ON GENERALIZED RICKART $*$-RINGS

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Abstract: A ring R with an involution $*$ is a generalized Rickart $*$-ring if for all $x \in R$ the right annihilator of x^n is generated by a projection for some positive integer n depending on x. In this work, we introduce generalized right projection of an element in a $*$-ring and prove that every element in a generalized Rickart $*$-ring has generalized right projection. Various characterizations of generalized Rickart $*$-rings are obtained. We introduce the concept of generalized weakly Rickart $*$-ring and provide a characterization of generalized Rickart $*$-rings in terms of weakly generalized Rickart $*$-rings. It is shown that generalized Rickart $*$-rings satisfy the parallelogram law. A sufficient condition is established for partial comparability in generalized Rickart $*$-rings. Furthermore, it is proved that pair of projections in a generalized Rickart $*$-ring possess orthogonal decomposition.
Keywords: generalized Rickart $*$-ring, generalized right projection, projections, generalized weakly Rickart $*$-ring.

1. Introduction

Kaplansky [5] introduced Baer rings and Baer $*$-rings to generalize various properties of AW^*-algebras (i.e., a C^*-algebra which is also a Baer $*$-ring), von Neumann algebras and complete $*$-regular rings. The concept of a Baer $*$-ring arises naturally from the study of functional analysis. For instance, every von Neumann algebra is a Baer $*$-algebra. For recent work on rings with involution, one can refer to [6, 7, 8, 9, 10, 11].

A ring R is said to be *reduced* if it does not contains nonzero nilpotent element. A ring R is said to be *abelian* if its every idempotent element is central. Let S be a nonempty subset of R. We write $r(S) = \{a \in R \,:\, sa = 0, \;\forall\; s \in S\}$, and is called the *right annihilator* of S in R, and $l(S) = \{a \in R \mid as = 0, \;\forall\; s \in S\}$, is the *left annihilator* of S in R. Let R be a ring and $a \in R$, then we write $r(\{a\}) = r(a)$ and $l(\{a\}) = l(a)$. A *$*$-ring* R is a ring equipped with an involution $x \to x^*$, that is, an additive anti-automorphism of the period at most two. An element e of a $*$-ring R is called a *projection* if it is self-adjoint (i.e. $e = e^*$) and idempotent (i.e. $e^2 = e$). Let P be a poset and $a, b \in P$. The join of a and b, denoted by $a \vee b$, is defined as $a \vee b = \sup\{a, b\}$. The meet of a and b, denoted by $a \wedge b$, is defined as $a \wedge b = inf\ \{a, b\}$. In a poset $(P, \leq)$, $a < b$ denotes $a \leq b$ with $a \neq b$. Let R be a $*$-ring and $e, f \in R$ be projections, we say that $e \leq f$ if $e = ef$, this defines a partial order on the set of all projections in R. A $*$-ring R is said to be a *Rickart $*$-ring*, if for each $x \in R$, $r(x) = eR$, where e is a projection in R. For each element x in a Rickart $*$-ring, there is unique projection e such that $xe = x$ and $xy = 0$ if and only if $ey = 0$, called the *right projection* of x, denoted by $RP(x)$. Similarly, the left projection $LP(x)$ is defined for each element x in Rickart $*$-ring. A $*$-ring R is said to be a *weakly Rickart $*$-ring*, if for any $x \in R$, there exists a projection e such that (1) $xe = x$, and (2) if for $y \in R$, $xy = 0$ then $ey = 0$.

In [3], Berberian posed the following open problem.

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Problem 1: Can every weakly Rickart $*$-ring be embedded in a Rickart $*$-ring with preservation of RP's?
In [3] Berberian provided a partial solution to this problem. Subsequently, in [12], authors offered another partial solution. In [10], a more general partial solution to Problem 1 is presented.

Let R be a $*$-ring. The projections e, f in R are said to be *equivalent* (written as $e \sim f$), if there exists $w \in R$ such that $w^*w = e$ and $ww^* = f$ (see [3]). By [3, Proposition 7, page 5], the relation $\sim$ is an equivalence relation on the set of projections in R. A $*$- ring is said to satisfy *parallelogram law* if $e - e \wedge f \sim e \vee f - f$ for every pair of projection e and f whose $e \wedge f$ and $e \vee f$ exists. Projections e and f in a $*$-ring R are said to be *partially comparable* if there exist non-zero projections e_0 and f_0 such that $e_0 \leq e$, $f_0 \leq f$ and $e_0 \sim f_0$. We say that a $*$-ring R has *partial comparability* (PC) if $eRf \neq 0$ implies e and f are partially comparable. Let R be a $*$-ring and e, f be projections in R. We say that e is *dominated* by f, written as $e \lesssim f$, if $e \sim g \leq f$ for some projection $g \in R$. Projections e and f in a $*$-ring R are said to be *generalized comparable* if there exists central projection h such that $he \lesssim hf$ and $(1-h)f \lesssim (1-h)e$. We say that R has *generalized comparability* (GC) if every pair of projection in R is generalized comparable. We say that a $*$-ring R has *orthogonal GC* if every pair of orthogonal projections are generalized comparable. Projections e and f in a $*$-ring R are said to be *very orthogonal* if there exists central projection h such that $he = e$ and $hf = 0$.

In [4], the authors introduced the concept of generalized Rickart $*$-ring. A $*$-ring R is called a generalized Rickart $*$-ring, if, for any $x \in R$, there exists a positive integer n such that $r(x^n) = gR$, for some projection g of R. In generalized Rickart $*$-rings, we also have $l(x^n) = Rh$ for some projection $h \in R$. This indicates that generalized Rickart $*$-rings are left-right symmetric. Generalized Rickart $*$-rings serve as a common generalization of both Rickart $*$-rings and generalized Baer $*$-rings. In [2], M. Ahmadi and A. Moussavi explored the behavior of the generalized Rickart $*$-condition under various constructions and extensions. They also provided examples of generalized Rickart $*$-rings and identified classes of finite and infinite-dimensional Banach $*$-algebras that are generalized Rickart $*$-rings but not Rickart $*$-rings.

In the second section of this paper, we introduce generalized right projection of an element in a $*$-ring and prove that every element of a generalized Rickart $*$-ring has a generalized right projection. Properties of generalized right projection of elements in a generalized Rickart $*$-ring are also studied. We introduce the concept of a generalized weakly Rickart $*$-ring and provide a characterization of generalized Rickart $*$-rings in terms of generalized weakly Rickart $*$-rings. It is shown that the center and corner of a generalized weakly Rickart $*$-ring are themselves generalized weakly Rickart $*$-ring. In section 3, we pose a problem for generalized Rickart $*$-rings analogous to Problem 1 for Rickart $*$-rings and provide a partial solution. In section 4, we prove that generalized Rickart $*$-rings satisfy the parallelogram law. A sufficient condition is established for generalized Rickart $*$-rings to exhibit partial comparability. Furthermore, it is shown that pairs of projections in a generalized Rickart $*$-ring that satisfying the parallelogram law possess orthogonal decomposition.

2. Weakly Generalized Rickart $*$-rings

It is evident that every Rickart $*$-ring is a generalized Rickart $*$-ring. In this section, we first recall examples and results from [2], which provides instances of generalized Rickart $*$-rings that are not Rickart $*$-rings. We then introduce generalized right projection of an element and generalized left projection of an element in a $*$-ring. We prove that every element of a generalized Rickart$*$-ring has a generalized right projection and discuss the properties of these projections. Furthermore, we introduce the class of generalized weakly Rickart $*$-rings and establish characterizations of generalized Rickart $*$-rings.

