr{U}$ by means of s* (see 5.1.11).

$\triangleleft$ Only necessity requires proving. An elementary verification shows that the formula

$$\iota\left(\sum_{i=1}^{n} \rho_i s_{\mathscr{U}} u_i\right) = \sum_{i=1}^{n} \rho_i S u_i \quad \left(u_i \in \mathscr{U},\ \rho_i \in \Pr(\mathscr{V})\right)$$

soundly defines a function $\iota : s\mathscr{U} \to \mathscr{V}$ that is the desired isometry. $\triangleright$

5.6.15. The following description of shift operators generalizes criterion 5.6.9 to the case of LNSs.

Theorem. *Fix order unities $1_{\mathscr{E}}$ and $1_{\mathscr{F}}$ in the K-spaces $\mathscr{E}$ and $\mathscr{F}$. Let $\mathscr{U}$ be a BKS over an order-dense ideal $E \subset \mathscr{E}$ and let $\mathscr{V}$ be an LNS over an order-dense ideal $F \subset \mathscr{F}$. An operator $S : \mathscr{U} \to \mathscr{V}$ is a shift operator if and only if it satisfies the following conditions:*

(a) *S is disjointness preserving;*

(b) *S is bounded;*

(c) *if $u \in \mathscr{U}$ and $|u|$ is a fragment of $1_{\mathscr{E}}$ then $|Su|$ is a fragment of $1_{\mathscr{F}}$;*

(d) *S is wide at $1_{\mathscr{E}}$.*

$\lhd$ Necessity of conditions (a)–(d) follows immediately from Theorem 5.6.9. Assume that an operator S satisfies conditions (a)–(d). Denote by $|S|$ the exact dominant of S and show first that $|S|: E \to F$ is a shift operator by verifying conditions (a)–(d) of Theorem 5.6.9. Condition (a) ensues from Corollary 5.4.4, condition (b) is ensured by the fact that $|S|$ is positive, condition (c) follows from Proposition 5.4.12, and condition (d) from Lemma 5.6.2. Thus, $|S|$ is a shift operator. Since the shadows of S and $|S|$ coincide (see Proposition 5.4.4), the operator $|S|$ is the restriction of the shift of S onto E.

Assign $\mathscr{U}_1 := \{u \in \mathscr{U} : |u| \text{ is a fragment of } 1_{\mathscr{E}}\}$, consider an arbitrary element $u \in \mathscr{U}_1$, and show that $|Su| = |S||u|$. For the sake of convenience, we assume that $|u| = 1_{\mathscr{E}}$ and $|S|1_{\mathscr{E}} = 1_{\mathscr{F}}$. This assumption does not restrict generality, since $S\big[\langle u\rangle\mathscr{U}\big] \subset \langle |S||u|\rangle\mathscr{V}$, and, therefore, we may regard S as an operator from $\langle u\rangle\mathscr{U}$ into $\langle |S||u|\rangle\mathscr{V}$. Denote the projection $\langle Su\rangle^{\perp}$ by ρ. Since $|Su|$ is a fragment of $1_{\mathscr{F}}$, it is sufficient to show that $\rho = 0$. Assume to the contrary that $\rho \neq 0$. Then, by Proposition 5.4.12, there is an element $u_1 \in \mathscr{U}$ such that $|u_1| = 1_{\mathscr{E}}$ and $\rho S u_1 \neq 0$. Assign $e := |u_1 + 3u|$. The equalities $|u| = |u_1| = 1_{\mathscr{E}}$ readily imply $2 1_{\mathscr{E}} \leqslant e \leqslant 4 1_{\mathscr{E}}$; hence, $\frac{1}{4}1_{\mathscr{E}} \leqslant 1/e \leqslant \frac{1}{2}1_{\mathscr{E}}$. The last inequality proves that the product $\bar{u} := (1/e)(u_1 + 3u)$ is defined in $\mathscr{U}$. By using Proposition 5.6.13 and the equality $\rho S u = 0$, we obtain:

$$
\begin{aligned}
\rho|S\bar{u}| &= \rho\big|S\big((1/e)(u_1+3u)\big)\big| = \rho|S|(1/e)\big|S(u_1+3u)\big| \\
&= |S|(1/e)|\rho S u_1 + 3\rho S u| = \rho|S|(1/e)|Su_1| = \langle \rho S u_1\rangle |S|(1/e).
\end{aligned}
$$

Observe that $|\bar{u}| = 1_{\mathscr{E}}$ and, consequently, $|S\bar{u}|$ is a fragment of $1_{\mathscr{F}}$. Therefore, the relations

$$
\rho|S\bar{u}| = \langle \rho S u_1\rangle |S|(1/e) \geqslant \langle \rho S u_1\rangle |S|\Big(\frac{1}{4}1_{\mathscr{E}}\Big) = \frac{1}{4}\langle \rho S u_1\rangle 1_{\mathscr{F}},
$$

yield the inequality $\rho|S\bar{u}| \geqslant \langle \rho S u_1\rangle 1_{\mathscr{F}}$, which contradicts the following relations:

$$
\langle \rho S u_1\rangle 1_{\mathscr{F}} \leqslant \rho|S\bar{u}| = \langle \rho S u_1\rangle |S|(1/e) \leqslant \langle \rho S u_1\rangle |S|\Big(\frac{1}{2}1_{\mathscr{E}}\Big) = \frac{1}{2}\langle \rho S u_1\rangle 1_{\mathscr{F}}.
$$

Thus, we established that $|Su| = |S||u|$ for all $u \in \mathscr{U}_1$. Denote by h the shadow of S. As is known, h coincides with the shadow of $|S|$. Then, applying Corollary 5.4.9 (2) to the operators $S\colon \mathscr{U} \to \mathscr{V}$ and $|S|_{\mathscr{U}}\colon \mathscr{U} \to |S|\mathscr{U}$, we obtain the equality $|Su| = |S||u|$ for all $u \in \mathscr{U}$ with norm in $h\mathscr{E}_1$. It remains to observe that $\{u \in \mathscr{U} : |u| \in h\mathscr{E}_1\} = \mathscr{U}$, since $E = \operatorname{dom}|S| \subset h\mathscr{E}_1$. $\triangleright$

5.7. Weighted Shift Operator

Weighted shift operators considered in this section are the compositions $W \circ S \circ w$ of two orthomorphisms w and W and a shift operator S. Representability of a disjointness preserving operator as such a composition is related to existence of a bounded set on which the operator is wide. In addition to this criterion, we also suggest some sufficient conditions for representability of an operator in the form $W \circ S \circ w$. The main result of the present section is representation of an arbitrary disjointness preserving operator as the strongly disjoint sum of weighted shift operators. Thus, operators of the form $W \circ S \circ w$ play the role of simple elements, from which wider classes of operators are constituted. In the sequel, this fact will allow us to construct one of analytic representations of disjointness preserving operators.

Throughout the section, E and F and order-dense ideals of the K-spaces $\mathscr{E}$ and $\mathscr{F}$. In the spaces $\mathscr{E}$ and $\mathscr{F}$, we fix order unities $1_{\mathscr{E}}$ and $1_{\mathscr{F}}$ and consider the multiplication that makes the spaces commutative ordered algebras with unities $1_{\mathscr{E}}$ and $1_{\mathscr{F}}$, respectively (see 5.1.12). We recall that orthomorphisms in the K-spaces under consideration are multiplication operators and we identify them with the corresponding multipliers (see 5.5.12). The ideal of the K-space $\mathscr{E}$ generated by the element $1_{\mathscr{E}}$ is denoted by $\mathscr{E}_1$. Observe that some notions introduced in this section depend on a concrete choice of unities $1_{\mathscr{E}}$ and $1_{\mathscr{F}}$.

5.7.1. We say that a linear operator $T\colon E \to F$ is a *weighted shift operator* if there exist order-dense ideals $E' \subset \mathscr{E}$ and $F' \subset \mathscr{F}$, orthomorphisms $w\colon E \to E'$ and $W\colon F' \to F$, and a shift operator $S\colon E' \to F'$ such that $T = W \circ S \circ w$, i.e., the diagram

$$\begin{array}{ccc} E & \xrightarrow{T} & F \\ {\scriptstyle w}\downarrow & & \uparrow{\scriptstyle W} \\ E' & \xrightarrow{S} & F' \end{array}$$

is commutative. The composition $W \circ S \circ w$ is called a *WSW-representation* of T, and the operators W, S, and w are respectively called the *outer weight*, the *shift*, and the *inner weight* of the representation $W \circ S \circ w$.

Observe that, in view of Theorem 5.4.1, a regular operator $T\colon E \to F$ is a weighted shift operator if and only if so is its modulus $|T|$. Moreover, if one of the operators T or $|T|$ admits a WSW-representation then the other one admits a WSW-representation with the same shift and inner weight. Thus, while discussing the question of whether an operator is a weighted shift operator, we may always assume the operator positive.

From the viewpoint of the above definition, the property of a mapping to be a weighted shift operator depends on the choice of $1_{\mathscr{E}}$ and $1_{\mathscr{F}}$. Actually, there is no such a dependence. Indeed, let an operator T admit a WSW-representation

$$Te = W * S(w * e) \quad (e \in E),$$

where " $*$ " is the multiplication corresponding to the unities $1_{\mathscr{E}}$ and $1_{\mathscr{F}}$. Then, after replacing $1_{\mathscr{E}}$ and $1_{\mathscr{F}}$ by $1'_{\mathscr{E}}$ and $1'_{\mathscr{F}}$ and introducing the new multiplication " $\cdot$ " in the K-spaces under consideration, the operator T remains a weighted shift operator and admits the WSW-representation

$$Te = W \cdot S'(w' \cdot e) \quad (e \in E),$$

where

$$S'x = (1'_{\mathscr{F}}/1_{\mathscr{F}}) \cdot S(1_{\mathscr{E}} \cdot x) \quad \big(x \in (\operatorname{dom} S)/1_{\mathscr{E}}\big)$$

and $w' = w/1^2_{\mathscr{E}}$ (here, the division and the power operation also correspond to the new unities). Thus, the notion of a weighted shift operator $T\colon E \to F$ makes sense for "pure" K-spaces E and F, without any dependence on their embedding into universally complete K-spaces and introducing a multiplicative structure. In particular, this implies that a positive operator $T\colon E \to F$ is a weighted shift operator if and only if it can be made a shift operator by an appropriate choice of unities $1_{\mathscr{E}}$ and $1_{\mathscr{F}}$.

Simple examples show that a single weighted shift operator can have different WSW-representations. However, variety of the components of a WSW-representation for a given operator T is naturally restricted by their connection with T and with each other. Two main aspects of this connection are reflected in the following proposition.

Proposition. *Let $T\colon E \to F$ be a weighted shift operator and let $W \circ S \circ w$ be a WSW-representation of it. Assign $\rho := \langle \operatorname{im} T \rangle$.*

(1) Denote the shift of T by S_T. Then S_T extends $\rho \circ S$ and the equality $W \circ S \circ w = W \circ S_T \circ w$ holds.

(2) Identify w and W with the corresponding elements of $\mathscr{E}$ and $\mathscr{F}$ and assign $W_T := o\text{-}\lim_{\pi \in \Pi} T\pi(1_{\mathscr{E}}/w) \in \mathscr{F}$, where $\Pi = \{\pi \in \operatorname{Pr}(\mathscr{E}) : \pi(1_{\mathscr{E}}/w) \in E\}$. Then $\rho W = W_T$ and $W \circ S \circ w = W_T \circ S \circ w$.

