MATRIX-VALUED METAPLECTIC GABOR FRAMES IN $L^2(\mathbb{R}^d, \mathbb{C}^{l\times l})$

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Abstract. The concept of matrix-valued metaplectic Gabor system is introduced in $L^2(\mathbb{R}^d, \mathbb{C}^{l\times l})$. We present that every matrix-valued metaplectic Gabor system is not a Gabor frame in $L^2(\mathbb{R}^d, \mathbb{C}^{l\times l})$. Necessary and sufficient conditions for matrix-valued metaplectic and Gabor systems are obtained to make them frame in $L^2(\mathbb{R}^d, \mathbb{C}^{l\times l})$. Also, the frame operator is represented in terms of trace in $L^2(\mathbb{R}^d, \mathbb{C}^{l\times l})$. Further, the concept of scalability for both the systems is introduced and some interesting properties of the Gabor frame and the scalable Gabor frame corresponding to a bounded linear operator $K-$ are obtained. Some necessary and sufficient conditions for the bounded linear metaplectic Gabor atom, generating a frame and $K-$ frame for $L^2(\mathbb{R}^d, \mathbb{C}^{l\times l})$, are given. Finally, we show that for some particular sequences, the scalable metaplectic Gabor frame $\{a_{(\mathrm{y},\eta)}\pi_A(\mathrm{y},\eta)g\}$ is a $K-$ metaplectic Gabor frame for $L^2(\mathbb{R}^d)$ and to illustrate the findings, examples and counter-examples are provided.

1. Introduction

The basic idea of signal breakdown in elementary signals was first observed by Gabor [12]. Based upon this incredible idea of Gabor, Duffin and Schaeffer [11] developed frames in the context of separable Hilbert spaces to study the nonharmonic expansion of functions. The concept of frame presented by Duffin and Schaeffer was later examined by Young [30]. Daubechies et al. [10] revived and revitalized Hilbert frames by demonstrating their significance for data processing. Since a frame being redundant, allows non-unique representations of signals with controlled bounds. This redundancy is helpful when there is noise, data loss, or other distortions, as it can reconstruct the signal even if some of the information is lost. A countable sequence $\{y_i\}_{i\in I}$ constitutes frame for a Hilbert space H, if the inequality $\alpha_0||y||^2 \leq \sum_{i=1}^{\infty}|\langle y, y_i\rangle|^2 \leq \beta_0||y||^2$ is satisfied $\forall y \in H$ and $0 < \alpha_0 \leq \beta_0 < \infty$. The sequence $\{y_i\}_{i\in I}$ is called a Bessel sequence if only the upper inequality is satisfied. Also, if both the bounds coincide then $\{y_i\}_{i\in I}$ is a tight frame. Furthermore, $\{y_i\}_{i\in I}$ is a Parseval's frame if $\alpha_0 = \beta_0 = 1$. For more information related to the frame, one can refer to [4, 5, 18–23]. The most commonly used frames in time-frequency analysis, compressed sensing, radar and sonar signal processing, communication systems, and biometric recognition; is the Gabor frame.

Traditionally, the concept of an orthonormal basis was used to retrieve a signal's information to receive all the information. Later, tight frames-a more adaptable system-were developed to represent any vector in terms of frame components. While it's true that every frame is not a tight frame, although some can be, particularly Parseval frames. Those frames which can be made Parseval are called scalable frames. Kutyniok et al. [17] presented the idea of the scalable frames, which are the scaled pieces themselves. However, Kutyniok's common scaling approach does not hold for

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many frames. Moreover, the approach is not appropriate for real-world circumstances due to the relatively large number of zero coefficients. As the real-world scenario requires the contribution of each vector used in the investigation. Later on, scalable frames were changed by Cassaza et al. [3], who also presented a novel concept for scaling frame components.

Some specific extensions of frames such as $K-$ frame [13], non-linear frames [14], $K-$atomic [17] and approximative $K-$ atomic decompostion [15], scalable frames [21], controlled frames [23], controlled $K-$ frames [22] and continuous $Kg-$ fusion frames [1] have been introduced for various theoretical and practical uses. $K-$ frames are generalized ordinary frames, and even though their span limit is limited to the range of K, their generality makes them practically significant.

In fields such as geophysics, medical imaging, image processing, and data frequently takes the form of multichannel signals, where each "sample" is a vector or matrix instead of a scalar. Matrix-valued Gabor and Wavelet frames allow us to analyze multichannel signals coherently by treating all channels together. These systems were firstly structured by Xia and Suter [29] and Antolin and Zalik [24] in $L^2(\mathbb{R}, \mathbb{C}^{l\times l})$. Vashisht [25] studied various properties of matrix-valued frames using a bounded linear operator in $L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$. In 2023, Jindal et al. [16] interpret the frame conditions in different groups including the extended affine and the Weyl-Heisenberg groups.

The most commonly used representation of a signal is STFT (Short Time Fourier Transform), as it can locate the local time-frequency behaviour of a signal. However, STFT often suffers from the trade-off between the time-frequency. To resolve this problem a more general $TF-$(Time-Frequency) distribution, that is, the Wigner distribution is used. Wigner distributions were introduced by Wigner [28] in 1932. Later, Ville [26] discovered that they had origins in signal analysis. To provide a better environment for the signal representation, more general distributions such as metaplectic Wigner distribution are used. Cordero and Rodino [8] introduced metaplectic Wigner distributions in 2022. In the context of metaplectic Wigner distributions, Cordero and Giacchi [7] introduced the metaplectic Gabor frames (MGF) as a generalization of Gabor frames in 2024.

Motivated by the research work of Antolin and Zalik [24], who introduced matrix-valued Gabor frames ($MVGF$) in $L^2(\mathbb{R}, \mathbb{C}^{l\times l})$, later studied by Vashisht et al. [16] in $L^2(\mathbb{R}, \mathbb{C}^{l\times l})$, and the MGF introduced by Cordero [6] in $L^2(\mathbb{R}^d)$, we extend the notion of $MVGF$ and MGF in $L^2(\mathbb{R}^d, \mathbb{C}^{l\times l})$. Therefore, in this article, we obtained some interesting results by interplaying between Gabor and MGF in $L^2(\mathbb{R}^d, \mathbb{C}^{l\times l})$. We obtain various necessary and sufficient conditions for the Gabor and metaplectic Gabor systems to be a frame for $L^2(\mathbb{R}^d, \mathbb{C}^{l\times l})$. The frame operator is represented in terms of trace in $L^2(\mathbb{R}^d, \mathbb{C}^{l\times l})$, and the concept of scalability for both the systems is introduced, and some new results are constructed using a bounded linear operator K. Lastly, we have shown that under a particular condition, scalable MGF becomes a $K-MGF$ frame for $L^2(\mathbb{R}^d)$. Examples and counter examples are also constructed.

The structure of the article is as follows: Section 2 includes basic terminologies and notations that will be used in the article. Section 3 offers the main findings of the article. This section starts with the $MVGF$ definition. Firstly, we prove the interplay between the $MVGF$ and the corresponding Gabor frame, followed by some examples and counter-examples. In the next theorem, we prove the result corresponding to Fourier coefficients in $L^2(\mathbb{R}^d, \mathbb{C}^{l\times l})$. Later, we provide the representation of the frame operator in $L^2(\mathbb{R}^d, \mathbb{C}^{l\times l})$. In Theorems 3.5 and 3.7 we obtain some interesting results of the Gabor frames. For a given example, we observe that every matrix-valued metaplectic Gabor system is not a Gabor frame in $L^2(\mathbb{R}^d, \mathbb{C}^{l\times l})$. In Theorem 3.9, the necessary and sufficient conditions are obtained for the MGF in $L^2(\mathbb{R}^d, \mathbb{C}^{l\times l})$. In Theorem 3.11, we discuss some properties of the MGF corresponding to a bounded linear operator. The required conditions for the matrix-valued metaplectic Gabor system to be a frame for $L^2(\mathbb{R}^d)$ are given. We discuss the conditions for which the scalable MGF is a $K-$ frame. Also, we provide some examples and counter-examples to enhance our findings. Finally, section 4 concludes the study by summarizing the obtained results and offering an overview of findings and insight of possible future work.