Example 2.1 ([2], Example 2.8)**.** (i) Let $S = C[x,y]/(x,y)^n$. Then S is a commutative local ring with unique maximal ideal $(\overline{x},\overline{y})$ having index of nilpotency n. The set S with the conjugate map as the involution, is a generalized Rickart $*$-ring but not Rickart $*$-ring.
(ii) Let G be a finite abelian p-group. Then the group algebra $T = F_pG$ is a finite commutative local ring with unique maximal ideal $rad(T)$ having index of nilpotency $|G|$. Let $*$ be the involution on the group ring T defined by $(\sum a_g g)^* = \sum a_g g^{-1}$. Hence T is a generalized Rickart $*$-ring but not Rickart $*$-ring.
(iii) Let S and T be the rings in (i) and (ii) respectively. Let n be a positive integer and let p be a prime. Then the $*$-ring $R = S \oplus T \oplus \mathbb{Z}_{p^n}$ is a generalized Rickart $*$-ring that is not Rickart $*$-ring.
(iv) Let $T = \{a_0 + a_1 i + a_2 j + a_3 k \ : \ a_i \in \mathbb{Z}_2 \text{ for } i = 1,2,3,4\}$ be the Hamilton quaternions over $\mathbb{Z}_2$. Then T is a commutative ring such that every element of T is either invertible or nilpotent. For $\alpha = a_0 + a_1 i + a_2 j + a_3 k$ with $a_0, a_1, a_2, a_3 \in \mathbb{Z}_2$, define $\alpha^* = a_0 - a_1 i - a_2 j - a_3 k$. Then $*$ is an involution for T. Then T is generalized Rickart $*$-ring but not a Rickart $*$ ring.

Definition 2.2. ([1], Definition 5.1). Let R be a ring with unity, and let $n \geq 2$ be an integer. Put $V_n = \sum_{i=1}^{n-1} E_{i,i+1}$. The triangular matrix rings $S_n(R), A_n(R), B_n(R), U_n(R)$, and $V_n(R)$ are define as follows:

$$A_n(R) = RI_n + \sum_{l=2}^{[\frac{n}{2}]} RV_n^{l-1} + \sum_{i=1}^{[\frac{n+1}{2}]} \sum_{j=[\frac{n}{2}]+i}^{n} RE_{ij}.$$

$$B_n(R) = RI_n + \sum_{l=3}^{[\frac{n}{2}]} RV_n^{l-2} + \sum_{i=1}^{[\frac{n+1}{2}]+1} \sum_{j=[\frac{n}{2}]+i-1}^{n} RE_{ij}.$$

$$U_n(R) = RI_n + \sum_{i=1}^{[\frac{n-1}{2}]} \sum_{j=[\frac{n}{2}]+1}^{n} RE_{ij} + \sum_{j=[\frac{n-1}{2}]+2}^{n} RE_{[\frac{n-1}{2}]+1,j}.$$

$$S_n(R) = RI_n + \sum_{i<j} RE_{i,j}, \ V_n(R) = RI_n + \sum_{l=2}^{n} RV_n^{l-1}.$$

Proposition 2.3 ([2], Theorem 4.6)**.** *Let A be an abelian C^*-algebra. If A is a generalized Rickart $*$-ring, then the Banach $*$-algebras $S_n(\mathcal{A}), A_n(\mathcal{A}), B_n(\mathcal{A}), U_n(\mathcal{A})$, and $V_n(\mathcal{A})$ are generalized Rickart $*$-rings but not Rickart $*$-rings. In particular, the Banach $*$-algebras $S_n(C), A_n(C), B_n(C), U_n(C)$, and $V_n(C)$ are generalized Rickart $*$-rings but not Rickart $*$-rings.*

Example 2.4 ([2], Example 4.7)**.** Let A be an infinite-dimensional commutative von Neumann algebra. Then A is a Baer $*$-ring (and hence a Rickart $*$-ring). Now let R be the ring $Sn(C)(An(C), Bn(C), Un(C), orVn(C))$ and put $T = A \oplus R$ (equipped with the max norm). Then T is an infinite-dimensional Banach $*$-algebra and T is a generalized Rickart $*$-ring but not a Rickart $*$-ring.

Following is an example of a finite generalized Rickart $*$-ring that is not a Rickart $*$-ring.

Example 2.5. Let $R = \mathbb{Z}_4$ be a $*$-ring with an identity involution . Here $0, 1$ are only projections in $\mathbb{Z}_4$. Now $r(2) = \{0, 2\} \neq 0\mathbb{Z}_4$ and $r(2) = \{0, 2\} \neq 1\mathbb{Z}_4$. Hence $r(2)$ is not generated by any projection in R. Therefore $\mathbb{Z}_4$ is not a Rickart $*$-ring. Observe that $r(0) = 1\mathbb{Z}_4, r(1) = 0\mathbb{Z}_4$, $r(2^2) = r(0) = 1\mathbb{Z}_4$, and $r(3) = 0\mathbb{Z}_4$. Hence for every $x \in \mathbb{Z}_4, \exists\ n \in \mathbb{N}$ and projection e in $\mathbb{Z}_4$ such that $r(x^n) = e\mathbb{Z}_4$. Therefore $\mathbb{Z}_4$ is a generalized Rickart $*$-ring.

The following is an example of a finite $*$-ring that is not a generalized Rickart $*$-ring.

Example 2.6. Let $R = M_4(\mathbb{Z}_4)$ with transpose as an involution. Then R is not a generalized Rickart $*$-ring (see Example 2.13).

In the following result, we prove that generalized Rickart $*$-rings, always contain the multiplicative identity (unity) element.

Proposition 2.7. *If R is a generalized Rickart $*$-ring then it has unity element.*

Proof. Let R be a generalized Rickart $*$-ring. Since for any $n \in \mathbb{N}$, $r(0^n) = r(0) = R = eR$ for some projection $e \in R$. Therefore for $x \in R$, we have $x = ey$ for some $y \in R$. This implies $ex = e^2y = ey = x$. Thus e is the unity element in R. □

Proposition 2.8. *If R is a generalized Rickart $*$-ring then for every $x \in R$ there exists $n \in \mathbb{N}$ and a projection $e \in R$ such that $x^n e = x^n$; and for $y \in R$, if $x^n y = 0$ then $ey = 0$.*

Proof. As R is a generalized Rickart $*$-ring, for every $x \in R, r(x^n) = gR$ for some $n \in \mathbb{N}$ and for some projection $g \in R$. Let $e = 1 - g$, then $x^n e = x^n(1 - g) = x^n - x^n g = x^n$. Let $y \in R$ and $x^n y = 0$. Then $y \in r(x^n) = gR$, this gives $y = gy$, that is $(1 - g)y = 0$. Therefore $ey = 0$. Thus $x^n y = 0$ implies that $ey = 0$. □

Observe that in the above proposition, the projection e is the smallest projection such that $x^n e = x^n$ (see Proposition 2.14). Now, we introduce the generalized right projection of an element in a $*$-ring.

Definition 2.9. Let R be a $*$-ring and $x \in R$. The projection $e \in R$ is said to be generalized right projection of x denoted by $GRP(x)$ if there exists $n \in \mathbb{N}$ such that $x^n e = x^n$; and for $y \in R$ if $x^n y = 0$ then $ey = 0$.

Similarly, we introduce the generalized left projection of an element in a $*$-ring.

Definition 2.10. Let R be a $*$-ring and $x \in R$. The projection $f \in R$ is said to be generalized left projection of x denoted by $GLP(x)$ if there exists $n \in \mathbb{N}$ such that $fx^n = x^n$; and for $y \in R$ if $yx^n = 0$ then $yf = 0$.