$\lhd$ Assertion (1) readily follows from 5.6.7 and 5.6.8. Let us prove (2). Due to the obvious equality $T \circ \langle w \rangle^{\perp} = 0$, we do not restrict generality by assuming that $\langle w \rangle = \langle 1 \rangle$. Then

$$o\text{-}\lim_{\pi \in \Pi} T\pi(1_{\mathscr{E}}/w) = o\text{-}\lim_{\pi \in \Pi} WS_T w\pi(1_{\mathscr{E}}/w) = o\text{-}\lim_{\pi \in \Pi} WS_T\pi 1_{\mathscr{E}} = \Big(\sup_{\pi \in \Pi} h(\pi)\Big)W,$$

where h is the shadow of T. Since $\rho = h(1)$, it is sufficient to show the relation $\sup_{\pi \in \Pi} h(\pi) = h(1)$. From $E \subset \operatorname{dom}(S_T \circ w)$ it follows that $w[E] \subset \operatorname{dom} S_T = h\mathscr{E}_1$ and, hence, $E \subset h\mathscr{E}_{1/w}$. It remains to employ Lemma 5.6.6. $\rhd$

Thus, a *WSW*-representation of a concrete operator determines to a great extent by the choice of the inner weight. Observe that every weighted shift operator admits a *WSW*-representation with positive inner weight. Indeed, consider an arbitrary *WSW*-representation $W \circ S \circ w$. Identifying the orthomorphism w with an element of $\mathscr{E}$ (see 5.5.12), denote the projection $\langle w^+ \rangle \in \operatorname{Pr}(E)$ by π and assign $\rho := \big\langle S(\pi 1_{\mathscr{E}}) \big\rangle$. Then

$$\begin{aligned} W \circ S \circ w &= W \circ S \circ \big(\pi|w| - \pi^{\perp}|w|\big) \\ &= W \circ \big(\rho \circ S \circ |w| - \rho^{\perp} \circ S \circ |w|\big) = (\rho W - \rho^{\perp} W) \circ S \circ |w|. \end{aligned}$$

Remark. If $W \circ S \circ w$ is a *WSW*-representation of an operator T with positive inner weight w, then the operators T^+, T^-, and $|T|$ admit the following *WSW*-representations: $T^+ = W^+ \circ S \circ w$, $T^- = W^- \circ S \circ w$, and $|T| = |W| \circ S \circ w$.

5.7.2. Theorem. *Let w be an arbitrary positive element of $\mathscr{E}$. A linear operator $T\colon E \to F$ admits a WSW-representation with inner weight w if and only if it is disjointness preserving, regular, and wide at the element $1_{\mathscr{E}}/w$.*

$\lhd$ Necessity ensues from Proposition 5.7.1 (2). Let us prove sufficiency. Suppose that a disjointness preserving operator $T\colon E \to F$ is wide at $1_{\mathscr{E}}/w$. Without loss of generality, we may assume that the operator T is positive. Assign $\Pi := \{\pi \in \mathrm{Pr}(\mathscr{E}) : \pi(1_{\mathscr{E}}/w) \in E\}$ and denote by W the orthomorphism of multiplication by $\sup_{\pi\in\Pi} T\pi(1_{\mathscr{E}}/w) \in \mathscr{F}$. Consider the composition $(1_{\mathscr{F}}/W)\circ T\circ(1_{\mathscr{E}}/w)$ as an operator from $w[E]$ into $\mathscr{F}$ and denote it by S. By proving that S is a shift operator, we will obtain the desired WSW-representation $W\circ S\circ w$ for T. In accordance with Theorem 5.6.9, it is sufficient to show that the operator S satisfies conditions (a)–(d) presented in the statement of that theorem. Verification of the conditions causes no difficulties. $\rhd$

We say that a subset of the K-space E is *$\mathscr{E}$-bounded* if it is bounded in $\mathscr{E}$. A subset $\mathscr{U}_0$ of an LNS over E is called *$\mathscr{E}$-bounded* if the set $\{|u_0| : u_0 \in \mathscr{U}_0\}$ is $\mathscr{E}$-bounded.

Corollary. *A linear operator $T\colon E \to F$ is a weighted shift operator if and only if it is disjointness preserving, regular, and wide on some $\mathscr{E}$-bounded subset of E.*

$\lhd$ If the operator T is wide on a set $D \subset E$ and an element $e \in \mathscr{E}$ is such that $|d| \leqslant e$ for all $d \in D$, then the operator T is wide at e and, in view of the last theorem, it admits a WSW-representation with inner weight $1_{\mathscr{E}}/e$. $\rhd$

5.7.3. Proposition. *Assume that regular operators $T, \overline{T}\colon E \to F$ are disjointness preserving and satisfy the inequality $|T| \leqslant |\overline{T}|$. Then T is a weighted shift operator if and only if so is $\overline{T}$. Moreover, the following assertions are true:*

(1) *If $\overline{W}\circ\overline{S}\circ\overline{w}$ is a WSW-representation of $\overline{T}$ then the operator T admits a WSW-representation of the form $W\circ\overline{S}\circ\overline{w}$, where $|W| \leqslant |\overline{W}|$.*

(2) *If $W\circ S\circ w$ is a WSW-representation of T then the operator $\overline{T}$ admits a WSW-representation of the form $\overline{W}\circ\overline{S}\circ w$, where $\langle \mathrm{im}\, T\rangle|W| \leqslant |\overline{W}|$.*

$\lhd$ Without loss of generality, we may assume that the operators T and $\overline{T}$ are positive.

(1) The claim is ensured by Corollary 5.5.19.

(2) Assume that T admits a WSW-representation $W\circ S\circ w$ and assign $\rho := \langle \mathrm{im}\, T\rangle$. According to Theorem 5.7.2, the operator T is wide at the element $1_{\mathscr{E}}/w$. Then the operator $\overline{T}$ also has this property and, by the same Theorem 5.7.2, it

admits a WSW-representation $\overline{W} \circ \overline{S} \circ w$. The desired interrelation between W and $\overline{W}$ ensues from Proposition 5.7.1. $\triangleright$

5.7.4. In accordance with Theorem 5.7.2, it seems interesting to study situations in which an operator $T: E \to F$ turns out to be wide on some $\mathscr{E}$-bounded subset of E. Without touching the general problem, we will only discuss several particular cases.

First of all, we point out a trivial corollary to Theorem 5.7.2: if $\{Te\}^{\perp\perp} = (\operatorname{im} T)^{\perp\perp}$ for some element $e \in E$ then T is a weighted shift operator (and it admits a WSW-representation with inner weight $1_{\mathscr{E}}/e$). In particular, the following assertion holds:

Proposition. *If there exists a strong order unity e in the K-space E then every disjointness preserving regular operator $T: E \to F$ is a weighted shift operator and admits a WSW-representation with inner weight $1_{\mathscr{E}}/e$.*

Of course, the indicated cases admit generalizations. For instance, since every set of pairwise disjoint elements in E is $\mathscr{E}$-bounded, we have the following assertion:

Proposition. *Let $T: E \to F$ be a disjointness preserving regular operator. If $\{Te_\xi : \xi \in \Xi\}^{\perp\perp} = (\operatorname{im} T)^{\perp\perp}$ for some family $(e_\xi)_{\xi\in\Xi}$ of pairwise disjoint elements in E, then T is a weighted shift operator.*

The condition stated in the last proposition is not necessary. indeed, let $\mathscr{E} = C_\infty(P)$, where P is an extremally disconnected compact space containing a nonisolated point $p \in P$. Denote by E the order-dense ideal $\{e \in \mathscr{E} : e(p) = 0\}$ of the K-space $\mathscr{E}$. Consider the set $Q := P\backslash\{p\}$ and let $\mathscr{F}$ be the K-space of all real-valued functions defined on Q. Define an operator $T: E \to \mathscr{F}$ as follows: $Te = e\big|_Q$. Obviously, the operator T is wide on the $\mathscr{E}$-bounded set $\{e \in E : |e| \leqslant 1\}$ (and, therefore, it is a weighted shift operator), but the family $(e_\xi)_{\xi\in\Xi}$ mentioned in the statement of the last proposition does not exist.

Another class of weighted shift operators resulted by combining Lemma 5.6.6 and Corollaries 5.4.6 and 5.6.5.

Theorem. *Every disjointness preserving sequentially o-continuous regular operator $T: E \to F$ is a weighted shift operator. Moreover, for every order unity $w \in \mathscr{E}$, such an operator T admits a WSW-representation with inner weight w.*

5.7.5. It is known that not every disjointness preserving regular operator is a weighted shift operator. For the sake of completeness, we will present here the corresponding example from [19], the more so as the example is, in a sense, typical (see below).

Let Q be an extremally disconnected compact space without isolated points. In this case, we can find an order-dense ideal $E \subset C_\infty(Q)$, a family $(e_\xi)_{\xi\in\Xi}$ in E, and a family $(q_\xi)_{\xi\in\Xi}$ in Q so that the following conditions be satisfied: the set $\{q_\xi : \xi \in \Xi\}$ is dense in Q, $e_\xi(q_\xi) = \infty$ for all $\xi \in \Xi$, and, for each $e \in E$, the number set $\{(e/e_\xi)(q_\xi) : \xi \in \Xi\}$ is bounded. Then the operator $T\colon E \to \ell^\infty(\Xi)$ acting by the rule $(Te)(\xi) = (e/e_\xi)(q_\xi)$ is disjointness preserving and regular (even positive), but is not a weighted shift operator.

The above construction of an operator T possesses the following property: if we denote by ρ_ξ the operator of multiplication by the characteristic function $\chi_{\{\xi\}}$, then we obtain a partition of unity $(\rho_\xi)_{\xi\in\Xi}$ in the algebra $\Pr(\ell^\infty(\Xi))$ such that all fragments of the form $\rho_\xi \circ T$ are weighted shift operators. It turns out that of all disjointness preserving regular operators are structured in the same way.

Theorem. *Let $T\colon E \to F$ be a disjointness preserving regular operator. Then there exists a partition of unity $(\rho_\xi)_{\xi\in\Xi}$ in the algebra $\Pr(F)$ such that, for each $\xi \in \Xi$, the composition $\rho_\xi \circ T$ is a weighted shift operator. Moreover, the projections ρ_ξ can be taken so that each composition $\rho_\xi \circ T$ admit a WSW-representation with inner weight $1_{\mathscr{E}}/e_\xi$, where e_ξ is a positive element of E. In this case, the operator T is decomposed into the strongly disjoint sum*

$$T = \bigoplus_{\xi\in\Xi} W \circ \rho_\xi S \circ (1_{\mathscr{E}}/e_\xi),$$

where S is the shift of T and $W\colon \mathscr{F} \to \mathscr{F}$ is the orthomorphism of multiplication by $o\text{-}\sum_{\xi\in\Xi} \rho_\xi T e_\xi$.

$\lhd$ By applying the exhaustion principle to the relation

$$\sup_{e\in E^+} \langle Te \rangle = \langle \operatorname{im} T \rangle,$$

we obtain a disjoint family $(\rho_\xi)_{\xi\in\Xi}$ in the algebra $\Pr(F)$ and a family $(e_\xi)_{\xi\in\Xi}$ of positive elements in E such that $\sup_{\xi\in\Xi} \rho_\xi \langle Te_\xi \rangle = \langle \operatorname{im} T \rangle$. After adding the projection

$\langle \operatorname{im} T\rangle^{\perp}$ to the family $(\rho_\xi)_{\xi\in\Xi}$ and the zero element to the family $(e_\xi)_{\xi\in\Xi}$, we make $(\rho_\xi)_{\xi\in\Xi}$ a partition of unity and preserve the relation $\sup_{\xi\in\Xi}\rho_\xi\langle Te_\xi\rangle = \langle \operatorname{im} T\rangle$. By Theorem 5.7.2, for each $\xi \in \Xi$, the composition $\rho_\xi \circ T$ is a weighted shift operator and admits a WSW-representation with inner weight $1_{\mathscr{E}}/e_\xi$. If S is the shift of T then the shift of $\rho_\xi \circ T$ is equal to $\rho_\xi S$ (see 5.6.8); thus, using Proposition 5.7.1, we conclude that $\rho_\xi \circ T = \rho_\xi Te_\xi \circ \rho_\xi S \circ (1_{\mathscr{E}}/e_\xi)$. ▷

5.7.6. Let $\mathscr{U}$ be a BKS over an order-dense ideal $E \subset \mathscr{E}$ and let $\mathscr{V}$ be a BKS over an order-dense ideal $F \subset \mathscr{F}$. We say that a linear operator $T\colon \mathscr{U} \to \mathscr{V}$ is a *weighted shift operator* if there exist a BKS $\mathscr{U}'$ over an order-dense ideal $E' \subset \mathscr{E}$, a BKS $\mathscr{V}'$ over an order-dense ideal $F' \subset \mathscr{F}$, orthomorphisms $w\colon \mathscr{U} \to \mathscr{U}'$ and $W\colon \mathscr{V}' \to \mathscr{V}$, and a shift operator $S\colon \mathscr{U}' \to \mathscr{V}'$ such that $T = W \circ S \circ w$, i.e., the diagram

$$\begin{array}{ccc} \mathscr{U} & \xrightarrow{T} & \mathscr{V} \\ {\scriptstyle w}\big\downarrow & & \big\uparrow{\scriptstyle W} \\ \mathscr{U}' & \xrightarrow{S} & \mathscr{V}' \end{array}$$

is commutative. As in the case of an operator in K-spaces, the composition $W\circ S\circ w$ is called a *WSW-representation* of T and the operators W, S and w are respectively called the *outer weight*, the *shift*, and the *inner weight* of the representation $W\circ S\circ w$. Of course, use of the terminology of 5.7.1 in the case of operators in LNSs is not quite correct, since a K-space is a particular case of an LNS. Therefore, in order to avoid confusion, we sometimes call a weighted shift operator *scalar* or *vector*, referring to definition 5.7.1 or 5.7.6, respectively. By analogous reasons, we speak about scalar or vector WSW-representations. A vector WSW-representation $W \circ S \circ w$ of an operator $T\colon \mathscr{U} \to \mathscr{V}$ will be called *semivector* if w is a scalar orthomorphism (see 5.5.17), i.e., $\mathscr{U}$ and $\mathscr{U}'$ are order-dense ideals of the same BKS over $\mathscr{E}$ and the orthomorphism w acts by the rule $u \mapsto eu$ for some fixed orthomorphisms $e \in \operatorname{Orth}(E, E')$.