2. Preliminaries

This section includes the basic terminologies relevant to the article. We will use the following notations:
$\Delta_{\mu,\nu} = \aleph\mu\varpi_\nu$ is used for the Gabor system where $\aleph\mu$ is the modulation operator with parameter μ and ϖ_ν is the translation operator with parameter ν. $\daleth$ is the normalized rescaling operator.
$\mathbb{Z}, \mathbb{C}, \mathbb{N}$ denotes the set of integers, complex and natural numbers respectively. $\mathbb{R}^d$ is d-dimensional Euclidean space. $S(\mathbb{R}^d)$ and $S'(\mathbb{R}^d)$ represents Schwartz function space and its topological dual respectively. $\mathcal{O}_{l\times l}$ and $\mathcal{I}_{l\times l}$ denotes the null and identity matrices of order l. $B(\mathcal{H}), GL(\mathbb{R}^{2d})$ denote the space of all bounded linear operators over $\mathcal{H}$ and general linear group of matrices over $\mathbb{R}^{2d}$ respectively. Throughout the article, the bold letters $\mathbf{f}, \mathbf{g}, \mathbf{x}, \mathbf{S}, \mathbf{J}, \boldsymbol{I}_l, \boldsymbol{f}_0$ are used for matrix-valued functions. I denotes a countable set and $\beth$ is a subset of $\mathbb{R}^{2d}$.
$L^2(\mathbb{R}^d, \mathbb{C}^{l\times l})$ is the matrix-valued space defined as:

$$L^2(\mathbb{R}^d, \mathbb{C}^{l\times l}) = \left\{ \mathbf{x}() = \begin{bmatrix} x_{11}() & \cdots & x_{1l}() \\ x_{21}() & \cdots & x_{2l}() \\ \vdots & \cdots & \vdots \\ x_{l1}() & \cdots & x_{ll}() \end{bmatrix} ; x_{\mathsf{ij}} \in L^2(\mathbb{R}^d), 1 \leq \mathsf{m}, \mathsf{n} \leq l \right\}.$$

The space $L^2(\mathbb{R}^d, \mathbb{C}^{l\times l})$, corresponding to the norm

$$||\mathbf{x}||_2^2 = \sum_{1\leq \mathsf{m},\mathsf{n}\leq l} \int_{\mathbb{R}^d} |x_{\mathsf{ij}}(y)|^2 dy, \ \mathbf{x} \in L^2(\mathbb{R}^d, \mathbb{C}^{l\times l}),$$

is a Banach space. The integral of the function $\mathbf{x} \in L^2(\mathbb{R}^d, \mathbb{C}^{l\times l})$ is defined as:

$$\int_{\mathbb{R}^d} \mathbf{x}(y) dy = \begin{bmatrix} \int_{\mathbb{R}^d} x_{11}(y)dy & \cdots & \int_{\mathbb{R}^d} x_{1l}(y)dy \\ \int_{\mathbb{R}^d} x_{21}(y)dy & \cdots & \int_{\mathbb{R}^d} x_{2l}(y)dy \\ \vdots & \cdots & \vdots \\ \int_{\mathbb{R}^d} x_{l1}(y)dy & \cdots & \int_{\mathbb{R}^d} x_{ll}(y)dy \end{bmatrix}.$$

The inner product $\langle \mathbf{f},\ \mathbf{g}\rangle$ for $\mathbf{f},\ \mathbf{g} \in L^2(\mathbb{R}^d, \mathbb{C}^{l\times l})$ is defined as:

$$\langle \mathbf{f},\ \mathbf{g}\rangle = \int_{\mathbb{R}^d} \mathbf{f}^* \mathbf{g} d\mathbf{x},$$

where $\mathbf{f}^*$ is the transpose of $\mathbf{f}$.
The translation and the modulation operators in $L^2(\mathbb{R}^d, \mathbb{C}^{l\times l})$ with their parameters $\mu,\ \nu \in \mathbb{R}^d$ are defined as $\aleph\mu\mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{x} - \mu)$ and $\varpi_\nu \mathbf{f}(\mathbf{x}) = e^{2\pi\iota\nu\mathbf{x}}\mathbf{f}(\mathbf{x})$ respectively.

Definition 2.1. [13] A sequence of vectors $\{y_i\}$ is said to be a $K-$frame for a Hilbert space H if it satisfies

$$\alpha_0 ||K^* y||^2 \leq \sum_{i\in I} |\langle y, y_i\rangle|^2 \leq \beta_0 ||y||^2, \ \forall y \in H,$$

where K is a bounded linear operator on the Hilbert space H.
If the sequence $\{y_i\}$ satisfies

$$\sum_{i\in I}|\langle y,y_i\rangle|^2=||K^*y||^2,$$

then the frame will be a Parseval $K-$ frame.

Definition 2.2. [7] A symplectic matrix is a matrix $\boldsymbol{P}\in\mathbb{R}^{2l\times 2l}$ if for some

$$\boldsymbol{J}=\begin{bmatrix}\mathcal{O}_{l\times l} & \mathcal{I}_{l\times l}\\ -\mathcal{I}_{l\times l} & \mathcal{O}_{l\times l}\end{bmatrix},$$

the equality $\boldsymbol{P^TJP}=\boldsymbol{J}$ holds.

Definition 2.3. [7] A group consisting of elements of the form

$$\begin{bmatrix}1 & a & b\\ 0 & 1 & c\\ 0 & 0 & 1\end{bmatrix}, where\ a,b,c\in\mathbb{R},$$

is called the Heisenberg group. A metaplectic operator $V:L^2(\mathbb{R}^d)\longrightarrow L^2(\mathbb{R}^d)$ is a unitary operator satisfying $V^{-1}\theta(z;\tau)V=\theta(\boldsymbol{P}z;\tau)$, where $z=(x,\xi)\in\mathbb{R}^{2d},\boldsymbol{P}$ is the symplectic matrix and $\theta=e^{2\pi\iota\tau-\iota\xi.x}\pi(z(x,\xi))$, is the Schrodinger representation of the Heisenberg group.

Definition 2.4. [7] The metaplectic Wigner distribution is defined as: $W_V(p,q)=\hat{V}(p\bigotimes\overline{q})$, where $p,q\in L^2(\mathbb{R}^d)$, $\hat{V}$ is the metaplectic operator and $p\bigotimes\overline{q}(z_1,z_2)=p(z_1)\overline{q}(z_2)$ is the tensor product of p and q.

Definition 2.5. [7] The operator $\pi_A(\mathrm{y},\eta):S(\mathbb{R}^d)\longrightarrow S'(\mathbb{R}^d)$ is called metaplectic atom if

$$\langle f,\pi_A(\mathrm{y},\eta)g\rangle=W_A(f,g)(\mathrm{y},\eta),(\mathrm{y},\eta)\in\beth\subseteq\mathbb{R}^{2d},\forall f,g\in S(\mathbb{R}^d).$$

The representation of metaplectic Gabor atom is given by Cordero [6] as

$$\pi_A(\mathrm{y},\eta)=\alpha_E e^{-2\pi\iota E_{11}E_{12}^{-\nu}\xi.x}\mu_{E_{22}^{-\nu}\xi}(\nu_{E_{12}E_{22}^{-1}(E_{11}-E_{12}E_{22}{}^{-1}E_{21})x})\daleth_{E_{22}E_{12}^{-1}},$$

where $\alpha_E=\sqrt{\frac{|E|}{|E_{22}E_{12}|}}$ and

$$E=\begin{bmatrix}E_{11} & E_{12}\\ E_{21} & E_{22}\end{bmatrix}, and\ |E|\neq 0.$$

Definition 2.6. [7] The metaplectic Gabor system $\{\pi_A(\mathrm{y},\eta)g\}$ is said to be a MGF for $L^2(\mathbb{R}^d)$ if it satisfies

$$\alpha_0||f||^2\leq\sum_{\mathrm{y},\eta\in\beth}|\langle f,\pi_A(\mathrm{y},\eta)g\rangle|^2\leq\beta_0||f||^2,$$

$\forall f\in L^2(\mathbb{R}^d)$ and $\alpha_0,\beta_0>0$.

Definition 2.7. [7] The normalized rescaling operator for some $E\in GL(d,\mathbb{R})$, $\daleth:L^2(\mathbb{R}^d)\longrightarrow L^2(\mathbb{R}^d)$ is defined as:

$$\daleth_E f(.)=|E|^{\frac{1}{2}}f(E.),\ \forall f\in L^2(\mathbb{R}^d).$$

Definition 2.8. [2] The sequence $\{b_n\}$ is said to be semi-normalized if for some positive numbers a,b, the following inequality:

$$a\leq b_n\leq b,\forall n\in\mathbb{N},$$

is satisfied.

Definition 2.9. [9] The Schwartz function space $S(\mathbb{R}^d)$ is the space of all those functions $f \in \mathbb{C}^\infty(\mathbb{R}^d)$ which decays rapidly such that

$$sup_{y\in\mathbb{R}^d}|y^c\partial^d f(y)| < \infty,\ \forall c,\ d \in \mathbb{N}^d,$$

where $\mathbb{C}^\infty(\mathbb{R}^d)$ is the space of smooth functions.
The topological dual of $S(\mathbb{R}^d)$ is the space $S'(\mathbb{R}^d)$ containing the tempered distributions.

Theorem 2.10. *[27] For some $\tau^d = [0, b]^d$, $b > 0$ and $f \in L^2(\tau^d) \subset L^2(\mathbb{R}^d)$, the following equality holds:*

$$\sum_{k\in\mathbb{Z}^d} |c_k|^2 = \frac{1}{b^d}|f|^2,\ \textit{where}\ c_k = \frac{1}{b^d}\int_{\tau^d} f(x)e^{-2\pi\iota k.x}dx.$$

For further studies on metaplectic Gabor frames and related concepts one can refer [6–9, 21].