Remark 2.11. By Proposition 2.8, every element of a generalized Rickart $*$-ring possesses a generalized right projection. Similarly, every element of a generalized Rickart $*$-ring possesses a generalized left projection.

Definition 2.12. A $*$-ring R is said to be generalized weakly Rickart $*$-ring if every $x \in R$ possesses a generalized right projection. That is $GRP(x)$ exists for every $x \in R$.

By Remark 2.11, every generalized Rickart $*$-ring is a generalized weakly Rickart $*$-ring. The following is an example of a $*$-ring which is not a weakly generalized Rickart $*$-ring.

Example 2.13. Let $R = M_4(\mathbb{Z}_4)$ and $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \in R$. Suppose $GRP(A) = E$. Then there exits $n \in \mathbb{N}$, such that $A^nE = A^n$; and $A^nB = 0$ implies $EB = 0$. That is $AE = A$; and $AB = 0$ implies $EB = 0$. But $AE = A$ implies $e_{11} + e_{21} + e_{31} + e_{41} = 1$. This gives $e_{11} + e_{12} + e_{13} + e_{14} = 1$, a contradiction. Thus $M_4(\mathbb{Z}_4)$ is not a generalized weakly Rickart $*$-ring.

Proposition 2.14. *Let R be a generalized Rickart $*$-ring.*

(1) *For $x, y \in R$, if $xy = 0$ then $GRP(x)GLP(y) = 0$.*
(2) *If $GRP(x) = e$ then e is the smallest projection such that $x^ne = x^n$ for some $n \in \mathbb{N}$.*

Proof. (1): Let $x, y \in R$ and $GRP(x) = e$, $GLP(y) = f$. Then for some $n \in \mathbb{N}$, $x^ne = x^n$; and $x^nz = 0$ implies $ez = 0$. Also, for some $m \in \mathbb{N}$, $fy^m = y^m$; and $zy^m = 0$ implies $zf = 0$. Suppose $xy = 0$. Then $x^{n-1}xy = 0$ implies $x^ny = 0$. This gives $ey = 0$. Therefore $eyy^{m-1} = 0$ implies $ey^m = 0$. Hence $ef = 0$. That is $GRP(x)GLP(y) = 0$.
(2): Let f be a projection such that $x^nf = x^n$. Then $x^n(e - f) = x^ne - x^nf = 0$. This gives $e(e - f) = 0$, that is $e = ef$. Hence $e \le f$. Therefore e is the smallest projection such that $x^ne = x^n$. □

The converse of Proposition 2.14 (1) is not true.

Example 2.15. Let $R = \mathbb{Z}_{12}$. Observe that $0, 1, 4, 9$ are the only projections in R. Let $x = 2$, $y = 3 \in R$. Since $2^2{\cdot}4 = 4 = 2^2$, we have $GRP(2) = 4$. Also, $9{\cdot}3^2 = 9 = 3^2$, gives $GLP(3) = 9$. Therefore $GRP(2)GLP(3) = 4 \cdot 9 = 0$, but $2 \cdot 3 = 6 \ne 0$. Thus $GRP(x)\ GLP(y) = 0$ but $xy \ne 0$.

In the following result, we provide a condition under which a $*$-subring of a generalized Rickart $*$-ring becomes a generalized Rickart $*$-ring.

Proposition 2.16. *Let R be a generalized Rickart $*$-ring and B be a $*$-subring of R satisfying the following conditions,*

(1) *B has unity element*
(2) *if $x \in B$ then $GRP(x) \in B$.*

Then B is a generalized Rickart $*$-ring.

Proof. Let $x \in B$ and $GRP(x) = e$. Therefore for some $n \in \mathbb{N}$, $x^ne = x^n$; and $x^ny = 0$ implies $ey = 0$. Since $e \in B$. for $y \in B$, $x^ny = 0$ if and only if $ey = 0$ if and only if $y = y - ey$ if and only if $y = (1-e)y$. Therefore $r(x^n) = (1-e)B$. Hence B is a generalized Rickart $*$-ring. □

Let R be a ring and S be a nonempty subset of R, the commutant of S in R, denoted S', is the set of elements of R that commute with every element of S, that is $S' = \{x \in R : xs = sx$, for all $s \in S\}$. We write $S'' = (S')'$, called the bicommutant of S.

In the following result, we provide a condition under which a $*$-subring of a generalized Rickart $*$-ring becomes a generalized Rickart $*$-ring, and generalized right projection of every element in the $*$-subring remains within it.

Proposition 2.17. *Let R be a generalized Rickart $*$-ring and B be a $*$-subring of R such that $B = B''$. Then*

(1) *$x \in B$ implies $GRP(x) \in B$.*
(2) *B is a generalized Rickart $*$-ring.*

Proof. (1): Suppose $x \in B$ thus $x \in R$. Let $GRP(x) = e$. Therefore there exists $n \in \mathbb{N}$ such that $x^n e = x^n$; and for $y \in R$ $x^n y = 0$ implies $ey = 0$. To prove $e \in B = B'' = (B')'$, it is enough to show $ey = ye$ for all $y \in B'$. Now $xy = yx$ gives $x^n y = yx^n$. Therefore $x^n(y - ye) = x^n y - x^n ye = x^n y - yx^n e = x^n y - yx^n = 0$. This implies $e(y - ye) = 0$ and hence $ey = eye$. Replace y by y^*. So $ey^* = ey^* e$, that is $ye = eye = ey$. Therefore $ey = ye$ for all $y \in B'$. Hence $e \in B$, that is $GRP(x) \in B$.
(2): We know for any nonempty subset S of a ring R, $1 \in S'$ and $S' = S'''$. Since $B = B''$ is equivalent to $B = S'$ for some $*$-subset S of R, we have $1 \in B$. By (1), $x \in B$ implies $GRP(x) \in B$. By Proposition 2.16, B is a generalized Rickart $*$-ring. □

Proposition 2.18. *Let R be a generalized Rickart $*$-ring and $x \in R$. Then there exists $n \in \mathbb{N}$ such that $r(x^n) = (1 - GRP(x))R$.*

Proof. Let $x \in R$ and $GRP(x) = e$. Therefore there exist $n \in \mathbb{N}$ such that $x^n e = x^n$; and for $z \in R$, $x^n z = 0$ implies $ez = 0$. We prove that $r(x^n) = (1 - e)R$. Let $y \in r(x^n)$, then $x^n y = 0$, and hence $ey = 0$. Therefore $y = y - ey = (1 - e)y \in (1 - e)R$. Hence $r(x^n) \subseteq (1 - e)R$. Let $w \in (1 - e)R$, then $w = (1 - e)w = w - ew$. Hence $ew = 0$ implies $x^n ew = 0$. Which gives $x^n w = 0$, and hence $w \in r(x^n)$. Therefore $(1 - e)R \subseteq r(x^n)$. Thus $r(x^n) = (1 - e)R = (1 - GRP(x))R$. □

In the following result, we provide a characterization of a generalized Rickart $*$-ring

Proposition 2.19. *A $*$-ring R is generalized Rickart $*$-ring if and only if R has unity element and for each $x \in R$ there exists a projection e such that $r(x^n) = r(e)$ for some $n \in \mathbb{N}$.*

Proof. Suppose R is a generalized Rickart $*$-ring. Let $x \in R$ and $GRP(x) = e$. Therefore for some $n \in \mathbb{N}$, $x^n e = x^n$; and $x^n y = 0$ if and only if $ey = 0$. Hence $y \in r(x^n)$ if and only if $y \in r(e)$ for some $n \in \mathbb{N}$. Thus $r(x^n) = r(e)$. Conversely, suppose R has unity element and for each $x \in R$ there exists a projection e such that $r(x^n) = r(e)$ for some $n \in \mathbb{N}$. Therefore $r(x^n) = r(e) = (1 - e)R$ for some $n \in \mathbb{N}$. Therefore for any $x \in R$, there exists a projection $1 - e$ such that $r(x^n) = (1 - e)R$ for some $n \in \mathbb{N}$. Hence R is a generalized Rickart $*$-ring. □

Following is a characterization of a generalized Rickart $*$-ring in terms of generalized weakly Rickart $*$-rings.