Theorem. *Let $\mathscr{U}$ be a BKS over an order-dense ideal $E \subset \mathscr{E}$ and let $\mathscr{V}$ be a BKS over an order-dense ideal $F \subset \mathscr{F}$. A linear operator $T\colon \mathscr{U} \to \mathscr{V}$ is a vector weighted shift operator if and only if it is dominated and its exact dominant $|T|\colon E \to F$ is a scalar weighted shift operator. Moreover, the following assertions hold:*

(1) *If $\overline{W} \circ \overline{S} \circ \overline{w}$ is a vector WSW-representation of T then $|T|$ admits a scalar WSW-representation $W \circ |\overline{S}| \circ |\overline{w}|$ such that $0 \leqslant W \leqslant |\overline{W}|$.*

(2) *Let $W \circ S \circ w$ be a scalar WSW-representation of $|T|$ with positive weights W and w. Then T admits a semivector WSW-representation $\overline{W} \circ \overline{S} \circ \overline{w}$ such that $|\overline{W}| = W$, $|\overline{S}| = S$, and $\overline{w}$ is the orthomorphism of multiplication by w.*

$\lhd$ (1) The claim readily follows from 5.7.3 (1).

(2) Suppose that $W \circ S \circ w$ is a scalar *WSW*-representation of $|T|$, where $w \colon E \to E'$, $S \colon E' \to F'$ and $W \colon F' \to F$. Let $m\mathscr{U}$ be the universal completion of $\mathscr{U}$, let $\mathscr{U}'$ be the ideal $\{u \in m\mathscr{U} : |u| \in E'\}$ of the BKS $m\mathscr{U}$, and let $\overline{w} \colon \mathscr{U} \to \mathscr{U}'$ be the orthomorphism of multiplication by w. Denote by $\mathscr{V}'$ the o-completion of the norm transformation of $\mathscr{U}'$ by means of S (see 5.1.11) and consider the corresponding operator of norm transformation $\overline{S} \colon \mathscr{U}' \to \mathscr{V}'$. Now, we are to construct an orthomorphism $\overline{W} \colon \mathscr{V}' \to \mathscr{V}$.

Assign $\mathscr{V}_0' := (\overline{S} \circ \overline{w})[\mathscr{U}]$ and define a linear operator $\overline{W}_0 : \mathscr{V}_0' \to \mathscr{V}$ as follows: $\overline{W}_0(\overline{S}\overline{w}u) := Tu$. Such a definition is sound, since the equality $\overline{S}\overline{w}u_1 = \overline{S}\overline{w}u_2$ implies

$$
\begin{aligned}
|Tu_1 - Tu_2| &\leqslant |T||u_1 - u_1| = WSw|u_1 - u_1| \\
&= WS|\overline{w}u_1 - \overline{w}u_2| = W\left|\overline{S}\overline{w}u_1 - \bar{S}\overline{w}u_2\right| = 0.
\end{aligned}
$$

Assign $\rho := \langle \operatorname{im} T \rangle$. Since $\rho \leqslant \langle (\overline{S} \circ \overline{w})[\mathscr{U}] \rangle$ and $\overline{w}[\mathscr{U}] = \{v' \in \mathscr{V}' : |v'| \in w[E]\}$, the operator $\rho \circ \overline{S}$ is wide on the ideal $w[E] \subset E'$. Consequently, by Proposition 5.6.2((2)$\Rightarrow$(3)), the set $\mathscr{V}_0' = (\rho \circ \overline{S})[\overline{w}[\mathscr{U}]]$ approximates $(\rho \circ \overline{S})[\mathscr{U}']$. The latter set, by the definition of the norm transformation $S\mathscr{U}'$, approximates the set $\rho[S\mathscr{U}']$, which in turn approximates $\rho[\mathscr{V}']$. Therefore, in view of 5.2.2, the set $\mathscr{V}_0'$ approximates $\rho[\mathscr{V}']$. Obviously, $|\overline{W}_0 v_0'| \leqslant W|v_0'|$ for all $v_0' \in \mathscr{V}_0'$. According to Corollary 5.4.8, the operator $\overline{W}_0$ admits a (unique) linear extension $\overline{W}_1 : \rho[\mathscr{V}'] \to \mathscr{V}$ such that $|\overline{W}_1 v'| \leqslant W|v'|$ for all $v' \in \mathscr{V}'$. Then the composition $\overline{W}_1 \circ \rho \colon \mathscr{V}' \to \mathscr{V}$ satisfies the inequality $|\overline{W}_1 \circ \rho| \leqslant W$ and, consequently, it is an orthomorphism. Thus, we have already constructed a *WSW*-representation $(\overline{W}_1 \circ \rho) \circ \overline{S} \circ \overline{w}$ of the operator T. However, we cannot assign $\overline{W} := \overline{W}_1 \circ \rho$ at this moment, since the equality $|\overline{W}| = W$ will not be guaranteed.

For all positive $e \in E$, we have

$$\begin{aligned}
\left|\overline{W}_1 \circ \rho\right| Swe &= \sup\left\{\left|\overline{W}_1 \rho v'\right| : v' \in \mathscr{V}',\ \left|v'\right| = Swe\right\} \\
&\geqslant \sup\left\{\rho\left|\overline{W}_0 v_0'\right| : v_0' \in \mathscr{V}_0',\ \left|v_0'\right| = Swe\right\} \\
&= \sup\left\{\rho\left|\overline{W}_0 \overline{S}\overline{w} u\right| : u \in \mathscr{U},\ \left|\overline{S}\overline{w}u\right| = Swe\right\} \\
&= \sup\left\{|Tu| : Sw|u| = Swe\right\} \\
&\geqslant \sup\left\{|Tu| : |u| = e\right\} = |T|e = WSwe,
\end{aligned}$$

whence $\left|\overline{W}_1 \circ \rho\right| Swe = WSwe$ by the inequality $\left|\overline{W}_1 \circ \rho\right| \leqslant W$. Thus, $W \circ S \circ w$ and $\left|\overline{W}_1 \circ \rho\right| \circ S \circ w$ are two *WSW*-representations of the operator $|T|$. Hence, according to Proposition 5.7.1 (2), the equality $\left|\overline{W}_1 \circ \rho\right| = \rho W$ holds. To ensure the equality $\left|\overline{W}\right| = W$, it is sufficient to define $\overline{W}$ as the sum of the orthomorphism $\overline{W}_1 \circ \rho$ and some "inactive" supplement with norm $\rho^{\perp} W$. Proposition 5.5.13 implies existence of an orthomorphism $\overline{W}_2 \in \mathrm{Orth}(\mathscr{U}, \mathscr{V})$ such that $\left|\overline{W}_2\right| = W$. We assign $\overline{W} := \overline{W}_1 \circ \rho + \overline{W}_2 \circ \rho^{\perp}$. $\triangleright$

REMARK. (1) The inequality $W \leqslant \left|\overline{W}\right|$ presented in assertion (1) of the last theorem can be strict. In other words, the equality $|T| = \left|\overline{W}\right| \circ \left|\overline{S}\right| \circ |\overline{w}|$ cannot be guaranteed for every *WSW*-representation $T = \overline{W} \circ \overline{S} \circ \overline{w}$. (A simple counterexample can be given in the case when $\mathscr{U}$ and $\mathscr{V}$ are Banach spaces.) However, (2) implies that every weighted shift operator $T \colon \mathscr{U} \to \mathscr{V}$ admits a *WSW*-representation $\overline{W} \circ \overline{S} \circ \overline{w}$ such that $|T| = \left|\overline{W}\right| \circ \left|\overline{S}\right| \circ |\overline{w}|$.

(2) From the last theorem it follows that each vector weighted shift operator admits a semivector *WSW*-representation. Moreover, if an operator admits a vector *WSW*-representation with inner weight w then it admits a semivector *WSW*-representation with inner weight the operator of multiplication by $|w|$.

(3) If we consider each of the K-spaces E and F as a BKS (over itself) then the exact dominant of every regular operator $T \colon E \to F$ coincides with its modulus $|T|$. This observation and the last theorem allow us to conclude the following: a mapping $T \colon E \to F$ is a vector weighted shift operator if and only if it is a scalar weighted shift operator. This fact justifies correctness of using the common term "weighted shift operator" for operators in BKSs as well as for operators in K-spaces.

5.7.7. Each of the assertions stated in the following theorem readily follows from a similar "scalar" assertion (see 5.7.1–5.7.4) and Theorem 5.7.6.

Theorem. *Let $\mathscr{U}$ be a BKS over an order-dense ideal $E \subset \mathscr{E}$ and let $\mathscr{V}$ be a BKS over an order-dense ideal $F \subset \mathscr{F}$.*

(1) *The property of a mapping $T\colon \mathscr{U} \to \mathscr{V}$ to be a weighted shift operator does not depend on choosing unities $1_{\mathscr{E}}$ and $1_{\mathscr{F}}$.*

(2) *A linear operator $T\colon \mathscr{U} \to \mathscr{V}$ is a weighted shift operator if and only if it is disjointness preserving, bounded, and satisfies the relation $T[\mathscr{U}_0]^{\perp\perp} = T[\mathscr{U}]^{\perp\perp}$ for some $\mathscr{E}$-bounded subset $\mathscr{U}_0 \subset \mathscr{U}$.*

(3) *Let w be an arbitrary positive element of $\mathscr{E}$. A linear operator $T\colon \mathscr{U} \to \mathscr{V}$ admits a WSW-representation with inner weight of norm w if and only if it is disjointness preserving, bounded, and wide at the element $1_{\mathscr{E}}/w$.*

(4) *Let $T\colon \mathscr{U} \to \mathscr{V}$ be a disjointness preserving bounded operator. If the relation $\{Tu\}^{\perp\perp} = (\operatorname{im} T)^{\perp\perp}$ holds for some element $u \in \mathscr{U}$ then T is a weighted shift operator and admits a WSW-representation with inner weight of norm $1_{\mathscr{E}}/|u|$.*

(5) *If there exists a strong order unity e in the K-space E then every disjointness preserving bounded operator $T\colon \mathscr{U} \to \mathscr{V}$ is a weighted shift operator and admits a WSW-representation with inner weight of norm $1_{\mathscr{E}}/e$.*

(6) *Every disjointness preserving sequentially o-continuous bounded operator $T\colon \mathscr{U} \to \mathscr{V}$ is a weighted shift operator. Moreover, for every order unity $w \in \mathscr{E}$, such an operator T admits a WSW-representation with inner weight of norm w.*

5.7.8. Theorem. *Suppose that $\mathscr{U}$ is a BKS over an order-dense ideal $E \subset \mathscr{E}$, $\mathscr{V}$ is a BKS over an order-dense ideal $F \subset \mathscr{F}$, $m\mathscr{U}$ and $m\mathscr{V}$ are universal completions of $\mathscr{U}$ and $\mathscr{V}$, and $T\colon \mathscr{U} \to \mathscr{V}$ is a disjointness preserving bounded operator. Then there exists a partition of unity $(\rho_\xi)_{\xi\in\Xi}$ in the algebra $\mathrm{Pr}(\mathscr{V})$ such that, for each $\xi \in \Xi$, the composition $\rho_\xi \circ T$ is a weighted shift operator.*

The projections ρ_ξ can be chosen so that each composition $\rho_\xi \circ T$ admit WSW-representation with inner weight of norm $1_{\mathscr{E}}/e_\xi$, where e_ξ is a positive element of E.

