What Is Boolean Valued Analysis

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Agenda

- The term "Boolean valued analysis" appeared within the realm of mathematical logic.
- It was Takeuti, a renowned expert in proof theory, who introduced the term. Takeuti defined Boolean valued analysis in [1, p. 1] as "an application of Scott–Solovay's Boolean valued models of set theory to analysis." Vopěnka invented similar models at the same time. That is how the question of the title receives an answer in zero approximation. However, it would be premature to finish at this stage. It stands to reason to discuss in more detail the following three questions.
- Why should we know anything at all about Boolean valued analysis?
- What need the working mathematician know this for?
- What do the Boolean valued models yield?

Why should we know anything at all about Boolean valued analysis?

- Curiosity often leads us in science, and oftener we do what we can. However we appreciate that which makes us wiser. Boolean valued analysis has this value, expanding the limits of our knowledge and taking off blinds from the eyes of the perfect mathematician, mathematician *par excellence*. To substantiate this thesis is the main target of this talk.

What need the working mathematician know this for?

- Part of the answer was given above: to become wiser. There is another, possibly more important, circumstance. Boolean valued analysis not only is tied up with many topological and geometrical ideas but also provides a technology for expanding the content of the already available theorems.
- Each theorem, proven by the classical means, possesses some new and nonobvious content that relates to "variable sets." Speaking more strictly, each of the available theorems generates a whole family of its next of kin in disguise which is enumerated by all complete Boolean algebras or, equivalently, nonhomeomorphic Stone spaces.

What do the Boolean valued models yield?

- The essential and technical parts of this survey are devoted to answering the question. We will focus on the general methods independent of the subtle intrinsic properties of the initial complete Boolean algebra. These methods are simple, visual, and easy to apply. Therefore they may be useful for the working mathematician.
- However, we must always keep in mind that the Boolean valued models were invented in order to simplify the exposition of Cohen's forcing [2]. Cohen was awarded a Fields medal in 1966 for his final step in settling Hilbert's problem No. 1.
- Hilbert [3] considered it plausible that "*as regards equivalence there are, therefore, only two assemblages of numbers, the countable assemblage and the continuum.*"

Boolean Algebra

- The stage of Boolean valued analysis is some Boolean valued model of ZFC. To define these models, we start with a complete Boolean algebra.
- A *Boolean algebraB* is an algebra with distinct unity $\mathbb{1}$ and zero $\mathbb{0}$, over the *two-point field* $2 := \{0, 1\}$, whose every element is idempotent. Given $a, b \in B$, put $a \leq b \leftrightarrow ab = a$. The completeness of B means the existence of the least upper and greatest lower bounds of each subset of B.
- For the sake of comfort we may view the elements $\mathbb{0}$ and $\mathbb{1}$ of the initial complete Boolean algebra B and the operations on B as an assemblage of some special symbols for discussing the validity of mathematical propositions.

A Boolean valued universe

- Given an ordinal α, put

$$\mathbb{V}_{\alpha}^{(B)} :=$$
$$\{x \mid \mathsf{Fnc}\,(x) \ \wedge\ (\exists\beta)(\beta < \alpha \ \wedge\ \mathsf{dom}(x) \subset \mathbb{V}_{\beta}^{(B)}) \ \wedge\ \mathsf{im}(x) \subset B\}.$$

- In more detail,

$$\mathbb{V}_{0}^{(B)} := \varnothing,$$
$$\mathbb{V}_{\alpha+1}^{(B)} :=$$
$$\{x \mid x \text{ is a function}, \mathsf{dom}(x) \subset \mathbb{V}_{\alpha}^{(B)},\ \mathsf{im}(x) \subset\ B\},$$
$$\mathbb{V}_{\alpha}^{(B)} := \bigcup_{\beta<\alpha} \mathbb{V}_{\beta}^{(B)} \quad (\alpha \text{ is a limit ordinal}).$$

- The class

$$\mathbb{V}^{(B)} := \bigcup_{\alpha\in\mathsf{On}} \mathbb{V}_{\alpha}^{(B)}$$

is a *Boolean valued universe*.