3. Main Results

Antolin and Zalik, [24] first introduced the concept of matrix-valued systems. Based upon this idea of matrix-valued systems, Vashisht et al. [16] proved that the frame conditions in $L^2(\mathbb{R})$, could be extended to $L^2(\mathbb{R}, \mathbb{C}^{l\times l})$ if we take a system of scalar matrix-valued functions and each function constitutes frame for $L^2(\mathbb{R})$. The conditions given by Vashisht et al. [16] are extended to the multi-dimensional matrix-valued Gabor and metaplectic Gabor systems and prove that the matrix-valued frame generating system need not be a scalar matrix. It can have different diagonal elements. We start the section by introducing $MVGF$ in $L^2(\mathbb{R}^d, \mathbb{C}^{l\times l})$.

Definition 3.1. A sequence of vectors $\{\Delta_{\mu,\nu}\boldsymbol{g}\boldsymbol{I}_l\}_{\mu,\nu\in\mathbb{Z}^d}$ is a $MVGF$ for $L^2(\mathbb{R}^d, \mathbb{C}^{l\times l})$ if it satisfies

$$\alpha_0||\mathbf{f}||^2 \leq \sum_{\mu,\nu\in\mathbb{Z}^d} ||\langle\Delta_{\mu,\nu}\boldsymbol{g}\boldsymbol{I}_l, \boldsymbol{f}\rangle||^2 \leq \beta_0||\mathbf{f}||^2, \forall\mathbf{f} \in L^2(\mathbb{R}^d, \mathbb{C}^{l\times l})\ \textit{and}\ \alpha_0,\ \beta_0 > 0.$$

Following is the necessary and sufficient condition for the matrix-valued Gabor system to be the $MVGF$ in $L^2(\mathbb{R}^d, \mathbb{C}^{l\times l})$.

Theorem 3.2. *The matrix-valued sequence $\{\Delta_{\mu,\nu}\boldsymbol{g}\boldsymbol{I}_l\}_{\mu,\nu\in\mathbb{Z}^d}=\{diag(\Delta_{\mu,\nu}g, \Delta_{\mu,\nu}g, ..., \Delta_{\mu,\nu}g)\}_{\mu,\nu\in\mathbb{Z}^d}$ is a MVGF in $L^2(\mathbb{R}^d, \mathbb{C}^{l\times l})$ iff $\{\Delta_{\mu,\nu}g\}_{\mu,\nu\in\mathbb{Z}^d}$ constitutes a Gabor frame in $L^2(\mathbb{R}^d)$.*

Proof. Let $\{\Delta_{\mu,\nu}g\}_{\mu,\nu\in\mathbb{Z}^d}$ be a Gabor frame for $L^2(\mathbb{R}^d)$ with frame bounds α_0, β_0. Then, for any $\boldsymbol{f} = [f_{\mathsf{mn}}]_{1\leq\mathsf{m},\mathsf{n}\leq l}$ in $L^2(\mathbb{R}^d, \mathbb{C}^{l\times l})$, we have

$$\sum_{\mu,\nu\in\mathbb{Z}^d} ||\langle\Delta_{\mu,\nu}\boldsymbol{g}\boldsymbol{I}_l, \boldsymbol{f}\rangle||^2 = \sum_{\mu,\nu\in\mathbb{Z}^d} ||\langle diag(\Delta_{\mu,\nu}g, \Delta_{\mu,\nu}g, ..., \Delta_{\mu,\nu}g), [\mathbf{f}]\rangle||^2 = \sum_{\mu,\nu\in\mathbb{Z}^d}\sum_{1\leq\mathsf{m},\mathsf{n}\leq l} |\langle\Delta_{\mu,\nu}g, f_{\mathsf{mn}}\rangle|^2.$$

Since, $\Delta_{\mu,\nu}g$ is given to be a Gabor frame with its bounds α_0, β_0 in $L^2(\mathbb{R}^d)$ then

$$\sum_{\mu,\nu\in\mathbb{Z}^d} ||\langle\Delta_{\mu,\nu}\boldsymbol{g}\boldsymbol{I}_l, \boldsymbol{f}\rangle||^2 \geq \alpha_0 \sum_{1\leq\mathsf{m},\mathsf{n}\leq l} ||f_{\mathsf{mn}}||^2 = \alpha_0||\boldsymbol{f}||^2.$$

Similarly,

$$\sum_{\mu,\nu\in\mathbb{Z}^d} ||\langle\Delta_{\mu,\nu}\boldsymbol{g}\boldsymbol{I}_l, \boldsymbol{f}\rangle||^2 \leq \beta_0||\boldsymbol{f}||^2.$$

Conversely, let $\{\Delta_{\mu,\nu}\boldsymbol{g}\boldsymbol{I}_l\}_{\mu,\nu\in\mathbb{Z}^d}$ be a $MVGF$ for $L^2(\mathbb{R}^d,\mathbb{C}^{l\times l})$ with its bounds α_0,β_0. Consider an arbitrary but fixed function $f\in L^2(\mathbb{R}^d)$. Then, for

$$\boldsymbol{f}_0=\begin{bmatrix}(f,0,...,0) & (0,0,...,0) & \cdots & (0,0,...,0)\\ (0,0,...,0) & (f,0,...,0) & \cdots & (0,0,...,0)\\ \vdots & \vdots & \cdots & \vdots\\ (0,0,...,0) & (0,0,...,0) & \cdots & (f,0,...,0)\end{bmatrix}.$$

We have

$$\alpha_0 l||f||^2=\alpha_0||\boldsymbol{f}_0||^2\le\sum_{\mu,\nu\in\mathbb{Z}^d}||\langle\Delta_{\mu,\nu}\boldsymbol{g}I_l,\boldsymbol{f}_0\rangle||^2=l\sum_{\mu,\nu\in\mathbb{Z}^d}|\langle\Delta_{\mu,\nu}g,f\rangle|^2\le\beta_0||\boldsymbol{f}_0||^2=\beta_0 l||f||^2.$$

$$\alpha_0||f||^2\le\sum_{\mu,\nu\in\mathbb{Z}^d}|\langle\Delta_{\mu,\nu}g,f\rangle|^2\le\beta_0||f||^2.$$

□

From Theorem 3.2, we obtain that every matrix-valued Gabor system is a Gabor frame for $L^2(\mathbb{R}^d,\mathbb{C}^{l\times l})$ iff it is a Gabor frame in $L^2(\mathbb{R}^d)$. Furthermore, Theorem 3.2 holds for the diagonal matrix-valued functions with non-zero values, and the upcoming example illustrates this result.

Example 3.1. *Consider the frame* $g=\{(1,0,0,0,1),(0,1,0,0,1),(0,0,1,0,1),(0,0,0,1,1),(0,0,0,0,2)\}$ *for* $L^2(\mathbb{R}^5)$ *and the corresponding matrix-valued function*

$$\boldsymbol{g}=\begin{bmatrix}(1,0,0,0,1) & (0,0,0,0,1) & (0,0,0,0,1) & (0,0,0,0,1) & (0,0,0,0,1)\\ (0,0,0,0,1) & (0,1,0,0,1) & (0,0,0,0,1) & (0,0,0,0,1) & (0,0,0,0,1)\\ (0,0,0,0,1) & (0,0,0,0,1) & (0,0,1,0,1) & (0,0,0,0,1) & (0,0,0,0,1)\\ (0,0,0,0,1) & (0,0,0,0,1) & (0,0,0,0,1) & (0,0,0,1,1) & (0,0,0,0,1)\\ (0,0,0,0,1) & (0,0,0,0,1) & (0,0,0,0,1) & (0,0,0,0,1) & (0,0,0,0,2)\end{bmatrix},$$

then for any diagonal matrix-valued function $\boldsymbol{f}$ *in* $L^2(\mathbb{R}^5,\mathbb{C}^{5\times 5})$

$$\boldsymbol{f}=\begin{bmatrix}(f,0,0,0,0) & (0,0,0,0,0) & (0,0,0,0,0) & (0,0,0,0,0) & (0,0,0,0,0)\\ (0,0,0,0,0) & (0,0,0,0,0) & (0,0,0,0,0) & (0,0,0,0,0) & (0,0,0,0,0)\\ (0,0,0,0,0) & (0,0,0,0,0) & (0,0,0,0,0) & (0,0,0,0,0) & (0,0,0,0,0)\\ (0,0,0,0,0) & (0,0,0,0,0) & (0,0,0,0,0) & (0,0,0,0,0) & (0,0,0,0,0)\\ (0,0,0,0,0) & (0,0,0,0,0) & (0,0,0,0,0) & (0,0,0,0,0) & (0,0,0,0,0)\end{bmatrix},$$

then the sequence $\{\Delta_{\mu,\nu}\boldsymbol{g}\boldsymbol{I}_l\}$ *constitutes a frame for any arbitrary but fixed translation parameter* $\nu=(0,0,0,0,1)$ *in* $L^2(\mathbb{R}^5,\mathbb{C}^{5\times 5})$ *with its frame bounds* $|f|^2$.