Proposition 2.20. *The following conditions on a $*$-ring R are equivalent.*

(1) *R is a generalized Rickart $*$-ring.*
(2) *R is a generalized weakly Rickart $*$-ring with unity.*

Proof. Suppose R is a generalized Rickart $*$-ring. By Proposition 2.7, R has unity element. Also, by Proposition 2.8, R is a generalized weakly Rickart $*$-ring. Therefore R is a generalized weakly Rickart $*$-ring with unity. Conversely suppose R is a generalized weakly Rickart $*$-ring with unity. Let $x \in R$. Then $GRP(x)$ exists in R. Therefore $r(x^n) = (1 - GRP(x))R$ for some $n \in \mathbb{N}$. Hence R is a generalized Rickart $*$-ring. □

Following is the example of weakly generalized Rickart $*$-ring which is not generalized Rickart $*$-ring.

Example 2.21. Let $R = \{x = (x_1, x_2, \cdots) \mid x_i \in \mathbb{C},\ for\ each\ x,\ there\ exists\ m \in \mathbb{N}\ such\ that\ x_k = 0\ for\ all\ k > m\}$. Then R is a ring with component-wise operations multiplication and addition. Define the involution $*$ on R as, for $x = (x_1, x_2, \cdots) \in R$, $x^* = (\bar{x_1}, \bar{x_2}, \cdots)$. Clearly $e = (e_1, e_2, \cdots) \in R$ is a projection if $e_i \in \{0, 1\}$. Also, the unity element is $u = (1, 1, \cdots)$ and $u \notin R$. Let $x = (x_1, x_2, \cdots) \in R$. Then there exists $m \in \mathbb{N}$ such that $x_k = 0$ for all $k > m$. Let us find $n \in \mathbb{N}$ and projection $e \in R$ such that $x^n e = x^n$ and $x^n y = 0$ implies $ey = 0$. Choose $n = 1$ and define $e = (e_1, e_2, \cdots)$ as $e_i = 1$ if $x_i \neq 0$ and $e_i = 0$ if $x_i = 0$. As x has only finitely many non-zero components, e also has finitely many non-zero components. So $e \in R$. Further, $e_i \in \{0, 1\}$ and $e_i = \bar{e_i}$, thus e is a projection in R. For each component $(xe)_i = x_i e_i$. If $x_i \neq 0$ then $e_i = 1$, so $(xe)_i = x_i \cdot 1 = x_i$. If $x_i = 0$ then $e_i = 0$, so $(xe)_i = 0 = x_i$. Thus, $xe = x$. Now suppose $xy = 0$. This means $x_i y_i = 0$ for all i. If $x_i \neq 0$ then y_i must be 0. If $x_i = 0$ then y_i can be anything. We have $(ey)_i = e_i y_i$. Thus, If $x_i \neq 0$ then $e_i = 1$. so $(ey)_i = 1 \cdot 0 = 0$. If $x_i = 0$ then $e_i = 0$ and hence $(ey)_i = 0$. Therefore, $ey = 0$. Thus R is a weakly generalized Rickart $*$-ring. To show R is not a generalized Rickart $*$-ring. We will find $x \in R$ such that for any $n \in \mathbb{N}$, $r(x^n) \neq eR$ for any projection $e \in R$. Let $x = (0, 1, 0, 0, \cdots) \in R$. We have $x^n = x$ for any $n \geq 1$. Let us find $r(x)$. Suppose $y = (y_1, y_2, \cdots) \in r(x)$. Hence, $xy = 0$. Therefore $(0, 1, 0, 0, \cdots) \cdot (y_1, y_2, \cdots) = (0, 0, \cdots)$. This gives $y_2 = 0$. Thus, $r(x) = \{(y_1, 0, y_3, y_4, \cdots) \mid y_i \in \mathbb{C}\}$. Suppose $r(x) = eR$ for some projection $e \in R$. Let $e_k = 0$ for $k > n_0$. If $y \in r(x)$, then $y \in eR$, thus $y_i = e_i z_i$ for some $z = (z_1, z_2, \cdots) \in R$. As $y_2 = 0$ for any $y \in r(x)$. So $(ez)_2 = e_2 z_2 = 0$. Hence $e_2 = 0$. Also, for $w = (1, 0, 0, \cdots) \in r(x)$, we have $e_1 = 1$. Similarly, $e_3 = 1$. Therefore $e = (1, 0, 1, 1, 1, \cdots)$. This contradicts the fact that e has only finitely many non-zero components. Hence, $r(x^n) \neq eR$ for any projection $e \in R$ and for any $n \in \mathbb{N}$. Thus, R is not a generalized Rickart $*$-ring.

In the following result, we prove that the center of a generalized weakly Rickart $*$-ring is itself a generalized weakly Rickart $*$-ring.

Proposition 2.22. *The center of a generalized weakly Rickart $*$-ring is generalized weakly Rickart $*$-ring.*

Proof. Suppose R is a generalized weakly Rickart $*$-ring. Let $C(R)$ denote the center of R and $x \in C(R)$. We will prove that $GRP(x)$ exists in $C(R)$. Since $x \in R$ and R is a generalized weakly Rickart $*$-ring. Therefore $GRP(x) = e$ exists in R. That is there exist $n \in \mathbb{N}$ such that $x^n e = x^n$; and $x^n y = 0$ implies $ey = 0$. Hence there exist $n \in \mathbb{N}$ such that $e(x^n)^* = (x^n)^*$; and $y^*(x^n)^* = 0$ implies $y^* e = 0$. Since $x \in c(R)$, we have $x^n(r - re) = x^n r - x^n re = x^n r - r x^n e = x^n r - r x^n = x^n r - x^n r = 0$. Therefore $e(r - re) = 0$, that is $er - ere = 0$, which gives $er = ere$. Also, $(r - er)(x^n)^* = r(x^n)^* - er(x^n)^* = r(x^n)^* - e(x^n)^* r = r(x^n)^* - (x^n)^* r = r(x^n)^* - r(x^n)^* = 0$. Hence $(r - er)e = 0$ implies $re = ere$. Therefore $er = re$ for all $r \in R$. Hence $e \in C(R)$, that is $GRP(x) = e \in C(R)$. Thus $C(R)$ is a generalized weakly Rickart $*$-ring. □

The involution $*$ of a ring R is called *weakly proper* if for any $x \in R$, $xx^* = 0$ implies $x^n = 0$ for some $n \in \mathbb{N}$.