B-valued sets

- The elements of $\mathbb{V}^{(B)}$ are *B-valued sets*. In particular, $\varnothing$ is the function with domain $\varnothing$ and range $\varnothing$.
- The three "lower" floors of $\mathbb{V}^{(B)}$ are composed as follows: $\mathbb{V}_0^{(B)} = \varnothing$, $\mathbb{V}_1^{(B)} = \{\varnothing\}$, and $\mathbb{V}_2^{(B)} = \{\varnothing, (\{\varnothing\}, b) \mid b \in B\}$.
- Observe also that $\alpha \leq \beta \to \mathbb{V}_\alpha^{(B)} \subset \mathbb{V}_\beta^{(B)}$ for all ordinals α and β.
- Moreover, $\mathbb{V}^{(B)}$ enjoys the *induction principle*

$$(\forall x \in \mathbb{V}^{(B)})\, ((\forall y \in \operatorname{dom}(x))\, \varphi(y) \to \varphi(x)) \to (\forall x \in \mathbb{V}^{(B)})\, \varphi(x),$$

with φ a formula of ZFC.

Truth-Values I

- Consider a formula φ of ZFC, where $\varphi = \varphi(u_1, \dots, u_n)$. Replacing $u_1, \dots, u_n$ with $x_1, \dots, x_n \in \mathbb{V}^{(B)}$, obtain a particular assertion about the objects $x_1, \dots, x_n$.
- Ascribe to φ some *truth value* $[\![\varphi]\!]$ so that $[\![\varphi]\!]$ be a member of B and all theorems of ZFC become "as valid as possible"; i.e., the *top* $\mathbb{1} := \mathbb{1}_B$ of B, the *unity* of B, serves as the truth value of a theorem of ZFC.

Truth-Values II

- The truth value of a well-formed formula must be determined by "double" recursion. If φ and ψ are already evaluated formulas of ZFC, and $[\![\varphi]\!] \in B$ and $[\![\psi]\!] \in B$ are the truth values of these formulas then put

$$
\begin{gathered}
[\![\varphi \wedge \psi]\!] := [\![\varphi]\!] \wedge [\![\psi]\!], \\
[\![\varphi \vee \psi]\!] := [\![\varphi]\!] \vee [\![\psi]\!], \\
[\![\varphi \to \psi]\!] := [\![\varphi]\!] \to [\![\psi]\!], \\
[\![\neg\varphi]\!] := \neg[\![\varphi]\!], \\
[\![(\forall x)\,\varphi(x)]\!] := \bigwedge_{x \in \mathbb{V}^{(B)}} [\![\varphi(x)]\!], \\
[\![(\exists x)\,\varphi(x)]\!] := \bigvee_{x \in \mathbb{V}^{(B)}} [\![\varphi(x)]\!].
\end{gathered}
$$

Truth-Values III

- To evaluate the atomic formulas $x \in y$ and $x = y$ for $x, y \in \mathbb{V}^{(B)}$, imagine that every B-valued set y is "fuzzy," i.e. "it contains each member z of $\mathrm{dom}(y)$ with probability $y(z)$."
- Attempting to preserve the logical tautology $x \in y \leftrightarrow (\exists z \in y)\,(z = x)$ alongside the axiom of extensionality, we arrive to the following recursive definition:

$$[\![x \in y]\!] := \bigvee_{z \in \mathrm{dom}(y)} y(z) \wedge [\![z = x]\!],$$

$$[\![x = y]\!] := \Bigg(\bigwedge_{z \in \mathrm{dom}(x)} x(z) \to [\![z \in y]\!]\Bigg) \wedge \Bigg(\bigwedge_{z \in \mathrm{dom}(y)} y(z) \to [\![z \in x]\!]\Bigg).$$

What Is True in $\mathbb{V}^{(B)}$ I

- A *formula* $\varphi(x_1, \dots, x_n)$ *is satisfied inside* $\mathbb{V}^{(B)}$ or the collection of $x_1, \dots, x_n$ *possesses the property* φ *inside* $\mathbb{V}^{(B)}$ provided that $[\![\varphi(x_1, \dots, x_n)]\!] = \mathbb{1}$. In this event we write $\mathbb{V}^{(B)} \models \varphi(x_1, \dots, x_n)$.
- If some formula φ of ZFC is expressed in the natural language of discourse then use quotes: $\mathbb{V}^{(B)} \models$ "φ." The *satisfaction mark* $\models$ implies the usage of the model-theoretic expressions of the kind "$\mathbb{V}^{(B)}$ is a Boolean valued model for φ" instead of $\mathbb{V}^{(B)} \models \varphi$.