For each $\xi \in \Xi$, assign $E_\xi := \{e/e_\xi : e \in E\}$ and $\mathscr{U}_\xi := \{u \in m\mathscr{U} : |u| \in E_\xi\}$, where $m\mathscr{U}$ is the universal completion of $\mathscr{U}$, and denote by $w_\xi : \mathscr{U} \to \mathscr{U}_\xi$ the scalar orthomorphism of multiplication by $1_{\mathscr{E}}/e_\xi$. Then there exist a BKS $\mathscr{V}'$ over $\mathscr{F}$, strongly disjoint shift operators $S_\xi\colon \mathscr{U}_\xi \to \mathscr{V}'$ ($\xi \in \Xi$), and an orthomorphism $W\colon \mathscr{V}' \to m\mathscr{V}$ such that the operators T and $|T|$ decompose into the following *strongly disjoint sums*:

$$T = \bigoplus_{\xi \in \Xi} W \circ S_\xi \circ w_\xi, \quad |T| = \bigoplus_{\xi \in \Xi} |W| \circ |S_\xi| \circ |w_\xi|.$$

$\lhd$ Consider an arbitrary disjointness preserving bounded operator $T\colon \mathscr{U} \to \mathscr{V}$. By Theorem 5.7.5, there exists a partition of unity $(\rho_\xi)_{\xi\in\Xi}$ in the algebra $\mathrm{Pr}(F)$ such that, for each $\xi \in \Xi$, the composition $\rho_\xi \circ |T|$ is a weighted shift operator and, moreover, admits a *WSW*-representation with inner weight $1_{\mathscr{E}}/e_\xi$, where e_ξ is a positive element of E. Define BKSs $\mathscr{U}_\xi$ and orthomorphisms $w_\xi\colon \mathscr{U} \to \mathscr{U}_\xi$ in the same way as in the statement of the theorem being proved. By Theorem 5.7.6, for each $\xi \in \Xi$, there exist a BKS $\mathscr{V}_\xi$ over an order-dense ideal $F_\xi \subset \rho_\xi[\mathscr{F}]$, a shift operator $S_\xi\colon \mathscr{U}_\xi \to \mathscr{V}_\xi$, and an orthomorphism $W_\xi\colon \mathscr{V}_\xi \to \rho_\xi[\mathscr{V}]$ such that $\rho_\xi \circ T = W_\xi \circ S_\xi \circ w_\xi$ and $\rho_\xi \circ |T| = |W_\xi| \circ |S_\xi| \circ |w_\xi|$. In order to complete the proof, it remains to construct the desired BKS $\mathscr{V}'$ and "glue" the orthomorphisms W_ξ together to obtain a single orthomorphism W.

Assign $\mathscr{V}_0' := \bigoplus_{\xi\in\Xi} \mathscr{V}_\xi$ (see 5.1.10) and denote by $\mathscr{V}'$ a universal completion of the BKS $\mathscr{V}_0'$. Naturally identifying $\mathscr{V}_\xi$ and $\rho_\xi[\mathscr{V}_0']$, we regard S_ξ as an operator from $\mathscr{U}_\xi$ into $\mathscr{V}'$. For each element $v_0' = (v_\xi)_{\xi\in\Xi} \in \mathscr{V}_0'$, assign $W_0(v') := o\text{-}\sum_{\xi\in\Xi} W_\xi(v_\xi) \in m\mathscr{V}$. Due to Corollary 5.4.8, the orthomorphism $W_0 : \mathscr{V}_0' \to m\mathscr{V}$ admits a unique extension to an orthomorphism $W\colon \mathscr{V}' \to m\mathscr{V}$. $\rhd$

5.8. Representation of Disjointness Preserving Operators

Constructing analytic representations of disjointness preserving operators is an old tradition. This question was studied by everyone who was interested in these operators from the abstract point of view. Representation of various classes of operators as composition and multiplication mappings is presented, for instance, in [2, 3, 8, 9, 14, 19–21, 41, 42]. According to Theorem 1.4.6 (3), an order-dense ideal of the K-space $C_\infty(Q)$, where Q is an extremally disconnected compact space, is the general form of a K-space. This fact provides a base for representation methods of studying operators in K-spaces. Analytic representations of operators are constructed in this section with the help of such operations as a continuous change of variable and the pointwise multiplication by a real-valued function.

Throughout the section, X and Y are totally disconnected, and P and Q extremally disconnected compact spaces. The symbol 1_M denotes the function on a set M which is identically equal to unity.

5.8.1. Assume that some "abstract" objects A and B (for instance, Boolean algebras, K-spaces, or BKSs) are represented via isomorphisms $\imath\colon A \to \widehat{A}$ and $\jmath : B \to \widehat{B}$ in the form of some "concrete" objects $\widehat{A}$ and $\widehat{B}$ (for instance, algebras of sets or spaces of functions). Then the ***interpretation*** of a mapping $f\colon A \to B$ (with respect to the representations $\imath$ and $\jmath$) is defined to be the composition $\jmath \circ f \circ \imath^{-1} : \widehat{A} \to \widehat{B}$.

5.8.2. Denote by $C_0(Y, X)$ the totality of all continuous functions $s\colon Y_0 \to X$ defined on various clopen subsets $Y_0 \subset Y$.

Proposition. *A mapping h: $\mathrm{Clop}(X) \to \mathrm{Clop}(Y)$ is a ring homomorphism if and only if there exists a function $s \in C_0(Y, X)$ such that $h(U) = s^{-1}[U]$ for all $U \in \mathrm{Clop}(X)$. For every ring homomorphism h, such a function s is unique.*

$\lhd$ The claim follows directly from the well-known theorem of R. Sikorski (see [18: § 11; 40]). $\rhd$

The relation $h(U) = s^{-1}[U]$ is called the ***representation*** of the ring homomorphism h by means of the function s. Observe that, due to the Stone theorem, the last proposition describes the structure of ring homomorphisms acting in arbitrary Boolean algebras.

5.8.3. The following proposition shows that every ring homomorphism (to within an isomorphism) is the mapping of intersection with a fixed set.

Proposition. *Let h: $\mathrm{Clop}(X) \to \mathrm{Clop}(Y)$ be a ring homomorphism. Then there exist a closed subset $Z \subset X$ and an order isomorphism $\imath$ of the Boolean algebra $\mathrm{Clop}(Z)$ onto $\mathrm{im}\, h$ such that $h(U) = \imath(U \cap Z)$ for all $U \in \mathrm{Clop}(X)$.*

$\lhd$ Let $h(U) = s^{-1}[U]$ be the representation of h by means of a function $s \in C_0(Y, X)$. Assign $Z := \mathrm{im}\, s$ and, for each element $W \in \mathrm{Clop}(Z)$, define the set $\imath(W) \in \mathrm{Clop}(Y)$ by the formula $\imath(W) := s^{-1}[W]$. Verification of the assertions of the theorem causes no difficulties. $\rhd$

5.8.4. ***Proposition.*** *Let E and F be order-dense ideals of $C_\infty(Q)$. A mapping W: $E \to F$ is an orthomorphism if and only if there exists a function $w \in C_\infty(Q)$ such that $W(e) = we$ for all $e \in E$. For every orthomorphism W, such a function w is unique.*

$\triangleleft$ The assertions stated are a reformulation of Theorem 5.5.12 with account taken of Theorem 1.4.6 (3). $\triangleright$

The relation $W(e) = we$ is called the ***representation*** of the orthomorphism W by means of the function w. Observe that, due to Theorem 1.4.6 (3), the last proposition describes the structure of orthomorphisms acting in arbitrary K-spaces.

5.8.5. Given arbitrary $s \in C_0(Q, P)$ and $e \in C_\infty(P)$, the function $e \bullet s : Q \to \overline{\mathbb{R}}$ is defined as follows:

$$(e \bullet s)(q) := \begin{cases} e\big(s(q)\big) & \text{if } q \in \operatorname{dom} s, \\ 0 & \text{if } q \in Q \setminus \operatorname{dom} s. \end{cases}$$

Of course, to ensure correctness, while using the notation $e \bullet s$, we must always have in mind a fixed set Q containing $\operatorname{dom} s$. Obviously, the function $e \bullet s$ is continuous but, in general, does not belong to $C_\infty(Q)$, since it can assume infinite values on a set with nonempty interior. The totality of all functions $e \in C_\infty(P)$ for which $e \bullet s \in C_\infty(Q)$ is denoted by $C_s(P)$.

Proposition. *Let $h \colon \operatorname{Pr}\big(C_\infty(P)\big) \to \operatorname{Pr}\big(C_\infty(Q)\big)$ be a ring homomorphism and let $hC(P)$ be the order-dense ideal of $C_\infty(P)$ defined in* 5.6.3. *Then $hC(P) = C_s(P)$, where $h(U) = s^{-1}[U]$ is the representation of h by means of an $s \in C_0(Q, P)$ $\big($with respect to the natural representations of $\operatorname{Pr}\big(C_\infty(P)\big)$ and $\operatorname{Pr}\big(C_\infty(Q)\big)\big)$.*

$\triangleleft$ The claim follows from Propositions 5.6.3 and 5.8.2. $\triangleright$

A continuous function $s \colon Q \to P$ is called *σ-exact*, if $s^{-1}[\operatorname{cl} G] = \operatorname{cl} s^{-1}[G]$ for every open σ-closed subset $G \subset P$. Below (see. 5.9.1), this property of a function is considered in more detail.

Lemma. *Denote the image of a function $s \in C_0(Q, P)$ by R.*

(1) *For every function $e \in C_s(P)$, the intersection $R \cap \operatorname{dom} e$ is dense in R, i.e., $C_s(P) \subset \big\{e \in C_\infty(P) : e|_R \in \overline{C}_\infty(R)\big\}$.*

(2) *If the function $s|^R$ is σ-exact then $C_s(P) = \big\{e \in C_\infty(P) : e|_R \in \overline{C}_\infty(R)\big\}$ and $\overline{C}_\infty(R) = \big\{e\big|_R : e \in C_s(P)\big\}$.*

$\triangleleft$ (1) Consider an arbitrary function $e \in C_s(P)$. If there were a nonempty open set $W \subset R$ disjoint from $\operatorname{dom} e$ then the function $e \bullet s$ would assume infinite values on the nonempty open set $s^{-1}[W]$, which would contradict the inclusion $e \bullet s \in C_\infty(Q)$. Consequently, the intersection $R \cap \operatorname{dom} e$ is dense in R.

(2) Let a function $e \in C_s(P)$ be such that the intersection $R \cap \operatorname{dom} e$ is dense in R. Then, using the fact that the function $s|^R$ is σ-exact and the intersection $R \cap \operatorname{dom} e$ is a σ-closed open subset of R, we obtain

$$\begin{aligned} \operatorname{cl}(e \circ s)^{-1}[\mathbb{R}] &= \operatorname{cl} s^{-1}[\operatorname{dom} e] = \operatorname{cl} s^{-1}[R \cap \operatorname{dom} e] \\ &= s^{-1}\big[\operatorname{cl}(R \cap \operatorname{dom} e)\big] = s^{-1}[R] = \operatorname{dom} s, \end{aligned}$$

and the first equality is established. The second equality follows from the first one due to the Tietze-Urysohn theorem. $\triangleright$

REMARK. The requirement in condition (2) of the lemma, that the function $s|^R$ be σ-exact, is essential, since the set $C_s(P)$ is not in general determined by the image of s. Indeed, suppose that $p \in P$ is not a P-point, i.e., the intersection of some sequence of neighborhoods of p is not a neighborhood of p. Let $\overline{P} := P \cup \{\infty\}$ be the enrichment of P by a new isolated point ∞. Then the identity function $s\colon P \to P$ and the function $\bar{s} := s \cup \big\{(\infty, p)\big\} : \overline{P} \to P$ have the same image, while the sets $C_s(P)$ and $C_{\bar{s}}(P)$ does not coincide.