The following example shows that Theorem 3.2, does not hold good if we take any diagonal entry zero.

Example 3.2. *Consider the functions* $\boldsymbol{f},\boldsymbol{g}\in L^2(\mathbb{R}^d,\mathbb{C}^{l\times l})$ *as*

$$\boldsymbol{f}=\begin{bmatrix}(0,0,...,0) & (0,0,...,0) & \cdots & (0,0,...,0)\\ (g,0,...,0) & (0,0,...,0) & \cdots & (0,0,...,0)\\ \vdots & \vdots & \vdots & \vdots\\ (0,0,...,0) & (0,0,...,0) & \cdots & (0,0,...,0)\end{bmatrix}_{l\times l},$$

and

$$\boldsymbol{g} = \begin{bmatrix} (1,0,...,0) & (1,0,...,0) & \cdots & (1,0,...,0) \\ (1,0,...,0) & (1,1,...,0) & \cdots & (1,0,...,0) \\ \vdots & \vdots & \vdots & \vdots \\ (1,0,...,0) & (1,0,...,0) & \cdots & (1,0,...,1) \end{bmatrix}_{l\times l},$$

where $g = \{(1,0,...,0),(1,1,...,0),...,(1,0,...,1)\}$ *is a frame in* $L^2(\mathbb{R}^d)$ *and the MVGF with translation parameter* $\nu = (1,0,...,0)$ *gives*

$$\sum_{\mu,\nu\in\mathbb{Z}^d} ||\langle \Delta_{\mu,\nu}\boldsymbol{g}\boldsymbol{I}_l, \boldsymbol{f}\rangle||^2 = \sum_{\mu\in\mathbb{Z}^d} \Bigg|\Bigg|\Bigg\langle \aleph\mu \begin{bmatrix} (0,0,...,0) & (0,0,...,0) & \cdots & (0,0,...,0) \\ (0,0,...,0) & (1,0,...,0) & \cdots & (0,0,...,0) \\ \vdots & \vdots & \vdots & \vdots \\ (0,0,...,0) & (0,0,...,0) & \cdots & (1,0,...,0) \end{bmatrix}_{l\times l} .$$

$$\begin{bmatrix} (0,0,...,0) & (0,0,...,0) & \cdots & (0,0,...,0) \\ (g,0,...,0) & (0,0,...,0) & \cdots & (0,0,...,0) \\ \vdots & \vdots & \vdots & \vdots \\ (0,0,...,0) & (0,0,...,0) & \cdots & (0,0,...,0) \end{bmatrix}_{l\times l} \Bigg\rangle\Bigg|\Bigg|^2 .$$

This implies

$$\sum_{\mu,\nu\in\mathbb{Z}^d} ||\langle \Delta_{\mu,\nu}\boldsymbol{g}\boldsymbol{I}_l, \boldsymbol{f}\rangle||^2 = 0.$$

This proves that $\{\Delta_{\mu,\nu}\boldsymbol{g}\boldsymbol{I}_l\}$ *is not a frame in* $L^2(\mathbb{R}^d, \mathbb{C}^{l\times l})$.

In the upcoming example we will observe that the Theorem 3.2 does not hold if we take a non-diagonal arbitrary matrix element in $L^2(\mathbb{R}^4, \mathbb{C}^{4\times 4})$.

Example 3.3. *Consider an arbitrary matrix-valued function*

$$\boldsymbol{f} = \begin{bmatrix} (0,0,0,0) & (0,0,0,0) & (0,0,0,0) & (0,0,0,0) \\ (0,g,0,0) & (0,0,0,0) & (0,0,0,0) & (0,0,0,0) \\ (0,0,0,0) & (0,0,0,0) & (0,0,0,0) & (0,0,0,0) \\ (0,0,0,0) & (0,0,0,0) & (0,0,0,0) & (0,0,0,0) \end{bmatrix},$$

and the matrix-valued frame for $L^2(\mathbb{R}^4)$ *as* $\{(1,0,0,1),(0,1,0,1),(0,0,1,1),(0,0,0,2)\}$ *and the corresponding matrix*

$$\boldsymbol{g} = \begin{bmatrix} (1,0,0,1) & (1,0,0,1) & (0,0,0,1) & (0,0,0,1) \\ (0,0,0,1) & (0,1,0,1) & (0,0,0,1) & (0,0,0,1) \\ (0,0,0,1) & (0,0,0,1) & (0,0,1,1) & (0,0,0,1) \\ (0,0,0,1) & (0,0,0,1) & (0,0,0,1) & (0,0,0,1) \end{bmatrix}.$$

Then for $L^2(\mathbb{R}^4, \mathbb{C}^{l\times l})$, *with* $\nu = (0,0,0,1)$ *we have*

$$\sum_{\mu,\nu\in\mathbb{Z}^d} ||\langle \Delta_{\mu,\nu}\boldsymbol{g}\boldsymbol{I}_l, \boldsymbol{f}\rangle||^2 = \sum_{\mu\in\mathbb{Z}^d} \Bigg|\Bigg|\Bigg\langle \aleph\mu \begin{bmatrix} (1,0,0,0) & (1,0,0,0) & (0,0,0,0) & (0,0,0,0) \\ (0,0,0,0) & (0,1,0,0) & (0,0,0,0) & (0,0,0,0) \\ (0,0,0,0) & (0,0,0,0) & (0,0,1,0) & (0,0,0,0) \\ (0,0,0,0) & (0,0,0,0) & (0,0,0,0) & (0,0,0,1) \end{bmatrix} .$$

$$\begin{bmatrix}(0,0,0,0) & (0,0,0,0) & (0,0,0,0) & (0,0,0,0)\\ (0,g,0,0) & (0,0,0,0) & (0,0,0,0) & (0,0,0,0)\\ (0,0,0,0) & (0,0,0,0) & (0,0,0,0) & (0,0,0,0)\\ (0,0,0,0) & (0,0,0,0) & (0,0,0,0) & (0,0,0,0)\end{bmatrix}\Bigg\rangle\Bigg|\Bigg|^2.$$

This gives

$$\sum_{\mu,\nu\in\mathbb{Z}^d}||\langle\Delta_{\mu,\nu}\boldsymbol{g}\boldsymbol{I}_l,\boldsymbol{f}\rangle||^2=0.$$

This proves that the result does not hold for a non-diagonal matrix-valued function.

Before representing the frame operator in $L^2(\mathbb{R}^d,\mathbb{C}^{l\times l})$ corresponding to the matrix-valued function, we generalizes Theorem 2.10 in the forecoming result, in $L^2(\mathbb{R}^d,\mathbb{C}^{l\times l})$ to obtain the relation between the matrix-valued function and its Fourier coefficients in $d-$ dimensions.

Theorem 3.3. *For some matrix-valued function $\boldsymbol{f}\in L^2(\tau^d,\mathbb{C}^{l\times l})\subset L^2(\mathbb{R}^d,\mathbb{C}^{l\times l})$, where $\tau^d=[0,b]^d, b>0$, the following equality holds:*

$$\sum_{k\in\mathbb{Z}^d}||c_k||^2=\frac{1}{b^d}||\boldsymbol{f}||^2,\ \textit{where } c_k=\frac{1}{b^d}\int_{\tau^d}\boldsymbol{f}(x)e^{-2\pi\iota k.x}dx.$$

Proof. For $\boldsymbol{f}=[f_{\mathsf{mn}}]_{1\le\mathsf{m},\mathsf{n}\le l}\in L^2(\tau^d,\mathbb{C}^{l\times l})\subset L^2(\mathbb{R}^d,\mathbb{C}^{l\times l})$ and $c_k=\frac{1}{b^d}\int_{\tau^d}\boldsymbol{f}(x)e^{-2\pi\iota k.x}dx$. We have

$$\begin{aligned}\sum_{k\in\mathbb{Z}^d}||c_k||^2=\sum_{k\in\mathbb{Z}^d}||\frac{1}{b^d}\int_{\tau^d}\boldsymbol{f}(x)e^{-2\pi\iota k.x}dx||^2&=\sum_{k\in\mathbb{Z}^d}\left|\left|\frac{1}{b^d}\int_{\tau^d}[f_{\mathsf{mn}}]e^{-2\pi\iota k.x}dx\right|\right|^2,\\ &=\sum_{k\in\mathbb{Z}^d}\left|\left|\frac{1}{b^d}\int_{\tau^d}[f_{\mathsf{mn}}.e^{-2\pi\iota k.x}]dx\right|\right|^2,\\ &=\sum_{1\le\mathsf{m},\mathsf{n}\le l}\sum_{k\in\mathbb{Z}^d}\left|\frac{1}{b^d}\int_{\tau^d}f_{\mathsf{mn}}.e^{-2\pi\iota k.x}dx\right|^2.\end{aligned}$$

Therefore, using Theorem 2.10, we have

$$\sum_{k\in\mathbb{Z}^d}||c_k||^2=\sum_{1\le\mathsf{m},\mathsf{n}\le l}\frac{1}{b^d}|f_{\mathsf{mn}}|^2=\frac{1}{b^d}\sum_{1\le\mathsf{m},\mathsf{n}\le l}|f_{\mathsf{mn}}|^2=\frac{1}{b^d}||\boldsymbol{f}||^2.$$

□

In the upcoming theorem, the representation of frame operator for $L^2(\mathbb{R}^d,\mathbb{C}^{l\times l})$ is given.