What Is True in $\mathbb{V}^{(B)}$ II

- All axioms of the first-order predicate calculus are obviously valid inside $\mathbb{V}^{(B)}$. In particular,

  (1) $[\![x = x]\!] = \mathbb{1}$,
  (2) $[\![x = y]\!] = [\![y = x]\!]$,
  (3) $[\![x = y]\!] \wedge [\![y = z]\!] \leq [\![x = z]\!]$,
  (4) $[\![x = y]\!] \wedge [\![z \in x]\!] \leq [\![z \in y]\!]$,
  (5) $[\![x = y]\!] \wedge [\![x \in z]\!] \leq [\![y \in z]\!]$.
  Observe that $\mathbb{V}^{(B)} \models x = y \wedge \varphi(x) \to \varphi(y)$ for every formula φ, i.e.,
  (6) $[\![x = y]\!] \wedge [\![\varphi(x)]\!] \leq [\![\varphi(y)]\!]$.

Precausion

- The equality $[\![x = y]\!] = \mathbb{1}$ holding in the Boolean valued universe $\mathbb{V}^{(B)}$ implies in no way that the functions x and y coincide in their capacities of members of $\mathbb{V}$. For instance, the function vanishing at one of the levels $\mathbb{V}_\alpha^{(B)}$, with $\alpha \geq 1$, plays the role of the empty set inside $\mathbb{V}^{(B)}$. This circumstance is annoying in a few constructions we use in the sequel.

The Separated Universe

- Put $\mathbb{V}^{(B)} := \overline{\mathbb{V}}^{(B)}$. To define the *separated Boolean valued universe* $\overline{\mathbb{V}}^{(B)}$, consider the equivalence $\{(x, y) \mid [\![x = y]\!] = \mathbb{1}\}$ on $\mathbb{V}^{(B)}$ and choose a member (representative of the least rank) in each class of equivalent functions.
- Note that the implication

$$[\![x = y]\!] = \mathbb{1} \to [\![\varphi(x)]\!] = [\![\varphi(y)]\!]$$

holds for every formula φ of ZFC and members x, y of $\mathbb{V}^{(B)}$. Therefore, we may evaluate formulas over the separated universe, disregarding the choice of representatives.

Transfer

- All theorems of ZFC are valid inside $\mathbb{V}^{(B)}$; i.e., in symbols,

$$\mathbb{V}^{(B)} \models \mathrm{ZFC}.$$

- The transfer principle is established by a bulky check that the truth value of every axiom of ZFC is $\mathbb{1}$ and all inference rules increase truth values.
- The transfer principle reads sometimes as follows: "$\mathbb{V}^{(B)}$ is a Boolean valued model of ZFC." That is how the term "Boolean valued model of set theory" enters the realms of mathematics.

Maxima

- To each formula φ of ZFC there is a member x_0 of $\mathbb{V}^{(B)}$ satisfying $[\![(\exists x)\,\varphi(x)]\!] = [\![\varphi(x_0)]\!]$.
- In particular, it is true inside $\mathbb{V}^{(B)}$ that if $\varphi(x)$ for some x then there exists a member x_0 of $\mathbb{V}^{(B)}$ (in the sense of $\mathbb{V}$) satisfying $[\![\varphi(x_0)]\!] = \mathbb{1}$. In symbols,

$$\mathbb{V}^{(B)} \models (\exists x)\,\varphi(x) \to (\exists x_0)\,\mathbb{V}^{(B)} \models \varphi(x_0).$$

- The *maximum principle* means that $(\exists x_0 \in \mathbb{V}^{(B)})\,[\![\varphi(x_0)]\!] = \bigvee_{x\in\mathbb{V}^{(B)}}[\![\varphi(x)]\!]$ for every formula φ of ZFC. The last formula illuminates the background behind the term "maximum principle." The proof of the principle consists in an easy application of the *mixing principle*.

