Simultaneous Linear Inequalities: Yesterday and Today

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Agenda

- Linear inequality implies linearity and order. When combined, the two produce an ordered vector space. Each linear inequality in the simplest environment of the sort is some half-space. Simultaneity implies many instances and so leads to the intersections of half-spaces. These yield polyhedra as well as arbitrary convex sets, identifying the theory of linear inequalities with convexity.

- Convexity reigns in the federation of geometry, optimization, and functional analysis. Convexity feeds generation, separation, calculus, and approximation. Generation appears as duality; separation, as optimality; calculus, as representation; and approximation, as stability [1].

- This talk addresses the origin and the state of the art of the relevant areas with a particular emphasis on the Farkas Lemma [2]. Our aim is to demonstrate how Boolean valued analysis may be applied to simultaneous linear inequalities with operators.

Founding Fathers

- Linearity, inequality, and convexity stem from the remote ages [3]–[5]. However, as the acclaimed pioneers who propounded these ideas and anticipated their significance for the future we must rank the three polymaths:

- JOSEPH-LOUIS LAGRANGE (January 25, 1736–April 10, 1813)

- JEAN-BAPTISTE JOSEPH FOURIER (March 21, 1768–May 16, 1830)

- HERMANN MINKOWSKI (June 22, 1864–January 12, 1909)

Joseph Lagrange (1736–1813)
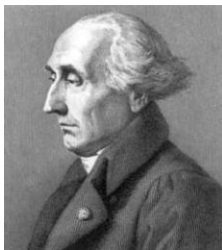
- In both research and exposition, he totally reversed the methods of his predecessors. They had proceeded in their exposition from special cases by a species of induction; his eye was always directed to the highest and most general points of view.... (Thomas J. McCormack [6])

Joseph Fourier (1768–1830)

- He [Fourier] himself was neglected for his work on inequalities, what he called "Analyse indéterminée." Darboux considered that he gave the subject an exaggerated importance and did not publish the papers on this question in his edition of the scientific works of Fourier. Had they been published, linear programming and convex analysis would be included in the heritage of Fourier. (Jean-Pierre Kahane [7])

Hermann Minkowski (1864–1909)

- Our science, which we loved above all else, brought us together; it seemed to us a garden full of flowers:.. He was for me a rare gift from heaven.... (David Hilbert [8])

Environment

- Assume that X is a real vector space, Y is a *Kantorovich space* also known as a complete vector lattice or a Dedekind complete Riesz space. Let $\mathbb{B} := \mathbb{B}(Y)$ be the *base* of Y, i.e., the complete Boolean algebras of positive projections in Y; and let $m(Y)$ be the universal completion of Y. Denote by $L(X, Y)$ the space of linear operators from X to Y. In case X is furnished with some Y-seminorm on X, by $L^{(m)}(X, Y)$ we mean the space of dominated operators from X to Y. As usual, $\{T \leq 0\} := \{x \in X \mid Tx \leq 0\}$; $\ker(T) = T^{-1}(0)$ for $T : X \to Y$. Also, $P \in \mathrm{Sub}(X, Y)$ means that P is *sublinear*, while $P \in \mathrm{PSub}(X, Y)$ means that P is *polyhedral*, i.e., finitely generated. The superscript $^{(m)}$ suggests domination.

Kantorovich's Theorem

- Find $\mathfrak{X}$ satisfying

$$X \xrightarrow{\ A\ } W$$
$$B \searrow \quad \downarrow \mathfrak{X}$$
$$Y$$
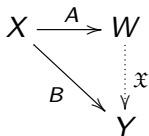
- **(1):** $(\exists \mathfrak{X})\ \mathfrak{X}A = B \leftrightarrow \ker(A) \subset \ker(B)$.
- **(2):** *If W is ordered by W_+ and $A(X) - W_+ = W_+ - A(X) = W$, then*[1]

$$(\exists \mathfrak{X} \geq 0)\ \mathfrak{X}A = B \leftrightarrow \{A \leq 0\} \subset \{B \leq 0\}.$$

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[1]Cp. [12, p. 51].