5.8.6. If $E \subset C_\infty(P)$ and $R \subset P$ then the set $\big\{e|_R : e \in E\big\}$ is denoted by $E|_R$.

Lemma. *Denote the image of a function $s \in C_0(Q, P)$ by R and assume that the function $s|^R$ is σ-exact. Then*

(1) *$\overline{C}_\infty(R)$ is a vector sublattice of $C_\infty(R)$;*

(2) *if E is an ideal of the K-space $C_s(P)$ then $E|_R$ is an ideal of the vector lattice $\overline{C}_\infty(R)$.*

$\triangleleft$ Assertion (1) readily follows from Lemma 5.8.5 (2). Let us prove (2). Assume that a function $g \in \overline{C}_\infty(R)$ satisfies the inequalities $0 \leqslant g \leqslant e|_R$ for some positive element $e \in E$. In view of Lemma 5.8.5 (2), there is a positive function $\bar{e} \in C_s(P)$ such that $g = \bar{e}|_R$. Then $\bar{e} \wedge e \in E$ and $g = (\bar{e} \wedge e)|_R$. $\triangleright$

5.8.7. *Proposition.* *Let E be an order-dense ideal of $C_\infty(P)$ and let F be an order-dense ideal of $C_\infty(Q)$. A mapping $S\colon E \to F$ is a shift operator if and only if there exists a function $s \in C_0(Q, P)$ such that $Se = e \bullet s$ for all $e \in E$.*

$\triangleleft$ Sufficiency can be easily established with the help of Theorem 5.6.9. Let us show necessity. Suppose that $S\colon E \to F$ is a shift operator and $h\colon \operatorname{Pr}(E) \to \operatorname{Pr}(F)$

is its shadow. Represent the algebras $\mathrm{Pr}(E)$ and $\mathrm{Pr}(F)$ as $\mathrm{Clop}(P)$ and $\mathrm{Clop}(Q)$ and consider the representation $\hat{h}(U) = s^{-1}[U]$ of the corresponding interpretation $\hat{h}\colon \mathrm{Clop}(P) \to \mathrm{Clop}(Q)$ of the homomorphism h by means of an $s \in C_0(Q,P)$. According to Proposition 5.8.5, the equality $hC(P) = C_s(P)$ holds. Since the operators $(e \mapsto e \bullet s)\colon C_s(P) \to C_\infty(Q)$ and $S_h\colon hC(P) \to C_\infty(Q)$ have the same shadow h and satisfy the equalities $1_P \bullet s = S_h(1_P) = h(1)1_Q$, they coincide in view of Proposition 5.6.7. Therefore, $Se = S_h e = e \bullet s$ for all $e \in E$. $\rhd$

5.8.8. The function s connected with the shift operator S in the way described in the last proposition is not unique in general. Indeed, assume that the compact space P contains two different nonisolated points p_1 and p_2, assign $E := \{e \in C_\infty(P) : e(p_1) = e(p_2) = 0\}$ and consider the functions $s_1, s_2\colon Q \to P$ identically equal to p_1 and p_2, respectively. Then $e \bullet s_1 = e \bullet s_2 = 0$ for all $e \in E$.

The following proposition clarifies the question about uniqueness of a representation of a shift operator.

Proposition. *Let E be an order-dense ideal of $C_\infty(P)$, let F be an order-dense ideal of $C_\infty(Q)$, and let $S\colon E \to F$ be a shift operator. Assign $Q_0 :=$ $\operatorname{supp}\operatorname{im} S = \operatorname{cl}\bigcup_{e\in E} \operatorname{supp} Se$.*

(1) If functions $s_1, s_2 \in C_0(Q,P)$ satisfy the equalities $Se = e \bullet s_1 = e \bullet s_2$ for all $e \in E$ then $Q_0 \subset \operatorname{dom} s_1 \cap \operatorname{dom} s_2$ and $s_1 = s_2$ on Q_0.

(2) There exists a unique function $s \in C(Q_0, P)$ such that $Se = e \bullet s$ for all $e \in E$. Furthermore, if s is such a function then $h(U) = s^{-1}[U]$ is a representation of the shadow h of the operator S.

$\lhd$ (1) Denote by D the totality of all points in P, at which some functions in E are nonzero. Obviously, the set $s_1^{-1}[D]$ is dense in Q_0; therefore, it is sufficient to establish the equality $s_1 = s_2$ on this set. Take an arbitrary point $q \in s_1^{-1}[D]$ and assume to the contrary that $s_1(q) \neq s_2(q)$. Since $s_1(q) \in D$, there exists a function $e \in E$ that satisfies the relations $e\big(s_1(q)\big) \neq 0$ and $e\big(s_2(q)\big) = 0$, which contradicts the equality $e \bullet s_1 = e \bullet s_2$.

(2) Existence of the function s follows from Proposition 5.8.7, and its uniqueness from assertion (1). The fact that s represents the shadow of S ensues from the proof of the Proposition 5.8.7. $\rhd$

If a function s satisfies the conditions stated in assertion (2) then the relation $Se = e \bullet s$ is called the *representation* of the shift operator S by means of the function s. Observe that, due to Theorem 1.4.6 (3), Propositions 5.8.7 and 5.8.8 describe the structure of shift operators acting in arbitrary K-spaces.

5.8.9. The following proposition shows that every shift operator (to within an isomorphism) is the operator of restriction onto a fixed set.

Proposition. *Let E be an order-dense ideal of $C_\infty(P)$, let F be an order-dense ideal of $C_\infty(Q)$, and let $S\colon E \to F$ be a shift operator. Then there exist a closed subset $R \subset P$ and a mapping $i\colon E|_R \to F$ such that*

(1) *$E|_R$ is a vector sublattice of the K-space $C_\infty(R)$;*
(2) *i is a linear and order isomorphism of $E|_R$ onto $\operatorname{im} S$;*
(3) *$Se = i(e|_R)$ for all $e \in E$.*

$\lhd$ Let $Se = e \bullet s$ be the representation of S by means of a function $s \in C_0(Q,P)$. Assign $R := \operatorname{im} s$ and, for each element $g \in E|_R$, define the function $i(g) \in C(Q,\overline{\mathbb{R}})$ by the formula $i(g) := g \bullet s$. Verification of assertions (1)–(3) causes no difficulties. $\rhd$

5.8.10. ***Theorem.*** *Let E be an order-dense ideal of $C_\infty(P)$ and let F be an order-dense ideal of $C_\infty(Q)$. A mapping $T\colon E \to F$ is a weighted shift operator if and only if there exist functions $s \in C_0(Q,P)$, $w \in C_\infty(P)$, and $W \in C_\infty(Q)$ such that $we \bullet s \in C_\infty(Q)$ and $Te = W(we \bullet s)$ for all $e \in E$.*

$\lhd$ The claim readily follows from Propositions 5.8.4 and 5.8.7. $\rhd$

5.8.11. Simple examples show that the components of a representation $Te = W(we \bullet s)$ of a weighted shift operator T are not unique. However, omitting certain details, we may say that the function s is unique and W is uniquely determined by the choice of w. This observation can be precisely stated as follows.

Proposition. *Let E be an order-dense ideal of $C_\infty(P)$, let F be an order-dense ideal of $C_\infty(Q)$, and let $T\colon E \to F$ be a disjointness preserving regular operator. Assign $Q_0 := \operatorname{supp} \operatorname{im} T$.*

(1) *Let functions $s_1, s_2 \in C_0(Q,P)$, $w_1, w_2 \in C_\infty(P)$ and $W_1, W_2 \in C_\infty(Q)$ be such that $Te = W_1(w_1 e \bullet s_1) = W_2(w_2 e \bullet s_2)$ for all $e \in E$. Then $Q_0 \subset \operatorname{dom} s_1 \cap \operatorname{dom} s_2$ and $s_1 = s_2$ on Q_0. If, in addition, $w_1 = w_2$ then $W_1 = W_2$ on Q_0.*

(2) *Let a positive function* $w \in C_\infty(P)$ *be such that the operator* T *is wide at* $1/w$ (see 5.6.6). *Then there exist unique functions* $s \in C(Q_0, P)$ *and* $W \in C_\infty(Q)$ *such that* $W = 0$ *outside* Q_0 *and* $Te = W(we \bullet s)$ *for all* $e \in E$. *Furthermore,* $\operatorname{supp} W = s^{-1}[\operatorname{supp} w] = Q_0$, $Se = e \bullet s$ *is a representation of the shift* S *of the operator* T, *and* $h(U) = s^{-1}[U]$ *is a representation of its shadow* h.

◁ Assertion (1) follows immediately from Proposition 5.7.1 (due to 5.8.4 and 5.8.8). Let us show (2). Existence of functions s and W ensues from Theorems 5.7.2 and 5.8.10, and their uniqueness from assertion (1). Connection of the function s with the shift and shadow of the operator T follows from Propositions 5.7.1 (1) and 5.8.8 (2). ▷

If s, w, and W satisfy the conditions stated in assertion (2), then the relation $Te = W(we \bullet s)$ is called the *representation* of the weighted shift operator T by means of the functions s, w, and W. Observe that, due to the Theorem 1.4.6 (3), assertions 5.8.10 and 5.8.11 describe the structure of weighted shift operators acting in arbitrary K-spaces.

REMARK. If $Te = W(we \bullet s)$ is a representation of a weighted shift operator T then the operators T^+, T^-, and $|T|$ admit the following representations: $T^+e = W^+(we \bullet s)$, $T^-e = W^-(we \bullet s)$, and $|T|e = |W|(we \bullet s)$.

5.8.12. Given arbitrary functions $f, g \in C(Q, \overline{\mathbb{R}})$, the product $fg \in C(Q, \overline{\mathbb{R}})$ is defined by the rule

$$(fg)(q) := \begin{cases} f(q)g(q) & \text{if the product } f(q)g(q) \text{ makes sense,} \\ & \text{i.e., does not have the form } 0 \cdot \pm\infty \text{ or } \pm\infty \cdot 0, \\ 0 & \text{if } f \equiv 0 \text{ or } g \equiv 0 \text{ in a neighborhood of } q \end{cases}$$

on a dense subset of Q and then extends onto the entire space Q by continuity.

Theorem. *Let* E *be an order-dense ideal of* $C_\infty(P)$, *let* F *be an order-dense ideal of* $C_\infty(Q)$, *and let* $T\colon E \to F$ *be a disjointness preserving regular operator. Consider the representation* $h(U) = s^{-1}[U]$ *of the shadow* h *of the operator* T *by means of a function* $s \in C_0(Q, P)$. *Then there exist a family* $(w_\xi)_{\xi \in \Xi}$ *of positive functions in* $C_\infty(P)$ *and a family* $(W_\xi)_{\xi \in \Xi}$ *of pairwise disjoint functions in* $C_\infty(Q)$ *such that* $1/w_\xi \in E$ *for all* $\xi \in \Xi$ *and*

$$Te = o\text{-}\sum_{\xi \in \Xi} W_\xi(w_\xi e \bullet s) \quad (e \in E). \tag{$*$}$$

◁ The assertion stated is a reformulation of Theorem 5.7.5 with account taken of Proposition 5.8.11 (2). ▷

Observe that the functions $w_\xi e \bullet s$ in the representation $(*)$, being continuous functions from Q into $\overline{\mathbb{R}}$, need not belong to $C_\infty(Q)$, while the products $W_\xi(w_\xi e \bullet s)$ do belong to $C_\infty(Q)$.

We call the relation $Te = o\text{-}\sum_{\xi\in\Xi} W_\xi(w_\xi e \bullet s)$ the *representation* of the operator T by means of the functions s, w_ξ, and W_ξ. Observe that, due to Theorem 1.4.6 (3), the last theorem describes the structure of disjointness preserving regular operators acting in arbitrary K-spaces.

REMARK. If $Te = o\text{-}\sum_{\xi\in\Xi} W_\xi(w_\xi e \bullet s)$ is a representation of the operator T then the operators T^+, T^-, and $|T|$ admit the following representations:

$$T^+ e = o\text{-}\sum_{\xi\in\Xi} W_\xi^+(w_\xi e \bullet s),$$

$$T^- e = o\text{-}\sum_{\xi\in\Xi} W_\xi^-(w_\xi e \bullet s),$$

$$|T| e = o\text{-}\sum_{\xi\in\Xi} |W_\xi|(w_\xi e \bullet s).$$

5.8.13. REMARK. It is known (see [16, 27]) that order-dense ideals of the LNS $C_\infty(Q, \mathscr{X})$ of extended continuous sections of an ample Banach bundle $\mathscr{X}$ over an extremally disconnected compact space Q exhaust all BKSs. Furthermore, every BKS over a Kantorovich-Pinsker space is isometric to an ideal subspace of $M(\Omega, \mathscr{X})$, the BKS of cosets of measurable sections of a measurable Banach bundle $\mathscr{X}$ with lifting over a measure space Ω (see [28]).

These facts allow us to construct analytic representations for operators in Banach-Kantorovich spaces which are analogous to those for operators in K-spaces (see, for instance, [31]). Unfortunately, we cannot present here the corresponding results for reasons of space.

5.9. Interpretation for the Properties of Operators

The representation theorems of § 5.8 allows us to interpret various properties of orthomorphisms, shift operators, weighted shift operators, and arbitrary disjointness preserving operators in terms of the properties of certain components of their representation. As an illustration, we consider order continuous operators, injective operators, and operators with ideal image.

Throughout the section, P and Q are extremally disconnected compact spaces.