Mixing

- Let $(b_\xi)_{\xi\in\Xi}$ be a *partition of unity* in B; i.e., a family of members of B such that

$$\bigvee_{\xi\in\Xi} b_\xi = \mathbb{1}, \quad (\forall \xi,\eta \in \Xi)\,(\xi \neq \eta \to b_\xi \wedge b_\eta = \mathbb{0}).$$

- For every family $(x_\xi)_{\xi\in\Xi}$ of elements of $\mathbb{V}^{(B)}$ and every partition of unity $(b_\xi)_{\xi\in\Xi}$ there is a unique mixing of (x_ξ) by (b_ξ) (or with probabilities (b_ξ)); i.e., a member x of the separated universe $\mathbb{V}^{(B)}$ satisfying $b_\xi \leq [\![x = x_\xi]\!]$ for all $\xi \in \Xi$.
- The *mixing* x of a family (x_ξ) by (b_ξ) is denoted as follows: $x = \mathrm{mix}_{\xi\in\Xi}(b_\xi x_\xi) = \mathrm{mix}\{b_\xi x_\xi \mid \xi \in \Xi\}$.
- Mixing is connected with the main particularity of a Boolean valued model, the procedure of collecting the highest truth value "stepwise."

Functional Realization I

- Let Q be the Stone space of a complete Boolean algebra B. Denote by $\mathfrak{U}$ the (separated) Boolean valued universe $\mathbb{V}^{(B)}$. Given $q \in Q$, define the equivalence $\sim_q$ on the class $\mathfrak{U}$ as follows: $u \sim_q v \ \leftrightarrow \ q \in [\![u = v]\!]$. Consider the bundle

$$V^Q := \left\{ \left(q, \sim_q(u)\right) \mid q \in Q,\ u \in \mathfrak{U} \right\}$$

  and agree to denote the pair $\left(q, \sim_q(u)\right)$ by $\widehat{u}(q)$. Clearly, for every $u \in \mathfrak{U}$ the mapping $\widehat{u} : q \mapsto \widehat{u}(q)$ is a section of V^Q.
- Note that to each $x \in V^Q$ there are $u \in \mathfrak{U}$ and $q \in Q$ satisfying $\widehat{u}(q) = x$. Moreover, we have $\widehat{u}(q) = \widehat{v}(q)$ if and only if $q \in [\![u = v]\!]$.

Functional Realization II

- Make each fiber V^q of V^Q into an algebraic system of signature $\{\in\}$ by letting $V^q \models x \in y \ \leftrightarrow\ q \in [\![u \in v]\!]$, where $u, v \in \mathfrak{U}$ are such that $\widehat{u}(q) = x$ and $\widehat{v}(q) = y$.
- Clearly, the class of the sets $\{\widehat{u}(A) \mid u \in \mathfrak{U}\}$, with A a clopen subset of Q, is a base for some topology on V^Q. This enables us to view V^Q as a continuous bundle called a *continuous polyverse*. A *continuous section* of V^Q is section that is a continuous function. Denote by $\mathfrak{C}$ the class of all continuous sections of V^Q.
- The mapping $u \mapsto \widehat{u}$ is a bijection between $\mathfrak{U}$ and $\mathfrak{C}$ in which we can clearly find grounds for a convenient functional realization of the Boolean valued universe $\mathbb{V}^{(B)}$. The details of this universal construction due to Gutman and Losenkov are presented in [8, Ch. 6].

Socialization

- Mathematics of the twentieth century exhibited many examples of the achievements obtained by socialization of objects and problems; i.e., their inclusion into a class of similar objects or problems.
- Hilbert said in his report [3]: "In dealing with mathematical problems, specialization plays, as I believe, a still more important part than generalization. Perhaps in most cases where we seek in vain the answer to a question, the cause of the failure lies in the fact that problems simpler and easier than the one in hand have been either not at all or incompletely solved. All depends, then, on finding out these easier problems, and on solving them by means of devices as perfect as possible and of concepts capable of generalization."
- Boolean valued models gain a natural status within category theory. The idea of a variable set becomes a cornerstone of the categorical analysis of logic which is accomplished in topos theory [4].