The Alternative

- Let X be a Y-seminormed real vector space, with Y a Kantorovich space. Assume that $A_1, \ldots, A_N$ and B belong to $L^{(m)}(X, Y)$. Then one and only one of the following holds:

  (1) There are $x \in X$ and $b, b' \in \mathbb{B}$ such that $b' \leq b$ and

  $$b'Bx > 0, \, bA_1x \leq 0, \ldots, bA_Nx \leq 0.$$

  (2) There are positive orthomorphisms $\alpha_1, \ldots, \alpha_N \in \mathrm{Orth}(m(Y))_+$ such that $B = \sum_{k=1}^{N} \alpha_k A_k$.

Reals: Hidden Dominance

- **Lemma 1.** *Let X be a vector space over some subfield R of the reals $\mathbb{R}$. Assume that f and g are R-linear functionals on X; in symbols, $f, g \in X^{\#} := L(X, \mathbb{R})$.*
  For the inclusion

  $$\{g \leq 0\} \supset \{f \leq 0\}$$

  to hold it is necessary and sufficient that there be $\alpha \in \mathbb{R}_+$ satisfying $g = \alpha f$.

Proof of Lemma 1

- Sufficiency is obvious.
- Necessity: The case of $f = 0$ is trivial. If $f \neq 0$ then there is some $x \in X$ such that $f(x) \in \mathbb{R}$ and $f(x) > 0$. Denote the image $f(X)$ of X under f by R_0. Put $h := g \circ f^{-1}$, i .e. $h \in R_0^{\#}$ is the only solution for $h \circ f = g$. By hypothesis, h is a positive R-linear functional on R_0. By the Bigard Theorem [12, p. 108] h can be extended to a positive homomorphism $\bar{h} : \mathbb{R} \to \mathbb{R}$, since $R_0 - \mathbb{R}_+ = \mathbb{R}_+ - R_0 = \mathbb{R}$. Each positive automorphism of $\mathbb{R}$ is multiplication by a positive real. As the sought α we may take $\bar{h}(1)$.
- The proof of Lemma 1 is complete.

Reals: Explicit Dominance

- **Lemma 2.** *Let X be an $\mathbb{R}$-seminormed vector space over some subfield R of $\mathbb{R}$. Assume that $f_1, \ldots, f_N$ and g are bounded R-linear functionals on X; in symbols, $f_1, \ldots, f_N, g \in X^* := L^{(m)}(X, \mathbb{R})$. For the inclusion*

$$\{g \leq 0\} \supset \bigcap_{k=1}^{N} \{f_k \leq 0\}$$

  to hold it is necessary and sufficient that there be $\alpha_1, \ldots, \alpha_N \in \mathbb{R}_+$ satisfying

$$g = \sum_{k=1}^{N} \alpha_k f_k.$$

Farkas: Explicit Dominance

- **Theorem 1.** *Assume that $A_1, \ldots, A_N$ and B belong to $L^{(m)}(X, Y)$. The following are equivalent:*

  (1) *Given $b \in \mathbb{B}$, the operator inequality $bBx \leq 0$ is a consequence of the simultaneous linear operator inequalities $bA_1x \leq 0, \ldots, bA_Nx \leq 0$, i.e.,*

  $$\{bB \leq 0\} \supset \{bA_1 \leq 0\} \cap \cdots \cap \{bA_N \leq 0\}.$$

  (2) *There are positive orthomorphisms $\alpha_1, \ldots, \alpha_N \in \text{Orth}(m(Y))$ such that*

  $$B = \sum_{k=1}^{N} \alpha_k A_k;$$

  i.e., B lies in the operator convex conic hull of $A_1, \ldots, A_N$.

Boolean Modeling

- Cohen's final solution of the problem of the cardinality of the continuum within ZFC gave rise to the Boolean valued models by Scott, Solovay, and Vopěnka.[2]

- Takeuti coined the term "Boolean valued analysis" for applications of the models to analysis.[3]

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[2]Cp. [9].
[3]Cp. [10].

Scott's Comments

- Scott forecasted in 1969:[4]

  We must ask whether there is any interest in these nonstandard models aside from the independence proof; that is, do they have any mathematical interest? The answer must be yes, but we cannot yet give a really good argument.