5.9.1. *Lemma.* *Let X and Y be totally disconnected compact spaces and let $s\colon X \to Y$ be a continuous function.*

(a) *The following assertions are equivalent:*

(1) $s^{-1}[\operatorname{int} F] = \operatorname{int} s^{-1}[F]$ *for every closed subset* $F \subset Y$;

(2) $s^{-1}[\operatorname{cl} G] = \operatorname{cl} s^{-1}[G]$ *for every open subset* $G \subset Y$;

(3) *if F is a closed subset of Y and* $\operatorname{int} F = \varnothing$ *then* $\operatorname{int} s^{-1}[F] = \varnothing$;

(4) *if G is an open subset of Y and* $\operatorname{cl} G = Y$ *then* $\operatorname{cl} s^{-1}[G] = X$;

(5) *the inverse image $s^{-1}[D]$ of every meager subset $D \subset Y$ is a meager subset of X;*

(6) *the inverse image $s^{-1}[D]$ of every comeager subset $D \subset Y$ is a comeager subset of X.*

(b) *The following assertions are equivalent:*

(1) $s^{-1}[\operatorname{int} F] = \operatorname{int} s^{-1}[F]$ *for every closed σ-open subset* $F \subset Y$;

(2) $s^{-1}[\operatorname{cl} G] = \operatorname{cl} s^{-1}[G]$ *for every open σ-closed subset* $G \subset Y$;

(3) *if F is a closed σ-open subset of Y and* $\operatorname{int} F = \varnothing$ *then* $\operatorname{int} s^{-1}[F] = \varnothing$;

(4) *if G is an open σ-closed subset of Y and* $\operatorname{cl} G = Y$ *then* $\operatorname{cl} s^{-1}[G] = X$.

A function s satisfying any of the conditions in (a) (in (b)) is called *exact* (*σ-exact*).

REMARK. If the compact space Y is extremally disconnected then the list (a) can be extended by the following equivalent assertions:

(7) *if U is a clopen subset of X then $s[U]$ is a clopen subset of Y;*

(8) *if U is an open subset of X then $s[U]$ is an open subset of Y.*

As is known, a function s satisfying condition (8) is called *open*. Thus, if the compact space Y is extremally disconnected then the classes of exact and open functions $s \in C(X, Y)$ coincide. The author does not know analogs of assertions (7) and (8) that are equivalent to the fact that the function s is σ-exact.

5.9.2. *Proposition.* *Let X and Y be totally disconnected compact spaces and let $h\colon \operatorname{Clop}(X) \to \operatorname{Clop}(Y)$ be a ring homomorphism. Consider the representation $h(U) = s^{-1}[U]$ of h by means of a function $s \in C_0(Y, X)$. The homomor-*

phism h is o-continuous (sequentially o-continuous) if and only if the function s is exact (σ-exact).

$\triangleleft$ There is a proof in [18: §22]. $\triangleright$

5.9.3. Let $\mathscr{U}$ and $\mathscr{V}$ be LNSs over order-dense ideals of the K-spaces $C_\infty(P)$ and $C_\infty(Q)$, respectively. If $T\colon \mathscr{U} \to \mathscr{V}$ is a disjointness preserving operator and $h(U) = s^{-1}[U]$ is the representation of the shadow h of the operator T by means of a function $s \in C_0(Q, P)$, then we say that s is the ***shift function*** of the operator T.

Theorem. *Suppose that E and F are order-dense ideals of $C_\infty(P)$ and $C_\infty(Q)$ (respectively), $\mathscr{U}$ is a BKS over E, $\mathscr{V}$ is an LNS over F, $T\colon \mathscr{U} \to \mathscr{V}$ is a disjointness preserving bounded operator, and $s \in C_0(Q, P)$ is its shift function. The operator T is o-continuous (sequentially o-continuous) if and only if the function s is exact (σ-exact).*

$\triangleleft$ Since the function s represents the shadow of T, the claim follows from 5.9.2 and 5.4.6. $\triangleright$

5.9.4. Proposition. *Let X and Y be totally disconnected compact spaces and let $h\colon \mathrm{Clop}(X) \to \mathrm{Clop}(Y)$ be a ring homomorphism. Consider the representation $h(U) = s^{-1}[U]$ of h by means of a function $s \in C_0(Y, X)$. The homomorphism h is injective if and only if the function s is surjective.*

5.9.5. Theorem. *Suppose that E and F are order-dense ideals of $C_\infty(P)$ and $C_\infty(Q)$ (respectively), $T\colon E \to F$ is a disjointness preserving regular operator, and $s \in C_0(Q, P)$ is its shift function. The operator T is injective if and only if the function s is surjective.*

$\triangleleft$ NECESSITY: In view of Proposition 5.9.4, it is sufficient to assume injectivity of the operator T and establish injectivity of its shadow $h\colon \mathrm{Pr}(E) \to \mathrm{Pr}(F)$. Consider an arbitrary projection $\pi \in \mathrm{Pr}(E)$ and suppose that $h(\pi) = 0$. Then $T\pi e = 0$ for all $e \in E$. Due to injectivity of T, the latter means that $\pi e = 0$ for all $e \in E$, i.e., $\pi = 0$.

SUFFICIENCY: Let $Te = \bigoplus_{\xi \in \Xi} W(w_\xi e \bullet s)|_{Q_\xi}$ be the representation of the operator T by means of $s \in C_0(Q, P)$, $w_\xi \in C_\infty(P)$, $Q_\xi \in \mathrm{Clop}(Q)$, and $W \in C_\infty(Q)$. Assume that the function s is surjective. For each $\xi \in \Xi$, assign $P_\xi := \operatorname{supp} w_\xi$. Consider an arbitrary functions $e \in E$ and suppose that $Te = 0$. Then $W(w_\xi e \bullet s)|_{Q_\xi} = 0$

for all $\xi \in \Xi$. The latter means that, for each $\xi \in \Xi$, the equality $w_\xi e \bullet s = 0$ holds on Q_ξ, which implies the equality $w_\xi e = 0$ on $s[Q_\xi]$ and, hence, the equality $e = 0$ on $s[Q_\xi] \cap P_\xi$. Thus, the function e is equal to zero on the union $D := \bigcup_{\xi \in \Xi} s[Q_\xi] \cap P_\xi$. It remains to show that the set D is dense in P.

Let a clopen set U be contained in the difference $P \backslash D$. Then, for all $e \in E$ and $\xi \in \Xi$, the equality $w_\xi \langle U \rangle e = 0$ holds on $U^\perp \cup P_\xi^\perp$. From the inclusion $s[Q_\xi] \cap P_\xi \subset U^\perp$ it follows that $w_\xi \langle U \rangle e = 0$ on $s[Q_\xi]$. Therefore, $(w_\xi \langle U \rangle e) \bullet s = 0$ on Q_ξ and, hence, $W\big((w_\xi \langle U \rangle e) \bullet s\big)|_{Q_\xi} = 0$. Arbitrariness of $\xi \in \Xi$ allows us to conclude that $T\langle U \rangle e = 0$, and arbitrariness of $e \in E$ yields the equality $h\langle U \rangle = 0$. According to injectivity of h (see Proposition 5.9.4), the latter means that $U = \varnothing$. $\rhd$

5.9.6. Remark. The author did not succeed in obtaining an adequate criterion for injectivity of an operator in BKSs. Simple examples show that the direct generalization of the last theorem to the case of an operator in BKSs is not true. Interpretation for injectivity of such operator must involve the outer weight of the representation.

5.9.7. Proposition. *Let X and Y be totally disconnected compact spaces and let $h \colon \mathrm{Clop}(X) \to \mathrm{Clop}(Y)$ be a ring homomorphism. Consider the representation $h(U) = s^{-1}[U]$ of the homomorphism h by means of a function $s \in C_0(Y, X)$. The equality $\operatorname{im} h = [0, h(1)]$ holds if and only if the function s is injective.*

5.9.8. Lemma. *A continuous function $s \colon Q \to P$ is injective if and only if the operator $(e \mapsto e \circ s) \colon C(P) \to C(Q)$ is surjective.*

$\lhd$ If the function s is injective then it is a homeomorphism of Q onto $\operatorname{im} s$. In this case, every function $f \in C(Q)$ can be represented as $g \circ s$, where $g \in C(\operatorname{im} s)$. By the Tietze-Urysohn theorem, the function g extends to an $e \in C(P)$.

If points $q_1, q_2 \in Q$ are different then there is a clopen set $V \subset Q$ that contains only one of them. If the operator $e \mapsto e \circ s$ is surjective then the characteristic function of V can be represented as $e \circ s$, whence $s(q_1) \neq s(q_2)$. $\rhd$

5.9.9. In the sequel, we discuss interpretation of the fact that an operator has ideal image. In order to clarify this property, we present a result established in [32: Lemma 2.7].

Lemma. *Let E and F be vector lattices and let $T \colon E \to F$ be a disjointness preserving regular operator. The following assertions are equivalent:*

(1) $\operatorname{im} T$ *is an ideal of* F;

(2) $\operatorname{im} |T|$ *is an ideal of* F;

(3) $|T|[0, e] = [0, |T|e]$ *for all positive* $e \in E$.

The list of equivalent properties (1)–(3) of the operator T can be extended by the following one: the operator T takes ideals of E into ideals of F, i.e., for every ideal $E_0 \subset E$, the set $T[E_0]$ is an ideal of F.

5.9.10. Proposition. *Suppose that* E *and* F *are order-dense ideals of* $C_\infty(P)$ *and* $C_\infty(Q)$ *(respectively),* $T\colon E \to F$ *is a disjointness preserving regular operator, and* $s \in C_0(Q, P)$ *is its shift function. Assume that* $\langle T\bar{e}\rangle = \langle \operatorname{im} T\rangle$ *for some element* $\bar{e} \in E$. *The image of* T *is an ideal of* F *if and only if the function* s *is injective.*

$\lhd$ Due to Theorem 5.4.1, we may assume that the operator T is positive and $\bar{e} \geqslant 0$. Moreover, for the sake of convenience, we assume that $\langle \operatorname{im} T\rangle = 1$, i.e., $\operatorname{dom} s = Q$.

Let the image of T be an ideal. In view of Lemma 5.9.8, to prove injectivity of s, it is sufficient to fix an arbitrary function $\beta \in C(Q)$, $0 \leqslant \beta \leqslant 1$, and represent it as $\alpha \circ s$, where $\alpha \in C(P)$. According to Lemma 5.9.9, the inequalities $0 \leqslant \beta T\bar{e} \leqslant T\bar{e}$ imply existence of an element $e \in E$ such that $0 \leqslant e \leqslant \bar{e}$ and $Te = \beta T\bar{e}$. Let a function $\alpha \in C(P)$ be such that $e = \alpha\bar{e}$. Then, according to 5.6.13, we have $(\alpha \circ s)T\bar{e} = T(\alpha\bar{e}) = Te = \beta T\bar{e}$, whence $\alpha \circ s = \beta$ due to the equality $\langle T\bar{e}\rangle = 1$.

Suppose now that the function s in injective. Fix arbitrary elements $e \in E$ and $f \in F$ satisfying the inequalities $0 \leqslant f \leqslant Te$ and show that $f \in \operatorname{im} T$. Let a function $\beta \in C(Q)$ be such that $f = \beta Te$. By injectivity of the operator $(e \mapsto e \circ s)\colon C(P) \to C(Q)$ (see 5.9.8), there exists a function $\alpha \in C(P)$ such that $\alpha \circ s = \beta$. Then $\alpha e \in E$ and, in view of 5.6.13, we have $T(\alpha e) = (\alpha \circ s)Te = \beta Te = f$. $\rhd$

5.9.11. Existence of an element $\bar{e} \in E$ satisfying the equality $\langle T\bar{e}\rangle = \langle \operatorname{im} T\rangle$ is an essential condition in the statement of Proposition 5.9.10. Without this requirement, the function s need not be injective even when T is a surjective shift operator. We will give a corresponding example in this subsection.

Lemma. *Consider functions* $s \in C_0(Q, P)$ *and* $f \in C_\infty(Q)$. *Suppose that there is an open set* $D \subset P$ *such that* s *is injective on* $s^{-1}[D]$ *and* f *is the identical zero outside* $s^{-1}[D]$. *Then* $f = e \bullet s$ *for some function* $e \in C_\infty(P)$. *For a positive* *and/or bounded function f, the corresponding function e can be chosen with the same property.*

$\lhd$ Denote the image of s by R and define a function $g\colon R \to \overline{\mathbb{R}}$ as follows:

$$g(p) := \begin{cases} f\big(s^{-1}(p)\big) & \text{if } p \in R \cap D, \\ 0 & \text{if } p \in R \backslash D. \end{cases}$$

Fix an arbitrary point $p \in R$ and show that the function g is continuous at p.