Theorem 3.4. *Consider the family $\{\Delta_{\mu,\nu}\boldsymbol{g}\}_{\mu,\nu\in\mathbb{Z}^d}$ and $\boldsymbol{g}=[g_{\mathsf{mn}}]$ where $g_{\mathsf{mn}}\in L^2(\mathbb{R}^d)$ such that g_{mn} is $\mathbb{Z}^d-$ periodic. If the family $\{\Delta_{\mu,\nu}\boldsymbol{g}\}$ constitutes a MVGF with bounds A,B then the inequality*

$$tr\langle A\boldsymbol{f},\boldsymbol{f}\rangle\le tr\langle\boldsymbol{f},\sum\boldsymbol{g}\boldsymbol{g}^*\boldsymbol{f}\rangle\le tr\langle B\boldsymbol{f},\boldsymbol{f}\rangle,$$

holds.

Proof. Since

$$S\boldsymbol{f}=\sum_{\mu,\nu\in\mathbb{Z}^d}\langle\boldsymbol{f},\Delta_{\mu,\nu}\boldsymbol{g}\rangle\Delta_{\mu,\nu}\boldsymbol{g}. \tag{3.1}$$

Therefore, using 3.1 we have

$$tr\langle \boldsymbol{f}, S\boldsymbol{f}\rangle = tr\left\langle \boldsymbol{f}, \sum_{\mu,\nu\in\mathbb{Z}^d} \langle \boldsymbol{f}, \Delta_{\mu,\nu}\boldsymbol{g}\rangle \Delta_{\mu,\nu}\boldsymbol{g}\right\rangle = tr\sum_{\mu,\nu\in\mathbb{Z}^d} |\langle \boldsymbol{f}, \Delta_{\mu,\nu}\boldsymbol{g}\rangle|^2.$$

So

$$\begin{aligned} tr\langle \boldsymbol{f}, S\boldsymbol{f}\rangle = \sum_{\mu,\nu\in\mathbb{Z}^d} ||\langle \Delta_{\mu,\nu}\boldsymbol{g}, \boldsymbol{f}\rangle||^2 &= \sum_{\mu,\nu\in\mathbb{Z}^d} \left\| \int_{\tau^d} e^{-2\pi\iota\mu.x}\boldsymbol{g}^*(x-\nu)\boldsymbol{f}(x)dx\right\|^2, \\ &= \sum_{\mu,\nu\in\mathbb{Z}^d} \left\| \int_{\tau^d} e^{-2\pi\iota\mu.x}\boldsymbol{g}^*(x)\boldsymbol{f}(x)dx\right\|^2. \end{aligned}$$

Using Theorem 3.3, we have

$$tr\langle \boldsymbol{f}, S\boldsymbol{f}\rangle = \sum_{\mu,\nu\in\mathbb{Z}^d} \left\| \int_{\tau^d} e^{-2\pi\iota\mu.x}\boldsymbol{g}^*(x)\boldsymbol{f}(x)dx\right\|^2 = \sum_{\nu\in\mathbb{Z}^d} ||\boldsymbol{g}^*\boldsymbol{f}||^2.$$

So

$$tr\langle \boldsymbol{f}, S\boldsymbol{f}\rangle = \sum_{\nu\in\mathbb{Z}^d} ||\boldsymbol{g}^*\boldsymbol{f}||^2 = \sum_{\nu\in\mathbb{Z}^d} tr\langle \boldsymbol{g}^*\boldsymbol{f}, \boldsymbol{g}^*\boldsymbol{f}\rangle = tr\langle \boldsymbol{f}, \sum_{\nu\in\mathbb{Z}^d} \boldsymbol{g}\boldsymbol{g}^*\boldsymbol{f}\rangle.$$

Thus, we have

$$S\boldsymbol{f} = \sum_{\nu\in\mathbb{Z}^d} \boldsymbol{g}\boldsymbol{g}^*\boldsymbol{f}.$$

□

In the upcoming theorems, the Gabor frame is characterized in $L^2(\mathbb{R}^d,\mathbb{C}^{l\times l})$ using a bounded linear operator K. The scalability of Gabor frames is also discussed.

Theorem 3.5. *For an invertible operator $K : L^2(\mathbb{R}^d,\mathbb{C}^{l\times l}) \longrightarrow L^2(\mathbb{R}^d,\mathbb{C}^{l\times l})$, $K \in B(\mathcal{H})$ satisfying $\langle K\boldsymbol{f}, K\boldsymbol{g}\rangle = \langle \boldsymbol{f}, \boldsymbol{g}\rangle$, $\forall \boldsymbol{f}, \boldsymbol{g} \in L^2(\mathbb{R}^d,\mathbb{C}^{l\times l})$, the sequence $\{\Delta_{\mu,\nu}\boldsymbol{g}\}_{\mu,\nu\in\mathbb{Z}^d}$ constitutes a Gabor frame in $L^2(\mathbb{R}^d,\mathbb{C}^{l\times l})$ iff $\{K\Delta_{\mu,\nu}\boldsymbol{g}\}$ is a Gabor frame.*

Proof. Suppose A and B are the frame bounds of the frame $\{\Delta_{\mu,\nu}\boldsymbol{g}\}$ in $L^2(\mathbb{R}^d,\mathbb{C}^{l\times l})$. Then, for some $\boldsymbol{f} \in L^2(\mathbb{R}^d,\mathbb{C}^{l\times l})$. Consider

$$\sum_{\mu,\nu\in\mathbb{Z}^d} ||\langle K\Delta_{\mu,\nu}\boldsymbol{g}, \boldsymbol{f}\rangle||^2 = \sum_{\mu,\nu\in Z^d} ||\langle \Delta_{\mu,\nu}\boldsymbol{g}, K^*\boldsymbol{f}\rangle||^2 \leq B||K^*\boldsymbol{f}||^2 \leq B||K^*||^2||\boldsymbol{f}||^2.$$

Similarly

$$\begin{aligned} ||\boldsymbol{f}||^2 = ||KK^{-1}\boldsymbol{f}||^2 &\leq \frac{||K||^2}{A} \sum_{\mu,\nu\in\mathbb{Z}^d} ||\langle \Delta_{\mu,\nu}\boldsymbol{g}, K^{-1}\boldsymbol{f}\rangle||^2, \\ &= \frac{||K||^2}{A} \sum_{\mu,\nu\in\mathbb{Z}^d} ||\langle K\Delta_{\mu,\nu}\boldsymbol{g}, \boldsymbol{f}\rangle||^2. \end{aligned}$$

This proves $\{K\Delta_{\mu,\nu}\boldsymbol{g}\}$ is a Gabor frame with its bounds $\frac{A}{||K||^2}$ and $B||K^*||^2$. Converse, can be easily proved if $K = I$. □

Definition 3.6. A Gabor frame $\{\Delta_{\mu,\nu}g\}$ is called a scalable Gabor frame for a Hilbert space H if for some sequence of reals $\{a_{\mu,\nu}\}$, the family $\{a_{\mu,\nu}\Delta_{\mu,\nu}g\}$ satisfies

$$\sum_{\mu,\nu\in\mathbb{Z}^d} |\langle f, a_{\mu,\nu}\Delta_{\mu,\nu}g\rangle|^2 = ||f||^2, \forall f \in H.$$

Theorem 3.7. *If $\{\Delta_{\mu,\nu}g : g \in L^2(\mathbb{R})\}$ is a scalable frame corresponding to the sequence $\{a_{\mu,\nu}\}$ for $L^2(\mathbb{R})$. Then, for some $K \in B(H)$ which preserves norm such that K^* is bounded below. Then, $\{Ka_{\mu,\nu}\Delta_{\mu,\nu}g\}$ will be a $K-$ frame.*

Proof. For some $h \in L^2(\mathbb{R})$, consider

$$\sum_{\mu,\nu\in\mathbb{Z}^d} |\langle h, Ka_{\mu,\nu}\Delta_{\mu,\nu}g\rangle|^2 = \sum_{\mu,\nu\in\mathbb{Z}^d} |\langle K^*h, a_{\mu,\nu}\Delta_{\mu,\nu}g\rangle|^2. \tag{3.2}$$