- In 2009 Scott wrote:[5]

  At the time, I was disappointed that no one took up my suggestion. And then I was very surprised much later to see the work of Takeuti and his associates. I think the point is that people have to be trained in Functional Analysis in order to understand these models. I think this is also obvious from your book and its references. Alas, I had no students or collaborators with this kind of background, and so I was not able to generate any progress.

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[4]Cp. [11].

[5]Letter of April 29, 2009 to S. S. Kutateladze.

Boolean Valued Universe

- Let $\mathbb{B}$ be a complete Boolean algebra. Given an ordinal α, put

$$V_\alpha^{(\mathbb{B})} := \{x \mid (\exists \beta \in \alpha) \; x : \mathrm{dom}(x) \to \mathbb{B} \; \& \; \mathrm{dom}(x) \subset V_\beta^{(\mathbb{B})}\}.$$

- The *Boolean valued universe* $\mathbb{V}^{(\mathbb{B})}$ is

$$\mathbb{V}^{(\mathbb{B})} := \bigcup_{\alpha \in \mathsf{On}} V_\alpha^{(\mathbb{B})},$$

  with On the class of all ordinals.

- The truth value $[\![\varphi]\!] \in \mathbb{B}$ is assigned to each formula φ of ZFC relativized to $\mathbb{V}^{(\mathbb{B})}$.

Descending and Ascending

- Given φ, a formula of ZFC, and y, a member of $\mathbb{V}^\mathbb{B}$; put
  $A_\varphi := A_{\varphi(\cdot,\,y)} := \{x \mid \varphi(x,\,y)\}$.
- The *descent* $A_\varphi\downarrow$ of a class A_φ is

$$A_\varphi\downarrow := \{t \mid t \in \mathbb{V}^{(\mathbb{B})} \ \& \ [\![\varphi(t,\,y)]\!] = \mathbb{1}\}.$$

- If $t \in A_\varphi\downarrow$, then it is said that t *satisfies* $\varphi(\cdot,\,y)$ *inside* $\mathbb{V}^{(\mathbb{B})}$.
- The *descent* $x\downarrow$ of $x \in \mathbb{V}^{(\mathbb{B})}$ is defined as

$$x\downarrow := \{t \mid t \in \mathbb{V}^{(\mathbb{B})} \ \& \ [\![t \in x]\!] = \mathbb{1}\},$$

  i.e. $x\downarrow = A_{\cdot\in x}\downarrow$. The class $x\downarrow$ is a set.
- If x is a nonempty set inside $\mathbb{V}^{(\mathbb{B})}$ then

$$(\exists z \in x\downarrow)[\![(\exists t \in x)\ \varphi(t)]\!] = [\![\varphi(z)]\!].$$

- The *ascent* functor acts in the opposite direction.

The Reals Within

- There is an object $\mathscr{R}$ inside $\mathbb{V}^{(\mathbb{B})}$ modeling $\mathbb{R}$, i.e.,

$$[\![\mathscr{R} \text{ is the reals}]\!] = \mathbb{1}.$$

- Let $\mathscr{R}\downarrow$ be the descent of the carrier $|\mathscr{R}|$ of the algebraic system $\mathscr{R} := (|\mathscr{R}|, +, \cdot, 0, 1, \leq)$ inside $\mathbb{V}^{(\mathbb{B})}$.

- Implement the descent of the structures on $|\mathscr{R}|$ to $\mathscr{R}\downarrow$ as follows:

$$x + y = z \leftrightarrow [\![x + y = z]\!] = \mathbb{1};$$
$$xy = z \leftrightarrow [\![xy = z]\!] = \mathbb{1};$$
$$x \leq y \leftrightarrow [\![x \leq y]\!] = \mathbb{1};$$
$$\lambda x = y \leftrightarrow [\![\lambda^\wedge x = y]\!] = \mathbb{1} \ (x, y, z \in \mathscr{R}\downarrow, \ \lambda \in \mathbb{R}).$$

- **Gordon Theorem.**[6] $\mathscr{R}\downarrow$ with the descended structures is *a universally complete vector lattice with base* $\mathbb{B}(\mathscr{R}\downarrow)$ *isomorphic to* $\mathbb{B}$.

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[6]Cp. [9, p. 349].