Distant Modeling

- Imagine that we are physically unable to compare the models pointwise. Happily, we take an opportunity to exchange information with the owner of the other model by using some means of communication, e.g., by having long-distance calls. While communicating, we easily learn that our partner uses his model to operate on some objects that are the namesakes of ours, i.e., sets, membership, etc. Since we are interested in ZFC, we ask him whether or not the axioms of ZFC are satisfied in his model. Manipulating the model, he returns a positive answer. After checking that he uses the same inference rules as we do, we cannot help but acknowledge his model to be a model of the theory we are all investigating.
- The novelty of distant modeling resides in our refusal to identify the universes of objects and admittance of some a priori unknown procedures of verification of propositions.

Technology

- Many possibilities for applying $\mathbb{V}^{(B)}$ base on the fact that irrespective of the choice of a Boolean algebra B, the universe is an arena for testing an arbitrary mathematical event. By transfer and maxima, every $\mathbb{V}^{(B)}$ has the objects that play the roles of numbers, groups, Banach spaces, manifolds, and whatever constructs of mathematics that are already introduced into practice or still remain undiscovered. These objects may be viewed as some nonstandard realizations of the relevant originals.
- All celebrated and not so popular theorems acquire interpretations for the members of $\mathbb{V}^{(B)}$, attaining the top truth value. We thus obtain a new technology of comparison between the interpretations of mathematical facts in the universes over various complete Boolean algebras. Developing the relevant tools is the crux of Boolean valued analysis.

Ascending and Descending

- No comparison is feasible without some dialog between $\mathbb{V}$ and $\mathbb{V}^{(B)}$. We need some sufficiently convenient mathematical toolkit for the comparative analysis of the interpretations of the concepts and facts of mathematics in various models. The relevant *technique of ascending and descending* bases on the operations of the canonical embedding, descent, and ascent to be addressed right away.

The Canonical Embedding

- Given $x \in \mathbb{V}$, denote by $x^{\wedge}$ the *standard name* of x in $\mathbb{V}^{(B)}$; i.e., the member of $\mathbb{V}^{(B)}$ that is defined by recursion as follows:

$$\varnothing^{\wedge} := \varnothing, \quad \mathrm{dom}(x^{\wedge}) := \{y^{\wedge} \mid y \in x\}, \quad \mathrm{im}(x^{\wedge}) := \{\mathbb{1}\}.$$

- Given a restricted formula φ of ZFC and a collection $x_1, \dots, x_n \in \mathbb{V}$, the following holds:

$$\varphi(x_1, \dots, x_n) \leftrightarrow \mathbb{V}^{(B)} \models \varphi(x_1^{\wedge}, \dots, x_n^{\wedge}).$$

This is the "restricted transfer."

Descent and Ascent

- Given an arbitrary member x of a (separated) Boolean valued universe $\mathbb{V}^{(B)}$, define the *descent* $x{\downarrow}$ of x as follows:

$$x{\downarrow} := \{y \in \mathbb{V}^{(B)} \mid [\![y \in x]\!] = \mathbb{1}\}.$$

- Assume that $x \in \mathbb{V}$ and $x \subset \mathbb{V}^{(B)}$; i.e., let x be some set composed of B-valued sets or, in other words, $x \in \mathscr{P}(\mathbb{V}^{(B)})$. Put $\varnothing{\uparrow} := \varnothing$, while $\mathrm{dom}(x{\uparrow}) := x$ and $\mathrm{im}(x{\uparrow}) := \{\mathbb{1}\}$ in case $x \neq \varnothing$. The member $x{\uparrow}$ of the (separated) universe $\mathbb{V}^{(B)}$; i.e., the distinguished representative of the class $\{y \in \mathbb{V}^{(B)} \mid [\![y = \ x{\uparrow}]\!] = \mathbb{1}\}$, is the *ascent* of x.