Proposition 2.23. *Let R be a generalized weakly Rickart $*$-ring. Then,*

(1) *for each* $x \in R$ *there exist* $n \in \mathbb{N}$ *such that* $r(x^n) \cap (x^*)^n R = \{0\}$.
(2) *the involution* $*$ *is weakly proper.*

Proof. (1): Let $x \in R$. Since R is a generalized weakly Rickart $*$-ring, there exists $n \in \mathbb{N}$ and a projection $e \in R$ such that $x^n e = x^n$; and for $y \in R$, $x^n y = 0$ implies $ey = 0$. We prove that $r(x^n) \cap (x^*)^n R = \{0\}$. Let $y \in r(x^n) \cap (x^*)^n R$. Therefore $x^n y = 0$ and $y = (x^*)^n s$ for some $s \in R$. This gives $y = (x^*)^n s = (x^n)^* s = (x^n e)^* s = e(x^n)^* s = e(x^*)^n s = ey = 0$. Hence $r(x^n) \cap (x^*)^n R = \{0\}$.
(2): Let $xx^* = 0$. Therefore $x^n (x^*)^n = 0$. This gives $e(x^*)^n = 0$. Hence $e(x^n)^* = 0$ implies $(x^n e)^* = 0$, and this gives $(x^n)^* = 0$. Therefore $x^n = 0$. □

Corollary 2.24 ([2], Proposition 2.11)**.** *Let R be a generalized Rickart $*$-ring. Then (i) for each $x \in R$, there exists an integer $n \geq 1$ such that $r(x^n) \cap (x^*)^n R = 0$; (ii) the involution $*$ is weakly proper.*

The following result provides the characterization of a generalized weakly Rickart $*$-ring.

Proposition 2.25. *The following conditions on a $*$-ring R are equivalent.*
(1) *R is generalized weakly Rickart $*$-ring.*
(2) *Involution $*$ is weakly proper and for every $x \in R$ there exist $n \in \mathbb{N}$ and a projection e in R such that $r(x^n) = r(e)$.*

Proof. Suppose R is a generalized weakly Rickart $*$-ring. By Proposition 2.23, involution on R is weakly proper. Let $x \in R$ and $GRP(x) = e$. Let $y \in r(x^n)$. Therefore $x^n y = 0$ implies $ey = 0$, and $y \in r(e)$. Thus $r(x^n) \subseteq r(e)$. Now let $y \in r(e)$. Therefore $ey = 0$, which implies $x^n y = x^n ey = x^n 0 = 0$. Hence $y \in r(x^n)$. Therefore $r(e) \subseteq r(x^n)$. Thus $r(x^n) = r(e)$. Conversely, suppose involution $*$ is weakly proper and for every $x \in R$ there exist $n \in \mathbb{N}$ and a projection e in R such that $r(x^n) = r(e)$. Let $x \in R$. Since $r(x^n) = r(e)$, we have $e(1-e) = 0$ implies $1 - e \in r(x^n)$. Therefore $x^n(1-e) = 0$ implies $x^n = x^n e$. If $x^n y = 0$ then $y \in r(x^n)$. This gives $y \in r(e)$ and hence $ey = 0$. Therefore $e = GRP(x)$ and R is a generalized weakly Rickart $*$-ring. □

In the following result, we prove that the corner of a generalized weakly Rickart $*$-ring is itself a generalized weakly Rickart $*$-ring.

Proposition 2.26. *Let R be a $*$-ring and e be a projection in R. If R is a generalized weakly Rickart $*$-ring then so is eRe.*

Proof. Let $x \in eRe$. Then $x \in R$. Since R is a generalized weakly Rickart $*$-ring, $GRP(x) = f$ exists in R. Therefore there exist $n \in \mathbb{N}$ such that $x^n f = x^n$; and for $y \in R$, $x^n y = 0$ implies $fy = 0$. Since $x \in eRe$, we have $x = exe$. So $x^n e = (exe)^n e = (exe)^n = x^n$.Therefore $x^n(e - f) = x^n e - x^n f = x^n e - x^n$. Hence $x^n(e - f) = 0$, which implies $f(e - f) = 0$ thus $fe = f$. We prove that $GRP(x) = f$ in eRe. As above $x^n f = x^n$. Suppose $x^n(eze) = 0$, then $feze = 0$. Note that $f = ef = efe \in eRe$. Therefore $GRP(x) = f$ in eRe. Thus eRe is a generalized weakly Rickart $*$-ring. □

Proposition 2.27. *Let R be a generalized weakly Rickart $*$-ring and S be a self-adjoint subset of R and $x \in S'$. If $GRP(x) = e$ then $se = ese = es$ for all $s \in S$.*

Proof. Since $x \in S'$, we have $xs = sx$ for all $s \in S$. As $GRP(x) = e$. Then there exists $n \in \mathbb{N}$ such that $x^n e = x^n$; and for $y \in R$, $x^n y = 0$ implies $ey = 0$. We have $x^n s = s x^n$. Now

$x^n(se - es) = x^n se - x^n es = sx^n e - x^n s = sx^n - x^n s = 0$. Therefore $e(se - es) = 0$ implies $ese = es$. Replacing s by s^* we get $es^*e = es^*$. Therefore $ese = se$. Thus $se = ese = es$. ☐

Lemma 2.28. *If R is a generalized weakly Rickart $*$-ring and S is self adjoint subset of R, then S' is a weakly generalized Rickart $*$-ring.*

Proof. Let $x \in S'$ and $GRP(x) = e$ in R. By Proposition 2.27, $se = es$ for all $s \in S$. Hence $e \in S'$. Therefore $GRP(x) = e$ in S'. Thus S' is a generalized weakly Rickart $*$-ring. ☐

3. Unitification of generalized weakly Rickart $*$-ring

Recall the definition of unitification of a $*$-ring given by Berberian [3]. Let R be a $*$-ring. We say that R_1 is a unitification of R, if there exists a ring K, such that,
1) K is an integral domain with involution (necessarily proper), that is, K is a commutative $*$-ring with unity and without divisors of zero (the identity involution is permitted),
2) R is a $*$-algebra over K (i.e., R is a left K-module such that, identically $1a = a$, $\lambda(ab) = (\lambda a)b = a(\lambda b)$, *and* $(\lambda a)^* = \lambda^* a^*$, for $\lambda \in K$ and $a, b \in R$).
3) R is torsion free K-module (that is $\lambda a = 0$ implies $\lambda = 0$ or $a = 0$).
Define $R_1 = R \oplus K$ (the additive group direct sum), thus $(a, \lambda) = (b, \mu)$ means, by the definition that $a = b$ and $\lambda = \mu$, and addition in R_1, is defined by the formula $(a, \lambda) + (b, \mu) = (a + b, \lambda + \mu)$. Define $(a, \lambda)(b, \mu) = (ab + \mu a + \lambda b, \lambda\mu)$, $\mu(a, \lambda) = (\mu a, \mu\lambda)$, $(a, \lambda)^* = (a^*, \lambda^*)$. Evidently, R_1 is also a $*$-algebra over K, has unity element $(0, 1)$ and R is a $*$-ideal in R_1.
Berberian has given a partial solution to Problem 1 as follows.

Theorem 3.1 ([3, Theorem 1, page 31])**.** *Let R be a weakly Rickart $*$-ring. If there exists an involutory integral domain K such that R is a $*$-algebra over K and it is a torsion-free K-module, then R can be embedded in a Rickart $*$-ring with preservation of RP's.*

After 1972, there was little progress made toward the solution of Problem 1. In [14], Thakare and Waphare provided partial solutions, where the condition on the underlying weakly Rickart $*$-rings was relaxed in two distinct ways. For the solution of this open problem, Berberian used the condition that R is a torsion-free left K-module, where K is an integral domain. Thakare and Waphare offered another solution in which the torsion-free condition was replaced with a different condition. They established the following.