Proof of Theorem 1

(2)$\to$ (1): If $B = \sum_{k=1}^{N} \alpha_k A_k$ for some positive $\alpha_1, \ldots, \alpha_N$ in $\text{Orth}(m(Y))$ while $bA_k x \leq 0$ for $b \in \mathbb{B}$ and $x \in X$, then

$$bBx = b \sum_{k=1}^{N} \alpha_k A_k x = \sum_{k=1}^{N} \alpha_k b A_k x \leq 0$$

since orthomorphisms commute and projections are orthomorphisms of $m(Y)$.

Proof of Theorem 1

$(1) \rightarrow (2)$:

- Consider the separated Boolean valued universe $\mathbb{V}^{(\mathbb{B})}$ over the base $\mathbb{B}$ of Y. By the Gordon Theorem the ascent $Y\uparrow$ of Y is $\mathscr{R}$, the reals inside $\mathbb{V}^{(\mathbb{B})}$.

- Using the canonical embedding, we see that $X^\wedge$ is an $\mathscr{R}$-seminormed vector space over the standard name $\mathbb{R}^\wedge$ of the reals $\mathbb{R}$.

- Moreover, $\mathbb{R}^\wedge$ is a subfield and sublattice of $\mathscr{R} = Y\uparrow$ inside $\mathbb{V}^{(\mathbb{B})}$.

Proof of Theorem 1

$(1) \rightarrow (2)$:

- Put $f_k := A_k\uparrow$ for all $k := 1, \dots, N$ and $g := B\uparrow$. Clearly, all $f_1, \dots, f_N, g$ belong to $(X^\wedge)^*$ inside $\mathbb{V}^\mathbb{B}$.

- Define the finite sequence

$$f : \{1, \dots, N\}^\wedge \rightarrow (X^\wedge)^*$$

as the ascent of $(f_1, \dots, f_N)$. In other words, the truth values are as follows:

$$[\![f_{k^\wedge}(x^\wedge) = A_k x]\!] = \mathbb{1}, \quad [\![g(x^\wedge) = Bx]\!] = \mathbb{1}$$

for all $x \in X$ and $k := 1, \dots, N$.

Proof of Theorem 1

$(1) \rightarrow (2)$:

- Put
$$b := [\![A_1 x \leq 0^\wedge]\!] \wedge \cdots \wedge [\![A_N x \leq 0^\wedge]\!].$$

  Then $bA_k x \leq 0$ for all $k := 1, \ldots, N$ and $bBx \leq 0$ by (1).

- Therefore,
$$[\![A_1 x \leq 0^\wedge]\!] \wedge \cdots \wedge [\![A_N x \leq 0^\wedge]\!] \leq [\![Bx \leq 0^\wedge]\!].$$

- In other words,
$$[\![(\forall k := 1^\wedge, \ldots, N^\wedge) f_k(x^\wedge) \leq 0^\wedge]\!]$$

$$= \bigwedge_{k := 1, \ldots, N} [\![f_{k^\wedge}(x^\wedge) \leq 0^\wedge]\!] \leq [\![g(x^\wedge) \leq 0^\wedge]\!].$$

Proof of Theorem 1

$(1) \to (2)$:

- By Lemma 2 inside $\mathbb{V}^{(\mathbb{B})}$ and the maximum principle of Boolean valued analysis, there is a finite sequence $\alpha : \{1^\wedge, \dots, N^\wedge\} \to \mathscr{R}_+$ inside $\mathbb{V}^{(\mathbb{B})}$ satisfying

$$\llbracket (\forall x \in X^\wedge) \; g(x) = \sum_{k=1^\wedge}^{N^\wedge} \alpha(k) f_k(x) \rrbracket = \mathbb{1}.$$

- Put $\alpha_k := \alpha(k^\wedge) \in \mathscr{R}_+\!\downarrow$ for $k := 1, \dots, N$.
- Multiplication by an element in $\mathscr{R}\!\downarrow$ is an orthomorphism of $m(Y)$. Moreover,

$$B = \sum_{k=1}^{N} \alpha_k A_k,$$

which completes the proof.

Counterexample: No Dominance

- Lemma 1, describing the consequences of a single inequality, does not restrict the class of functionals under consideration.

- The analogous version of the Farkas Lemma simply fails for two simultaneous inequalities in general.