But the sequence $\{\Delta_{\mu,\nu}g\}$ is scalable, so $\{a_{\mu,\nu}\Delta_{\mu,\nu}g\}$ is a Parseval frame, therefore from 3.2, we have

$$\sum_{\mu,\nu\in\mathbb{Z}^d} |\langle h, Ka_{\mu,\nu}\Delta_{\mu,\nu}g\rangle|^2 = ||K^*h||^2 \leq ||K||^2||h||^2.$$

Since K is bounded below, so $a||h||^2 \leq ||K^*h||^2$ for some $a > 0$.
Therefore, we have

$$a||h||^2 \leq ||K^*h||^2 = \sum_{\mu,\nu\in\mathbb{Z}^d} |\langle h, Ka_{\mu,\nu}\Delta_{\mu,\nu}g\rangle|^2.$$

This gives

$$a||h||^2 \leq \sum_{\mu,\nu\in\mathbb{Z}^d} |\langle h, Ka_{\mu,\nu}\Delta_{\mu,\nu}g\rangle|^2.$$

Since K preserves isometry, so using $||Kh||^2 = ||h||^2$, we get

$$a||Kh||^2 \leq \sum_{\mu,\nu\in\mathbb{Z}^d} |\langle h, Ka_{\mu,\nu}\Delta_{\mu,\nu}g\rangle|^2.$$

Also $||K^*h||^2 = ||Kh||^2$, therefore, we get

$$a||K^*h||^2 \leq \sum_{\mu,\nu\in\mathbb{Z}^d} |\langle h, Ka_{\mu,\nu}\Delta_{\mu,\nu}g\rangle|^2 \leq ||K||^2||h||^2.$$

This proves that $\{Ka_{\mu,\nu}\Delta_{\mu,\nu}g\}$ is K frame. □

Cordero [7] generalizes the Gabor frames to MGF. These frames are generated by metaplectic Gabor atom defined in [6]. Inspired from the work of Cordero [6, 7] and L. K. Vashisht [16, 25], here we have defined the matrix-valued MGFs and observed that every matrix-valued metaplectic Gabor system is not a frame for $L^2(\mathbb{R}^d, \mathbb{C}^{l\times l})$. First, we introduce the matrix-valued MGFs.

Definition 3.8. A sequence of vectors $\{\pi_A(\mathrm{y}, \eta)\boldsymbol{gI}_l\}$ is said to be a matrix-valued MGF in $L^2(\mathbb{R}^d, \mathbb{C}^{l\times l})$ if it satisfies

$$\alpha_0||\mathbf{f}||^2 \leq \sum_{\mathrm{y},\eta\in\beth} ||\langle \mathbf{f}, \pi_A(\mathrm{y}, \eta)\mathbf{g}\boldsymbol{I}_l||^2 \leq \beta_0||\mathbf{f}||^2, \forall \mathbf{f} \in L^2(\mathbb{R}^d, \mathbb{C}^{l\times l}).$$

Following is the example which shows that the matrix-valued metaplectic system is not a frame for $L^2(\mathbb{R}^d, \mathbb{C}^{l\times l})$.

Example 3.4. *The metaplectic Gabor atom is defined as:*

$$\pi_A(\mathrm{y},\eta)=\alpha_E e^{-2\pi\iota E_{11}E_{12}^{-T}\eta\cdot\mathrm{y}}\mu_{E_{22}^{-T}\eta}(\nu_{E_{12}E_{22}^{-1}(E_{11}-E_{12}E_{22}{}^{-1}E_{21})\mathrm{y}})\daleth_{E_{22}E_{12}^{-1}},\ \ \mathrm{y},\eta\in\beth,$$

where $\alpha_E=\sqrt{\frac{|E|}{|E_{22}E_{12}|}}$ *and*

$$E=\begin{bmatrix}E_{11} & E_{12}\\ E_{21} & E_{22}\end{bmatrix}.$$

First, we construct a metaplectic atom with

$$E=\begin{bmatrix}2 & 1\\ 0 & 1\end{bmatrix},$$

where

$$E_{11}=\begin{bmatrix}2\end{bmatrix},\qquad E_{12}=\begin{bmatrix}1\end{bmatrix},\qquad E_{21}=\begin{bmatrix}0\end{bmatrix},\qquad E_{22}=\begin{bmatrix}1\end{bmatrix}.$$

Here, $|E|=2,\ |E_{22}E_{12}|=1$*. So,* $\alpha_E=\sqrt{2}$*.*
We observe that $\nu_{E_{12}E_{22}^{-1}(E_{11}-E_{12}E_{22}{}^{-1}E_{21})\mathrm{y}}=\nu_{E_{12}E_{22}^{-1}E_{11}\mathrm{y}}=\nu_{2\mathrm{y}}$ *which gives*

$$\daleth_{E_{22}E_{12}^{-1}}g=\sqrt{|E|}g(E)=\sqrt{2}g\begin{bmatrix}2 & 1\\ 0 & 1\end{bmatrix}.$$

Substituting all

$$\pi_A(\mathrm{y},\eta)g=\sqrt{2}e^{-2\pi\iota\times2\times1\eta\cdot\mathrm{y}}\mu_\eta\nu_{2\mathrm{y}}g\begin{bmatrix}2 & 1\\ 0 & 1\end{bmatrix}.$$

Since $E\in GL(\mathbb{R}^{2d})$*, we have* $d=1$*, so* $(\mathrm{y},\eta)\in\mathbb{R}\times\mathbb{R}$*.*
Let $\mathrm{y}=\frac{1}{2}$ *and* $\eta=1$*, then for some translation invariant function* g*, we have* $\pi_A(\frac{1}{2},1)g=\sqrt{2}e^{-2\pi\iota}\mu_1 g\begin{bmatrix}1 & 0\\ -1 & 0\end{bmatrix}$*.*
Now, for some $f=\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$*,*

$$\langle f,\pi_A(\mathrm{y},\eta)g\rangle=\langle\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix},\sqrt{2}e^{-2\pi\iota}e^{2\pi\iota}g\begin{bmatrix}1 & 0\\ -1 & 0\end{bmatrix}\rangle=\sqrt{2}\langle\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix},\begin{bmatrix}g & 0\\ -g & 0\end{bmatrix}\rangle,$$

$$=\sqrt{2}tr\left\{\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}g & -g\\ 0 & 0\end{bmatrix}\right\}=\sqrt{2}tr\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}=0.$$

This implies $\sum|\langle f,\pi_A(\mathrm{y},\eta)g\rangle|^2=0$*, which means this can not be a frame.*

The preceding analysis demonstrates that not all matrix-valued metaplectic atoms are capable of generating a matrix-valued MGF in $L^2(\mathbb{R}^d,\mathbb{C}^{l\times l})$. This holds for the diagonal matrix-valued functions with non-zero values. Next, we have given a necessary and sufficient condition for matrix-valued MGF.

Theorem 3.9. *The sequence of functions* $\{\pi_A(\mathrm{y},\eta)\boldsymbol{g}\boldsymbol{I}_l\}=\{diag(\pi_A(\mathrm{y},\eta)g,\pi_A(\mathrm{y},\eta)g,...,\pi_A(\mathrm{y},\eta)g)\}$ *is a frame for* $L^2(\mathbb{R}^d,\mathbb{C}^{l\times l})$ *iff* $\{\pi_A(\mathrm{y},\eta)g\}$ *is a MGF for* $L^2(\mathbb{R}^d)$*.*

Proof. Let $\{\pi_A(\mathrm{y},\eta)g\}$ be a *MGF* with A and B as its frame bounds in $L^2(\mathbb{R}^d)$. Consider

$$\sum_{\mathrm{y},\eta\in\beth}||\langle\boldsymbol{f},\pi_A(\mathrm{y},\eta)\mathbf{g}\boldsymbol{I}_l\rangle||^2=\sum_{\mathrm{y},\eta\in\beth}||\langle[f_{\mathsf{mn}}],diag\pi_A(\mathrm{y},\eta)g,...,\pi_A(\mathrm{y},\eta)g\rangle||^2=\sum_{\mathrm{y},\eta\in\beth}\sum_{1\le\mathsf{m},\mathsf{n}\le l}|\langle f_{\mathsf{mn}},\pi_A(\mathrm{y},\eta)g\rangle|^2,$$

$$\le B\sum_{1\le\mathsf{m},\mathsf{n}\le l}|f_{\mathsf{mn}}|^2=B||\boldsymbol{f}||^2.$$

Similarly

$$\sum_{\mathrm{y},\eta\in\beth} ||\langle \boldsymbol{f}, \pi_A(\mathrm{y},\eta)\mathbf{g}\boldsymbol{I}_l\rangle||^2 \geq A||\boldsymbol{f}||^2.$$

Therefore

$$A||\mathbf{f}||^2 \leq \sum_{\mathrm{y},\eta\in\beth} |\langle \mathbf{f}, \pi_A(\mathrm{y},\eta)\mathbf{g}\mathbf{I}_l\rangle|^2 \leq B||\mathbf{f}||^2.$$

Converse one can easily prove. □

Definition 3.10. The $K-MGF$ in $L^2(\mathbb{R}^d)$ is a family of elements $\{\pi_A(\mathrm{y},\eta)g\}$ that satisfies

$$\alpha_0||K^*f||^2 \leq \sum_{\mathrm{y},\eta\in\beth} |\langle f, \pi_A(\mathrm{y},\eta)g\rangle|^2 \leq \beta_0||f||^2,\ \forall f \in L^2(\mathbb{R}^d),\ \alpha_0, \beta_0 > 0.$$

Next, we discuss various properties of $MGFs$ in terms of bounded linear operators.