(1) Suppose that $p \in R \cap D$. Since the set D is open, we thus have a clopen set $U \subset P$ such that $p \in U \subset D$. From injectivity of s on $s^{-1}[D]$ it follows that the restriction $s|^U$ is a homeomorphism of $s^{-1}[U]$ onto $R \cap U$. Therefore, the function $g|_U = f \circ \left(s|^U\right)^{-1}$ is continuous.

(2) Suppose now that $p \in R\backslash D$. Fix an arbitrary number $\varepsilon > 0$ and show that $|g| < \varepsilon$ in a neighborhood of p. Assign $Q_\varepsilon := \big\{q \in Q : |f(q)| \geqslant \varepsilon\big\}$. Taking account of the fact that $f = 0$ outside $s^{-1}[D]$, we have the inclusion $Q_\varepsilon \subset s^{-1}[D]$; hence, $s[Q_\varepsilon] \subset D$. Since $|f| < \varepsilon$ outside Q_ε, we conclude that $|g| < \varepsilon$ outside $s[Q_\varepsilon]$. It remains to observe that $R\backslash s[Q_\varepsilon]$ is a neighborhood of p in the space R.

Thus, the function g is continuous. Obviously, $g \bullet s = f$. This implies that $g \in \overline{C}_\infty(R)$ $\big($if $|g| = \infty$ on a nonempty open set $W \subset R$ then $|f| = |g \bullet s| = \infty$ on the nonempty open set $s^{-1}[W]$, which contradicts the containment $f \in C_\infty(Q)\big)$. According to the Tietze-Urysohn theorem, there exists a function $e \in C_\infty(P)$ such that $e = g$ on R. Obviously, e is the desired function. Observe that positiveness and/or boundedness of the function f implies the same property of g, which in turn allows us to choose a function e with the appropriate property. $\rhd$

Example. As is known, the difference $\beta\mathbb{N}\backslash\mathbb{N}$ contains a discreet set D of cardinality continuum (see [4: Chapter IV, Problem 52]). Denote by $s\colon \beta D \to \beta\mathbb{N}$ the continuous extension of the identity mapping of D. Introduce the notation $\overline{D} := \mathrm{cl}_{\beta\mathbb{N}} D$, $E := \{e \in C(\beta\mathbb{N}) : e = 0 \text{ on } \overline{D}\backslash D\}$, $F := \{f \in C(\beta D) : f = 0 \text{ on } \beta D\backslash D\}$, and assign $Se := e \circ s$ for all $e \in E$. Then $S\colon E \to F$ is a surjective shift operator, while its shift function s is not injective.

$\lhd$ First of all, show that s is actually the shift function of the operator S. To this end, we should establish the equality $\operatorname{supp}\operatorname{im} S = \beta D$ (see 5.8.8). Since the subset $D \subset \beta\mathbb{N}$ is discreet, each point $q \in D$ has a neighborhood $U \subset \beta\mathbb{N}$ such that $U \cap D = \{q\}$. Then $\chi_U \in E$ and $(S\chi_U)(q) = \chi_U(s(q)) = \chi_U(q) = 1$. Thus, $D \subset \operatorname{supp}\operatorname{im} S$, whence $\operatorname{supp}\operatorname{im} S = \beta D$.

Now, show that the operator S is surjective. Fix an arbitrary element $f \in F$ and assign $\mathscr{D} := \beta\mathbb{N}\backslash(\overline{D}\backslash D)$. Then $\mathscr{D}$ is an open subset of $\beta\mathbb{N}$, $s^{-1}[\mathscr{D}] = s^{-1}[D] = D$, s is injective on D, and f is the identical zero outside D. Therefore, in view of the last lemma, there exists a function $e \in C(\beta\mathbb{N})$ such that $f = e \circ s$. It is clear that $e \in E$ and, therefore, $f \in \operatorname{im} S$.

It remains to observe that the function $s\colon \beta D \to \beta\mathbb{N}$ is not injective, since (see [4: Chapter VI, Problem 180])

$$|\beta D| = 2^{2^{|D|}} > 2^{2^{|\mathbb{N}|}} = |\beta\mathbb{N}|,$$

where $|X|$ stands for the cardinality of a set X. $\triangleright$

5.9.12. Theorem. *Suppose that E and F are order-dense ideals of $C_\infty(P)$ and $C_\infty(Q)$ (respectively), $T\colon E \to F$ is a disjointness preserving regular operator and $s \in C_0(Q,P)$ is its shift function. The image of T is an ideal of F if and only if, for every element $e \in E$, the function s is injective on the set* $\operatorname{supp} Te$. *The last property of the function s is equivalent to its injectivity on the union* $\bigcup\{\operatorname{supp} Te : e \in E\}$ *(which is an open dense subset of* $\operatorname{dom} s$*).*

$\triangleleft$ Necessity: Suppose that the image of T is an ideal and consider an arbitrary element $e \in E$. It is clear that the image of the composition $\langle Te\rangle \circ T$ is an ideal too and, in view of Proposition 5.9.10, its shift function is injective. It remains to observe that the shift function of the operator $\langle Te\rangle \circ T$ coincides with the restriction of s onto $\operatorname{supp} Te$.

Sufficiency: Theorem 5.4.1 allows us to assume that the operator T is positive. Fix arbitrary positive elements $e \in E$ and $f \in F$ satisfying the inequality $f \leqslant Te$ and show that $f \in \operatorname{im} T$. Since the function s is injective on the set $\operatorname{supp} Te$, in view of Proposition 5.9.10, the image of the composition $\langle Te\rangle \circ T$ is an ideal of F. According to Lemma 5.9.9, the inequalities $0 \leqslant f \leqslant \langle Te\rangle Te$ imply existence of an element $e_0 \in E$ such that $0 \leqslant e_0 \leqslant e$ and $\langle Te\rangle Te_0 = f$; whence $Te_0 = f$.

Injectivity of the function s on each set of the form $\operatorname{supp} Te$ $(e \in E)$ implies injectivity of s on the union $\bigcup\{\operatorname{supp} Te : e \in E\}$, since the containments $q_1 \in \operatorname{supp} Te_1$ and $q_2 \in \operatorname{supp} Te_2$ yield $q_1, q_2 \in \operatorname{supp} T(|e_1| \vee |e_2|)$. $\triangleright$

REMARK. Under the hypotheses of the last theorem, injectivity of the function s on the union $\cup\{\operatorname{Supp} Te : e \in E\}$ is not sufficient for the image of T to be an ideal (here $\operatorname{Supp} f = \{q \in Q : f(q) \neq 0\}$). Indeed, assign $P = Q = \beta\mathbb{N}$, fix a point $p \in P\backslash\mathbb{N}$, and, naturally identifying $C(Q)$ and ℓ^∞, consider the operator $T: C(P) \to C(Q)$ acting by the rule

$$(Te)(n) = \begin{cases} e(p) & \text{if } n = 1, \\ e(n)/n & \text{if } n > 1 \end{cases} \qquad (n \in \mathbb{N})$$

for all $e \in C(P)$. The image of T is not an ideal, since, for instance, $(1, \frac{1}{2}, \dots, \frac{1}{n}, \dots)$ belongs to $\operatorname{im} T$, but $(1, 0, 0, \dots)$ does not. However, the shift function s of the operator T is injective on the set $\cup\{\operatorname{Supp} Te : e \in E\} = \mathbb{N}$, since $s(1) = p$ and $s(n) = n$ whenever $n \in \mathbb{N}\backslash\{1\}$.

5.9.13. As is known (see 5.5.17), every BKS over an order-dense ideal of $C_\infty(P)$ is a module over $C(P)$. A subset $\mathscr{U}_0$ of such BKS is called a $C(P)$-*submodule* of it, if $\alpha u \in \mathscr{U}_0$ for all $u \in \mathscr{U}_0$ and $\alpha \in C(P)$.

Lemma. *Suppose that E and F are order-dense ideals of $C_\infty(P)$ and $C_\infty(Q)$ (respectively), $\mathscr{U}$ is a BKS over E, and $\mathscr{V}$ is a BKS over F. The following properties of an operator $T: \mathscr{U} \to \mathscr{V}$ are equivalent:*

(1) *T takes $C(P)$-submodules of $\mathscr{U}$ into $C(Q)$-submodules of $\mathscr{V}$;*

(2) *for every $u \in \mathscr{U}$ and every $\beta \in C(Q)$, there exists a function $\alpha \in C(P)$ such that $T(\alpha u) = \beta T u$.*

$\lhd$ It is sufficient to observe that the set $\{\alpha u : \alpha \in C(P)\}$ is a $C(P)$-submodule of $\mathscr{U}$. $\rhd$

5.9.14. *Proposition.* *Suppose that E and F are order-dense ideals of $C_\infty(P)$ and $C_\infty(Q)$ (respectively), $\mathscr{U}$ is a BKS over E, $\mathscr{V}$ is a BKS over F, $T: \mathscr{U} \to \mathscr{V}$ is a disjointness preserving bounded operator, and $s \in C_0(Q, P)$ is its shift function. Assume that $\langle T\bar{u}\rangle = \langle \operatorname{im} T\rangle$ for some element $\bar{u} \in \mathscr{U}$. The operator T takes $C(P)$-submodules of $\mathscr{U}$ into $C(Q)$-submodules of $\mathscr{V}$ if and only if the function s is injective.*

$\lhd$ For the sake of convenience, we assume that $\langle \operatorname{im} T\rangle = 1$, i.e., $\operatorname{dom} s = Q$. Suppose that T takes $C(P)$-submodules of $\mathscr{U}$ into $C(Q)$-submodules of $\mathscr{V}$. In view of 5.9.8, to prove injectivity of s, it is sufficient to fix an arbitrary function $\beta \in C(Q)$ and represent it as $\alpha \circ s$, where $\alpha \in C(P)$. According to Lemma 5.9.13, there exists a function $\alpha \in C(P)$ such that $T(\alpha\bar{u}) = \beta T\bar{u}$. Then, due to 5.6.13, we have

$$|\alpha \circ s - \beta||T\bar{u}| = \big|(\alpha \circ s)T\bar{u} - \beta T\bar{u}\big| = \big|T(\alpha\bar{u}) - \beta T\bar{u}\big| = 0;$$

whence $\alpha \circ s = \beta$ in view of the equality $\langle T\bar{u}\rangle = 1$.

Now, such that the function s is injective. Fix arbitrary elements $u \in \mathscr{U}$ and $\beta \in C(Q)$. According to surjectivity of the operator $(e \mapsto e \circ s)\colon C(P) \to C(Q)$ (see 5.9.8), there exists a function $\alpha \in C(P)$ such that $\alpha \circ s = \beta$. Then, due to 5.6.13, we have $T(\alpha u) = (\alpha \circ s)Tu = \beta Tu$. It remains to employ Lemma 5.9.13. $\rhd$

5.9.15. Lemma. *Let $\mathscr{U}$ be a BKS over an order-dense ideal of $C_\infty(Q)$. For any elements $u, v \in \mathscr{U}$, there is a function $f \in C(Q)$ such that $\langle u + fv\rangle = \langle u\rangle \vee \langle v\rangle$.*

$\lhd$ As f we can take any function that is different from $|u|/|v|$ everywhere. For instance, we may let

$$f := \big\langle |u|/|v| \leqslant 2\big\rangle 3 + \big\langle |u|/|v| > 2\big\rangle 1.$$

Then the equality $\langle u + fv\rangle = \langle u\rangle \vee \langle v\rangle$ ensues from the following relations:

$$\langle u\rangle \vee \langle v\rangle \leqslant \big\langle |u| \neq f|v|\big\rangle \leqslant \langle u + fv\rangle \leqslant \langle u\rangle \vee \langle v\rangle. \quad \rhd$$

Theorem. *Suppose that E and F are order-dense ideals of $C_\infty(P)$ and $C_\infty(Q)$ (respectively), $\mathscr{U}$ is a BKS over E, $\mathscr{V}$ is a BKS over F, $T\colon \mathscr{U} \to \mathscr{V}$ is a disjointness preserving bounded operator, and $s \in C_0(Q, P)$ is its shift function. The operator T takes $C(P)$-submodules of $\mathscr{U}$ into $C(Q)$-submodules of $\mathscr{V}$ if and only if, for every element $u \in \mathscr{U}$, the function s is injective on the set* $\operatorname{supp}|Tu|$. *This property of the function s is equivalent to its injectivity on the union $\bigcup\{\operatorname{supp}|Tu| : u \in \mathscr{U}\}$ (which is an open dense subset of* $\operatorname{dom} s$*).*

$\lhd$ NECESSITY: Suppose that T takes $C(P)$-submodules of $\mathscr{U}$ into $C(Q)$-submodules of $\mathscr{V}$ and consider an arbitrary element $u \in \mathscr{U}$. It is clear that the composition $\langle Tu\rangle \circ T$ preserves submodules too and, in view of Proposition 5.9.14, its shift function is injective. It remains to observe that the shift function of the operator $\langle Tu\rangle \circ T$ coincides with the restriction of s onto $\operatorname{supp}|Tu|$.