Descending 2

- By example, descend the two-point Boolean algebra:
- Take two arbitrary members $0,\ 1 \in \mathbb{V}^{(B)}$ satisfying the condition $[\![0 \neq 1]\!] = \mathbb{1}_B$. For instance, put $0 := \mathbb{0}_B^{\wedge}$ and $1 := \mathbb{1}_B^{\wedge}$. The descent D of the two-point Boolean algebra $\{0, 1\}^B$ inside $\mathbb{V}^{(B)}$ is a complete Boolean algebra isomorphic to B.
- The formulas $[\![\chi(b) = 1]\!] = b$ and $[\![\chi(b) = 0]\!] = \neg b$, with $b \in B$, yield the isomorphism $\chi : B \to D$.

Descending the Reals

- By maxima and transfer there is a member $\mathscr{R}$ of $\mathbb{V}^{(B)}$ such that $\mathbb{V}^{(B)} \models$ "$\mathscr{R}$ is an ordered field of the reals."
- Clearly, the field $\mathscr{R}$ is unique up to isomorphism inside $\mathbb{V}^{(B)}$; i.e., if $\mathscr{R}'$ is another field of the reals inside $\mathbb{V}^{(B)}$ then $\mathbb{V}^{(B)} \models$ "the fields $\mathscr{R}$ and $\mathscr{R}'$ are isomorphic." It is easy that $\mathbb{R}^{\wedge}$ is an Archimedean ordered field inside $\mathbb{V}^{(B)}$.
- Therefore, $\mathbb{V}^{(B)} \models$ "$\mathbb{R}^{\wedge} \subset \mathscr{R}$ and $\mathscr{R}$ is a (metric) completion of $\mathbb{R}^{\wedge}$." Given the usual unity 1 of $\mathbb{R}$, note that $\mathbb{V}^{(B)} \models$ "$1 := 1^{\wedge}$ is an order unit of $\mathscr{R}$."

Gordon's Theorem

- The algebraic system $\mathscr{R}{\downarrow}$ is a universally complete K-space. Moreover, there is a (canonical) isomorphism χ of the Boolean algebra B to the base $\mathfrak{P}(\mathscr{R}{\downarrow})$ of $\mathscr{R}{\downarrow}$ such that

$$\chi(b)x = \chi(b)y \leftrightarrow b \leq [\![x = y]\!],$$

$$\chi(b)x \leq \chi(b)y \leftrightarrow b \leq [\![x \leq y]\!]$$

for all $x, y \in \mathscr{R}{\downarrow}$ and $b \in B$.

- This remarkable result, establishing the immanent connection between Boolean valued analysis and vector-lattice theory, belongs to Gordon [7]. The Gordon theorem has demonstrated that each universally complete Kantorovich space provides a new model of the field of the reals, and all these models have the same rights in mathematics. Moreover, each Archimedean vector lattice, in particular, an arbitrary L_p-space, $p \geq 1$, ascends into a dense sublattice of the reals $\mathbb{R}$ inside an appropriate Boolean valued universe.

New Vistas

- Descending the basic scalar fields opens a turnpike to the intensive application of Boolean valued models in functional analysis. The technique of Boolean valued analysis demonstrates its efficiency in studying Banach spaces and algebras as well as lattice-normed spaces and modules. The corresponding results are collected and elaborated in [8, Chs. 10–12].

Logic and Freedom

- The twentieth century is marked with deep penetration of the ideas of mathematical logic into many sections of science and technology. Logic is a tool that not only organizes and orders our ways of thinking but also liberates us from dogmatism in choosing the objects and methods of research. Logic of today is a major instrument and institution of mathematical freedom. Boolean valued analysis serves as a brilliant confirmation of this thesis.
- Boolean valued analysis is a special mathematical technique based on validating truth by means of a nontrivial Boolean algebra. From a category-theoretic viewpoint, Boolean valued analysis is the theory of Boolean toposes. From a topological viewpoint, it is the theory of continuous polyverses over Stone spaces.

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