Theorem 3.11. *Let $\{\pi_A(\mathrm{y},\eta)g\}$ be a MGF in $L^2(\mathbb{R}^d)$ and K is a bounded linear operator on $L^2(\mathbb{R}^d)$ that preserves isometry. Then, $\{K\pi_A(\mathrm{y},\eta)g\}$ is a $K-$ MGF for $L^2(\mathbb{R}^d)$.*

Proof. Let $\{\pi_A(\mathrm{y},\eta)g\}$ be MGF with its frame bounds A and B in $L^2(\mathbb{R}^d)$. Then, for some $f \in L^2(\mathbb{R}^d)$. Consider

$$\sum_{\mathrm{y},\eta\in\beth} |\langle f, K\pi_A(\mathrm{y},\eta)g\rangle|^2 = \sum_{\mathrm{y},\eta\in\beth} |\langle K^*f, \pi_A(\mathrm{y},\eta)g\rangle|^2 \geq A||K^*f||^2. \tag{3.3}$$

Similarly

$$\sum_{\mathrm{y},\eta\in\beth} |\langle f, K\pi_A(\mathrm{y},\eta)g\rangle|^2 \leq B||K^*||^2||f||^2. \tag{3.4}$$

From 3.3 and 3.4, we obtain

$$A||K^*f||^2 \leq \sum_{\mathrm{y},\eta\in\beth} |\langle f, K\pi_A(\mathrm{y},\eta)g\rangle|^2 \leq B||K^*||^2||f||^2.$$

Therefore, $\{K\pi_A(\mathrm{y},\eta)g\}$ is a $K-$ MGF for $L^2(\mathbb{R}^d)$ with its frame bounds A and $B||K^*||^2$. □

Theorem 3.12. *The matrix-valued metaplectic Gabor system*

$$\{K\pi_A(\mathrm{y},\eta)\boldsymbol{g}\boldsymbol{I}_l\} = \{diag(K\pi_A(\mathrm{y},\eta)g, K\pi_A(\mathrm{y},\eta)g, ..., K\pi_A(\mathrm{y},\eta)g)\},$$

where K preserves isometry, is a Gabor frame for $L^2(\mathbb{R}^d, \mathbb{C}^{l\times l})$ with its bound A and $||BK^*||^2$ iff $\{\pi_A(\mathrm{y},\eta)g\}$ is a MGF for $L^2(\mathbb{R}^d)$.

Proof. Let $\{\pi_A(\mathrm{y},\eta)g\}$ be a MGF in $L^2(\mathbb{R}^d)$ with its frame bounds A and B. Then, for some $\boldsymbol{f} \in L^2(\mathbb{R}^d, \mathbb{C}^{l\times l})$, consider

$$\sum_{\mathrm{y},\eta\in\beth} ||\langle \boldsymbol{f}, K\pi_A(\mathrm{y},\eta)\boldsymbol{g}\rangle||^2 = \sum_{1\leq \mathsf{m},\mathsf{n}\leq l}\sum_{\mathrm{y},\eta\in\beth} |\langle f_{\mathsf{mn}}, K\pi_A(\mathrm{y},\eta)g\rangle|^2 = \sum_{1\leq \mathsf{m},\mathsf{n}\leq l}\sum_{\mathrm{y},\eta\in\beth} |\langle K^*f_{\mathsf{mn}}, \pi_A(\mathrm{y},\eta)g\rangle|^2. \tag{3.5}$$

Therefore, using 3.5 we have

$$A\sum_{1\leq \mathsf{m},\mathsf{n}\leq l} |K^*f_{\mathsf{mn}}| \leq \sum_{\mathrm{y},\eta\in\beth} ||\langle \boldsymbol{f}, K\pi_A(\mathrm{y},\eta)\boldsymbol{g}\rangle||^2 \leq B\sum_{1\leq \mathsf{m},\mathsf{n}\leq l} |K^*f_{\mathsf{mn}}|^2 = B||K^*f||^2. \tag{3.6}$$

This gives

$$A||K^*\boldsymbol{f}||^2 \leq \sum_{\mathrm{y},\eta\in\beth} ||\langle \boldsymbol{f}, K\pi_A(\mathrm{y},\eta)\boldsymbol{g}\rangle||^2 = B||K^*f||^2.$$

But K preserves isometry, therefore, we have

$$A||\boldsymbol{f}||^2 \leq \sum_{\mathrm{y},\eta\in\beth} ||\langle \boldsymbol{f}, K\pi_A(\mathrm{y},\eta)\boldsymbol{g}\rangle||^2 = B||K^*\mathbf{f}||^2.$$

Converse one can easily prove. □

In the context of Hilbert spaces, it has been determined that no frame can be universally categorized as either a scalable frame or $K-$ frame. This concept is illustrated in the following examples. Additionally, the study presents the necessary condition for a scalable MGF to be considered as a $K - MGF$. First, we introduce the concept of scalable MGF.

Definition 3.13. The MGF $\{\pi_A(\mathrm{y},\eta)g\}$ for H is said to be scalable if for some sequence of reals $\{a_{(\mathrm{y},\eta)}\}$, the MGF generated by the sequence $\{a_{(\mathrm{y},\eta)}\pi_A(\mathrm{y},\eta)g\}$ satisfies

$$\sum_{\mathrm{y},\eta\in\beth} |\langle f, a_{(\mathrm{y},\eta)}\pi_A(\mathrm{y},\eta)g\rangle|^2 = ||f||^2, \forall f \in H.$$

The forecoming example shows that for every sequence $\{a_{\mathrm{y},\eta}\}$ the scalable frame $\{\pi_A(\mathrm{y},\eta)g\}$ is not a $K - MGF$.

Example 3.5. *Let $\{\pi_A(\mathrm{y},\eta)g\}$ be a $K-MGF$ for $L^2(\mathbb{R}^d)$ with its frame bounds α_0 and β_0. Then, the scalable MGF $\{a_{(\mathrm{y},\eta)}\pi_A(\mathrm{y},\eta)g\}$ is not a $K-$ frame corresponding to the sequence*

$$\{a_{(\mathrm{y},\eta)}\} = \begin{cases} ||\mathrm{y}+\eta||, if\ \mathrm{y} = (n', 0, ..., 0),\ and\ \eta = (0, n', ...0),\ and\ (n' = 1, 2, ...\infty), \\ 0,\ otherwise. \end{cases}$$

Here, in this example, $a_{(\mathrm{y},\eta)} = ||\mathrm{y}+\eta|| = ||(n', n', 0, ..., 0)|| = \sqrt{n'^2 + n'^2} = \sqrt{2}n'$. It is given that

$$\alpha_0||K^*f||^2 \leq \sum_{\mathrm{y},\eta\in\beth} |\langle f, \pi_A(\mathrm{y},\eta)g\rangle|^2 \leq \beta_0||f||^2, \forall f \in L^2(\mathbb{R}^d). \tag{3.7}$$

So, we have

$$\sum_{\mathrm{y},\eta\in\beth} |\langle f, a_{(\mathrm{y},\eta)}\pi_A(\mathrm{y},\eta)g\rangle|^2 = \sum_{\mathrm{y},\eta\in\beth} |\langle f, \sqrt{2}n'\pi_A(\mathrm{y},\eta)g\rangle|^2 = 2n'^2 \sum_{\mathrm{y},\eta\in\beth} |\langle f, \pi_A(\mathrm{y},\eta)g\rangle|^2,\ where\, n' \in \mathbb{N}. \tag{3.8}$$

Thus, from 3.7 and 3.8, we get

$$2n'^2\alpha_0||K^*f||^2 \leq \sum_{(\mathrm{y},\eta)} |\langle f, a_{(\mathrm{y},\eta)}\pi_A(\mathrm{y},\eta)g\rangle|^2 \leq 2n'^2\beta_0||f||^2 \leq 2(n'+1)^2B||f||^2. \tag{3.9}$$

As $n' \longrightarrow \infty$ in 3.9, we observe that $\{a_{(\mathrm{y},\eta)}\pi_A(\mathrm{y},\eta)g\}$ is not a $K - MGF$.
The above discussion leads to the following result that for certain sequences $\{a_{\mathrm{y},\eta}\}$, the scalable MGF $\{a_{(\mathrm{y},\eta)}\pi_A(\mathrm{y},\eta)g\}$ constitutes a $K-$frame.