Theorem 3.2 ([12, Theorem 2])**.** *A weakly Rickart $*$-ring R can be embedded into a Rickart $*$-ring, provided there exists a ring K such that*

(1) *K is an integral domain with involution,*
(2) *R is $*$-algebra over K, and*
(3) *For any $\lambda \in K - \{0\}$, there exist a projection e_λ that is an upper bound for the set of left projections of the right annihilators of λ, that is if $x \in R$ and $\lambda x = 0$ then $LP(x) \leq e_\lambda$.*

Based on the theory developed in Section 2, we pose the following problem for generalized Rickart $*$-rings, similar to Problem 1.

Problem 2: Can every generalized weakly Rickart $*$-ring be embedded in a generalized Rickart $*$-ring? with preservation of GRP.

For a partial solution to Problem 2, the following results are required.

Proposition 3.3. *If $(a, 0) \in R_1 = R \oplus K$ then $(a, 0)^n = (a^n, 0)$ for all $n \in \mathbb{N}$.*

Proof. We prove the result by using mathematical induction on n. Clearly result holds for $n = 1$. Suppose result is true for $n = k$. That is $(a,0)^k = (a^k,0)$. Consider $(a,0)^{k+1} = (a,0)(a,0)^k = (a,0)(a^k,0) = (aa^k,0) = (a^{k+1},0)$. Thus by method of induction $(a,0)^n = (a^n,0)$ for all $n \in \mathbb{N}$. □

Proposition 3.4. *If* $(a,\lambda) \in R_1$ *and* $n \in \mathbb{N}$ *then* $(a,\lambda)^n = (a^n + \binom{n}{1}a^{n-1}\lambda + \binom{n}{2}a^{n-2}\lambda^2 + \cdots + \binom{n}{n-1}a\lambda^{n-1}, \lambda^n)$.

Proof. We prove the result by using mathematical induction on n. Clearly result is true for $n = 1$. Suppose the result is true for $n = k$. That is $(a,\lambda)^k = (a^k + \binom{k}{1}a^{k-1}\lambda + \cdots + \binom{k}{k-1}a\lambda^{k-1}, \lambda^k)$. Consider $(a,\lambda)^{k+1} = (a,\lambda)^k(a,\lambda) = (a^k+\binom{k}{1}a^{k-1}\lambda+\cdots+\binom{k}{k-1}a\lambda^{k-1}, \lambda^k)(a,\lambda)$ $= (a^{k+1}+\binom{k}{1}a^k\lambda+\cdots+\binom{k}{k-1}a^2\lambda^{k-1}+a\lambda^k+a^k\lambda+\binom{k}{1}a^{k-1}\lambda^2+\cdots+\binom{k}{k-1}a\lambda^k, \lambda^{k+1}) = (a^{k+1}+[\binom{k}{0}+\binom{k}{1}]a^k\lambda+\cdots+[\binom{k}{k-1}+\binom{k}{k}]a\lambda^k, \lambda^{k+1}) = (a^{k+1}+\binom{k+1}{1}a^k\lambda+\binom{k+1}{2}a^{k-1}\lambda^2+\cdots+\binom{k+1}{k}a\lambda^k, \lambda^{k+1})$. Hence by induction, $(a,\lambda)^n = (a^n + \binom{n}{1}a^{n-1}\lambda + \binom{n}{2}a^{n-2}\lambda^2 + \cdots + \binom{n}{n-1}a\lambda^{n-1}, \lambda^n)$ for all $n \in \mathbb{N}$. □

Lemma 3.5. *If a ring* R *has weakly proper involution then the involution on* R_1 *is weakly proper.*

Proof. Since the involution in R is weakly proper. Therefore for $x \in R$, $xx^* = 0$ implies $x^n = 0$ for some $n \in \mathbb{N}$. Let $(a,\lambda) \in R_1$ and $(a,\lambda)(a,\lambda)^* = 0$. This gives $(a,\lambda)(a^*,\lambda^*) = 0$. Therefore $(aa^* + \lambda^*a + \lambda a^*, \lambda\lambda^*) = 0$. Since K is an integral domain, $\lambda\lambda^* = 0$ implies $\lambda = 0$. Hence $(aa^*,0) = 0$. Therefore $aa^* = 0$. Thus $a^n = 0$ for some $n \in \mathbb{N}$. Hence $(a,\lambda)^n = (a,0)^n = (a^n,0) = (0,0) = 0$. So R_1 has weakly proper involution. □

In the following result, we provide a partial solution to Problem 2.

Theorem 3.6. *A generalized weakly Rickart $*$-ring can be embedded in a generalized Rickart $*$-ring provided there exists a ring* K *such that*

1. *K is an integral domain with involution.*
2. *R is $*$-algebra over K.*
3. *For any nonzero $\lambda \in K$, there exists a projection e_λ such that $\lambda x = 0$ implies $GRP(x) \le e_\lambda$.*

Proof. Let $R_1 = R \oplus K$ (the additive group direct sum) with operations as defined above. First we prove that for any self-adjoint element $a \in R$ and $0 \ne \lambda \in K$ there exists largest projection g such that $(ag + \lambda g)^m = 0$ for some $m \in \mathbb{N}$. Let $GRP(a) = e_0$. Then $a^m e_0 = a^m$ and $a^m y = 0$ implies $e_0 y = 0$ for some $m \in \mathbb{N}$. Let e_λ be a projection which exists by the assumption (3). Let e be the largest projection in $\{e_0, e_\lambda\}$. Let $GRP(ae + \lambda e) = h$. Hence there exists $m \in \mathbb{N}$ such that $(ae+\lambda e)^m h = (ae+\lambda e)^m$ and $(ae+\lambda e)^m y = 0$ implies $hy = 0$. Now $e_0 \le e$ implies $e_0 = e_0 e = ee_0$. Let $g = e-h$. Since $(ae+\lambda e)^m e = (ae+\lambda e)^m$, we have $h \le e$, that is $h = he = eh$. Therefore $eg = e - eh = e - h = g$. This gives $g \le e$ and hence $g = eg = ge$. Thus $(ag + \lambda g)^m = (aeg + \lambda eg)^m = (ae + \lambda e)^m g = (ae + \lambda e)^m(e - h) = (ae + \lambda e)^m - (ae + \lambda e)^m h = 0$. To prove g is largest. Suppose $(ak + \lambda k)^m = 0$. We have $e_0 = e_0 e = ee_0$. Since $a^m e_0 = a^m$, $a^m e_0 e = a^m e$, which implies $a^m e_0 = a^m e$. Therefore $ea^m = a^m$, which gives $kea^m k = ka^m k$. Hence $(ke-k)a^m k = 0$. Since $(ak+\lambda k)^m = 0$, we have $(ke-k)\{-\binom{m}{1}a^{m-1}k\lambda k - \cdots - \lambda^m k\} = 0$. Equating coefficient of $a^{m-1}k$, we get $\lambda m(ke - k) = 0$. Therefore $\lambda(ke - k) = 0$. Let $GRP(ke - k) = f$. Then $(ke - k)^n f = (ke - k)^n$ and $(ke - k)^n y = 0$ implies $fy = 0$. Since $(ke - k)^n e = 0$, we have $fe = 0$. Further $\lambda(ke - k) = 0$ implies $\lambda(ke - k)^n = 0$. Which gives