- The inclusion $\{f = 0\} \subset \{g \leq 0\}$ equivalent to the inclusion $\{f = 0\} \subset \{g = 0\}$ does not imply that f and g are proportional in the case of an arbitrary subfield of $\mathbb{R}$. It suffices to look at $\mathbb{R}$ over the rationals $\mathbb{Q}$, take some discontinuous $\mathbb{Q}$-linear functional on $\mathbb{Q}$ and the identity automorphism of $\mathbb{Q}$.

Reconstruction: No Dominance

- **Theorem 2.**

  Take A and B in $L(X, Y)$. The following are equivalent:

  (1) $(\exists \alpha \in \mathrm{Orth}(m(Y)))\ B = \alpha A$;

  (2) There is a projection $\varkappa \in \mathbb{B}$ such that

  $$\{\varkappa b B \leq 0\} \supset \{\varkappa b A \leq 0\}; \quad \{\neg \varkappa b B \leq 0\} \supset \{\neg \varkappa b A \geq 0\}$$

  for all $b \in \mathbb{B}$.[7]

- PROOF. Boolean valued analysis reduces the claim to the scalar case. Applying Lemma 1 twice and writing down the truth values, complete the proof.

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[7] As usual, $\neg \varkappa := \mathbb{1} - \varkappa$.

Interval Operators

- Let X be a vector lattice. An *interval operator* **T** from X to Y is an order interval $[\underline{T}, \overline{T}]$ in $L^{(r)}(X, Y)$, with $\underline{T} \leq \overline{T}$.[8]
- The interval equation $\mathbf{B} = \mathfrak{X}\mathbf{A}$ has a *weak interval solution* provided that $(\exists \mathfrak{X})(\exists A \in \mathbf{A})(\exists \mathbf{B} \in \mathbf{B})\ \mathbf{B} = \mathfrak{X}\mathbf{A}$.
- Given an interval operator **T** and $x \in X$, put

$$P_\mathbf{T}(x) = \overline{T}x_+ - \underline{T}x_-.$$

- Call **T** *adapted* in case $\overline{T} - \underline{T}$ is the sum of finitely many disjoint addends.
- Put $\sim (x) := -x$ for all $x \in X$.

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[8]Cp. [14]

Interval Equations

- **Theorem 3.** *Let X be a vector lattice, and let Y be a Kantorovich space. Assume that $\mathbf{A}_1, \ldots, \mathbf{A}_N$ are adapted interval operators and $\mathbf{B}$ is an arbitrary interval operator in the space of order bounded operators $L^{(r)}(X, Y)$.*
  The following are equivalent:

- (1) *The interval equation*

$$\mathbf{B} = \sum_{k=1}^{N} \alpha_k \mathbf{A}_k$$

  has a weak interval solution $\alpha_1, \ldots, \alpha_N \in \mathrm{Orth}(Y)_+$.
  (2) *For all $b \in \mathbb{B}$ we have*

$$\{b\mathfrak{B} \geq 0\} \supset \{b\mathfrak{A}_1^{\sim} \leq 0\} \cap \cdots \cap \{b\mathfrak{A}_N^{\sim} \leq 0\},$$

  where $\mathfrak{A}_k^{\sim} := P_{\mathbf{A}_k} \circ \sim$ for $k := 1, \ldots, N$ and $\mathfrak{B} := P_{\mathbf{B}}$.

Inhomogeneous Inequalities

- **Theorem 4.** Let X be a Y-seminormed real vector space, with Y a Kantorovich space. Assume given some dominated operators $A_1, \ldots, A_N, B \in L^{(m)}(X, Y)$ and elements $u_1, \ldots, u_N, v \in Y$. The following are equivalent:

  (1) For all $b \in \mathbb{B}$ the inhomogeneous operator inequality $bBx \le bv$ is a consequence of the consistent simultaneous inhomogeneous operator inequalities $bA_1x \le bu_1, \ldots, bA_Nx \le bu_N$, i.e.,

  $$\{bB \le bv\} \supset \{bA_1 \le bu_1\} \cap \cdots \cap \{bA_N \le bu_N\}.$$

  (2) There are positive orthomorphisms $\alpha_1, \ldots, \alpha_N \in \mathrm{Orth}(m(Y))$ satisfying

  $$B = \sum_{k=1}^{N} \alpha_k A_k; \quad v \ge \sum_{k=1}^{N} \alpha_k u_k.$$

Inhomogeneous Matrix Inequalities

- **Theorem 5.**[9] *Let X be a Y-seminormed real vector space, with Y a Kantorovich space. Assume that $A \in L^{(m)}(X, Y^s)$, $B \in L^{(m)}(X, Y^t)$, $u \in Y^s$, and $v \in Y^t$, where s and t are some naturals.*
  The following are equivalent:

- (1) *For all $b \in \mathbb{B}$ the inhomogeneous operator inequality $bBx \leq bv$ is a consequence of the consistent inhomogeneous inequality $bAx \leq bu$, i.e., $\{bB \leq bv\} \supset \{bA \leq bu\}$.*

- (2) *There is some $s \times t$ matrix with entries positive orthomorphisms of $m(Y)$ such that $B = \mathfrak{X}A$ and $\mathfrak{X}u \leq v$ for the corresponding linear operator $\mathfrak{X} \in L_+(Y^s, Y^t)$.*

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[9]Cp. [13].

Complex Scalars

- **Theorem 6.** Let X be a Y-seminormed complex vector space, with Y a Kantorovich space. Assume given some $u_1, \ldots, u_N, v \in Y$ and dominated operators $A_1, \ldots, A_N, B \in L^{(m)}(X, Y_\mathbb{C})$ from X into the complexification $Y_\mathbb{C} := Y \otimes iY$ of Y.[10] Assume further that the inhomogeneous simultaneous inequalities $|A_1 x| \leq u_1, \ldots, |A_N x| \leq u_N$ are consistent. Then the following are equivalent:

  (1) $\{b|B(\cdot)| \leq bv\} \supset \{b|A_1(\cdot)| \leq bu_1\} \cap \cdots \cap \{b|A_N(\cdot)| \leq bu_N\}$ for all $b \in \mathbb{B}$.

  (2) There are complex orthomorphisms $c_1, \ldots, c_N \in \mathrm{Orth}(m(Y)_\mathbb{C})$ satisfying

  $$B = \sum_{k=1}^{N} c_k A_k; \quad v \geq \sum_{k=1}^{N} |c_k| u_k.$$

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[10] Cp. [3, p. 338].

Inhomogeneous Sublinear Inequalities

- **Lemma 3.** *Let X be a real vector space. Assume that $p_1, \ldots, p_N \in \mathrm{PSub}(X) := \mathrm{PSub}(X, \mathbb{R})$ and $p \in \mathrm{Sub}(X)$. Assume further that $v, u_1, \ldots, u_N \in \mathbb{R}$ make consistent the simultaneous sublinear inequalities $p_k(x) \leq u_k$, with $k := 1, \ldots, N$.*
  The following are equivalent:
  (1) $\{p \geq v\} \supset \bigcap\limits_{k=1}^{N} \{p_k \leq u_k\}$;
  (2) there are $\alpha_1, \ldots, \alpha_N \in \mathbb{R}_+$ satisfying

$$(\forall x \in X) \ p(x) + \sum_{k=1}^{N} \alpha_k p_k(x) \geq 0, \quad \sum_{k=1}^{N} \alpha_k u_k \leq -v.$$

Proof of Lemma 3

- $(2) \rightarrow (1)$: If x is a solution to the simultaneous inhomogeneous inequalities $p_k(x) \leq u_k$ with $k := 1, \ldots, N$, then

$$0 \leq p(x) + \sum_{k=1}^{N} \alpha_k p_k(x) \leq p(x) + \sum_{k=1}^{N} \alpha_k u_k(x) \leq p(x) - v.$$

- $(1) \rightarrow (2)$: Given $(x, t) \in X \times \mathbb{R}$, put $\bar{p}_k(x, t) := p_k(x) - t u_k$, $\bar{p}(x, t) := p(x) - tv$ and $\tau(x, t) := -t$. Clearly, $\tau, \bar{p}_1, \ldots, \bar{p}_N \in \mathrm{PSub}(X \times \mathbb{R})$ and $\bar{p} \in \mathrm{Sub}(X \times \mathbb{R})$. Take

$$(x, t) \in \{\tau \leq 0\} \cap \bigcap_{k=1}^{N} \{\bar{p}_k \leq 0\}.$$

If, moreover, $t > 0$; then $u_k \geq p_k(x/t)$ for $k := 1, \ldots, N$ and so $p(x/t) \leq v$ by hypothesis. In other words $(x, t) \in \{\bar{p} \leq 0\}$. If $t = 0$ then take some solution $\bar{x}$ of the simultaneous inhomogeneous polyhedral inequalities under study.