SUFFICIENCY: Fix arbitrary elements $u \in \mathscr{U}$ and $\beta \in C(Q)$. Since the function s is injective on the set $\operatorname{supp} |Tu|$, in view of Proposition 5.9.14, the composition $\langle Tu \rangle \circ T$ takes $C(P)$-submodules of $\mathscr{U}$ into $C(Q)$-submodules of $\mathscr{V}$. According to Lemma 5.9.13, there exists a function $\alpha \in C(P)$ such that $\langle Tu \rangle T(\alpha u) = \beta Tu$; whence, due to the relations $\langle T(\alpha u) \rangle = \langle (\alpha \bullet s) Tu \rangle \leqslant \langle Tu \rangle$, we have $T(\alpha u) = \beta Tu$.

Show that injectivity of the function s on each set of the form $\operatorname{supp} |Tu|$ ($u \in \mathscr{U}$) implies injectivity of s on the union $\cup \{\operatorname{supp} |Tu| : u \in \mathscr{U}\}$. To this end, it is sufficient to fix arbitrary elements $u_1, u_2 \in \mathscr{U}$ and find a $u \in \mathscr{U}$ such that $\operatorname{supp} |Tu| = \operatorname{supp} |Tu_1| \cup \operatorname{supp} |Tu_2|$. According to the last lemma, there is a function $\beta \in C(Q)$ that satisfies the relation $\operatorname{supp} |Tu_1 + \beta Tu_2| = \operatorname{supp} |Tu_1| \cup \operatorname{supp} |Tu_2|$. Injectivity of s on the set $\operatorname{supp} |Tu_2|$, in view of Lemma 5.9.8, implies existence of a function $\alpha \in C(P)$ such that $\alpha \circ s = \beta$ on $\operatorname{supp} |Tu_2|$. It remains to observe that $T(u_1 + \alpha u_2) = Tu_1 + (\alpha \bullet s) Tu_2 = Tu_1 + \beta Tu_2$. $\triangleright$

Comments

It is worth noting that as a rule we confine ourselves to considering K-spaces and Banach-Kantorovich spaces. Generalizations of the obtained results to the case of arbitrary vector lattices and lattice-normed spaces will appear elsewhere.

5.1. Section 5.1 only contains the information about vector lattices and lattice-normed spaces which was not exposed in the previous chapters. For the basic definitions and facts about the objects under consideration, we refer the reader to Chapter 1.

The description of Boolean homomorphisms stated in Proposition 5.1.2 is obtained in [29]. Propositions 5.1.3–5.1.6 are well known (see [10, 18]). The schema of a formal mixing employed in the proof of Proposition 5.1.9 steams from [5, 6, 16]. The notion of the disjoint sum of a family of LNSs (see 5.1.10) is introduced to be employed in the main result 5.7.8 on decomposition of a disjointness preserving operator into weighted shift operators. The new notion of the norm transformation of an LNS (see 5.1.11) is used for describing vector shift operators in Section 5.6.

5.2. Naturally, the notion of order convergence or o-convergence plays an important role in the theory of lattice-normed spaces. However, certain key problems related to this notion were not solved for a long time. Among them, the following natural question deserves mentioning: Is the o-closure of a subset of an LNS

o-closed, i.e., is it true that the second o-closure does not add new elements? Since o-convergence is not topological, this question is not trivial. While constructing the o-closed hull, authors had to consider Borel classes, i.e., to enrich a given set with o-limits of its elements, then with o-limits of o-limits, etc. (see [12]). Induction (and even transfinite induction) thus resulted made the proofs and constructions rather intricate (see, for instance, [13: 1.4.11, 4.1.8 (b); 16: 3.6–3.11]).

The concept of order approximation which is developed in Section 5.2 allows us to solve the indicated problems. All the results in this section appeared for the first time in [29]. The section also contains a generalization of the concept of order approximation to the case of h-convergence, which is useful in studying questions of continuity for disjointness preserving operators.

5.3. The notion of dominated operator is based on a simple idea ascending from Cauchy's method of dominants. Loosely speaking, the idea can be expressed as follows: if an operator is dominated by another operator, called a dominant, then the properties of the latter have a substantial influence on the properties of the former. Thus, an operator possessing a dominant qualified in a sense must be qualified itself.

A mathematical apparatus providing a natural shape for the idea of a dominant was suggested by L. V. Kantorovich in 1935–36 (see [11, 12]). Later, many authors studied various particular cases of dominated operators within the theory of vector and normed lattices. Recent achievements in studying dominated operators in lattice-normed spaces belong to A. G. Kusraev and his students (see [13, 14, 16, 33–36]).

The notion of dominated operator is closely related to so-called order-bounded operators. In many cases, the two classes of operators coincide (see, for instance, Proposition 5.4.12). From the theory of topological vector spaces it is well known that some continuity of a linear operator is often equivalent to its boundedness of relevant type. The indicated idea seems to be left without attention in the theory of lattice-normed spaces. In particular, this explains the fact that connection between different types of continuity of operators in LNSs is also not studied in its essence.

This circumstance is expressed for instance in the general delusion that r-continuity is sequential. It is a simple matter that r-convergence is sequential, but this fails for r-continuity. In Section 5.3, we show in particular that r-continuity and sequential r-continuity differ. Moreover, countable r-continuity differs from each of the two properties and occupies a strictly intermediate position between them. All the results and examples in this section, except 5.3.7–5.3.9, appeared for the first time [29].

5.4. The shadow of an operator as a Boolean homomorphism (without introducing the corresponding term) was first considered in [14: Theorems 1.Γ and 1.6] for lattice homomorphisms and disjointness preserving operators in lattice-normed spaces.

In Section 5.4, we develop this notion and show that many properties of disjointness preserving operators can be expressed in terms of their shadows. In particular, this is true of certain questions of continuity. The results stated in 5.4.1–5.4.9 are published for the first time.

The problem of finding sufficient conditions for an operator to be bounded or dominated is traditionally studied for disjointness preserving operators (see [14: 6.5]). Yu. A. Abramovich's condition (R) [19: Theorem A] was the first equivalent for boundedness of disjointness preserving operators weaker than sequential r-o-continuity. Later, this condition was also weakened. P. T. N. MacPolin and A. W. Wickstead showed [37: Theorem 2.1] that, for a disjointness preserving operator in vector lattices to be bounded, it is sufficient that the operator under test be r-semicontinuous (the latter term is introduced in 5.3.3 and the result is presented in 5.4.10).

Attempts at generalizing the Abramovich-MacPolin-Wickstead criterion to the case of operators in lattice-normed spaces cannot lead to a success, since all the four types of boundedness considered in 5.3.3 are pairwise different for that class of operators (the corresponding examples are presented in 5.3.4–5.3.6). Thus the main problem about sufficient conditions for boundedness remains open for disjointness preserving operators in LNSs. A small step in this direction is made in 5.4.13.

5.5. An orthomorphism is a band preserving operator that is order-bounded. The problem of finding sufficient conditions for boundedness of disjointness preserving operators is actually solved for operators in vector lattices (see [19: Theorem A; 37: Theorem 2.1] and Theorem 5.4.10). However, the problem remains actual for operators in lattice-normed spaces (see the commentary for Section 5.4). Our Theorem 5.5.11 asserts that, for band preserving operators in LNSs, all the types of boundedness coincide. However, this result does not answer the natural question, whether every band preserving operator must be bounded automatically. This question, raised for the first time by A. W. Wickstead in [41], admits different answers depending on spaces in which the operator in question acts. There are many results that guarantee automatic boundedness for a band preserving operator acting in concrete classes of vector lattices (see [2: Theorem 2; 3: Theorem 3.2; 3: Theorem 3.3; 37: Corollary 2.3]).

For the first time, existence of an unbounded band preserving operator was announced in [2: Theorem 1]. Later, it was clarified that the situation described in the paper is, in a sense, typical. Namely, it was established (see [3: Theorem 2.1; 37: Theorem 3.2]) that all band preserving operators in a universally complete K-space are automatically bounded if and only if the K-space is locally one-dimensional. (The definition of a locally one-dimensional K-space is presented in [37: 3.1] with a preliminary analysis. See also 5.5.1 and 5.5.2.)

Thus, A. W. Wickstead's question about boundedness of band preserving operators was given an exhaustive answer. However, a new notion of locally one-dimensional K-space crept into the answer. Unfamiliarity of this notion resulted in the conjecture about its coincidence with the notion of discrete (= atomic) K-space. Interesting events are connected with the conjecture. In 1981, Yu. A. Abramovich, A. I. Veksler, and A. V. Koldunov [3: Theorem 2.1] gave a proof for existence of an unbounded band preserving operator in every nondiscrete universally complete K-space, thus corroborating the conjecture of coincidence locally one-dimensional and discrete K-spaces (see also [19: Section 5]). However, the proof occurred to be erroneous. Later, in 1985, P. T. N. MacPolin and A. W. Wickstead [37: Section 3] gave an example of a nondiscrete locally one-dimensional K-space, confuting the conjecture this time. However, an error was discovered in the example. Finally, in 1993, A. W. Wickstead [22] fixed the conjecture as an open question.

It is interesting that the same superstition (naturally, expressed in other terms) was popular among the specialists in Boolean-valued analysis (see 5.5.3).

The conjecture under discussion is confuted in 5.5.4–5.5.9. This is made with the help of describing a locally one-dimensional K-space in terms of its base (Theorem 5.5.7). The mentioned results belong to A. E. Gutman [30].

Subsections 5.5.12–5.5.19 are devoted to a study of the module structure in a Banach-Kantorovich space and its relation to the notion of orthomorphism. The results presented here are essentially known (see, for instance, [13]).

5.6. The study of multiplicative operators in vector lattices was initiated by B. Z. Vulikh [7, 9] who proved that o-continuous shift operators in K-spaces with unity are multiplicative. Theorem 5.6.10 generalizes this result to the case of arbitrary shift operators in arbitrary K-spaces. The idea of considering the shift of a disjointness preserving operator is not new. Analogs of this notion occur, for instance, in [32] and in many papers about isometries of L^p-spaces.

5.7. The main criterion for WSW-representability stated in 5.7.2 is close to [21: 3.12]. Some of the criteria presented in 5.7.4 and 5.7.7 are also known (see [1, 20, 21]). Note that one of the sufficient conditions for WSW-representability (the second proposition in 5.7.4) was mistakenly regarded as necessary in [1: Theorem 5]. The corresponding counterexample is given in 5.7.4. Existence of a similar example due to A. V. Koldunov is mentioned in [21: 3.14].

It is worth observing that our notion of weighted shift operator differs slightly from the analogous construction in the literature. The classical construction does not contain an inner weight (see [2: Theorem 6; 3: Theorem 4.1; 14: Theorems 2.8 and 2.9; 1: Theorem 6; 21: 3.8–3.18]). We regard this circumstance is a small demerit of the theory which, in particular, restricts the class of representations of vector lattices providing the WSW-representability and makes the problem of a global WSW-representation more difficult.

None of the known results ensured representation of an arbitrary bounded disjointness preserving operator on the entire domain of definition. Each representation theorem either restricted the class of operators under consideration (for instance by requiring order continuity), or restricted the class of spaces (for instance, by considering only Banach lattices), or did not guarantee a representation on the entire domain of definition (but only, for instance, on its principal ideals). In our opinion, the failure in searching for a global representation of disjointness preserving operators is mainly determined by the absence of an inner weight in the definition of a weighted shift operator. Involving an inner weight allows us to decompose an arbitrary bounded disjointness preserving operator in lattice-normed spaces into the strongly disjoint sum of weighted shift operators (Theorem 5.7.8).

This result is new even for the case of operators in K-spaces (Theorem 5.7.5).

5.8. Many facts presented in Section 5.8 are not new. Some of them just repeat Yu. A. Abramovich's results and treat the corresponding representations in more detail. Only Theorem 5.8.12 is new. This theorem interprets the decomposition in 5.7.5 of a disjointness preserving operator into the sum of weighted shift operators in terms of their functional representations.

5.9. The global representation of 5.8.12 for a disjointness preserving operator, as well as the notions of the shift of an operator and the corresponding shift function, allows us to interpret the abstract properties of the operator in terms of its concrete function representation or in terms of the properties of its shift function. Examples of similar interpretations can be found, for instance, in [1, 20, 21].

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