$\lambda(ke-k)^n f=0$. Hence $(ke-k)^n\lambda f=0$. Therefore $f(\lambda f)=0$. Thus $\lambda f=0$. By (3) $GRP(f)\leq e_\lambda\leq e$. Therefore $f\leq e$, that is $f=fe=ef$ (since $GRP(f)=f$). Hence $f=0$. As $(ke-k)^n=(ke-k)^n f$ implies $(ke-k)^n=0$. But $(ke-k)^n=\pm(ke-k)$. Hence $\pm(ke-k)=0$ gives $ke=k$. So $(ae+\lambda e)^m k=(aek+\lambda ek)^m=(ak+\lambda k)^m=0$. Therefore $hk=0$. Hence $kg=k(e-h)=ke-kh=k-0=k$. Thus $k\leq g$. Hence g is the largest projection such that $(ag+\lambda g)^m=0$. Since $(0,1)$ is the unity element of R_1. By above Proposition 2.20 it is enough to show that R_1 is a generalized weakly Rickart $*$-ring. Let $(a,\lambda)\in R_1$.
Suppose $\lambda=0$. Since $(a,0)\in R_1$, we have $a\in R$. As R is a generalized weakly Rickart $*$-ring, $GRP(a)=e$ exists in R. That is for some $n\in\mathbb{N}$, $a^n e=a^n$; and for $y\in R$, $a^n y=0$ implies $ey=0$. We will prove that $GRP(a,0)=(e,0)$. Since $(a,0)^n(e,0)=(a^n,0)(e,0)=(a^n e,0)=(a^n,0)=(a,0)^n$. Suppose $(a,0)^n(b,\mu)=0$. Then $(a^n,0)(b,\mu)=0$. This implies $(a^n b+\mu a^n,0)=0$, that is $a^n b+\mu a^n=0$. This gives $a^n b+\mu a^n e=0$, and hence $a^n(b+\mu e)=0$. Since $a^n y=0$ implies $ey=0$, we have $e(b+\mu e)=0$. Therefore $eb+\mu e=0$. That is $(e,0)(b,\mu)=0$. Hence $GRP(a,0)=(e,0)$.
Now, suppose $\lambda\neq 0$. Then there exists a largest projection g in R such that $(ag+\lambda g)^t=0$ for some $t\in\mathbb{N}$. Note that $(-g,1)$ is a projection in R_1. We prove that $GRP(a,\lambda)=(-g,1)$. Consider $(a,\lambda)^t(-g,1)=(a^t+\binom{t}{1}a^{t-1}\lambda+\cdots+\binom{t}{t-1}a\lambda^{t-1},\lambda^t)(-g,1)=(-a^t g-\binom{t}{1}a^{t-1}\lambda g-\cdots-\binom{t}{t-1}a\lambda^{n-1}g-\lambda^t g+a^t+\binom{t}{1}a^{t-1}\lambda+\cdots+\binom{t}{t-1}a\lambda^{t-1},\lambda^t)=(-(ag+\lambda g)^t+a^t+\binom{t}{1}a^{t-1}\lambda+\cdots+\binom{t}{t-1}a\lambda^{t-1},\lambda^t)=(a^t+\binom{t}{1}a^{t-1}\lambda+\cdots+\binom{t}{t-1}a\lambda^{t-1},\lambda^t)=(a,\lambda)^t$. To prove $(a,\lambda)^t(b,\mu)=0$ implies $(-g,1)(b,\mu)=0$. Let $(a,\lambda)^t(b,\mu)=0$. Then $(a^t+\binom{t}{1}a^{t-1}\lambda+\cdots+\binom{t}{t-1}a\lambda^{t-1},\lambda^t)(b,\mu)=0$. This implies $\lambda^t\mu=0\Rightarrow\mu=0$ (since $\lambda\neq 0$). Therefore $(a^t+\binom{t}{1}a^{t-1}\lambda+\cdots+\binom{t}{t-1}a\lambda^{t-1},\lambda^t)(b,0)=0$.

To prove $(a,\lambda)^t(b,0)^m=0$ implies $(-g,1)(b,0)^m=0$ for some $m\in\mathbb{N}$. Let $GLP(b)=f$, then there exist $m\in\mathbb{N}$ such that $fb^m=b^m$; and for $y\in R$, $yb^m=0$ implies $yf=0$. Therefore $(a,\lambda)^t(b,0)^m=0$. This implies that $(a^t+\binom{t}{1}a^{t-1}\lambda+\cdots+\binom{t}{t-1}a\lambda^{t-1},\lambda^t)(b^m,0)=0$. Therefore $(\{a^t+\binom{t}{1}\lambda a^{t-1}+\cdots+\binom{t}{t-1}\lambda^{t-1}a\}b^m+\lambda^t b^m,0)=0$. Hence $(\{a^t f+\binom{t}{1}\lambda a^{t-1}f+\cdots+\binom{t}{t-1}\lambda^{t-1}af+\lambda^t f\}b^m,0)=0$. This gives $(af+\lambda f)^t b^m=0$. Since $yb^m=0$ implies $yf=0$, we have $(af+\lambda f)^t f=0$. This implies $(af+\lambda f)^t=0$. But g is the largest projection such that $(ag+\lambda g)^t=0$. Therefore $f\leq g$, which gives $f(1-g)=0$. Hence $(-g,1)(b,0)^m=(-g,1)(b^m,0)=(-gb^m+b^m,0)=((1-g)b^m,0)=((1-g)fb^m,0)=(0,0)=0$.

Hence $GRP(a,\lambda)=(-g,1)$. Thus R_1 is a generalized Rickart $*$-ring. □

4. Parallelogram law, Generalized comparability and partial comparability

In this section, we prove that generalized Rickart $*$-ring satisfies parallelogram law. A sufficient condition is provided for a generalized Rickart $*$-ring to exhibit partial comparability. It is shown that a pair of projections in a generalized Rickart $*$-ring that satisfy the parallelogram law possesses orthogonal decomposition.

Proposition 4.1. *Let R be a generalized Rickart $*$-ring such that $GLP(x)\sim GRP(x)$ for all $x\in R$. Then R satisfies the parallelogram law.*

Proof. Let $x=e-ef=e(1-f)$. Then $e\vee f=f+GRP(e(1-f))$. Therefore $GRP(e(1-f))=e\vee f-f$. Also $GLP(e(1-f))=e-e\wedge f$. Since $GLP(x)\sim GRP(x)$, we have $GLP(e(1-f))\sim GRP(e(1-f))$. Therefore $e-e\wedge f\sim e\vee f-f$. Thus R satisfies the parallelogram law. □

Projections e and f are said to be in a *position* p' in case $e \wedge (1-f) = 0$ and $e \vee (1-f) = 1$, that is e and $1-f$ are complementary.

Proposition 4.2. *Let R be a generalized Rickart $*$-ring and e, f are projections in R. Then the following are equivalent.*

(1) *e, f are in position p'.*
(2) *$GLP(ef) = e$ and $GRP(ef) = f$.*

Proof. Suppose e, f are in position p'. Therefore $e \wedge (1-f) = 0$ and $e \vee (1-f) = 1$. Note that $ef = e(1-(1-f))$. Since $e \vee f = f + GRP(e(1-f))$, we have $GRP(ef) = GRP(e(1-(1-f))) = e \vee (1-f) - (1-f) = e \vee (1-f) - 1 + f = f$. Similarly, $GL(ef) = GLP(e(1-(1-f))) = e - e \wedge (1-f) = e$. Conversely suppose $GLP(ef) = e$ and $GRP(ef) = f$. Therefore $GRP(e(1-(1-f))) = f$ implies $e \vee (1-f) - (1-f) = f$. This gives $e \vee (1-f) - 1 + f = f$ that is $e \vee (1-f) = 1$. Similarly $GLP(ef) = e$ implies $e \wedge (1-f) = 0$. Thus e, f are in a position p'. □