Theorem 3.14. *If $\{\pi_A(\mathrm{y},\eta)g\}$ is a $K - MGF$ in $L^2(\mathbb{R}^d)$ with α_0, β_0 its bounds then for a semi-normalized sequence $\{a_{(\mathrm{y},\eta)}\}$, the scalable frame generated by the sequence $\{a_{(\mathrm{y},\eta)}\pi_A(\mathrm{y},\eta)g\}$ is a $K-$ frame.*

Proof. Since $\{\pi_A(\mathrm{y},\eta)g\}$ is a $K-MGF$ with its frame bounds α_0,β_0, we have

$$\alpha_0||K^*f||^2 \leq \sum_{\mathrm{y},\eta\in\daleth} |\langle f,\pi_A(\mathrm{y},\eta)g\rangle|^2 \leq \beta_0||f||^2, \forall f\in L^2(\mathbb{R}^d). \tag{3.10}$$

Consider

$$\sum_{\mathrm{y},\eta\in\daleth} |\langle f,a_{(\mathrm{y},\eta)}\pi_A(\mathrm{y},\eta)g\rangle|^2 = \sum_{\mathrm{y},\eta\in\daleth} |a_{(\mathrm{y},\eta)}|^2|\langle f,\pi_A(\mathrm{y},\eta)g\rangle|^2. \tag{3.11}$$

But the sequence $\{a_{(\mathrm{y},\eta)}\}$ is semi-normalized, therefore according to 2.8, there exists constants $a,b>0$ such that $a\leq |a_{(\mathrm{y},\eta)}|^2\leq b$ for all y,η.
So, from 3.11, we have

$$a\sum_{\mathrm{y},\eta\in\daleth} |\langle f,\pi_A(\mathrm{y},\eta)g\rangle|^2 \leq \sum_{\mathrm{y},\eta\in\daleth} |\langle f,a_{(\mathrm{y},\eta)}\pi_A(\mathrm{y},\eta)g\rangle|^2 \leq b\sum_{\mathrm{y},\eta\in\daleth} |\langle f,\pi_A(\mathrm{y},\eta)g\rangle|^2.$$

Now using 3.10 and 3.11, we get

$$a\alpha_0||K^*f||^2 \leq a\sum_{\mathrm{y},\eta\in\daleth} |\langle f,\pi_A(\mathrm{y},\eta)g\rangle|^2 \leq \sum_{\mathrm{y},\eta\in\daleth} |\langle f,a_{(\mathrm{y},\eta)}\pi_A(\mathrm{y},\eta)g\rangle|^2 \leq b\sum_{\mathrm{y},\eta\in\daleth} |\langle f,\pi_A(\mathrm{y},\eta)g\rangle|^2 \leq b\beta_0||f||^2.$$

This gives

$$a\alpha_0||K^*f||^2 \leq \sum_{\mathrm{y},\eta\in\daleth} |\langle f,a_{(\mathrm{y},\eta)}\pi_A(\mathrm{y},\eta)g\rangle|^2 \leq b\beta_0||f||^2.$$

Therefore, we have

$$A'||K^*f||^2 \leq \sum_{\mathrm{y},\eta\in\daleth} |\langle f,a_{(\mathrm{y},\eta)}\pi_A(\mathrm{y},\eta)g\rangle|^2 \leq B'||f||^2,\ \textit{where } A'=a\alpha_0 \textit{ and } B'=b\beta_0. \tag{3.12}$$

Also, if $A'=B'$ in 3.12 then the sequence $\{a_{(\mathrm{y},\eta)}\pi_A(\mathrm{y},\eta)g\}$ is a tight $K-MGF$. It will be Parseval $K-MGF$ if $K=I$. ☐

The following example illustrates that if we take a semi-normalized sequence $\{a_{\mathrm{y},\eta}\}$, then the Theorem 3.14, holds.

Example 3.6. *Consider the semi-normalized sequence*

$$\{a_{\mathrm{y},\eta}\} = \begin{cases} ||\mathrm{y}+\eta||,\ \textit{if } \mathrm{y}=(1,0,...0),\ \textit{and } \eta=(1,0,...,0), \\ 1,\ \textit{otherwise.} \end{cases}$$

and the $K-MGF$ $\{\pi_A(\mathrm{y},\eta)g : g\in L^2(\mathbb{R}^d)\}$, *with its bounds* α_0,β_0. *Then, the sequence* $\{a_{(\mathrm{y},\eta)}\pi_A(\mathrm{y},\eta)g\}$ *generates a* $K-MGF$ *for* $L^2(\mathbb{R}^d)$.

Since

$$1\leq \{a_{(\mathrm{y},\eta)}\}\leq 2, \forall\ \mathrm{y},\eta.$$

It is given that

$$\alpha_0||K^*f||^2 \leq \sum_{\mathrm{y},\eta\in\daleth} |\langle f,\pi_A(\mathrm{y},\eta)g\rangle|^2 \leq \beta_0||f||^2, \forall f\in L^2(\mathbb{R}^d). \tag{3.13}$$

Consider

$$\sum_{\mathrm{y},\eta\in\daleth} |\langle f,a_{\mathrm{y},\eta}\pi_A(\mathrm{y},\eta)g\rangle|^2 = \sum_{\mathrm{y},\eta\in\daleth} |a_{\mathrm{y},\eta}|^2|\langle f,\pi_A(\mathrm{y},\eta)g\rangle|^2. \tag{3.14}$$

Now using equation 3.14, we have

$$\sum_{\mathrm{y},\eta\in\beth} |\langle f, \pi_A(\mathrm{y},\eta)g\rangle|^2 \leq \sum_{\mathrm{y},\eta\in\beth} |\langle f, a_{\mathrm{y},\eta}\pi_A(\mathrm{y},\eta)g\rangle|^2 \leq 4\sum_{\mathrm{y},\eta\in\beth} |\langle f, \pi_A(\mathrm{y},\eta)g\rangle|^2, \forall f \in L^2(\mathbb{R}^d).$$

Therefore, using equation 3.13

$$\alpha_0||K^*f||^2 \leq \sum_{\mathrm{y},\eta\in\beth} |\langle f, a_{\mathrm{y},\eta}\pi_A(\mathrm{y},\eta)g\rangle|^2 \leq 4\beta_0||f||^2.$$

This implies $\{a_{(\mathrm{y},\eta)}\pi_A(\mathrm{y},\eta)g\}$ is $K-MGF$.

4. CONCLUSION

In this article, we studied Gabor and metaplectic Gabor systems in $L^2(\mathbb{R}^d, \mathbb{C}^{l\times l})$. Inspired from the recent research work in the field of $MVGF$ in $L^2(\mathbb{R}, \mathbb{C}^{l\times l})$, MGF, scalable frames, we observe that for the real life problems it would be interesting to observe things in multiple dimensions. Based upon this idea, firstly we define the $MVGF$ in $L^2(\mathbb{R}^d, \mathbb{C}^{l\times l})$. The necessary and sufficient conditions for the matrix-valued Gabor systems to be a frame $L^2(\mathbb{R}^d, \mathbb{C}^{l\times l})$ are obtained. Later on, the frame operator in $L^2(\mathbb{R}^d, \mathbb{C}^{l\times l})$ is defined. Metaplectic Gabor atoms are defined to introduce the $MGFs$, which are a natural extension of Gabor frames [6]. But we observe that most of the problems in signal analysis works on matrix valued functions as an input or output. Therefore, these $MGFs$ are generalized as matrix-valued $MGFs$ in $L^2(\mathbb{R}^d, \mathbb{C}^{l\times l})$. We proved that not every matrix-valued metaplectic Gabor atom generates a frame for $L^2(\mathbb{R}^d, \mathbb{C}^{l\times l})$. Also, the necessary and sufficient conditions are proved for the matrix-valued metaplectic Gabor system to be a frame. We also discussed some properties of Gabor and $MGFs$ under the action of some bounded linear operator K. The necessary and sufficient conditions are proved for the scalable Gabor frame to be $K-$ frame. The importance of scalable frames is very well known in the fields such as image and signal analysis, where stable reconstruction of signals is required. Here, we introduce scalable $MGFs$ and noticed that the $K-$ scalable $MGFs$ are not always a $K-$frame unless for some particular type of sequences. Lastly, the conditions for the sequence $\{a_{(\mathrm{y},\eta)}\}$ is obtained such that $\{a_{(\mathrm{y},\eta)}\pi_A(\mathrm{y},\eta)g\}$ is $K-$ MGF for $L^2(\mathbb{R}^d)$. Some examples and counter-examples are provided to demonstrate the outcomes.

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