Proof of Lemma 3

Since $x \in K := \bigcap_{k=1}^{N} \{p_k \leq 0\}$; therefore, $p_k(\bar{x} + x) \leq p(x) + p_k(x) \leq u_k$ for all $k := 1, \ldots, N$. Hence, $p(\bar{x} + x) \geq v$ by hypothesis. So the sublinear functional p is bounded below on the cone K. Consequently, p assumes only positive values on K. In other words, $(x, 0) \in \{\bar{p} \leq 0\}$. Thus

$$\{\bar{p} \geq 0\} \supset \bigcap_{k=1}^{N} \{\bar{p}_k \leq 0\}$$

and by Lemma 2.2. of [1] there are positive reals $\alpha_1, \ldots, \alpha_N, \beta$ such that for all $(x, t) \in X \times \mathbb{R}$ we have

$$\bar{g}(x) + \beta\tau(x) + \sum_{k=1}^{N} \alpha_k \bar{p}_k(x) \geq 0.$$

Clearly, the so-obtained parameters $\alpha_1, \ldots, \alpha_N$ are what we sought for. The proof of Lemma 3 is complete.

Sublinear Inequalities

- **Theorem 7.** *Let X be a Y-seminormed real vector space, with Y a Kantorovich space. Given are some dominated polyhedral sublinear operators $P_1, \ldots, P_N \in \mathrm{PSub}^{(m)}(X, Y)$ and a dominated sublinear operator $P \in \mathrm{Sub}^{(m)}(X, Y)$. Assume further that $u_1, \ldots, u_N, v \in Y$ make consistent the simultaneous inhomogeneous inequalities $P_1(x) \leq u_1, \ldots, P_N(x) \leq u_N$.*
  The following are equivalent:
  (1) for all $b \in \mathbb{B}$ the inhomogeneous sublinear operator inequality $bP(x) \geq bv$ is a consequence of the simultaneous inhomogeneous sublinear operator inequalities $bP_1(x) \leq bu_1, \ldots, bP_N(x) \leq bu_N$, i.e.,

  $$\{bP \geq bv\} \supset \{bP_1 \leq bu_1\} \cap \cdots \cap \{bP_N \leq bu_N\};$$

  (2) there are positive $\alpha_1, \ldots, \alpha_N \in \mathrm{Orth}(m(Y))$ satisfying

  $$(\forall x \in X) \; P(x) + \sum_{k=1}^{N} \alpha_k P_k(x) \geq 0, \quad \sum_{k=1}^{N} \alpha_k u_k \leq -v.$$

Lagrange's Principle

- The finite value of the constrained problem

$$P_1(x) \leq u_1, \ldots, P_N(x) \leq u_N, \quad P(x) \to \inf$$

is the value of the unconstrained problem for an appropriate Lagrangian without any constraint qualification other that polyhedrality.

- The Slater condition allows us to eliminate polyhedrality as well as considering a unique target space. This is available in a practically unrestricted generality [12].

- About the new trends relevant to the Farkas Lemma see [15]–[19].

Freedom and Inequality

- Convexity is the theory of linear inequalities in disguise.

- Abstraction is the freedom of generalization. Freedom is the loftiest ideal and idea of man, but it is demanding, limited, and vexing. So is abstraction. So are its instances in convexity, hence, in simultaneous inequalities.

- The freedom of set theory empowered us with the Boolean valued models yielding a lot of surprising and unforeseen visualizations of the ingredients of mathematics.

- Mathematics becomes logic. Logic organizes and orders our ways of thinking, manumitting us from conservatism in choosing the objects and methods of research. Logic of today is a fine instrument and institution of mathematical freedom.

- Freedom presumes liberty and equality. Inequality paves way to freedom.

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