Proposition 4.3. *Let R be a generalized Rickart $*$-ring. Then the following are equivalent.*

(1) *R satisfies the parallelogram law.*
(2) *If e, f are in position p' then $e \sim f$.*

Proof. Suppose R satisfies the parallelogram law. Let e, f be projections in a position p'. Thus $e \wedge (1-f) = 0$ and $e \vee (1-f) = 1$. By the parallelogram law $e - e \wedge f \sim e \vee f - f$. Replacing f by $1-f$, we get $e - e \wedge (1-f) \sim e \vee (1-f) - (1-f)$. Hence $e - 0 \sim 1 - 1 + f$, that is $e \sim f$. Conversely, suppose e, f are in a position p' implies $e \sim f$. Let e, f be a pair of projections. Let $GRP(ef) = f'$ and $GLP(ef) = e'$. Since $eff = ef$, we have $f' \leq f$ that is $f' = f'f$. Similarly, since $eef = ef$, we have $e' \leq e$ that is $e' = e'e$. Therefore $ef = e'(ef)f' = (e'e)ff' = e'f'$. Thus $GRP(e'f') = f'$ and $GLP(e'f') = e'$. By Proposition 4.2, e' and f' are in a position p'. Therefore $e' \sim f'$. Note that $e \wedge f = e - GLP(e(1-f))$. Replacing f by $1-f$, we get $e \wedge (1-f) = e - GLP(ef) = e - e'$. Hence $e' = e - e \wedge (1-f)$. Similarly $f' = e \vee (1-f) - 1 + f$. Therefore $e' \sim f'$, which gives $e - e \wedge (1-f) \sim e \vee (1-f) - 1 + f = f - (1-e) \wedge f$. Hence $e - e \wedge (1-f) \sim f - (1-e) \wedge f$. Thus R satisfies the parallelogram law. □

Projections e and f in a $*$-ring R are said to be generalized comparable if there exists central projection h such that $he \lesssim hf$ and $(1-h)f \lesssim (1-h)e$. We say that R has generalized comparability (GC) if every pair of projections is generalized comparable. Projections e and f in a $*$-ring R are said to be very orthogonal if there exists central projection h such that $he = e$ and $hf = 0$

Proposition 4.4. *If projection e and f are very orthogonal in a generalized Rickart $*$-ring R, then e, f are orthogonal, $GRP(e)GLP(f) = 0$ and $eRf = 0$.*

Proof. Suppose e and f are very orthogonal. Therefore their exists central projection h such that $he = e$ and $hf = 0$. Hence $ef = hef = ehf = 0$. Therefore e and f are orthogonal. Further, $GRP(e) = e$ and $GLP(f) = f$. Hence $GRP(e)GLP(f) = ef = 0$. Also, $eRf = heRf = eRhf = 0$. □

Example 4.5. Orthogonal projections need not be very orthogonal. In $\mathbb{Z}_{12}$, the projections $0, 1, 4$, and 9 are all central. Since $2 \cdot 6 = 0$, we have 2 and 6 are orthogonal. But $h \cdot 2 = 2$ and $h \cdot 6 = 0$ does not hold for any central projection h in $\mathbb{Z}_{12}$. Therefore $2, 6$ are not very orthogonal.

Projections e and f in a $*$-ring R are said to be partially comparable if there exists nonzero projections e_0 and f_0 such that $e_0 \leq e, f_0 \leq f$ and $e_0 \sim f_0$. We say that R has partial comparability (PC) if $eRf \neq 0$ implies e, f are partially comparable. A $*$-ring is said to have orthogonal GC if every pair of orthogonal projections is generalized comparable.

Proposition 4.6. *If R is a generalized Rickart $*$-ring with $GRP(x) \sim GLP(x)$ for all $x \in R$ then R has PC.*

Proof. Let e and f be projections in R such that $eRf \neq 0$. Let $x = eaf \in eRf$ be such that $x \neq 0$. Let $GRP(x) = f_0$ and $GLP(x) = e_0$. Then there exists $n \in \mathbb{N}$ such that $x^n f_0 = x^n$; and for $y \in R$, $x^n y = 0$ implies $f_0 y = 0$. Now $xf = x$ gives $x^n f = x^n$. Since $GRP(x) = f_0$, we have $f_0 \leq f$. Similarly $e_0 \leq e$. As $GLP(x) \sim GRP(x)$, we have $e_0 \sim f_0$. Therefore there exist e_0, f_0 such that $e_0 \leq e$, $f_0 \leq f$ and $e_0 \sim f_0$. Therefore R has PC. □

Example 4.7. Let $R = M_2(\mathbb{Z}_3)$ and $e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $f = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \in R$. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in R$ be such that $A^*A = e$ and $AA^* = f$. Therefore $\begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$. This gives $a^2 + c^2 = 1, ab + cd = 0, b^2 + d^2 = 0$ and $a^2 + b^2 = 2, ac + bd = 2, c^2 + d^2 = 2$. If $a = 0$ then $c = 1$ or 2. In both cases $b = 0,\ d = 0$. If $a = 1$ then $b = c = d = 0$. If $a = 2$ then $b = 0,\ c = 0,\ d = 0$. Observe that none of the solution satisfy $a^2+b^2 = 2$. Hence $A^*A = e$ and $AA^* = f$ do not hold for any $A \in R$. Therefore $e \nsim f$.

Example 4.8. Let $R = M_2(\mathbb{Z}_3)$, $e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $f = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$, $1 - e = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $1 - f = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \in R$. Note that $e(1 - f) = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$, $ef = \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}$, $f(1 - e) = \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix}$, $e(1 - e) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $f(1 - f) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Hence $e, f, 1 - e, 1 - f$ are incomparable. For the pair $e, 1 - f$, we have $e - e \wedge (1 - f) = e$ and $e \vee (1 - f) - 1 + f = f$. But $e \nsim f$. Hence for $e, 1 - f$ parallelogram law does not hold. Note that $0, 1$ are only central projections in R. Let e, f as above. For $h = 1, he \not\lesssim hf$ that is $e \not\lesssim f$, because if $e \lesssim f$ then $e \sim g \leq f$ this implies $g = 0$ or $f = 0$ but $e \sim 0$ gives $e = 0$, a contradiction and $g = f$ gives $e \sim f$ a contradiction. Therefore $e \not\lesssim f$. For $h = 0, (1 - h)f \not\lesssim (1 - h)e$ that is $f \not\lesssim e$ as above. Thus R does not have GC. We have $eRf \neq 0$ but e, f do not have non-zero sub-projections e_0, f_0 such that $e_0 \sim f_0$. In fact e and f are only non-zero sub-projections of e, f respectively but $e \nsim f$. Thus R does not have PC.

Proposition 4.9. *Let R be a generalized Rickart $*$-ring satisfying parallelogram law. If e, f are projections in R, then there exists orthogonal decomposition $e = e' + e''$, $f = f' + f''$ with $e' \sim f'$ and $ef'' = fe'' = 0$.*

Proof. Let $GLP(ef) = e'$ and $GRP(ef) = f'$. Then $ef = e'f'$. Therefore $GLP(e'f') = e'$ and $GRP(e'f') = f'$. By Proposition 4.2, we have e' and f' are in position p'. Therefore $e' \sim f'$. Let $e'' = e - e'$, $f'' = f - f'$. Then $e''(ef) = (e - e')ef = ef - e'ef = ef - ef = 0$. Therefore $(e''e)f = 0$. Since $e'' \leq e$, we have $e''f = 0$. Similarly $(ef)f'' = 0$ gives $ef'' = 0$. □

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